NASA Manned Spacecraft Center R&D Procurement Branch

Effects of Digital Approximation In Statistical Estimation

by

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FINAL TECHNICAL REPORT

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DEPARTMENT OF MATHEMATICS AND STATISTICS
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Chapter I

Introduction

I. Background and Introduction

To say that a random variable X is continuous means that the values {x} which X can achieve can be measured to a countable infinity of decimal places. In practice, one never deals with continuous observations, but rather with a discrete variable Y, measured to some finite number of decimals, (the number of places being specified either by choice or the experimenter's limited measuring ability). Generally, one assumes that X and Y are equivalent.

It is this assumption which this paper will explore in part, i.e., we shall try to quantify the error due to Y being only approximately equal to X. There are several ways of "rounding off" in measurement problems, (e.g., rounding to the nearest integer or nearest .1 of an integer, etc.). We shall consider here the case where the value of X is truncated to the right, as explained specifically below. We shall furthermore confine ourselves to errors arising in parameter estimation, i.e., suppose that X is a function of a parameter θ, (θ may be vector valued), then we shall ask, how are the sampling properties of θ^*, an estimate of θ, affected by using Y instead of X? These problems will be considered in some detail for the normal distribution and in somewhat less detail for the Exponential, T, Chi-square and F-distributions.

II. Preliminaries

Before discussing the above-mentioned problem for particular density functions we include the following necessary preliminary information. Random variables will always be denoted by capital Roman letters while values of such variables will be represented by lower case letters. Thus X will denote a random variable while x denotes a value of X. If the range of X is any non-negative real value, then for some set of a_i

$$x = \sum_{-\infty}^{\infty} a_i 10^i ,$$

where

$$a_i \in \{N | N \text{ is an integer and } 0 \leq N \leq 9\}.$$

As mentioned previously observed values of X will be truncated to the right, i.e. if y is an observed value or observable value of X then y is of the form

$$y = \sum_{i=c}^{\infty} a_i 10^i .$$

Each observed value of a set of values of a random variable X will be truncated in the same manner. If $h = 10^c$, $c = \pm 0, \pm 1, \pm 2, \ldots$, the set of observable values of the random variable X will be

$$\left(y_k \middle| y_k = \begin{cases} kh & \text{when } kh \leq x < kh + h, \quad k = 0,1,2,\ldots \\ kh + h & \text{when } kh < x \leq kh + h, \quad k = -1,-2,\ldots \end{cases} \right) \tag{1}$$

This therefore defines a discrete random variable Y whose values are defined by (1). Clearly the probability density function, g_Y(pdf) for Y is defined by

$$g_Y(y_k) = \int_{A_k} f_X(x)\,dx, \tag{2}$$

where f_x is the pdf for X and $A_k = \begin{cases} [kh,kh+h), & y > 0,\ k = 1,2,\ldots \\ (-h,h), & y_k = 0 \\ (kh-h,kh], & y_k < 0, k = -1,-2,\ldots \end{cases}$

If X has a pdf f_x or simply f we will often say $X \sim f_x$ or

The notation "$\sim$" should be read "is distributed as".

Chapter II

The Normal Distribution

I. Introduction

Throughout this chapter we will assume $X \sim N(\mu,\sigma^2)$ and consider the effects of truncation on the estimator $\bar{X}$ and its associated confidence intervals. In order to accomplish this we shall first consider a sequence of preliminary theorems. The following forms of $E[X^n]$ and $E[Y^n]$ will be useful.

$$E(X^n) = \sum_{k=0}^{\infty} \int_{kh}^{kh+h} \frac{x^n}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$+ \sum_{k=0}^{\infty} \int_{-kh-h}^{-kh} \frac{x^n}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad (2.1)$$

$$E(Y^n) = \sum_{k=0}^{\infty} k^n h^n \int_{kh}^{kh+h} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$+ \sum_{k=0}^{\infty} (-kh)^n \int_{-kh-h}^{-kh} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad (2.2)$$

Theorem 2.1. If $X \sim N(0,\sigma^2)$ and Y is distributed as the truncation of X, then

$$E(Y^n) \begin{cases} = 0 & \text{if } n \text{ is odd} \\ > 0 & \text{if } n \text{ is even}. \end{cases}$$

Proof.

Consider n to be odd first.

$$E(Y^n) = \sum_{k=0}^{\infty} k^n h^n \int_{kh}^{kh+h} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx$$

$$\sum_{k=0}^{\infty} k^n h^n \int_{-kh-h}^{-kh} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx$$

$$\sum_{k=0}^{\infty} k^n h^n \int_{kh}^{kh+h} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx$$

$$\sum_{k=0}^{\infty} k^n h^n \int_{kh}^{kh+h} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(-\frac{t}{\sigma}\right)^2} dx$$

$$= 0$$

if the series $\sum_{k=0}^{\infty} k^n h^n \int_{kh}^{kh+h} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx$ converges.

Since

$$k^n h^n \int_{kh}^{kh+h} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx \le k^n h^{n+1} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{kh}{\sigma}\right)^2}$$

and $\sum_{k=0}^{\infty} k^n h^{n+1} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{kh}{\sigma}\right)^2}$ converges by the integral test,

then the desired series converges.

Now consider n to be even. It is easily seen that

$$E(Y^n) = 2 \sum_{k=0}^{\infty} k^n h^n \int_{kh}^{kh+h} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx ,$$

and this series is a convergent positive term series. Thus

$$E(Y^n) > 0 .$$

Corollary 2.1. If $X \sim N(0,\sigma^2)$ then $\overline{Y}$ is an unbiased estimator for 0, i.e., $E[\overline{Y}] = 0$.

Theorem 2.2. Let $X \sim N(\mu,\sigma^2)$ and Y is distributed as the truncation of X. If $\mu > 0$, then $E(Y^n) > 0$.

Proof.

Consider n to be odd first.

$$E(Y^n) = \sum_{k=0}^{\infty} k^n h^n \int_{\frac{kh-\mu}{\sigma}}^{\frac{kh+h-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$- \sum_{k=0}^{\infty} k^n h^n \int_{\frac{kh+\mu}{\sigma}}^{\frac{kh+h+\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$= \sum_{k=0}^{\infty} k^n h^n \left\{ \int_{\frac{kh-\mu}{\sigma}}^{\frac{kh+h-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \right.$$

$$\left. - \int_{\frac{kh+\mu}{\sigma}}^{\frac{kh+h-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \right\}$$

since the first two series converge. Examining the limits on each integral for each value of k, it is seen that the quantity inside the braces is positive for each value of k. Thus $E(Y^n) > 0$ when n is odd.

Now consider n to be even.

$$E(Y^n) = \sum_{k=0}^{\infty} k^n h^n \int_{\frac{kh-\mu}{\sigma}}^{\frac{kh+h-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$+ \sum_{k=0}^{\infty} k^n h^n \int_{\frac{kh+\mu}{\sigma}}^{\frac{kh+h+\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt .$$

Each of the above series are convergent positive term series. Thus $E(Y^n) > 0$ when n is even.

Theorem 2.3. If $X \sim N(\mu,\sigma^2)$, Y is distributed as the truncation of X, and $\mu > 0$, then $E(X^n) > E(Y^n)$.

Proof.

Consider n to be odd first.

$$E(X^n) - E(Y^n) = \sum_{k=0}^{\infty} \int_{kh}^{kh+h} \frac{x^n}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$+ \sum_{k=0}^{\infty} \int_{-kh-h}^{-kh} \frac{x^n}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$- \sum_{k=0}^{\infty} k^n h^n \int_{kh}^{kh+h} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$+ \sum_{k=0}^{\infty} k^n h^n \int_{-kh-h}^{-kh} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \sum_{k=0}^{\infty} \int_{kh}^{kh+h} \frac{x^n - k^n h^n}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$+ \sum_{k=0}^{\infty} \int_{-kh-h}^{-kh} \frac{x^n + k^n h^n}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

since all four series converge. Observe that

$$\int_{-kh-h}^{-kh} \frac{x^n + k^n h^n}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= - \int_{kh}^{kh+h} \frac{x^n - k^n h^n}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{-x-\mu}{\sigma}\right)^2} dx$$

Thus

$$E(X^n) - E(Y^n) = \sum_{k=0}^{\infty} \int_{kh}^{kh+h} \frac{x^n - k^n h^n}{\sqrt{2\pi}\,\sigma} \left\{ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} - e^{-\frac{1}{2}\left(\frac{x+\mu}{\sigma}\right)^2} \right\} dx$$

$$= \sum_{k=0}^{\infty} \int_{kh}^{kh+h} \frac{x^n - k^n h^n}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x^2+\mu^2}{\sigma^2}\right)} \left\{ e^{\frac{x\mu}{\sigma^2}} - e^{\frac{-x\mu}{\sigma^2}} \right\} dx$$

Since the integrand is nonnegative for each value of k, then

$$E(X^n) - E(Y^n) > 0 \quad \text{or} \quad E(X^n) > E(Y^n)$$

when n is odd.

Now consider n to be even.

$$E(X^n) - E(Y^n) = \sum_{k=0}^{\infty} \int_{kh}^{kh+h} \frac{x^n - k^n h^n}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$+ \sum_{k=0}^{\infty} \int_{-kh-h}^{-kh} \frac{x^n - k^n h^n}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \sum_{k=0}^{\infty} \int_{kh}^{kh+h} \frac{x^n - k^n h^n}{\sqrt{2\pi}\,\sigma} \left\{ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \right.$$

$$\left. + \, e^{-\frac{1}{2}\left(\frac{x+\mu}{\sigma}\right)^2} \right\} dx \, .$$

Since the integrand is nonnegative for each value of k, then

$$E(X^n) - E(Y^n) > 0 \quad \text{or} \quad E(X^n) > E(Y^n)$$

when n is even.

Theorem 2.4. If $X \sim N(\mu, \sigma^2)$, Y is distributed as the truncation of X, and if $\mu < 0$, then $E(Y^n) < 0$ when n is odd and $E(Y^n) > 0$ when n is even.

Proof.

Consider n to be odd first.

$$E(Y^n) = -\left\{\sum_{k=0}^{\infty} k^n h^n \int_{\frac{kh+\mu}{\sigma}}^{\frac{kh+h+\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt\right.$$

$$\left. - \sum_{k=0}^{\infty} k^n h^n \int_{\frac{kh-\mu}{\sigma}}^{\frac{kh+h-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \right\}.$$

The quantity inside the braces is $E(Y'^n)$ where Y' is distributed as the truncation of X' and $X' \sim N(-\mu,\sigma)$. By Theorem 3.2, $E(Y'^n) > 0$.

Thus $E(Y^n) < 0$ when n is odd.

Now consider n to be even.

$$E(Y^n) = \sum_{k=0}^{\infty} k^n h^n \int_{\frac{kh-\mu}{\sigma}}^{\frac{kh+h-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$+ \sum_{k=0}^{\infty} k^n h^n \int_{\frac{kh+\mu}{\sigma}}^{\frac{kh+h+\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt .$$

The right side is $E(Y'^n)$ where Y' is distributed as above. Thus

$$E(Y^n) > 0 \text{ when n is even .}$$

Theorem 2.5. If $X \sim N(\mu,\sigma^2)$, Y is distributed as the truncation of

X, and $\mu < 0$, then

$$E(X^n) - E(Y^n) \begin{cases} > 0 & \text{if } n \text{ is even} \\ < 0 & \text{if } n \text{ is odd} . \end{cases}$$

Proof.

Let $X' \sim N(-\mu,\sigma^2)$ and Y' is distributed as the truncation of X'. Consider n to be odd first. It is well-known that

$$E(X^n) = -E(X'^n) \quad \text{if } n \text{ is odd.}$$

By the proof of Theorem 2.4,

$$E(Y^n) = -E(Y'^n) .$$

Thus

$$E(X^n) - E(Y^n) = -\{E(X'^n) - E(Y'^n)\} .$$

Since $-\mu > 0$, then $E(X'^n) - E(Y'^n) > 0$ by Theorem 2.3. Thus

$$E(X^n) - E(Y^n) < 0 \quad \text{or} \quad E(X^n) < E(Y^n)$$

when n is odd.

Now consider n to be even. It is also well-known that

$$E(X^n) = E(X'^n) \quad \text{if } n \text{ is even.}$$

By the proof of Theorem 2.4,

$$E(Y^n) = E(Y'^n).$$

Thus

$$E(X^n) - E(Y^n) = E(X'^n) - E(Y'^n).$$

Now $E(X^n) - E(Y^n) > 0$ since $E(X'^n) - E(Y'^n) > 0$ by Theorem 2.3. Hence $E(X^n) > E(Y^n)$ when n is even.

Corollary 2.2. If $X \sim N(\mu,\sigma^2)$, $\mu \neq 0$ then $\bar{Y}$ is a biased estimator for μ.

From the above theorems it is clear that $\bar{X}$ and $\bar{Y}$ do not in general enjoy the same statistical properties. Moreover, since all estimates for μ are based on $\bar{Y}$ it seems logical to investigate its properties as an estimator

II. Truncated Estimators

We now prove the following important result.

Theorem 2.6. Let Y_i, $i = 1,2,\ldots,m$ be independent truncated normal random variables with distinct means μ_i. Then the maximum likelihood estimates of μ_i, $i = 1,\ldots,n$ based on a sample of size one from each population are given by

$$\hat{\mu}_i = \begin{cases} 0 & , \text{ if } y_i = 0 \\ y_i + \frac{h}{2} , & \text{if } y_i > 0 \\ y_i - \frac{h}{2} , & \text{if } y_i < 0 . \end{cases} \tag{2.3}$$

Proof: Let $f(x_i;\mu_i,\sigma_i)$, $i = 1,2,\ldots,n$ be density functions corresponding to distinct normal populations. Let a sample of size one be taken from each population and for convenience let $y_1,\ldots,y_p$, $y_{p+1},\ldots,y_r,y_{r+1},\ldots,y_n$ be the zero, positive, and negative observations, respectively. The joint likelihood function for the above observations is

$$L(y_1,\ldots,y_n;\mu_1,\ldots,\mu_n,\sigma_1,\ldots,\sigma_n)$$

$$= \prod_{j=1}^{p} \int_{-h}^{h} f(x_j;\mu_j,\sigma_j)dx_j \prod_{j=p+1}^{r} \int_{y_j}^{y_j+h} f(x_j;\mu_j,\sigma_j)dx_j \cdot$$

$$\prod_{j=r+1}^{n} \int_{y_j-h}^{y_j} f(x_j;\mu_j,\sigma_j)dx_j.$$

Now let

$$L^*(y_1,\ldots,y_n;\mu_1,\ldots,\mu_n,\sigma_1,\ldots,\sigma_n) = \log L(y_1,\ldots,y_n;\mu_1,\ldots,\mu_n,\sigma_1,\ldots,\sigma_n).$$

Thus

$$L^*(y_1,\ldots,y_n;\mu_1,\ldots,\mu_n,\sigma_1,\ldots,\sigma_n) = \sum_{i=1}^{p} \log \int_{-h}^{h} f(u_i;\mu_i,\sigma_1)du_i$$

$$+ \sum_{i=p+1}^{r} \log \int_{y_i}^{y_i+h} f(u_i;\mu_i,\sigma_i)du_i + \sum_{i=r+1}^{n} \log \int_{y_i-h}^{y_i} f(u_i;\mu_i,\sigma_i)du_i.$$

But if $1 \le i \le p$, then

$$\frac{\partial L^*}{\partial \mu_i} = \frac{1}{m(\mu_i)} \int_{-h}^{h} \frac{1}{\sqrt{2\pi}\,\sigma_i^3} (u_i - \mu_i)\, e^{-\frac{1}{2\sigma^2}(u_i-\mu_i)^2} du_i$$

$$= \frac{1}{m(\mu_i)\sqrt{2\Pi}\ \sigma_i^3}\left[e^{-\frac{1}{2\sigma_i^2}(-h-\mu_i)^2} - e^{-\frac{1}{2\sigma_i^2}(h-\mu_i)^2} \right]$$

where $m(\mu_i) = \int_{-h}^{h} f(u_i;\mu_i,\sigma_i)\,du_i.$

Setting $\frac{\partial L^*}{\partial \mu_i} = 0$ we obtain

$$(-h - \hat{\mu}_i)^2 = (h - \hat{\mu}_i)^2$$

or

$$\hat{\mu}_i = 0.$$

Now if $p + 1 \leq i \leq r$, then

$$\frac{\partial L^*}{\partial \mu_i} = \frac{1}{m(\mu_i)} \int_{y_i}^{y_i+h} \frac{1}{\sqrt{2\Pi}\ \sigma_i^3} (u_i - \mu_i)\, e^{-\frac{1}{2\sigma_i^2}(u_i-\mu_i)^2} du_i$$

$$= e^{-\frac{1}{2}\left(\frac{y_i-\mu_i}{\sigma_i}\right)^2}\left[1 - e^{-\frac{h(y_i-\mu_i)}{\sigma_i^2} - \frac{h^2}{2\sigma_i^2}}\right]$$

where

$$m(\mu_i) = \int_{y_i}^{y_i+h} f(u_i;\mu_i,\sigma_i)\,du_i.$$

Setting

$$\frac{\partial L^*}{\partial \mu_i} = 0$$

we obtain $$1 - e^{-\frac{h(y_i-\mu_i)}{\sigma_i^2} - \frac{h^2}{2\sigma_i^2}} = 0$$

or $$\hat{\mu}_i = y_i + \frac{h}{2}$$

Similarly it can be shown that if $r+1 \leq i \leq n$, then

$$\hat{\mu}_i = y_i - \frac{h}{2}$$

Since $\frac{\partial L^*}{\partial \mu} = 0$ is also sufficient in this case the theorem is proved.

Corollary 2.3. Let Y be a truncated normal random variable with mean μ and $Y_1, \ldots, Y_m$ be a corresponding random sample. Then the maximum likelihood estimate of μ is

$$\hat{\mu} = \frac{1}{n} \sum \hat{\mu}_i$$

where the $\hat{\mu}_i$'s are given by (2.3).

Note that $\hat{\mu}$ is given by

$$\hat{\mu} = \overline{y} + \left(\frac{t - s}{2n}\right) h$$

where t is the number of positive observations and s is the number of negative observations in the sample.

A method for defining the random variable Z has been established by theorem 2.6. That is, let z denote values of Z, then

$$z = \begin{cases} y + \frac{1}{2} h & \text{if } y > 0 \\ y - \frac{1}{2} h & \text{if } y < 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Properties of this new random variable Z will now be investigated. In particular, sampling properties will be investigated, i.e. if a random sample of size n is taken from a normal population and the samples are truncated (as described previously), will the truncated values give more accurate results than the associated values of the random variable Z? In order to perform such an investigation, normal random variables must be generated.

In the following discussion X will be a normal random variable generated to 16 decimal places of accuracy. Since values of Y approach values of X and $h \to 0$ this degree of accuracy will be assumed sufficient to consider values of these 16 place random variables as values of a continuous random variable X. For clarity, we now reformulate the problem. Suppose $X \sim N(\mu,\sigma^2)$. Then the maximum likelihood estimator $\hat{\mu}$ for the mean μ and the unbiased estimator σ^2 for the variance σ^2 is

$$\hat{\mu}_X = \frac{1}{n} \sum_{i=1}^{n} X_i = \overline{X} \tag{2.4}$$

and

$$\hat{\sigma}_X{}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 = S_X^2. \tag{2.5}$$

In any physical problem values of the quantities (2.4) and (2.5) cannot be used. Instead one uses the corresponding truncated random variables Y and their corresponding estimators, i.e.

$$\hat{\mu}_Y = \frac{1}{n} \sum_{i=1}^{n} Y_i = \overline{Y} \tag{2.6}$$

$$\hat{\sigma}_Y{}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2 = S_Y^2. \tag{2.7}$$

However theorem 2.6 suggests that some of the error in using (2.6) and (2.7) rather than (2.4) and (2.5) may be compensated for by use of the random variable Z. Thus we define

$$\hat{\mu}_Z = \frac{1}{n} \sum_{i=1}^{n} Z_i = \overline{Z} \tag{2.8}$$

$$\hat{\sigma}_Z^{\,2} = \frac{1}{n-1} \sum_{i=1}^{n} (Z_i - \overline{Z})^2 \tag{2.9}$$

Table 1 gives values of the estimators $\hat{\mu}_X$, $\hat{\mu}_Y$, $\hat{\mu}_Z$ generated by the IBM 360/50 for different values of n, h and μ/σ. Similarly, Table 2 gives values of estimators $\hat{\sigma}_X^{\,2}$, $\hat{\sigma}_Y^{\,2}$, $\hat{\sigma}_Z^{\,2}$. In both tables the estimates have been rounded to four decimal places. The quantity $\overline{Z}$ is, of course, the maximum likelihood estimate for μ established in theorem 2.6. Table 1 shows that $|\hat{\mu}_X - \hat{\mu}_Z| < |\hat{\mu}_X - \hat{\mu}_Y|$ in almost all cases. Table 2 shows that in most cases $|\hat{\sigma}_X^{\,2} - \hat{\sigma}_Z^{\,2}| \leq |\hat{\sigma}_X^{\,2} - \hat{\sigma}_Y^{\,2}|$. In the case of the variance the difference between the estimators is apparently of less significance than in the case of the mean. In fact, in Table 2, if $\frac{\mu}{\sigma} \geq 2.4$ we have $\hat{\sigma}_Z^{\,2} = \hat{\sigma}_Y^{\,2}$. This latter observation is expected if one recalls that if $Z_i = Y_i + \frac{1}{2}h$ or $Z_i = Y_i - \frac{1}{2}h$ for every i then $\hat{\sigma}_Z^{\,2} = \hat{\sigma}_Y^{\,2}$.

MAXIMUM LIKELIHOOD ESTIMATES FOR E[X]

Table I

n	h	μ/σ	$\hat{\mu}_X$	$\hat{\mu}_Y$	$\hat{\mu}_Z$
5	1.0	0.0	1.4660	1.0000	1.5000
5	1.0	0.6	.6115	.2000	.7000
5	1.0	1.2	1.3894	1.0000	1.3000
5	1.0	1.8	1.5840	1.0000	1.3000
5	1.0	2.4	2.2831	2.0000	2.5000
5	1.0	3.0	3.5106	3.0000	3.5000
5	1.0	3.6	2.9853	2.4000	2.9000
5	1.0	4.2	3.9782	3.6000	4.1000
5	1.0	4.8	4.9379	4.4000	4.9000
10	1.0	0.0	-.0162	-.1000	-.1000
10	1.0	0.6	.6596	.4000	-.7000
10	1.0	1.2	1.2340	.9000	1.3000
10	1.0	1.8	1.8355	1.4000	1.8000
10	1.0	2.4	2.5124	2.0000	2.5000
10	1.0	3.0	3.3555	2.8000	3.3000
10	1.0	3.6	3.8254	3.2000	3.7000
10	1.0	4.2	4.5703	4.2000	4.7000
10	1.0	4.8	4.8492	4.5000	5.0000
15	1.0	0.0	-.6337	-.4000	-.6333
15	1.0	0.6	.3256	.2667	-.3667
15	1.0	1.2	1.5785	1.2667	1.7000
15	1.0	1.8	1.5877	1.1333	1.6333
15	1.0	2.4	2.3390	1.8667	2.3667
15	1.0	3.0	3.0308	2.4667	2.9667
15	1.0	3.6	3.3319	2.8000	3.3000
15	1.0	4.2	4.5010	3.9333	4.4333
15	1.0	4.8	4.9872	4.4667	4.9667
25	1.0	0.0	-.0764	-.0400	-.1000
25	1.0	0.6	1.3227	.8800	1.2600
25	1.0	1.2	1.2671	.8800	1.2200
25	1.0	1.8	1.6569	1.1200	1.6200

n	h	μ/σ	$\hat{\mu}_X$	$\hat{\mu}_Y$	$\hat{\mu}_Z$
25	1.0	2.4	2.3928	1.8000	2.3000
25	1.0	3.0	3.3946	2.9600	3.4600
25	1.0	3.6	3.6589	3.0800	3.5800
25	1.0	4.2	4.6561	4.1200	4.6200
25	1.0	4.8	5.0107	4.5600	5.0600
5	0.1	0.0	-.1187	-.1200	-.1300
5	0.1	0.6	1.1194	1.0800	1.1300
5	0.1	1.2	.7236	.6600	.6900
5	0.1	1.8	2.1749	2.1000	2.1500
5	0.1	2.4	1.5941	1.5400	1.5900
5	0.1	3.0	2.9730	2.9000	2.9500
5	0.1	3.6	3.1898	3.1600	3.2100
5	0.1	4.2	3.9630	3.9000	3.9500
5	0.1	4.8	4.8721	4.8400	4.8900
10	0.1	0.0	-.0295	-.0200	-.0300
10	0.1	0.6	.5333	.4900	.5300
10	0.1	1.2	1.5307	1.4800	1.5300
10	0.1	1.8	2.4269	2.3800	2.4300
10	0.1	2.4	2.0422	1.9800	2.0300
10	0.1	3.0	2.8020	2.7500	2.8000
10	0.1	3.6	3.7158	3.6600	3.7100
10	0.1	4.2	3.4514	3.4000	3.4500
10	0.1	4.8	5.4254	5.3800	5.4300
15	0.1	0.0	-.2928	-.2867	-.3033
15	0.1	0.6	.7276	.6867	.7300
15	0.1	1.2	.9755	.9333	.9833
15	0.1	1.8	2.0947	2.0467	2.0967
15	0.1	2.4	2.1571	3.1133	2.1633
15	0.1	3.0	2.9307	2.8867	2.9367
15	0.1	3.6	3.4697	3.4200	3.4700
15	0.1	4.2	4.3364	4.2867	4.3367
15	0.1	4.8	4.9480	4.9000	4.9500
25	0.1	0.0	-.2530	-.2360	-.2460
25	0.1	0.6	.4131	.3880	.4140
25	0.1	1.2	1.2729	1.2400	1.2740

n	h	μ/σ	$\hat{\mu}_X$	$\hat{\mu}_Y$	$\hat{\mu}_Z$
25	0.1	1.8	1.6631	1.6000	1.6500
25	0.1	2.4	2.9079	2.8600	2.9100
25	0.1	3.0	3.0240	2.9760	3.0260
25	0.1	3.6	3.7020	3.6480	3.6980
25	0.1	4.2	4.1684	4.1160	4.1660
25	0.1	4.8	4.6623	4.6120	4.6620
5	0.01	0.0	.4135	.4120	.4150
5	0.01	1.2	1.0885	1.0800	1.0850
5	0.01	2.4	3.3280	3.3240	3.3290
5	0.01	3.6	3.1048	3.0980	3.1030
5	0.01	4.8	5.2367	5.2320	5.2370
10	0.01	0.0	.3544	.3520	.3540
10	0.01	1.2	1.1704	1.1660	1.1690
10	0.01	2.4	2.2828	2.2780	2.2830
10	0.01	3.6	3.0954	3.0900	3.0950
10	0.01	4.8	4.9910	4.9850	4.9900
15	0.01	0.0	-.0264	-.0273	-.0270
15	0.01	1.2	1.0635	1.0600	1.0637
15	0.01	2.4	1.9583	1.9540	1.9590
15	0.01	3.6	3.2765	3.2707	3.2757
15	0.01	4.8	4.5027	4.4987	4.5037
25	0.01	0.0	-.2450	-.2396	-.2406
25	0.01	1.2	1.4557	1.4504	1.4550
25	0.01	2.4	2.5060	2.5016	2.5066
25	0.01	3.6	3.4165	3.4116	3.4166
25	0.01	4.8	4.7123	4.7072	4.7122

MAXIMUM LIKELIHOOD ESTIMATES FOR σ^2

Table II

n	h	μ/σ	$\hat{\sigma}_X^2$	$\hat{\sigma}_Y^2$	$\hat{\sigma}_Z^2$
5	1.0	0.0	3.1316	2.0000	2.0000
5	1.0	0.6	.3693	.2000	.2000
5	1.0	1.2	.8442	.5000	1.2000
5	1.0	1.8	1.8945	1.0000	1.7000
5	1.0	2.4	.2746	.5000	.5000
5	1.0	3.0	1.1376	1.5000	1.5000
5	1.0	3.6	.7501	1.3000	1.3000
5	1.0	4.2	.5452	.8000	.8000
5	1.0	4.8	1.5254	1.3000	1.3000
10	1.0	0.0	.5864	.1000	.4889
10	1.0	0.6	.7484	.2667	.6222
10	1.0	1.2	.9964	.9889	1.2889
10	1.0	1.8	2.1678	1.8222	2.2333
10	1.0	2.4	.4334	.6667	.6667
10	1.0	3.0	1.8417	1.7333	1.7333
10	1.0	3.6	.6291	.8444	.8444
10	1.0	4.2	1.3802	1.0667	1.0667
10	1.0	4.8	.6630	.7222	.7222
15	1.0	0.0	.7931	.5429	1.1238
15	1.0	0.6	.6231	.2095	.6952
15	1.0	1.2	1.9020	1.6381	2.0286
15	1.0	1.8	1.1923	1.2667	1.2667
15	1.0	2.4	1.0113	.9810	.9810
15	1.0	3.0	1.2595	1.4096	1.4096
15	1.0	3.6	.8299	1.0286	1.0286
15	1.0	4.2	.8482	1.2095	1.2095
15	1.0	4.8	1.6305	1.8381	1.8381
25	1.0	0.0	.9593	.4567	1.0833
25	1.0	0.6	1.2041	1.0267	1.3467
25	1.0	1.2	.9973	.7767	1.2100
25	1.0	1.8	.9711	.8600	.8600

n	h	μ/σ	$\hat{\sigma}^2_X$	$\hat{\sigma}^2_Y$	$\hat{\sigma}^2_Z$
25	1.0	2.4	.8442	.9167	.9167
25	1.0	3.0	1.4118	1.5400	1.5400
25	1.0	3.6	.9172	.9100	.9100
25	1.0	4.2	1.0527	1.2767	1.2767
25	1.0	4.8	.6279	.8400	.8400
5	0.1	0.0	.3205	.2770	.3320
5	0.1	0.6	.5773	.5920	.5920
5	0.1	1.2	.4264	.3880	.4330
5	0.1	1.8	1.8729	1.8250	1.8250
5	0.1	2.4	.5257	.5130	.5130
5	0.1	3.0	1.0717	1.0600	1.0600
5	0.1	3.6	2.0959	2.0800	2.0800
5	0.1	4.2	3.0849	3.0650	3.0650
5	0.1	4.8	.8228	.8480	.8480
10	0.1	0.0	.3239	.2929	.3373
10	0.1	0.6	1.3361	1.1433	1.2622
10	0.1	1.2	.9031	.8618	.8618
10	0.1	1.8	1.0725	1.0484	1.0484
10	0.1	2.4	.9318	.9129	.9129
10	0.1	3.0	.8588	.8494	.8494
10	0.1	3.6	.7137	.6938	.6938
10	0.1	4.2	1.1942	1.1978	1.1978
10	0.1	4.8	1.0557	1.0796	1.0796
15	0.1	0.0	1.4357	1.3399	1.4427
15	0.1	0.6	.4298	.4298	.4446
15	0.1	1.2	.3695	.3638	.3638
15	0.1	1.8	1.0058	.9998	.9998
15	0.1	2.4	1.1309	1.1184	1.1184
15	0.1	3.0	.9529	.9541	.9541
15	0.1	3.6	.3931	.3886	.3886

III. Confidence Intervals

As was seen in the previous section the quantity $\overline{Y}$ has significantly different properties than $\overline{X}$ and $\overline{Z}$. Moreover, in most cases we found the values of $\overline{Z}$ to be closer to those of $\overline{X}$ than were the values of $\overline{Y}$. This immediately leads us to consider these effects on well known confidence intervals which depend on $\overline{X}$. To accomplish this, values of the random variable $\overline{X}$ were generated under the same conditions and assumptions as the previous section. The confidence intervals studied were 90% confidence intervals associated with the mean of a normal distribution, i.e.,

$$\left.\begin{array}{ll}
\text{(i)} & (\overline{X} - 1.645\sigma/\sqrt{n},\ \overline{X} + 1.645\sigma/\sqrt{n}) \\
\text{(ii)} & (\overline{Y} - 1.645\sigma/\sqrt{n},\ \overline{Y} + 1.645\sigma/\sqrt{n}) \\
\text{(iii)} & (\overline{Z} - 1.645\sigma/\sqrt{n},\ \overline{Z} + 1.645\sigma/\sqrt{n})
\end{array}\right\} \quad (3.1)$$

and

$$\left.\begin{array}{ll}
\text{(i)} & \overline{X} - a\sqrt{\sum_{i=1}^{n} (X_i-\overline{X})^2}\Big/\sqrt{n(n-1)},\quad \overline{X} + a\sqrt{\sum_{i=1}^{n} (X_i-\overline{X})^2}\Big/\sqrt{n(n-1)} \\
\text{(ii)} & \overline{Y} - a\sqrt{\sum_{i=1}^{n} (Y_i-\overline{Y})^2}\Big/\sqrt{n(n-1)},\quad \overline{Y} + a\sqrt{\sum_{i=1}^{n} (Y_i-\overline{Y})^2}\Big/\sqrt{n(n-1)} \\
\text{(iii)} & \overline{Z} - a\sqrt{\sum_{i=1}^{n} (Z_i-\overline{Z})^2}\Big/\sqrt{n(n-1)},\quad \overline{Z} + a\sqrt{\sum_{i=1}^{n} (Z_i-\overline{Z})^2}\Big/\sqrt{n(n-1)}\ ,
\end{array}\right\} \quad (3.2)$$

where a is a constant such that

$$\int_{a}^{\infty} f(t;n-1)\,dt = .05$$

and f is pdf of a random variable with a t-distribution. Tables 3 and 4 give the results of this study. Each line in these tables represent 100 different random samples of size n from a $N(\mu,\sigma^2)$ population. The columns X, Y and Z give the number of times that the mean is in the corresponding confidence interval. Column M gives the number of times out of 100 that the confidence interval using $\overline{Z}$ approximated the confidence interval using $\overline{X}$ better than the confidence interval using $\overline{Y}$ did. In almost all cases the $\overline{Z}$ confidence interval is a better representation of the $\overline{X}$ confidence interval.

COMPARISON OF 90% CONFIDENCE INTERVALS WHEN σ^2 IS KNOWN

Table IV

n	h	μ	σ	X	Y	Z	M
10	1.0	3	1	88	49	91	100
10	1.0	4	1	90	56	88	100
10	1.0	5	1	93	58	92	100
10	1.0	10	1	87	46	88	100
10	1.0	15	5	84	87	86	99
10	1.0	20	5	91	88	91	100
10	1.0	25	5	95	93	95	100
10	1.0	50	5	92	87	92	100
10	1.0	30	10	90	92	91	100
10	1.0	40	10	89	92	89	100
10	1.0	50	10	92	92	93	100
10	1.0	100	10	89	92	89	100
10	.1	3	1	90	90	90	99
10	.1	4	1	94	95	95	100
10	.1	5	1	91	91	91	99
10	.1	10	1	87	87	88	100
20	1.0	3	1	84	29	85	100
20	1.0	4	1	90	26	86	100
20	1.0	5	1	86	27	87	100
20	1.0	10	1	88	25	89	100
20	1.0	15	5	89	87	89	100
20	1.0	20	5	89	82	90	100
20	1.0	25	5	89	90	90	100
20	1.0	50	5	87	83	86	100
20	1.0	30	10	93	88	93	100
20	1.0	40	10	88	90	88	100
20	1.0	50	10	92	92	92	100
20	1.0	100	10	89	85	89	100
20	.1	3	1	92	88	92	100

n	h	μ	σ	X	Y	Z	M
20	.1	4	1	90	83	89	100
20	.1	5	1	85	86	85	100
20	.1	10	1	94	94	94	100
30	1.0	3	1	92	15	92	100
30	1.0	4	1	86	11	85	100
30	1.0	5	1	91	15	89	100
30	1.0	10	1	89	14	92	100
30	1.0	15	5	94	88	94	100
30	1.0	20	5	88	79	89	100
30	1.0	25	5	92	88	93	100
30	1.0	50	5	91	84	91	100
30	1.0	30	10	89	88	89	100
30	1.0	40	10	92	93	92	100
30	1.0	50	10	88	85	89	100
30	1.0	00	10	95	92	95	100
30	.1	3	1	88	87	89	100
30	.1	4	1	92	90	92	100
30	.1	5	1	93	97	93	100
30	.1	10	1	94	94	94	100
41	1.0	3	1	91	9	89	100
41	1.0	4	1	91	11	89	100
41	1.0	5	1	85	9	88	100
41	1.0	10	1	91	5	90	100
41	1.0	15	5	90	80	91	100
41	1.0	20	5	92	83	91	100
41	1.0	25	5	88	73	86	100
41	1.0	50	5	89	80	92	100
41	1.0	30	10	89	91	90	100
41	1.0	40	10	87	87	88	100
41	1.0	50	10	84	82	84	100
41	1.0	100	10	91	89	90	100
41	.1	3	1	91	91	91	100
41	.1	4	1	89	89	88	100
41	.1	5	1	91	89	91	100
41	.1	10	1	88	92	87	100

n	h	μ	σ	X	Y	Z	M
61	1.0	3	1	93	1	92	100
61	1.0	4	1	88	4	91	100
61	1.0	5	1	85	1	84	100
61	1.0	10	1	91	2	90	100
61	1.0	15	5	91	88	91	100
61	1.0	20	5	88	76	89	100
61	1.0	25	5	92	84	93	100
61	1.0	50	5	87	76	86	100
61	1.0	30	10	92	91	92	100
61	1.0	40	10	91	86	91	100
61	1.0	50	10	89	91	89	100
61	.1	3	1	87	88	87	100
61	.1	4	1	90	87	89	100
61	.1	5	1	90	88	89	100
61	.1	10	1	91	89	90	100
121	1.0	3	1	92	0	92	100
121	1.0	4	1	86	0	87	100
121	1.0	5	1	96	0	93	100
121	1.0	10	1	93	0	91	100
121	1.0	15	5	90	72	90	100
121	1.0	20	5	84	61	85	100
121	1.0	25	5	94	75	94	100
121	1.0	50	5	89	67	89	100
121	1.0	30	10	88	78	88	100
121	1.0	40	10	86	84	86	100
121	.1	3	1	89	83	89	100
121	.1	4	1	90	89	90	100
121	.1	5	1	86	74	86	100
121	.1	10	1	94	85	94	100
121	.1	15	5	94	92	94	100
121	.1	20	5	90	92	90	100
121	.1	25	5	92	92	92	100
121	.1	50	5	96	95	96	100
121	.01	3	1	90	90	90	100
121	.01	4	1	93	92	93	100

n	h	σ	μ	X	Y	Z	M
121	.01	5	1	93	92	93	100
121	.01	10	1	86	86	86	100

COMPARISON OF 90% CONFIDENCE INTERVALS
WHEN σ^2 IS NOT KNOWN.

Table V

n	h	μ	σ	X	Y	Z	M
10	1.0	3	1	91	49	91	99
10	1.0	4	1	90	60	93	100
10	1.0	5	1	93	62	93	99
10	1.0	10	1	89	67	91	100
10	1.0	15	5	90	92	91	100
10	1.0	20	5	89	86	88	98
10	1.0	25	5	88	88	87	99
10	1.0	50	5	94	91	95	100
10	1.0	30	10	91	94	41	98
10	1.0	40	10	92	92	93	99
10	1.0	50	10	91	93	92	100
10	1.0	100	10	87	87	85	100
10	.1	3	1	89	90	89	98
10	.1	4	1	92	90	92	99
10	.1	5	1	86	84	85	100
10	.1	10	1	89	89	89	99
20	1.0	3	1	87	31	87	100
20	1.0	4	1	89	35	90	100
20	1.0	5	1	88	33	91	100
20	1.0	10	1	93	33	92	100
20	1.0	15	5	90	89	89	100
20	1.0	20	5	90	88	90	100
20	1.0	25	5	94	90	94	100
20	1.0	50	5	92	88	91	100
20	1.0	30	10	93	87	93	100
20	1.0	40	10	84	81	86	100
20	1.0	50	10	90	89	90	100
20	1.0	100	10	89	88	89	100
20	.1	3	1	92	93	92	100

n	h	μ	σ	X	Y	Z	M
20	.1	4	1	89	91	90	100
20	.1	5	1	84	85	84	100
20	.1	10	1	89	90	90	100
30	1.0	3	1	90	20	89	100
30	1.0	4	1	91	16	90	100
30	1.0	5	1	92	16	91	100
30	1.0	10	1	93	21	90	100
30	1.0	15	5	95	89	95	100
30	1.0	20	5	90	83	89	100
30	1.0	25	5	93	80	93	100
30	1.0	50	5	94	86	84	100
30	1.0	30	10	87	86	88	100
30	1.0	40	10	88	84	88	100
30	1.0	50	10	93	86	92	100
30	1.0	00	10	92	92	92	100
30	.1	3	1	90	91	90	100
30	.1	4	1	96	96	96	100
30	.1	5	1	91	89	91	100
30	.1	10	1	91	87	91	100
41	1.0	3	1	97	39	97	100
41	1.0	4	1	94	49	94	100
41	1.0	5	1	95	53	95	100
41	1.0	10	1	92	49	92	100
41	1.0	15	5	94	92	94	100
41	1.0	20	5	95	91	95	100
41	1.0	25	5	95	94	95	100
41	1.0	50	5	98	95	97	100
41	1.0	30	10	93	95	93	100
41	1.0	40	10	95	94	95	100
41	1.0	50	10	98	97	98	100
41	1.0	100	10	98	97	98	100
41	.1	3	1	90	90	90	100
41	.1	4	1	93	93	93	100
41	.1	5	1	98	97	98	100
41	.1	10	1	97	96	97	100

n	h	μ	σ	X	Y	Z	M
61	1.0	3	1	92	37	93	100
61	1.0	4	1	95	39	94	100
61	1.0	5	1	95	38	95	100
61	1.0	10	1	95	37	94	100
61	1.0	15	5	90	81	89	100
61	1.0	20	5	96	96	96	100
61	1.0	25	5	91	87	92	100
61	1.0	50	5	92	87	92	100
61	1.0	30	10	95	92	96	100
61	1.0	40	10	91	94	91	100
61	1.0	50	10	94	94	95	100
61	1.0	100	10	83	88	84	100
61	.1	3	1	94	90	95	100
61	.1	4	1	93	88	93	100
61	.1	5	1	95	93	95	100
61	.1	10	1	94	92	94	100
121	1.0	3	1	93	0	95	100
121	1.0	4	1	92	0	92	100
121	1.0	5	1	90	0	92	100
121	1.0	10	1	95	0	97	100
121	1.0	15	5	94	76	94	100
121	1.0	20	5	92	65	93	100
121	1.0	25	5	87	71	88	100
121	1.0	50	5	88	69	87	100
121	1.0	30	10	87	80	88	100
121	1.0	40	10	87	90	87	100
121	1.0	50	10	88	82	87	100
121	1.0	100	10	91	86	91	100
121	.1	3	1	91	88	91	100
121	.1	4	1	85	82	85	100
121	.1	5	1	89	77	89	100
121	.1	10	1	97	83	97	100

Summary: if a random sample of size n is taken from a normal population and each sample value is truncated as described in this report, then the associated random variable Z will yield better results than the truncated values. In particular confidence intervals for the mean are more accurate if the values of the random variable Z are used. The sample mean obtained by using Z variables are better estimates of the mean. The sample variance is better or at least as good if the Z variables are used.

Chapter III

LINEAR MODEL

I. Introduction

In the preceding chapter the effect of truncation on estimators for the mean and variance in a normal distribution was considered. In this chapter those same effects will be investigated with regard to the so called BLUE estimator in a linear model. To be more specific let X be an "observable" random variable and assume

$$X = H\beta + e \ , \tag{3.1}$$

where X is a $n \times 1$ "observation" vector.[1] H is an $n \times p$ non random matrix, β is a $p \times 1$ parameter vector and e is an $n \times 1$ vector of random errors.

Then, as is well known, if $E[e] = 0$, $E[ee^T] = \sigma^2 I$, and H is of rank p the BLUE estimator for β is given by

$$\hat{\beta} = (H^T H)^{-1} H^T X \ , \tag{3.2}$$

where T denotes transpose. The question we shall now consider is the effect of truncation on the estimator $\hat{\beta}$ given by (3.2).

II. BLUE Estimators When Observations are Truncated

We shall use the same notation as in the previous chapters. That is Y will denote the truncation of X and the extent of this truncation is given by $h = 10^c$, $c = 0, \pm 1, \pm 2, \ldots$ Since in

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[1] We include the quotations on the words observable and observations to recall to the reader's mind that in reality it is truncated values of these random variables that are observed and not true values.

reality the estimator (3.2) takes the form

$$\hat{\beta} = (H^T H)^{-1} H^T Y \tag{3.3}$$

the following theorem is significant.

Theorem 3.1. If $\beta \neq 0$ then $\hat{\beta} = (H^T H)^{-1} H^T Y$ is a biased estimator for β.

Proof: Let $H = (h_i)$, $\beta = (\beta_i)$ where h_i is a $1 \times p$ vector and β_i is a scaler. Then

$$E[\hat{\beta}] = (H^T H)^{-1} H^T E[Y] ,$$

and by theorems 2.3 and 2.5 $E[Y_i] > E[X_i]$ if $h_i\beta > 0$ and $E[Y_i] < E[X_i]$ if $h_i\beta < 0$. Since $E[X] = H\beta$ the result is established. Since (3.3) is a biased estimator for β we will now determine an estimator which essentially compensates for this bias.

Theorem 3.2. If $Z = (Z_i)$ is the random vector defined by

$$z_i = \begin{cases} y_i + \frac{h}{2} & \text{if } y_i > 0 \\ y_i - \frac{h}{2} & \text{if } y_i < 0 \\ 0 & \text{if } y_i = 0 \end{cases} \tag{3.4}$$

$X \sim N(0,\sigma^2 I)$ and Y is distributed as the truncation of X, then the maximum likelihood estimator for β is given by

$$\hat{\beta} = (H^T H)^{-1} H^T Z \tag{3.5}$$

where H is of full column rank.

Proof: For convenience let $y_1,\ldots,y_p,y_{p+1},\ldots,y_r,y_{r+1},\ldots,y_n$ be zero, positive and negative observations, respectively. Then the likelihood functions for the n observations are

$$L(y_1, y_2, \ldots, y_n; \mu_1, \mu_2, \ldots, \mu_n)$$

$$= \prod_{i=1}^{p} \int_{-h}^{h} f(u_i;\mu_i)du_i \prod_{i=p+1}^{r} \int_{y_i}^{y_i+h} f(u_i;\mu_i)du_i \prod_{i=r+1}^{n} \int_{y_i-h}^{y_i} f(u_i;\mu_i)du_i, \tag{3.6}$$

where

$$f(u_i;\mu_i) = \frac{1}{\sqrt{2\Pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(u_i - \mu_i)^2}, \quad \text{and} \quad \mu_i = h_i\beta .$$

Now let

$$L^*(y_1, \ldots, y_n; \mu_1, \ldots, \mu_n) = \log L(y_1, \ldots, y_n; \mu_1, \ldots, \mu_n).$$

Then

$$L^*(y_1, \ldots, y_n; \mu_1, \ldots, \mu_n)$$

$$= \sum_{i=1}^{p} \log \int_{-h}^{h} f(u_i;\mu_i)du_i + \sum_{i=p+1}^{r} \log \int_{y_i}^{y_i+h} f(u_i;\mu_i)du_i$$

$$+ \sum_{i=r+1}^{n} \log \int_{y_i-1}^{y_i} f(u_i;\mu_i)du_i \tag{3.7}$$

But if $1 \leq j \leq p$

$$\frac{\partial L^*}{\partial \mu_j} = \frac{1}{m(\mu_j)} \int_{-h}^{h} \frac{1}{\sqrt{2\Pi}\,\sigma^3} (u_j - \mu_j)\, e^{-\frac{1}{2\sigma^2}(u_j - \mu_j)^2} du_j$$

$$= \frac{1}{m(\mu_j)\sqrt{2\Pi}\,\sigma} \left[e^{-\frac{1}{2\sigma^2}(-h - \mu_j)^2} - e^{-\frac{1}{2\sigma^2}(h - \mu_j)^2} \right]$$

where $m(\mu_j) = \int_{-h}^{h} f(u_j;\mu_j)du_j$.

Setting $\frac{\partial L^*}{\partial \mu_j} = 0$ we have

$$(-h - \hat{\mu}_j)^2 = (h - \hat{\mu}_j)^2$$

or

$$\hat{\mu}_j = \widehat{h_j \beta} = 0 \qquad (3.8)$$

Now if $p+1 \leq j \leq r$ we have

$$\frac{\partial L^*}{\partial \mu_j} = \frac{1}{m(\mu_j)} \int_{y_j}^{y_j+h} \frac{1}{\sqrt{2\Pi}\,\sigma^3} (u_j - \mu_j)\, e^{-\frac{1}{2\sigma^2}(u_j - \mu_j)^2} du_j$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{y_j - \mu_j}{\sigma}\right)^2}}{m(\mu_j)\sqrt{2\Pi}\,\sigma} \left[1 - e^{-\frac{h(y_j - \mu_j)}{\sigma^2} - \frac{h^2}{2\sigma^2}} \right]$$

where $m(\mu_j) = \int_{y_j}^{y_j+h} f(u_j;\mu_j)du_j$.

Setting $\frac{\partial L^*}{\partial \mu_j} = 0$ we get

$$1 - e^{-\frac{h(y_j - \hat{\mu}_j)}{\sigma^2} - \frac{h^2}{2\sigma^2}} = 0$$

or $$\hat{\mu}_j = y_j + \frac{h}{2} \,. \qquad (3.9)$$

Similarly if $r+1 \leq j \leq n$ we obtain

$$\hat{\mu}_j = y_j - \frac{h}{2} \qquad (3.10)$$

By trivial considerations of the slopes $\frac{\partial L^*}{\partial \mu_j}$, $j = 1,\ldots,n$, it is easy to see that (3.8), (3.9) and (3.10) maximize the corresponding term in the sums of (3.7). Hence if $h_j\hat{\beta} = \hat{\mu}_j$, $\hat{\beta}$ maximizes (3.7) and the theorem is established. To show that $h_j\hat{\beta} = \hat{\mu}_j$, where $\hat{\beta}$ is given by (3.3), note that $\hat{\mu}_j = h_j\beta$ implies $Z = (\hat{\mu}_j)$ is in the range of H. Hence

$$\hat{\beta} = (H^T H)^{-1} H^T Z$$

implies

$$H^T H\hat{\beta} = H^T Z$$

$$H\hat{\beta} = HH^+ Z = Z,$$

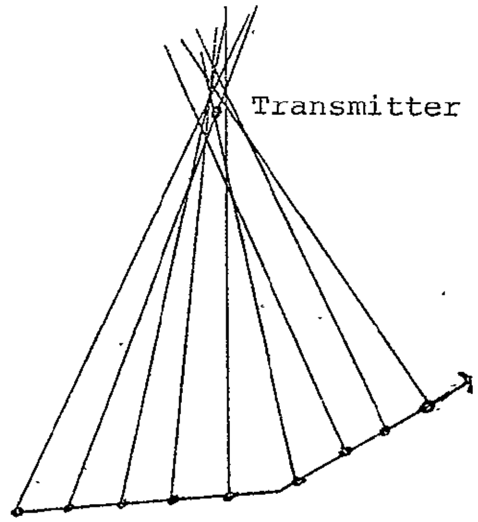
where H^+ is the Penrose pseudoinverse of H.

Theorem 3.2 establishes (3.5) as a maximum likelihood estimator for β. We conjecture that $\hat{\beta}$ given by (3.5) is also unbiased. This has been shown if $h_i\beta$ is divisible by the truncation factor h.

In the next section we apply the results of theorem 3.2 to a physical problem with good results. These results also tend to support the conjecture that $\hat{\beta}$ is unbiased.

III. Application

In order to show the significance of the preceding results we will now consider an application to one of the common problems in the field of electronic warfare. The problem is as follows: During flight an aircraft obtains bearing readings on an unfriendly electromagnetic transmitter. The problem is to estimate the latitude and longitude of this transmitter from the bearing data. The following figure shows a two dimensional version of the data.



Aircraft Receiving Passive Bearing Data

Fig. 3.1.

By assuming the earth is a sphere one can obtain the following mathematical model for the geometry of Figure 3.1.

$$\cos(B_i+H_i) = \frac{\phi_e - \phi_i}{\sqrt{(\phi_e - \phi_i)^2 + (\lambda_e - \lambda_i)^2 \cos^2\phi_i}} \qquad (3.11)$$

where

B_i = relative bearing to the transmitter on the i-th observation

H_i = heading of the aircraft at the i-th observation

ϕ_e = latitude of the transmitter

λ_e = longitude of the transmitter

ϕ_i = latitude of the aircraft at the i-th observation

λ_i = longitude of the aircraft at the i-th observation

Now let

$$\tilde{B}_i = B_i + \Delta B_i \qquad \tilde{\phi}_i = \phi_i + \Delta\phi_i$$

$$\tilde{H}_i = H_i + \Delta H_i \qquad \hat{\lambda}_{oe} = \lambda_e + \Delta\lambda_e$$

$$\tilde{\lambda}_i = \lambda_i + \Delta\lambda_i \qquad \hat{\phi}_{oe} = \phi_e + \Delta\phi_e \quad ,$$

where ˜ denotes observed, Δ denotes random error and $\hat{\lambda}_{oe}$ and $\hat{\phi}_{oe}$ denote initial estimates for λ_e and ϕ_e. Then we can write from (3.11)

$$\tilde{B}_i - \Delta B_i = -\tilde{H}_i + \Delta H_i + \text{Cos}^{-1}\left\{\frac{\hat{\phi}_{oe} - \tilde{\phi}_i - \Delta\phi_e + \Delta\phi_i}{[(\hat{\phi}_e - \tilde{\phi}_i - \Delta\phi_e + \Delta\phi_i)^2 + (\hat{\lambda}_e - \Delta\lambda_e - \tilde{\lambda}_i + \Delta\lambda_i)^2 \text{Cos}(\tilde{\phi}_i - \Delta\phi_i)]^{\frac{1}{2}}}\right\}$$

$$= F(\hat{\lambda}_{oe}, \hat{\phi}_{oe}, \tilde{\lambda}_i, \tilde{\phi}_i, \tilde{H}_i, \Delta\lambda_e, \Delta\phi_e, \Delta\lambda_i, \Delta\phi_i, \Delta H_i)$$

Now by expanding F about the zero error vector and ignoring higher order terms one obtains the linear model

$$B = HM + U \ , \tag{3.12}$$

where

$B = (\Delta B_i)$ and $\Delta B_i = \tilde{B}_i - F(\hat{\lambda}_{oe}, \hat{\phi}_{oe}, \tilde{\lambda}_i, \tilde{\phi}_i, \tilde{H}_i, 0,0,0,0,0)$

$H = (h_i)$ and $h_i = \left(\frac{\partial F}{\partial(\Delta\phi_e)} \Bigg|_{\Delta_i=0} \ , \ \frac{\partial F}{\partial(\Delta\lambda_e)} \Bigg|_{\Delta_i=0} \right)$

$M = \begin{pmatrix} \Delta\phi_e \\ \Delta\lambda_e \end{pmatrix}$

and U = Random errors such that $E[U] = 0$ and $E[U\,U^T] = \sigma^2 I$

Clearly estimating M is equivalent to estimating ϕ_e and λ_e if an initial estimate is known. However, there are, from our previous work, three estimators associated with (3.12), the theoretical one

$$\hat{M} = (H^T H)^{-1} H^T B \ , \tag{3.13}$$

the truncated one

$$\hat{M} = (H^T H)^{-1} H^T B_T \ , \tag{3.14}$$

where B_T denotes the observation vector resulting from bearing and heading readings read to a finite number of places, and the estimator based on the vector B_T, i.e.

$$\hat{M} = (H^T H)^{-1} H^T Z \tag{3.15}$$

where in Z, $\frac{h}{2}$, $-\frac{h}{2}$ or 0 is added to bearing and heading readings depending on whether they are positive, negative or zero. To evaluate the estimators (3.13), (3.14) and (3.15) the problem was simulated on the IBM 360/50. A straight flight path was assumed on a sphere and all noise was normal. The standard deviation of bearing and heading was $1/2^\circ$ while aircraft latitude and longitude had a standard deviation of $.01^\circ$. The flight path was at a distance of approximately 100 miles from the transmitter at boresight (or normal distance) and was 150 miles long. Tables 3.1 and 3.2 below show typical results. In all simulations the estimates given by (3.15) were closer to the estimate given by (3.13) than was the estimate given by (3.14).

Table 3.1.

Bearings Read to Nearest Degree

Error in $(H^TH)^{-1}H^TB$	Error in $(H^TH)^{-1}H^TB_T$	Error in $(H^TH)^{-1}H^TZ$
.1132	1.1152	.1046
.0561	.9376	.0572
.0931	1.0345	.0893
.1765	1.2189	.1675
.1378	1.1157	.1401
.2026	1.3268	.1928
.0743	.8573	.0788
.1322	.9901	.1411
.1647	1.1043	.1523
.2039	1.2365	.1987
.1114	.9875	.1175
.1050	1.0001	.1123
.0635	.9005	.0715
.2111	1.2377	.2005
.1763	1.1211	.1832

Location Errors in Miles
Using Three Different Estimators

Table 3.2

Bearings Read to Nearest .1 Degree

Error in $(H^TH)^{-1}H^TB$	Error in $(H^TH)^{-1}H^TB_T$	Error in $(H^TH)^{-1}H^TZ$
.1132	.1563	.1091
.0561	.1032	.0577
.0931	.1247	.0999
.1765	.1966	.1788
.1378	.1888	.1421
.2026	.2431	.1998
.0743	.1011	.0695
.1322	.1537	.1300
.1647	.1892	.1713
.2039	.2333	.2108
.1114	.1431	.1097
.1050	.1287	.1055
.0635	.0890	.0682
.2111	.2400	.2245
.1763	.2036	.1834

Location Errors in Miles
Using Three Different Estimators

Chapter IV

The Exponential Frequency Function

Suppose X is distributed exponentially and Y is distributed as the truncation of X. This chapter shall investigate the relationship between the random variables X and Y.

The frequency function of the random variable X is given by

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{\theta} e^{-\frac{x}{\theta}} & x \geq 0. \end{cases}$$

It is well-known that the mean and variance of the random variable X are $\mu_X = \theta$ and $\sigma_X^2 = \theta^2$, respectively. It is also well-known that the moment generating function (m.g.f.) of the random variable X is given by

$$M_X(t) = 1/(1-t\theta).$$

Theorem 4.1. If X is distributed exponentially and Y is distributed as the truncation of X, then the m.g.f. of Y is given by

$$M_Y(t) = (e^{h/\theta}-1)/(e^{h/\theta}-e^{ht}).$$

The mean and variance of Y are given by

$$\mu_Y = h/(e^{h/\theta}-1) \quad \text{and} \quad \sigma_Y^2 = h^2 e^{h/\theta}/(e^{h/\theta}-1)^2,$$

respectively. Moreover, $\mu_Y \to \mu_X$ and $\sigma_Y^2 \to \sigma_X^2$ as $h \to 0$.

Proof.

The frequency function of the random variable Y is given by

$$g(y_k) = \int_{kh}^{kh+h} \frac{1}{\theta} e^{-\frac{x}{\theta}} dx \qquad k = 0,1,2,\ldots$$

$$= e^{-\frac{kh}{\theta}}(1 - e^{-h/\theta}) \qquad k = 0,1,2,\ldots$$

$$= e^{-\frac{y_k}{\theta}}(1 - e^{-h/\theta}) \qquad k = 0,1,2,\ldots$$

The m.g.f. of Y is given by

$$M_Y(t) = \sum_{k=0}^{\infty} e^{kht} e^{-kh/\theta}(1 - e^{-h/\theta})$$

$$= (1 - e^{-h/\theta}) \sum_{k=0}^{\infty} e^{-(h/\theta - ht)k}$$

$$= (1 - e^{-h/\theta})/(1 - e^{-(h/\theta - ht)})$$

$$(e^{h/\theta}-1)/(e^{h/\theta}-e^{ht}).$$

$$M_Y'(t) = he^{ht}(e^{h/\theta}-1)/(e^{h/\theta}-e^{ht})^2.$$

$$M_Y''(t) = h^2e^{ht}(e^{h/\theta}-1)(e^{2ht}+e^{h/\theta}e^{ht})/(e^{h/\theta}-e^{ht})^3.$$

$$\mu_Y = M_Y'(0)$$

$$= h/(e^{h/\theta}-1).$$

$$\sigma_Y^2 = M_Y''(0) - \mu_Y^2$$

$$= h^2(e^{h/\theta}+1)/(e^{h/\theta}-1)^2 - h^2/(e^{h/\theta}-1)^2$$

$$= h^2e^{h/\theta}/(e^{h/\theta}-1)^2.$$

$$\lim_{h\to 0} \mu_Y = \lim_{h\to 0} h/(e^{h/\theta}-1)$$

$$= \lim_{h\to 0} 1/\tfrac{1}{\theta}\, e^{h/\theta} \qquad \text{by L'Hospital's rule}$$

$$= \theta.$$

Thus $\mu_Y \to \mu_X$ as $h \to 0$.

$$\lim_{h\to 0} \sigma_Y^2 = \lim_{h\to 0} h^2e^{h/\theta}/(e^{h/\theta}-1)^2$$

$$= \lim_{h\to 0} (2+2h/\theta)/(2e^{h/\theta}\div\theta^2) \qquad \text{by L'Hospital's rule}$$

$$= \theta^2.$$

Hence, $\lim_{h\to 0} \sigma_Y^2 = \sigma_X^2$.

It is well-known that the maximum likelihood estimator for θ is given by

$$\hat{\theta} = \bar{X}.$$

Theorem 4.2. If X is distributed exponentially and Y is distributed as the truncation of X, then the maximum likelihood estimator for θ is given by

$$\tilde{\theta} = h[\log\ (\bar{Y}(\bar{Y}-h))]^{-1}.$$

Furthermore $\tilde{\theta} \to \hat{\theta}$ as $h \to 0$.

Proof.

The likelihood function is given by

$$L = (1 - e^{-h/\theta})^n \prod_{p=1}^{n} e^{-y_p/\theta} .$$

Hence $$\log L = n \log\ (1-e^{-h/\theta}) - \sum_{p=1}^{n} y_p/\theta,$$

And $$\frac{\partial}{\partial\theta} \log L = -hn/(\theta^2(1-e^{-h/\theta})) + \sum_{p=1}^{n} y_p/\theta^2.$$

Setting

$\frac{\partial}{\partial\theta} \log L = 0$ yields

$$hn/(1-e^{-h/\theta}) = \sum_{p=1}^{n} y_p$$

or

$$h/(1-e^{-h/\theta}) = \bar{Y} .$$

Hence

$$e^{h/\theta} = \overline{Y}/(\overline{Y}-h)$$

or

$$h/\theta = \log\ (\overline{Y}/(\overline{Y}-h)).$$

Thus

$$\tilde{\theta} = h/\log\ (\overline{Y}/(\overline{Y}-h)),$$

and
$$\lim_{h\to 0} \tilde{\theta} = \lim_{h\to 0} h/\log\ (\overline{Y}/(\overline{Y}-h))$$

$$= \lim_{h\to 0} (\overline{Y}-h) \qquad \text{by L'Hospital's rule}$$

$$= \overline{X},$$

since $\overline{Y} \to \overline{X}$ as $h \to 0$.

Theorem 4.3. If X is distributed exponentially and Y is distributed as the truncation of X, then $E(Y^n) < E(X^n)$.

Proof.

$$E(X^n)-E(Y^n) = \int_0^\infty x^n \frac{1}{\theta} e^{-x/\theta}dx - \sum_{k=0}^{\infty} k^n h^n \int_{kh}^{kh+h} \frac{1}{\theta} e^{-x/\theta}dx$$

$$= \sum_{k=0}^{\infty} \int_{kh}^{kh+h} (x^n - k^n h^n) \frac{1}{\theta} e^{-x/\theta} dx$$

$$> 0 \ ,$$

since the integrand is nonnegative for each value of k. Thus $E(X^n) > E(Y^n)$.

If X is distributed exponentially and a random sample of size n is taken from this population, then $\overline{X}$ is the maximum likelihood estimate of the parameter θ. If the sample values are truncated as described in this paper, then $\overline{Y}$ is used instead of $\overline{X}$ to determine this estimate. Since

$$E(\overline{Y}) = E(Y) \quad \text{and} \quad E(\overline{X}) = E(X),$$

then

$$E(\overline{X}) - E(\overline{Y}) = E(X) - E(Y)$$

$$> 0$$

by Theorem 4.3. Thus $\overline{Y}$ is a biased estimate of θ. The next theorem is concerned with the bias due to the truncated sample mean $\overline{Y}$.

Theorem 4.4. If X is distributed exponentially and Y is distributed as the truncation of X, then

$$E(X) - E(Y) = \frac{h}{2} - \frac{h^2}{12\theta} + \frac{h^4}{\;\;\;\;3} - \cdots$$

Proof.

$$E(X) - E(Y) = \theta - \mu_Y$$

$$= \theta - h/(e^{h/\theta}-1)$$

$$= \theta - \{\theta - \frac{h}{2} + \frac{h^2}{12\theta} - \frac{h^3}{720\theta^3} + \ldots\}$$

$$= \frac{h}{2} - \frac{h^2}{12\theta} + \frac{h^4}{720\theta^3} - \ldots .$$

Chapter V

The Chi-Square Frequency Function

Let X have the chi-square distribution and Y be distributed as the truncation of X. This chapter will be concerned with various properties of the random variable Y and its frequency function. Several lemmas are necessary in order to pursue this subject.

Lemma 5.1. If p is a nonnegative integer, then

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} j^p = \begin{cases} 0 & p < n \\ (-1)^n n! & p = n. \end{cases}$$

Lemma 5.2. If p is a nonnegative integer, then

$$\sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (k-j)^p = \begin{cases} 0 & p < k+1 \\ (k+1)! & p = k+1. \end{cases}$$

Proof.

$$\sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (k-j)^p$$

$$= \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (-1)^p \left\{ j^p + \sum_{i=1}^{p} \binom{p}{i} j^{p-i} (-k)^i \right\}$$

$$= \begin{cases} 0 & p < k+1 \\ (-1)^{k+1}(-1)^{k+1}(k+1)! & p = k+1 \end{cases}$$

by Lemma 5.1.

Since $(-1)^{k+1}(-1)^{k+1} = 1$, the proof is complete.

Lemma 5.3. Let

$$D(h) = \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (e^{h/2})^{k-j}.$$

Then $\frac{h^n}{D(h)} \to 0$ as $h \to 0$ if $k+1 < n$ and $\frac{h^n}{D(h)} \to 2^n$ as $h \to 0$ if $k+1 = n$.

Proof.

Suppose $k+1 < h$. Applying L'Hospital's rule $p = k+1$ times yields

$$\lim_{h\to 0} \frac{h^n}{D(h)} = \frac{\lim\limits_{h\to 0} (n)(n-1)\ldots(n-p+1)h^{n-p}}{\lim\limits_{h\to 0} \sum\limits_{j=0}^{k+1} (-1)^j \binom{k+1}{j} \left(\frac{k-j}{2}\right)^p (e^{h/2})^{k-j}}$$

$$= \frac{0}{\sum\limits_{j=0}^{k+1} (-1)^j \binom{k+1}{j} \left(\frac{k-j}{2}\right)^p}$$

$$= \frac{0}{2^{-p}(k+1)!} \qquad \text{by lemma 5.2.}$$

Now suppose $k+1 = n$. Applying L'Hospital's rule $p = k+1$ times yields

$$\lim_{h\to 0} \frac{h^n}{D(h)} = \frac{\lim_{h\to 0} n!}{\lim_{h\to 0} \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} \left(\frac{k-j}{2}\right)^p (e^{h/2})^{k-j}}$$

$$= \frac{n!}{2^{-n} \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (k-j)^n}$$

$$= \frac{n!}{2^{-n} n!} \qquad \text{by lemma 5.2}$$

$$= 2^n.$$

Lemma 5.4.

$$\sum_{k=0}^{\infty} k^p x^k = \sum_{k=1}^{p} (-1)^k \frac{x^k}{(1-x)^{k+1}} \sum_{a=1}^{k} (-1)^a \binom{k}{a} a^p.$$

Proof.

$$\sum_{k=0}^{\infty} k^p x^k = \left(x \frac{d}{dx}\right)^p \sum_{k=0}^{\infty} x^k$$

$$= \left(x \frac{d}{dx}\right)^p \frac{1}{1-x}$$

$$= \sum_{k=1}^{p} (-1)^k \frac{x^k}{(1-x)^{k+1}} \sum_{a=1}^{k} (-1)^a \binom{k}{a} a^p . \quad *$$

Lemma 5.5. Let $n > 1$. Then

$$\frac{2^{1-n} h^n}{(n-1)!} \sum_{k=0}^{\infty} k^{n-1} e^{-hk/2} \to 2 \quad \text{as} \quad h \to 0.$$

Proof.

By the preceding lemma,

$$a(n,h) = \frac{2^{1-n} h^n}{(n-1)!} \sum_{k=0}^{\infty} k^{n-1} e^{-hk/2}$$

$$= \frac{2^{1-n} h^n}{(n-1)!} \sum_{k=1}^{n-1} (-1)^k \frac{(e^{-h/2})^k}{(1-e^{-h/2})^{k+1}} \sum_{a=1}^{k} (-1)^a \binom{k}{a} a^{n-1}$$

$$= \frac{2^{1-n} h^n}{(n-1)!} \sum_{k=1}^{n-1} (-1)^k \frac{1}{(e^{h/2}-1)^k (1-e^{-h/2})} \sum_{a=1}^{k} (-1)^a \binom{k}{a} a^{n-1}.$$

Observe

$$(e^{h/2}-1)^k (1-e^{-h/2}) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (e^{h/2})^{k-j} \{1 - e^{-h/2}\}$$

$$= \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (e^{h/2})^{k-j}.$$

* I.J. Schwatt - An Introduction to the Operations with Series, Univ. of Penn. Press, p. 93.

Thus

$$a(n,h) = \frac{2^{1-n}}{(n-1)!} \sum_{k=1}^{n-1} \frac{(-1)^k h^n}{\sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (e^{h/2})^{k-j}} \sum_{a=1}^{k} (-1)^a \binom{k}{a} a^{n-1}.$$

By lemma 5.3

$$\lim_{h\to 0} a(n,h) = \frac{2^{1-n}}{(n-1)!} \sum_{k=n-1}^{n-1} (-1)^k 2^n \sum_{a=1}^{n-1} (-1)^a \binom{n-1}{a} a^{n-1}$$

$$= \frac{2^{1-n}}{(n-1)!} (-1)^{n-1} 2^n (-1)^{n-1} (n-1)!$$

by lemma 5.1.

Thus $\lim_{h\to 0} a(n,h) = 2$.

If $X \sim \chi^2(2n)$, then the frequency function of X can be integrated. Because of this, this chapter will only be concerned with a chi-square variable X with an even number of degrees of freedom.

Theorem 5.6. Let $X \sim \chi^2(2n)$ and Y be distributed as the truncation of X. Then the frequency function of Y is given by

$$g(y_k) = \sum_{p=0}^{n-1} 2^{p+1-n} \frac{1}{(n-1-p)!} e^{-hk/2} (hk)^{n-1-p}$$

$$- \sum_{p=0}^{n-1} 2^{p+1-n} \frac{1}{(n-1-p)!} e^{-(hk+h)/2} (hk+h)^{n-1-p}$$

where $y_k = kh$, $k = 0,1,2,\ldots$.

Proof.

The frequency function of X is given by

$$f(x) = \frac{2^{-n}}{(n-1)!} x^{n-1}e^{-x/2} .$$

Thus

$$g(y_k) = \int_{kh}^{kh+h} \frac{2^{-n}}{(n-1)!} x^{n-1}e^{-x/2}dx, \quad k = 0,1,2,\ldots .$$

The results are immediate after applying Theorem 1A of the appendix.

Theorem 5.7. Let $X \sim \chi^2(2n)$ and Y be distributed as the truncation of X. The moment generating function of Y is given by

$$M_Y(t) = \sum_{p=0}^{n-1} 2^{p+1-n} \frac{1}{(n-1-p)!} h^{n-1-p} \sum_{k=0} k^{n-1-p}e^{-(h(1-2t)k)/2}$$

$$- \sum_{p=0}^{n-1} 2^{p+1-n} \frac{1}{(n-1-p)!} e^{-h/2}h^{n-1-p} \sum_{k=0}^{\infty} (k+1)^{n-1-p}e^{-(h(1-2t)k)/2}.$$

Proof.

$$M_Y(t) = \sum_{k=0}^{\infty} e^{kht}g(y_k)$$

$$= \sum_{k=0}^{\infty} e^{kht} \left\{ \sum_{p=0}^{n-1} e^{p+1-n} \frac{1}{(n-1-p)!} e^{-hk/2} (kh)^{n-1-p} \right.$$

$$\left. - \sum_{p=0}^{n-1} 2^{p+1-n} \frac{1}{(n-1-p)!} e^{-(hk+h)/2} (hk+h)^{n-1-p} \right\}$$

The result is immediate if the order of summation is interchanged.

If $X \sim x^2(2n)$ and Y is distributed as the truncation of X, let $M_Y'(0)_n$ denote the evaluation of the first derivative of the moment generating function of Y at $t=0$. Similarly for $M_Y''(0)_n$.

Theorem 5.8. Let $n > 1$. Then

$$M_Y'(0)_n = \frac{2^{1-n}h^n}{(n-1)!} \sum_{k=0}^{\infty} k^{n-1} e^{-hk/2} + M_Y'(0)_{n-1}$$

and

$$M_Y''(0)_n = \frac{2^{1-n}h^{n+1}}{(n-1)!} \left\{ 2 \sum_{k=0}^{\infty} k^n e^{-hk/2} - \sum_{k=0}^{\infty} k^{n-1} e^{-hk/2} \right\}$$

$$+ M_Y''(0)_{n-1} \, .$$

Proof.

By Theorem 5.7

$$M_Y'(0)_n = \sum_{p=0}^{n-1} 2^{p+1-n} \frac{1}{(n-1-p)!} h^{n-1-p} \sum_{k=0}^{\infty} k^{n-1-p} hke^{-hk/2}$$

$$- \sum_{p=0}^{n-1} 2^{p+1-n} \frac{1}{(n-1-p)!} e^{-h/2} h^{n-1-p} \sum_{k=0}^{\infty} (k+1)^{n-1-p} hke^{-hk/2}$$

$$= \frac{2^{1-n}h^{n}}{(n-1)!} \sum_{k=0}^{\infty} k^{n} e^{-hk/2}$$

$$- \frac{2^{1-n}h^{n}}{(n-1)!} e^{-h/2} \sum_{k=0}^{\infty} (k+1)^{n-1} e^{-hk/2}$$

$$+ \sum_{p=1}^{n-1} 2^{p+1-n} \frac{1}{(n-1-p)!} h^{n-1-p} \sum_{k=0}^{\infty} k^{n-1-p} hke^{-hk/2}$$

$$- \sum_{p=1}^{\infty} 2^{p+1-n} \frac{1}{(n-1-p)!} e^{-h/2} h^{n-1-p} \sum_{k=0}^{\infty} (k+1)^{n-1-p} hke^{-hk/2}$$

$$= \frac{2^{1-n}h^{n}}{(n-1)!} \sum_{k=0}^{\infty} \{k^{n} - (k+1)^{n-1} ke^{-h/2}\} e^{-hk/2}$$

$$+ \sum_{q=0}^{n-2} 2^{q-2-n} \frac{1}{(n-2-q)!} h^{n-2-q} \sum_{k=0} k^{n-2-q} hke^{-hk/2}$$

$$- \sum_{q=0}^{n-2} 2^{q+2-n} \frac{1}{(n-2-q)!} e^{-h/2} h^{n-2-q} \sum_{k=0} (k+1)^{n-2-q} hke^{-hk/2}$$

$$= \frac{2^{1-n}h^{n}}{(n-1)!} \sum_{k=0}^{\infty} k^{n} - (k+1)^{n-1} ke^{-h/2} e^{-hk/2} + M'_{Y}(0)_{n-1}$$

$$= \frac{2^{1-n}h^{n}}{(n-1)!} \sum_{k=0}^{\infty} k^{n-1} e^{-hk/2} + M'_{Y}(0)_{n-1}$$

The second equality follows in a similar manner.

Corollary 5.9. Let $X \sim \chi^2(2)$ and Y be distributed as the truncation of X. The mean and variance of Y are given by

$$\mu_Y = he^{-h/2}/(1-e^{-h/2})$$

$$\sigma_Y^2 = h^2e^{-h/2}/(1-e^{-h/2})^2$$

Proof.

$$\mu_Y = M_Y'(0)$$

$$= \sum_{k=0}^{\infty} hke^{-hk/2} - e^{-h/2} \sum_{k=0}^{\infty} hke^{-hk/2}$$

$$= h(1-e^{-h/2}) \sum_{k=0}^{\infty} ke^{-hk/2}$$

$$= h(1-e^{-h/2})e^{-h/2}/(1-e^{-h/2})^2 \qquad \text{by lemma 5.4}$$

$$= he^{-h/2}/(1-e^{-h/2})$$

$$M_Y''(0) = \sum_{k=0}^{\infty} (hk)^2e^{-hk/2} - e^{-h/2} \sum_{k=0}^{\infty} (hk)^2e^{-hk/2}$$

$$= h^2(1-e^{-h/2}) \sum_{k=0}^{\infty} k^2e^{-hk/2}$$

$$= h^2(1-e^{-h/2})e^{-h/2}(1+e^{-h/2})/(1-e^{-h/2})^3$$

by lemma 5.4. Thus

$$\sigma_Y^2 = M_Y''(0) - \mu_Y^2$$

$$= h^2e^{-h/2}(1+e^{-h/2})/(1-e^{-h/2})^2 - h^2e^{-h}/(1-e^{-h/2})^2$$

$$= h^2e^{-h/2}/(1-e^{-h/2})^2 .$$

These results could have been obtained by observing that a chi-square variable with 2 degrees of freedom is also an exponential variable where the parameter $\theta = 2$. Theorem 4.1 would have yielded the results. If μ_X and σ_X^2 are the mean and variance of the random variable X, respectively, then Theorem 4.1 also yields

$$\mu_Y \to \mu_X \quad \text{and} \quad \sigma_Y^2 \to \sigma_X^2 \text{ as } \quad h \to 0.$$

Theorem 5.10. If $X \sim \chi^2(2n)$ and Y is distributed as the truncation of X, then

$$\mu_Y \to \mu_X \quad \text{and} \quad \sigma_Y^2 \to \sigma_X^2 \text{ as } \quad h \to 0.$$

Proof.

By Theorem 5.8

$$\mu_Y(2n) = \frac{2^{1-n}h^n}{(n-1)!}\sum_{k=0}^{\infty} k^{n-1}e^{-hk/2} + \mu_Y(2n-2).$$

Applying Theorem 5.8 repeatedly yields

$$\mu_Y(2n) = \sum_{j=0}^{n-2} \frac{2^{1-n+j} n^{n-j}}{(n-j-1)!} \sum_{k=0}^{\infty} k^{n-j-1} e^{-hk/2} + \mu_Y(2).$$

Applying lemma 5.5 yields

$$\lim_{h\to 0} \mu_Y(2n) = \lim_{h\to 0} \sum_{j=0}^{n-2} \frac{2^{1-n+j} h^{n-j}}{(n-j-1)!} \sum_{k=0}^{\infty} k^{n-j-1} e^{-hk/2} + \lim_{h\to 0} \mu_Y(2)$$

$$= 2(n-1) + 2$$

$$= 2n = \mu_X.$$

In order to show that $\sigma_Y^2 \to \sigma_X^2$ as $h \to 0$, consider σ_Y^2 in the form

$$\sigma_Y^2 = M''_{Y_n}(0) - [M'_Y(0)_n]^2.$$

Thus

$$\lim_{h\to 0} \sigma_Y^2 = \lim_{h\to 0} M''_Y(0) - 4n^2.$$

Since $\sigma_X^2 = 4n$, it must be shown that

$$\lim_{h\to 0} M''_Y(0)_n = 4n^2 + 4n.$$

By Theorem 5.8, applied n-1 times

$$M_Y''(0)_n = \sum_{j=0}^{n-2} \frac{2^{2-n+j}h^{n-j+1}}{(n-j-1)!} \sum_{k=0}^{\infty} k^{n-j}e^{-hk/2}$$

$$- h \sum_{j=0}^{n-2} \frac{2^{1-n+j}h^{n-j}}{(n-j-1)!} \sum_{k=0}^{\infty} k^{n-j-1}e^{-hk/2} + M_Y''(0)_2.$$

Applying lemma 5.4 to the first term on the right yields

$$\frac{2^{2-n+j}h^{n-j+1}}{(n-j-1)!} \sum_{k=0}^{\infty} k^{n-j}e^{-hk/2}$$

$$= \frac{2^{2-n+j}h^{n-j+1}}{(n-j-1)!} \sum_{k=1}^{n-j} \frac{(-1)^k(e^{-h/2})^k}{(1-e)^{-h/2})^{k+1}} \sum_{a=1}^{k} (-1)^a \binom{k}{a} a^{n-j}$$

$$= \frac{2^{2-n+j}}{(n-j-1)!} \sum_{k=1}^{n-j} \frac{(-1)^k h^{n-j+1}}{\sum_{j=0}^{k+1} (-1)^j \frac{k+1}{j} (e^{h/2})^{k-j}} \sum_{a=1}^{k} (-1)^a \binom{k}{a} a^{n-j}$$

$$= b(n,h).$$

By lemma 5.3

$$\lim_{h\to 0} b(n,h) = \frac{2^{2-n+j}}{(n-j-1)!} \sum_{k=n-j}^{n-j} (-1)^k 2^{n-j+1} \sum_{a=1}^{k} (-1)^a \binom{k}{a} a^{n-j}$$

$$= \frac{2^{2-n+j}}{(n-j-1)!} (-1)^{n-j} 2^{n-j+1} (-1)^{n-j} (n-j)!$$

$$= 8(n-j) \qquad \text{by lemma 5.1.}$$

Thus

$$\lim_{h \to 0} M_Y''(0)_n = \sum_{j=0}^{n-2} 8(n-j) - \lim_{h \to 0} h(2n-2) + \lim_{h \to 0} M_Y''(0)_2$$

$$= \sum_{j=0}^{n-2} 8(n-j) + 0 + 8$$

$$= \sum_{j=0}^{n-1} 8(n-j)$$

$$= 8n^2 - 4n^2 + 4n$$

$$= 4n^2 + 4n.$$

Thus $\sigma_Y^2 \to \sigma_X^2$ as $h \to 0$.

Theorem 5.11. If $X \sim \chi^2(2n)$ and Y is distributed as the truncation of X, then

$$E(X^n) > E(Y^n).$$

Proof.

$$E(X^n) - E(Y^n) = \int_0^{\infty} x^n \frac{2^{-n}}{(n-1)!} x^{n-1}e^{-x/2}dx$$

$$- \sum_{k=0} (kh)^n \int_{kh}^{kh+h} \frac{2^{-n}}{(n-1)!} x^{n-1}e^{-x/2}dx$$

$$= \sum_{k=0}^{\infty} \int_{kh}^{kh+h} (x^n - k^n h^n) \frac{2^{-n}}{(n-1)!} x^{n-1}e^{-x/2}dx$$

$$> 0$$

since the integrand is nonegative on $[kh, kh+h]$ for all values of k. Thus $E(X^n) > E(Y^n)$.

Using Theorem 5.8, Tables VII, VIII, IX and X were evaluated. In examining the tables, it is seen that the mean of Y is approximately $\frac{1}{2}$ h units less than the mean for X. The variance of Y is very close to the variance of X except when $h = 1.0$. It is consistently .08 larger than the variance of X then.

MEAN AND VARIANCE WHEN h IS 1.000

AND N DEGREES OF FREEDOM

N	MEAN	VARIANCE
2	0.1541494D 01	0.3917698D 01
4	0.3500343D 01	0.8078566D 01
6	0.5499832D 01	0.1208528D 02
8	0.7499996D 01	0.1608342D 02
10	0.9500001D 01	0.2008332D 02
12	0.1150000D 01	0.2408333D 02
14	0.1350000D 02	0.2808333D 02
16	0.1550000D 02	0.3208333D 02
18	0.1750000D 02	0.3608333D 02
20	0.1950000D 02	0.4008333D 02
22	0.2150000D 02	0.4408333D 02
24	0.2350000D 02	0.4808333D 02
26	0.2550000D 02	0.5208333D 02
28	0.2750000D 02	0.5608333D 02
30	0.2950000D 02	0.6008333D 02

TABLE VII

MEAN AND VARIANCE WHEN h IS 0.100

AND N DEGREES OF FREEDOM

N	MEAN	VARIANCE
2	0.1950417D 01	0.3999167D 01
4	0.3950000D 01	0.8000833D 01
6	0.5950000D 01	0.1200083D 02
8	0.7950000D 01	0.1600083D 02
10	0.9950000D 01	0.2000083D 20
12	0.1195000D 02	0.2400083D 02
14	0.1395000D 02	0.2800083D 02
16	0.1595000D 02	0.3200083D 02
18	0.1795000D 02	0.3600083D 02
20	0.1995000D 02	0.4000083D 02
22	0.2195000D 02	0.4400083D 02
24	0.2395000D 02	0.4800083D 02
26	0.2595000D 02	0.5200083D 02
28	0.2795000D 02	0.5600083D 02
30	0.2995000D 02	0.6000083D 02

TABLE VIII

MEAN AND VARIANCE WHEN h IS 0.010

AND N DEGREES OF FREEDOM

N	MEAN	VARIANCE
2	0.1995004D 01	0.3999992D 01
4	0.3995000D 01	0.8000008D 01
6	0.5995000D 01	0.1200001D 02
8	0.7995000D 01	0.1600001D 02
10	0.9950000D 01	0.2000001D 02
12	0.1199500D 02	0.2400001D 02
14	0.1399500D 02	0.2800001D 02
16	0.1599500D 02	0.3200001D 02
18	0.1799500D 02	0.3600001D 02
20	0.1999500D 02	0.4000001D 02
22	0.2199500D 02	0.4400001D 02
24	0.2399500D 02	0.4800001D 02
26	0.2599500D 02	0.5200001D 02
28	0.2799500D 02	0.5600001D 02
30	0.2999500D 02	0.6000001D 02

TABLE IX

MEAN AND VARIANCE WHEN h IS 0.001

AND N DEGREES OF FREEDOM

N	MEAN	VARIANCE
2	0.1999500D 01	0.4000000D 01
4	0.3999500D 01	0.8000000D 01
6	0.5999500D 01	0.1200000D 02
8	0.7999500D 01	0.1600000D 02
10	0.9999500D 02	0.2000000D 02
12	0.1199950D 02	0.2400000D 02
14	0.1399950D 02	0.2800000D 02
16	0.1599950D 02	0.3200000D 02
18	0.1799950D 02	0.3600000D 02
20	0.1999950D 02	0.4000000D 02
22	0.2199950D 02	0.4400000D 02
24	0.2399950D 02	0.4800000D 02
26	0.2599950D 02	0.5200000D 02
28	0.2799950D 02	0.5600000D 02
30	0.2999950D 02	0.6000000D 02

TABLE X

Chapter VI

THE t FREQUENCY FUNCTION

Let X have the t distribution and Y be distributed as the truncation of X. As in Chapter V, properties of the random variable Y and its frequency function will be studied.

Theorem 6.1. Let $X \sim t(p)$ and Y be distributed as the truncation of X. Then $g(y_k) = g(y_{-k})$, $k = 1,2,3,\ldots$, where g is the frequency function of Y.

Proof. By Theorem 1C of the appendix,

$$g(y_k) = \frac{[(p-1)/2]!}{\sqrt{p\pi}\ [(p-2)/2]!} \int_{kh}^{kh+h} [1+x^2/p]^{-\frac{p+1}{2}}\, dx$$

$$= \frac{[(p-1)/2]!}{\sqrt{p\pi}\ [(p-2)/2]!} \int_{-kh}^{-kh-h} [1+\mu^2/p]^{-\frac{p+1}{2}}\, (-du)$$

$$= \frac{[(p-1)/2]!}{\sqrt{p\pi}\ [(p-2)/2]!} \int_{-kh-h}^{-kh} [1+\mu^2/p]^{-\frac{p+1}{2}}\, du$$

$$= g(y_{-k}).$$

In the next theorem, the following identity will be used

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\ .$$

Theorem 6.2. Let $X \sim t(p)$ and Y be distributed as the truncation of X. If $p = 2n+1$, $n = 0,1,2,\ldots$, then the density of Y is given by

$$g(0) = \frac{1}{\sqrt{(2n+1)\pi}} \sum_{i=0}^{n-1} \frac{(n-i-1)!}{[(2n-2i-1)/2]!} \frac{h}{[1+h^2/(2n+1)]^{n-i}}$$

$$+ \frac{2}{\pi} \tan^{-1} \frac{h}{\overline{2n+1}}$$

$$g(y_k) = \frac{1}{\pi} \tan^{-1} \frac{h\sqrt{2n+1}}{2n+1+kh(kh+h)}$$

$$+ \frac{1}{2\sqrt{(2n+1)\pi}} \sum_{i=0}^{n-1} \frac{(n-i-1)!}{[(2n-2i-1)/2]!} G(h) \quad k = 1,2,3,\ldots$$

where

$$G(h) = \frac{(k+1)h}{[1+(kh+h)^2/(2n+1)]^{n-i}} - \frac{kh}{[1+(kh)^2/(2n+1)]^{n-i}} .$$

If $p = 2n$, $n = 1,2,3,\ldots$, then the density of Y is given by

$$g(0) = \frac{1}{\sqrt{2n\pi}} \sum_{i=0}^{n-1} \frac{[(2n-2i-3)/2]!}{(n-i-1)!} \frac{1}{[1+h^2/2n]^{\frac{2n-2i-1}{2}}}$$

$$g(y_k) = \frac{1}{2\sqrt{2n\pi}} \sum_{i=0}^{n-1} \frac{[(2n-2i-3)/2]!}{(n-i-1)!} H(h) \qquad k = 1,2,3,\ldots$$

where

$$H(h) = \frac{(k+1)h}{[1+(kh+h)^2/2n]^{\frac{2n-2i-1}{2}}} - \frac{kh}{[1+(kh)^2/2n]^{\frac{2n-2i-1}{2}}}$$

Proof.

Consider $p = 2n+1$, $n = 0,1,2,\ldots$.

$$g(0) = \frac{n!}{\sqrt{(2n+1)\pi}\ [(2n-1)/2]!} \int_{-h}^{h} \frac{dx}{[1+x^2/(2n+1)]^{n+1}}$$

$$= \frac{2n!}{\sqrt{(2n+1)\pi}\ [(2n-1)/2]!} \int_{0}^{h} \frac{dx}{[1+x^2/(2n+1)]^{n+1}}$$

$$= \frac{2n!}{\sqrt{(2n+1)\pi}\ [(2n-1)/2]!} \left\{ \frac{[(2n-1)/2]!\sqrt{2n+1}}{\sqrt{\pi}\ \ n!} \tan^{-1} \frac{x}{\sqrt{2n+1}} \right.$$

$$\left. + \frac{1}{2n} \sum_{i=0}^{n-1} \frac{[(2n-1)/2]!(n-i-1)!}{[(2n-2i-1)/2]!(n-1)!} \frac{x}{[1+x^2/(2n+1)]^{n-i}} \right\} \Bigg|_0^h$$

$$= \frac{2}{\pi} \tan^{-1} \frac{h}{\sqrt{2n+1}} + \frac{1}{\sqrt{(2n+1)\pi}} \sum_{i=0}^{n-1} \frac{(n-i-1)!}{[(2n-2i-1)/2]!} \frac{h}{[1+h^2/(2n+1)]^{n-i}}.$$

$$g(y_k) = \frac{n!}{\sqrt{(2n+1\pi}\ [(2n-1)/2]!} \int_{kh}^{kh+h} \frac{dx}{[1+x^2/(2n+1)]^{n+1}}$$

$$= \frac{n!}{\sqrt{(2n+1)\pi}\ [(2n-1)/2]!} \left\{ \frac{[(2n-1)/2]!\sqrt{2n+1}}{\sqrt{\pi}\ n!} \tan^{-1} \frac{x}{\sqrt{2n+1}} \right.$$

$$\left. + \frac{1}{2n} \sum_{i=0}^{n-1} \frac{[(2n-1)/2]!(n-i-1)!}{[(2n-2i-1)/2]!(n-1)!} \frac{x}{[1+x^2/(2n+1)]^{n-i}} \right\} \Bigg|_{kh}^{kh+h}$$

The result is immediate if the previously mentioned identity is applied after evaluating at the upper and lower limits.

If $p = 2n$, $n = 1,2,3,\ldots$, the result follows in a similar manner.

Corollary 6.3. $\sum_{k=0}^{\infty} \tan^{-1} \frac{h}{1+h^2k(k+1)} = \frac{\pi}{2}$ where $h = 10^{-j}$, $j = 0,$

Proof.

Suppose $X \sim t(1)$. If Y is distributed as the truncation of X, then the density of Y is given in Theorem 6.2 to be

$$g(0) = \frac{2}{\pi} \tan^{-1} h$$

$$g(y_k) = g(y_{-k}) = \frac{1}{\pi} \tan^{-1} \frac{h}{1+h^2k(k+1)} \qquad k = 1,2,3,\ldots$$

Since g is a density, then

$$g(0) + \sum_{k=1}^{\infty} g(y_k) + \sum_{k=1}^{\infty} g(y_{-k}) = 1 .$$

But

$$g(0) + \sum_{k=1}^{\infty} g(y_k) + \sum_{k=1}^{\infty} g(y_{-k}) = \sum_{k=0}^{\infty} \frac{2}{\pi} \tan^{-1} \frac{h}{1+h^2k(k+1)}$$

Thus

$$\sum_{k=0}^{\infty} \tan^{-1} \frac{h}{1+h^2k(k+1)} = \frac{\pi}{2}$$

When h=1 in the preceding corollary, the sum of the series is given by Konrad Knopp in his book "Theory and Application of Infinite Series", page 267, problem 102b.

Suppose $X \sim t(p)$. Then it can easily be seen that $E(X) = 0$ if $p \geq 2$ and $E(X^2) = p/(p-2)$ if $p \geq 3$. When $p = 1$, the t density does not possess a mean or variance. When $p = 2$, the t density does not possess a variance. The following theorem considers the same problem for the random variable Y where Y is distributed as the truncation of X.

Theorem 6.4. If $X \sim t(p)$ and Y is distributed as the truncation of X, then $E(Y)$ does not exist for $p = 1$ and $E(Y^2)$ does not exist for $p = 1,2$.

Proof.

Consider $p = 1$.

$$E(Y) = \sum_{k=0}^{\infty} kh \frac{2}{\pi} \tan^{-1} \frac{h}{1+kh(kh+h)} - \sum_{k=0}^{\infty} kh \frac{2}{\pi} \tan^{-1} \frac{h}{1+kh(kh+h)} .$$

It appears that $E(Y)$ should be 0 when $p = 1$, but the series above turns out to be divergent. Thus it will be necessary to show that the series does indeed diverge. Since the series is positive, a comparison test will be used. It is well-known that

$$\tan^{-1} x > \frac{x}{x^2+1} \quad \text{for every } x > 0.$$

Thus

$$\tan^{-1} \frac{h}{1+kh(kh+h)} > \frac{h[1+kh(kh+h)]}{h^2+[1+kh(kh+h)]^2}$$

and

$$kh \tan^{-1} \frac{h}{1+kh(kh+h)} > \frac{kh^2[1+kh(kh+h)]}{h^2+[1+kh(kh+h)]^2}$$

$$= \frac{1}{k} \left\{ \frac{k^2h^2[1+kh(kh+h)]}{h^2+[1+kh(kh+h)]^2} \right\}$$

$$= \frac{1}{k} b_k$$

where $b_k = \dfrac{k^2h^2[1+kh(kh+h)]}{h^2+[1+kh(kh+h)]^2}$

Observe $\lim\limits_{k\to\infty} b_k = 1.$

Thus $\sum\limits_{k=1}^{\infty} \frac{1}{k} b_k$ diverges since $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges. Since the given series is term by term larger than $\sum_{k=1}^{\infty} \frac{1}{k} b_k$, then the given series diverges

$$E(Y^2) = 2 \sum_{k=0}^{\infty} (kh)^2 \frac{2}{\pi} \tan^{-1} \frac{h}{1+kh(kh+h)}$$

$$= \frac{4h^2}{\pi} \sum_{k=1}^{\infty} k^2 \tan^{-1} \frac{h}{1+kh(kh+h)} .$$

But

$$k^2 \tan^{-1} \frac{h}{1+kh(kh+h)} > \frac{hk^2[1+kh(kh+h)]}{h^2+[1+kh(kh+h)]^2}$$

If $c_k = \frac{hk^2[1+kh(kh+h)]}{h^2+[1+kh(kh+h)]^2}$, then $\lim_{k\to\infty} c_k = \frac{1}{h} \neq 0$.

Thus $\sum_{k=1}^{\infty} c_k$ diverges, and so does $\sum_{k=1}^{\infty} k^2 \tan^{-1} \frac{h}{1+kh(kh+h)}$

Thus when $p = 1$, $E(Y)$ and $E(Y^2)$ do not exist.

Consider $p = 2$.

$$E(Y^2) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (kh)^2 \sqrt{\pi} \left\{ \frac{(k+1)h}{[1+(kh+h)^2/2]^{1/2}} - \frac{kh}{[1+(kh)^2/2]^{1/2}} \right\}$$

$$= \frac{h^3}{\sqrt{2}} \sum_{k=1}^{\infty} k^2 \left\{ \frac{[k+1][1+(kh)^2/2]^{1/2} - k[1+(kh+h)^2/2]^{1/2}}{[1+(kh+h)^2/2]^{1/2}[1+(kh)^2/2]^{1/2}} \right\}$$

$$= \frac{h^3}{\sqrt{2}} \sum_{k=1}^{\infty} a_k b_k$$

where $a_k = k^2/\{[1+(kh+h)^2/2]^{1/2}[1+(kh)^2/2]^{1/2}\}$

$$b_k = [k+1][1+(kh)^2/2]^{1/2} - k[1+(kh+h)^2/2]^{1/2}.$$

Now $\lim_{k\to\infty} a_k = \frac{2}{h^2} \neq 0$. $\sum_{k=1}^{\infty} b_k$ diverges since $\sum_{k=1}^{\infty} b_k$ can be written

$\sum_{k=1}^{\infty} b_k = 2\sum_{k=1}^{\infty} [1+(kh)^2/2]^{1/2}$ which diverges. Thus $E(Y^2)$ is represented by a divergent series, i.e., $E(Y^2)$ does not exist when $p = 2$.

Theorem 6.5. Let $X \sim t(p)$ and Y be distributed as the truncation of X. If $p \geq 2$, then $E(Y) = 0$. If $p \geq 3$, then $E(Y^2)$ exists, and $E(X^2) > E(Y^2)$.

Proof.

$$E(Y) = \sum_{k=0}^{\infty} kh\ g(y_k) + \sum_{k=0}^{\infty} (-kh)\ g(y_{-k})$$

$$= \sum_{k=1}^{\infty} kh\ g(y_k) - \sum_{k=1}^{\infty} kh\ g(y_k)$$

Thus $E(Y) = 0$ if $\sum_{k=1}^{\infty} kh\ g(y_k)$ is a convergent series.

$$\sum_{k=1}^{\infty} kh\ g(y_k) = \sum_{k=1}^{\infty} kh \frac{[(p-1)/2]!}{\sqrt{p\pi}\ [(p-2)/2]!} \int_{kh}^{kh+h} \frac{dx}{[1+x^2/p]^{\frac{p+1}{2}}}$$

$$= c_p \sum_{k=1}^{\infty} a_k$$

where $c_p \doteq \dfrac{h[(p-1)/2]!}{\sqrt{p\pi}\;[(p-2)/2]!}$

and $$a_k = k \int_{kh}^{kh+h} \frac{dx}{[1+x^2/p]^{\frac{p+1}{2}}}$$

Observe $$a_k \le \frac{hk}{[1+(kh)2/p]^{\frac{p+1}{2}}}$$

$$< \frac{hk}{[(kh)^2/p]^{\frac{p+1}{2}}}$$

$$= h \left[\frac{p}{h^2}\right]^{\frac{p+1}{2}} \Big/ k^p$$

$$= h_p b_k$$

where $h_p = h[p/h^2]^{\frac{p+1}{2}}$

and $b_k = \dfrac{1}{k^p}$.

Since $p \ge 2$, then $\sum_{k=1}^{\infty} b_k$ converges. Thus $h_p \sum_{k=1}^{\infty} b_k$ converges, and this then implies that $\sum_{k=1}^{\infty} a_k$ converges. Thus $E(Y) = 0$ if $p \ge 2$.

Now consider $E(Y^2)$ when $p \ge 3$.

$$E(Y^2) = \sum_{k=0}^{\infty} (kh)^2\, g(y_k) + \sum_{k=0}^{\infty} (-kh)^2\, g(y_{-k})$$

$$= 2 \sum_{k=1}^{\infty} (kh)^2\, g(y_k).$$

Thus $E(Y^2)$ will exists if the above series converges.

$$\sum_{k=1}^{\infty} (kh)^2 \, g(y_k) = \sum_{k=1}^{\infty} (kh)^2 \frac{[(p-1)/2]!}{\sqrt{p\pi}\,[(p-2)/2]!} \int_{kh}^{kh+h} \frac{dx}{[1+x^2/p]^{\frac{p+1}{2}}}$$

$$= d_p \sum_{k=1}^{\infty} a_k$$

where $d_p = \dfrac{h^2[(p-1)/2]!}{\sqrt{p\pi}\,[(p-2)/2]!}$

and $\quad a_k = k^2 \displaystyle\int_{kh}^{kh+h} \frac{dx}{[1+x^2/p]^{\frac{p+1}{2}}}$

Observe $a_k \le \dfrac{k^2 h}{[1+(kh)^2/p]^{\frac{p+1}{2}}}$

$$\le \frac{k^2 h}{[(kh)^2/p]^{\frac{p+1}{2}}}$$

$$= h\left[\frac{p}{h^2}\right]^{\frac{p+1}{2}} \Big/ k^{p-1}$$

$$= h_p b_k$$

where $h_p = h\left[\dfrac{p}{h^2}\right]^{\frac{p+1}{2}}$

and $\quad b_k = \dfrac{1}{k^{p-1}}$.

Since $p \geq 3$, then $\sum_{k=1}^{\infty} b_k$ converges, and this implies $h_p \sum_{k=1}^{\infty} b_k$ converges. Thus $\sum_{k=1}^{\infty} a_k$ converges. Thus $E(Y^2)$ exists if $p \geq 3$.

The lemma will be proved if it can be shown that $E(X^2) > E(Y^2)$ when $p \geq 3$.

$$E(X^2) = \frac{2[(p-1)/2]!}{\sqrt{p\pi}\ [(p-2)/2]!} \int_0^{\infty} \frac{x^2\ dx}{[1+x^2/p]^{\frac{p+1}{2}}}$$

$$= \frac{2[(p-1)/2]!}{\sqrt{p\pi}\ [(p-2)/2]!} \sum_{k=0}^{\infty} \int_{kh}^{kh+h} \frac{x^2\ dx}{[1+x^2/p]^{\frac{p+1}{2}}}$$

$$E(Y^2) = \frac{2[(p-1)/2]!}{\sqrt{p\pi}\ [(p-2)/2]!} \sum_{k=0}^{\infty} (kh)^2 \int_{kh}^{kh+h} \frac{dx}{[1+x^2/p]^{\frac{p+1}{2}}}$$

If $p \geq 3$, both of the above series converge. Thus

$$E(X^2) - E(Y^2) = \frac{2[(p-1)/2]!}{\sqrt{p\pi}\ [(p-2)/2]!} \sum_{k=0}^{\infty} \int_{kh}^{kh+h} (x^2-k^2h^2) \frac{dx}{[1+x^2/p]^{\frac{p+1}{2}}}$$

$$> 0$$

since the integrand is positive on $(kh,kh+h)$.

Thus $E(X^2) > E(Y^2)$ when $p \geq 3$.

Theorem 6.6. Let $X \sim N(0,1)$ and Y be distributed as the truncation of X. Let $X' \sim t(p)$ and Y' is distributed as the truncation of X'. The distribution of Y' approaches the distribution of Y as the number of degrees of freedom becomes infinite.

Proof.

Let g and g' be the desnities of Y and Y', respectively. Then

$$g(y_k) = g(y_{-k}) = \int_{kh}^{kh+h} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \qquad k = 1,2,3,\ldots ,$$

$$g'(y_k) = g'(y_{-k}) = \int_{kh}^{kh+h} \frac{[(p-1)/2]!}{\sqrt{p\pi}\,[(p-2)/2]!} \frac{dx}{[1+x^2/p]^{\frac{p+1}{2}}} \qquad k = 1,2,3,\ldots ,$$

$$g(0) = \int_{-h}^{h} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx, \qquad \text{and}$$

$$g'(0) = \int_{-h}^{h} \frac{[(p-1)/2]!}{\sqrt{p\pi}\,[(p-2)/2]!} \frac{dx}{[1+x^2/p]^{\frac{p+1}{2}}} .$$

$$\lim_{p\to\infty} g'(y_k) = \lim_{p\to\infty} \int_{kh}^{kh+h} \frac{[(p-1)/2]!}{\sqrt{p\pi}\,[(p-2)/2]!} \frac{dx}{[1+x^2/p]^{\frac{p+1}{2}}}$$

$$= \int_{kh}^{kh+h} \lim_{p\to\infty} \left\{ \frac{[(p-1)/2]!}{\sqrt{p\pi}\,[(p-2)/2]!} \frac{1}{[1+x^2/p]^{\frac{p+1}{2}}} \right\} dx$$

$$= \int_{kh}^{kh+h} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = g(y_k).$$

The last equality is obtained since the distribution of X' approaches the distribution of X as the number of degrees of freedom becomes infinite. Taking the limit inside the integral is permissible since the integrand is a continuous function of the variables x and p, i.e., the integral is a continuous function of p, and the integral is bounded as a function of p.

Theorem 6.7. Let $X \sim t(p)$ and Y be distributed as the truncation of X. If $p \geq 3$, then $\sigma_Y^2 \to \sigma_X^2$ as $h \to 0$.

Proof.

By Theorem 6.5, $E(X^2) \geq E(Y^2)$ or $E(X^2) - E(Y^2) \geq 0$.

Since $E(X^2)$ and $E(Y^2)$ are represented by convergent series if $p \geq 3$, then $E(X^2) - E(Y^2)$ is a convergent series, too.

$$E(X^2) - E(Y^2) = \sum_{k=0}^{\infty} \frac{2[(p-1)/2]!}{\sqrt{p\pi}\ [(p-2)/2]!} \int_{kh}^{kh+h} \frac{x^2-k^2h^2}{[1+x^2/p]^{\frac{p+1}{2}}}\, dx$$

$$= a_p \sum_{k=0}^{\infty} b_k$$

where $a_p = \dfrac{2[(p-1)/2]!}{\sqrt{p\pi}\ [(p-2)/2]!}$

and $b_k = \displaystyle\int_{kh}^{kh+h} \frac{x^2-k^2h^2}{[1+x^2/p]^{\frac{p+1}{2}}}\, dx.$

Now

$$b_k = \int_{kh}^{kh+h} (x-kh)(x+kh) \frac{dx}{[1+x^2/p]^{\frac{p+1}{2}}}$$

$$\leq h \int_{kh}^{kh+h} \frac{2x\ dx}{[1+x^2/p]^{\frac{p+1}{2}}}$$

$$= -\frac{2hp}{p-1} \frac{1}{[1+x^2/p]^{\frac{p-1}{2}}} \Bigg|_{kh}^{kh+h}$$

$$= \frac{2hp}{p-1} \left\{ \frac{1}{[1+(kh)^2/p]^{\frac{p-1}{2}}} - \frac{1}{[1+(kh+h)^2/p]^{\frac{p-1}{2}}} \right\}$$

$$= c_k$$

The n^{th} partial sum of the series $\sum_{k=0}^{\infty} c_k$ is

$$S_n = \frac{2hp}{p-1} \left\{ 1 - \frac{1}{[1+(nh+h)^2/p]^{\frac{p-1}{2}}} \right\}$$

and $\lim_{n \to \beta} S_n = \frac{2hp}{p-1}$ since $p \geq 3$.

Thus $\sum_{k=0}^{\infty} b_k \leq \frac{2hp}{p-1}$, and

$$0 \leq E(X^2) - E(Y^2) \leq \frac{2hp}{p-1} a_p$$

$$= 4h \frac{p}{p-1} \frac{[(p-1)/2]!}{\sqrt{p\pi}\ [(p-2)/2]!}$$

Thus $E(X^2) - E(Y^2) \to 0$ as $h \to 0$, or $E(Y^2) \to E(X^2)$ as $h \to 0$, or $\sigma_Y^2 \to \sigma_X^2$ as $h \to 0$, since $E(X) = E(Y) = 0$ if $p \geq 3$.

Chapter VII

THE F FREQUENCY FUNCTION

Let X have the F distribution and Y be distributed as the truncation of X. As in the preceding two chapters, properties of the random variable Y and its frequency function will be studied.

Theorem 7.1. Let $X \sim F(2m,2n)$ and Y be distributed as the truncation of X. Then the density of Y is given by

$$g(y) = \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!\,(n-1)!} (kh)^{m-p} \left(1 + \frac{mkh}{n}\right)^{-m-n+p}$$

$$- \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!\,(n-1)!} (kh+h)^{m-p} \left(1 + \frac{m(k+1)h}{n}\right)^{-m-n+p}$$

where $y = kh$, $k = 0,1,2,\ldots$.

Proof.

Since $Y = kh$ when $kh \le X < kh+h$, then by Theorem 1D of the appendix

$$P[Y = kh] = P[kh \le X < kh+h]$$

$$\int_{kh}^{kh+h} \frac{(m+n-1)!}{(m-1)!\,(n-1)!} \left(\frac{m}{n}\right)^{m} \frac{F^{m-1}}{\left(1 + \frac{mF}{n}\right)^{m+n-p}}\, dF$$

$$\sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+1-p-1)!}{(m-p)!\,(n-1)!} \frac{F^{m-p}}{\left(1 + \frac{mF}{n}\right)^{m+n-p}} \Bigg|_{kh}^{kh+h}$$

$$= \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-1-p)!}{(m-p)!(n-1)!} \frac{(kh)^{m-p}}{\left(1 + \frac{mkh}{n}\right)^{m+n-p}}$$

$$- \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \frac{(kh+h)^{m-p}}{\left[1 + \frac{m(k+1)h}{n}\right]^{m+n-p}}.$$

Theorem 7.2. Let $X \sim F(2m,2n)$ and Y be distributed as the truncation of X. Then the moment generating function of Y is given by

$$M_Y(t) = \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} (1 - e^{-ht}) \sum_{k=0}^{\infty} e^{kht} (kh)^{m-p} \left(1+\frac{mkh}{n}\right)^{-m-n+p}$$

Proof.

$$M_Y(t) = E(e^{tY})$$

$$= \sum_{k=0}^{\infty} e^{kht} \cdot g(y_k)$$

$$= \sum_{k=0}^{\infty} e^{kht} \left\{ \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} (kh)^{m-p}\ 1+\frac{mkh}{n}\ ^{-m-n+p} \right.$$

$$\left. - \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} (kh+h)^{m-p} \left(1+\frac{m(k+1)h}{n}\right)^{-m-n+p} \right\}$$

$$= \sum_{p=1}^{m} \ \frac{m}{n}\ ^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \left\{ \sum_{k=0}^{\infty} e^{kht} (kh)^{m-p} \left(1+\frac{mkh}{n}\right)^{-m-n+p} \right.$$

$$- \sum_{k=0} e^{kht} (kh+h)^{m-p} \left(1+\frac{m(k+1)h}{n}\right)^{-m-n+p} \Bigg\}$$

$$\sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} (1 - e^{-ht}) \sum_{k=0} e^{kht} (kh)^{m-p}$$

$$\left(1+\frac{mkh}{n}\right)^{-m-n+p}$$

Theorem 7.3. Let $X \sim F(2m,2n)$ and Y be distributed as the truncation of X. Then

$$M_Y'(0) = \left(\frac{n}{m}\right)^{n} h^{1-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p}$$

and

$$M_Y''(0) = 2\left(\frac{n}{m}\right)^{n} h^{2-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p}$$

$$- \left(\frac{n}{m}\right)^{n} h^{2-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p}$$

Proof.

$$M_Y'(0) = \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \left\{ \sum_{k=0}^{\infty} (kh)^{m-p+1} \left(1+\frac{mkh}{n}\right)^{-m-n+p} \right.$$

$$- \sum_{k=0}^{\infty} kh(kh+h)^{m-p} \left(1+\frac{m(k+1)h}{n}\right)^{-m-n+p} \Bigg\}$$

$$= \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \Bigg\{ \sum_{k=0}^{\infty} (kh)^{m-p+1} \left(1+\frac{mkh}{n}\right)^{-m-n+p}$$

$$- \sum_{k=1}^{\infty} (kh-h)(kh)^{m-p} \left(1+\frac{mkh}{n}\right)^{-m-n+p} \Bigg\}$$

$$= \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \Bigg\{ h \sum_{k=1}^{\infty} (kh)^{m-p} \left(1+\frac{mkh}{n}\right)^{-m-n+p} \Bigg\}$$

$$= \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{-n} h^{1-n} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p}$$

$$= \left(\frac{n}{m}\right)^{n} h^{1-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p}$$

$$M_Y''(0) = \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \Bigg\{ \sum_{k=0}^{\infty} (kh)^{m-p+2} \left(1+\frac{mkh}{n}\right)^{-m-n+p}$$

$$- \sum_{k=0}^{\infty} (kh)^{2}(kh+h)^{m-p} \left(1+\frac{m(k+1)h}{n}\right)^{-m-n+p} \Bigg\}$$

$$= \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \Bigg\{ \sum_{k=0}^{\infty} (kh)^{m-p+2} \left(1+\frac{mkh}{n}\right)^{-m-n+p}$$

$$- \sum_{k=1}^{\infty} (kh-h)^{2}(kh)^{m-p} \left(1+\frac{mkh}{n}\right)^{-m-n+p}$$

$$= \sum_{p=1}^{m} \left(\frac{m}{n}\right)^{m-p} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \left\{ 2h \sum_{k=1} (kh)^{m-p-1} \left(1+\frac{mkh}{n}\right)^{-m-n+p} \right.$$

$$\left. - h^2 \sum_{k=1} (kh)^{m-p} \left(1+\frac{mkh}{n}\right)^{-m-n+p} \right\}$$

$$= 2 \left(\frac{n}{m}\right)^{n} h^{2-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p+1} \left(k+\frac{n}{mh}\right)^{-m-n+p}$$

$$- \left(\frac{n}{m}\right)^{n} h^{2-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p}$$

If $X \sim F(n_1,n_2)$, it is well-known that $E(X) = n_2/(n_2-2)$ where $n_2 > 2$, and $\sigma_X^2 = 2n_2^2\,(n_1+n_2-2)/(n_1)(n_2-2)^2(n_2-4)$ where $n_2 > 4$. The following theorem considers the existence of $E(Y)$ and σ_Y^2 where Y is distributed as the truncation of X, and X has an F distribution.

Theorem 7.4. Let $X \sim F(2m,2n)$ and Y be distributed as the truncation of X. Then $E(Y)$ exists when $n > 1$, and σ_Y^2 exists when $n > 2$.

Proof.

By Theorem 7.3,

$$M_Y'(0) = E(Y)$$

$$= \left(\frac{n}{m}\right)^{n} h^{1-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p}$$

Thus $E(Y)$ will exists if the series

$$\sum_{k=1}^{\infty} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p}$$

converges for all values of p. The series may be written in the form

$$\sum_{k=1}^{\infty} c_k a_k$$

where

$$c_k = \frac{k^{m-p}}{(k+n/mh)^{m-p}}$$

and

$$a_k = \frac{1}{(k+n/mh)^n} \quad .$$

Now

$$\lim_{k\to\infty} c_k = 1, \quad \text{and the series} \sum_{k=1}^{\infty} a_k$$

converges for $n > 1$. Thus $E(Y)$ exists for $n > 1$. In order to show that σ_Y^2 exists when $n > 2$, it must be shown that $M_Y''(0)$ exists for $n > 2$. By Theorem 7.3,

$$M_Y''(0) = 2\left(\frac{n}{m}\right)^n h^{2-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p+1}\left(k+\frac{n}{mh}\right)^{-m-n+p}$$

$$- \left(\frac{n}{m}\right)^n h^{2-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p}\left(k+\frac{n}{mh}\right)^{-m-n+p}$$

$M_Y''(0)$ will exist if the two infinite series converge. The second infinite series converges for $n > 1$ as previously shown. Now

$$\sum_{k=1}^{\infty} k^{m-p+1} \left(k+\frac{n}{mh}\right)^{-m-n+p} = \sum_{k=1}^{\infty} c_k a_k$$

where

$$c_k = k^{m-p+1} \Big/ \left(k+\frac{n}{mh}\right)^{m-p+1}$$

and

$$a_k = 1 \Big/ \left(k+\frac{n}{mh}\right)^{n-1}$$

Since $\lim_{k\to\infty} c_k = 1$, and $\sum_{k=1}^{\infty} a_k$ converges for $n > 2$, then $M_Y''(0)$ exists for $n > 2$. Since $M_Y'(0)$ and $M_Y''(0)$ exists when $n > 2$, then σ_Y^2 will exist when $n > 2$.

Lemma 7.5. If $x > 0$, then

$$\sum_{i=0}^{n} \binom{n}{i}(-1)^{i} \frac{1}{i+x} = x^{-1} \Big/ \binom{x+n}{n}$$

where n is any positive integer.

Proof.

The proof will be by induction on n. Let $n = 1$. Then

$$\sum_{i=0}^{1} \binom{1}{i}(-1)^{i} \frac{1}{x+i} = \frac{1}{x} - \frac{1}{x+1}$$

$$= \frac{1}{x(x+1)}$$

$$= x^{-1} / \binom{x+1}{1}$$

Assume the equation is true for n = j, i.e.,

$$\sum_{i=0}^{j} \binom{j}{i} \left(-1\right)^{i} \frac{1}{x+i} = \frac{1}{x} / \binom{x+j}{j} .$$

Now

$$\sum_{i=0}^{j+1} \binom{j+1}{i} \left(-1\right)^{i} \frac{1}{x+i}$$

$$= \frac{1}{x} + \sum_{i=1}^{j} \left[\binom{j}{i} + \binom{j}{i-1}\right] (-1)^{i} \frac{1}{x+i} + (-1)^{j+1} \frac{1}{x+j+1}$$

$$= \frac{1}{x} + \sum_{i=1}^{j} \binom{j}{i} (-1)^{i} \frac{1}{x+i} + \sum_{i=1}^{j} \binom{j}{i-1} (-1)^{i} \frac{1}{x+i} + (-1)^{j+1} \frac{1}{x+j+1}$$

$$= \sum_{i=0}^{j} \binom{j}{i} (-1)^{i} \frac{1}{x+i} + \sum_{i=0}^{j-1} \binom{j}{i} (-1)^{i+1} \frac{1}{x+i+1} + (-1)^{j+1} \frac{1}{x+j+1}$$

$$= \sum_{i=0}^{j} \binom{j}{i} (-1)^{i} \frac{1}{x+i} - \sum_{i=0}^{j} \binom{j}{i} (-1)^{i} \frac{1}{x+i+1}$$

$$= x^{-1} / \binom{x+j}{j} - (x+1)^{-1} / \binom{x+1+j}{j}$$

$$= \frac{j!x!}{x(x+j)!} - \frac{j!(x+1)!}{(x+1)(x+1+j)!}$$

$$= \frac{j!x!(x+1+j-x)}{x(x+1+j)!}$$

$$= \frac{1}{x} \frac{(j+1)!x!}{(x+1+j)!}$$

$$= x^{-1} / \binom{x+1+j}{j+1}.$$

Since the equation is true for n = j+1 when it is true for n = j, then the equation is true for all natural numbers n by mathematical induction.

The following lemma will be needed for the theorem following it.

Lemma 7.6. If $p > 1$, then

$$\frac{1}{(p-1)(n+1)^{p-1}} \leq \sum_{k=1}^{\infty} \frac{1}{k^p} - \sum_{k=1}^{n} \frac{1}{k^p} \leq \frac{1}{(p-1)n^{p-1}}$$

Proof.

The inequality follows immediately from the inequality

$$\int_{n+1}^{\infty} \frac{dx}{x^p} \leq \sum_{k=n+1}^{\infty} \frac{1}{k^p} \leq \int_{n}^{\infty} \frac{dx}{x^p}$$

Theorem 7.7. Let $X \sim F(2m,2n)$ and Y be distributed as the truncation of X. If $n = jm$ where j is a positive integer, then $\mu_Y \to \mu_X$ (for $n > 1$) and $\sigma_Y^2 \to \sigma_X^2$ (for $n > 2$) as $h \to 0$.

Proof.

The restriction $n = jm$ is necessary in order to use lemma 7.6.

It will be shown that $M_Y'(0) \to \mu_X$ as $h \to 0$.

By Theorem 7.3

$$M_Y'(0) = \left(\frac{n}{m}\right)^n h^{1-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p}$$

Observe

$$k^{m-p} = \left(k+\frac{n}{mh} - \frac{n}{mh}\right)^{m-p}$$

$$= \sum_{i=0}^{m-p} \binom{m-p}{i} \left(k+\frac{n}{mh}\right)^{m-p-i} \left(-\frac{n}{mh}\right)^{i}$$

Thus

$$\sum_{k=1}^{\infty} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p} = \sum_{k=1}^{\infty} \left\{ \sum_{i=0}^{m-p} \binom{m-p}{i} \left(-\frac{n}{mh}\right)^{i} \Big/ \left(k+\frac{n}{mh}\right)^{n+i} \right\}$$

$$= \sum_{i=0}^{m-p} \binom{m-p}{i} \left(-\frac{n}{mh}\right)^{i} \sum_{k=1}^{\infty} \left(k+\frac{n}{mh}\right)^{-n-i}$$

$$= \sum_{i=0}^{m-p} \binom{m-p}{i} \left(-\frac{n}{mh}\right)^{i} \left\{ \sum_{k=1}^{\infty} \frac{1}{k^{n+i}} - \sum_{k=1}^{n/mh} \frac{1}{k^{n+i}} \right\}$$

Since n > 1, Lemma 7.6 can be used on the terms within the braces.

Thus

$$(n+i-1)^{-1}\left(\frac{n}{mh}+1\right)^{-n-i+1} \le \sum_{k=1}^{\infty}\frac{1}{k^{n+i}} - \sum_{k=1}^{n/mh}\frac{1}{k^{n+i}}$$

$$\le (n+i-1)^{-1}\left(\frac{n}{mh}\right)^{-n-i+1}$$

Thus

$$\left(\frac{n}{m}\right)^{n} h^{1-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!\,(n-1)!} \sum_{i=0}^{m-p} \binom{m-p}{i}\left(-\frac{n}{mh}\right)^{i} (n+i-1)^{-1}\left(\frac{n}{mh}+1\right)^{-n-i+1}$$

$$\le M_Y'(0) \le$$

$$\left(\frac{n}{m}\right)^{n} h^{1-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!\,(n-1)!} \sum_{i=0}^{m-p} \binom{m-p}{i}\left(-\frac{n}{mp}\right)^{i} (n+i-1)^{-1}\left(\frac{n}{mh}\right)^{-n-i+1}$$

or

$$\frac{n}{m} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!\,(n-1)!} \sum_{i=0}^{m-p} \binom{m-p}{i} (-1)^{i} (n+i-1)^{-1} \left(\frac{n}{mh}\right)^{n+i-1}\left(\frac{n}{mh}+1\right)^{-n-i+1}$$

$$\le M_Y'(0) \le$$

$$\frac{n}{m} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{i=0}^{m-p} \binom{m-p}{i} (-1)^{i} (n+i-1)^{-1} \left(\frac{n}{mh}\right)^{n+i-1} \left(\frac{n}{mh}\right)^{-n-i+1}$$

or

$$a(m,n) \leq M_Y'(0) \leq b(m,n)$$

where

$$a(m,n) = \frac{n}{m} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{i=0}^{m-p} \binom{m-p}{i} (-1)^{i} (n+i-1)^{-1} \left(\frac{mh}{n}+1\right)^{-n-i+1}$$

$$b(m,n) = \frac{n}{m} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{i=0}^{m-p} \binom{m-p}{i} (-1)^{i} (n+i-1)^{-1}.$$

Observe

$$\lim_{h \to 0} a(m,n) = b(m,n)$$

Thus $M_Y'(0) \to b(m,n)$ as $h \to 0$.

Thus it must be shown that $b(m,n) = \mu_X$, that is

$$b(m,n) = \frac{n}{n-1} = \frac{2n}{2n-2}$$

$$b(m,n) = \frac{n}{m} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{i=0}^{m-p} \binom{m-p}{i} (-1)^{i} (i+n-1)^{-1}$$

$$= \frac{n}{m} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!\,(n-1)!} \quad \frac{1}{n-1} \quad \frac{(m-p)!\,(n-1)!}{(m-p+n-1)!}$$

by Lemma 7.5

$$= \frac{n}{m(n-1)} \sum_{p=1}^{m}$$

$$= \frac{n}{n-1}$$

Lemma 7.5 can be applied since $n > 1$ in this case.

In order to show that $\sigma_Y^2 \to \sigma_X^2$ (for $n > 2$) as $h \to 0$, it will be shown that

$$M_Y''(0) - [M_Y'(0)]^2 \to \sigma_X^2 \quad \text{as} \quad h \to 0.$$

If it can be shown that $M_Y''(0) \to \frac{n^2(m+1)}{m(n-1)(n-2)}$ as $h \to 0$, then the proof will be complete since

$$\frac{n^2(m+1)}{m(n-1)(n-2)} - \frac{n^2}{(n-1)^2} = \frac{n^2}{(n-1)^2} \left[\frac{(m+1)(n-1)}{m(n-2)} - 1\right]$$

$$\frac{n^2}{(n-1)^2} \quad \frac{n+m-1}{m(n-2)}$$

$$= \frac{n^2(n+m-1)}{m(n-2)(n-1)^2}$$

$$= \sigma_X^2 .$$

By Theorem 7.3

$$M_Y''(0) = 2\left(\frac{n}{m}\right)^n h^{2-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p+1}\left(k+\frac{n}{mh}\right)^{-m-n+p}$$

$$- \left(\frac{n}{m}\right)^n h^{2-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{k=1}^{\infty} k^{m-p}\left(k+\frac{n}{mh}\right)^{-m-n+p}$$

$$= 2A(m,n) \sum_{k=1}^{\infty} k^{m-p+1}\left(k+\frac{n}{mh}\right)^{-m-n+p}$$

$$- A(m,n) \sum_{k-1}^{\infty} k^{m-p}\left(k+\frac{n}{mh}\right)^{-m-n+p}$$

where

$$A(m,n) = \left(\frac{n}{m}\right)^n h^{2-n} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!}$$

It is easily seen that

$$A(m,n) \sum_{i=0}^{m-p} \binom{m-p}{i}\left(-\frac{n}{mh}\right)^i (n+i-1)^{-1}\left(\frac{n}{mh}+1\right)^{-n-i+1}$$

$$\leq A(m,n) \sum_{k=1}^{\infty} k^{m-p}\left(k+\frac{n}{mh}\right)^{-m-n+p} \leq$$

$$A(m,n) \sum_{i=0}^{m-p} \binom{m-p}{i}\left(-\frac{n}{mh}\right)^i (n+i-1)^{-1}\left(\frac{n}{mh}\right)^{-n-i+1}$$

or

$$\frac{nh}{m} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!\,(n-1)!} \sum_{i=0}^{m-p} \binom{m-p}{i} (-1)^i (n+i-1)^{-1} \left(\frac{n}{mh}\right)^{n+i-1} \left(\frac{n}{mh}+1\right)^{-n-i+1}$$

$$\leq A(m,n) \sum_{k=1}^{\infty} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p}$$

$$\frac{nh}{m} \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!\,(n-1)!} \sum_{i=0}^{m-p} \binom{m-p}{i} (-1)^i (n+i-1)^{-1}$$

Thus

$$A(m,n) \sum_{k=1}^{\infty} k^{m-p} \left(k+\frac{n}{mh}\right)^{-m-n+p} \to 0 \quad \text{as} \quad h \to 0$$

since the first and third parts of the inequality approach 0 as $h \to 0$.

In a similar manner

$$2A(m,n) \sum_{i=0}^{m-p+1} \binom{m-p+1}{i} \left(-\frac{n}{mh}\right)^i (n+i-2)^{-1} \left(\frac{n}{mh}+1\right)^{-n-i+2}$$

$$\leq 2A(m,n) \sum_{k=1}^{\infty} k^{m-p+1} \left(k+\frac{n}{mh}\right)^{-m-n+p} \leq$$

$$2A(m,n) \sum_{i=0}^{m-p+1} \binom{m-p+1}{i} \left(-\frac{n}{mh}\right)^{i} (n+i-2)^{-1} \left(\frac{n}{mh}\right)^{-n-i+2}$$

or

$$2\left(\frac{n}{m}\right)^2 \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{i=0}^{m-p+1} \binom{m-p+1}{i} (-1)^i (n+i-2)^{-1} \left(\frac{n}{mh}\right)^{n+i-2} \left(\frac{n}{mh}+1\right)^{-n-i+2}$$

$$A(m,n) \sum_{k=1} k^{m-p+1} \left(k+\frac{n}{mh}\right)^{-m-n+p} \leq$$

$$2\left(\frac{n}{m}\right)^2 \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{i=0}^{m-p+1} \binom{m-p+1}{i} (-1)^i (n+i-2)^{-1}$$

Thus

$$A(m,n) \sum_{k=1} k^{m-p+1} \left(k+\frac{n}{mh}\right)^{-m-n+p} \longrightarrow$$

$$2\left(\frac{n}{m}\right)^2 \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{i=0}^{m-p+1} \binom{m-p+1}{i} (-1)^i (n+i-2)^{-1}$$

as $h \to 0$ since the first and third parts of the inequality approach the above value as $h \to 0$.

By Lemma 7.5

$$2\left(\frac{n}{m}\right)^2 \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \sum_{i=0}^{m-p+1} \binom{m-p+1}{i} (-1)^i (n+i-2)^{-1}$$

$$= 2\left(\frac{n}{m}\right)^2 \sum_{p=1}^{m} \frac{(m+n-p-1)!}{(m-p)!(n-1)!} \frac{1}{n-2} \frac{(m-p+1)!(n-2)!}{(n+m-p-1)!}$$

$$= 2\left(\frac{n}{m}\right)^2 \sum_{p=1}^{m} \frac{m-p+1}{(n-1)(n-2)}$$

$$= \frac{2n^2}{m^2(n-1)(n-2)} \sum_{p=1}^{m} (m+1-p)$$

$$= \frac{2n^2}{m^2(n-1)(n-2)} \left\{ m(m+1) - \frac{m(m+1)}{2} \right\}$$

$$= \frac{n^2(m+1)}{m(n-1)(n-2)}$$

Thus $M_Y''(0) \to \dfrac{n^2(m+1)}{m(n-1)(n-2)}$ as $h \to 0$ which is what was desired.

Chapter VIII

SUMMARY

In this report we established that theoretical best estimators are in reality not best estimators. In chapters two and three, which are probably the most important chapters, a simple method was developed which essentially compensates for this discrepancy between practical and theoretical results in the case of normal random variables. The useful effects of this correction factor are demonstrated on point and interval estimators.

In chapter 4 similar results were established for the exponential distribution. Chapters 5,6, and 7 establish the necessary ground work for investigation of the distributional relationships between the truncated Chi-Square, T, and F-distributions.

In conclusion the authors would like to make the comment that in their opinion the results of this study are of some practical importance. Moreover, it is our feeling that a question has been raised which appears to be of sufficient importance to stimulate many people in the area of mathematical statistics to make some additional and worthwhile contribution in this area.

APPENDIX

Theorem 1A:

$$\int x^{n-1}e^{-x/2}dx = -\sum_{p=0}^{n-1} 2^{p+1}\frac{(n-1)!}{(n-1-p)!}e^{-x/2}x^{n-1-p}$$

Proof.

An induction proof easily establishes the results.

Theorem 1B:

If n = 2k, k = 1,2,3,..., then

$$\int \cos^n x\ dx = \frac{\sin x}{n}\left[\cos^{n-1}x + \sum_{i=1}^{k-1}\frac{(n-1)(n-3)\ldots(n-2i+1)}{(n-2)(n-4)\ldots(n-2i)}\cos^{n-2i-1}x\right]$$

$$+\frac{(n-1)(n-3)\ldots5.3.1}{n(n-2)\ldots6.4.2}x.$$

If n = 2k-1, k = 1,2,3,..., then

$$\cos^n x\ dx = \frac{\sin x}{n}\left[\cos^{n-1}x + \sum_{i=1}^{k-1}\frac{(n-1)(n-3)\ldots(n-2i+1)}{(n-2)(n-4)\ldots(n-2i)}\cos^{n-2i-1}x\right].$$

Proof.

The integrals can be obtained by applying the equation

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

Theorem 1C.

If $p = 2n+1$, $n = 1,2,3,\ldots,$ then

$$\int [1 + ax^2]^{-\frac{p+1}{2}} dx = \int [1 + ax^2]^{-(n+1)} dx$$

$$= \frac{1}{2n} \sum_{i=0}^{n-1} \frac{[(2n-1)/2]!(n-i-1)!}{[(2n-2i-1)/2]!(n-1)!} \frac{x}{(1+ax^2)^{n-i}}$$

$$+ \frac{[(2n-1)/2]!}{\sqrt{a\pi}\; n!} \tan^{-1} x\sqrt{a} \quad .$$

If $p = 2n$, n $1,2,3,\ldots,$ then

$$\int [1 + ax^2]^{-\frac{p+1}{2}} dx = \int [1 + ax^2]^{-\frac{2n+1}{2}} dx$$

$$= \frac{1}{2n-1} \sum_{i=0}^{n-1} \frac{(n-1)![(2n-]i-3)/2]!}{(n-i-1)![(2n-3)/2]!} x(1 + ax^2)^{-\frac{2n-2i-1}{2}}$$

Proof.

Letting $\tan\theta = x\sqrt{a}$, the results can be obtained from Theorem 1B.

Theorem 1D:

$$\int F^{m-1}[1 + \frac{mF}{n}]^{-m-n} dF = -\sum_{p=1}^{m} \left(\frac{n}{m}\right)^p \frac{(m-1)!(m+n-p-1)!}{(m-p)!(m+n-1)!} F^{m-p}[1 + \frac{mF}{n}]^{-m-n+p}$$

where m and n are positive integers.

Proof.

Integrating by parts m-1 times will yield the desired result.