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A Bound for the Spectral Radius of a Matrix

by

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It is the purpose of this note to derive a bound for the eigenvalues of a complex matrix  $A = (a_{ij})$  in terms of  $a_{ij}$ . The estimate will be derived by constructing an integral equation with degenerate kernel for which each finite eigenvalue is the reciprocal of a non-zero eigenvalue of  $A$ . The precise result is:

Theorem: Let  $A = (a_{ij})$  be an  $n \times n$  complex matrix and  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of  $A$  (not necessarily distinct) but one of which is  $\neq 0$ , then

$$(1) \quad |\lambda|_{\max}^2 = \max |\lambda_i|^2$$

$$\leq G^2(a_{ij}) \equiv \sum_{l=1}^n (2l-1)^{-1} \{\alpha_{ll} + \beta_{ll}\} + \sum_{\substack{l,k=1 \\ l \neq k}}^n (l+k-1)^{-1} \{\alpha_{lk} + \beta_{lk}\}$$

with

$$(2) \quad \alpha_{lk} = \sum_{j=1}^n (2j-1)^{-1} q_{jl} \bar{q}_{jk}, \quad \beta_{lk} = \sum_{\substack{j,m=1 \\ j \neq m}}^n (j+m-1)^{-1} q_{jl} \bar{q}_{mk}$$

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where

$$(3) \quad a_{k\ell} = \sum_{m=1}^n a_{\ell m} b_{mk}, \quad 1 \leq j, \ell \leq n,$$

with

$$(4) \quad b_{mk} = (k+m-1) \prod_{\substack{p=1 \\ p \neq m}}^n \frac{p+k-1}{p-m} \prod_{\substack{r=1 \\ r \neq k}}^n \frac{r+m-1}{r-k}.$$

Remark:  $(b_{mk})$  is a universal matrix in the sense it is independent of  $A$ .

Before we prove the theorem we shall prove the following.

Lemma: Consider the integral equation

$$(5) \quad \varphi(s) = \mu \int_0^1 K(s, t) \varphi(t) dt$$

where

$$(6) \quad K(s, t) = \sum_{\ell=1}^n s^{\ell-1} q_{\ell}(t)$$

with

$$(7) \quad q_{\ell}(t) = \sum_{j=1}^n q_{j\ell} t^{j-1}$$

and  $q_{j\ell}$  is given by (3). If  $\lambda \neq 0$  is an eigenvalue of  $A$ , then  
 $\mu = \lambda^{-1}$  is an eigenvalue of (5).

Proof of Lemma: First we note that

$$(8) \quad \int_0^1 t^{j-1} q_\ell(t) dt = a_{\ell j}, \quad 1 \leq j, \ell \leq n.$$

This follows since

$$(9) \quad \int_0^1 t^{j-1} q_\ell(t) dt = \sum_{k=1}^n q_{k\ell} \int_0^1 t^{j+k-2} dt \quad \text{by (7)}$$

$$= \sum_{k=1}^n (j+k-1)^{-1} q_{k\ell}$$

$$= \sum_{k=1}^n (j+k-1)^{-1} \sum_{m=1}^n a_{\ell m} b_{mk} \quad \text{by (3)}$$

$$= \sum_{k=1}^n (j+k-1)^{-1} \sum_{m=1}^n (k+m-1) \prod_{\substack{p=1 \\ p \neq m}}^n \frac{p+k-1}{p-m} \prod_{\substack{r=1 \\ r \neq k}}^n \frac{r+m-1}{r-k} a_{\ell m}$$

$$= \sum_{m=1}^n a_{\ell m} \left\{ \sum_{k=1}^n (j+k-1)^{-1} (k+m-1) \prod_{\substack{p=1 \\ p \neq m}}^n \frac{p+k-1}{p-m} \prod_{\substack{r=1 \\ r \neq k}}^n \frac{r+m-1}{r-k} \right\}.$$

Case 1  $m = j$ . The  $j^{\text{th}}$  term of the sum in  $m$  becomes:

$$a_{\ell j} \sum_{k=1}^n \prod_{\substack{p=1 \\ p \neq j}}^n \frac{p+k-1}{p-j} \prod_{\substack{r=1 \\ r \neq k}}^n \frac{r+j-1}{r-k}.$$

If we let

$$\psi_{n,j}(x) = \prod_{\substack{p=1 \\ p \neq j}}^n \frac{p-x}{p-j}, \quad \varphi_{n,k}(x) = \prod_{\substack{r=1 \\ r \neq k}}^n \frac{r-x}{r-k},$$

then the above becomes

$$a_{lj} \sum_{k=1}^n \psi_{n,j}[-(k-1)] \varphi_{n,k}[-(j-1)]$$

in which  $\psi_{n,j}(x)$  is a polynomial of degree  $\leq n-1$  in  $x$  and thus

$$\psi_{n,j}[-(k-1)] = P_{n,j}(k)$$

is a polynomial of degree  $\leq n$  in  $k$ . By the Lagrange identities

$$\sum_{j=1}^n j^m \varphi_{n,j}(x) = x^m, \quad m = 0, 1, \dots, n.$$

Thus:

$$\begin{aligned} (10) \quad \sum_{k=1}^n \psi_{n,j}[-(k-1)] \varphi_{n,k}[-(j-1)] &= \sum_{k=1}^n P_{n,j}(k) \varphi_{n,k}[-(j-1)] \\ &= P_{n,j}[-(j-1)] \\ &= \psi_{n,j}(j) \\ &= 1. \end{aligned}$$

Case 2  $m \neq j$ . For any term, not  $j$ , of the sum in  $m$  we get:

$$(11) \quad \sum_{k=1}^n a_{\ell_m}(j+k-1)^{-1}(k+m-1) \psi_{n,m}[-(k-1)] \varphi_{n,k}[-(m-1)].$$

Note that

$$(j+k-1)^{-1} \psi_{n,m}[-(k-1)] = (j+k-1)^{-1} \prod_{\substack{p=1 \\ p \neq m}} \frac{p+k-1}{p-m}$$

is a polynomial in  $k$  of degree  $\leq n-2$  since  $m \neq j$  and thus  $p = j$  occurs in the product. As in Case 1, since

$$(j+k-1)^{-1}(k+m-1) \psi_{n,m}[-(k-1)]$$

is a polynomial of degree  $\leq n-1$  in  $k$ , we get, using the Lagrange identities that (11) becomes:

$$(12) \quad a_{\ell_m}(j-m)^{-1}(m-m) \psi_{n,m}[m] = 0.$$

Thus combining the results for Case 1 and Case 2 above we get (8).

To complete the proof of the lemma, let  $x = (x_1, x_2, \dots, x_n)^T$  be an eigenvector of  $A$  corresponding to the eigenvalue  $\lambda$ , i.e.

$$(13) \quad Ax = \lambda x.$$

Let

$$(14) \quad \varphi(s) = \sum_{j=1}^n x_j s^{j-1},$$

then  $\varphi(s)$  is  $\neq 0$  and is a solution of

$$\varphi(s) = \lambda^{-1} \int_0^1 K(s,t) \varphi(t) dt.$$

To see this note that by (6)

$$\begin{aligned} \lambda^{-1} \int_0^1 K(s,t) \varphi(t) dt &= \lambda^{-1} \int_0^1 K(s,t) \sum_{j=1}^n x_j t^{j-1} dt \\ &= \lambda^{-1} \sum_{\ell=1}^n s^{\ell-1} \sum_{j=1}^n x_j \int_0^1 q_{\ell}(t) t^{j-1} dt \\ &= \lambda^{-1} \sum_{\ell=1}^n s^{\ell-1} \sum_{j=1}^n a_{\ell j} x_j && \text{by (8)} \\ &= \lambda^{-1} \sum_{\ell=1}^n s^{\ell-1} \lambda x_{\ell} && \text{by (13)} \\ &= \varphi(s) && \text{by (14).} \end{aligned}$$

This completes the proof of the lemma.

Proof of theorem: As is well known from the theory of integral equations all eigenvalues of

$$(15) \quad \varphi(s) = \mu \int_0^1 K(s,t) \varphi(t) dt$$

lie outside the disc

$$|\mu| \|K\|_2 < 1$$

where

$$\|K\|_2^2 = \int_0^1 \int_0^1 |K(s,t)|^2 ds dt,$$

i.e., if  $\mu$  is an eigenvalue of (15) then

$$|\mu|^{-1} \leq \|K\|_2.$$

Thus in particular for

$$|\lambda|_{\max} = \max |\lambda_i| \neq 0$$

of (1) we have

$$|\lambda|_{\max} \leq \|K\|_2.$$

In order to calculate  $\|K\|_2$  note that

$$(16) \quad \sum_{\ell=1}^n s^{\ell-1} q_{\ell}(t) \sum_{k=1}^n s^{k-1} \overline{q_k(t)} = \sum_{\ell=1}^n s^{2\ell-2} q_{\ell}(t) \overline{q_{\ell}(t)} + \sum_{\substack{\ell, k=1 \\ \ell \neq k}}^n s^{\ell+k-2} q_{\ell}(t) \overline{q_k(t)}.$$

But by (7),

$$(17) \quad q_{\ell}(t) \overline{q_{\ell}(t)} = \sum_{j=1}^n q_{j\ell} t^{j-1} \sum_{m=1}^n \overline{q_{m\ell}} t^{m-1} \\ = \sum_{j=1}^n q_{j\ell} \overline{q_{j\ell}} t^{2j-2} + \sum_{\substack{j, m=1 \\ j \neq m}}^n t^{j+m-2} q_{j\ell} \overline{q_{m\ell}},$$

and for  $\ell \neq k$

$$\begin{aligned}
 (18) \quad q_\ell(t) \overline{q_k(t)} &= \sum_{j=1}^n q_{j\ell} t^{j-1} \sum_{m=1}^n \bar{q}_{mk} t^{m-1} \\
 &= \sum_{j=1}^n q_{j\ell} \bar{q}_{jk} t^{2j-2} + \sum_{\substack{j,m=1 \\ j \neq m}}^n t^{j+m-2} q_{j\ell} \bar{q}_{mk}.
 \end{aligned}$$

Thus combining (16), (17) and (18) gives

$$\begin{aligned}
 \|K\|_2^2 &= \int_0^1 dt \int_0^1 ds K(s,t) \overline{K(s,t)} = \\
 &= \int_0^1 dt \left\{ \sum_{\ell=1}^n (2\ell-1)^{-1} q_\ell(t) \overline{q_\ell(t)} + \sum_{\substack{\ell,k=1 \\ \ell \neq k}}^n (\ell+k-1)^{-1} q_\ell(t) \bar{q}_k(t) \right\} \\
 &= \sum_{\ell=1}^n (2\ell-1)^{-1} \left\{ \sum_{j=1}^n (2j-1)^{-1} q_{j\ell} \bar{q}_{j\ell} + \sum_{\substack{j,m=1 \\ j \neq m}}^n (j+m-1)^{-1} q_{j\ell} \bar{q}_{m\ell} \right\} \\
 &\quad + \sum_{\substack{\ell,k=1 \\ \ell \neq k}}^n (\ell+k-1)^{-1} \left\{ \sum_{j=1}^n (2j-1)^{-1} q_{j\ell} \bar{q}_{jk} + \sum_{\substack{j,m=1 \\ j \neq m}}^n (j+m-1)^{-1} q_{j\ell} \bar{q}_{mk} \right\}
 \end{aligned}$$

which concludes the proof of the theorem.

Example: We shall give an estimate, using the above theorem, for the matrix

$$A = \begin{bmatrix} 8 & 4 \\ 1 & \frac{2}{3} \end{bmatrix},$$

$$B = (b_{mk}) = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix},$$



$$Q = (q_{kl}) = (AB)^T = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix},$$

$$\alpha = (\alpha_{lk}) = \begin{bmatrix} 64 & 0 \\ 0 & \frac{4}{3} \end{bmatrix},$$

$$\beta = (\beta_{lk}) = \begin{bmatrix} 0 & 8 \\ 8 & 0 \end{bmatrix},$$

and

$$G^2(a_{ij}) = 64 + \frac{4}{9} + 8 = 72\frac{4}{9} \approx 72.444.$$

Thus  $\lambda_{\max} = \max |\lambda_i| \leq G(a_{ij}) \approx 8.511$  where the actual eigenvalues are

$$\lambda = \frac{13 \pm \sqrt{157}}{3}, \quad |\lambda|_{\max} \approx 8.510.$$

Comparing estimates with some of the familiar bounds [1] for this matrix we see that

$$|\lambda|_{\max} \leq \sum_{i,j=1}^2 |a_{ij}| = 13\frac{2}{3}$$

or

$$|\lambda|_{\max} \leq \left( \sum_{1 \leq i,j \leq 2} |a_{ij}|^2 \right)^{1/2} = 9.024$$

or

$$|\lambda|_{\max} \leq \max_{1 \leq i,j \leq 2} |a_{ij}| \cdot 2 = 16.$$

This shows, for this example, that the estimate of the theorem is the best of the above estimates.

Bibliography

- [1] Taussky, O., and Marcus, M., Hermitian forms and eigenvalues, article in Survey of Numerical Analysis, edited by J. Todd, McGraw-Hill (1962), pp. 283.