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A GENERALIZATION OF THE WEIBULL DISTRIBUTION

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Athens, Georgia

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16. ABSTRACT

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The problem of parameter estimation is considered and three separate sets of estimators are presented. These are i) moment estimators, ii) maximum likelihood estimators and iii) alternate estimators based on the first three moments and the first order statistic.

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A GENERALIZATION OF THE WEIBULL DISTRIBUTION

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SUMMARY

The two-parameter Weibull distribution with origin at zero and the three-parameter version of this distribution with origin at γ are widely employed as statistical models in connection with life testing. In this paper a four-parameter generalization of this distribution is introduced in order to provide a more versatile model for use in life studies and in related investigations.

The problem of parameter estimation is considered and three separate sets of estimators are presented. These are, i) moment estimators, ii) maximum likelihood estimators and iii) alternate estimators based on the first three moments and the first order statistic.

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1. INTRODUCTION

It is well known that the three-parameter gamma distribution (i.e. the Pearson Type III distribution) may be obtained from the one-parameter gamma distribution by making the transformation $z = (x - \gamma)/\beta$, where the density of the single parameter gamma distribution is written as

$$f(z;\alpha) = \begin{cases} \dfrac{1}{\Gamma(\alpha + 1)} z^{\alpha} e^{-z} ; & z > 0, \\ \\ 0, \text{ elsewhere} . & \end{cases} \tag{1}$$

Numerous writers have studied the gamma distribution in its various forms. Karl Pearson of course studied it extensively and his name is firmly attached to the Type III version of this distribution. A development of the Type III distribution together with tables of areas was presented in NASA CR-61266 by the writer in collaboration with Helm and Sugg [3]. Harter [7,8] and Harter and Moore [6] have recently given considerable attention to the gamma distribution and to its various special cases as well as to certain generalizations.

2. THE FOUR-PARAMETER DISTRIBUTION

A four parameter generalized Weibull distribution may be obtained from (1) by making the transformation

$$z = (x - \gamma)^{\delta}/\beta \quad . \tag{2}$$

The Pearson Type III distribution may then be considered as a special case of the 4-parameter Weibull distribution in which $\delta = 1$. Using the transformation (2), it follows that

$$g(x) = f(z)\left|\frac{\partial z}{\partial x}\right| = \frac{\delta(x - \gamma)^{\delta-1}}{\beta} f(z) \quad ,$$

and subsequently , we have

$$g(x;\alpha,\beta,\gamma,\delta) = \begin{cases} \dfrac{\delta}{\Gamma(\alpha + 1)\ \beta^{\alpha+1}} (x - \gamma)^{\delta(\alpha+1)-1} e^{-(x-\gamma)^{\delta}/\beta} \ ; & x > \gamma \ . \\ \\ 0, \text{ elsewhere.} & \end{cases} \tag{3}$$

If we translate the origin in (3) to the point $x = \gamma$; i.e. if we make the transformation

$$w = x - \gamma, \tag{4}$$

the density function (3) becomes

$$h(w;\alpha,\beta,\delta) = \begin{cases} \dfrac{\delta}{\Gamma(\alpha+1)\ \beta^{\alpha+1}}\, w^{(\alpha+1)-1} e^{-w^{\delta}/\beta} \ ; & w > 0\ , \\ \\ 0, \text{ elsewhere}\ . & \end{cases} \tag{5}$$

The kth moment of w about the origin may then be written as

$$\mu'_{k:w} = \frac{\delta}{\Gamma(\alpha+1)\ \beta^{\alpha+1}} \int_0^{\infty} w^k w^{\delta(\alpha+1)-1}\ e^{-w^{\delta}/\beta}\ dw. \tag{6}$$

On integrating (6), we obtain

$$\mu'_{k;w} = \frac{\beta^{k/\delta}}{\Gamma(\alpha+1)} \Gamma[(k/\delta) + \alpha + 1]\ . \tag{7}$$

The mean, variance, and the third and fourth standard moments follow as

$$
\left.
\begin{aligned}
\mu_w &= \beta^{1/\delta}\, \Gamma[(1/\delta) + \alpha + 1] \,/\, \Gamma(\alpha + 1) \;, \\[2ex]
\sigma_w^2 &= \beta^{2/\delta}\, \frac{\{\Gamma(\alpha+1)\Gamma[(2/\delta) + \alpha + 1] - \Gamma^2[(1/\delta) + \alpha + 1]\}}{\Gamma^2(\alpha + 1)} \;, \\[2ex]
\alpha_{3:w} &= B(\alpha,\delta) \,/\, A^{3/2}(\alpha,\delta) \;, \\[2ex]
\alpha_{4:w} &= C(\alpha,\delta) \,/\, A^{2}(\alpha,\delta) \;,
\end{aligned}
\right\} \quad (8)
$$

where ,

$$
\left.
\begin{aligned}
A(\alpha,\delta) &= \Gamma(\alpha+1)\Gamma[(2/\delta) + \alpha + 1] - \Gamma^2[(1/\delta) + \alpha + 1] \;, \\[2ex]
B(\alpha,\delta) &= \Gamma^2(\alpha+1)\Gamma[(3/\delta) + \alpha + 1] \;, \\[1ex]
&\quad -3\Gamma(\alpha+1)\Gamma[(2/\delta) + \alpha + 1]\Gamma[(1/\delta) + \alpha + 1] \;, \\[1ex]
&\quad +2\Gamma^3[(1/\delta) + \alpha + 1] \;,
\end{aligned}
\right\} \quad (9)
$$

$$
\left.\begin{aligned}
C(\alpha,\delta) = {} & \Gamma^3(\alpha+1)\Gamma[(4/\delta) + \alpha + 1] \\
& -4\Gamma^2(\alpha+1)\Gamma[(3/\delta) + \alpha + 1]\Gamma[(1/\delta) + \alpha + 1] \\
& +6\Gamma(\alpha+1)\Gamma[(2/\delta) + \alpha + 1]\Gamma^2[(1/\delta) + \alpha + 1] \\
& -3\Gamma^4[(1/\delta) + \alpha + 1] \quad .
\end{aligned}\right\} \quad \text{(9) (continued)}
$$

The square of the coefficient of variation may be written as

$$
C.V.^2(w) = \left[\frac{\Gamma(\alpha + 1)\Gamma[(2/\delta) + \alpha + 1]}{\Gamma^2[(1/\delta) + \alpha + 1]} - 1 \right] . \qquad (10)
$$

Since $w = x - \gamma$, and since σ^2 , α_3 and α_4 are invariant under translation of the origin, it follows that

$$
\left.\begin{aligned}
\mu_x &= \gamma + \beta^{1/\delta}\Gamma[(1/\delta) + \alpha + 1] / \Gamma(\alpha + 1) \quad , \\
\sigma_x^2 &= \sigma_w^2 \quad , \\
\alpha_{3:x} &= \alpha_{3:w} \quad , \\
\alpha_{4:x} &= \alpha_{4:w} \quad .
\end{aligned}\right\} \qquad (11)
$$

The distribution function of the random variable W may be written as

$$H(w;\alpha,\beta,\delta) = \int_o^w \frac{\delta}{\Gamma(\alpha + 1)\ \beta^{\alpha+1}}\ y^{\delta(\alpha+1)-1}\ e^{-y^\delta/\beta}\ dy\ ,$$

$$= \frac{1}{\Gamma(\alpha + 1)} \int_o^{w^\delta/\beta} z^\alpha\ e^{-z}\ dz\ ,$$

which becomes

$$H(w;\alpha,\beta,\delta) = I[(w^\delta/\beta);\ \alpha + 1]\ /\ \Gamma(\alpha + 1)\ , \tag{12}$$

where $I[(w^\delta/\beta);\ \alpha + 1]$ is the incomplete gamma function.

3. MOMENT ESTIMATION

In the general four-parameter distribution, the moment estimates are found by simultaneously solving the following

$$\left.\begin{aligned}
\bar{x} &= \gamma + \beta^{1/\delta}\ \Gamma[(1/\delta)+\alpha+1]\ /\ \Gamma(\alpha+1)\ ,\\
s_x^2 &= \beta^{2/\delta}\ A(\alpha,\delta)\ /\ \Gamma^2(\alpha+1)\ ,\\
a_3 &= B(\alpha,\delta)\ /\ A^{3/2}(\alpha,\delta)\ ,\\
a_4 &= C(\alpha,\delta)\ /\ A^2(\alpha,\delta)\ ,
\end{aligned}\right\}\tag{13}$$

where $A(\alpha,\delta)$, $B(\alpha,\delta)$ and $C(\alpha,\delta)$ are given in equations (9).

The last two equations of (13) may be solved simultaneously for α^* and δ^*. With the values thus found, we estimate β from the second equation as

$$\beta^* = \left[\frac{s_x\ \Gamma(\alpha^*+1)}{\sqrt{A(\alpha^*,\delta^*)}}\right]^{\delta^*} = s_x^{\delta^*}\ D(\alpha^*,\delta^*)\ ,\tag{14}$$

where

$$D(\alpha,\delta) = \left[\frac{\Gamma\ (\alpha+1)}{\sqrt{A(\alpha,\delta)}}\right]^{\delta}.\tag{15}$$

Finally we estimate γ from the first equation of (13) as

$$\gamma^* = \bar{x} - \beta^{*1/\delta^*}\ \Gamma[(1/\delta^*)+\alpha^*+1]\ /\ \Gamma(\alpha^*+1)\ ,\tag{16}$$

which may be written as

$$\gamma^* = \overline{x} - \mu(\alpha^*, \beta^*, \delta^*) , \tag{17}$$

where

$$\mu(\alpha^*,\beta^*,\delta^*) = \beta^{*1/\delta^*} \Gamma[(1/\delta^*) + \alpha^* + 1] / \Gamma(\alpha^* + 1) . \tag{18}$$

An <u>alternate set</u> of estimates may be obtained by replacing the fourth standard moment of (13) with the smallest sample (i.e. the first order statistic) observation. Thus we have,

$$\left.\begin{aligned} x_{min} &= \gamma , \\ \overline{x} &= \gamma + \beta^{1/\delta} \Gamma[(1/\delta) + \alpha + 1] / \Gamma(\alpha + 1) , \\ s_x^2 &= \beta^{2/\delta} A(\alpha,\delta) / \Gamma^2(\alpha + 1) , \\ a_3 &= B(\alpha,\delta) / A^{3/2}(\alpha,\delta) , \end{aligned}\right\} \tag{19}$$

From the first equation of (19) we have

$$\gamma^{**} = x_{min} . \tag{20}$$

From the second and third equations we have

$$\frac{s_x^2}{(\bar{x} - x_{min})^2} = \frac{A(\alpha,\delta)}{\Gamma^2[(1/\delta) + \alpha + 1]} \quad . \tag{21}$$

We solve (21) and the fourth equation of (19) simultaneously for α^{**} and δ^{**} . It then follows as with the basic moment estimates that

$$\beta^{**} = s_x^{\delta^{**}} D(\alpha^{**},\delta^{**}) \ . \tag{22}$$

4. MAXIMUM LIKELIHOOD ESTIMATION

The likelihood function of a random sample of n observations from the four-parameter distribution with density (3) may be written as

$$L(x_1, \ldots , x_n; \alpha,\beta,\gamma,\delta) =$$

$$\left(\frac{\delta}{\Gamma(\alpha + 1)\ \beta^{\alpha+1}}\right)^n \prod_{i=1}^{n} (x_i - \gamma)^{\delta(\alpha+1)-1} e^{-(x_i - \gamma)^{\delta}/\beta} \ . \tag{23}$$

The logarithm of (23) follows as

$$\ln L = n \ln \delta - n(\alpha + 1) \ln \beta - n \ln \Gamma(\alpha + 1)$$

$$+ [\delta(\alpha + 1) - 1] \sum_1^n \ln (x_i - \gamma) - \sum_1^n (x_i - \gamma)^{\delta} / \beta \quad . \tag{24}$$

On differentiating (24) with respect to β, δ, γ, and α in turn and equating to zero, we obtain the following estimating equations

$$\frac{-n(\alpha + 1)}{\beta} + \frac{1}{\beta^2} \sum_1^n (x_i - \gamma)^{\delta} = 0 \ ,$$

$$\frac{n}{\delta} - \frac{1}{\beta} \sum_1^n (x_i - \gamma)^{\delta} \ln(x_i - \gamma) + (\alpha + 1) \sum_1^n \ln(x_i - \gamma) = 0 \ ,$$

$$\sum_1^n \left(\frac{1 - \delta(\alpha + 1)}{x_i - \gamma} \right) + \frac{\delta}{\beta} \sum_1^n (x_i - \gamma)^{\delta - 1} = 0 \ , \tag{25}$$

$$-n \, \psi(\alpha + 1) - n \ln \beta + \delta \sum_1^n \ln(x_i - \gamma) = 0 \ ,$$

where $\psi(\alpha + 1) = \dfrac{\partial \ln \Gamma(\alpha + 1)}{\partial \alpha}$ is the Digamma function.

The required maximum likelihood estimates are to be obtained by simultaneously solving the four equations of (25). This task may be accomplished using various iterative techniques. Before discussing a general solution, let us examine certain possible simplifications.

From the first equation of (25) , it follows that

$$\beta = \frac{\sum_{1}^{n} (x_i - \gamma)^{\delta}}{n(\alpha + 1)} . \tag{26}$$

On substituting (26) into the second equation of (25), we have

$$H(\alpha,\delta,\gamma) = \frac{1}{\alpha+1} \left[\frac{(\alpha+1) \sum_{1}^{n} (x_i-\gamma)^{\delta} \ln (x_i-\gamma)}{\sum_{1}^{n} (x_i - \gamma)^{\delta}} - \frac{1}{\delta} \right]$$

$$- \frac{1}{n} \sum_{1}^{n} \ln(x_i - \gamma) = 0 . \tag{27}$$

On substituting (26) into the fourth equation of (25), we have

$$G(\alpha,\delta,\gamma) = \frac{1}{\delta} \ln \left[\frac{\sum_{1}^{n} (x_i - \gamma)^{\delta}}{n(\alpha + 1)} \right] - \frac{1}{\delta} \psi (\alpha + 1)$$

$$- \frac{1}{n} \sum_{1}^{n} \ln(x_i - \gamma) = 0 . \tag{28}$$

On making a similar substitution of (26) into the third equation of (25), we have

$$J(\alpha,\delta,\gamma) = \sum_1^n \left(\frac{1 - \delta(\alpha + 1)}{x_i - \gamma} \right) + \frac{n\delta\,(\alpha + 1)\,\sum_1^n (x_i - \gamma)^{\delta-1}}{\sum_1^n (x_i - \gamma)^{\delta}} = 0. \qquad (29)$$

Our task has now been reduced to that of solving (27), (28) and (29) simultaneously for the required estimates $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$. With this solution accomplished, $\hat{\beta}$ will then follow from (26) on replacing α, δ, and γ in that equation with their respective estimates. Although the solution of these equations in the general case might be considered rather formidable, the special case in which γ is known is much simpler. Suppose for example that γ is known to be zero. Should γ have some value other than zero, as long as it is known we can introduce a new variable $Y = X - \gamma$ and then Y has the range $Y \geq 0$.

With γ known, we need only solve the pair of equations $H(\alpha,\delta) = 0$ and $G(\alpha,\delta) = 0$ where G and H are as defined in (28) and (27) using the known value for γ. We might choose a value α_o, and with α_o fixed, solve $H(\alpha_o,\delta) = 0$ for δ_o. Next calculate $G(\alpha_o,\delta_o)$ and if the calculated value is zero, our task is finished. Otherwise, we choose a second value α_1 and solve $H(\alpha_1, \delta) = 0$ for δ_1. Now calculate $G(\alpha_1,\delta_1)$. If $G(\alpha_o,\delta_o)$ and $G(\alpha_1,\delta_1)$ are of opposite signs, we may interpolate for the required estimate $\hat{\alpha}$, and in turn for $\hat{\delta}$ as summarized below.

INTERPOLATION FOR $\hat{\alpha}$ and $\hat{\delta}$

α	$G(\alpha,\delta)$	δ
α_0	$G(\alpha_0,\delta_0) \gtrless 0$	δ_0
$\hat{\alpha}$	0	$\hat{\delta}$
α_1	$G(\alpha_1,\delta_1) \lessgtr 0$	δ_1

With $\hat{\alpha}$ and $\hat{\delta}$ thus determined, $\hat{\beta}$ follows on substituting these values into (26). Obviously, it is desirable to choose values α_0 and α_1 that are good approximations to $\hat{\alpha}$. If the first choices are not sufficiently close to $\hat{\alpha}$, then it may be necessary to make several trials before arriving at a suitable pair which in the notation employed here would be labeled α_0 and α_1.

In the general case with γ to be estimated, we might begin by choosing a value (approximation) γ_0 and proceed as above to find values $\hat{\alpha}|\gamma_0$ and $\hat{\delta}|\gamma_0$ which are to be thought of as conditional estimates given that $\gamma = \gamma_0$. We then calculate $J[(\hat{\alpha}|\gamma_0), (\hat{\delta}|\gamma_0),\gamma_0]$. Similarly, we select a value γ_1 and repeat the process until we have evaluated $J[(\hat{\alpha}|\gamma_1), (\hat{\delta}|\gamma_1), \gamma_1]$. We continue until our final choice for γ_0 and γ_1 are such that our two evaluations of J are of opposite signs. We then interpolate as summarized below.

INTERPOLATION FOR $\hat{\gamma}$

γ	$J[\alpha,\delta,\gamma]$	α	δ
γ_o	$J[(\hat{\alpha}\vert\gamma_o),\ (\hat{\delta}\vert\gamma_o),\gamma_o] \gtrless 0$	$\hat{\alpha}\vert\gamma_o$	$\hat{\delta}\vert\gamma_o$
$\hat{\gamma}$	0	$\hat{\alpha}$	$\hat{\delta}$
γ_1	$J[(\hat{\alpha}\vert\gamma_1),\ (\hat{\delta}\vert\gamma_1),\gamma_1] \lessgtr 0$	$\hat{\alpha}\vert\gamma_1$	$\hat{\delta}\vert\gamma_1$

With $\hat{\alpha}$, $\hat{\delta}$ and $\hat{\gamma}$ thus determined, $\hat{\beta}'$ follows as before on substituting these values into (26). Linear interpolation can be expected to yield satisfactory estimates if the absolute value $|\gamma_1 - \gamma_o|$ is sufficiently small.
However, several attempts may be necessary before obtaining a pair of approximations that meet any reasonable definition of "sufficiently small" as used here.

5. THE THREE-PARAMETER WEIBULL DISTRIBUTION

The standard three-parameter Weibull distribution may be viewed as a special case of the four-parameter distribution. Its density function follows from (3) on setting $\alpha = 0$. We thus have

$$g(x;\ \beta,\ \gamma,\ \delta) = \left(\frac{\delta}{\beta}\right) (x-\gamma)^{\delta-1}\, e^{-(x-\gamma)^{\delta}/\beta} \quad ; \ x > \gamma \tag{30}$$

$= 0$, elsewhere.

In this case, the moment estimating equations of (13) reduce to

$$\bar{x} = \gamma + \beta^{1/\delta}\Gamma[(1/\delta) + 1] ,$$

$$s_x^2 = \beta^{2/\delta}\{\Gamma[(2/\delta) + 1] - \Gamma^2[(1/\delta) + 1]\} , \tag{31}$$

$$a_3 = \frac{\Gamma[3/\delta) + 1] - 3\Gamma\ [(2/\delta) + 1]\ \Gamma[(1/\delta) + 1] + 2\Gamma^3[(1/\delta) + 1]}{\{\Gamma[(2/\delta) + 1] - \Gamma^2[(1/\delta) + 1]\}^{3/2}} .$$

The third standard moment is a function of the shape parameter δ only. Therefore the third equation of (31) can be solved for δ^*, and the remaining moment estimates follows as

$$\beta^* = s_x^{\delta^*}/\{\Gamma[(2/\delta^*) + 1] - \Gamma^2[(1/\delta^*) + 1]\}^{\delta^*/2}$$

$$\gamma^* = \bar{x} - (\beta^*)^{1/\delta^*}\ \Gamma[(1/\delta^*) + 1] . \tag{32}$$

Alternate estimates can be obtained by setting

$$\gamma^{**} = x_{min} , \tag{33}$$

i.e. setting γ equal to the first order statistic, and then dropping the third equation of (31). On eliminating β between the two remaining equations of (31) we have

$$\frac{s_x^2}{(\bar{x} - x_{min})^2} = \frac{\Gamma[(2/\delta) + 1] - \Gamma^2[(1/\delta) + 1]}{\Gamma^2[(1/\delta) + 1]} . \tag{34}$$

Equation (34) can be solved for δ^{**}, and β^{**} then follows from the second equation of (31), when δ^{**} is substituted for δ.

6. ILLUSTRATIVE EXAMPLES

To illustrate application of results obtained here, estimates have been calculated for two separate samples from three-parameter Weibull distributions. The first sample consists of 250 observations selected with the aid of a random number generator from a population in which $\beta = 16$, $\delta = 2$, and $\gamma = 10$. The second consists of 200 observations selected in the same manner from a population in which $\beta = 8$, $\delta = 3$, and $\gamma = 4$.

Example 1:

Following is a summary of the sample data for this example:

Class Interval			Frequency
10.00	-	10.75	7
10.75	-	11.50	23
11.50	-	12.25	39
12.25	-	13.00	40
13.00	-	13.75	36
13.75	-	14.50	38
14.50	-	15.25	29
15.25	-	16.00	16
16.00	-	16.75	12
16.75	-	17.50	7
17.50	-	18.25	2
18.25	-	19.00	1
Total			250

$\bar{x} = 13.494$

$s^2 = 2.853$

$a_3 = 0.408$

$x_{min} = 10.227$

$n = 250$

$$s^2 = \sum_{1}^{250} (x_i - \bar{x})^2/250.$$

The summary statistics were calculated directly from the raw data and are thus not subject to grouping errors which might be present in corresponding values calculated from the preceding frequency table. Estimates calculated from these data along with the true population values are given below:

SUMMARY OF ESTIMATES

Type Estimator	β	δ	γ	$\beta^{1/\delta}$	μ	σ^2	α_3	α_4
Moment	33.599	2.408	9.678	4.305	13.494	2.853	0.408	2.902
Maximum Likelihood	20.169	2.188	9.998	3.948	13.494	2.842	0.516	3.051
Two Moments and 1st Order Stat.	14.041	2.025	10.227	3.687	13.494	2.853	0.615	3.215
True Values	16	2	10	4	13.545	3.434	0.631	3.245

This example, of course, is for the special case in which $\alpha = 0$, and the moment estimates were obtained by simultaneously solving the three equations of (31) using the given sample data. The maximum likelihood estimates were calculated by simultaneously solving the first three equations of (25) with $\alpha = 0$, while the estimates based the first two moments and the first order statistic were obtained by simultaneously solving the first three equations of (19) with $\alpha = 0$. For comparison, the true population values are also included.

Example 2

The second sample was obtained in the same manner as the first and the estimates based on this sample were likewise obtained using the same procedures as for the first example. Following are summaries of both the sample data and the estimates obtained for this example.

Class Interval			Frequency
4.25	-	4.50	5
4.50	-	4.75	4
4.75	-	5.00	9
5.00	-	5.25	19
5.25	-	5.50	23
5.50	-	5.75	25
5.75	-	6.00	27
6.00	-	6.25	28
6.25	-	6.50	23
6.50	-	6.75	18
6.75	-	7.00	9
7.00	-	7.25	5
7.25	-	7.50	3
7.50	-	7.75	2
Total			200

$\bar{x} = 5.886$

$s^2 = 0.450$

$a_3 = 0.023$

$x_{min} = 4.266$

$n = 200$

SUMMARY OF ESTIMATES

Type Estimator	β	δ	γ	$\beta^{1/\delta}$	μ	σ^2	α_3	α_4
Moment	20.407	3.510	3.761	2.361	5.886	0.450	0.023	2.713
Maximum Likelihood	20.073	3.503	3.767	2.354	5.886	0.449	0.024	2.713
Two Moment and 1st Order Stat.	4.758	2.594	4.266	1.824	5.886	0.450	0.318	2.820
True Values	8	3	4	2	5.786	0.421	0.168	2.729

Calculations for these two illustrative examples were carried out by Mr. Russell Helm.

Maximum likelihood estimating equations applicable here follow from those given in equation (25) for the general four-parameter distribution when we set $\alpha = 0$. We accordingly retain the first three equations of (25) with $\alpha = 0$ to be solved simultaneously for $\hat{\gamma}$, $\hat{\delta}$, and $\hat{\beta}$, Solution of these estimating equations in the case where $\gamma = 0$ have been rather fully discussed in reference [2] and elsewhere.

7. SOME CONCLUDING REMARKS

Since the range over which our random variable is defined depends on γ , the estimation of this parameter presents certain difficulties not encountered in estimating other parameters. With $\gamma \leq x$, it seems natural to employ the first order statistic as an estimate. This estimate is of course biased since x_1 constitutes an upper bound on γ. Furthermore in certain distributions, the likelihood function becomes infinitely large when we set $\gamma = x_1$. As a way out of these difficulties in connection with the log-normal distribution in which the lower limit is one of the parameters to be estimated, the writer [1] in 1951 suggested using the following relation

$$\frac{k}{n} = \int_{\gamma}^{x_o} f(x)\ dx\ , \tag{35}$$

in which

$$x_o = x_1 + \eta/2 \ , \qquad (36)$$

where x_1 is the first order statistic and η is the interval of precision with which x_1 has been measured. We let k designate the number of times that x_1 occurs in a sample of size n. In most samples of concern to us, $k = 1$.

Since the integration is easy to carry out in the case of the Weibull and related distributions, it would appear that estimating equation (35) might be quite useful in estimating γ in these distributions as well as in the log-normal distribution. Further investigation of the resulting estimators and their properties is planned as a future project.

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