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SENSITIVITY REDUCTION THROUGH REOPTIMIZATION

By Fredrick James Taylor

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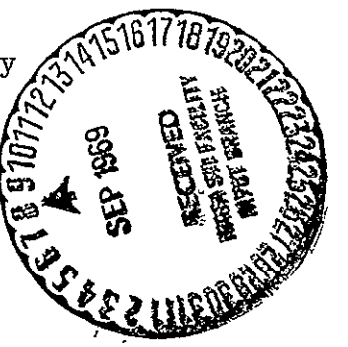

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by

Fredrick James Taylor

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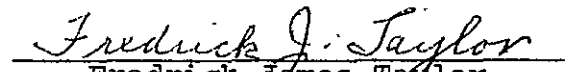

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SENSITIVITY REDUCTION THROUGH REOPTIMIZATION

Abstract

The ability to maintain an optimal solution independent of parameter variations is philosophically appealing. Unfortunately, whenever the internal system parameters change in value, the system usually no longer operates in an optimal fashion. Therefore, the purpose of this research is to devise a scheme which will continually adjust its control strategy in such a manner that the control remains optimum for a set of parameter variations. To accomplish this, the Maximum Principle is applied to a truncated Taylor series representation of the Hamiltonian of the system with parameter variations. An adaptive control strategy is thereby derived. However, the adaptive structure requires plant identification, so special attention is given to this problem.

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CHAPTER I

INTRODUCTION

1.1 Historical Background

Progress in the sensitivity problem has not in general kept pace with the general advances achieved within the framework of optimal control. Briefly, the optimization problem involves transferring the state of a given system from some initial state to a given target set under various constraints, in such a manner as to minimize a given cost functional. The need for a sensitivity analysis results from the structure of the system's dynamical model $\dot{x}(t) = f(x(t),u(t),\alpha)$, where $x(t)$ is the state of the system, $u(t)$ the control effort, and α the parameter vector of the system. This model represents an ideal plant where the parameters are assumed to be known exactly. Such parameters shall be called the "nominal parameters." However, the plant parameters may change in value during their life span, or even if fixed, their precise values may not be known. Therefore, the dependence of the state dynamical model upon parameter values gives rise to a performance functional, state, and optimal control strategy which are dependent upon those parameter values.

Considerable progress has been made in the design of linear autonomous systems with controlled sensitivity to parameter uncertainty. The foundation work for this class of problems is due to Bode[1], with applications and exten-

sions by Horowitz[2,3]. Some methods have been presented in the field of optimal control systems which are subject to parameter variations. There are essentially two mutually exclusive philosophies being pursued in this field. They are[4]:

The study of

(1) Performance Sensitivity

Sensitivity reduction by considering the dependence of the cost functional on the parameters.

(2) Trajectory Sensitivity

Sensitivity reduction by considering the dependence of the states on the parameters.

1.2 Performance Sensitivity

Consider any optimal control law which may be implemented in either an open-loop or closed-loop structure. The question of which structure offers the smallest variations in cost for given parameter variations was posed by Dorato[5]. From the study of classical control systems, the notion of sensitivity reduction to parameter variations with the implementation of particular feedback laws was developed. The sensitivity index considered was usually equivalent to

$$S_a^T = \frac{\partial T}{T} \Big/ \frac{\partial a}{a} \qquad (1\text{-}1)$$

which is a ratio of the change in transfer function T per change in parameter value α. For the open-loop configuration $S_a^T = 1$.

For certain feedback mechanizations S_α^T can be made less than unity implying a reduction in sensitivity. Pagurek[6] investigated an analogue of the classical sensitivity problem. He considered the cost functional

$$C(u,\alpha) = \int_{t_o}^{T} L(x(t),u(t),\alpha)\,dt \qquad (1\text{-}2)$$

Let $\delta C(u,\alpha)$ represent the first variation of the cost functional. If the plant's initial conditions are known and the target set is the whole state space, then Pagurek stated that

$$\delta C(u,\alpha)_o - \delta C(u,\alpha)_c = 0 \qquad (1\text{-}3)$$

where the subscripts o and c denote open- and closed-loop quantities. That is, the cost index sensitivity to parameter change is the same for both open- and closed-loop transformations provided, of course, the parameter variations are infinitesimally small. The question of how much the cost index changes from the nominally optimal cost was left unresolved. Although Pagurek's results were for a special case, a more general result, according to Sorbal, was given by Kokotonic and Sannuit[7]. It was found that (1-3) need not be zero in general.

Another performance sensitivity idea considered a game theoretic approach. Rohrer and Sobral[8] defined the "relative sensitivity index" for a control u(t) to be

$$S^{R}(u(t),\alpha) = \frac{C(u(t),\alpha) - C(u^{o}(t),\alpha)}{|C(u^{o}(t),\alpha)|} \tag{1-4}$$

where $C(u(t),\alpha)$ is defined by (1-2) and $u^{o}(t)$ is the optimal control for the plant having parameters α. That is,

$$C(u^{o}(t),\alpha) = \min_{u}\left\{C(u(t),\alpha)\right\}. \tag{1-5}$$

The "plant sensitivity" is chosen to be a quantitative index of the deviations of $C(u(t),\alpha)$ from $C(u^{o}(t),\alpha)$ and is defined to be

$$S^{P}(u(t)) = \max_{\alpha}\left\{S^{R}(u(t),\alpha)\right\}. \tag{1-6}$$

The optimal design criterion becomes $u(t) = u^{*}(t)$, where

$$\begin{aligned} S_{p}(u^{*}(t)) &= \min_{u}\left\{S^{P}(u(t))\right\} \\ &= \min_{u}\max_{\alpha}\left\{S^{R}(u(t),\alpha)\right\}. \end{aligned} \tag{1-7}$$

Essentially what is being accomplished is $u(t)$ is chosen to make $C(u(t))$ as close as possible to the optimal value of $C(u(t))$ at all values of α.

1.3 Trajectory Sensitivity

In this method there is generated a trajectory in the solution space which is "least" sensitive to parameter variations. This is accomplished by constructing an augmented cost index. Instead of the cost functional being a function of $x(t)$, $u(t)$, t, the augmented cost functional is a function of $x(t)$, $u(t)$, t, and a term which relates the change in plant trajectories to parameter variations. For example, the cost index which is a function of $x(t)$, $u(t)$, t, and $\partial x(t)/\partial\alpha$. A trajectory from the

solution space which is least sensitive to parameter variations will not in general produce a trajectory which is optimal with respect to the original unaugmented cost criterion. Trajectory sensitivity methods have been investigated by Kahne[9], D'Angelo, Moe, and Hendricks[10], Bradt[11], and others.

As the two sensitivity classes differ philosophically, so do their applications. Performance sensitivity methods maintain the identity of the original cost functional and therefore the results achieved relate to some optimal solution. Trajectory sensitivity concerns itself with minimizing, in a sense, a cone of trajectories about a trajectory which is optimal with respect to an augmented cost criterion.

The last technique described in the performance sensitivity section is closely allied to an idea proposed by Kokotovic and Heller*. They have adopted an approach which preserves the concept of optimality in the sense that the control law minimizes some given unaugmented cost index

$$C(u) = \int_{t_o}^{T} L(x(t), u(t), t)\, dt. \qquad (1\text{-}8)$$

The cost index defined by (1-8) is aesthetically pleasing in that most physically meaningful optimal control problems have a cost index of this form. Their objective was to develop a system which attempts to be optimal for "small"

*Notes, 1967

parameter variations. Their approach postulated the a-priori control law

$$* \quad \delta u = C_1 \delta x + C_2 \Delta a$$

where C_1 and C_2 are obtained via the Maximum (Minimum) Principle. That is, the control δu, which was called the "optimally sensitive control" will tend to minimize (1-8) under the influence of small parameter variations. Besides the loss in generality due to requiring that the feedback control have the structure of (*), the authors left several important facets of the problem unanswered. They are:

(1) How "large" may the allowable parameter variations be?

(2) Is the system's cost of operation (i.e., (1-8)) less sensitive to parameter variations than its non-adaptive counterpart (i.e., using the fixed nominally optimal control only)?

(3) Under what conditions will the adaptive structure result in a cost which is equal to, or close to, the true minimal cost of operation for a system subject to parameter variations?

Also, the authors' implementation restricted the number of parameters considered to be equal to the number of states considered. In such cases (see Figure 1.1)

$$\Delta a = f(\delta x).$$

However, the authors state that this dimensionality

restriction may be overcome.

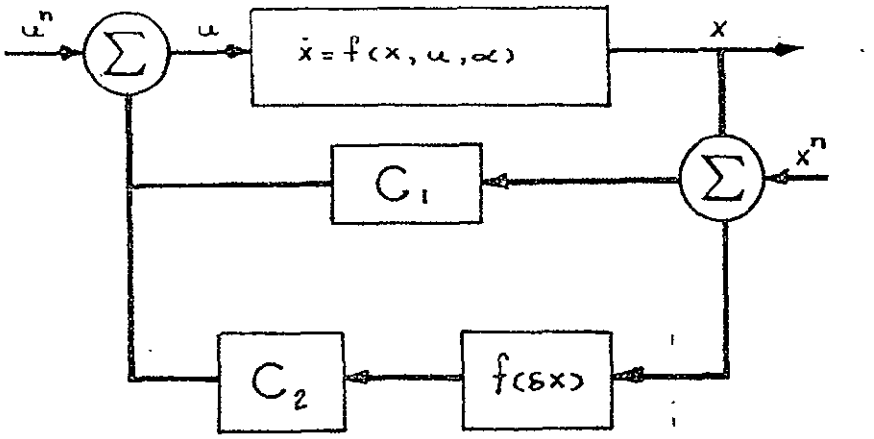

Figure 1-1

Adaptive Controller

1.4 Objectives and Methods

The objective of this investigation is to develop a design method which will operate optimally (or if not, arbitrarily close to optimal) over some allowable set of parameter variations. Various types of parameter variations will be investigated and given a unified analysis. The cost functional considered will be that cost functional associated with the nominal problem.

The problem will be formulated in a general mathematical sense with all structural forms (i.e., constraints and control strategy) being a derived result instead of being assumed a priori. Arguments pertaining to known optimal solutions and their neighboring solutions will be used to develop a general Hamiltonian system of equations valid over a set of admissible parameters. The desired control law will be obtained from a set of necessary

conditions placed on the general Hamiltonian system. This control law will be found to minimize (1-8) over the set of well-defined admissible parameter variations.

Parameter estimation will be found necessary to mechanize the derived system. Therefore, questions relative to parameter estimation and dimensional restrictions will be explored. Various computational devices will be developed to satisfy the parameter estimation condition.

Error analysis techniques will be employed to establish bounds on the allowable parameter variations and cost index variations. Finally, it will be shown that if certain conditions are satisfied, the derived system is less sensitive to parameter variations (in the performance sensitivity sense) than its nominally optimal controlled counterpart.

1.5 Notation and Symbols

The following symbols will be used throughout the study:

x	state vector
α	parameter vector
α^n	nominal parameter vector
α^*	actual parameter vector
α^e	estimated parameter vector
α_i	parameter, real
p	costate vector
u	control vector

y	augmented state vector
$H(\cdot)$	Hamiltonian function
$C(u)$	cost index (cost function or performance index)
$\dot{z}$	$\frac{dz}{dt}$

Spaces

E^q	Euclidian q dimensional space
$C^i\|I\|$	The class of all functions with i continuous partial derivatives with respect to all arguments on the real interval $t \varepsilon I$
$P\|I\|$	The class of all piecewise continuous functions
$\|\ \|$	Norm
$<,>$	non-degenerate inner product
$'$	Transpose
$o(\varepsilon)$	"order" of ε
$[\![v]\!]$	Largest integer smaller than or equal to v

Variable identification (superscripts)

*	optimal variable
e	adaptive (approximate optimum) variable
n	nominal variable
-	actual system variable

CHAPTER II

FORMULATION OF THE PROBLEM

2.1 Necessary Conditions for Optimization

Consider the autonomous plant

$$\dot{x}(t) = f(x(t), u(t), \alpha), \quad t \in [t_o, T]$$

where: $x(t)\ E^n$, $x(t_o) = x_o$ (2-1)

$$\Omega \triangleq \left\{ u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{bmatrix} \;\middle|\; \begin{array}{l} u_i(t) \text{ is an admissible measurable} \\ \text{control function on } t \in [t_o, T], \\ i = 1, \ldots, r \end{array} \right\} *$$

α, the parameter vector will be formally defined as:

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} \in \mathcal{A} \subset E^m$$

and $\mathcal{A} = \{\alpha_i \mid \alpha_i$ is an admissible measurable parameter of the system defined in (2-1), $i = 1, \ldots, m\}$. The performance index $C(u)$ is defined to be:

$$C(u) = \tfrac{1}{2} \langle x(T), T x(T) \rangle + \int_{t_o}^{T} L(x(t), u(t))\, dt \qquad (2\text{-}2)$$

where $\langle x(T), T\, x\, (T) \rangle$ is the terminal cost functional and T is a diagonal positive definite matrix (notice it is assumed to be quadratic).

Necessary conditions for finding a $u(t) \in \Omega$, which transfers the state of system defined by (2-1) from x_o to some target set S at the terminal time T, T given, with:

$$f \in C[t_o, T],\ f:\ E^n \times \Omega \times \mathcal{A} \to E^n,\ L \in C[t_o, T],\ L:\ E^n \times \Omega \to E^n$$

---

*Measurable functions will be considered to be bounded. This does not allow for the existence of the Dirac delta distributions because distributions are not functions.

such that C(u) is minimized, subject to the differential side constraint (2-1), are well known from Maximum (Minimum) Principle[12]. The Maximum Principle states that for a u*(t) to be an optimal control, in that min C(u) = C(u*), is that there exists a nontrivial p*(t) such that for

$$H(x,p,u;a) \triangleq L(x,u)+\langle p, f(x,u;a)\rangle \qquad (2\text{-}3)$$

and $t\in[t_o,T]$, $p(t)\ \varepsilon E^n$

(i) p*(t), x*(t), and u*(t) are solutions of the canonical equations

$$\dot{x}^*(t)=\frac{\partial H(x^*(t),p^*(t),u^*(t))}{\partial p}\quad, x^*(t_o)=x_o \qquad (2\text{-}4)$$

$$\dot{p}^*(t)=-\frac{\partial H(x^*(t),p^*(t),u^*(t))}{\partial x} \qquad (2\text{-}5)$$

(ii) $\min_{u\varepsilon\Omega} H\ (x^*(t),\ p^*(t),\ u(t)) = H(x^*(t),\ p^*(t),\ u^*(t))$ (2-6)

(iii), and $p^*(t)$ satisfies the usual transversality conditions dictated by the target set S.*

Conditions i, ii, and iii are only necessary and not sufficient. For example, there exists cases where i, ii, and iii are satisfied but the satisfying u(t) is not optimal but a singular control belonging to Ωis. However, for the purposes of this investigation it shall be assumed that a unique nonsingular optimal control always exists and that it satisfies the Maximum Principle.

To establish a more definite understanding about the character of the plant considered in (2-1), and therefore the problem, the set of admissible parameters must be

---

*For a listing of transversality conditions for the defined problem see pages 306-307 of 12.

explored more fully. The set of admissible parameters should accept a wide spectrum of interpretations. By this it is meant that the admissible parameters should be applicable to all systems where the parameters are considered to be, for example, initial conditions, or plant coefficients, or perhaps a combination of the two. Fortunately, there exists a property of ordinary differential equations which will establish an equivalency between these three parameter cases.

2.2 Set of Admissible Parameters

Consider the continuity aspects of a system of first order ordinary differential equations for the following three cases:

For $x \in E^n$

(i) $\dot{x}(t) = f(t,x(t))$, $x(t_o) = y$

y considered to be a parameter vector, $y \in E^n$.

(ii) $\dot{x}(t) = f(t,x,\alpha)$, $x(t_o) = x_o$ fixed

$\alpha = (\alpha_1, \ldots, \alpha_m)$ to be considered as a parameter vector, $\alpha \in E^m$.

(iii) Combinations of (i) and (ii)

It is known that under the proper reduction routine[13]:

$$(i) \Longleftrightarrow (ii) \Longleftrightarrow (iii)$$

Moreover, these proper reductions preserve all the continuity properties of the original system considered.

Proof:

$$(i) \Longrightarrow (ii)$$

Let $x = z - y$, $x(t)$ a solution of (i). Then $\dot{z} = \dot{x}$ for y

constant. Therefore, $\dot{z} = \dot{x} = f(t,z(t) - y) \triangleq g(t,z,y)$ and $z(t_o) = 0$. Continuity and the smoothness of (i) are preserved under this linear transformation.

(ii) $\Longrightarrow$ (i). Let $z = (x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m)'$ and

$$z(t_o) = (x_1^o, \ldots, x_n^o, \alpha_1, \ldots, \alpha_m)'$$

define $\dot{z} = \begin{bmatrix} f(t,x,\alpha) \\ 0 \end{bmatrix} \triangleq g(t,z)$.

Again smoothness is preserved; i.e., if $f \in C^q[t_o,T]$ then $g \in C^q[t_o,T]$.

The other implications follow: #

Because there exists an equivalence between (i), (ii), and (iii), a problem formulated in any of the three classes may be reduced to any chosen class. Therefore, this investigation will only consider problems posed in class (ii).

For the sake of completeness, t_o may also be considered to be a parameter in (i), and therefore in class (ii) or (iii), by increasing the dimension of (i) by one.

Proof:

First convert the nonautonomous n^{th} order ordinary differential system of equations to a (n+1) order ordinary autonomous system of differential equations by letting

$$\dot{x}_{n+1}(t) = 1, \; x_{n+1}(t_o) = t_o.^{\dagger}$$

Define: $z(t) = (x_{n+1}(t), x_1(t), \ldots, x_n(t))'$

Define: $g(t,x) = (1, f(t,x))'$

---

†By virtue of this transformation only autonomous systems need be considered, realizing that a nonautonomous system can be reduced to an autonomous equivalent.

Then $z(\tau=0) = (0, x_1(0), \ldots, x_n(0))$

$$\dot{z}(\tau) = (1, \dot{x}_1(\tau), \ldots, \dot{x}_n(\tau))' = g(z(\tau))$$

$$z(\tau = t_o) = (x_{n+1}(\tau = t_o), x(\tau = t_o)) = (t_o, x_o)$$

Thus, the continuity properties of a system of ordinary differential equations with respect to parameters admits a specialized analysis of a class (ii) problem without ignoring study of all other possible cases.

However, if one considers a plant to be parameterized by vectors $\alpha \in \mathcal{A}$, then the optimization problem, over $\mathcal{A}$, would have to be accomplished the cardinal number of $\mathcal{A}$ times. This means that for every parameter vector in $\mathcal{A}$ the optimal control would have to be computed and this computation performed for all such vectors in $\mathcal{A}$. For example, if $\alpha \in \mathcal{A} \subset E^1$ such that $\mathcal{A} = [0,1]$, then the optimization would have to be performed $\mathcal{c}$ times, where $\mathcal{c}$ is the power of the continuium. Therefore, one is motivated to seek a technique which would allow for the analysis of the optimal problem but which would significantly reduce the number of computations required if α is allowed to range over some set. The most obvious approach would be to hope for the existence of an extension of a known solution into a neighborhood of that solution. Or in other words, the hope is that the behavior of an optimal control $u_{\tilde{\alpha}}$ corresponding to a parameter vector $\tilde{\alpha}$ is related to the optimal control $u_{\bar{\alpha}}$ for $\bar{\alpha}$ in a straightforward manner. For example,

$\mathcal{A} = [0,1]$ and for any $\alpha \varepsilon \mathcal{A}$ the "optimal" solution of $\dot{w} = \alpha w$, $w(t_o) = w_o$, is defined to be $\phi(t,\alpha)$. Suppose also that it only be required that if $\tilde{\alpha}$ belongs to a small δ neighborhood of $\bar{\alpha}$ the "optimal solution $\phi(t,\tilde{\alpha})$ belongs to some ε neighborhood of $\phi(t,\bar{\alpha})$, $\varepsilon > 0$, $\delta > 0$. This statement requires, for ε small, small parameter variations result only in small trajectory variations. Obviously, $\phi(t,\alpha)$ is unique for all $\alpha \varepsilon \mathcal{A}$. There exists a continuous mapping G, such that $G: \mathcal{A} \to E^1 X [t_o, T]$. But $\mathcal{A}$ is compact and therefore has a finite open δ cover. Then, under G, the solution space $\psi(t,\alpha) = \{\phi(t,\alpha)\} = \{\phi(t,\alpha_j) | \phi(t,\alpha_j)$ is the solution of $\dot{w} = \alpha_j w$, $w(t_o) = w_o$ for all $\alpha_j \varepsilon \mathcal{A}\}$ has a finite open ε cover. And furthermore, there exists a $\delta > 0$ such that for $|\alpha_j - \tilde{\alpha}| < \delta$ and $\varepsilon > 0$ given, $|\phi(t,\alpha_j) - \phi(t,\tilde{\alpha}_j| = |\exp(-\alpha_j t) - \exp(-\tilde{\alpha} t)| < \varepsilon$. Therefore, instead of considering $\mathcal{C}$ exact computations, one need only consider a finite number of calculations if the ε error criterion is acceptable to the designer.

The extension mechanism to be investigated will be an expansion of the Hamiltonian by a truncated Taylor Series about some known solution. In particular, the known solution considered will be the nominal solution. The nominal solution is that optimal solution which corresponds to $\alpha = \alpha^n$, where α^n is the nominal design parameter vector. The Hamiltonian is dependent upon the given plant (2-1) in which the plant parameters are imbedded into its definition. Therefore, it is desirable, before attempting

to extend the Hamiltonian in a truncated Taylor Series, to introduce a more flexible notation which will give the parameter vector a more explicit representation in the Hamiltonian. This is accomplished as follows:

Define a new state vector (augmented state vector) y such that

$$y \in E^n \times \mathcal{A} \subset E^{n+M}$$

where

$$y \triangleq \begin{bmatrix} x \\ \alpha \end{bmatrix}, \quad x \in E^n, \quad \alpha \in \mathcal{A} \subset E^m$$

It should be noted that the α defined above is not time varying. It was stated earlier that only autonomous systems need be considered by virtue of a reduction technique displayed in footnote†. This technique is to be thought of as a device which will eliminate the explicit time dependence from the plant. For example:

Consider the nonautonomous system

$$\dot{x} = (\alpha_1 t + \alpha_2 \sin t)x + u \qquad t \in [0,1]$$

$$x_{(o)} = 0$$

then under the transformation $t = x_{n+1} = x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} (\alpha_1 x_2 + \alpha_2 \sin x_2)\, x_1 + u \\ 1 \end{bmatrix}, \quad x_{(o)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The parameter vector of this autonomous system is $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, where α_1 and α_2 are constants. As a practical consideration, one may wish to allow a weak parameter time dependence to exist. This may take the form of a drift or

parameter aging. This situation will be considered in more detail in Chapter 3. An example of this proposition may be an extension of the previous illustration for $\alpha_1(t) \cong \alpha_1$ and $\alpha_2(t) \cong \alpha_2$, then

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} (\alpha_1(t)x_2 + \alpha_2(t)\sin x_2)x_1 + u \\ 1 \end{bmatrix}$$

Now, also define

$$y(t_o) = \begin{bmatrix} x_o \\ \alpha_o \end{bmatrix}$$

where α_o is the initial parameter vector, which without any a priori knowledge of its value will be assumed to be the nominal parameter vector α^n. With this notation the Hamiltonian may now be expanded in a truncated Taylor Series. The development proceeds as follows:

2.3 Extended Hamiltonian Systems

Define the nominal Hamiltonian to be:

$$H^n(y(t), p(t), u(t)) = H(y^n(t), p^n(t), u^n(t))$$

for

$p(t) \in E^{n+m}$

$y^n(t), p^n(t), u^n(t)$ being solutions of (2-4, 5, and 6) for $\alpha = \alpha^n$, and

$$\dot{y}^n(t) = \begin{bmatrix} \dot{x}^n(t) \\ \dot{\alpha}^n(t) \end{bmatrix} = \begin{bmatrix} f(x^n(t), u^n(t), \alpha \\ 0 \end{bmatrix}$$

$$y(t_o) = \begin{bmatrix} x_o^n \\ \alpha^n \end{bmatrix} = \begin{bmatrix} x_o \\ \alpha^n \end{bmatrix}$$

Theorem 2-1: Let $H(y(t), p(t), u(t))$ and all its partial derivatives up through order k be continuous in some neighborhood N of $(y^n(t), p^n(t), u^n(t))$. Then for $(y(t), p(t), u(t)) \in N$ (dropping the t dependence)

$$H(y,p,u) = \sum_{i=0}^{k-1} \frac{1}{i!} \langle (y-y^n, p-p^n, u-u^n), \nabla \rangle^i H^n(y,p,u) + \frac{1}{k!} \langle (y-y^n, p-p^n, u-u^n), \nabla \rangle^k H^q(y,p,u) \tag{2-7}$$

where

$$H^q(y,p,u) = H(y^q, p^q, u^q), \quad (y^q, p^q, u^q) \in N$$

and ∇ is the gradient operator[14].

The vector (y^q, p^q, u^q) is a point on the line segment connecting $(y,p,u) \in N$ to (y^n, p^n, u^n) N. Because the neighborhood N is a convex subset of $E^{2(n+m)+r}$ (i.e. it is a $2(n+m)+r$ dimensional ball). The interior of N, i(N) is either convex or empty. If i(N) is not empty, then the following is true:

> Given two points (y,p,u) and $y^n, p^n, u^n)$ in N with (y^n, p^n, u^n) obviously in the interior of N then every point on the line segment between (y,pu) and (y^n, p^n, u^n) (with the possible exception of (y,p,u) itself) is an interior point of N.

Because N is convex, $N = \overline{co}(N)$ ($\overline{co}(N)$ denotes the convex hull of N). So equivalently, $(y^q, p^q, u^q) \in (\overline{co}(N))$ except for the previous noted exception.

Example: $(y,p,u)\varepsilon E^3$

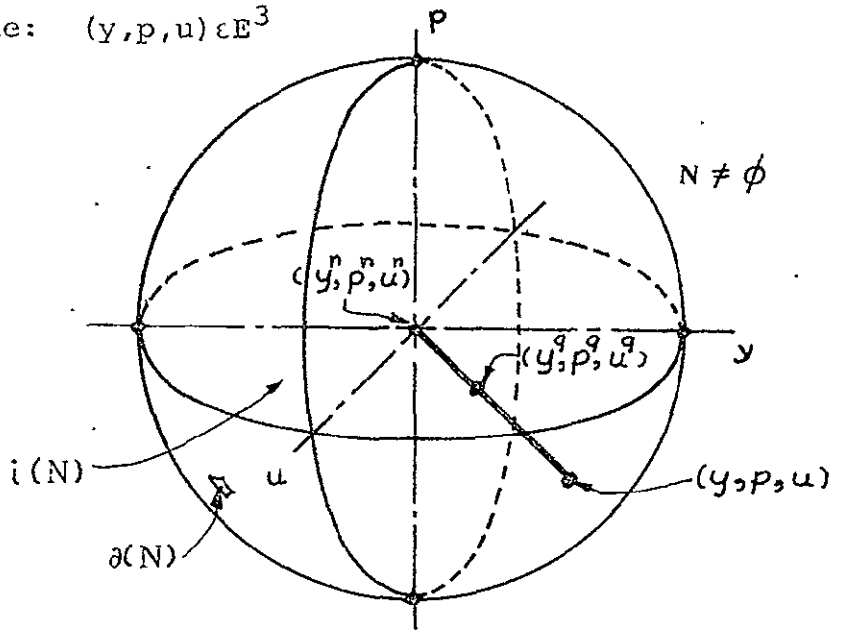

Figure 2-1

CONVEX N

By (2-7), the Hamiltonian can be represented in a neighborhood N of a particular vector (y^n,p^n,u^n) by a truncated finite series if the Hamiltonian satisfies the conditions stated in the theorem. The last term on the right hand side of (2-7) has special significance. This term, which is evaluated at some point in N, (not necessarily (y^n,p^n,u^n)), represents the error in approximating H(y,p,u) by

$$\sum_{i=0}^{k-1} \frac{1}{i!} \langle (y-y^n, p-p^n, u-u^n), \nabla \rangle^i H^n(y,p,u).$$

The term

$$\frac{1}{k!} \langle (y-y^n, p-p^n, u-u^n), \nabla \rangle^k H^q(y,p,u)$$

shall henceforth be referred to as the truncation error and denoted as $o(\varepsilon^k)$. The argument ε will later be used to denote the vector difference between (y,p,u) and

(y^n,p^n,u^n). The necessary condition that $H(y,p,u)\varepsilon C^k(t_o,T)$ is really not to sever. In the cases to be investigated it will always be assumed that $H(y,p,u)$ does indeed possess k continuous partial derivatives. If it does not, then there still remains another alternative. From the Theory of Mollifiers[15] a function can be approximated, in norm to within an arbitrary error ε, $\varepsilon > 0$, by a $C^\infty[t_o,T]$ function. Therefore, even though the Hamiltonian may not have a desired number of partial derivatives existing, it can be approximated by a function which has the desired derivatives. Furthermore, the derivatives which do exist in the original function will also be found in the approximating function.

The study shall now concentrate on a special truncated Taylor Series representation of the Hamiltonian

$$H(y,p,u) = \sum_{i=0}^{2} \frac{1}{i!} \langle (y-y^n, p-p^n, u-u^n), \nabla \rangle^i H^n(y,p,u) + \frac{1}{3!} \langle (y-y^n, p-p^n, u-u^n), \nabla \rangle^3 H^q(y,p,u). \tag{2-8}$$

Therefore, (2-7) is being considered for k = 3 and $H(y,p,u) \varepsilon C^3[t_o,T]$ at least, in some neighborhood of (y^n,p^n,u^n). This property will be used later for purposes of uniqueness of solutions.

Let: $C(u) = \frac{1}{2} < x(T), Tx(T) > + \int_{t_o}^{T} L(x,u)dt$ again, and impose once more the differential side constraint

$$\dot{y} \triangleq g(y,u) = \left[\begin{array}{c} f(y,u) \\ \hline 0 \end{array}\right], \quad y_o \text{ given}$$

Notation: Let

$$\left[\frac{\partial^K H^n(y,p,u)}{\partial y^j \partial p^i \partial u^h}\right] = H^n_{y^j p^i u^h}$$

where $j + i + h = k$.

Then the necessary conditions for the minimization of $H(y,p,u)$ given by (2-4,5 and 6) become

$$\begin{aligned}\dot{y} &= \frac{\partial H(y,p,u)}{\partial p} \\ &= H^n_p(y,p,u) + H^n_{p^2}(y,p,u)(p-p^n) + \\ &+ H^n_{py}(y,p,u)(y-y^n) + H^n_{pu}(y,p,u)(u-u^n) + \\ &+ o_p(\epsilon^3)\end{aligned} \tag{2-9}$$

$$\begin{aligned}\dot{p} &= -\frac{\partial H(y,p,u)}{\partial y} \\ &= -H^n_y(y,p,u) - H^n_{y^2}(y,p,u)(y-y^n) + \\ &- H^n_{yp}(y,p,u)(p-p^n) - H^n_{yu}(y,p,u)(u-u^n) + \\ &- o_y(\epsilon^3)\end{aligned} \tag{2-10}$$

and where u^* in (2-6) is assumed to exist and be unique. Also, equation (2-6) shall be weakened for computational purposes as follows:

For $H(y,p,u) \in C^3[t_o,T]$ given,

the optimal control u^* satisfies †

$$(\dagger)\quad H_u(y,p,u)\Big|_{u=u^*} = 0$$

For some $u^* \in N \subset i(\Omega)$*.

It shall also be required that the second derivative type test $H_u2(y,p,u^*) > 0$ thereby insuring a minimum. It should be noted that $H_u(y,p,u)$ and $H_u2(y,p,u)$ exist over N.

Summary of Assumptions: $u^* \in i(\Omega)$

i) u^* is obtainable from $H_u(y,p,u) = 0$

ii) $H_u2(y,p,u^*) > 0$

By the assumed uniqueness of u^*, u^* is a globally optimal control over N.

Therefore, u^* satisfies

$$H_u(y,p,u)\Big|_{u=u^*} = H_u^n(y,p,u) + H_{u^2}^n(y,p,u)(u-u^n) + H_{uy}^n(y,p,u)(y-y^n) + H_{up}^n(y,p,u)(p-p^n) + O_u(\epsilon^3)\Big|_{u=u^*} \equiv 0 \quad \text{over } t \in [t_o, T]. \tag{2-11}$$

The assumptions placed on u^*, that is (i) and (ii), are really not too restrictive if the class of control functions to be implemented possess some natural smoothness. However, such control strategies as bang-bang control would naturally be excluded from this analysis.

From the extended canonical equations (2-9, 10, and 11) several important properties pertaining to the

---

* If u^* satisfies (2-6) and $u^* \in \partial\Omega$, then finding u^* by setting † equal to zero would not make sense. Therefore, it shall be required that the u satisfying † belongs to $i(\Omega)$.

extended Hamiltonian system become apparent. First, it is desirable to insure that the solutions to the extended canonical equations are well behaved for an arbitrary parameter vector α "close" to α^n.

Lemma 2-1

For $\alpha \to \alpha^n$, the solution of (2-9)

$$\dot{y} = H_p^n(y,p,u) + H_{p^2}^n(y,p,u)(p-p^n) + H_{py}^n(y,p,u)(y-y^n) + H_{pu}^n(y,p,u)(u-u^n) + O_p(\epsilon^3)$$

$$\text{ie: } H_p(y,p,u)$$

converges uniquely to $y^n(t)$, the solution of (2-10)

$$\dot{p} = -H_y^n(y,p,u) - H_{y^2}^n(y,p,u)(y-y^n) - H_{yp}^n(y,p,u)(p-p^n) - H_{yu}^n(y,p,u)(u-u^n) - O_y(\epsilon^3)$$

$$\text{ie: } -H_y(y,p,u)$$

converges uniquely to $p^n(t)$,

and the u which satisfies (2-11)

$$\text{ie: } H_u(y,p,u) \equiv 0$$

converges uniquely to $u^n(t)$.

Proof:

By hypothesis there exists a unique control effort, call it $u_\alpha(t)$, which satisfies the original optimization problem with α being an arbitrary plant parameter and

$\alpha \in \mathcal{A}$. Remembering (2-9) is an exact equality and for $(y,u) \in N$, $t \in [t_o, T]$

(i)

$$\dot{x}(t) = f(x(t), u_\alpha(t), \alpha)$$

$$x^n(t_o) = x_o^n$$

Because for $\alpha = \alpha^n$, $f(x(t), \alpha^n, u^n(t))$ is Lipschitizian in $x(t)$, and $u^n(t)$ (unique by hypothesis), the solution of (i), for $x(t_o) = x_o$, is unique and was defined to be $x^n(t)$. The function $f(x(t), \alpha, u_\alpha(t))$ is also continuous on Π_α.

$$\Pi_\alpha = \left\{ (t, x(t), \alpha \mid t \in [t_o, T], (x, \alpha) \in N \right\}.$$

Also, suppose $|f(x(t), \alpha, u_\alpha(t)| \leq M$ on Π_α.

Claim:

There exists a $\delta > 0$ such that for any fixed α (which implies a fixed $u_\alpha(t)$) with $||(\alpha - \alpha^n, u_\alpha - u^n)|| < \delta$, every solution ϕ_α of (i) exists uniquely on $[t_o, T]$ and as $\alpha \to \alpha^n$ (correspondingly $u_\alpha \to u^n$ from the continuity of $H_u(y,p,u)$), $\phi_\alpha(t) \to x^n(t)$.

Proof:

The proof will be developed locally then extended. Choose λ sufficiently small so that

$$\Gamma = \left\{ (t,x) \in E^{n+1} \mid |t - t_o| \leq \lambda, |x - x_o| \leq M\lambda, \right.$$
$$\left. \text{where } M \text{ is a Lipschitiz constant} \right\} \subset N$$

Then, for any α, $u_\alpha \in N$ a solution of (i) exists on $[t - t_o] < \lambda$. Let ϕ_α be any solution of (i) for a

parameter vector α and corresponding control u_α.* Then $\{\phi_\alpha\}$ forms a bounded equicontinuous family. That is, for each $\varepsilon \to 0$ there exists a $\delta > 0$ such that

$$||\phi_\alpha(t) - \phi_\alpha(t')|| < \varepsilon$$

where $|t - t'| < \delta$. This implies, by Ascoli's Theorem[16], $\{\phi\alpha_{(\cdot)}\}$ has a uniformly convergent subsequence $\{\phi_{\alpha(k)}(\cdot)\}$. But from the uniqueness of the solution at $a^n, u_a^n = u^n$,

$$\phi_{\alpha(k)} \to \phi_\alpha n = x^n(t) \text{ as } \alpha(k) \to \alpha^n.$$

This gives a local result over $|t - t_o| < \lambda$. Extend the results to $[t_o, T]$. Notice also, only the uniqueness of ϕ_α at a point was required.

#(Claim)

Thus, from the claim the solution of (i) tends to the unique solution $x^n(t)$ as $\alpha \to \alpha^n$. Obviously the solution of

$$\dot{\alpha}(t) = 0, \ \alpha(t_o) = \alpha$$

tends to α^n as $\alpha \to \alpha^n$.

Now all that remains is to show $p(t) \to p^n(t)$ as $\alpha \to \alpha^n$

(ii) $\dot{p}(t) = - H_y(y(t), p(t), u(t))$, $p(T)$ given and $H_y(y(t), p(t), u(t)$ is Lipschitzian in p. Therefore, there exists a unique solution of (ii) for $y(t)$, $u(t)$ given. Let $y(t)$, $u(t)$ assume the role of the parameter vector α in the claim. Then, from the claim

$$p(t) \to p^n(t) \text{ as } \alpha \to \alpha^n$$

---

*Fundamental Lemma[13]. ϕ is a Caratheadory solution of $\dot{q}(t) = h(t, q(t))$, $q(t_o) = q_o$ on $\tau\varepsilon(t_o, t)$ if and only if $\phi(t) = q_o + \int_{t_o}^{t} h(t', \phi(t'))dt'$.

The previous results can be used to show how the cost index $C(u)$ reacts as $\alpha \to \alpha^n$. The obvious question to ask is "does $C(u)$ converge to $C(u^n)$ as $\alpha \to \alpha^n$?"

Theorem 2-2: As $\tilde{\alpha} \to \alpha^n$, for $\tilde{\alpha}$, $\alpha^n \varepsilon \mathcal{A}, C(\tilde{u}) \to C(u^n)$, where u denotes the unique solution of (2-11) for $\alpha = \tilde{\alpha}$.

Proof:

All that need be proven is that $x(t) \to x^n(t)$ and $\tilde{u}(t) \to u^n(t)$, where $\tilde{x}$ is the solution of (2-9) for $\alpha = \tilde{\alpha}$. Lemma 2-1 satisfies this demand. Then $(\tilde{x},\tilde{u}) \to (x^n,u^n)$ as $\tilde{\alpha} \to \alpha^n$, thus $C(\tilde{u}) \to C(u^n)$

#

What of the case where $\alpha \neq \alpha^n$? If the optimization problem is to make sense, one would intuitively expect the following to be trivially true.

Theorem 2-3, $\min C(u) = C(u^*) \leq C(u^n)$, where u^* denotes the unique solution of (2-11) for $\alpha^* \neq \alpha^r$.

Proof:

Suppose not. Then there exists a $u^n \in N$ such that

$$C(u^n) < C(u^*).$$

But, for $(y^*, p^*, u^*) \varepsilon N$ the $\min_{u \varepsilon \Omega} H(y^*,p^*,u) = H(y^*,p^*,u^*)$,

#

It should be noted that a weak inequality is used in Theorem 2-3. Even if u^* is a globally optimal control over N, it is possible to construct a cost functional whose arguments $\tilde{x}(t)$ and $\tilde{u}(t)$ are independent of parameter variations, when $\tilde{x}(t) \in E^n(t)$ and $\tilde{u}(t) \in E^r(t)$.

For the sake of notational convenience the following notational convention shall be established.

Let

$$\Delta y = y - y^n;\ y, y^n \varepsilon N$$

$$\Delta p = p - p^n;\ p, p^n \varepsilon N$$

$$\Delta u = u - u^n;\ u, u^n \varepsilon N.$$

Also, in (2-9), (2-10), or (2-11) it should be noted that $H^n_u(y,p,u) = 0$ and $H^n_{p2}(y,p,u) = 0$ (due to p appearing linearly in H(y,p,u)). From (2-11), Δu may be explicitly solved in terms of Δy and Δp because $H^n_{u2}(y,p,u) > 0$ by hypothesis; therefore, $H^n_{u2}(y,p,u)^{-1}$ exists.

$$\Delta u = -\left\{ \left[H^n_{u^2}(y,p,u)\right]^{-1} \left[H^n_{py}(y,p,u)\,\Delta y + H^n_{up}(y,p,u)\,\Delta p + o_u(\epsilon^3)\right] \right\}. \tag{2-12}$$

The results of (2-12) may now be used to eliminate Δu from (2-9) and (2-10) by direct substitution.

$$\Delta \dot{y} = \left[H^n_{py}(y,p,u) - H^n_{pu}(y,p,u)\left[H^n_{u^2}(y,p,u)\right]^{-1} H^n_{uy}(y,p,u)\right]\Delta y - \left[H^n_{pu}(y,p,u)\left[H^n_{u^2}(y,p,u)\right]^{-1} H^n_{up}(y,p,u)\right]\Delta p + - H^n_{pu}(y,p,u)\left[H^n_{u^2}(y,p,u)\right]^{-1} o_u(\epsilon^3) + o_p(\epsilon^3) \tag{2-13}$$

where: $\Delta y(t_o) = \begin{bmatrix} 0 \\ \hline \gamma \end{bmatrix}$,

γ arbitrary.

The arbitrariness of γ will not affect the desired result as will be shown shortly, and

$$\Delta\dot{p} = \left\{ H^n_{y^2}(y,p,u) - H^n_{yu}(y,p,u) \left[H^n_{u^2}(y,p,u) \right]^{-1} H^n_{uy}(y,p,u) \right\} \Delta y - \left\{ H^n_{yp}(y,p,u) - H^n_{yu}(y,p,u) \left[H^n_{u^2}(y,p,u) \right]^{-1} H^n_{up}(y,p,u) \right\} \Delta p + \quad (2\text{-}14)$$

$$- H^n_{yu}(y,p,u) \left[H^n_{u^2}(y,p,u) \right]^{-1} o_u(\epsilon^3) - o_y(\epsilon^3),$$

$\Delta p(T)$ given.

For N sufficiently small such that the higher order terms $o_u(\epsilon^3)$, $o_p(\epsilon^3)$ and $o_y(\epsilon^3)$ are negligible, then (2-13) and (2-14) can be represented as a system of 2(n+m) linear ordinary differential equations. (Error bounds on $o_u(\epsilon^3)$, $o_p(\epsilon^3)$ and $o_y(\epsilon^3)$ will be established later). Suppose such a non-empty neighborhood exists.

Then:

$$\begin{bmatrix} \Delta\dot{y} \\ \hline \Delta\dot{p} \end{bmatrix} = \left[\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & -A'(t) \end{array}\right] \begin{bmatrix} \Delta y \\ \hline \Delta p \end{bmatrix} = G(t) \begin{bmatrix} \Delta y \\ \hline \Delta p \end{bmatrix} \quad (2\text{-}15)$$

Δy_o, $\Delta p(T)$ given

where A, B, and C are defined in the following manner:

For $H(y,p,u) = L(x,u) + \langle \bar{p}, f(x,u,\alpha) \rangle + \langle \hat{p}, 0 \rangle$

where $p = \begin{bmatrix} \bar{p}_{(n\times 1)} \\ \hline \hat{p}_{(m\times 1)} \end{bmatrix}$, $p(T) = \begin{bmatrix} \text{given} \\ \hline 0 \end{bmatrix}$

Then, from (2-13)

$$A(t) = \left(\left[\begin{array}{c|c} f^n_x(x,u,a) & f^n_a(x,u,a) \\ \hline 0 & 0 \end{array} \right] - \left[\begin{array}{c} f^n_u(x,u,a) \\ \hline 0 \end{array} \right] \left[L^n_{u^2}(x,u) + \left[f'_u(x,u,a) \right]^n_u \right]^{-1} \left[L^n_{ux}(x,u) + \left[f'_u(x,u,a)\bar{p} \right]^n_x \right. \right.$$
$$\left. \left. \left[f'_u(x,u,a)\bar{p} \right]^n_a \right] \right) \qquad (2\text{-}13i)$$

$$B(t) = -\left(\left[\begin{array}{c} f^n_u(x,u,a) \\ \hline 0 \end{array} \right] \left[L_{u^2}(x,u) + \left[f'_u(x,u,a)\bar{p} \right]^n_u \right]^{-1} \left[f^{n\prime}_u(x,u,a) \,\middle|\, 0 \right] \right) \qquad (2\text{-}13ii)$$

and from (2-14)

$$C(t) = -\left\{ \left[\begin{array}{c|c} L^n_{x^2}(u,x) + \left[f'_x(x,u,a)\bar{p} \right]^n_u & \left[f'_x(x,u,a)\bar{p} \right]^n_a \\ \hline \left[f_a(x,u,a)\bar{p} \right]^n_u & 0 \end{array} \right] + \right.$$
$$\left. - \left[\begin{array}{c} L^n_{xu}(x,u) + \left[f'_x(x,u,a)\bar{p} \right]^n_u \\ \hline \left[f'_a(x,u,a) \right]^n_u \end{array} \right] \left[L^n_{u^2}(x,u) + \right. \qquad (2\text{-}14i)$$
$$\left. + \left[f'_u(x,u,a)\bar{p} \right]^n_u \right]^{-1} \left[L^n_{xa}(x,u) + \left[f'_x(x,u,a)\bar{p} \right]^n_u \,\middle|\, \left[f_a(x,u,a)\bar{p} \right]^n_u \right] \Big\}$$

A rather interesting phenomenon occurs in equation (2-15). The rows of its $\Delta\alpha$ partition are equal to zero. In fact, the truncation errors associated with the $\Delta\alpha$ rows are also zero. This implies that the autonomy of the parameters have been analytically preserved. Therefore, the parameter variations considered, of the form $\alpha - \alpha^n$, equal a constant. For a physical viewpoint, however, it is desirable to allow for a previously noted weak time varying of the parameter. If so, such qualities as parameter drift and

aging may be considered. This quality shall be introduced into the problems structure as follows. Assume the existence of a $\xi(t)$ which will appear in the $\Delta\dot{\alpha}$ equations in place of a truncation error. Then $\Delta\dot{\alpha} = \xi(t), \Delta\alpha(o)$ arbitrary has a solution which is equal to a constant plus a low amplitude, or slowly varying time dependent term if $||\xi(t)||$ is small. It will be demanded that $||\xi(t)||$ be sufficiently small so that it may be dismissed as were the legitimate truncation errors. This is a reasonable demand because one would not expect a set of parameters to exist which are rapidly changing or strongly varying in magnitude, to admit a truncated Taylor series representation of $H(y,p,u)$.

Eq. (2-15) is recognized to be a matrix Riccati type equation. It can be recognized as such because of the structure of $G(t)$. The solution of (2-15) is given by a $(n+m)\times(n+m)$ nonlinear system of differential equations. That is, there exists a symmetric $K(t)$ such that

$$* \quad -\dot{K}(t) = A'(t)K(t) + K(t)A(t) - K(t)B(t)K(t) + C(t)$$

$$K(T) = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \qquad (2\text{-}16)$$

and $\Delta p(t) = K(t)\Delta y(t)$.

The solution of * will usually involve a computer mechanization of the problem. Some techniques of solving a Riccati equation can be found in Tuel[17], Friedland[18], and Bass with an example supplied by Stoner, Taylor, and Bass[19]. If any of the $2(n+m)$ solutions of (2-15) are

known in closed form, then partioning routines can be used to reduce the computational difficulties associated with finding K(t)[13]. Partioning techniques also suggest another method of realizing the elements of K(t) for this particular problem.

Partition K(t) follows:

$$K(t) \quad \left[\begin{array}{c|c} K_1(t) & K_2(t) \\ \hline K_2'(t) & K_3(t) \end{array}\right]$$

where K_1 is nxn, K_2 is nxm, K_3 is mxm. Then Δu defined by (2-12) can be expressed as

$$\Delta u = -\left[H^n_{u^2}(y,p,u)\right]^{-1}\left[H^n_{uy}(y,p,u) + \left[f^{n'}_u(y,u,a)K_1(t) \vdots f^{n'}_u(y,u,a)K_2(t)\right]\right]\Delta y \qquad (2\text{-}12i)$$

Therefore, one needs only to focus attention on calculating $K_1(t)$ and $K_2(t)$. From (2-13i, ii), (2-14i), and (2-15) the differential equations defining $K_1(t)$ and $K_2(t)$ are established as:

$$A(t) = \left[\begin{array}{c|c} A_1(t) & A_2(t) \\ \hline A_3(t) & A_4(t) \end{array}\right] \qquad \begin{array}{l} A_1 \text{ is } n\times n,\ A_2 \text{ is } n\times m \\ A_3 \text{ is } m\times n,\ A_4 \text{ is } m\times m. \end{array}$$

$$A_1(t) = f^n_x(x,u,a) - f^n_u(x,u,a)\left[L^n_{u^2}(x,u) + \left(f'_u(x,u,a)\bar{p}\right)^n_a\right]^{-1}\left(L^n_{ux}(x,u) + \left(f'_u(x,u,a)\bar{p}\right)^n_x\right)$$

$$A_2(t) = f^n_a(x,u,a) - f^n_u(x,u,a)\left[L^n_{u^2}(x,u) + \left(f'_u(x,u,a)\bar{p}\right)^n_a\right]^{-1}\left(f'_u(x,u,a)\bar{p}\right)^n_a$$

$$A_3(t) = A_4(t) = 0$$

$$B(t) \triangleq \begin{bmatrix} B_1(t) & B_2(t) \\ B_3(t) & B_4(t) \end{bmatrix}, \quad \begin{matrix} B_1(t) \text{ is } n\times n,\ B_2(t) \text{ is } n\times m \\ B_3(t) \text{ is } m\times n,\ B_4(t) \text{ is } m\times m \end{matrix}$$

$$B_1(t) = f^n_u(x,u,a)\left[L^n_{u^2}(x,u) + \left(f'_u(x,u,a)\bar{p}\right)^n_u\right]^{-1} f^n_u(x,u,a)$$

$$B_2(t) = B_3(t) = B_4(t) = 0$$

$$C(t) = \begin{bmatrix} C_1(t) & C_2(t) \\ C_3(t) & C_4(t) \end{bmatrix}, \quad \begin{matrix} C_1(t) \text{ is } n\times n,\ C_2(t) \text{ is } n\times m \\ C_3(t) \text{ is } m\times n,\ C_4(t) \text{ is } m\times m \end{matrix}$$

$$C_1(t) = L^n_{x^2}(x,u) + \left(f'_x(x,u,a)\bar{p}\right)^n_x - \left[L^n_{xu}(x,u) + \left(f'_x(x,u,a)\bar{p}\right)^n_u\right]\left[L^n_{u^2}(x,u) + \left(f'_u(x,u,a)\bar{p}\right)^n_a\right]^{-1}\left[L^n_{ux}(x,u) + \left(f'_u(x,u,a)\bar{p}\right)^n_x\right]$$

$$C_2(t) = \left(f'_x(x,u,a)\bar{p}\right)^n_a - \left[L^n_{xu}(x,u) + \left(f'_x(x,u,a)\bar{p}\right)^n_u\right]\left[L^n_{u^2}(x,u) + \left(f'_u(x,u,a)\bar{p}\right)^n_a\right]^{-1}\left(f'_u(x,u,a)\bar{p}\right)^n_a$$

$$C_3(t) = \left(f'_a(x,u,a)\bar{p}\right)^n_u - \left(f'_a(x,u,a)\bar{p}\right)^n_u\left[L_{u^2}(x,u) + \left(f'_u(x,u,a)\bar{p}\right)^n_u\right]^{-1}\left(L^n_{ux}(x,u) + \left(f'_u(x,u,a)\bar{p}\right)^n_x\right)$$

$$C_4(t) = \left(f'_a(x,u,a)\bar{p}\right)^n_u\left[L_{u^2}(x,u) + \left(f'_u(x,u,a)\bar{p}\right)^n_u\right]^{-1}\left(f'_u(x,u,a)\bar{p}\right)^n_a$$

Therefore:

$$(i)\quad -\dot{K}_1(t) = K_1(t)A_1(t) + A_1'(t)K_1(t) - K_1(t)B_1(t)K_1(t) + C_1(t),\ K_1(T) = T$$

and

$$(ii)\quad -\dot{K}_2(t) = K_1(t)A_2(t) + A_1'(t)K_2(t) - K_1(t)B_1(t)K_2(t) + C_2(t),\ K_2(T) = 0.$$

Comments:

Equation (i) is a homogeneous nxn nonlinear differential system of equations. Its boundary data is specified at t = T. It is independent of the initial data $\Delta y(t_o)$.

Equation (ii) is a nonhomogeneous (nxm) linear differential system of equations. Its boundary data is specified at t = T. It is independent of the initial data $\Delta y(t_o)$, thus leaving $\Delta\alpha_o$ arbitrary as acceptable.

Consider now the solution to (2-12) to be of the form

$$\Delta u(t) = \tilde{G}\Delta y(t) \qquad (2\text{-}17)$$

where G(t) is calculated through one of the suggested methods. A remarkable observation may be abstracted from (2-17). The implementation of the adaptive control need only require that $x^n(t)$ need be stored in some memory device. The simplest example of such a device would be a tape. Thus, after $\tilde{G}(t)$ has been precomputed, $\tilde{G}(t)$, being a function of the nominal variables $y^n(t)$, $p^n(t)$, and $u^n(t)$, it will be multiplied in real time by $(y(t) - y^n(t))$. But $y^n(t)$ equals $(x^n(t), \alpha^n)$, and α^n is nothing more than

some known biasing term. Thus only n known time varying functions must be introduced into the system from memory in real time instead of $2(n+m)+r$ functions which might have been earlier predicted.

The schematized realization of the adaptive system is given in Figure 2-2, where α^e is an estimation of α^* and α^* is the actual plant parameter vector. Parameter estimation will be treated in Chapter 3.

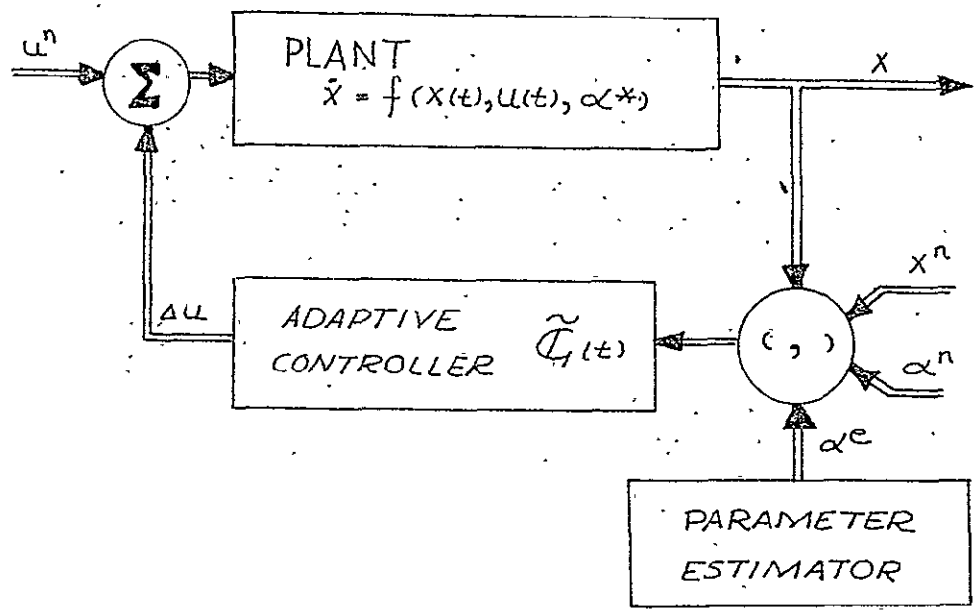

Figure 2-2

Adaptive Structure

2.4 Truncation Errors

To complete this section, the structure of the truncation error terms will be investigated. Rather than an exhibit the results in tensor notation, the equivalent scalar triple sum will be analyzed.

Let $z \triangleq (y,p,u)\ E^{2(n+m)+r}$

Then the truncation error of (2-8) becomes

$$(i)\quad \frac{1}{3!}\sum_{i,j,k=1}^{2(n+m)+r}(z_i-z_i^n)(z_j-z_j^n)(z_k-z_k^n)\frac{\partial^3 H^q(z)}{\partial z_i\,\partial z_j\,\partial z_k}\,.$$

Evaluating $\dfrac{\partial^3 H^q(z)}{\partial z_i\,\partial z_j\,\partial z_k}$ for special cases it can be noted that α and p appear linearly in $H(y,p,u)$. Therefore, for $\gamma \triangleq (z_i, z_j, z_k)$ having any two or three of its element being an element, or elements of p, or the last m elements of p,

$$\frac{\partial^3 H^q(z)}{\partial z_i\,\partial z_j\,\partial z_k} \equiv 0.$$

For $\lambda \triangleq (z_i, z_j, z_k)$ having two or three of its elements being an element or elements, of α

$$\frac{\partial H^q(z)}{\partial z_i\,\partial z_j\,\partial z_k} \equiv 0.$$

It may be noted that the truncation error, defined in (i), is dependent upon z^q, $z^q \varepsilon N$. The requirement of finding such a z^q which establishes the equality in (2-8) may be relaxed by redefining the truncation error. Let the bound on the truncation error, denoted as ϵ, be defined as follows:

$$(ii)\quad \epsilon = \sup_{z\in N}\frac{1}{3!}\left[\sum_{i,j,k=1}^{2(n+m)+r}(z_i-z_i^n)(z_j-z_j^n)(z_k-z_k^n)\frac{\partial^3 H(z)}{\partial z_i\,\partial z_j\,\partial z_k}\right]$$

$$\geq \sup_{z\in N}\left[\sum_{i,j,k=1}^{2(n+m)+r}(z_i-z_i^n)(z_j-z_j^n)(z_k-z_k^n)\frac{\partial^3 H^q(z)}{\partial z_i\,\partial z_j\,\partial z_k}\right].$$

Also, the truncation error associated with (2-12), (2-13), and (2-14) may be interpreted similarly.

Let $\eta(z)$ be the middle term of (ii).

Then

$$\text{(iii)} \quad \left\{ \begin{array}{l} o_u(\epsilon^3) \leq \sup_{z \in N} \left\{ \dfrac{\partial \eta}{\partial u} \right\} \\ o_p(\epsilon^3) \leq \sup_{z \in N} \left\{ \dfrac{\partial \eta}{\partial p} \right\} \\ o_y(\epsilon^3) \leq \sup_{z \in N} \left\{ \dfrac{\partial \eta}{\partial y} \right\} \end{array} \right\}$$

Therefore, requiring $o_u(\epsilon^3)$, $o_p(\epsilon^3)$, and $o_y(\epsilon^3)$ being small can be insured by requiring the right hand side of (iii) is small.

2.5 Example Problem I

Example: Linear Regular (single channel control)

Consider the plant

(i) $\dot{X}(t) = A(\alpha)X(t)+Bu(t)$, $X(t_o)=C$

and the cost index

(ii) $C(u) = \frac{1}{2}\langle X(T), \bar{T}X(T)\rangle + \frac{1}{2}\int_{t_o}^{T} \langle X(t), QX(t)\rangle + \langle u(t), Ru(t)\rangle \, dt$

where

The terminal time T is specified

$\bar{T}$ is a 2x2 positive semidefinite matrix

Q is a 2x2 diagonal positive semidefinite matrix

R is a 1x1 diagonal positive definite matrix

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A(\alpha) = \begin{bmatrix} 0 & 1 \\ -\alpha_1 & -\alpha_2 \end{bmatrix}$

The nominally optimal solution is found as follows: Define the nominal Hamiltonian to be

$$H(X(t),\alpha^n,p(t),u(t)) = \tfrac{1}{2}\langle X(t), QX(t)\rangle + \tfrac{1}{2}\langle u(t), Ru(t)\rangle + \langle p(t), A(\alpha^n)X(t)+Bu(t)\rangle$$

and

$$\text{(iii)} \qquad \dot{p}(t) = -\frac{\partial H}{\partial y} = -QX(t) - A'(\alpha^n)p(t)$$

$$\frac{\partial H}{\partial u(t)} = 0 \rightarrow Ru(t)+B'p(t) = 0$$

then

$$u(t) = -R^{-1}B'p(t)$$

Note also

$$\frac{\partial^2 H}{\partial u(t)^2} = R > 0.$$

Then

$$\text{(iv)} \qquad \begin{bmatrix} \dot{x}^n(t) \\ \dot{p}^n(t) \end{bmatrix} = \left(\begin{array}{c|c} A(\alpha^n) & -BR^{-1}B' \\ \hline -Q & -A'(\alpha^n) \end{array}\right) \begin{bmatrix} x^n(t) \\ p^n(t) \end{bmatrix}$$

and $P^n(T) = \bar{T}X^n(T)$.

Assume the solution of (iv) to exist and be of the form

$$\begin{bmatrix} p_1^n(t) \\ p_2^n(t) \end{bmatrix} = \left(\begin{array}{c|c} k_{11}^n(t) & k_{12}^n(t) \\ \hline k_{12}^n(t) & k_{22}^n(t) \end{array}\right) \begin{bmatrix} x_1^n(t) \\ x_2^n(t) \end{bmatrix}.$$

Now construct G of (2-15), remembering

$$y(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \alpha_1 \\ \alpha_2 \end{bmatrix}, \quad p(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix}$$

$$G = \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_1 & -a_2^n & x_1^n & x_2^n & 0 & -1/r_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -q_{11} & 0 & p_2^n & 0 & 0 & a_1^n & 0 & 0 \\ 0 & -q_{22} & 0 & p_2^n & -1 & a_2^n & 0 & 0 \\ p_2^n & 0 & 0 & 0 & 0 & -x_1^n & 0 & 0 \\ 0 & p_2^n & 0 & 0 & 0 & -x_2^n & 0 & 0 \end{array}\right]$$

Note:

$p_2^n(t)$ could be redefined in terms of $X^n(t)$.

$$\text{ie: } p_2^n(t) = \sum_{i=1}^{2} k_{2i}^n(t) x_i^n(t) \triangleq \langle \lambda, x^n(t) \rangle.$$

Therefore,

$$-\dot{K}(t) = \begin{bmatrix} 0 & -a_1^n & 0 & 0 \\ 1 & -a_2^n & 0 & 0 \\ 0 & x_1^n & 0 & 0 \\ 0 & x_2^n & 0 & 0 \end{bmatrix} K(t) + K(t) \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_1^n & -a_2^n & x_1^n & x_2^n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$- K(t) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1/r_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} K(t) + \begin{bmatrix} -q_{11} & 0 & p_2^n & 0 \\ 0 & -q_{22} & 0 & p_2^n \\ p_2^n & 0 & 0 & 0 \\ 0 & p_2^n & 0 & 0 \end{bmatrix}$$

where $K(t) = [k_{ij}]$, $i, j = 1, \ldots, 4$

$$K(T) = \left[\begin{array}{c|c} \bar{T} & \\ \hline 0 & 0 \end{array}\right]$$

Solve for K(t) and construct $\tilde{G}(t)$ of (2-17). The bound

on the truncation error for this example is:

$$\varepsilon = \sup_{z\varepsilon N}[(\alpha_1 - \alpha_1^n)(X_1 - X_1^n)(p_2 - p_2^n)+(\alpha_2 - \alpha_2^n)(x_2 - x_2^n)(p_2 - p_2^n)]$$

For example, if $z\varepsilon N$, $N = \left\{(x,\alpha,u,p) \mid \| Z \| \leq r\right\}$

then $\varepsilon = 2r^3$.

Also

$$o_u(\epsilon^3) = 0$$

$$o_p(\epsilon^3) = \begin{bmatrix} o_{p_1}(\epsilon) \\ o_{p_2}(\epsilon) \\ o_{p_3}(\epsilon) \\ o_{p_4}(\epsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ (\alpha - \alpha_1^n)(x_1 - x_1^n)+(\alpha_2 - \alpha_2^n)(x_2 - x_2^n) \\ 0 \\ 0 \end{bmatrix}$$

$$o_y(\epsilon^3) = \begin{bmatrix} o_{x_1}(\epsilon) \\ o_{x_2}(\epsilon) \\ o_{\alpha_1}(\epsilon) \\ o_{\alpha_2}(\epsilon) \end{bmatrix} = \begin{bmatrix} (\alpha_1 - \alpha_1^n)(p_2 - p_2^n) \\ (\alpha_2 - \alpha_2^n)(p_2 - p_2^n) \\ (x_1 - x_1^n)(p_2 - p_2^n) \\ (x_2 - x_2^n)(p_2 - p_2^n) \end{bmatrix}$$

Under the previous example N,

$$o_u(\epsilon^3) = 0$$

$$o_p(\epsilon^3) = 2\begin{bmatrix} 0 \\ r^2 \\ 0 \\ 0 \end{bmatrix}$$

$$o_y(\epsilon^3) = 2\begin{bmatrix} r^2 \\ r^2 \\ r^2 \\ r^2 \end{bmatrix}$$

Example Problem 2:

Consider the plant

$$\dot{X}(t) = \alpha X(t) + u(t),\ X(0) = 1$$

Let the nominal value of a be $a^n = -1$. Also, consider the cost index given by

$$C(u) = \frac{1}{2}\int_0^\infty (x^2(t) + u^2(t))\,dt.$$

The nominal system is optimized by solving

$$\dot{x}^n(t) = -x^n(t) - p^n(t)$$
$$\dot{p}^n(t) = -x^n(t) + p^n(t)$$
$$u^n(t) = -p^n(t) = -k^n x^n(t)$$

where k^n is the nominal Riccati gain and is equal to

$$k^n = -1+\sqrt{2} > 0$$

Thus $p^n(t) = (-1+\sqrt{2})x^n(t)$

$$u^n(t) = (1-\sqrt{2})x^n(t)$$

and $x^n(t) = \exp(-\sqrt{2}t)$

Now consider the system subject to the parameter variation $a = a^n - \epsilon$, $u(t) = u^n(t)$.

$$\dot{\tilde{x}}(t) = (a^n - \epsilon)\tilde{x}(t) + (1-\sqrt{2})x^n(t)$$

$$= -(1+\epsilon)\tilde{x}(t) + (1-\sqrt{2})\exp(\sqrt{2}t)$$

$$\tilde{x}(0) = 1$$

$$\tilde{x}(t) = \exp\left[-(1+\epsilon)t\right]\left[1 + \frac{(1-\sqrt{2})}{(1+\epsilon-\sqrt{2})}\left\{\exp\left[(1+\epsilon-\sqrt{2})t\right] - 1\right\}\right].$$

Note: for $\epsilon = 0$, $\tilde{x}(t) = x^n(t)$

For $a = a^n - \epsilon = a^*$, the true optimal strategy is

$$\dot{x}^*(t) = -(1+\epsilon)x^*(t) - p^*(t),\ x^*(0) = 1$$
$$\dot{p}^*(t) = -x^*(t) + (1+\epsilon)p^*(t)$$
$$u^*(t) = -p^*(t) = -k^*x^*(t)$$

where $k^* = -(1+\epsilon) + \sqrt{(1+\epsilon)^2+1} > 0$,

and

$$X^*(t) = \exp\left[-\sqrt{(1+\varepsilon)^2+1}\,t \right].$$

Now the adaptive dynamics are:

for $a_1 = -1$, $a_2 = X^n(t)$, $b_1 \mp 1$, $c_1 = 1$, $c_2 = p^n(t)$

Therefore

$$k_1^e(-1)+(-1)k_1^e - k_1^e(1)k_1^e+1 = -\dot{k}_1^e$$

But this is a homogeneous constant coefficient differential equation evaluated over $t_o = 0$, $T \to \infty$. Therefore, from a steady state argument $\dot{k}_1^e = 0$. Thus

$$k_1^e = -1+\sqrt{2} = k^n.$$

Also:

$$-\dot{k}_2^e = k_1^e x^n - k_2^e - k_1^e k_2^e + p^n, \quad k_2^e(t\to\infty) = 0.$$

Integrating backwards in time one finds

$$k_2^e(t) = 2(1-\sqrt{2})\exp(-\sqrt{2}t).$$

Therefore

$$u^e(t) = u^n(t) - \left[(-1+\sqrt{2})(x-x^n) + \exp(-\sqrt{2}t)2(1-\sqrt{2})\Delta\alpha\right]$$

Letting $\Delta\alpha = \alpha - \alpha^n = \alpha^n - \varepsilon - \alpha^n = -\varepsilon$

$$\dot{x}^e(t) = -(1+\epsilon)x^e(t) - (-1+\sqrt{2})x^e(t) - \exp(-\sqrt{2}t)2(-1+\sqrt{2})\epsilon$$

Thus

$$\dagger\quad \dot{x}^e(t) = \exp(-\sqrt{2}-\epsilon)t - \exp\left[-(\sqrt{2}+\epsilon)t\right]\int_o^t (\exp(\sqrt{2}+\epsilon)T)$$

$$2(-1+\sqrt{2})\epsilon\exp(-\sqrt{2}T)\,dT$$

$$= \exp\left[-(\sqrt{2}+\epsilon)t\right]\left\{1 - 2(-1+\sqrt{2})\left[\exp(\epsilon t)-1\right]\right\}$$

Notice, for $\varepsilon = 0$, $X^e(t) = X^n(t)$ and for εt small $X^e(t) \cong X^*(t)$. Consider now a 10% parameter variation (i.e. $\varepsilon = .1$). One finds

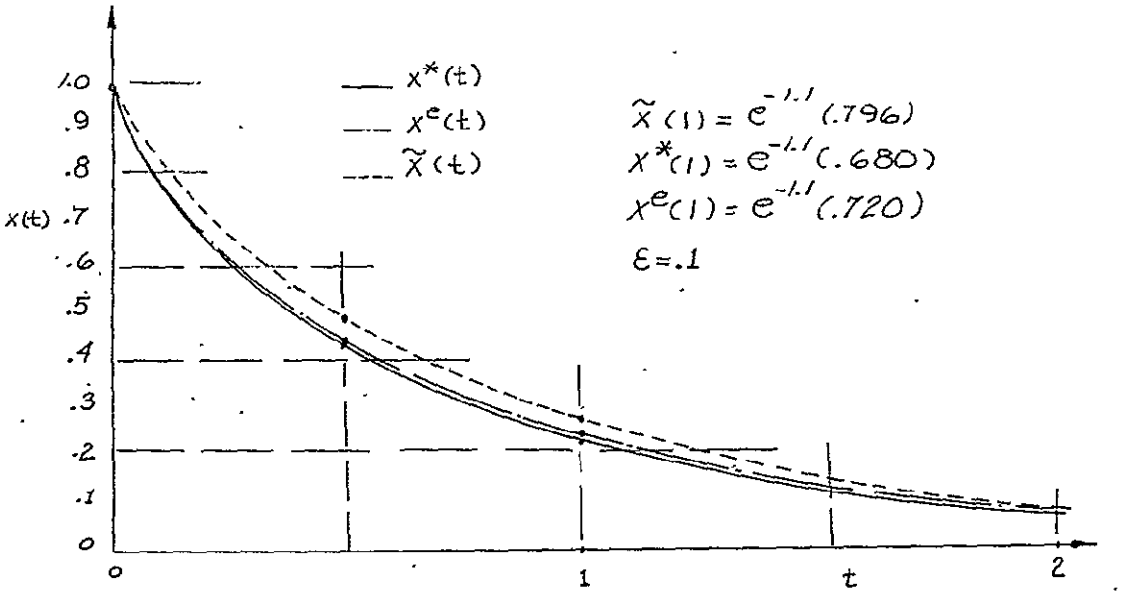

Figure 2-3

STATE SPACE $\varepsilon = .1$

Now considering the various control efforts considered for $\varepsilon = 1$.

$$u^*(t) = (1.1 - \sqrt{2.21}) \exp(-\sqrt{2.21}\, t)$$

$$u^n(t) = (-.414) \exp(-\sqrt{2}\, t)$$

$$u^e(t) = u^n(t) - [(-1+\sqrt{2})(x^e(t) - x^n(t)) - \exp(-\sqrt{2}t)2(-1+\sqrt{2})(.1)]$$

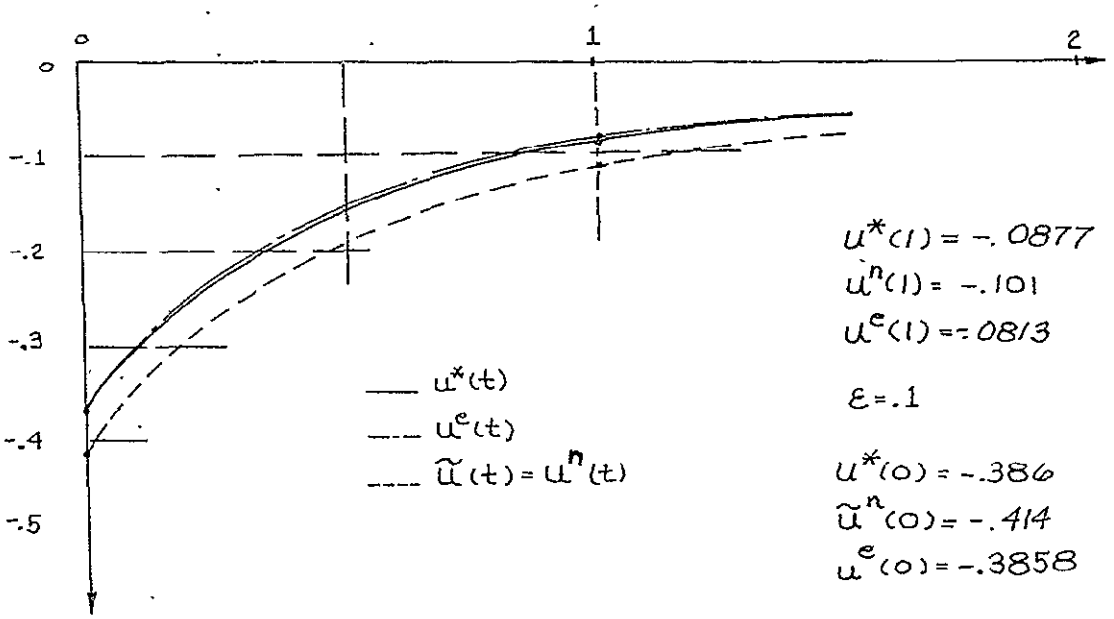

Figure 2-4
CONTROL SPACE $\varepsilon = .1$

From (†) it can be noted for (εt) small (i.e., the internal where $X^e(t)$ has its largest values) $X^e(t)$ acts optimally. The non-optimal characteristics of $X^e(t)$ occur when (εt) is large. But over this internal $X^e(t)$ is small and its contribution to $C(u^e)$ is minor.

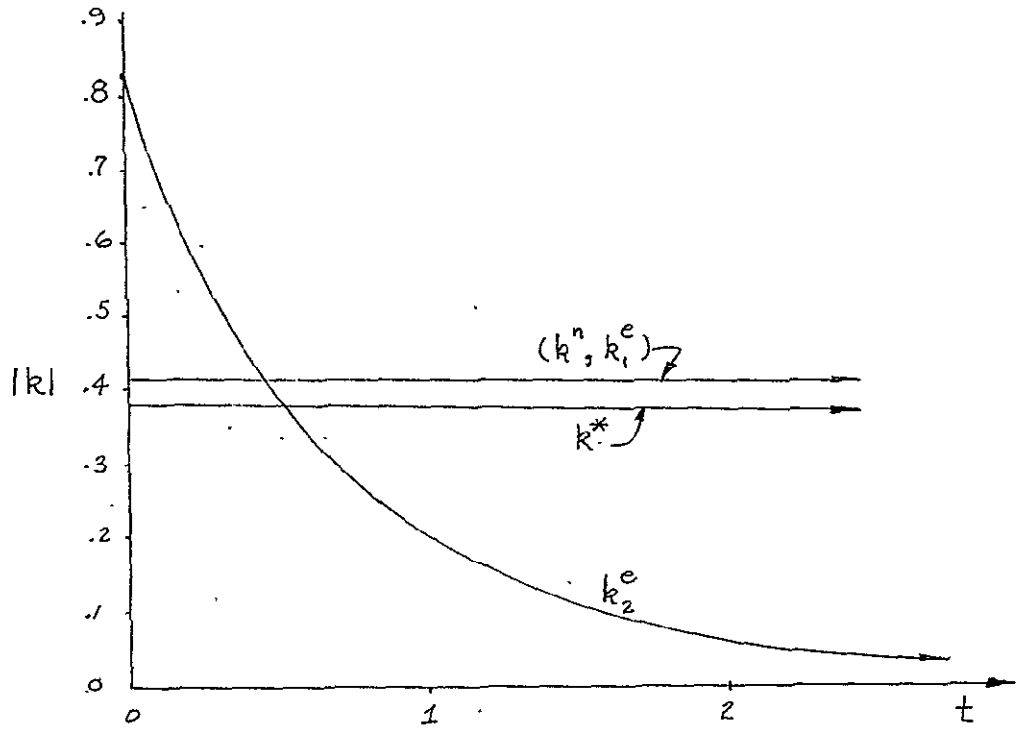

Figure 2-5
GAIN PLOTS

It is easy to see from Figures 2-2 and 2-3 that

$$C(u^*) < C(u^e) < C(u^r)$$

In fact, Figure (2-6) exhibits a significant reduction in cost by using $u^e(t)$ over $u^n(t)$.

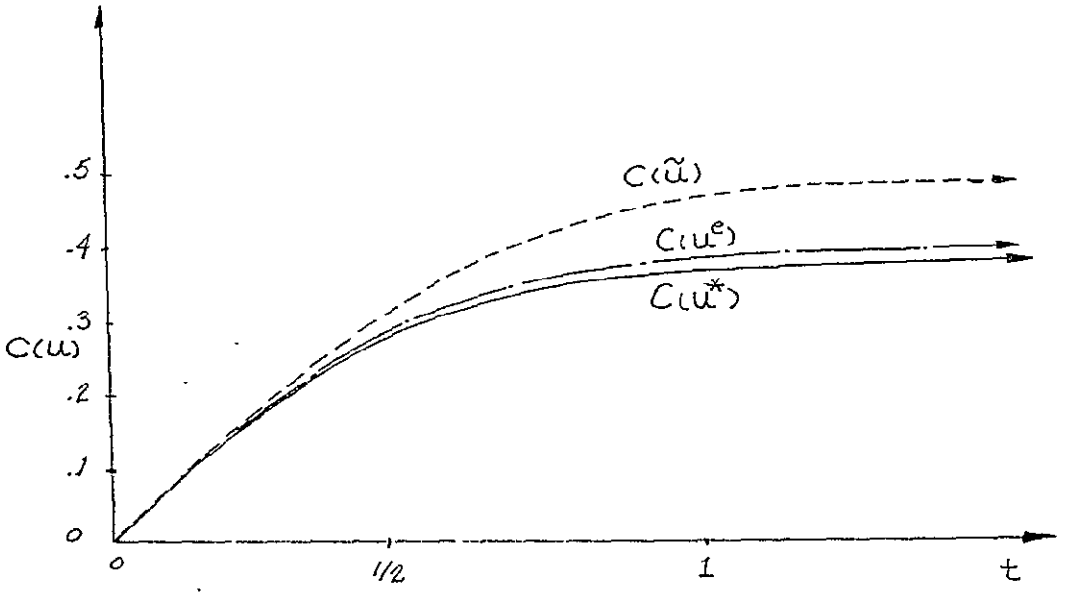

Figure 2-6

COST INCURRED

2.6 Reoptimization

It is assumed:

(i) that the actual state X(t) can be measured in real time in the absence of noise. If the observations are noisy, then a state estimation technique might have to be considered. (Examples -- nonlinear stochastic filters, Kalman filter, etc.)

(ii) $X^n(t)$ and α^n can be loaded into the system in real time. This could be accomplished by loading these vectors from a tape which is synchronized in real time with the physical system.

and

(iii) α^e can be generated (estimated).

Methods which satisfy (iii) (i.e., the parameter vector α

need be estimated) and will be treated later in this investigation.

It may be noted that the feedback loop is adaptive and produces a control effort which is in such a direction as to minimize the cost incurred. This preservation of the concept of optimality will be called "reoptimization."

In order that the adaptive structure be of practical use it must possess the quality that:

$$|C(u^*) - C(u^e)| \leq |C(u^*) - C(u^n)|$$

for u^e being the adaptive control and for a set of parameter vectors in some neighborhood of α^n.

Theorem 2-4: For an adaptive control u^e satisfying (2-17) $u^e \varepsilon i(\Omega)_\eta$ there exists $\varepsilon \geq 0$, and sufficiently small, such that for $||\alpha^e - \alpha^*|| < \varepsilon$; $\alpha^e, \alpha \varepsilon \mathcal{A}$,

$$|C(u^*) - C(u^e)| \leq |C(u^*) - C(u^n)|.$$

Proof:

For $\alpha = \alpha^* \neq \alpha^n$, $C(u^*) \leq C(u^n)$ from Theorem 2-3. From (2-13) and the matricies defined by (2-15) (α^e being the approximation of α^*)

$$(*) \quad \dot{y}^* - \dot{y}^e = A(y^* - y^e) + B(p^* - p^e) + \gamma(y^* - y^n, p^* - p^n, u^* - u^n),$$

$$(y_o^* - y_o^e) = \begin{bmatrix} o \\ \sigma \end{bmatrix}$$

where $\gamma(y^* - y^n\ p^* - p^n, u^* - u^n)$ is the truncation error of (2-13) for (*). From (2-14) and the matrices defined by (2-15)

$$\dot{p}^* - \dot{p}^e = C(y^* - y^e) - A'(p^* - p^e) +$$
$$+ \bar{\gamma}(y^* - y^n, p^* - p^n, u^* - u^n),$$
$$(p^*(T) - p^e(T)) = \text{given}$$

Now choose $N_\varepsilon \neq \phi$ such that the truncation errors γ and $\bar{\gamma}$ are neglectable. Either N_ε is the point set (y^n, p^n, u^n) or otherwise. If N_ε is the point set (which is the result of insufficient smoothness of $H(y,p,u)$ about $H^n(y,p,u)$), finished. If N_ε is not the point set, then there exists α^e such that $||\alpha^* - \alpha^e|| < ||\alpha^* - \alpha^n||$.

In an argument similar to that of Lemma 2-1, except $\alpha^e \to \alpha^*$ instead of α^n;

i.e.: $\begin{cases} x^e(t_o) = x^*(t_o) \\ p^e(T) = p^*(T) \end{cases}$, $\alpha^e \to \alpha^*$

the solution of

(i) $$\begin{bmatrix} \dot{y}^* - \dot{y}^e \\ \dot{p}^* - \dot{p}^e \end{bmatrix} = \begin{bmatrix} A(t) & B(t) \\ C(t) & -A'(t) \end{bmatrix} \begin{bmatrix} y^* - y^e \\ p^* - p^e \end{bmatrix}$$

tends to the trivial solution (i.e., $(y^*(t) - y^e(t)) \to 0$ and $(p^*(t) - p^e(t)) \to 0$. The trivial solution is the unique solution of (i) for $\alpha^e = \alpha^*$ (i.e., $\varepsilon = 0$). It is apparent that u^e tends to u^* for this local argument because of their linear structure.

Therefore, for α^* and $\alpha^e \varepsilon N_\varepsilon$ and α^e sufficiently close to α^*, $L(x^e,u^e)$ defined in (2-2) can be made arbitrarily close or equal to $L(x^*,u^*)$. But u^n is a fixed function of time and for $\alpha^* \neq \alpha^n$ is some constant non-zero vector λ.

Consider the sequence of parameter vectors α_i^n, $\alpha_i^n \varepsilon N_\varepsilon$, such that $||\alpha_i^n - \alpha^n|| = 0$ for $i = 1, 2, \ldots$. Then for α^* and $\alpha_i^n \varepsilon N_\varepsilon$, and $||\alpha^* - \alpha_i^n|| = \lambda$, $L(x_o^n, u_i^n)$ need

not converge to $L(x^*,u^*)$ as $i \to \infty$. Then $||C(u^*) - C(u^e)|| \leq ||C(u^*) - C(u^n)||$ for a set of α^e's sufficiently to α^*.

The adaptive structure now formally possessed the qualities demanded of a reoptimization scheme (see Theorem (2-2), (2-3, (2-4)). Namely, $C(u^e) \to C(u^n)$ as $\alpha^e \to \alpha^n$, $C(u^*) \leq C(u^n)$, and $C(u^n) \to C(u^e)$ as $\alpha^e \to \alpha^*$. Also, subject to the constraints imposed on the problem, the derived adaptive control is the "best" minimizing control for the given problem. One intrinsic feature still needs attention, namely the computational requirements imposed by the necessity of generating α^e (see Figure 2-2).

CHAPTER III

PARAMETER ESTIMATION

3.1 Parameter Estimation

The objective of a parameter estimator is to approximate the actual parameters while hopefully satisfying as many of the following qualities as possible.

Qualities to be achieved:

(1) Compute uniquely the unknown parameters.

(2) The estimation should converge rapidly to an accurate solution.

(3) The computational routine and its physical mechanization should not be complex.

(4) The parameter estimator should be free of dimensional restrictions.

The efficiency of a parameter estimator may be thought of as a trade-off between the first three qualities. Quality 4, however, is an often neglected property of estimation techniques and will be given special attention. After all, any method used must allow for all the desired parameters independent of their number of be computed. Therefore, the estimation device is truly an engineering problem.

If the plant parameters can be obtained by direct measurement, then the reoptimization problem is direct. In general, however, one cannot hope that all, if any, of the parameters are monitorable.* The estimation of

---

*Monitorable implies obtainable by direct measurement

non-monitorable parameters will vary in complexity and accuracy as a function of the computational technique considered. It is for this reason that numerous selected computational techniques will be developed leaving the choice of which method to mechanize a matter of personal preference.

To motivate some of the numerical and gradient techniques to be developed in this Chapter, the effect of parameter variations upon the systems' trajectories will be explored. With suitable modifications, some due to H. Hermes, a method similar to one developed by L. S. Pontryagin, V. G. Boltyanski, R. V. Gamkrilidze, and E. F. Mischenko[20] will be employed to examine the parameter-trajectory variational properties. (Pontryagin's work was used to show how a given trajectory was disturbed by variations in the control effort.)

Suppose $x(t)$ and $u(t)$ are monitorable, $u(t)$ measurable (in the sense of Lebesque), then α can be estimated as follows:

Consider

$$\dot{x}(t) = f(x,u;\ \alpha^*) \qquad (3\text{-}1)$$

$$x(t_o) = x_o$$

α not monitorable over $t\varepsilon[t_o,T]$

Assume the solution of (3-1) exists over $t\varepsilon[t_1 - \ell\varepsilon,\ t_1] \subset [t_o,T]$,

$$\ell > 0$$

$$\varepsilon > 0.$$

Consider the model of (3-1) to be:

$$\dot{x}^e = f(x^e, u^e, a^e)$$

$$x^e(t_1 - \ell\epsilon) = x(t_1 - \ell\epsilon) \qquad (3\text{-}2)$$

over

$$t \in [t_1 - \ell\epsilon, t_1].$$

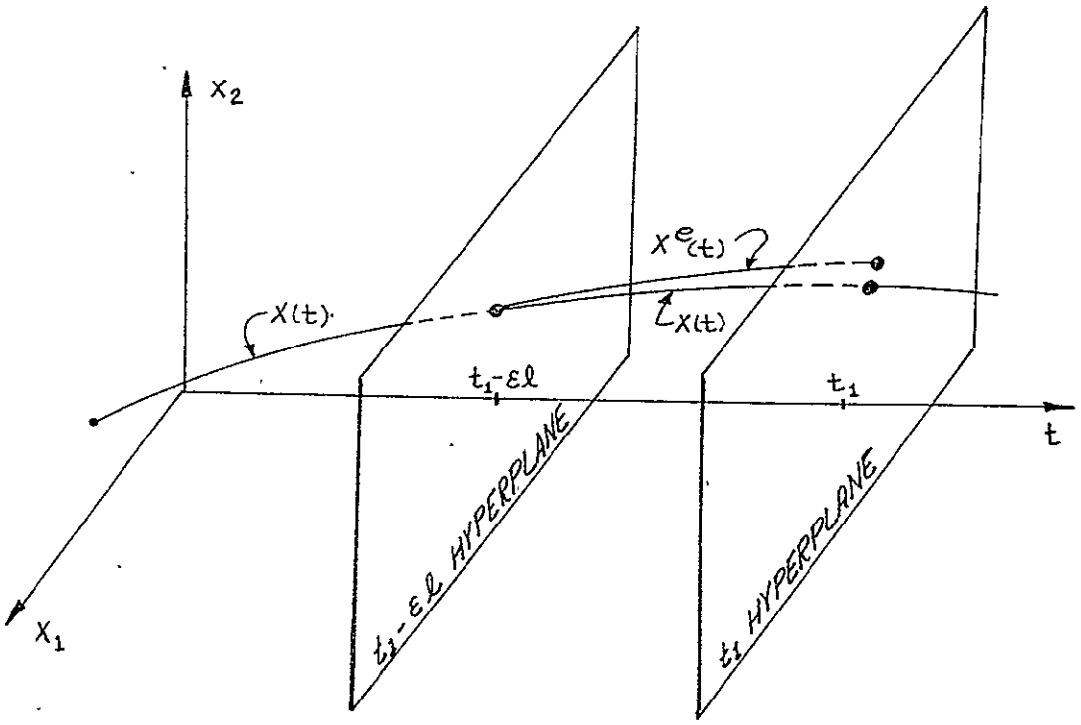

Figure 3-1

Trajectory Variations

For $x^e(t)$, $u^e(t)$ measurable, the game is to find an α^e such that (3-2) models (3-1) in that it minimizes some normed difference in their trajectories*. Define x_i^e to be the solution of (3-2) for $\alpha = \alpha_i^e$, $\alpha_i^e \varepsilon \mathcal{A}$, or i being the cardinality of $\mathcal{A}$. Define:

$$\Gamma_e(t) = \bigcup_{\forall \alpha_i^e \in \mathcal{A}} x_i^e ,$$

Obviously

$$X(t_1) \cap \Gamma_e(t_1) \neq \phi.$$

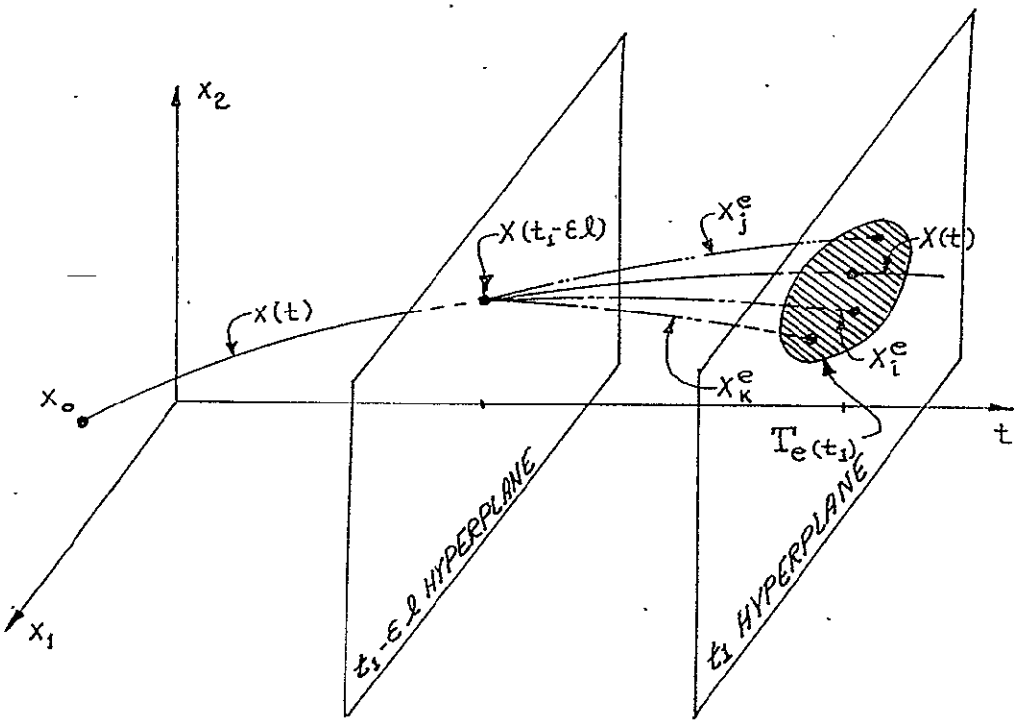

Figure 3-2

$\Gamma e(t_1)$

*An E^n norm will be used. While a $L^2[t_1 - \ell\varepsilon, t_1]$ norm would be desirable, the computational problems become prohibitive for this development.

Let it be demanded that t_1 is a Lebesgue point of α, α measurable (i.e.: if

$$\int_{t_1-\ell\epsilon}^{t_1} f(\gamma, a^i)\,dt = f(\gamma(t_1), a^i)\ell\epsilon + o(\epsilon)$$

where $o(\epsilon)$ denotes "order of ϵ").

Define an elementary perturbation of α as follows:

$$a_{\pi_i} = \left\{ \begin{array}{l} a_i^e \text{ on } t_1-\ell\epsilon \le t \le t_1 \\ a^* \text{ elsewhere} \end{array} \right\} \tag{3-3}$$

Consider $\phi(t) \triangleq x^i(t,\varepsilon)$, $u(t)\varepsilon\Omega$ and assumed known (i.e., monitorable), and $x_i^e(t,\varepsilon) = x(t,\alpha_{\pi_i})$.

Now compute $\dot{\phi}(o)$ as follows, using (3-1), (3-2), (3-3),

$$\begin{aligned} &\lim_{\epsilon\to o} \frac{1}{\epsilon}\left[x_i^e(t_1,\epsilon) - x(t_1)\right] \\ &= \lim_{\epsilon\to o} \frac{1}{\epsilon}\int_{t_1-\ell\epsilon}^{t_1} \left(f(x_i^e(t,\epsilon), a_i^e) - f(x(t), a^*)\right)dt \\ &= \lim_{\epsilon\to o} \frac{1}{\epsilon}\left\{\left[f(x(t_1,\epsilon), a_i^e) - f(x(t_1), a^*)\right]\ell\epsilon + o(\epsilon)\right\} \\ &= \left[f(x(t_1), a_i^e) - f(x(t_1), a^*)\right]\ell \\ &= \dot{\Phi}(o) \end{aligned} \tag{3-4}$$

Define (i)

$$\nu_{\pi_i}(t_1) = \frac{d}{d\epsilon} x_{\pi_i}(t_1, \epsilon)\Big|_{\epsilon=0} \tag{3-5}$$

(ii)

$$\mathcal{A}_{\pi_i} = \left\{ a_{\pi_i} \,\middle|\, a_{\pi_i} \text{ close to } a^* \right\} \tag{3-6}$$

Then

$$x_{\pi_i}(t_1, \epsilon) = x(t_1) - \nu_{\pi}(t_1)\epsilon + o(\epsilon), \tag{3-7}$$

$$\text{for } a_{\pi_i} \in \mathcal{A}_{\pi_i} \subset \mathcal{A},$$

is a point function.

Let $\sigma(t_1)$ be a hypersurface of all solutions of $\dot{x} = f(x,u;\alpha\pi_i)$ at $t=t_1$. Notice also $\nu_{\pi_i}(t_1)$ is tangent to $\sigma(t_1)$ at the point $x_{\pi_i}(t_1)$.

If $o(\epsilon)$ is sufficiently small

$$x_{\pi_i}(t_1, \epsilon) \cong x(t_1) - \nu_{\pi_i}(t_1)\epsilon. \tag{3-8}$$

The error in approximating $x(t_1)$ with $x_{\pi_i}(t_1, \epsilon)$ is $\nu_{\pi_i}(t_1)$. Minimizing $||\nu_{\pi_i}(t_1)||$ for ϵ fixed and sufficiently small, one achieves by using (3-7),

$$\min \|\nu_{\pi_i}(t_1, \epsilon)\| = \min \|\left[f(x,t_1,a^*) - f(x,t_1,a_i^e)\right]\ell\|$$

which can be written for a minimum occurring at $\alpha_i^e = \alpha^e$ as

$$\min \|\nu_{\pi_i}(t_1, \epsilon)\| = \min \|\left[\dot{x}(t_1) - f(x(t_1); a_i^e\right]\ell\|. \tag{3-9}$$

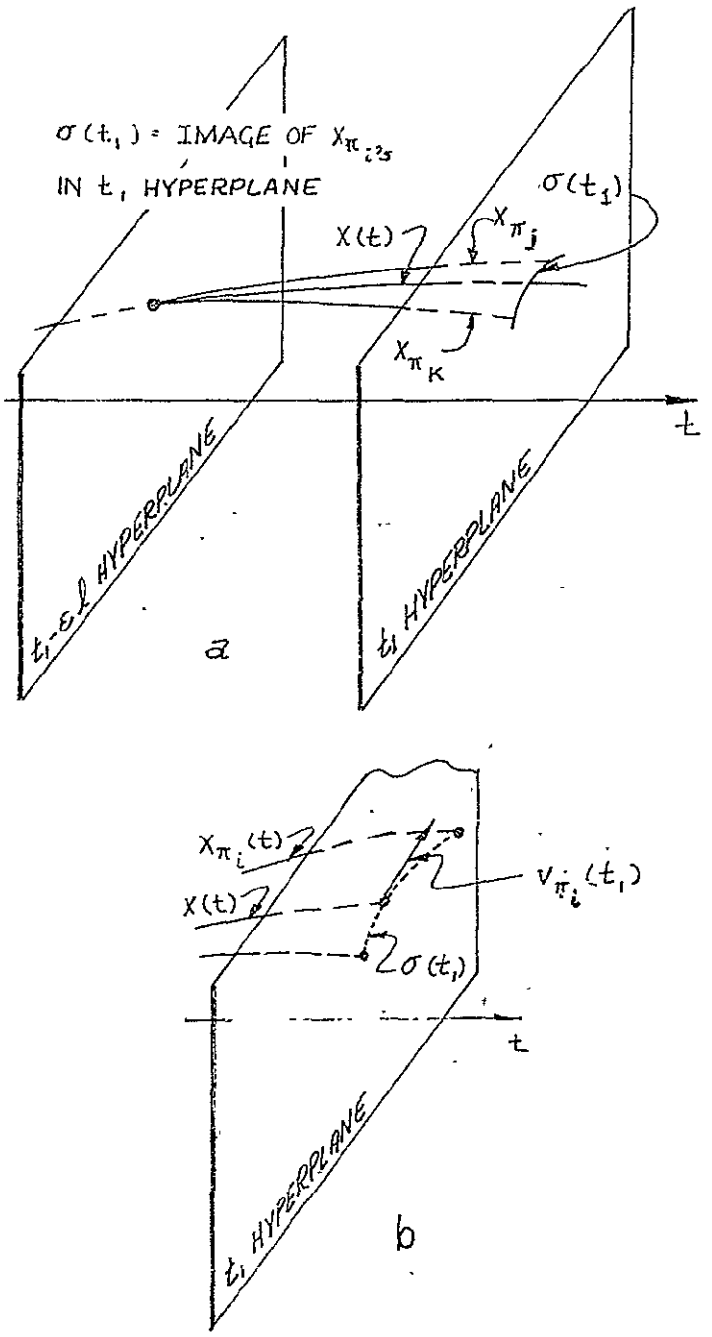

Figure 3-3 (a,b)
σ Hypersurface

The result of (3-9) lends itself to an intuitive interpretation based on triangles. This oversimplification will investigate the solution to a two-dimensional system of ordinary differential equations.

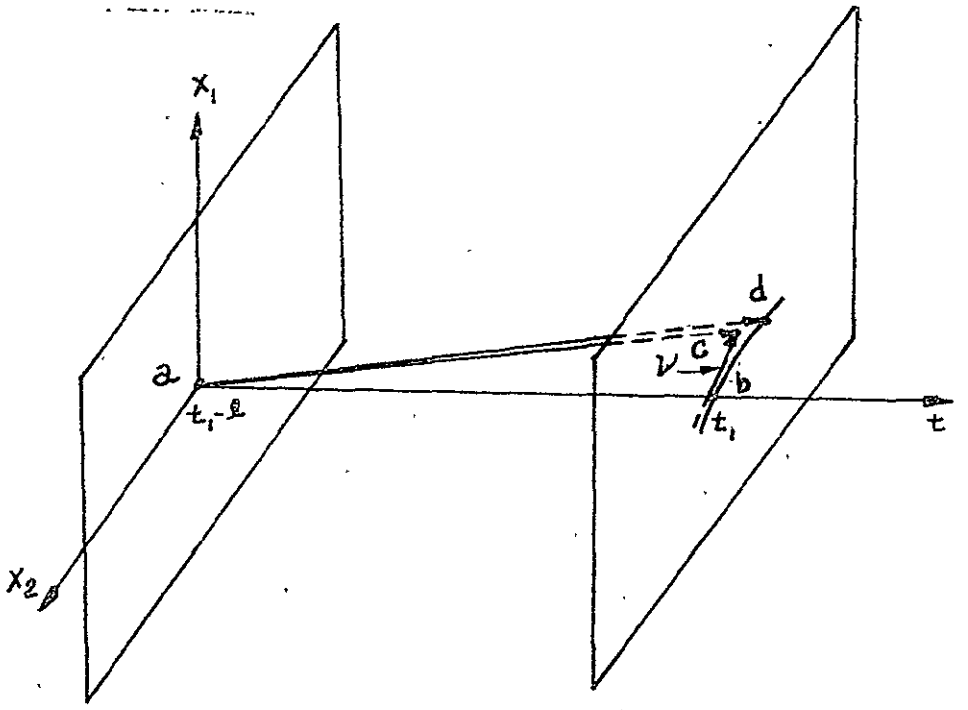

Figure 3-4
Simple Example

Let b - a = ℓ, c - b = ν and the line segment ab represent the solution (3-1) (i.e., $\alpha = \alpha^*$) and the line segment ad represent $x_{\pi_i}(t)$ for $\alpha = \alpha_{\pi_i}$. The slope of the line segment ac is approximately that of ad (i.e., $\dot{x}_{\pi_i}(t)$). Therefore, as $\nu \to 0$

$$\left\| \frac{\nu}{\ell} \right\| \to \left\| \dot{x}(t_1) - f(x, t; \alpha_i^e) \right\| \to 0.$$

Several engineering problems must be resolved if (3-9) is to be developed further. They are:

i) Is $\dot{x}(t_1)$ monitorable?

ii) What are, if any, the dimensionality restrictions on the problem?

If $\dot{x}(t)$ is monitorable, the α^e which minimizes (3-9) may, in some cases, be directly computed. The vector elements $\dot{x}_i(t_1)$, $1 \leq i \leq n$, which are not monitorable must be calculated from monitorable information. Some of the methods which will facilitate this are:

a) One-sided derivatives for $x(t)$ sufficiently smooth and $\ell\varepsilon$ sufficiently small, ℓ proportional to an a priori smoothness judgment on $x(t)$.

b) If any element of elements, $\dot{x}_i(t)$ of $\dot{x}(t)$ are independent of parameters, $\dot{x}_i(t)$ can be computed directly. (i.e., $\dot{x}_i(t)=G(x(t))$, $x(t)$ monitorable)

c) Mesh refinement methods where observations are obtainable at times t_i, $t_i \varepsilon [t_1-\varepsilon\ell, t_1]$, $i = 1, \ldots, k$, t_i monotonically increasing. The following methods will establish derived derivatives at a point, or points interior to $[t_1 - \varepsilon\ell, t_1]$.

(1) Derivative formulas from difference operations.

(2) Central difference formulas [21].

(3) Modified Euler's Formula.

(4) Weighted averaging of a sequence of $\dot{x}_i(t_{i,s})$ where $\dot{x}(t_i)$ is found by any applicable method. This will smooth the data and reduce the effect of data points which have a large variance from the mean.

The techniques of finding $\dot{x}(t_1)$ will not be pursued any further. The problem of numerically approximating a derivative is that it may be noisy. Other approximation methods shall be developed which will not have need of a $\dot{x}(t_1)$ computation. Therefore, it will be assumed that if $\dot{x}(t_1)$ is a required computation, it may be accomplished with sufficient accuracy so as not to introduce significant errors into the system. The question of dimensional requirement will now be treated.

3-2. Dimensional Restrictions

In Equation (3-9) it shall be assumed that $\dot{x}(t_1)$ has been satisfactorily computed. Also, $x(t_1)$ is known from direct measurements and the mapping f of (3-1) is well defined. Then the problem of finding an α^e which satisfies (3-9) becomes a problem of solving a linear system of non-homogeneous equations. Because α^e appears linearly in (3-9), $\dot{x}(t_1)$, $x(t_1)$ being known constants, one is interested in the solution to an algebraic equation of the form:

$$A(t_1)\alpha^e = b(t_1) \qquad (3\text{-}10)$$

where $b(t_1)$ is a n vector whose components are $b_i(t_1)$, $i = 1, \ldots, n$, and $b_i(t_1) = \dot{x}_i(t_1)$ plus any term in the ith row of $f(x(t_1); \alpha^e)$ which does not multiply an element of α^e. The matrix $A(t_1)$ is n x n.* If there does exist a zero row in $A(t_1)$ remove it by reducing the dimension of $A(t_1)$ and $b(t_1)$ by some appropriate amount (all remaining arguments will be applicable). The matrix $A(t_1)$ is formed by those fixed lumped system parameters and known state variables which multiply any element of α^e. The construction of $A(t_1)$ will have at least one entry per row. For example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{pmatrix} a_1 x_1 + x_2^2 \\ a_2 x_2^3 + x_3 \\ a_3 x_1 + a_4 x_3 \end{pmatrix} \Rightarrow A(t_1) = \begin{pmatrix} x_1(t_1) & 0 & 0 & 0 \\ 0 & x_2^3(t_1) & 0 & 0 \\ 0 & 0 & x_1(t_1) & x_3(t_1) \end{pmatrix}$$

A matrix $A(t_1)$ which has only one entry per row is exemplified by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{pmatrix} a_1 x_1 + x_2^2 \\ a_2 x_2^3 + x_3 \\ a_3 x_1 + x_3 \end{pmatrix} \Rightarrow A(t_1) = \begin{pmatrix} x_1(t_1) & 0 & 0 \\ 0 & x_2^3(t_1) & 0 \\ 0 & 0 & x_1(t_1) \end{pmatrix}$$

Let the rank of $A(t_1)$ be q.

*$A(t_1)$ will be assumed to be free of zero rows. A zero row would be the result of some $b_i(t) = 0$ in (3-10) being independent of α^e.

If $q = m = n$ the solution of (3-10) is trivial and is

$$\alpha^e = A^{-1}(t_1)b(t_1)$$

If $q = m<n$ and $x(t_1)$ is monitorable, and $\dot{x}(t_1)$ is monitored and/or computed exactly, (3-10) represents a system of m consistent linearly independent equations. Then construct

$$\tilde{A}(t_1)\alpha^e = \tilde{b}(t_1)$$

where

$\tilde{A}(t_1)$ is $m \times m$, rank m.

$\alpha^e \in E^m$

$\tilde{b}(t_1) \in E^m$

the solution becomes

$$\alpha^e = \tilde{A}^{-1}(t_1)\tilde{b}(t_1).$$

Otherwise, two types of degeneracy can occur. R. E. Mortensen [22] listed them as:

1. If $x(t_1)$ and/or $\dot{x}(t_1)$ is not monitored or computed exactly, and if $b(t_1) \varepsilon Im(A(t_1))$ then no exact solution is possible. $Im(A(t_1))$ denotes the image space of $A(t_1)$. The image space of $A(t_1)$ is the set of vectors $b(t_1) \varepsilon E^m$ of the form $b(t_1) = A(t_1)\alpha^e$ for some $\alpha^e \varepsilon E^m$. $Im(A(t_1))$ is a subspace of E^n. Case one is like having more equations than unknowns. The special case to be

considered is one where $b(t_1)$ is considered noisy, and one may choose to take redundant measurements. "Noisy" shall be interpreted as the uncertainty associated with a measurement or a calculation. In this case, one may request a solution in a "best lease mean squares fit" sense. (See end of case 2 for further results.)

2. If ker $(A(t_1)) \neq \phi$, then a solution, if it exists, is not unique. Here ker $(A(t_1))$ denotes the kernel of $A(t_1)$. Suppose E^m and E^n are the spaces to be considered; and let $A(t_1)$ be the linear mapping $A(t_1) : E^m \rightarrow E^n$. Then the kernel of $A(t_1)$ is the subset of vectors $\alpha^e \varepsilon E^m$ such that $A(t_1)\alpha^e = 0$. It follows that ker $(A(t_1))$ is a subspace of E^m. The case where ker $(A(t_1)) \neq \phi$ is like having more unknowns than equations. This particular formulation may easily be the class of problems that (3-9) is imbedded into. However, there are ways to circumvent a type 2 degeneracy to achieve a usable solution, if one does exist. One way of achieving the solution is with quadratic programming with an ordered vector cost functional.[22]

Because uniqueness cannot be achieved, it is desirable to at least guarantee that the computed vector α^e has properties which make sense in the physical system. Let it be required that α^e be close to α^n. This will exclude solutions which require the system parameters to have large variations from the nominal. It is reasonable to suggest this because the adaptive problem considered local parameter variations about the nominal.

Let

$$A(t_1)\alpha^e = b(t_1) = \lambda$$

and

$$\|\alpha^e - \alpha^n\|^2 = \langle \alpha^e - \alpha^n , \alpha^e - \alpha^n \rangle$$

$$\|\lambda\| = \langle \lambda , \lambda \rangle$$

Define $\tilde{\lambda} \triangleq \lambda - A(t_1)\alpha^n$,

$$\Gamma = \min \left\{ \|\tilde{\lambda}\| \leq \|\alpha^e - \alpha^n\| \right\}$$

This means, first determine an α^e such that $||\tilde{\lambda}||$ is minimized. If $b(t_1) \varepsilon \mathrm{Im}(A(t_1))$, then $||\tilde{\lambda}|| = 0$, otherwise minimize $||\tilde{\lambda}|| > 0$. If ker $(A(t_1)) = \emptyset$ then the solution is unique and the problem is finished. If ker $(A(t_1)) \neq \emptyset$, then it is possible to minimize not only $||\tilde{\lambda}||$ but $||\alpha^e - \alpha^n||$ also. This results is a "best" solution with respect to the given Γ. This can be more compactly stated by the following adaptation of a theorem found in [22].

Theorem 4-1: The solution to min $\{||\tilde{\lambda}|| \leftarrow ||\alpha^e - \alpha^n||\}$, where $A(t_1)\alpha^e - b(t_1) = \lambda$, and $A(t_1), b(t_1), \alpha^n$ are given, is $\alpha^e = A^+(t_1)b(t_1) + \alpha^n$, where A^+ is the pseudoinverse of A.†

The notion of a pseudoinverse is not limited to a type 2 problem. Consider a type 1 problem again with $m \leq n$ (i.e.: an overdetermined, or overspecified system of linear equations).

Define:

$$P = \|A(t_1)\alpha^e - b(t_1)\|^2 = \langle A(t_1)\alpha^e - b(t_1), A(t_1)\alpha^e - b(t_1)\rangle.$$

The parameter vector α^e will be said to be the "best solution in the least mean square sense" if it minimizes P. The minimization is accomplished by setting the gradient of $||A(t_1)\alpha^e - b(t_1))|| = 0$.

$$\nabla_\alpha \|A(t_1)\alpha^e - b(t_1)\| = 2A'(t_1)A(t_1)\alpha^e - 2A'(t_1)b(t_1)$$

$$\Rightarrow \alpha^e = (A'(t_1)A(t_1))^{-1}A'(t_1)\,b(t_1).$$

But, Penrose in [23] established the following equalities:

(i) $A\,A^+A = A$

(ii) $(A\,A^+)' = A\,A^+$

Therefore

from (i) $A'(t_1)(A(t_1)A^\dagger(t_1))' = A'(t_1)$

from (ii) $A'(t_1)A(t_1)A^\dagger(t_1) = A'(t_1)$,

†The pseudoinverse is also known by the name generalized inverse.

then $$A^{\dagger}(t_1) = (A'(t_1)A(t_1))^{-1}A'(t_1),$$

where $A'(t_1)A(t_1)$ is m x m.

If $A^{\dagger}(t_1)$ exists (i.e.: rank $A(t_1)$ = m) then the minimizing α^e is defined by:

$$\alpha^e = A^{\dagger}(t_1)\,b(t_1).$$

The parameter estimation problem will now turn to modeling techniques. Unfortunately, modeling techniques also live in the shadow of dimensional requirements. Therefore, the methods to be developed will be applicable only when certain restrictions upon the state and parameter vectors have been satisfied.

3-3. Modeling and Implicit Estimation:

Consider a system model of the form

$$\begin{aligned}\dot{x}^e(t) &= f(x^e(t), \bar{u}(t), \alpha^e(t)) \\ &= f(x^e(t), \alpha^e(t)) \qquad (3\text{-}11)\\ x^e(t_j) &\text{ given}, \; t_j \in [t_o, T]\end{aligned}$$

(superscript "-" denotes actual system variables)

where $\bar{u}$ is known from measurements (thus its functional dependence in (3-11) is omitted), and α^e is arbitrary, $\alpha^e \varepsilon \mathcal{A}$. Assume (3-11) is completely controllable and observable. Also, it will be demanded that $\dot{x}^e(t)$, $x^e(t)$, and $\alpha^e(t)$ are monitorable over $t\varepsilon[t_o,T]$. This is reasonable since (3-11) is to be synthesized (example: RLC circuit or computer program), and the monitorability can be built into the model. The model will share whatever

continuity properties (2-1) possesses. The model, which is a parameterized version of the plant, will be used to estimate the actual plant parameter vector through either (1) numerical methods or (2) gradient techniques.

The computational techniques to be developed will be philosophically different than those methods required by (3-9). While the solution of (3-9) involves the general solution of a linear system of equations (i.e.: (3-10)), the modeling technique will use iterative routines.

Formally, a model will be subject to the same control effort as that acting on the plant. The output of the model will be compared to the output of the plant in some a priori manner and an estimate of the plant parameter vector will then be made. The model will be updated, and the process repeated again using the original control. The process of iteration will be continued until terminated by some decision device.

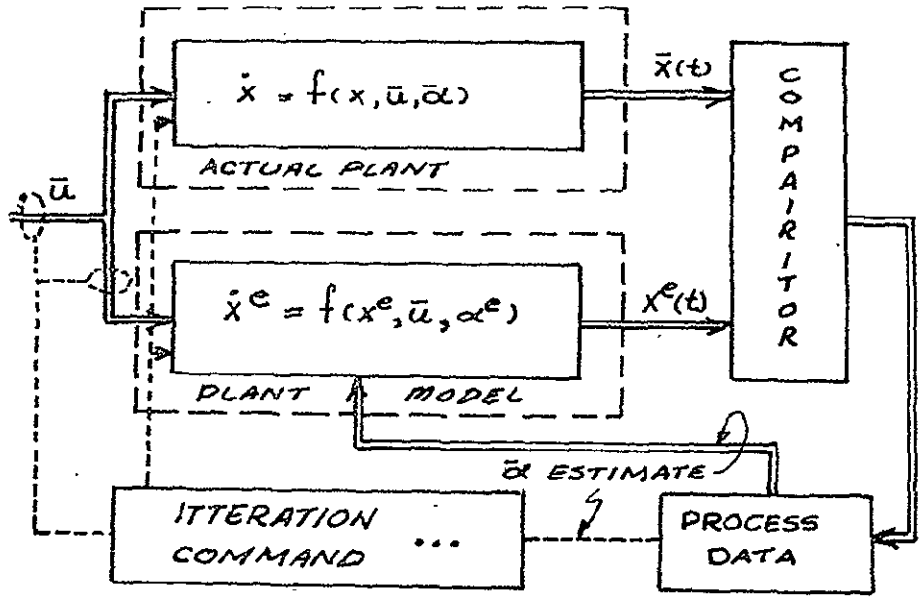

Figure 3-5
Formal System Model

Before the computational devices can be properly introduced, some important mapping requirements must be analyzed. It is desirable to insure that the parameter vectors achieved by either method (1) or (2) are unique. The Implicit Function Theorem will satisfy this goal. This well-known theorem establishes some very useful properties of a system of the form $F(x,\alpha) = z$ for a mapping F that is continuously differentiable with respect to α. A theorem to be found in [24] states that F is continuously differentiable with respect to α if and only if $[\partial F_i/\partial \alpha_j]$ exists, and is continuous for all i and j.

Theorem 3-2: (Implicit Function Theorem) [13]

Let z, α F, g be m dimensional vectors, and x an n dimensional vector. Let $F(x, \alpha)$ be continuous for (x,α) near $(\bar{x},\bar{\alpha})$ and have a continuous partial derivative with respect to the components of α. Let the Jacobian with respect to α be nonsingular (i.e.: $\det[\partial F^j/\partial \alpha_i] \neq 0$) at $(\bar{x},\bar{\alpha})$. Let $\bar{z} = F(\bar{x},\bar{\alpha})$. Then there exists $\varepsilon>0$ and $\delta>0$ such that if z and α are fixed, $||z-\bar{z}|| < \delta$ and $||x-\bar{x}||<\delta$, the equation $z = F(x,\alpha)$ has a unique solution $\alpha = g(x,z)$ satisfying $||\alpha-\bar{\alpha}||<\varepsilon$. Furthermore, $g(x,z)$ is continuous for $||z-\bar{z}||<\delta$, $||x-\bar{x}||<\delta$ and has continuous partial derivatives with respect to the components of z.

It is apparent from Theorem 3-2 that $F(x,\alpha)$ is a restricted mapping (i.e.: $E^{m+n} \rightarrow E^m$). Therefore, if the Implicit Function Theorem and model (3-11) are used together, special attention must be given to all defined mappings. Specifically, $F(x,\alpha)$ will serve as the previously noted state and model output comparator whose argument, in part, will be the model (3-11).

Assume for the moment that F is well defined (i.e.: $F:E^{n+m} \rightarrow E^m$). How then may $F(x,\alpha)$ be constructed so as to yield a computable α? The mappings to be considered are:

(i) $F(x,\alpha) = (K(t_i) - \phi(t_i,\alpha))$

(ii) $F(x,\alpha) = (K(t_i) - f(x(t_i),\alpha)$

for $t \in [t_o, T]$

$$K(t_i) \in E^q \text{ known.}$$

Several consequences to be found in these defined mappings are:

They are point functions.[*]

Also, the solution of $F(x,\alpha) = 0$ involves dimensionality considerations. (See the Dimensionality Restrictions Section)

The engineer may also use some of his intuitive powers to enrich the solution space of $F(x,\alpha) = 0$.

For example, the case where $m > n$, which implies $m > q$, it may be possible to consider $(m-q)$ parameters to be essentially nominal over some interval of time. In such a case, one may choose to fix these $(m - q)$ parameters equal to their nominal value over this interval. This would admit a $\tilde{F}: E^{n+q} \to E^q$ over this interval, and the uniqueness of the q parameters would be the result of a satisfied Implicit Function Theorem. The interval

---

*If F was defined to be a bounded linear operator taking a Banach (or Hilbert) space into a Banach (Hilbert) space, then Theorem 3-2 would require an investigation with respect to Frechet [Strong] differentials[25]. This will be considered later in this chapter.

$[t_o, T]$ can be covered by a union of closed connected intervals. For each of these intervals, $(m - q)$ parameters would be chosen to be essentially constant if possible.

Another notion of essentially constant parameter vectors can be associated with $F(x,\alpha)$. Instead of investigating the parameter alone, consider

$$F(x,\alpha) \cong F^n(x,\alpha) + F_x^n(x,\alpha)(x - x^n) + F_\alpha^n(x,\alpha)(\alpha - \alpha^n).$$

for x and α sufficiently close to x^n and α^n. Consider now the invariance of $F(x,\alpha)$ to parameter variations. Define:

$$\xi_j = F_{\alpha_j}^n(x,\alpha), \quad 1 \leq j \leq m.$$

Suppose $(m-q)$ of the $\xi_j(\alpha_j - \alpha_j^n)$ are small, and furthermore are small with respect to the remaining q, then these $(m-q)$ terms may be ignored. That is, their contribution to $F(x,\alpha)$ is negligible. The parameters associated with these $(m-q)$ terms may be, for convenience, fixed at their nominal value; thereby admitting a map $\tilde{F}: E^{m+q} \to E^q$.

Example:

Consider the following

$$F(x,\alpha) = \begin{bmatrix} K_1 - 100\,\alpha_1 x_2 \\ K_2 - 100\,\alpha_2 x_1 - \alpha_3 x_2 \end{bmatrix}$$

Also, require that $(\alpha_1,\alpha_2,\alpha_3)$ and (x_1,x_2) are restricted to some set about their nominal values.

Let:

$$X_1 = 10 \pm 1 \qquad a_1 = 1 \pm .1$$
$$X_2 = 1 \pm .1 \qquad a_2 = 1 \pm 1$$
$$a_3 = 1 \pm .1$$

Here α_1 and α_3 would be the most likely candidates to be essentially nominal. Also, K_1 and K_2 are known, and

$$-110 \leq K_1 \leq -90 \;,\; K_1^n = -100$$
$$-1101.1 \leq K_2 \leq -900.9 \;,\; K_2^n = -100.1$$

$$\xi_1(a_1 - a_1^n) = -100 x_2(a_1 - a_1^n)$$
$$\xi_2(a_2 - a_2^n) = -100 x_1(a_2 - a_2^n)$$
$$\xi_3(a_3 - a_3^n) = -x_2(a_3 - a_3^n)$$

For this example, α_3 would be the most likely candidate to have its variational effect ignored. A technique may also be developed using a mixture of both of these methods. Another suggestion relates to the adaptive gain defined by (2-17), namely $\tilde{G}$. The matrix $\tilde{G}$ is a n+m x r and time varying. Partition $\tilde{G}$ as follows:

$$r \left\{ \left[\underbrace{\tilde{G}_1}_{n} \;\vdots\; \underbrace{\tilde{G}_2}_{m} \right] \right.$$

$$\tilde{G} \triangleq \left[\left\{ \tilde{g}_{ij} \right\} \right] \qquad \begin{matrix} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, r \end{matrix}$$

If (m-q) vectors, say γ_i

$$\gamma_i \triangleq \begin{bmatrix} g_{1i}\,a_i \\ \vdots \\ g_{ri}\,a_i \end{bmatrix} \quad , i = 1, 2, \cdots, m$$

are negligible, component by component, compared to the remaining q vectors, then the corresponding α_i, of which there are (m-q), may be assumed nominal. This is because they have a minimal influence in the adaptive control. Then a mapping $\bar{F}:E^{n+q} \to E^q$ may be postulated. The problem of constructing a mapping F by intuitive means, rather than by techniques discussed in the Dimensional Restriction section, relies on ad hoc methods and the engineer must <u>think</u> seriously before ignoring totally the effects of (m-q) plant parameters. Assume that $F(x,\alpha)$ is now the well-defined map $F:E^{n+q} \to E^q$. For simplicity, let q = m, remembering the original maps F may have been defined for $q \neq m$.

3-4. Numerical Techniques

The numerical device to be investigated is the Newton Raphson method. Alternative methods posed by Todd and Ward, and Rich and Shaw, are also applicable to this section[21].

Consider $F(x,\alpha) = (\bar{x}(t_i) - \phi(t_i;\alpha^e))$

where:

The vector $\bar{x}(t_i)$ is a point in the state output space at $t = t_i$, $t_i \varepsilon [t_o, T]$.

$\phi(t_i, \alpha^e)$ is the solution of the model equation (3-11) at $t = t_i$, $x^e(t_j) = \bar{x}(t_j)$ given, $t_j < t_i$, and α^e given.

$F(x,\alpha)$ has the following properties:

(i) $F(\bar{x},\bar{\alpha}) = 0$

(ii) $F(x,\alpha) \varepsilon C^3[t_o,T]$ implies $F_\alpha(x,\alpha)$ is continuous on $||\bar{x}(t_i) - x^e(t_i), \bar{\alpha} - \alpha^e|| \leq \varepsilon$.

Now, if $F_\alpha(\bar{x},\bar{\alpha})$ is nonsingular, then there exists a unique α such that $\alpha^e = g(\bar{x}(t_i), x^e(t_i))$ in a neighborhood of $\bar{x}(t_i)$, $\bar{\alpha}(t_i)$. To compute α, let $\bar{\alpha} = \alpha^{e(o)} + g$, where $\alpha^{e(o)}$ is an initial guess of $\bar{\alpha}$. By Newton Raphson's method

$$g^{(k+1)} = g^{(k)} - [F_\alpha(x^e(t_i), \alpha^{e(o)} + g^{(k)}]^{-1}$$

$$[F(x^e(t_i), \alpha^{e(o)} + g^{(k)}]$$

with $g^{(o)} = 0$. At each iteration step k a g^k is computed, and if $\alpha^{e(o)}$ is sufficiently close to $\bar{\alpha}$, then $g^{(k)} \to g$ always as $k \to \infty$.

For example, suppose the plant is defined to be

(i) $\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t); \bar{\alpha})$

$= \bar{\alpha}\bar{x}(t) + \bar{u}(t)$,

$\bar{x}(o) = 0$,

and $\left.\begin{array}{l} \bar{a} \equiv 1 \\ \bar{u}(t) \equiv 1 \end{array}\right\}$ over $t \in [0,1]$

then $\bar{x}(1) = 1 - \exp(-1)$

Consider the model

(ii) $$\dot{x}^e(t) = a^e x^e(t) + \bar{u}(t)$$
$$x^e(0) = \bar{x}(0)$$
a^e arbitrary.

The solution to (ii) is:

$$x^e(t) = \frac{1}{a^e}\left(1 - \exp(-a^e t)\right)$$

and

$$F_{a^e}(x^e(1), a^e) = \frac{\partial\left[\bar{x}(1) + (1/a^e)(\exp(-a^e) - 1)\right]}{\partial a^e}$$

$\left[F_{a^e}(x^e(1), a^e(1))\right]^{-1}$ is nonsingular for all finite α and

$$\left[F_{a^e}(x^e(1), a^e(1))\right]^{-1} = \frac{(a^e)^2}{(-1+(1+a^e)\exp(-a^e))}$$

For $g^{(0)} = 0$ and the initial guess of $\bar{\alpha}$ being $\alpha^{e(0)} = 1/2$

$$g^{(1)} = 0 + .45 = .45 \Rightarrow a^{e(1)} = .95$$

$$g^{(2)} = .45 + .0695 = .5195 \Rightarrow a^{e(2)} = 1.095$$

$$g^{(3)} = .5195 - .0190 = .5005 \Rightarrow a^{e(3)} = 1.005$$

After three iterations, convergence becomes apparent. For $g^k \to 1/2$, for some k sufficiently large;

$$g^{(k+1)} = \frac{1}{2} - \frac{1-\exp(-1)-(1-\exp(-1))}{(-1+2\exp(-1))} = \frac{1}{2} = g^{(k)}$$

which implies

$$a^{e(0)} + g^{(k+1)} = \bar{a}$$

In this example, the model's trajectories were known in closed form. The quality was used strongly in the computation of α^e. If a closed form solution was not available, then the iterative method would have been based totally on numerical results.

A special case of the technique just developed would allow for one iteration, then again sample the plant data, namely $\bar{u}(t)$, $\bar{x}_o$, $\bar{x}(t)$ or $\dot{\bar{x}}(t)$ for $t\varepsilon[t_j,t_i]\varepsilon[t_o,T]$ and continue. The data from the second sampling then would be influenced by the updated α^e (achieved by one iteration of the last sampled data) in the adaptive control loop. This process may have merit if significant convergence is made through one iteration in that the adaptive control would be operating with a "fresher" parameter estimate. However, one must realize that frequent sampling will reduce the possible number of iterations which may be imbedded into a fixed real time interval.

Termination of these routines will be introduced as follows:

1. Completion of a fixed number of iterations, say N. This can be related to a fixed time interval over which iterating will be allowed.
2. Satisfying the $\epsilon>0$ condition,
   $$\| g^{i+p} - g^{i} \| < \epsilon$$
   p a positive integer
3. Similar to 2, except to minimize the effect of possible oscillatory convergence of g, require that the variation
   $$\sum_{l=1}^{p} \| g^{l+1} - g^{l} \| < \epsilon$$
   p a positive integer.
4. Combinations of 1, 2, or 3.

The termination device remains rather arbitrary. A few further comments on its structure will be offered as a guide to the designer. Suppose a mean (or maximum) estimate of the real time required by the computer to complete one iteration is Δt. Let ΔT denote the amount of real time that the computer will be allowed to operate on a specific set of plant data. Then $[\![\Delta T/\Delta t]\!]$ (see Symbols, Chapter 1) represents the mean (or minimum) integer number of iterations processed for the specified plant data. To add an additional degree of design freedom, consider ΔT to be either variable or fixed over

$[t_o, T]$. For example, let ΔT be proportional to an a priori smoothness assumption on the monitored (or derived) plant data.

For a particular t_j, t_j being the initiating time for a sampling interval, the basic structure of a typical "fixed" sampling interval is given in figure 3-6.

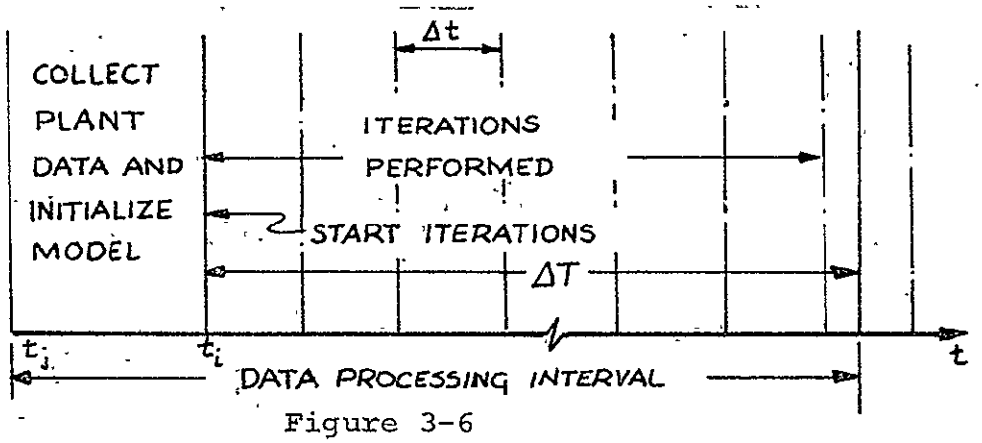

Figure 3-6
Fixed Iteration Interval

The iteration devices proposed by methods 2 and 3 (i.e.: satisfy an $\varepsilon>0$ condition) may be though of as a "rate of change" test, in that

$$\frac{\| q^{(i+1)} - q^{i} \|}{|(i+1) - i|} = \| q^{(i+1)} - q^{i} \|$$

The designer must establish a workable trade off between too large an ε, which might sacrifice convergence information, and too small an ε, which may never be achieved in a reasonable amount of time.

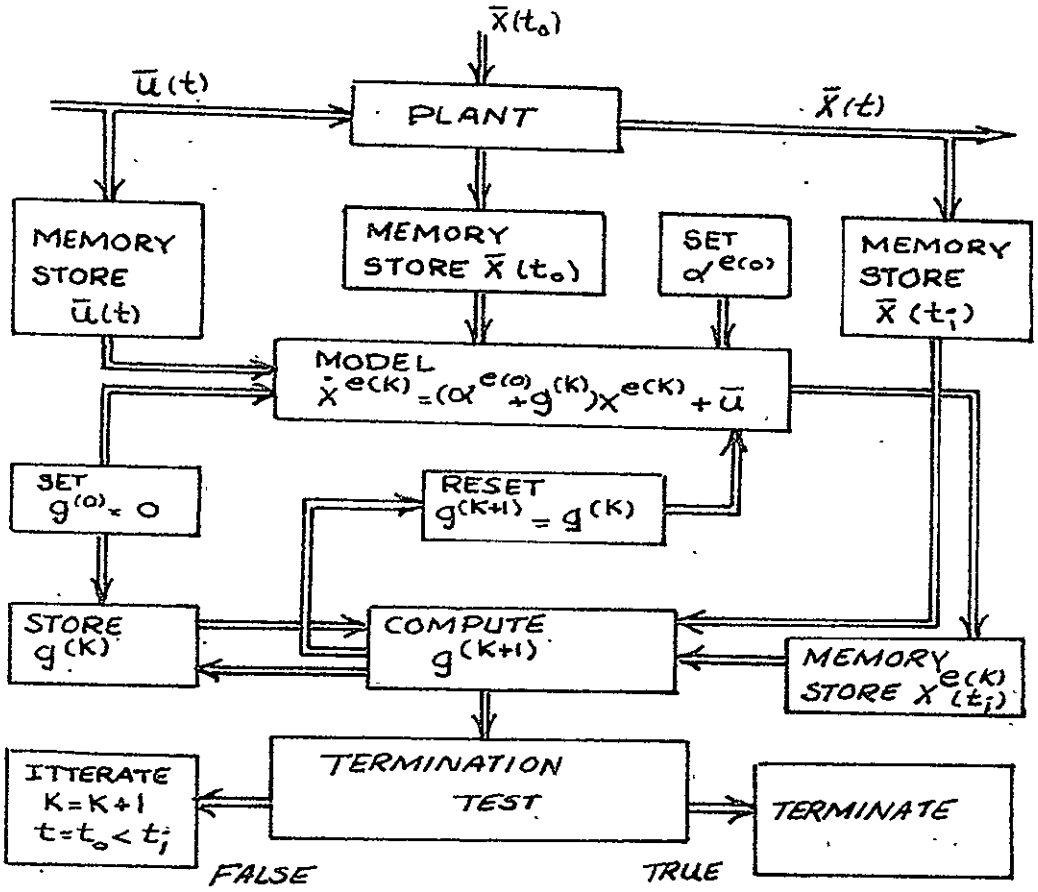

Figure 3-7
Plant - Model Diagram for Example

Earlier, the notion of acknowledging the existence of a weak time dependent component of the parameter vector $\bar{\alpha}$ was established. (example: drift, aging) The parameter estimators thus far developed were point function estimators, which implies that the estimated parameters were constants. An approximation of the time varying nature of the parameter vector can be accomplished with generalized step functions.

Definition: A generalized step function is one in which a measurable function takes on only a finite number of different real values. (i.e.: simple functions).

The claim is that for some sequence of generalized step functions

$$\left\{ a_n^e \right\} ,$$

$$\bar{a}(t) = \lim_{n \to \infty} \left\{ a_n^e \right\} .$$

Proof: Halmes[26].

This is to say that a generalized step function with a finite collection of values can be found which is arbitrarily close to $\bar{\alpha}(t)$.

Consider for example:

$$\bar{a}(t) = t.$$

Define:

$$\phi_n(t) = \left\{ \begin{array}{ll} \frac{1}{n} [\![nt]\!] & \text{if } |t| \leq n \\ 0 & \text{if } |t| > n \end{array} \right\}$$

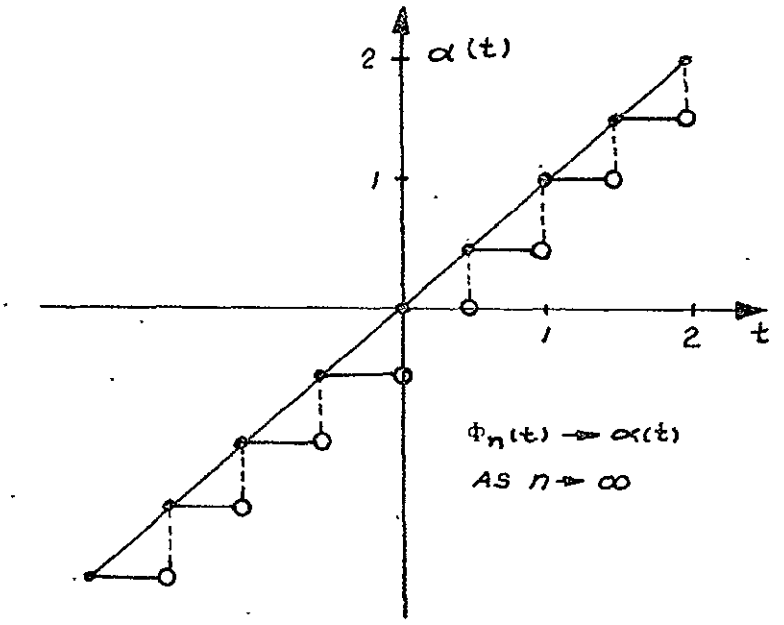

Figure 3-8

A Generalized Step Function

Therefore, one is motivated to find a "good" approximation of $\bar{\alpha}(t)$ with some refinement of the sampling interval.

Define: The kth parameter estimation to be that parameter vector based upon the kth set of independent plant data, k = 1, ..., s, s finite. (Independence to be defined later.)

Define: The data sampling interval (see figure 3-6) for the kth parameter estimation to be $[t_j^k, t_i^k]$.

Definition: The plant data is said to be independent if for all ℓ, k=1, ..., s

$$[t_j^k, t_i^k] \cap [t_j^\ell, t_i^\ell] = \phi$$

if $k \neq \ell$.

Define: The kth iteration interval (see figure 3-6) for the kth parameter estimation to be $[t_i^k, \Delta T^k + t_i^k]$, remembering ΔT^k may be fixed or variable depending on the termination strategy employed.

Define: The kth data processing interval (see figure 3-6) for the kth parameter estimation to be P^k.

$$P^k = [t_j^k, t_i^k + \Delta T].$$

The data processing scheme proceeds as follows:

1. If operating with only one complete model (with memory) in the loop (see figure 3-5), perform operation P^1 for some $t_j^1 \geq t_o$. Follow P^1 with P^2 ... P^s, $P^k \cap P^\ell = \emptyset$ for k, ℓ = 1, ..., s, $k \neq \ell$ and for the fixed terminal time T, $[t_i^s, t'] \cap T = \emptyset$ for $t' \in (t_i^s, t_i^s + \Delta T^s)$, but $\Delta T_i^s \cap T$ need not be empty. The spacing of these intervals is a matter of the designer's judgement. For example, one may choose to initiate

the data sampling to begin at $t_j^1 = t_o$ or at some delayed time $t_j^1 > t_o$. Also, one may choose to follow P^k immediately with P^{k+1}, $k+1 \leq s$, or delay it in time. The greater the expected rate of parameter change, the more densely packed should be the P's. Obviously considering a P^{s+1}, such that $[t_j^{s+1}, t_i^{s+1} + \Delta T^{s+1}] \cap T = T$

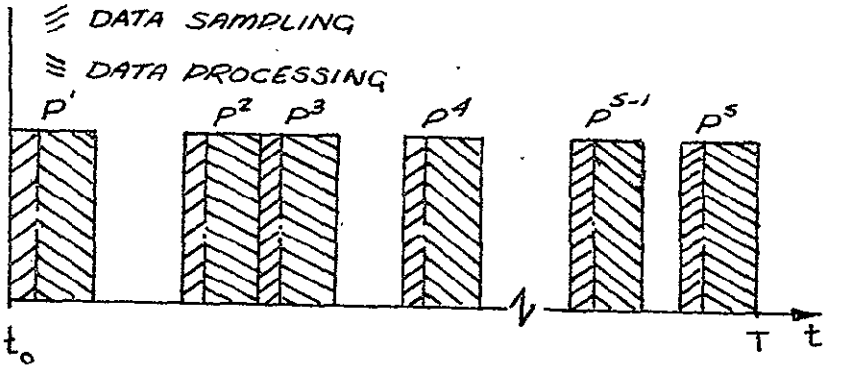

Figure 3-9
Example Sampling Interval

2. If operating with several complete models (with memory) in the loop (see figure 3-5) follow the outline of events established in (1) for each model. However, require that the intersections of small neighborhoods about the $t_i^{k's}$ associated with each model be empty.

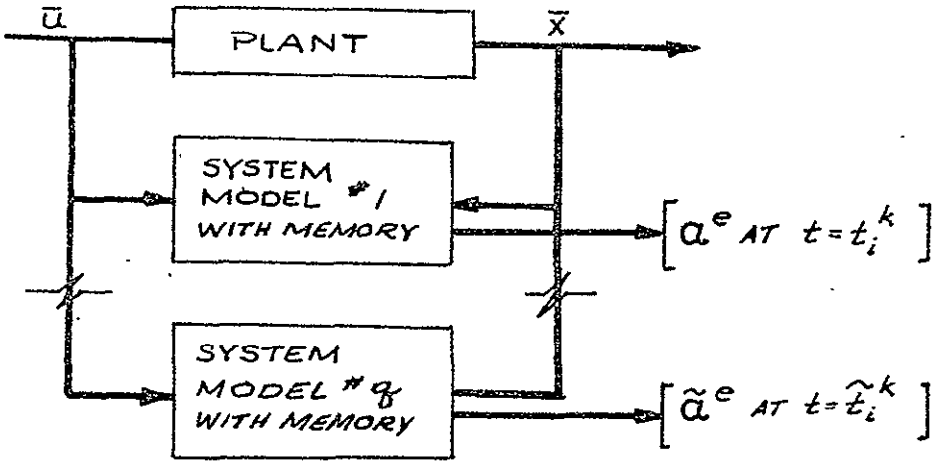

Figure 3-10

Parallel Models

Consider a second example

$$F(X, a) = (\dot{\bar{X}}(t_i) - f(x^e(t_i), a^e))$$

where $\dot{x} \in E^n$ and $f(x^e, a^e)$ is given by (3-11).

F(x,α) again satisfies the following

(i) $F(\bar{X}(t_i), \bar{a}) = 0$

(ii) $F_{a^e}(x^e(t_i), a^e(t_i))$ is continuous on

$$\| \bar{X}(t_i) - X^e(t_i), \bar{a} - a^e(t_i) \| \leq \epsilon$$

and if $F_\alpha e(\bar{x}, \bar{\alpha})$ is nonsingular, then there exists a unique α^e such that

$$a^e = g(x^e(t_i), \dot{\bar{X}}(t_i)).$$

The desired α is again obtained from Newton Raphson's method. It should be noted that $\dot{\bar{x}}(t_i)$ may not be monitorable and may have to be computed numerically.

Consider the numerical example:

$$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{u}(t); \bar{a})$$
$$= \bar{a}\bar{x}(t) + \bar{u}(t)$$

and

$$\bar{x}(t_o) = 0$$

$$\left.\begin{array}{l} \bar{a} \equiv 1 \\ \bar{u}(t) \equiv 1 \end{array}\right| \qquad t \in [t_o, T]$$

Then

$$\dot{\bar{x}}(1) = \exp(-1).$$

Define the plant model to be

$$\dot{x}^e(t) = a^e x^e(t) + \bar{u}(t)$$

for arbitrary α^e and $x^e(t_o) = 0$

$$F_{a^e}(x^e, a^e) = \frac{\partial(\dot{\bar{x}}(t_1) - a^e x^e(t_1) - 1)}{\partial a^e} = -x^e(t_1).$$

Again, if $\alpha^{e(o)} = 1/2$, $g^{(o)} = 0$

$$g^{(1)} = .143 \Rightarrow a^{e(1)} = .643$$
$$g^{(2)} = .225 \Rightarrow a^{e(2)} = .725$$
$$g^{(3)} = .280 \Rightarrow a^{e(3)} = .780$$

for $g^{(k)} = 1/2, g^{(k+1)} = 1/2 = g^{(k)} \Rightarrow \alpha^{e(k)} = 1.$

The closed form solution of $F(x,\alpha)$ was again available, which will not always be the case. Therefore, $F_\alpha e(x^e, \alpha^e)$ as well as $f(x^e(t_i), \alpha^e)$ may require numerical computations.

The truncation devices which are applicable to this problem; as well as data processing intervals, have been previously considered. (See figure 3-11)

It may be noted that the rate of convergence of numerical example one was superior to that of numerical example two. This is not to be considered a general rule. The rate of convergence is a property of the problem (i.e.: $F(x,\alpha)$) considered. Another computational device which has parameter estimation applications is the method of steepest descent.

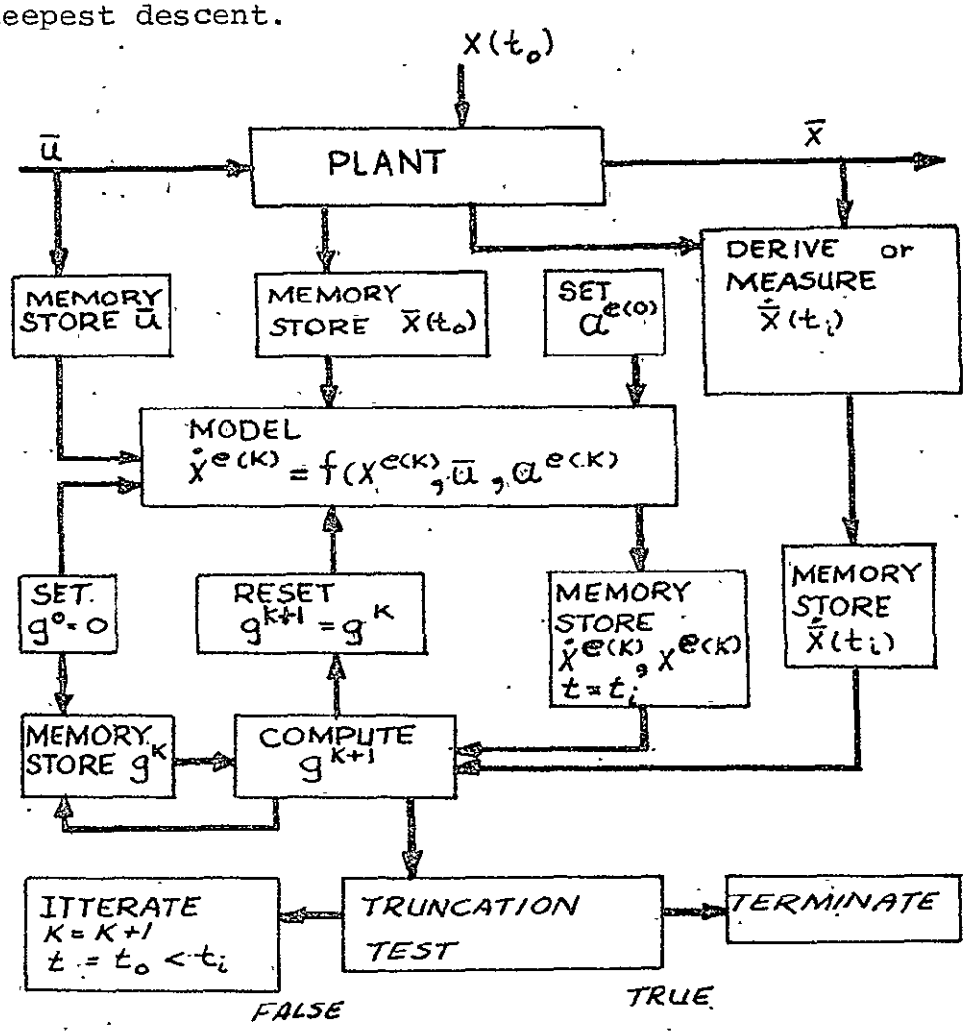

Figure 3-11

Plant - Model Diagram for Example

3-5. Gradient Techniques

The particular problems to be investigated are:

1. The minimization of a function on E^q.
2. The minimization of a function on E^q with side constraints.
3. The minimization of functionals.
4. The minimization of functionals with side constraints.

These methods will vary in their utility from problem to problem. In some instances, approximations will be required. In all cases, the required function, or functional, will be assumed to be well defined.

The first method answers the following question:

> Find a local minimum of a real valued function defined on E^q. Let $z = (z_1,\ldots,z_q)$ be a point in some region $\mathcal{T} \subseteq E^q$. Let $C(z)$ be a real valued, continuously differentiable function defined on $\mathcal{T}$. Pick a z_o, and consider all curves $z(s)$ parameterized by arc length s, $z: |dz(s)/ds| \triangleq |u'(s)| = 1 \Rightarrow$ unit velocity. For such curves passing through z_o, find a curve which minimizes $C(z(s))$ as rapidly as possible.

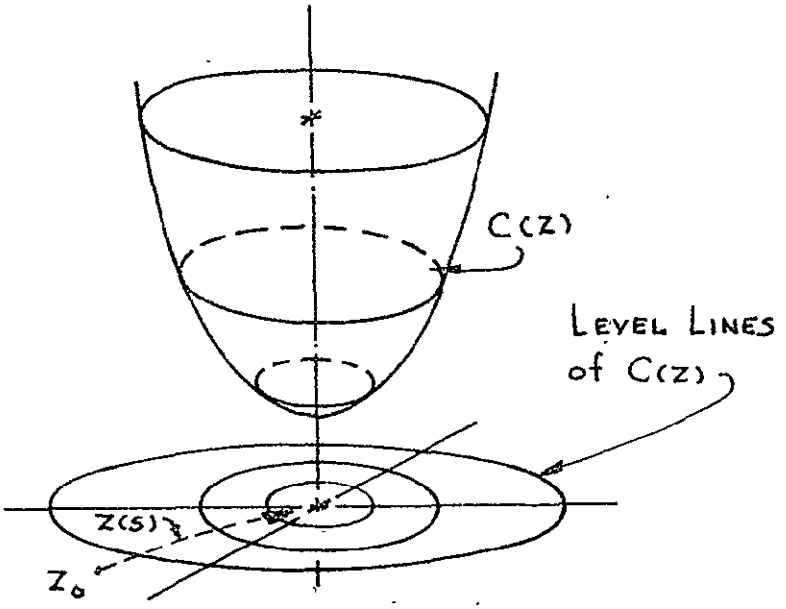

Figure 3-12
Minimization Process

The solution is found thusly:

$$\frac{dC(z(s))}{ds} \triangleq \nabla_z C(z(s)) \cdot z'(s)$$

choose

$$z'(s) = -\frac{\nabla_z C(z(s))}{|\nabla_z C(z(s))|}, \quad z(0) = z_0$$

for $\nabla_z C(z(s)) \neq 0$. Therefore $dC(z(s))/ds = -|\nabla_z C(z(s))|$. This defines a curve which is always normal to the level lines of C(z(s)) as implied by figure 3-14. If C(z) has a local minimum, then the scheme proceeds to it with unit velocity. In application

one may choose the following:

$$z'(s) = -k \nabla_z C(z(s)), \quad z(0) = z_0$$

where k adjusts the rate of convergence. If k is too large, then the minimizing trajectory may pass by the local minimum with sufficient velocity so as to tend towards another possible local minimum. If k is too small then the convergence rate is needlessly too slow.

Example:

Consider a plant

$$\dot{\bar{x}} = \bar{a}x + \bar{u}, \quad \bar{x}(0) = 0$$

and $\bar{u} \equiv 1$, $\bar{a} \equiv 1$ over $t \in [0,1]$.

Therefore

$$\bar{x}(t) = 1 - \exp(-t).$$

Consider the model

$$\dot{x}^e = a^e x^e + \bar{u}, \quad x^e(0) = 0.$$

Then $x^e(t) = (1 - \exp(-a^e t))/a^e$ and let $2C(a^e) = (\phi(1, a^e) - \bar{x}(1))^2$, where $\phi(t, a^e)$ is the solution to the model equation with arbitrary a^e.

Then $\dfrac{\partial C(a^e)}{\partial a^e} = (\phi(1, a^e) - \bar{x}(1)) \dfrac{\quad}{\partial a^e}$, $\bar{x}(1)$ monitored.

Note: In this particular example, the above partial derivative can be directly computed. But if $\phi(t, a^e)$ was not in closed form, then the partial derivative would have to

be approximated by the derived partial

$$\frac{\partial \phi(t,a^e)}{\partial a^e} \cong \frac{\phi(t,a^e) - \phi(t,\tilde{a}^e)}{(a^e - \tilde{a}^e)}$$

for $\tilde{a}^e$ sufficiently close to a^e. Another technique of determining this required partial derivative is given by Margolis[27]. He considered the ordinary differential equation.

$$\text{(i)} \quad \dot{X}(t) + aX(t) = \beta(t) \; ; \; X(t_o) = X_o$$

which can be written trivially as

$$\text{(ii)} \quad \frac{\partial X(t)}{\partial t} + aX(t) = \beta(t) \quad , \text{ ie: } \frac{\partial X(t)}{\partial t} = \frac{dX(t)}{dt}$$

Now take the partial of (ii) with respect to α and note $\beta(t) \neq f(\alpha,t)$.

$$\text{(iii)} \quad \frac{\partial^2 X(t)}{\partial a \partial t} + a\frac{\partial X(t)}{\partial a} + X(t) = 0$$

Let $v(t) = \frac{\partial X(t)}{\partial a}$, then (iii) becomes

$$\text{(iv)} \quad \dot{v}(t) = a\,v(t) = -X(t)$$

where the solution of (iv), for x(t) monitorable is $v(t) = \frac{\partial X(t)}{\partial a}$ if α is constant.

However, if α is time varying, then $\partial x/\partial t = \partial x/\partial t + (\partial x/\partial \alpha)(\partial \alpha/\partial t)$. So the validity of (iv) is predicated on α being either constant or slowly varying in time. If this be the case, then the solution to (iv) is a good approximation of $\partial x(t)/\partial \alpha$.

Therefore:

$$\frac{\partial C(a^e)}{\partial a^e} = \left(\frac{1}{a^e}(1-\exp(-a^e)-\bar{x}(1)\right)\left(\frac{1-(1+a^e)\exp(-a^e)}{a^e}\right)$$

and, $$(a^{e}(s))' = -k\,\frac{\partial C(a^{e}(s))}{\partial a^{e}(s)}\,, \quad \text{for } a^{e}(s)\neq 0$$

and $\lim_{s\to\infty} a^{e}(s) \to \bar{a}$ and if $a^{e}(0)=\bar{a}$, $a^{e}(s)=\bar{a} \iff (a^{e}(s))'=0$.

The solution of the above equation is nonlinear, and can be solved numerically.

Example:

Consider $2C(a) = (\dot{x}^{e}(1, a^{e}) - \dot{\bar{x}}(1))^2$.

Then, with computations similar to those found in the previous example

$$(a^{e}(s))' = -k(1-\exp(-a^{e}(s)) - \dot{\bar{x}}(1))\exp(-a(s)),$$

$a(0)=a_0$, $\dot{\bar{x}}(1)$ monitored.

Again $\lim_{s\to\infty} \alpha^e(s) \to \bar{\alpha}$, and for $\alpha^e(o) = \bar{\alpha}$,

$$\alpha^e(s) = \bar{\alpha} \iff (\alpha^e(s))' = 0.$$

A solution diagram of the previous example is given in Figure 3-15.

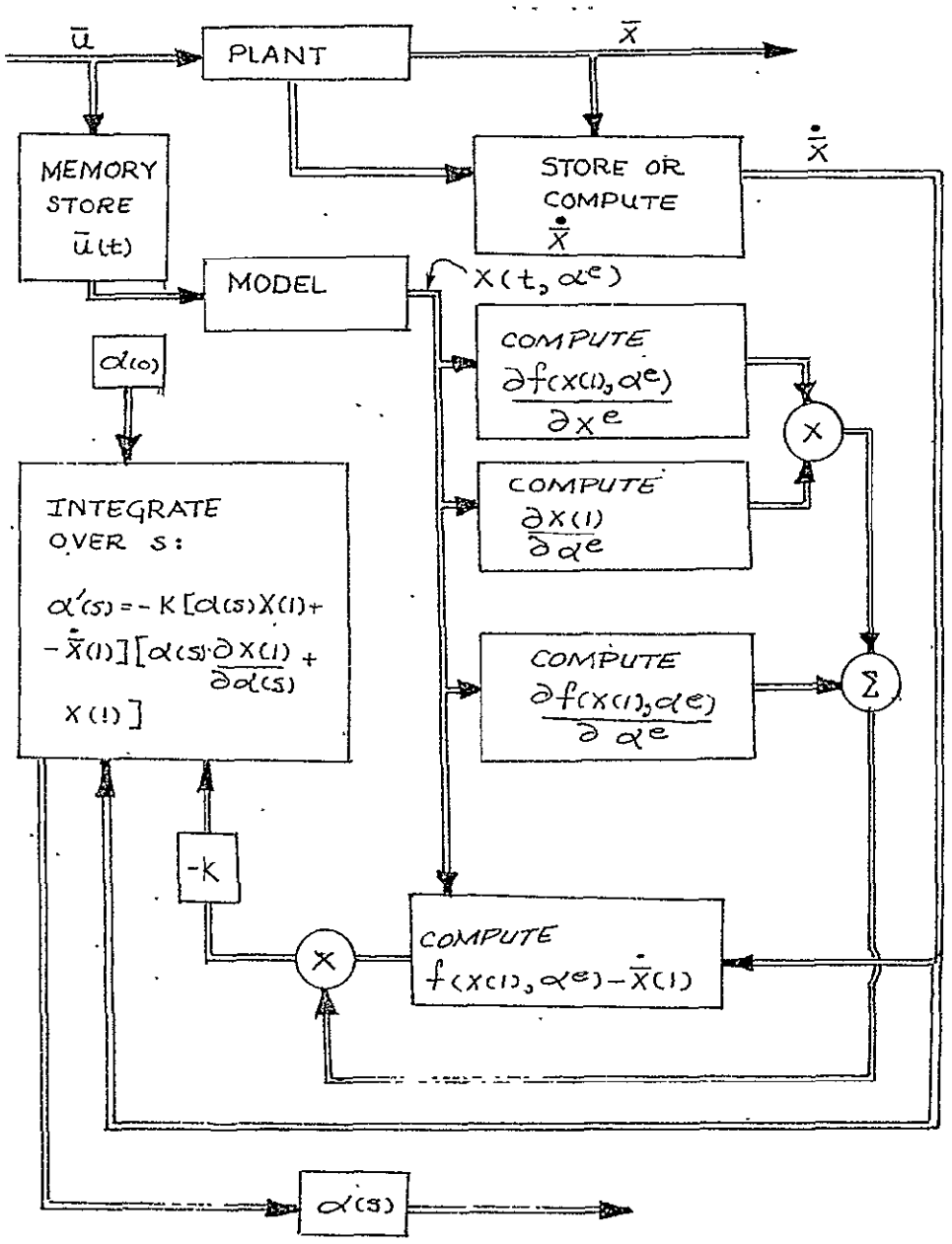

Figure 3-13
Second Example

Method 2 has the following formulation:

Let $C(\alpha)$ be given and $C(\alpha)\varepsilon C^1[t_o,T]$. Consider the minimization of $C(\alpha)$ subject to $r<m$ constraints $g_i(\alpha) = 0$, $i = 1, \ldots, r$, where $g_i(\alpha)\varepsilon C^1[t_o,T]$. Consider all curves along $\alpha(\sigma)$, with $\alpha(o) = \alpha_o$ which satisfies $\nabla_\alpha g_i(\alpha(\sigma))\cdot\alpha'(\sigma) = 0$ (see figure 3-14).

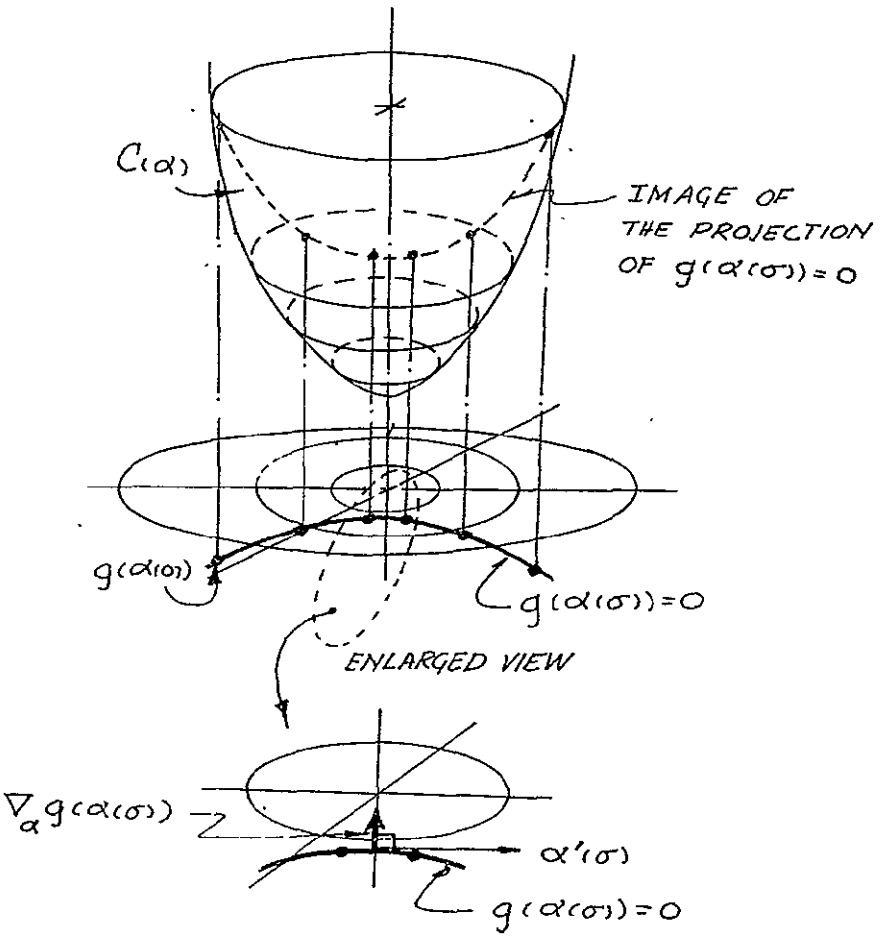

Figure 3-14
Side Constraint Problem

Among these $\alpha(\sigma)$, find the one which minimizes $d((\alpha(\sigma))/d\sigma$. It can be shown[28] that the solution to this problem is:

$$\alpha'(\sigma) = -\nabla_a C(\alpha(\sigma)) + \sum_{k=1}^{r} \lambda_k \nabla_a g_k(\alpha(\sigma))$$

where $\lambda' = (\lambda_1, \ldots, \lambda_r)$, $\lambda = G^{-1}y$ if G is nonsingular,

$$G = [\{G_{ik}\}] \text{ where } G_{ik} = \nabla_a g_i(a) \cdot \nabla_a g_k(a),$$

(G is a Grammin matrix)

and

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix}, \quad y_i = (\nabla_a g_i(a), \nabla_a C(a)).$$

However, if one investigates the possible choices for $C(\alpha^e)$ or $g(\alpha^e)$, namely:

(a) $(x^e(t_i, a^e) - \bar{x}(t_i))^p = 0$

(b) $(\dot{x}^e(t_i, a^e) - \dot{\bar{x}}(t_i))^q = 0$

for p, q positive integers, one would soon find that $\alpha'(\sigma) \equiv 0$ if $x^e(t_i)$, $\bar{x}(t_i)$, $\dot{\bar{x}}^e(t_i)$ are monitored exactly, and $\dot{\bar{x}}(t_i)$ is monitored or computed exactly. If $\alpha'(\sigma) = 0$ then $\alpha(\sigma) = \alpha(o)$. This situation is a result of $g(\alpha^e) = 0$ being equivalent to $C(\alpha^e) \equiv 0$.

In other words, (a) or (b) used as a side condition will only introduce redundant information into the parameter estimator. Then the minimization process can only yield a $\alpha(\sigma)$ which equals the fixed α^e and thus $\alpha'(\sigma) = 0$. Therefore, this method is of little use without some modifications. The modifications to be considered will relax the side constraints. This will serve to allow uncertainties in the monitoring or computing of $x^e(t_i)$, $\bar{x}(t_i)$, and $\dot{x}^e(t_i)$. Also, the relaxation will suppress the equivalence relation between $g(\alpha^e) = 0$ and $C(\alpha^e)$. This condition will be artificially introduced with "Valentine's Device"[29]. Valentine's claim was that inequality constraints can be reduced to equality constraints by increasing the dimension of the constraint space.

Suppose $\alpha^{e'} = (\alpha^e, \ldots, \alpha^e_m)$ and there are $\ell < m$ constraints $g_i(\alpha^e) \leq 0$ given.

Define:

$$\hat{g}_j[(\hat{\alpha}^e)'] = \hat{g}_j[(\alpha^e_1, \cdots, \alpha^e_m, \alpha^e_{m+1}, \cdots, \alpha^e_{m+\ell})']$$

$$= g_j[(\alpha^e_1, \cdots, \alpha^e_m)'] + [(\alpha^e_{m+j})']^2$$

$$j = 1, \ldots, \ell.$$

Consider minimizing $\hat{C}(\hat{\alpha}^e) = C(\alpha^e)$ in $E^{m+\ell}$ subject to constraints $\hat{g}_j(\hat{\alpha}^e) = 0$. Notice that $\hat{g}_j(\hat{\alpha}^e)=0 \Rightarrow g_j(\alpha^e) \leq 0$. Then, for $\hat{C}(\hat{\alpha}^e) = C(\alpha^e)$, the minimum of $\hat{C}(\hat{\alpha}^e)$ is the minimum of $C(\alpha^e)$. In particular, each of the $g_j(\alpha^e)$ shall be expressed as $g_j(\alpha^e) \leq \varepsilon_j$, where ε_j represents an a priori error estimate associated with the j constraint. Thus, ε_j associates a number with, in a sense, an allowable error tolerance resulting from inaccurate measurements or computations. For example, let $g_j(\alpha^e) = (\dot{\bar{x}}_j(t_i) - \alpha_j^e x^e(t_i) - 1)^2 \leq \varepsilon_j$. If ε_j was to equal zero, which would be used if perfect system information was available, then $\alpha_j^e = (\dot{\bar{x}}_j(t_i) - 1) / x^e(t_i) \Big|_{\substack{\epsilon_j = 0 \\ \alpha_j^e \geq 0}}$. For $\varepsilon_j > 0$, implying system uncertainty,

$$\alpha_j^e = \left[\pm \sqrt{\epsilon_j} / x^e(t_i) + (\dot{\bar{x}}_j(t_i) - 1) / x^e(t_i) \right] \Big|_{\substack{\epsilon_j = 0 \\ \alpha_j^e \geq 0}} .$$

If the system is slowly evolving in time, the techniques which determine the minimization of a function at a point have been shown applicable. Data can be collected and processed over some interval of time, say ξ. If the parameters do not change significantly, then the generated parameter vector at the end of this ξ interval is a good approximation of $\bar{\alpha}$ for some future time.

One may desire to base the estimate of $\bar{\alpha}$ based on more information than used for point estimation. For

example, let $\alpha^e \in \mathcal{L}_2[t_o,T]$. * In other words, one wishes to investigate $\alpha^e(\cdot)$ rather that $\alpha^e(t_i)$. This is accomplished with a functional format placed on the problem. There are several dissimilarities existing between a steepest descent problem, using a functional structure, and a E^q structure. First of all, the functional problem will be worked in Hilbert space, denote H. Secondly, the notion of a directly computed gradient no longer exists. Instead, one must consider Frechet (strong) differentials[25]. However, the problem will be simplified by requiring that the weak differential exists and equals the strong[25]. The existence of a strong differential implies the existence of the weak differential, but the converse is not necessarily true. Then by the Riesz Representation Theorem[30], a gradient operator may be defined.

Definition: (gradient operator)

For a Hilbert space H, and a continuous linear functional $C'(\alpha_o^e)$ on H, there exists a unique element $\nabla_{\alpha_o^e} C(\alpha_o^e)$ (the gradient of $C(\alpha^e)$ at α_o^e in H) such that:

$$C'(\alpha_o^e)h = (\nabla_{\alpha_o^e} C(\alpha_o^e), h)$$

---

*Definition: $\mathcal{L}_2[\sigma]$ is the set of all measurable functions that are square integrable over $[\sigma]$.

where (,) is the Hilbert space inner product and

$$C'(a_o^e)h = \lim_{\lambda \to 0} \frac{C(a_o^e + \lambda h) - C(a_o^e)}{\lambda}$$

for a_o^e and h in H, λ a scaler.

The functional minimization problems are:

1. Minimize a given $C(\alpha^e)$ for $\alpha^e \varepsilon H$ with no side constraints.

   Solution: Path of steepest descent.

$$(a^e(\sigma))' = -\nabla_{a^e} C(a^e(\sigma)).$$

2. Minimize a given $C(\alpha^e)$ for $\alpha^e \varepsilon H$ and constraints

$$g_i(a^e) = 0 \ , \ i = 1, \cdots, k.$$

   Solution: Path of steepest descent.

$$(a^e(\sigma))' = -\nabla_{a^e} \left(C(a^e(\sigma)) + \sum_{i=1}^{k} \lambda_i \nabla g_i(a^e(\sigma)),\right.$$

   λ_i previously defined.

The construction of $C(\alpha^e)$ must demand that the minimization of $C(\alpha^e)$ would imply $\alpha^e \to \bar{\alpha}$ over some interval of time. For example:

$$a \in \mathcal{L}_2[t_j, t_i] \ , \ [t_j, t_i] \subset [t_o, T]$$

$$C(a^e) = \int_{t_j}^{t_i} (\bar{x}(t) - \phi(t, a^e), \bar{x}(t) - \phi(t, a^e))\, dt.$$

Example:

This problem is motivated by the linear regulator problem. The problem will be offered primarily as a device to exhibit the techniques required of (1) or (2). Consider the special plant and the computations paralleling a technique used in the solution of the linear regulator problem, with appropriate modifications.

Let the linear plant be defined to be:

$$\dot{\bar{x}}(t) = C(t)\bar{\alpha}(t) + A(t)\bar{x}(t) + f(t),$$
$$\bar{x}(t_j) = x_o$$

where $f(t)$, $A(t)$, $C(t)$ are known on $t\varepsilon[t_j, t_i]$.

This particular plant is of the form $\dot{x} = Ax + B\bar{u}$ except for the $C(t)\bar{\alpha}(t)$ term. The term $f(t)$ will play the role of $B(t)\bar{u}(t)$ and $C(t)\bar{\alpha}(t)$ will serve as a biasing term for the plant's input forcing function. One may consider $f(t)$ to be the external forcing function, and $C(t)\bar{\alpha}(t)$ to be an internal forcing function

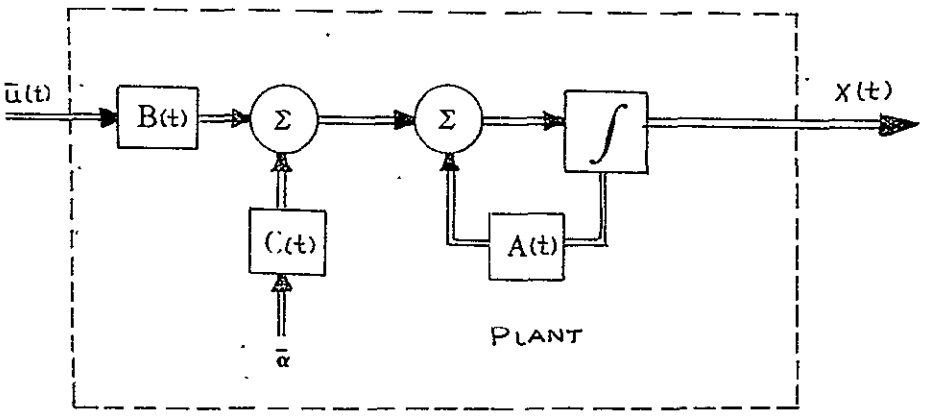

Figure 3-15
Example Problem Plant

Let the model be:

(i) $$\dot{x}^e(t) = C(t)a^e(t) + A(t)x^e(t) + f(t)$$

$$a^e \in \mathcal{L}_2[t_j, t_i]$$

$$x^e(t_j) = \bar{X}_o$$

Then, for $x^e(t_i) = \bar{X}_f$ given,

(ii) $$x^e(t_i) = \Phi(t_i, t_j)x^e(t_j) + \int_{t_j}^{t_i} \Phi(t_i, \tau)\,[C(\tau)a^e(\tau) + f(\tau)]\,d\tau$$

where $\Phi(t, t_j)$ is the fundamental matrix for the homogeneous part of (i).

The cost functional to be minimized will force the parameters to vary only in a neighborhood of their nominal values. Let

(iii) $$C(a^e) = \int_{t_j}^{t_i} \frac{(a^e(t) - a^n,\ a^e(t) - a^n)}{2}\,dt$$

$$\triangleq \frac{\|a^e(t) - a^n\|}{2}.$$

Also, the following constraints will be imposed:

(iv) $$g_k(a^e) = (\psi, a^e) - \sigma^k = 0$$

$$k = 1, \cdots, q$$

for $x(t) \in E^q$, where:

(v) $$\psi(t) = \Sigma(t_i)\Sigma(t)C(t)$$

$\psi^k(t)$ is the kth row of $\psi(t)$

and

(vi) σ^k is the kth element of

$$\bar{x}(t_i) - \Sigma(t_i,t_j)\bar{x}(t_j) - \int_{t_j}^{t_i} \Sigma(t_i,\tau) f(\tau)\, d\tau.$$

Therefore $g_i(a^e)$ becomes:

$$\int_{t_i}^{t_j} (\Sigma(t_i,\tau)\, C(\tau)\, a^e(\tau) - \bar{x}(t_i) + \Sigma(t_i,t_j)\bar{x}(t_j))\, d\tau +$$

$$+ \int_{t_j}^{t_i} \Sigma(t_i,\tau)\, f(\tau)\, d\tau = 0$$

This implies the model will be required to satisfy the terminal state conditions of the actual plant. One might also choose to use Valentine's method to represent some allowable terminal error.

Let $DC(a^e,h) = (\nabla_a C(a^e),h)$

$$= \frac{d}{d\lambda} C(a^e + \lambda h)\Big|_{\lambda=0}$$

$$= 1/2 \int_{t_j}^{t_i} \frac{d(a^e + \lambda h - a^n)^2}{d\lambda}\, d\tau \Big|_{\lambda=0}$$

$$= \int_{t_j}^{t_i} (a^e + \lambda h - a^n)\, h\, d\tau \Big|_{\lambda=0}$$

$$= (a^e - a^n, h) \Rightarrow \nabla_{a^e} C(a^e) = (a^e - a^n),$$

similarly

$$\nabla_a g_{j_k}(a^e) = \psi^k$$

(vii) $$G = [(g_{k\ell})] \quad , k, \ell = 1, \cdots, q$$

where:

$$g_{k\ell} = (\nabla_a g_k(a^e), \nabla_a g_\ell(a^e)) = (\psi^k, \psi^\ell)$$
$$= \int_{t_i}^{t_j} \psi^k(t) \psi^\ell(t)\, dt,$$

then

$$G = X(t_i, t_j) X^{-1}(t_i, t_j)\, C(t_i) C'(t_i) [X^{-1}(t_i, t_j)]'$$
$$X'(t_i, t_j) = X(t_i, t_j) M X(t_i, t_j)$$

where M is a controllability matrix, therefore a nonsingular[†].

The path of steepest descent becomes the solution to

(viii)

$$\frac{da^e(\epsilon)}{d\epsilon} = -\nabla_{a^e} C(a^e) + \sum_{k=1}^{q} \lambda_k \nabla g_k(a^e)$$
$$= -(a^e - a^n) + \sum_{k=1}^{q} \lambda_k \psi^k$$

[†] A necessary and sufficient condition that (i) be controllable (given originally by Kalman, and to be found on pages 187-188 of 28) is that:

$$M = \int_{t_i}^{t_j} X^{-1}(\tau) C(\tau) C'(\tau) (X^{-1}(\tau))\, d\tau$$

is nonsingular. In particular, if $X'(t)C(t)C'(t)(X^{-1}(t))'$ is nonsingular at just one t, then M is nonsingular for all t.

where

$$\lambda_k \triangleq (G^{-1}y)_k$$

$$y_k \triangleq (\nabla_{a^e} C, \nabla g_k)$$

$$= (a_k^e - a_k^n, \psi^k)$$

$$= (a_k^e, \psi^k) - (a_k^n, \psi^k)$$

$$= \sigma^k - \int_{t_j}^{t_i} X(t_i,\tau) C(\tau) a_k^n \, d\tau$$

$$= \sigma^k - N_k a_k^n$$

Therefore (viii) becomes

$$\frac{d a^e(\epsilon,t)}{d\epsilon} = -[a^e(\epsilon,t) - a^n] + C'(t)[X^{-1}(t)]' X'(t_i) G^{-1}[\sigma + N a^n],$$

or

$$a^e(\epsilon,t) = \exp(-\epsilon) a^e(o,t_j) + \exp(-\epsilon)\int_o^{\epsilon} \exp(\tau)\,[C'(t)$$

$$[X^{-1}(t)]X'(t_i)G^{-1}(\sigma + Na^n) + a^n]\, d\tau$$

$$= \exp(-\epsilon) a^e(o,t) + [1 - \exp(-\epsilon)]\,[C'(t)[X^{-1}(t)]'$$

$$X'(t_i) G^{-1}(\sigma + N a^n) + a^n].$$

The α^e which satisfies the minimization of $C(\alpha^e)$ with the given side constraints is found by letting $\epsilon \to \infty$. Also note that because $\exp(-\epsilon) \to 0$, the final α^e is independent of $\alpha^e(o,t)$.

3-6. Non-deterministic Parameter Estimation

So far the various parameter estimation devices have had a deterministic structure. There has been some active research associated with the problem of estimating the "best" parameter with respect to some given cost index with sufficient a priori statistics known, or assumed known. This research paper will not attempt to enter a detailed analysis of non-deterministic problems. The techniques involved in the stochastic problem are manifold. A useful device, which will find the parameter vector, minimizing a cost index of the form $||\bar{x}(t) - x^e(t)||$, is an extension of a scalar problem solved by Aoki[31]. It essentially uses the property of a sufficient statistic, a differential difference representation of the plant, a normally distributed parameter vector with unknown (but constant) mean, and a Baysian decision rule to update the estimate of the parameters. The parameter estimation is accomplished by minimizing the expected cost, starting with the last stage, then working backwards in time, and using previous observation to update the estimate of the parameters.

Although no definite parameter estimation procedure has been established, it shall be assumed that the techniques offered are sufficiently rich in number and variety so as to produce a "good" parameter estimate, if one does exist.

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CHAPTER IV

ERROR ANALYSIS

4.1 Approximating Parameters

Consider again the vector differential equation

$$\begin{bmatrix} \Delta y \\ \hline \Delta p \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & -A' \end{array}\right] \begin{bmatrix} \Delta y \\ \hline \Delta p \end{bmatrix}$$

with its 2(n+m) boundary conditions given. The partitioned Δy vector was defined to be $\left[\Delta x \middle| \Delta\alpha\right]'$, where $\Delta\alpha = \alpha - \alpha^n$. Ideally, the estimated α^e should equal the actual parameter vector α^*. (note: In Chapter III the actual parameter vector was denoted $\bar{\alpha}$). Suppose α^e has a value other than α^*. Such an estimation error would give rise to two solutions of the Riccati equation (2-15) corresponding to $\Delta\alpha^e \triangleq \alpha^e - \alpha^n$ and $\Delta\alpha^* \triangleq \alpha^* - \alpha^n$. Nothing has been said at this point about the two solutions being dissimilar.

The system parameters by assumption belong to some differentiable class of functions which implies some degree of smoothness. However, the class of functions which the estimated parameters belong to has been left to the designer to choose. This can be exemplified by the case where the parameter vector is a composition of constant vectors over disjoint intervals. That is, a piecewise constant vector. One would desire that $\alpha^e \equiv \alpha^*$, but this would necessitate α^e belongs to some smooth class of functions. The exhibited computational methods would, however, forbid in

general the realization of α^e in a smooth manner. As a result, α^e will be necessity be imbedded into a larger class of functions. For example, simple functions or piecewise continuous functions. This "large" class of functions would allow the designer a high degree of flexibility in choosing which parameter estimating device may be used. The choice of measurable vector valued functions will serve as the "large" class of functions to be considered. This class will be denoted

$$M[t_o, T]$$

such that

$$a^e \in M[t_o, T]$$

$$M[t_o, T] = \left\{ a(t) \,\middle|\, a \text{ measurable on } [t_o, T] \right\}.$$

This class of funtions is very rich in estimated parameter vector candidates. For example:

Let

$$a^e \in C^3[0,2] \subset M[0,2]$$

$$a^e(t) = \left| \begin{matrix} t^3/6 & , t \in [0,1] \\ t^2/2 - 1/3 & , t \in (1,2] \end{matrix} \right|$$

then $$\frac{d^3 a^e(t)}{dt^3} = \left| \begin{matrix} 1 & , t \in [0,1] \\ 0 & , t \in (1,2] \end{matrix} \right|$$

or, let $a^e \in P[0,1] \subset M[0,1]$ for example, and

$$a^n = .1$$

$$a^* = 1 + t$$

or, let $\alpha^e \in P[0,1] \subset M_{[0,1]}$ for example, and

$$\alpha^e = \begin{cases} 1 & , t \in [0, 1/2) \\ 1\,1/2 & , t = 1/2 \\ 2 & , t \in (1/2, 1] \end{cases}$$

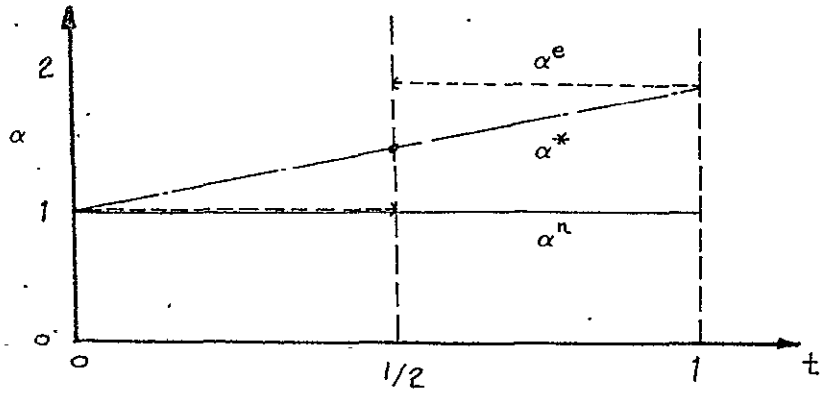


Figure 4-1a

PARAMETER PLOT

Then form

$$\Delta\alpha^e = \alpha^e - \alpha^n$$
$$\Delta\alpha^* = \alpha^* - \alpha^n$$

one achieves

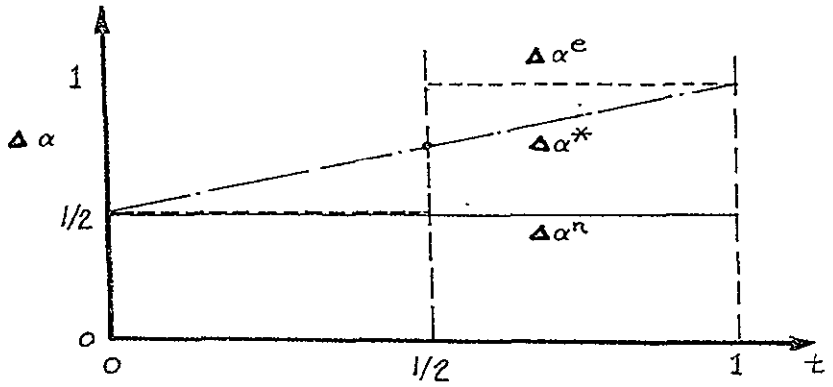


Figure 4-1b

Δ PARAMETER PLOT

Because $\Delta\alpha^e$ is an m dimensional subspace of $[\Delta y | \Delta p]' \in E^{2(n+n)}$, the righthand side of (2-15) is allowed to be discontinuous. One is interested in how this discontinuity affects the solution of (2-15). With α^e or α^* defining $\Delta\alpha$ and α^n fixed, it is possible to determine the variational bounds on the solutions of (2-15) for variations in $\Delta\alpha$ using calculus in the sense of Lebesque. But to generate insight into the problem, the study of discontinuous vector fields will be pursued.

4.2 Discontinuous Vector Fields

To motivate such arguments one can study a common application of discontinuous vector fields in control theory using closed loop state feedback control belonging to some constrained set. Problems of this class involve such notions as solutions in the sense of Filippov and stability with respect to measure. H. Hermes[32] discussed such concepts and developed some additional extensions. He chose to consider a control system of the form $\dot{x} = X(x)$

$$\text{ie: } \dot{x} = g(x, u(x)),\ x \in E^n,\ u \in E^r. \tag{4-1}$$

It should be noted that this forms a special class of problems in that the control effort is a function of the states of the system only. The control u(x) is chosen from some control set Ω and the target set consists of a manifold π in $[0,\infty) \times E^n$. If g is bounded and Lipschitzian in both arguments and u is a given Lipschitzian control,

then (4-1) has a unique solution for $x(0) = x_o$ given. Define the solution of (4-1) to be $\phi(t,0,x_o)$, where the three arguments of ϕ are to be read as the solution at time t initiating at time t equal to zero with initial data x_o. A solution which reaches the target set at $t = t_1$ will be represented as $\phi(t_1,0,x_o)\varepsilon\pi$. A reasonable question to ask is: If the target set π has a dimension less than n in E^{n+1}, can one ever expect to find a collection of x's belonging to some nontrivial neighborhood of the initial data x_o, say $\eta(x_o) \subset E^n$, and a time, say $t(x)$, $0 \leq t(x) < \infty$, such that the solution $\phi(t(x),0,x))\varepsilon\pi$? This is to say, can one show a solution starting from some n dimensional manifold reaches the target set π in some time $t(x)$ and has a dimension less than n. The answer intuitively seems to be no. In fact, it is no. The reason that the answer is false is that u(x) is Lipschitzian. If, however, the Lipschitz condition on u(x) was removed and u(x) allowed to be discontinuous, the answer may be affirmative. For example:

Suppose u is Lipschitzian, then consider (4-1) in E^{2+1}, and

$$\prod \triangleq \left[x_2, x_1 = \text{constant}, t = \text{constant}\right].$$

Then for no $t(x)$ there exists $\phi(t(x),0,x)) \in \prod$ for $x\varepsilon\, \eta(x_o)$.

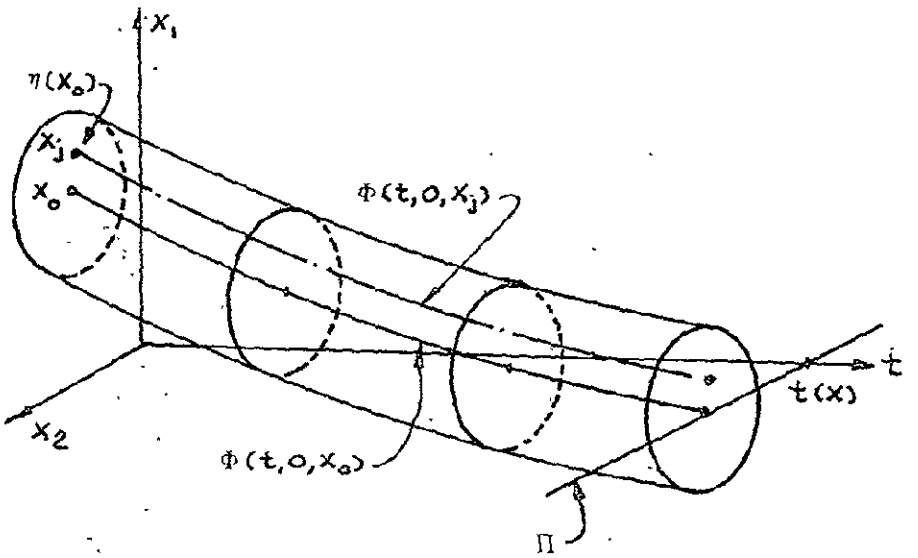


Figure 4-2

u Lipschitzian

If u is allowed to be discontinuous, then the following could occur:

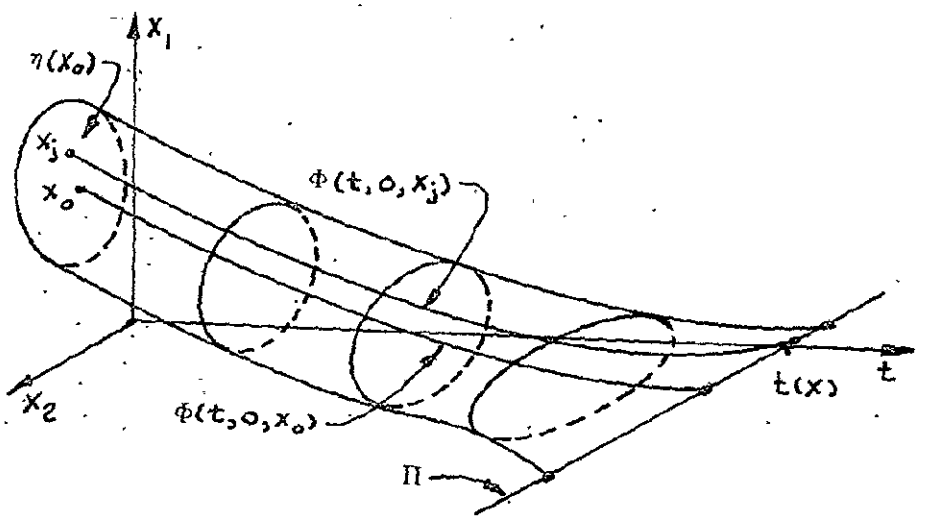


Figure 4-3

u Discontinuous

To develop the concept of solutions of differential equations whose righthand sides are discontinuous solutions in the sense of Filippov will be investigated.

4.3 Solutions in the Sense of Filippov

Let X be a measurable function defined almost everywhere in some domain $G \subset E^n$. Define:

$$K\{X(x)\} = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{co}\{X(U(x,\delta)-N)\}$$

where

$\overline{co}$ denotes convex hull

$U(x,\delta)$ denotes a closed δ neighborhood of x

N denotes an arbitrary set of E^n

μ denotes Lebesque measure

Definition: An absolutely continuous vector valued function ϕ on $[0,T]$ is called a solution in the sense of Filippov of $\dot{x} = X(x)$ if for almost all t, $\dot{\phi}(t) \in K\{X(\phi(t))\}$

Here $K(X(x))$, in a sense, determines how the derivative $\dot{x} = X(x)$ behaves locally. For example:

Consider the following graph of trajectories leaving a neighborhood of a point x, $x \in E^2$.

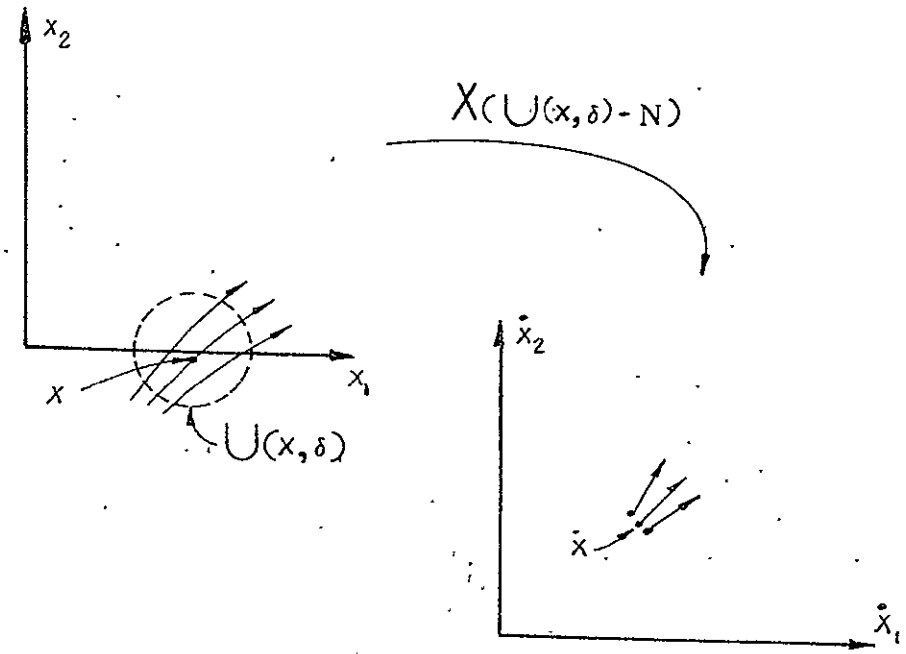


Figure 4-4

VECTOR FIELD ONE

As $U(x,\delta)$ becomes small, (i.e., $\delta \to 0$), $K\{X(x)\}$ becomes $\dot{x}$, or $K\{X(x)\} = X(x)$. However, the following extreme case may occur. Consider the following graph of trajectories.

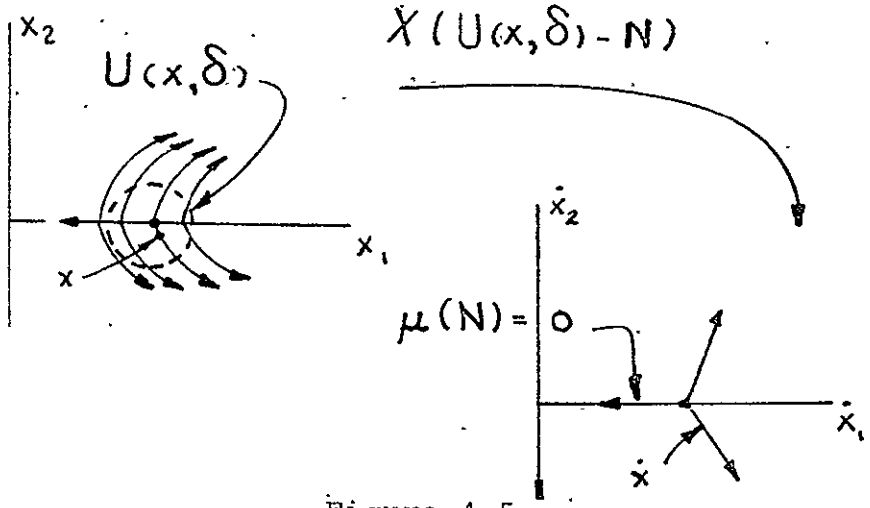


Figure 4-5

VECTOR FIELD TWO

Here $K\{X(x)\}$ becomes a wedge in E^{2+1}. This can be interpreted to mean the following. If a noisy measurement of x is taken, then one cannot be sure, even locally, that this measurement error will not admit a totally different trajectory than the one associated with the true x. Therefore, one may generate a whole family of different trajectories in some convex hull, even as $\delta \to 0$. If X is continuous, as a special case, $K\{X(x)\} = X(x)$.

Definition: If there is an absolutely continuous function ϕ for the real variable t which satisfies some initial data and $\dot{\phi}(t) = X(\phi(t))$ almost everywhere, we call ϕ a classical solution.

If ψ is a Filippov solution of $\dot{x} = X(x)$, $x(0) = x_o$, then for any $\varepsilon, \delta > 0$ there exists a measurable function $\mathcal{E}: [0,T] \to E^n$ with $||\mathcal{E}|| < \delta$ such that a classical solution ϕ exists on $[0,T]$ for the problem $\dot{x} = X(x + \mathcal{E}(t))$, $x(0) = x_o$, and satisfies $||\phi - \psi|| < \varepsilon$.

Here

$$\|\mathcal{E}\| = \text{ess. sup}\{|\mathcal{E}(t)|, t \in [t_o, T]\}$$

$$= \inf\{M \mid |\mathcal{E}(t)| \leq M \text{ almost everywhere on } [0,T]\}.$$

So, if one can show (2-15) has a solution in the sense of Filippov, then there exists an allowably small measurable error $\mathcal{E}(t)$ such that the resultant solution $\psi(t)$ differs from the classical solution $\phi(t)$ by no more than ε.

Claim

Equation (2-15) has a solution in the sense of Filippov.

Proof:

$$\text{(i)} \begin{bmatrix} \Delta \dot{y} \\ \Delta \dot{p} \end{bmatrix} = \begin{bmatrix} A & B \\ C & -A' \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta p \end{bmatrix}$$

$$\text{let} \quad D \triangleq \begin{bmatrix} A & B \\ C & -A' \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

whose elements d_{ij}, $i,j = 1, \ldots, 2(n+m)$ are bounded. Also, define

$$\Delta a \triangleq \Delta a^* = a^* - a^n \ .$$

Because (i) is a Riccati equation which has a unique solution which is expressable in terms of $K(t)$, ($K(t)$ a continuous bounded linear operator) and establishes the relationship

$$\Delta p(t) = K(t) \Delta y(t) \qquad \text{(see 2-16)}$$

$\Delta \dot{y}$ may be described as follows:

$$\text{(ii)} \quad \Delta \dot{y} = E(\Delta y) = D_1(\Delta y, K \Delta y) = (A + BK) \Delta y \ .$$

Therefore, all that need be shown to prove the claim is that (ii) has a solution in the sense of Filippov. Call the solution of (ii) $\tilde{\psi}(t)$. Notice from the continuity of $\tilde{\psi}(t)$ on $[t_o, T]$ that $\tilde{\psi}$ is a solution of (2-15) in the sense of Filippov.

Therefore, for any $\varepsilon, \delta > 0$ there exists a measurable function $\mathcal{E}(t)$ on $[t_o, T]$ with $||\mathcal{E}|| < \delta$ such that a ψ satisfying $\Delta \dot{y} - E(\Delta y + \mathcal{E}(t))$ exists and furthermore it satisfies

$||\phi - \psi|| < \varepsilon$. Although an $\mathcal{E}(t)$ has only been postulated, one may hope that there exists such an $\mathcal{E}(t)$ which can be used to represent the difference between α^* and α^e. If one considers such a representation, and if $\mathcal{E}(t)$ is small in norm, it shall be shown that the difference between the solutions of (ii) corresponding to $\Delta\alpha^* = \alpha^* = \alpha^n$ and $\Delta\alpha^e = \alpha^e - \alpha^n$ is also small. Therefore, one's attention is directed toward relating the magnitude of the parameter estimation error to the errors found in the solution space (i.e., $\phi - \psi$). This shall be accomplished through the study of "stability with respect to measure."

4.4 Stability with Respect to Measure

Definition:[32] A vector field E for which a classical solution ϕ of $\dot{z} = E(z)$ exists with arbitrary initial data z_o, is said to be stable with respect to measure if given an $\varepsilon>0$ and $T>0$ finite, there exists a $\delta>0$ such that whenever $\mathcal{E}$ is a measurable function on $[t_o,T]$ with values in E^n and $||\mathcal{E}|| < \delta$, for which a corresponding solution of

$$\dot{z} = E(z+\mathcal{E}(t))$$

$$z(t_o) = z_o$$

exists on $t\varepsilon[t_o,T]$, then $||\phi - \psi|| < \varepsilon$.

The last definition is a canonical definition for this problem. By that it is meant that this definition would have evolved from a straight calculus attack on the problem. However, this approach would probably not have developed fully the powerful property which is nested in the given definition. First of all, the whole, or only

part of the vector z may be considered to represent measurement errors. Also, the measurement error is additive which will lend itself to intuitive arguments. Another important feature is that one is working with a given error bound placed on the normed difference of ϕ and ψ, namely ε. This allows the designer to establish some maximum tolerable error on the solutions of (i) in some a priori fashion.

Claim:

The vector field defined by D_1 is stable with respect to measure.

Proof:

Consider $\varepsilon>0$ given and $t\varepsilon\left[t_o,T\right]$. Let ϕ be the classical solution of (ii). Let ψ be a solution in the classical sense of

(iii) $$\Delta\dot{y} = D_1(\Delta y + \mathcal{E}, K(\Delta y+\mathcal{E}))$$
$$\Delta y(t_o) \text{ given}$$
or $$\Delta\dot{y} = (A+BK)\Delta y + (A+BK)\mathcal{E}$$

Let $\Phi(t)$ be the fundamental matrix of (ii). Let $\phi(t,t_o,y)$ and $\psi(t,t_o,y)$ denote the solutions of (ii) and (iii) respectively at time t and arbitrary initial data y_o given at t_o. Then

$$\phi(t,t_o,y) = \Phi(t,t_o)y$$

and

$$\psi(t,t_o,y) = \Phi(t,t_o)y + \int_{t_o}^{T}\Phi(t,T)(A(T)+B(T)K(T))\mathcal{E}(T)dT.$$

Therefore

$$\| \phi(t,t_o,z) - \psi(t,t_o,z) \| =$$

$$= \| \int_{t_o}^{t} \Phi(t,\tau)(A(\tau)+B(\tau)K(\tau))\,\mathcal{E}(\tau)\,d\tau \| \leq$$

$$\leq |T-t_o| \, \| \Phi(t,\tau)(A(\tau)+B(\tau)K(\tau)) \| \| \mathcal{E}(\tau) \|,$$

$$\tau \in [t_o, T].$$

Therefore choose

$$\delta \quad \frac{\epsilon}{|T-t_o| \, \| \Phi(t,\tau)(A(\tau)+B(\tau)K(\tau)) \|} \quad \# \qquad (4\text{-}2)$$

If a more delicate δ was desired, then a closer examination of $\| \Phi(t,\tau)(A(\tau)+B(\tau)K(\tau)) \|$ for a given plant need be performed.

Example: (scalar)

Let $f(x;u;\alpha) = \alpha x(t) + bu(t)$

and

$$C(u) = \frac{1}{2} \int_{t_o}^{T} (x^2(t) + u^2(t))\,dt$$

Then

$$\left[\begin{array}{c} D_1 \\ \hline D_2 \end{array}\right] = \left[\begin{array}{cc:cc} a^n & x^n(t) & -(b^n)^2 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & -\bar{p}^n(t) & -a^n & 0 \\ -\bar{p}^n(t) & 0 & -x^n(t) & 0 \end{array}\right]$$

For $\Delta\alpha = \alpha^* - \alpha^n = c$, c a constant under perfect measurement.

From (ii):

$$\Delta \dot{x}(t) = a^n \Delta x(t) + x^n(t)c - (b^n)^2 [k_{11} \Delta x(t) + k_{12} c]$$

$$= (a^n - (b^n)^2 k_{11}) \Delta x(t) + (x^n(t) - (b^n)^2 k_{12}) c$$

$$\Delta \dot{a} = 0;$$

Suppose $\Delta\alpha$ is subject to a measurement error such that

$$\Delta y(t) = \begin{bmatrix} \Delta x(t) \\ \Delta a \end{bmatrix} = \begin{bmatrix} \Delta x(t) \\ c + \mathcal{E}(t) \end{bmatrix}$$

The term $c + \mathcal{E}(t)$ represents $(\alpha^* - \alpha^n) + (\alpha^e - \alpha^*)$. Suppose α^e equals α^* almost everywhere on $[t_o, T]$. Then the "bad" data points belong to a set of zero measure. Therefore, $||\mathcal{E}(t)|| = 0$ and $\delta = 0 \Rightarrow ||\phi - \psi|| = 0$. This result correlates with the fact that two differential equations, namely,

(a) $\dot{\omega} = f(\omega, o)$
$\omega(t_o) = \omega_o$

and

(b) $\dot{\omega} = f(\omega, v)$
$\omega(t_o) = \omega_o$

and v measurable,

where f is bounded and Lipschitzian in the first argument and measurable in the second, will have solutions equal everywhere if v differs from zero only on a set of zero measure, If $\mathcal{E}(t)$ is some finite (in norm) measurement error, then a bound δ can be calculated from (4-2). In this example only parameter vector measurement and/or estimation errors were considered. The δ resulting from an a priori ϵ will influence which of the possible choices of a parameter estimator may be selected. That is, there will be a prescribed δ precision required from the

parameter estimator implemented.

4.5 Cost Index Error

What variations exist in the cost index due to measurement (i.e., approximation errors)? Consider a neighborhood of $y^n(t)$ sufficiently small so as to admit a truncated Taylor series representation of $H(y,p,u)$ with negligible truncation terms. Let $C(u)$ be again defined as

$$C(u) = \tfrac{1}{2}\langle x(T), Tx(T)\rangle + \int_{t_0}^{T} L(x,u)\,dt$$

where L is Lipschitz continuous. From D_1 being stable with respect to measure and $\varepsilon>0$ such that if $||\mathcal{E}(t)||<\delta$, then $||\phi - \psi||<\varepsilon$. The vector $\phi(t)$ and $\psi(t)$ are members of $E^{(n+m)}$ and were previously defined. By Minkowski's inequality it is apparent that

$$\|\phi(t)-\psi(t)\| \le \|(\phi_1(t),\cdots,\phi_n(t))'-(\psi_1(t),\cdots,\psi_n(t))'\| + \|(\phi_{n+1}(t),\cdots,\phi_{n+m}(t))'-(\psi_{n+1}(t),\cdots,\psi_{n+m}(t))'\|.$$

Even with the postulated small measurement, or estimation errors, the adaptive control Δu will have its original structure.

$$\Delta u(t) = G(t)\Delta y(t) + H(t)\Delta p(t)$$

where $G(t)$ and $H(t)$ are defined by the matraices of (2-12) in an obvious manner. But by virtue of the fact that $\Delta p(t) = K(t)\Delta y(t)$, $\Delta u(t)$ becomes

$$\Delta u(t) = \left[G(t) + H(t)K(t)\right]\Delta y(t).$$

Because L is Lipschitzian, $C(u_1) - C(u_2) =$

$$= \tfrac{1}{2}\langle x_1(T), Tx_2(T)\rangle - \tfrac{1}{2}\langle x_2(T), Tx_2(T)\rangle + \int_{t_0}^{T}\left[L(x_1,u_1) - L(x_2,u_2)\right]dt$$

can be written as

$$\|C(u_1) - C(u_2)\| \leq |T-t_0|\,\bar{K}\,\|X_1-X_2, u_1-u_2\| + \\ + \bar{K}\,\|X_1(T)-X_2(T)\|$$

where $\bar{K}$ is Lipschitz constant, u_1 corresponds to the control strategy with $\alpha_1 = \alpha^*$ and u_2 corresponds to the control effort with $\alpha_2^2 = \alpha^* - \mathcal{E}$.

By Minkowski's inequality

$$\|C(u_1) - C(u_2)\| \leq |T-t_0|\,\bar{K}\,\left[\|X_1-X_2\| + \|u_1-u_2\|\right] + \\ + \bar{K}\,\|X_1(T)-X_2(T)\|$$

or

$$\|C(u_1) - C(u_2)\| \leq |T-t_0|\,\bar{K}\,\left[1 + 1/|T-t_0| + \|G+HK\|\right]\epsilon = \epsilon'$$

for ϵ given and $\delta>0$ sufficiently small.

Thus, for the ϵ, δ conditions satisfied, the normed difference in the cost indicies based upon the ϕ and ψ solutions are nothing more than a scalar multiple of ϵ. If maintaining ϵ' below some maximal value was of paramount importance, then one would solve for the last equation for ϵ and this ϵ would then define a required δ.

The error bounds found in this chapter are not necessarily the sharpest available, but will always be satisfied. Therefore, the designer may look upon them as a maximal guide.

4.6 Truncation Error

Consider now (2-13) and (2-14) with regard to their solution with and without ignoring the contributions of

truncation errors. That is

(i)

$$\Delta \dot{y} = \left[\begin{array}{c|c} A & B' \end{array}\right] \left[\begin{array}{c} \Delta y \\ \hline \Delta p \end{array}\right] + \left[\begin{array}{c} \lambda_1 \\ \hline 0 \end{array}\right], \quad \lambda_1 : n \times 1$$

$$\Delta y(t_o) \text{ given}$$

(ii)

$$\Delta \dot{p} = \left[\begin{array}{c|c} C & -A' \end{array}\right] \left[\begin{array}{c} \Delta y \\ \hline \Delta p \end{array}\right] + \left[\begin{array}{c} \lambda_2 \end{array}\right], \quad \lambda_2 : n+m \times 1$$

$$\Delta p(T) \text{ given}$$

for some previously defined λ_1, and λ_2. The vectors λ_1 and λ_2 will be used to represent the lumped truncation errors suggested by (2-13) and (2-14). Furthermore, for λ_1 and λ_2 being small bounded truncation errors defined over $(y,p,u) \epsilon N$ (N defined in Theorem 2-1) there exists an ϵ_1 and ϵ_2 such that

$$\left.\begin{array}{l} \sup \left\| \left[\begin{array}{c} \lambda_1 \\ \hline 0 \end{array}\right] \right\| \leq \epsilon_1 \\ \sup \left| \lambda_2 \right| \leq \epsilon_2 \end{array}\right\} \quad t \in [t_o, T] \text{ and for all } (y,p,u) \in N$$

Let $\Omega(t,t_o)$ be the $2(n+m) \times 2(n+m)$ fundamental matrix of

(iii)
$$\begin{bmatrix} \Delta \dot{y} \\ \Delta \dot{p} \end{bmatrix} = \begin{pmatrix} A & B \\ C & -A' \end{pmatrix} \begin{bmatrix} \Delta y \\ \Delta p \end{bmatrix},$$

that is

$$\begin{bmatrix} \Delta y(t) \\ \Delta p(t) \end{bmatrix} = \Omega(t,t_o) \begin{bmatrix} \Delta y(t_o) \\ \Delta p(t_o) \end{bmatrix}$$

where Δp_o is the unknown initial costate vector. Partition $\Omega(t,t_o)$ as

$$\Omega(t,t_0)=\left(\begin{array}{c|c}\Omega_{11}(t,t_0) & \Omega_{12}(t,t_0)\\ \hline \Omega_{21}(t,t_0) & \Omega_{22}(t,t_0)\end{array}\right).$$

Define $\tilde{T}$ to be ($\tilde{T}$ being the weighting positive semidefinite matrix in (2-2),

$$\tilde{T}=\left(\begin{array}{c|c}T & 0\\ \hline 0 & 0\end{array}\right) \qquad (n+m)\times(n+m)$$

which results in

$$\Delta p(T)=\tilde{T}\ \Delta y(T).$$

This condition requires that at the terminal time T the estimated parameter may belong to all of E^m. (Remember, Δy actually belongs to some restricted subset of E^{n+m}). Also

$$\Delta y(T)=\Omega_{11}(T,t)\Delta y(t)+\Omega_{12}(T,t)\Delta p(t)+$$
$$+\int_t^T\left(\Omega_{11}(T,\tau)\left[\begin{array}{c}\lambda_1\\ \hline 0\end{array}\right]+\Omega_{12}(T,\tau)\left[\lambda_2\right]\right)d\tau$$

and

$$\Delta p(T)=\Omega_{21}(T,t)\ \Delta y(t)+\ \Omega_{22}(T,t)\Delta p(t)+$$
$$+\int_t^T\left(\ \Omega_{21}(T,\tau)\left[\begin{array}{c}\lambda_1\\ \hline 0\end{array}\right]+\Omega_{22}(T,\tau)\left[\lambda_2\right]\right)d\tau$$

$$=\tilde{T}\ \Delta y(T)$$

After some algebra, $\Delta p(t)$ becomes

$$\Delta p(t)=\left[\Omega_{22}(T,t)-\tilde{T}\,\Omega_{12}(T,t)\right]^{-1}\left[\tilde{T}\Omega_{11}(T,t)-\ \Omega_{21}(T,t)\right]\Delta y(t)+$$
$$+\int_t^T\left(\left[\Omega_{22}(T,\tau)-\tilde{T}\,\Omega_{12}(T,\tau)\right]^{-1}\left(\tilde{T}\Omega_{11}(T,\tau)-\Omega_{21}(T,\tau)\left[\begin{array}{c}\lambda_1\\ \hline 0\end{array}\right]+\left[\lambda_2\right]\right)d\tau$$

But

$$K(t) = \left[\Omega_{22}(T,t) - \tilde{T}\Omega_{12}(T,t)\right]^{-1}\left[\tilde{T}\Omega_{11}(T,t) - \Omega_{21}(T,t)\right],$$
$$K(T) = \tilde{T}.^{12}$$

The matrix K(t) has been previously defined. Upon substitution

$$\Delta p(t_o) = K(t_o)\Delta y(t_o) + \int_{t_o}^{T}\left(K(\tau)\begin{bmatrix}\lambda_1\\ \overline{0}^{-1}\end{bmatrix} + [\lambda_2]\right)d\tau$$

Therefore,

$$\Delta y(t) = \left[\Omega_{11}(t,t_o) + \Omega_{12}(t,t_o)K(t_o)\right]\Delta y(t_o) +$$
$$+ \int_{t_o}^{t}\left(\left[\Omega_{11}(t,\tau) + \Omega_{12}(t,\tau)K(\tau)\begin{bmatrix}\lambda_1\\ \overline{0}^{-1}\end{bmatrix} + 2\Omega_{12}(t,\tau)[\lambda_2]\right)d\tau$$

Let $\Delta\bar{y}(t)$ and $\Delta\bar{p}(t)$ be the solutions of (iii), $\Delta\bar{y}(t_o)$ given, $\Delta\bar{p}(T) = \Delta p(T)$ given. Then, for $\Delta\bar{p}(t) = K(t)\Delta\bar{y}(t)$,

$$\Delta\bar{y}(t) = \left[\Omega_{11}(t,t_o) + \Omega_{12}(t,t_o)K(t_o)\right]\Delta\bar{y}(t_o).$$

Therefore,

$$\|\Delta y(t) - \Delta\bar{y}(t)\| = \left\|\int_{t_o}^{T}\left(\left[\Omega_{12}(t,\tau)K(\tau) + \Omega_{11}(t,\tau)\right]\begin{bmatrix}\lambda_1\\ \overline{0}^{-1}\end{bmatrix} + \right.\right.$$
$$\left.\left. + 2\,\Omega_{12}(t,\tau)[\lambda_2]\right)d\tau\right\|$$
$$\leq |T - t_o|\left[\|\Omega_{12}(t,\tau)K(\tau) + \Omega_{11}(t,\tau)\|\epsilon_1 + 2\|\Omega_{12}(t,\tau)\|\epsilon_2\right]$$

for $t,\tau\in[t_o T]$,

or

$$\text{(iv)}\quad \|\Delta y(t) - \Delta\bar{y}(t)\| \leq M\epsilon_1 + N\epsilon_2.$$

Therefore, the computable error introduced in (i) by ignoring the truncation is always less than a weighted linear combination of the largest truncation error expected as y and p range over some restricted domain. This domain was

investigated in Chapter II.

4.7 General Performance Error

Equation (iv) may also be used to establish an error bound on the cost function. The difference between $C(u^*)$ (with truncation error considered) and $C(u^e)$ (ignoring truncation errors) can be established as follows.

With ϵ_1 and ϵ_2 defined previously, consider ϵ_3 to be

$$\epsilon_3 \leq \sup |o_u(\epsilon^3)| : t \in [t_o, T] \text{ and for all } (y, p, u) \in N, o_u(\epsilon^3) \text{ given by 2-12.}$$

Therefore, for $\tilde{G}(t)$ of (2-17) represented as

$$\tilde{G}(t) = [R(t) \vdots S(t)],$$

R(t) is rxn

S(t) is rxm,

$$\|u^*(t) - u^e(t)\| \leq \|R(t)\| \|x^*(t) - x^e(t)\| + \|S(t)\| \|\alpha^* - \alpha^e\| + \epsilon_3.$$

Also, it can be noted that

$$\langle x^*(T), T x^*(T) \rangle - \langle x^e(T), T x^e(T) \rangle = \sum_{i=1}^{n} T_{ii} (x_i^{*2}(T) - x_i^{e2}(T)) \leq \Gamma \|x^*(T) - x^e(T)\|$$

where

$$\Gamma = \max_{i=1,\cdots,n} T_{ii}$$

define k to be a constant such that

$$\frac{\|x^*(T) - x^e(T)\|}{k} \leq 1$$

Then $k\Gamma^2 \|x^*(T) - x^e(T)\| \geq \Gamma \|x^*(T) - x^e(T)\|^2$.

The difference in the cost indicies $C(u^*)$ and $C(u^e)$ becomes

$\tilde{K}$ is a Lipschitz constant)

$$\| C(u^*) - C(u^e) \| \leq |T-t_o| \tilde{K} (\| x^*(t) - x^e(t) \| + \| R(t) \| \| x^*(t) - x^e(t) \| + \| S(t) \| \| a^* - a^e \| + \epsilon_3) + k\Gamma \| x^*(T) - x^e(T) \|$$

$$\leq |T-t_o| \tilde{K} \left[\left(1 + \| R(t) \| + \frac{k\Gamma}{|T-t_o|\tilde{K}} \right) \| x^* - x^e \| + \| S(t) \| \| a^* - a^e \| + \epsilon_3 \right]$$

from (iv)

$$\leq |T-t_o| \tilde{K} (1 + \| R(t) \| + (k\Gamma / |T-t_o| \tilde{K})(M\epsilon_1 + N\epsilon_2) + \| S(t) \| \| a^* - a^e \| + \epsilon_3 \triangleq \Delta(k, \epsilon_1, \epsilon_2, \epsilon_3, a^*, a^e, \Gamma)$$

for $\alpha^* = \alpha^e$,

$$\Delta(k, \epsilon_1, \epsilon_2, \epsilon_3, a^*, a^e, \Gamma) = \Delta(k, \epsilon_1, \epsilon_2, \epsilon_3, \Gamma)$$

For a prescribed maximum allowable value of $\Delta(k, \epsilon_1, \epsilon_2, \epsilon_3, \Gamma)$, say λ, $\| C(u^*) - C(u^e) \| \leq \lambda$. Then k, ε_1, ε_2, ε_3 may be chosen (not uniquely) in such a way as to satisfy the λ constraint. Choosing a maximum acceptable terminal difference (i.e., $\| x^*(T) - x^e(T) \|$), thus fixing k, one can find the neighborhood N (of Theorem 2-1) such that the ε_1, ε_2, ε_3 candidates are satisfied.

Because of the general treatment given to error bounds in this chapter, the error bounds generated were, in some sense, maximal. Tighter bounds may be established for particular problems through a more detailed analysis of its structure

CHAPTER V

SENSITIVITY

5.1 Sensitivity Index

The previously developed machinery will be used to show that under certain local restrictions a reduction in the systems sensitivity to parameter variations will be accomplished. The question of sensitivity will be developed qualitatively. The sensitivity index chosen should satisfy the following intuitive ideas:

1. The ideal adaptive control is one which equals the optimal control for a particular problem. If the adaptive control for certain parameter variations equals the optimal control over those same variations, then the system's cost is insensitive to those parameter variations.
2. If the adaptive control does not produce a cost $C(u)$, ($C(u)$ defined by (2-2)) which equals the minimum of $C(u)$ over a set of parameter variations, then the system should be considered sensitive to those parameter variations. The greater the disparity between $C(u)$ and the minimum $C(u) = C(u^*)$, the more sensitive to parameter variations is the system's cost.

A sensitivity index S^γ chosen will compare the cost of operation of a system subject to parameter variations to the optimal cost of operation $C(u^*)$. Or more succinctly

$$S^\gamma = |C(u^*) - C(u^\gamma)| \tag{5-1}$$

where the properties of $C(u^*)$ and $C(u^\gamma)$ follow directly from (2-1) and (2-2). Although the construction of a sensitivity index is arbitrary, (5-1) shall be assumed to be the canonical sensitivity index. With this definition of S^γ and Theorem (2-4) in mind, the following is easily shown.

5.2 Sensitivity Reduction

Correlary (2-4)

Under the conditions stated in Theorem 2-4

$$S^e \leq S^n$$

where:

u^*, u^e, and u^n have been previously defined.

Proof:

Apply Theorem 2-4 to S^e and S^n.

This result guarantees that under certain local conditions given in Theorem 2-4, a system operating with an adaptive control strategy is less sensitive to parameter variations than its nominally optimal counterpart. In fact, as $\alpha^e \to \alpha^*$, $S^e \to 0$. These results are not philosophically disturbing in light of the fact that the adaptive control was derived to have a reoptimizing quality. One would naturally expect a control effort which is constantly tending to minimize a given cost index to be closer (or equal) to the optimal cost than a system operating with a fixed nominally optimal control.

5.3 Alternative Sensitivity Index

To reinforce these ideas geometrically a more indirect attack on the sensitivity problem will be performed. Suppose for the moment C(u) is a monotically increasing function of time. One might then be motivated to consider a quadratic sensitivity index. Because the original cost index is a function of x(t) and u(t), the new sensitivity shall also be. Define the sensitivity index N^i as follows:

$$N^i = \|X^*(T) - X^i(T)\|^2 + \int_{t_o}^{T} \langle x^*(t) - x^i(t), S(t)(x^*(t) - x^i(t)) \rangle + \langle u^*(t) - u^i(t), R(t)(u^*(t) - u^i(t)) \rangle \, dt \tag{5-2}$$

where S(t) and R(t) are positive definite and

$$\dot{x}^i(t) = f(x^i(t), u^i(t), \alpha^*)$$
$$x^i(t_o) = x_o \tag{5-3}$$

Because N^i is positive definite, zero is the minimal value of N^i. Consider, for example, the case where the parameter variations from the nominal are sufficiently small and known exactly (i.e., $\alpha^e = \alpha^*$) such that $u^*(t) = u^e(t)$. Then $x^*(t) = x^e(t)$ and $N^e = N^* = 0$, which implies the system is insensitive to such parameter variations. If two systems operating with control efforts $u^i(t)$ and $u^k(t)$ are compared for a given set of admissible parameter variations, and if $N^j > N^k$ over this set of parameter variations, then the system associated with u^k shall be considered to be the "less sensitive" of the two for the given parameter variations with respect to N^i.

One may compare the sensitivity, in the N^i sense, of the derived adaptive system with the nominally optimal

controlled system as follows:

Consider the dynamical n dimensional system

$$\dot{x}(t) = f(x(t), u(t), a^*), \quad x(t_o) = x_o$$

with $u^*(t)$ and $u^n(t)$ previously defined.

Condition (i)

For sufficiently small parameter variations it has been established that $u^e(t)$ and $u^*(t)$ may be represented as:

$$u^e(t) = u^n(t) + G(t)(x(t) - x^n(t)) + H(t)(a - a^n) \tag{5-4}$$

$$u^*(t) = u^n(t) + G(t)(x(t) - x^n(t)) + H(t)(a - a^n) + o(\epsilon^2). \tag{5-5}$$

Combining (5-4) and (5-5)

$$u^e(t) = u^*(t) - o(\epsilon^2).$$

Consider the following admissible variations

$$x(t) = x^*(t) + \epsilon \delta x(t), \quad a = a^* + \epsilon \delta a$$

For the adaptive system (i.e. $u(t) = u^e(t)$), one achieves the variational differential equation

$$\lim_{\epsilon \to o} \frac{\partial \dot{x}^*(t) + \epsilon \delta \dot{x}(t)}{\partial \epsilon} = \delta \dot{x}(t)$$

(ii)
$$\delta \dot{x}(t) = \lim_{\epsilon \to o} \frac{\partial (f(x^*(t) + \epsilon \delta x(t), a^* + \epsilon \delta a, u^*(t) - o(\epsilon^2))}{\partial \epsilon} \tag{5-6}$$

$$= f_x(x^*(t), a^*, u^*(t)) \delta x(t) + f_a(x^*(t), a, u^*(t)) \delta a$$

$$\delta x(t_o) = 0$$

where all the partial derivatives exist, are well defined and continuous over $t \in [t_o, T]$. Define the solution of (5.6) to be $\xi(t)$.

$$\xi(t) = \int_{t_o}^{T} \Phi(t,\tau) f_a(x^*(\tau), a^*, u(\tau)) \delta a \, d\tau$$

where

$$\dot{\Phi}(t,\tau) = f_x(x^*(t), a^*, u^*(t)) \Phi(t,\tau)$$

$$\Phi(\tau,\tau) = I.$$

For $u(t) = u^n(t)$

(iii)

$$\begin{aligned} \delta\dot{x}(t) = \lim_{\epsilon \to 0} \big(& f(x^*(t) + \epsilon \delta x(t), a^* + \epsilon \delta a, \\ & u^*(t) - G(t)(x^*(t) + \epsilon \delta x(t) - x^n(t)) + \\ & - H(t)(a^* + \epsilon \delta a - a^n) - o(\epsilon^2) \big) \end{aligned} \tag{5-10}$$

$$\begin{aligned} = & f_x(x^*(t), a^*, u^*(t)) \delta x(t) + f_a(x^*(t), a^*, \\ & u^*(t)) \delta a - f_u(x^*(t), a^*, u^*(t) \big[G(t) \\ & \delta x(t) + H(t) \delta a \big] \end{aligned}$$

$$\delta x(t_o) = 0$$

Define the solution of (5-10) to be $\eta(t)$.

$$\eta(t) = \int_{t_o}^{t} \Phi(t,\tau)\, f_a(x^*(\tau), a^*, u(\tau))\delta a \, d\tau +$$

$$- \int_{t_o}^{t} \Phi(t,\tau)\, f_u(x^*(\tau), a^*, u^*(\tau)) \left[G(\tau)\delta x(\tau) + \right.$$

$$\left. + H(\tau)\delta a \right] d\tau$$

or equivalently

$$\eta(t) = \xi(t) - L_1(\delta\eta(t)) - L_2(a(t)) \qquad (5\text{-}11)$$

where

$$L_1 \triangleq \int_{t_o}^{t} \Phi(t,\tau)\, f_u(x^*(\tau), a^*, u^*(\tau))\, G(\tau)$$

$$L_2 \triangleq \int_{t_o}^{t} \Phi(t,\tau)\, f_u(x^*(\tau), a^*, u^*(\tau)) H(\tau)$$

and L_1 and L_2 are bounded linear operators.

It is now possible to compare the following sensitivity indicies:

N^e: the sensitivity index for the adaptive (reoptimization) scheme.

N^n: the sensitivity index for the nominally optimal controlled scheme.

If the reoptimization scheme is to be "less sensitive" than the nominal scheme for a given set of admissible parameter varations

(iv) $$N^e - N^n \leq 0.$$

This implies

$$\int_{t_o}^{T} [\langle \delta\xi(t), S(t)\delta\xi(t)\rangle + \langle \delta u^e(t), R(t)\delta u^e(t)\rangle + - \langle \delta\eta(t), S(t)\delta\eta(t)\rangle - \langle \delta u^n(t), R(t)\delta u^n(t)\rangle]\, dt + + \langle \delta\xi(T), S(T)\delta\xi(T)\rangle - \langle \delta\eta(T), R(T)\delta\eta(T)\rangle .$$

Theorem 5-1: For the sensitivity index given by (5-2) and Condition (i) satisfied, a sufficient condition that $N^e - N^n \leq 0$ is that

(v) $$\frac{2\langle \xi(t), S(t)[L_1(\delta\eta(t)) + L_2(\delta\alpha)]\rangle}{\| L_1(\delta\eta(t)) + L_2(\delta\alpha)\|^2_{S(t)}} \leq 1$$

Proof:

One notes $\delta u^e(t) = \lim_{\varepsilon \to 0} \frac{\partial}{\partial \varepsilon} o(\varepsilon^2) \to 0$,

and $\delta u^n(t) = -G(t)\delta x(t) - H(t)\delta\alpha$ (from 5-5)

$$\therefore \int_{t_o}^{t} \langle \delta u^e(t), R(t)\delta u^e(t)\rangle dt \leq \int_{t_o}^{t} \langle \delta u^n(t), R(t)\delta u^n(t)\rangle dt.$$

Therefore, for (iv) to hold true, it is sufficient to show

$$\int \{ 2\langle \xi(t), S(t)[L_1(\delta\eta(t)) + L_2(\delta\alpha)]\rangle +$$

$$-\langle [L_1(\delta\eta(t)) + L_2(\delta a)], S(t)[L_1(\delta\eta(t)) + L_2(\delta a)]\rangle \Big\} dt + 2\langle \xi(T), S(T)[L_1(\delta\eta(T)) + L_2(\delta a)]\rangle - \langle [L_1(\delta\eta(T)) + L_2(\delta a), S(T)[L_1(\delta\eta(T)) + L_2(\delta a)]\rangle \leq 0.$$

A sufficient condition is

$$\frac{2\langle \xi(t), S(t)[L_1(\delta\eta(t)) + L_2(\delta a)]\rangle}{\| L_1(\delta\eta(t)) + L_2(\delta a)\|} \leq 1$$

for all $t \in [t_o, T]$.

Although the structure of the scalar sufficiency appears innocent, its computation may be a very complex problem. However, the sufficiency condition does reinforce some intuitive ideas one may have about $\xi(t)$ and $\eta(t)$. Consider the following state trajectories in E^3 and their intersection with a $t = t'$ hyperplane.

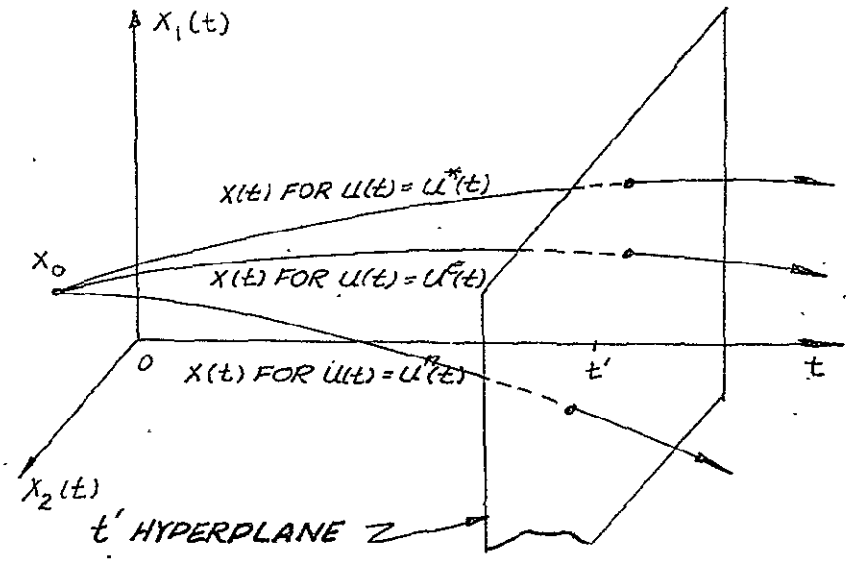


Figure 5-1

STATE TRAJECTORIES

In the t´ hyperplane the following vectors may be identified:

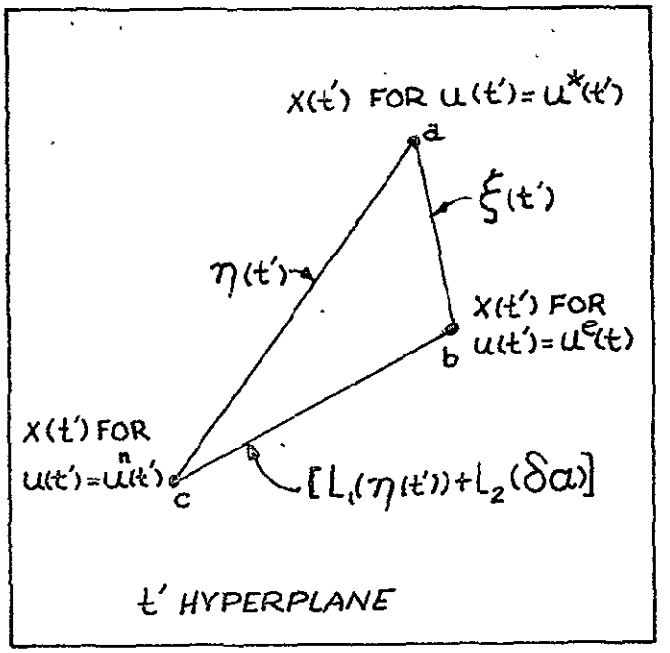


Figure 5-2

t´ HYPERPLANE

The vector $\eta(t')$ is the solution to the state variational equation with $u(t) = u^n(t)$, $t \in [t_o, t']$, at $t = t'$. The vector $\xi(t)$ is the solution to the state variational equation with $u(t)$ defined by (5-4), $t \in [t_o, t']$ at $t = t'$. The vector $[L_1(\eta(t')) + L_2(\delta\alpha)]$ is defined by (5-11) at $t = t'$. Inequality (v) (Theorem 5-1) places some restrictions on the magnitude and orientation of $\xi(t)$. For example, some acceptable values of $\xi(t)$ are diagramed below.

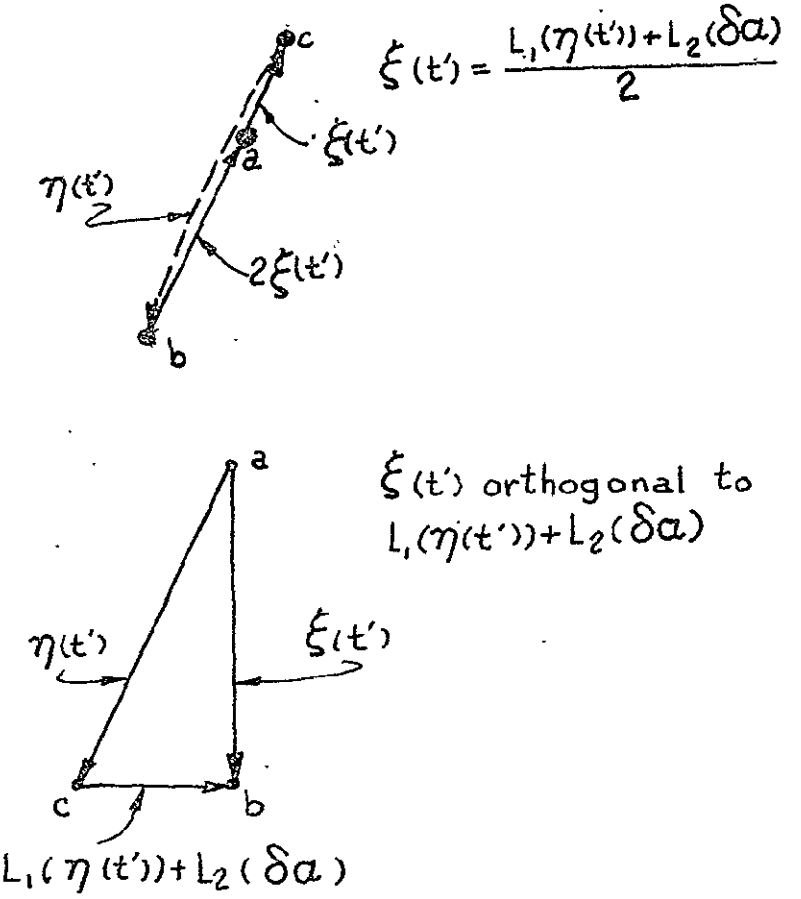
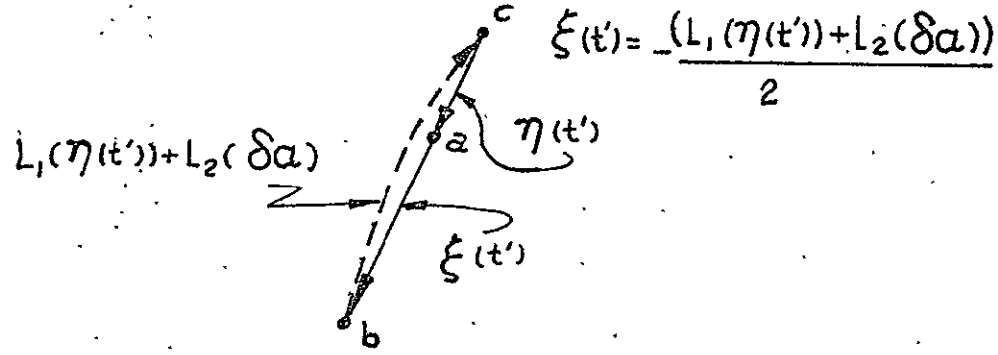


Figure 5-3

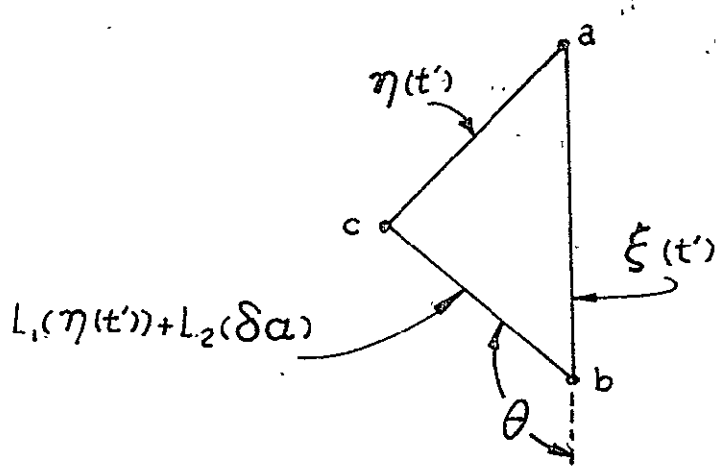
ACCEPTABLE $\xi(t')$

Examples where the sufficiency condition is violated are (note: $||\eta(t)|| < ||\xi(t)||$)



(a)

The angle θ between $\xi(t)$ and $-(L_1(\eta(t)) + L_2(\delta\alpha))$ belongs to $(90^\circ, 270^\circ)$.



(b)

Figures 5-4a,b

UNACCEPTABLE VALUES OF $\xi(t')$

Therefore, the first variational argument goes beyond requiring that the variations in the adaptive control, from the true optimal control, be small, or neglected. It requires that $\xi(t)$ be small and have a proper direction. Because of the continuity properties associated with the variational equations one would expect $\xi(t)$ to have a direction similar to the direction associated with $\eta(t)$. Therefore, (v) becomes basically a magnitude restriction of the form $||\xi(t)|| \leq ||\eta(t)||$.

Chapter VI.

SUMMARY

6.1 Summary

The problem of sensitivity with respect to parameter variations has been studied under the philosophical requirement that optimality be preserved. The device which accomplishes this has the property of generating a control effort which minimizes

$$C(u) = \tfrac{1}{2}\langle X(T), TX(T)\rangle + \int_{t_o}^{T} L(X(t), u(t))\,dt$$

for parameter values belonging to a set of admissible parameter variations. The mathematical machinery used to develop such a device was a truncated Taylor Series representation of the system's Hamiltonian system of equations. Once certain local smoothness and partial derivative tests had been satisfied in a neighborhood of the nominally optimal solution, the Maximum (Minimum) Principle was applied to the problem. As a result of this action, a set of canonical equations were generated from which the matrix Riccati equation evolved as a by-product. As a derived result, the adaptive control abstracted from these equations was found to be a linear combination of the system's states and parameters.

It was shown that for parameter estimates sufficiently close to the system's actual parameters, the cost incurred using the adaptive control structure was less than, or at worst equal to, that incurred by a system operating under a nominally optimal control policy only. That is, under

certain local restrictions the adaptive system's control effort was optimal, or near optimal, over a set of admissible parameter variations. By virtue of this fact, and a sensitivity index given by (5-1),

$$S^{\gamma} \triangleq |C(u^*) - C(u^{\gamma})|$$

it was shown that the adaptive system was less sensitive to parameter variations than its nominally optimal controlled counterpart.

Numerous techniques were offered to resolve the "curse of an adaptive system" which is parameter estimation. They were basically of two deterministic classes; namely, numerical and gradient techniques. Also, an introduction to a particular class of non-deterministic parameter estimation scheme was explored with suggestions and computational techniques given to satisfy this requirement.

6.2 Suggestions for Further Research

The most obvious area requiring additional research is that of parameter estimation. The devices and techniques which may be developed in this area, because they are after all approximation schemes, are limited only by the designer's imagination.

Numerical experimentation should also prove useful in determining a larger class of admissible parameter variations than that achieved by analytical means. Another numerical study might be to form a linearized approximation of a particular given plant to achieve a system of the form

$$\dot{x} = A(a)x + Bu.$$

Then mechanize an adaptive controller for this system, and perform numerical experiments on it over a set of parameter variations, to establish whether or not this technique has any merit.

An interesting analytical study might involve determining an optimal selection policy which would define a finite set of fixed parameter vectors. It would be required that for each parameter vector, say α_i, a successful reoptimization can be accomplished with an adaptive controller with an acceptable degree of accuracy. A set of fixed dissimilar parameter vectors will be called the optimal set of parameter vectors if it satisfies the criterion that for an open neighborhood ψ_{α_i} of parameter vectors about α_i admitting the previously discussed reoptimization, and some given set of N of parameter variations about the nominal parameter vector α^n, ℓ is minimum, where

$$\bigcup_{i=1}^{\ell} \psi_{\alpha_i} \subset N.$$

That is, one wishes to find the minimal number of open set (i.e., ψ_{α_i}) which forms an open cover of N.

Also, controllability and observability were essential in constructing the reoptimizing system. A study pertaining to the loss of controllability or observability, if indeed it should occur, would prove beneficial. Such an analysis would be straightforward.

The optimization problem considered was of a fixed

terminal time class. Extensions of this result into the other basic classes of optimal control problems should be pursued. Also, the terminal cost index was defined to be quadratic. It was assumed to be quadratic to reduce some of the computational problems encountered without too great a loss of generality. Therefore, it is suggested that this terminal cost index might be generalized.

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