R.P.I. Technical Report MP-5

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Analysis and Design of a Capsule Landing System
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ABSTRACT

The guidance problem in the entry phase of a Martian mission is complicated by the uncertainty which exists in the values of the atmospheric density parameters. Trajectories obtained from aerodynamic breaking are very sensitive to deviations in the atmospheric parameters. For the controls in a guidance scheme, this investigation is concerned with using (i) a discrete change in the ballistic coefficient (m/C_DA) by altering the effective area of drag (A), and (ii) a sequence of small discrete changes in the flight path angle by applying an impulsive force perpendicular to the direction of the velocity vector.

With a knowledge of reference atmospheric parameters and by tracking the actual parameters, the problem is to find a correct altitude for changing the drag surface. This operation should minimize the deviation of the actual terminal conditions from the final terminal conditions. The sequence of impulsive changes in the flight path angle can then be employed as a trim to reduce the errors in the actual terminal conditions.

A reference value for the ballistic coefficient was obtained from NASA. To determine the effect of a change in the drag surface on the final values of the state variables, the area ratio was varied as a function of altitude. Simulating this procedure over the range of possible atmospheres determines the value of A/A_o necessary to compensate for deviations from the reference set

of atmospheric parameters. It will also determine the altitude at which the ballistic coefficient is changed as a function of the actual atmosphere.

However, the accuracy in this scheme is dependent on the accuracy from an updating scheme when the change in area is made. If improved values of the parameters are obtained after the ballistic coefficient is changed, a sequence of discrete changes in the flight path angle may be used to minimize the resulting state variable errors.

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I. INTRODUCTION

The entry guidance system for a Martian capsule must meet certain terminal constraints at a terminal altitude if the soft landing of a capsule on the surface is to be successful. For Earth re-entry, the atmosphere is considered a well known quantity, and the guidance system need only correct for errors in the entry conditions to meet the final constraints. However, the Mars entry guidance problem is complicated by the uncertainty which exists in the characteristics of the Martian atmosphere. A guidance scheme must be able to compensate for variations in the atmosphere as well as in the entry conditions if the capsule is to meet the predetermined terminal conditions.

At present there are approximately ten models of the Martian atmosphere used in engineering design. These models are all of the same mathematical form and the only variation is the parameter values used in them[2,3] (see Appendix B). A reference trajectory will have to be designed on earth based on one of the models and the guidance system will have to respond to variations in the reference values of the atmospheric parameters. Therefore, an on-board system will be needed to update the actual values of the atmospheric parameters encountered when the craft arrives[4].

The trajectory errors due to deviations between the actual and reference values of the Mars atmospheric parameters are very important. The trajectories obtained from aerodynamic breaking are particularly sensitive to deviations in these parameters and according to present estimates [5] the parameter deviations may be quite large. The surface pressure alone is considered ranging from 5 to 10 millibars.

Now the density of Mars is much lower than that found on Earth and the aerodynamic drag of the capsule is directly dependent on the density. To compensate for this lessened atmospheric drag, two techniques have been suggested.[1] These are (i) altering the ballistic coefficient of the vehicle, and (ii) lengthening the flight path in the atmosphere.

Variation of the ballistic coefficient has been suggested as a control technique in papers by Phillips and Cohen[6], and by Warden[7]. However, both papers made the use of simplifying assumptions which are valid for Earth entry. Thru these assumptions, analytical solutions of the equations of motion were found and applied for the analysis of drag controlled entry. However, these assumptions are not valid for the Mars entry problem, but the use of the ballistic coefficient as a control seems particularly applicable to Mars entry due to the small drag forces.

In addition, it is possible to lengthen (or shorten) the flight range of the capsule by changing the flight path angle at some point on the trajectory. This type of control might be used to make the range error zero thru a sequence of small corrections; or it might be used to decrease the terminal velocity by extending the length of the flight.

Therefore, it is the aim of this report to discuss the use of a discrete change in the ballistic coefficient and a sequence of small discrete changes in the flight path angle as the controls in a guidance scheme which compensates for the atmospheric uncertainty of Mars.

An estimate of the terminal constraints that we wish to satisfy for the entry phase of a Martian landing mission has been outlined by NASA[1]. For the purpose of the investigation, the desired final velocity constraint is taken as wanting to reduce the speed of the craft from approximately fifteen thousand feet per second at entry to less than one thousand feet per second at an altitude of twenty thousand feet above the surface. The final range constraint is taken as minimizing its deviation from the reference value chosen.

II. SUMMARY

It is shown that the value of the ballistic coefficient has a distinct effect on the terminal conditions of an entry capsule. If the ballistic coefficient is always less than .30 slugs/ft^2, the final velocity is less than one thousand feet per second at twenty thousand feet for all proposed models. If the composition of the actual atmosphere is similar to that of the reference atmosphere, the change in the drag surface may be used to make the vehicle reach a reference range by implementing the change at an altitude determined by the actual atmosphere.

If the area change is made at the wrong altitude, the range error is non-zero. A sequence of small discrete changes in the flight path angle is used to reduce this error. The size of the discrete change is computed through a numerical scheme based on the system's sensitivity to atmospheric parameter deviations. If the allowable step change in the flight path angle is bounded, the required velocity changes of the craft are considerably smaller than the velocity changes necessary for the unbounded allowable angle change.

III DISCRETE CHANGE OF DRAG SURFACE

A. Method of Analysis

The ballistic coefficient (m/C_DA) is a geometric characteristic of an entry vehicle. For a vehicle of fairly constant mass, a large value of the coefficient represents a sharp, pointy, needle-like vehicle whereas a small value corresponds to a blunt vehicle. The parameter A is the cross-section of the vehicle's frontal area which is perpendicular to the velocity vector of the capsule. This is the effective drag surface which is related to the magnitude of the drag force on the vehicle by the equation:

$$D = \frac{1}{2} \rho(y) \; V^2 (C_D A) \tag{2-1}$$

Different values of A will result in difference trajectories; and if A is change during entry, the terminal state variables of the altered trajectory will be different from those of the unaltered trajectory. Our interest is the actual effect of this change in the drag surface on the terminal state of the vehicle. First the size of the change in area must be determined which compensates for atmospheric parameter deviations. It is also necessary to find the correct altitude for changing the drag surface based upon the parameter deviations.

The equations of motion for two dimensional entry into a non-rotating atmosphere of a spherical planet by a non-lifting vehicle are given as follows where the normalized altitude is the independent variable (see Appendix A).

$$\frac{d\Theta}{dx} = \frac{h}{N}\frac{1}{g_oR_o}\frac{1}{v^2\sin\Theta}\left[g(x) - \frac{v^2g_oR_o}{R_o+h(1-x/N)}\right]\cos\Theta$$

$$\frac{dv}{dx} = \frac{h}{N}\frac{1}{g_oR_o}\frac{1}{v^2\sin\Theta}\left[g(x)\sin\Theta - \frac{1}{2}\rho(x)g_oR_ov\left(\frac{C_DA_o}{m}\right)\frac{A}{A_o}\right]$$

$$\frac{d\Omega}{dx} = \frac{h}{N}\frac{\cot\Theta}{R_o+h(1-x/N)} \qquad (2\text{-}2)$$

where
- Θ = flight path angle
- Ω = range angle
- v = normalized velocity
- x = normalized altitude
- h = reference altitude
- N = scale factor
- g_o = surface gravitational acceleration
- R_o = radius of Mars
- A = actual area
- A_o = reference area

m= mass of the capsule

C_D= coefficient of drag

ρ= local density

Since analytic solutions for these differential equations do not exist over the entire range of flight being considered, the terminal condtions must be found through numerical integration [11,12].

Now $m/C_D A_o$ represents the reference value for the ballistic coefficient as given by NASA[1]. By varing the ratio of (A/A_o) in equation (2-2) the actual ballistic coefficient in the integration will change. Therefore this ratio (A/A_o) is our parameter of interest.

A discussion of the density models used is in Appendix B. Presently, NASA is using a reference atmosphere whose composition consists predominantly of carbon dioxide. Therefore, unless stated different, the results given are for such atmospheres. However a comparison of effects of carbon dioxide verses nitrogen atmosphere will be included for completeness.

B. Results of Trajectory Study

In Martian entry, the region of importance for aerodynamic breaking is below on hundred thousand feet in altitude since the density of the atmosphere is considered so low (see fig. 3-9). Above this altitude the trajectory is independent of the actual atmosphere. The drag force is effectively zero.

Plotting the terminal velocity as a function of the ballistic coefficient shows that for $A/A_o = 1.0$ (the reference value), the terminal velocity is below one thousand feet per second for all the atmospheres (see fig. 3-1). For any value of the ballistic coefficient less than or equal to the reference value, the terminal velocity constraint also seems to be satisfied. If we assume a high density reference atmosphere of VM-4 (10 mb), the drag surface must be approximately doubled for an actual VM-8 (5 mb) atmosphere if the terminal velocities are to match.

Similarly, plotting the terminal range as a function of the ballistic coefficient shows that taking a VM-4 reference and a VM-8 actual atmosphere requires the drag surface also too be approximately doubled if the terminal range error is to be zero (see fig. 3-2).

Therefore, it appears that the ballistic coefficient must change approximately by a factor of two if we are to compensate for the possible parameter deviations. The question now is how to determine the altitude at which this discrete change in the shape of the vehicle is to be implemented.

Let us now consider the effect of drag control at specific altitudes of the flight range of the capsule (see fig. 3-3 & 3-4). The terminal altitude is taken as twenty thousand feet. The flight range obtained for a change of the drag surface at this altitude is the reference value. The significant point from these range curves is their symmetry. If the ballistic coefficient is changed at a specific altitude, the amount of range deviation from the original trajectory is independent of the actual atmosphere. The deviation is only a function of the size of the change in area.

Similarly, consider the effect of drag control at specific altitudes of the terminal velocity of the vehicle (fig. 3-5 & 3-6). As expected, the final velocity is less than the reference terminal velocity (1000 ft/sec), independent of the atmospheric model. Also, for a specific model, the value of the final velocity is invariant if the ballistic coefficient is changed above a threshold altitude of approximately fifty thousand feet.

Now consider the joint effect on the final velocity and the flight range for a specific area change. As predicted, a doubling of the drag surface area produces an overlapping region (for the atmospheres considered) of both the final velocity and the flight range (see fig. 3-4 & 3-6). Assume for example, that a VM-4 atmospheric model without any change in the ballistic coefficient is chosen for a reference trajectory. Then if a different atmosphere is encountered, it is possible to find an altitude at which doubling the drag surface area will result in the terminal velocity of the reference trajectory. Similarly, it is possible to match the reference flight range by doubling the drag surface at an appropriate altitude.

However, the altitude of change for achieving each terminal condition is not the same. Now from basic control theory, the change of the ballistic coefficient can only be used to control one state variable. Consider restricting the terminal velocity to less than one thousand feet per second at twenty thousand feet and not to any specific value. Using a ballistic coefficient less than or equal to the given reference satisfies this velocity requirement. Therefore, the change in the drag surface can be used to make the vehicle attain a reference range by implementing the change at the proper altitude.

So far, this discussion has considered only the case of increasing the reference drag surface. This type of change increases the drag force on the vehicle and tends to shorten the range of the capsule. A high density reference atmosphere must be chosen for this type of change, and the area change is made if the actual atmosphere is of lower density.

However, let us consider the alternate approach of starting with an area greater than A_o and then decreasing the actual area to the reference value at some altitude. The values of the ballistic coefficient would still be small enough to make the terminal velocity less than our reference of one thousand feet per second. Therefore, it should be possible to still use the area change to reach a reference range. Indeed, the flight range characteristics do overlap if the area at entry is twice the reference value ($A=2A_o$) (see fig. 3-13 & 3-14).

Therefore, changing the drag surface by a factor of two (A_o to $2A_o$ or $2A_o$ to A_o) allows the vehicle to reach a reference range value. However, decreasing the area will lengthen the range of the vehicle whereas increasing the area shortens the range. Also, the reference trajectory must be chosen for a low density atmosphere and the area change made if a higher density is encountered.

Now a comparison of figures 3-4 & 3-13 show that the amount of overlap is governed by the size of the change in the drag surface. For the same size change in the area (whether increased or decreased), the conclusion on the ability of the capsule to reach a reference range value is the same. Therefore, any following results shown are only for an increase in the drag surface. But similar results and conclusions could be obtained for an identical decrease in the area.

Now doubling the area of the drag surface is a marginal case for obtaining an overlapping region in the flight range characteristics. A VM-8 actual atmosphere trajectory only reaches a VM-4 reference range if the change is implemented very early in the important region of the atmosphere (below 100,000 ft.). It might be desireable to obtain a greater amount of overlap in the range curves. This could be accomplished by using a larger change in the ballistic coefficient (see fig. 3-7). However, this is not a very appealling solution for physical reasons. Implementing a large change of area with the structural strength necessary for re-entry may be quite difficult to accomplish and it may also involve an excessive amount of weight.

As a possible solution, it was thought that lowering the final altitude might allow for greater overlapping in the flight range characteristics due to the longer time of flight. However, it was found that the amount of overlap for a specific area change did not vary with a change in the final altitude (see fig. 3-8). This result substantiates the previous suggestion that the ability of the capsule to reach a reference range is governed only by the size of the area change made.

Now as stated previously, these results, so far, have been for atmospheric models whose main constituent is carbon dioxide. However, as of the date of this report, it is still possible that the main component of the atmosphere may instead be nitrogen. It is this uncertainty in the atmospheric composition that actually makes the Martian entry problem difficult. For a nitrogen atmosphere, the important region of a nitrogen atmosphere extends much higher than the region for a carbon dioxide atmosphere (see fig. 3-9). The density in a nitrogen atmosphere at two hundred thousand feet is equivalent to the density in a carbon dioxide atmosphere at one hundred thousand feet. Therefore, the vehicle's trajectory will begin to be effected at approximately two hundred thousand feet by a nitrogen atmosphere whereas a trajectory in a carbon dioxide atmosphere is not effected until approximately one hundred thousand feet.

For a comparison of results, consider the effect of drag control at specific altitudes on the terminal conditions for nitrogen atmospheres. Again the final velocity is less than one thousand feet per second for a ballistic coefficient less than or equal to the reference value (see fig. 3-10). The final velocity also shows the characteristic of being invariant if changed above a threshold altitude. The flight range characteristics show overlapping for a doubling of the drag surface (see fig. 3-11). Therefore, it is possible to use a doubling of the drag surface to make the vehicle reach a reference range. However, the altitude of change is generally much higher for a specific range correction in nitrogen atmospheres than in carbon dioxide atmospheres.

The flight range in nitrogen atmospheric models is also considerably shorter than the range in carbon dioxide models (see fig. 3-11 & 3-12). If the reference trajectory is chosen for one type of composition and the actual atmosphere is of the other type of composition, it is not possible to reach the reference range by changing the area by a factor of two. The area change must be much greater than a factor of two, or else some additional type of control must also be implemented.

In brief summary, the value of the ballistic coefficient has a distinct effect on the terminal conditions of an entry capsule. If the composition of the actual atmosphere is similar to that of the reference atmosphere, it is possible to meet pre-specified terminal conditions if the drag surface is changed by a factor of two. In fact, the range error can be made zero if the area is changed at an appropriate altitude. However, if the actual atmospheric composition differs greatly from the reference atmosphere, the reference flight range cannot be reached by a reasonable change in the ballistic coefficient.

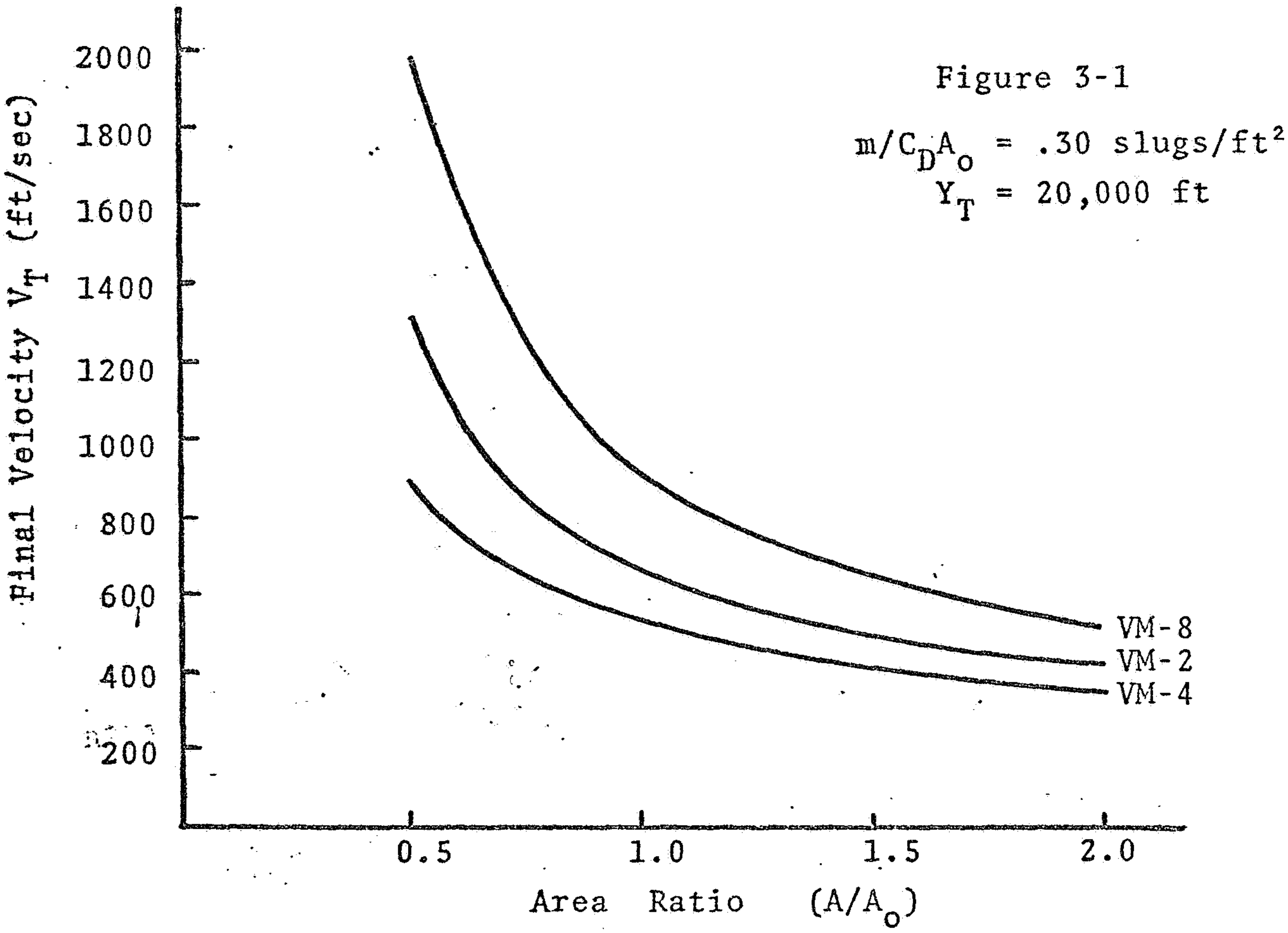


Figure 3-1
m/C_DA_o = .30 slugs/ft²
Y_T = 20,000 ft
Final Velocity V_T (ft/sec)
2000
1800
1600
1400
1200
1000
800
600
400
200
VM-8
VM-2
VM-4
0.5
1.0
1.5
2.0
Area Ratio (A/A_o)

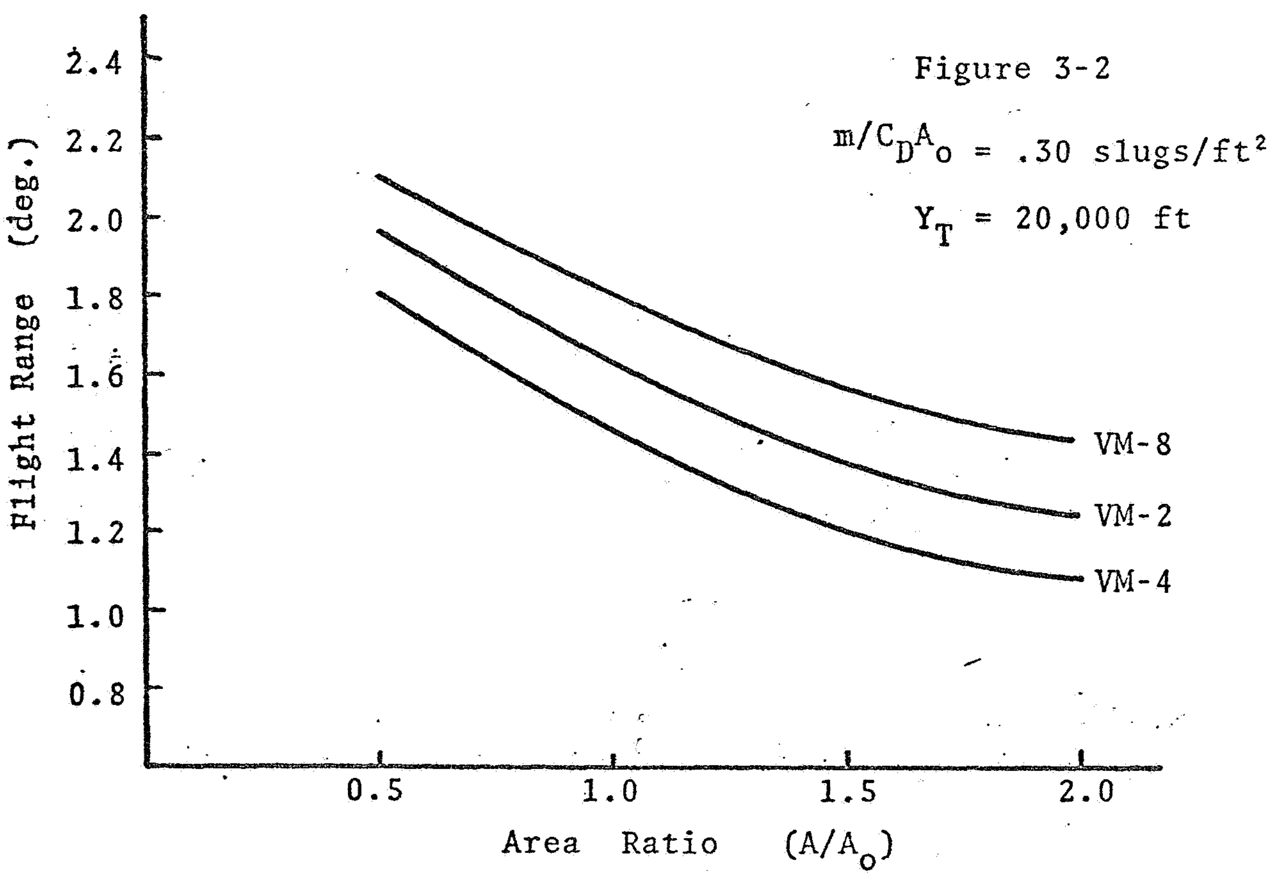


Figure 3-2
m/C_DA_o = .30 slugs/ft²
Y_T = 20,000 ft
Flight Range (deg.)
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
VM-8
VM-2
VM-4
0.5
1.0
1.5
2.0
Area Ratio (A/A_o)

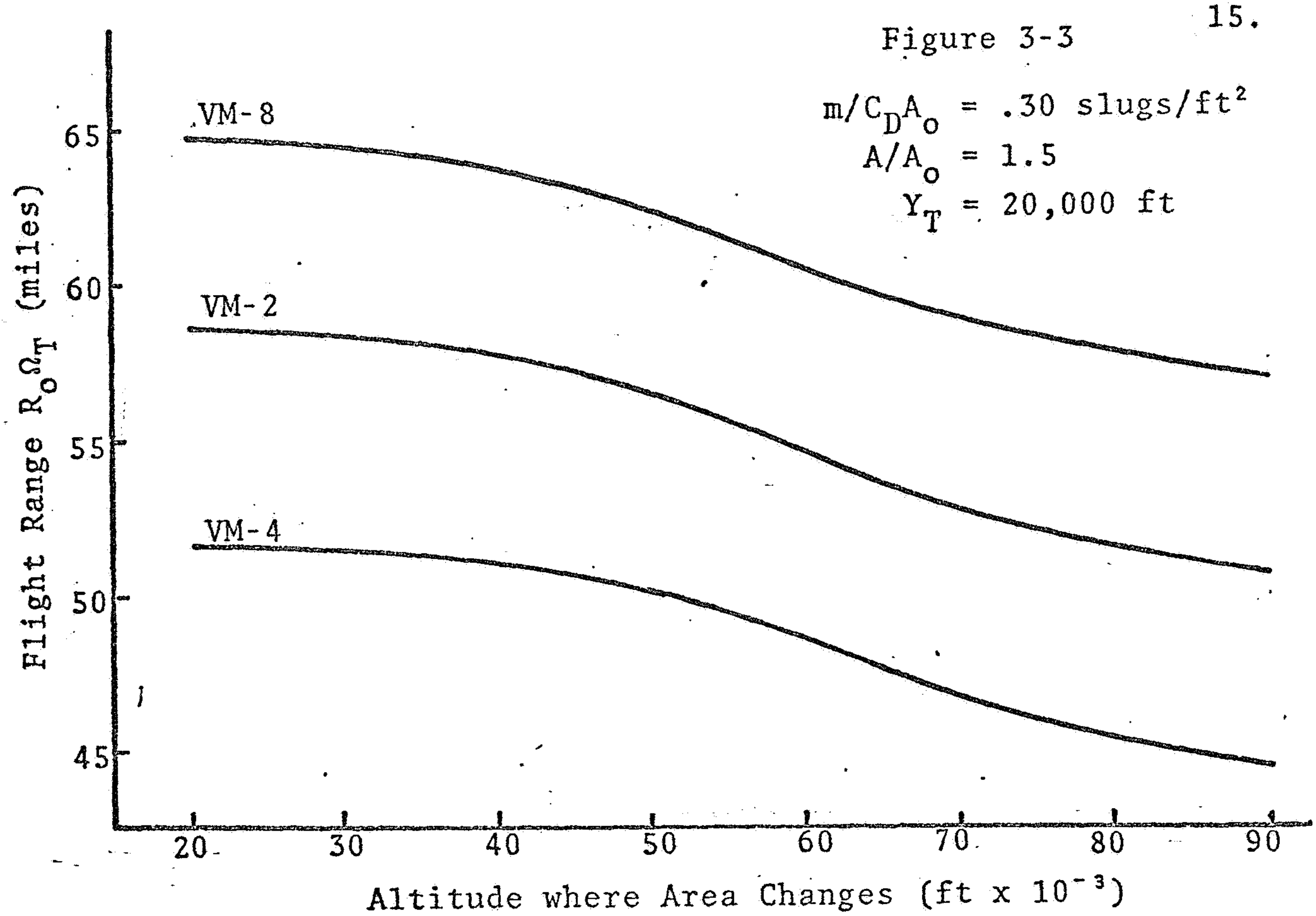


Figure 3-3
m/C_DA_o = .30 slugs/ft²
A/A_o = 1.5
Y_T = 20,000 ft
VM-8
VM-2
VM-4
65
60
55
50
45
Flight Range $R_o\Omega_T$ (miles)
20
30
40
50
60
70
80
90
Altitude where Area Changes (ft x 10^{-3})

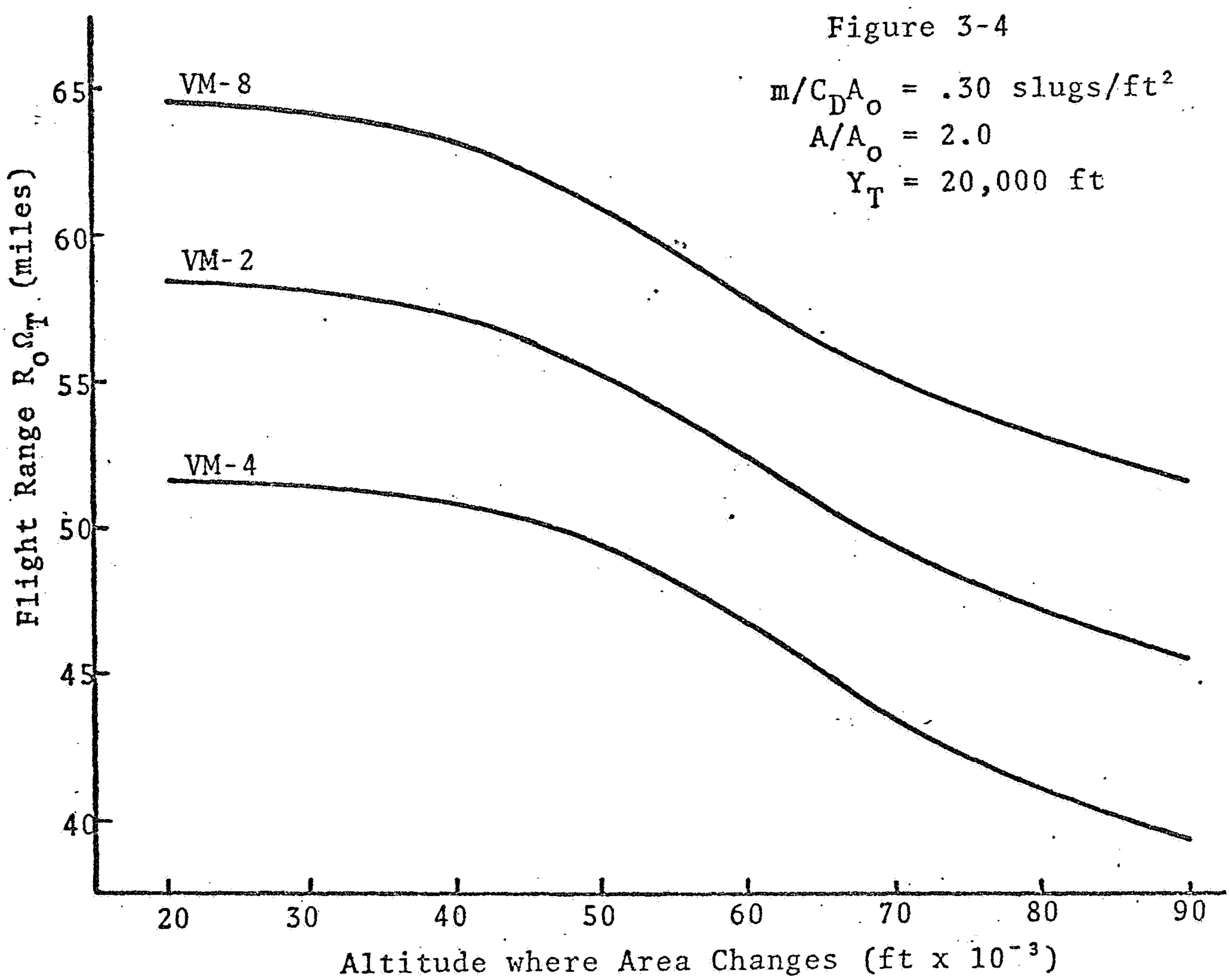


Figure 3-4
m/C_DA_o = .30 slugs/ft²
A/A_o = 2.0
Y_T = 20,000 ft
VM-8
VM-2
VM-4
65
60
55
50
45
40
Flight Range $R_o\Omega_T$ (miles)
20
30
40
50
60
70
80
90
Altitude where Area Changes (ft x 10^{-3})

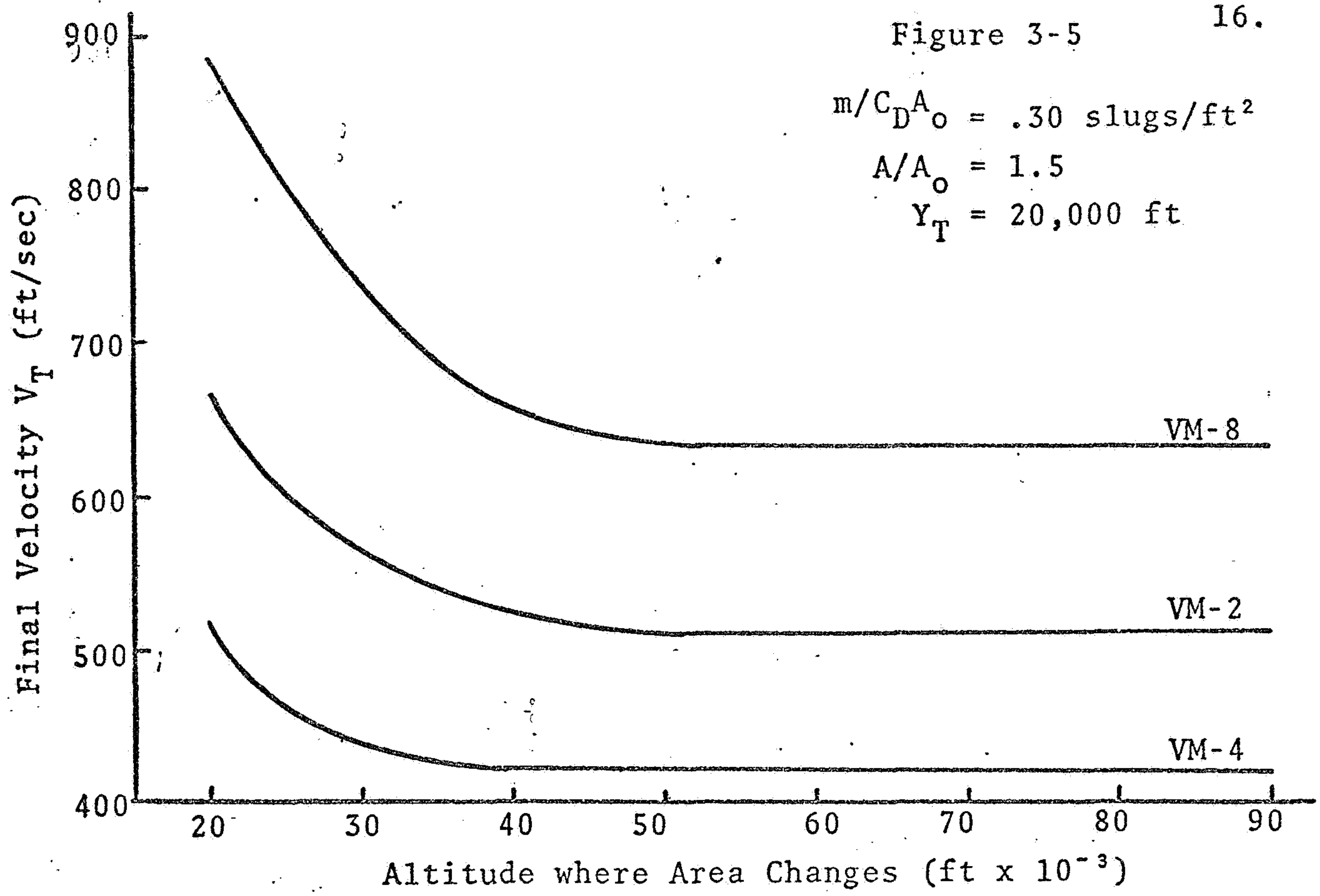


Figure 3-5
m/C_DA_o = .30 slugs/ft²
A/A_o = 1.5
Y_T = 20,000 ft
Final Velocity V_T (ft/sec)
900
800
700
600
500
400
VM-8
VM-2
VM-4
20
30
40
50
60
70
80
90
Altitude where Area Changes (ft x 10⁻³)

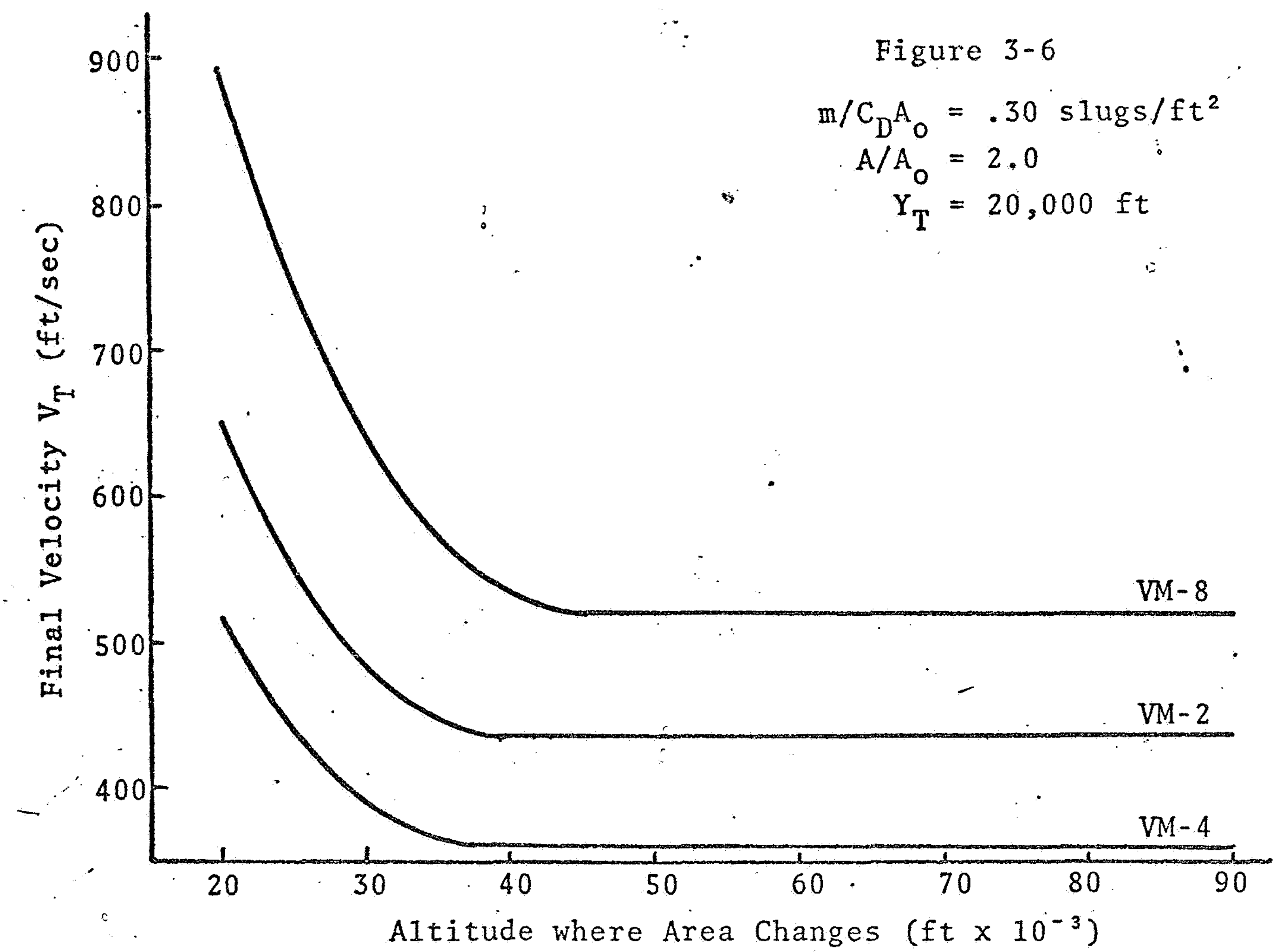


Figure 3-6
m/C_DA_o = .30 slugs/ft²
A/A_o = 2.0
Y_T = 20,000 ft
Final Velocity V_T (ft/sec)
900
800
700
600
500
400
VM-8
VM-2
VM-4
20
30
40
50
60
70
80
90
Altitude where Area Changes (ft x 10⁻³)

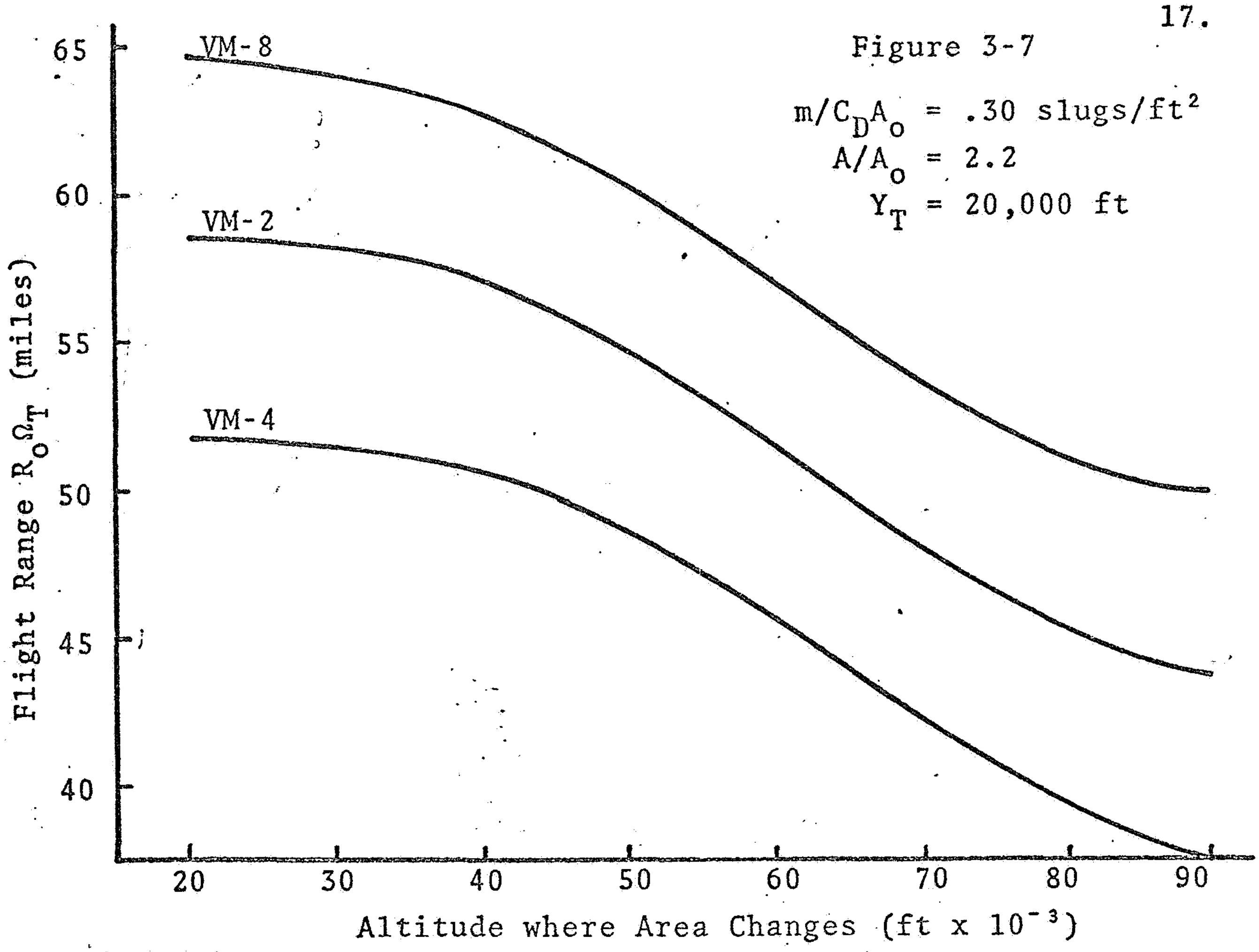


Figure 3-7
m/C_DA_o = .30 slugs/ft²
A/A_o = 2.2
Y_T = 20,000 ft
VM-8
VM-2
VM-4
Flight Range R_oΩ_T (miles)
65
60
55
50
45
40
20
30
40
50
60
70
80
90
Altitude where Area Changes (ft x 10⁻³)

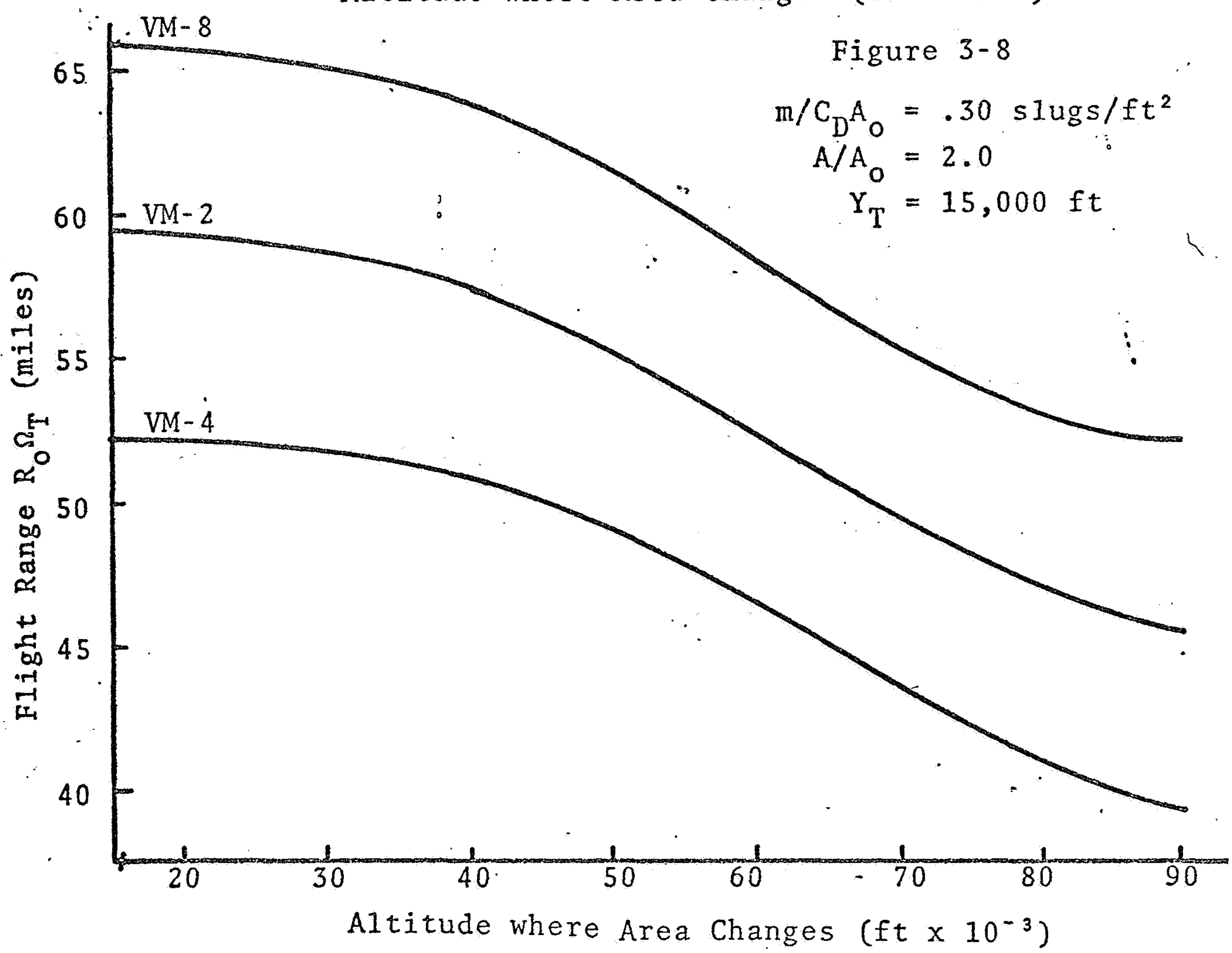


Figure 3-8
m/C_DA_o = .30 slugs/ft²
A/A_o = 2.0
Y_T = 15,000 ft
VM-8
VM-2
VM-4
Flight Range R_oΩ_T (miles)
65
60
55
50
45
40
20
30
40
50
60
70
80
90
Altitude where Area Changes (ft x 10⁻³)

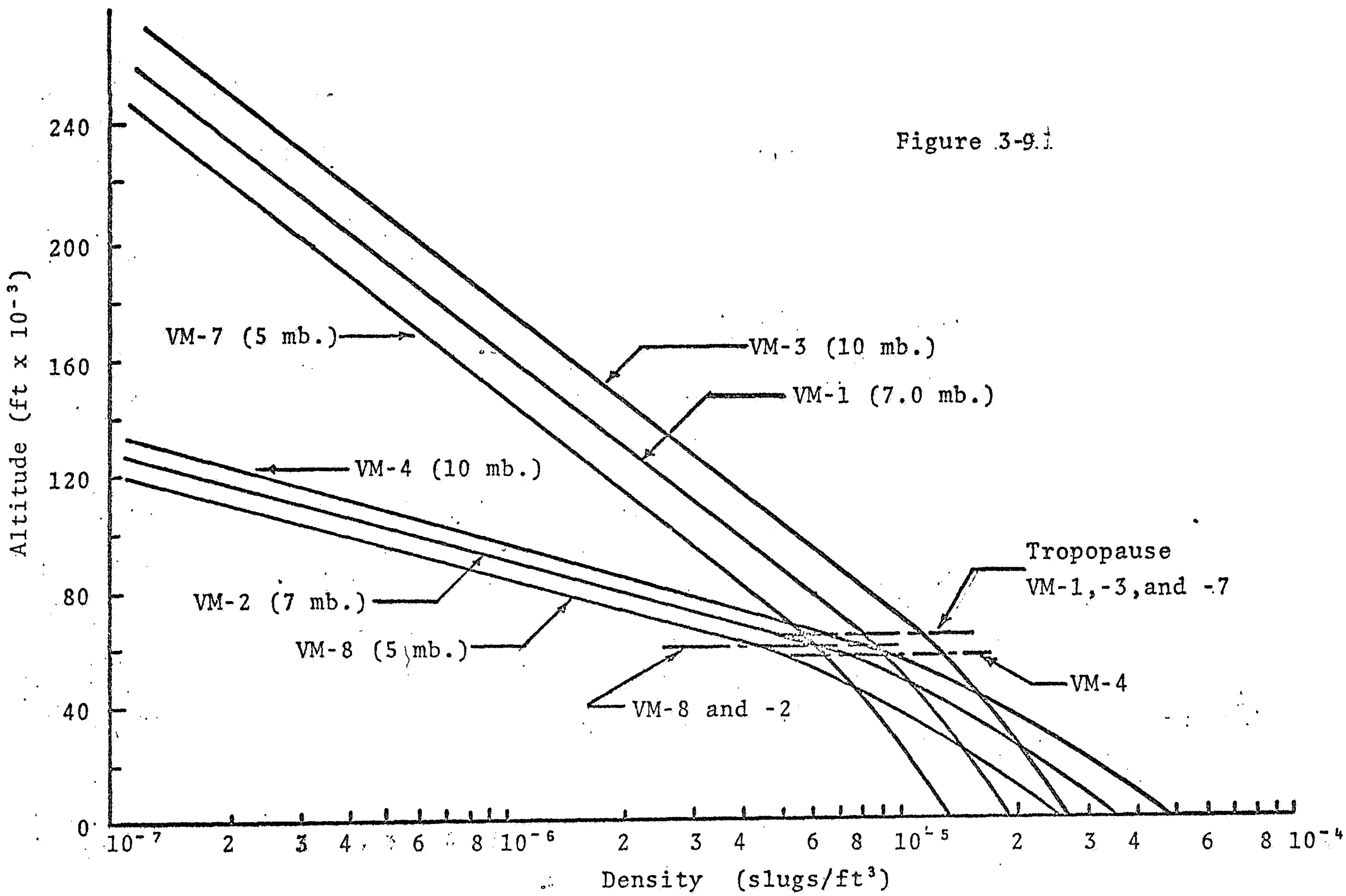

VM-7 (5 mb.)
VM-3 (10 mb.)
VM-1 (7.0 mb.)
VM-4 (10 mb.)
VM-2 (7 mb.)
VM-8 (5 mb.)
Tropopause
VM-1, -3, and -7
VM-4
VM-8 and -2
Altitude (ft x 10^-3)
Density (slugs/ft^3)
240
200
160
120
80
40
0
10^-7
2
3
4
6
8
10^-6
10^-5
10^-4

Figure 3-9.1

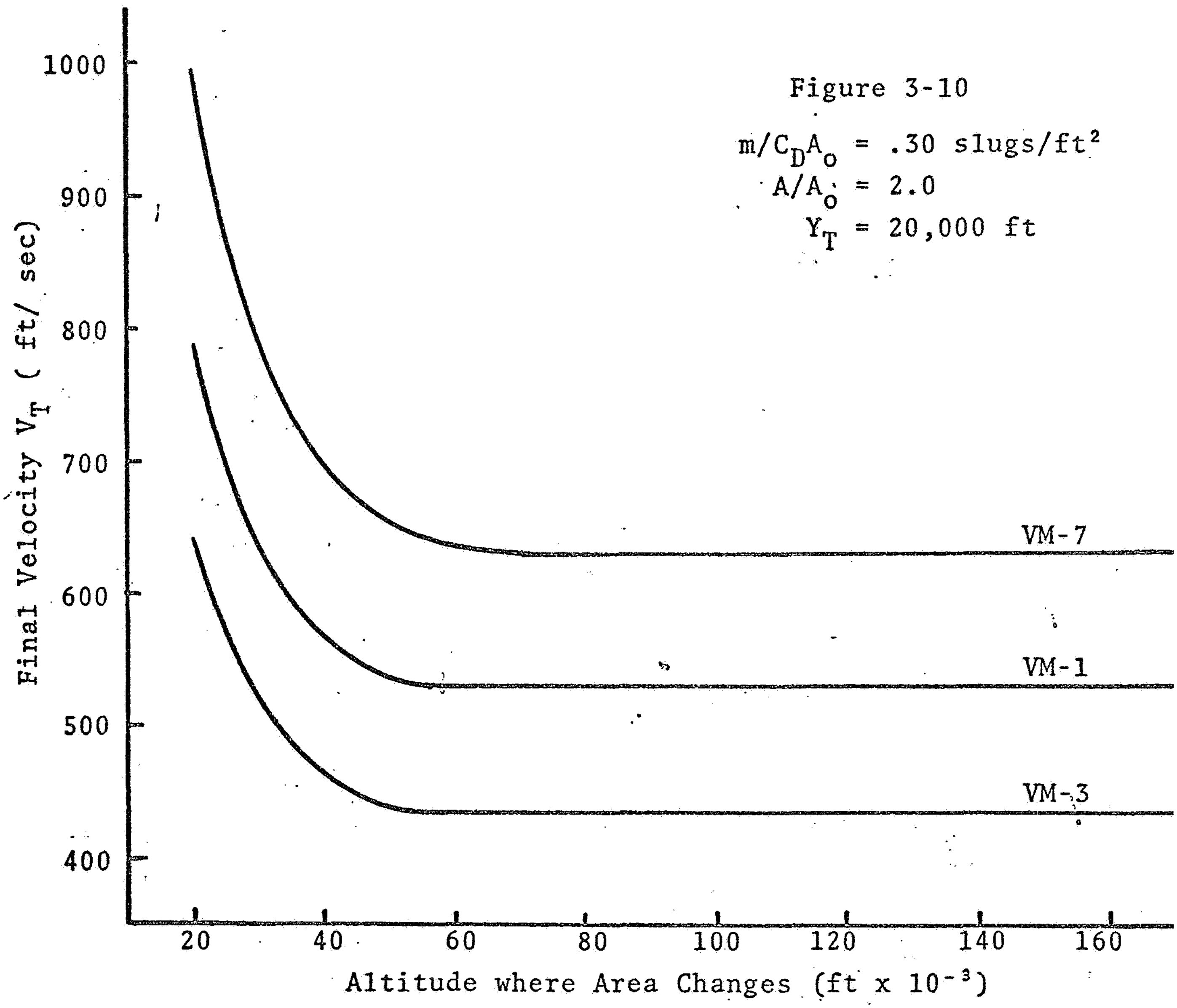


Figure 3-10
m/C_DA_o = .30 slugs/ft²
A/A_o = 2.0
Y_T = 20,000 ft
1000
900
800
700
600
500
400
Final Velocity V_T (ft/ sec)
VM-7
VM-1
VM-3
20
40
60
80
100
120
140
160
Altitude where Area Changes (ft x 10^{-3})

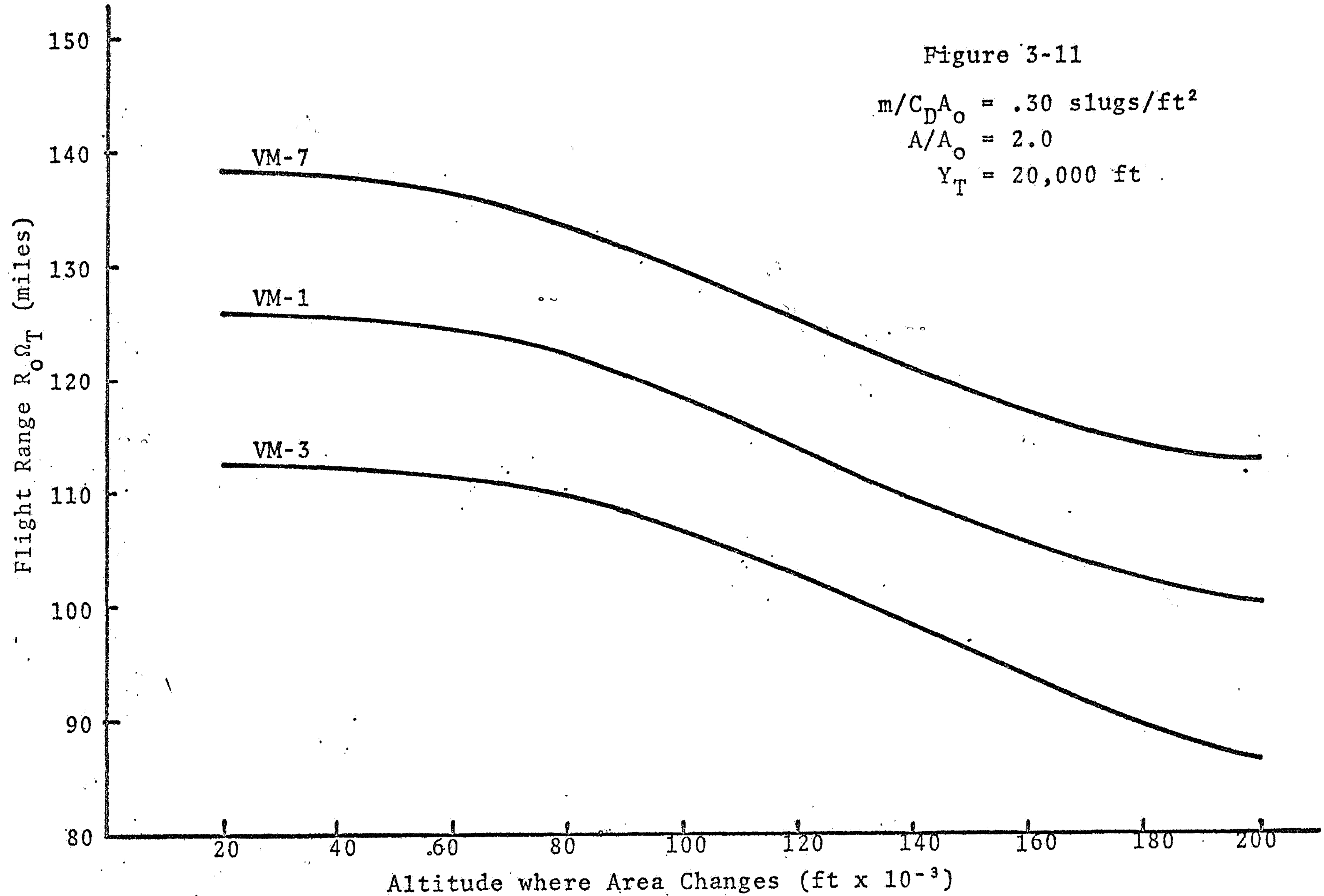
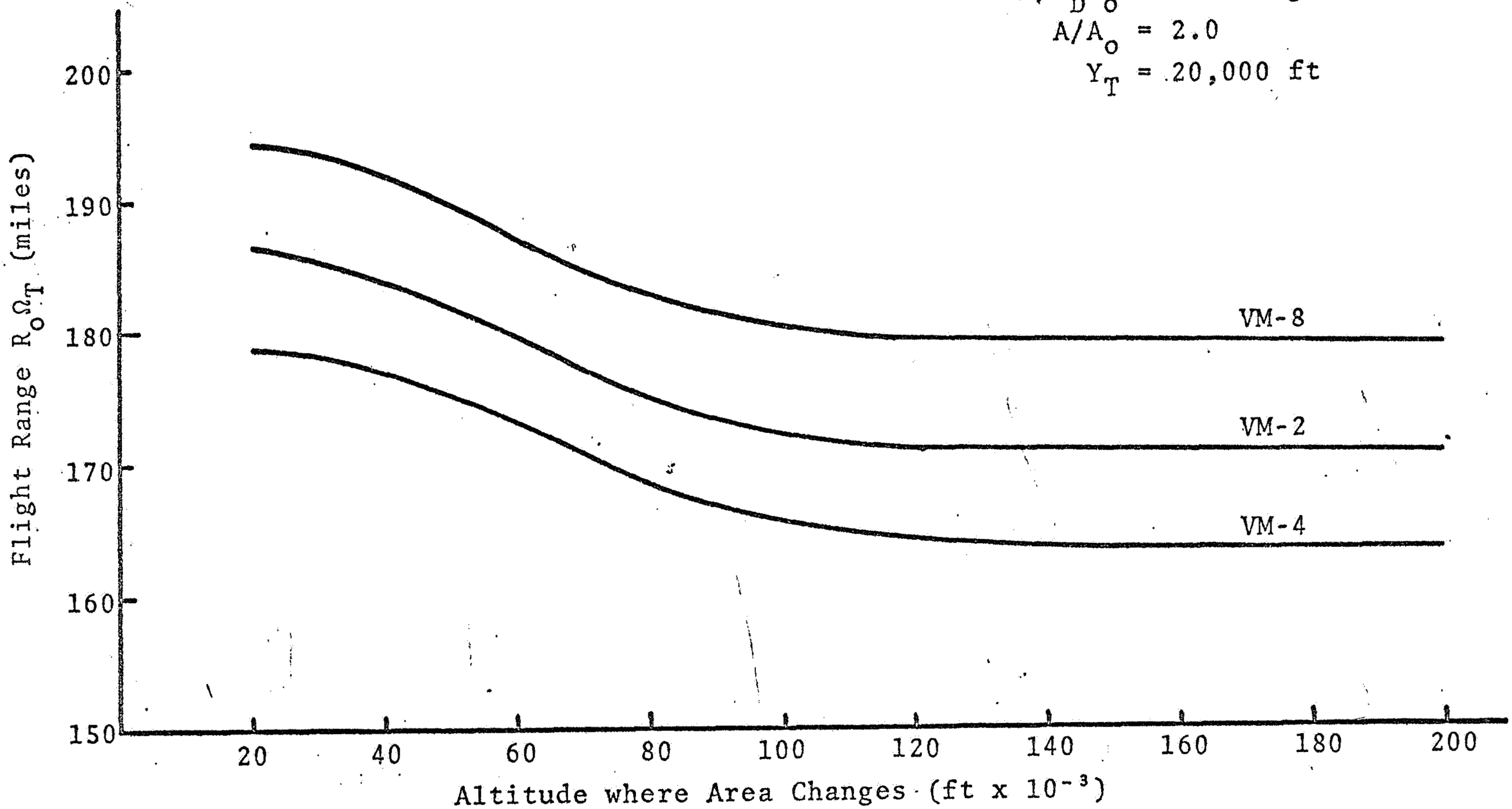


Figure 3-11

m/C_DA_o = .30 slugs/ft^2
A/A_o = 2.0
Y_T = 20,000 ft

Figure 3-12

m/C_DA_o = .30 slugs/ft²
A/A_o = 2.0
Y_T = 20,000 ft



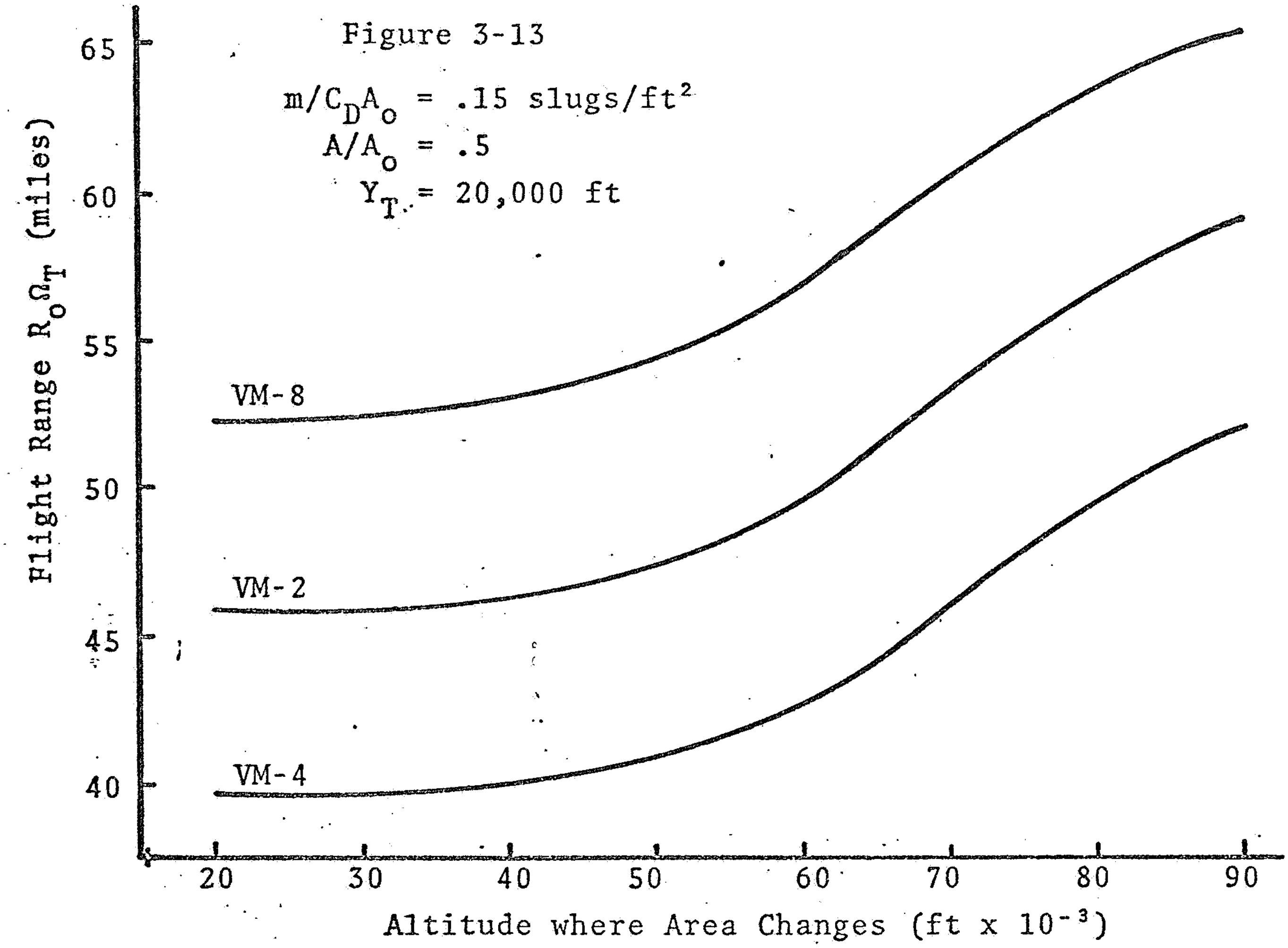


Figure 3-13
m/C_DA_o = .15 slugs/ft²
A/A_o = .5
Y_T = 20,000 ft
Flight Range R_oΩ_T (miles)
65
60
55
50
45
40
VM-8
VM-2
VM-4
20
30
40
50
60
70
80
90
Altitude where Area Changes (ft x 10⁻³)

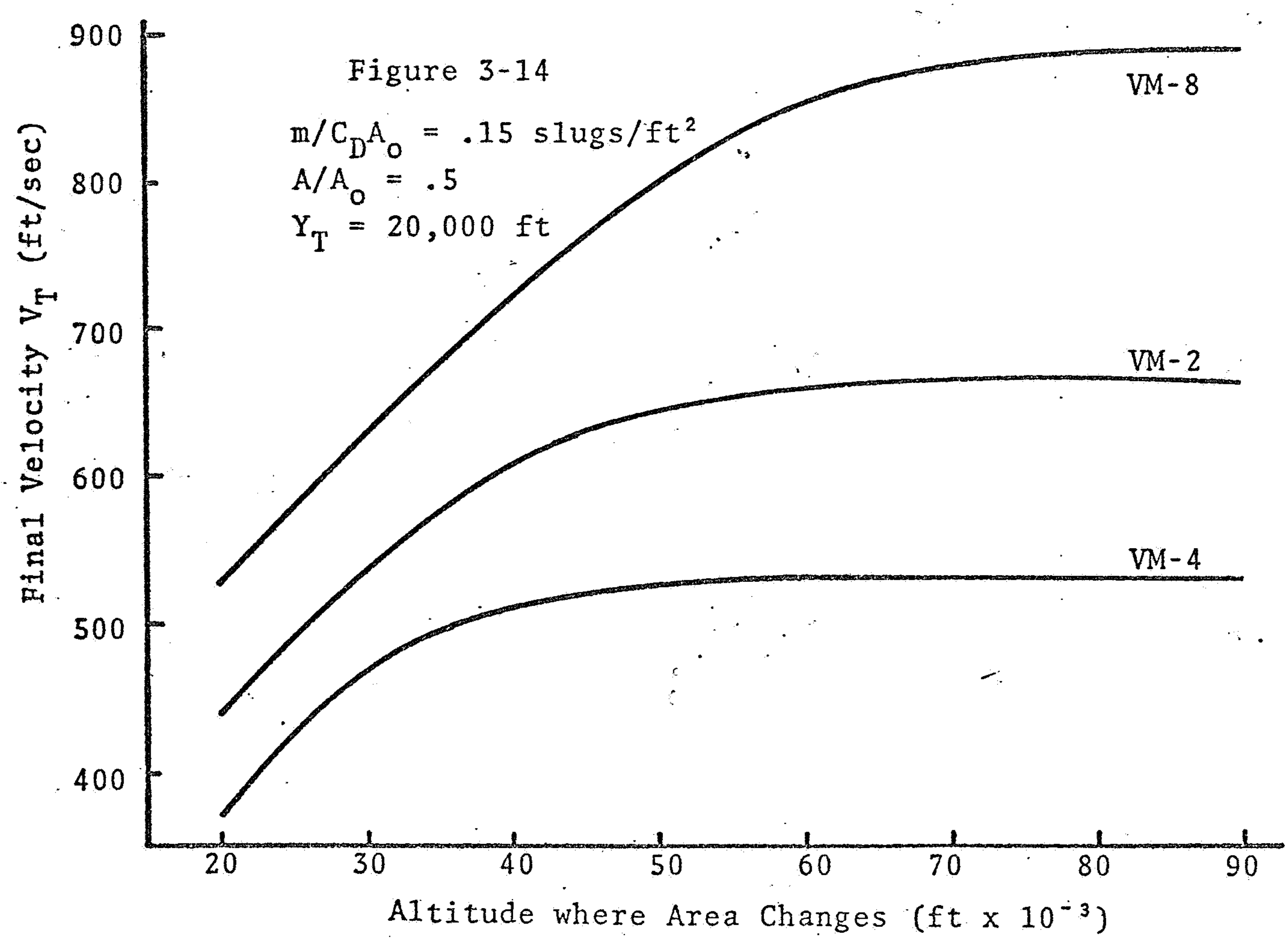


Figure 3-14
m/C_DA_o = .15 slugs/ft²
A/A_o = .5
Y_T = 20,000 ft
Final Velocity V_T (ft/sec)
900
800
700
600
500
400
VM-8
VM-2
VM-4
20
30
40
50
60
70
80
90
Altitude where Area Changes (ft x 10⁻³)

IV. DISCRETE CHANGE IN FLIGHT PATH ANGLE

A. Method of Analysis

Based on estimates of the actual atmospheric parameters, the altitude for changing the drag surface may be chosen to reduce the final range error. However, if the estimates from the atmospheric updating scheme[4] have not yet converged on the true values of the actual parameters, the ballistic coefficient will not be changed at the altitude which minimizes the range error. Hence, for improved values of the parameters obtained after the drag surface is changed, one would like to make small changes in the direction of flight to compensate for the error induced by changing the effective area at the wrong altitude.

By applying an impulsive force perpendicular to the direction of the velocity vector, the flight path angle, or direction of flight, can be discretely changed. For small changes in flight angle the velocity of the vehicle is changed in direction only. The magnitude is unaltered. Therefore, by using a sequence of small impulses, the change in flight path angle can be employed as a trim to reduce the error in the actual terminal range.

The problem is then to determine the size of the change in the actual flight path angle necessary to reduce the flight range error. Viewing this as an initial condition type of problem, one must determine what the flight path should be at the present altitude to minimize the final range error. The size of this discrete change is based on (i) the deviations of the actual state variables from the reference trajectory, (ii) the deviations of the atmospheric parameters from the reference atmosphere, and (iii) the magnitude of the change in the ballistic coefficient- all at the present altitude. Sensitivity analysis can be used to predict the terminal range error based on the deviations between actual and reference conditions; and from this error, the size of the discrete change which will reduce the predicted range error to zero can be computed.

B. Derivation of Sensitivity Coefficients

To estimate the range error and the flight path angle change necessary to compensate for the error, we will make use of sensitivity coefficients. This section presents a method for finding these coefficients. This method was derived by Dr. P. J. Cefola previously and is presented here for completeness[9]. The next section (IV C.) will present a new method of analysis using these coefficients.

Sensitivity coefficients give the ratio of a particular state variable deviation (as a function of altitude) to deviations in the reference parameters of the system (at a particular altitude). For an exact definition of the sensitivity coefficients, consider a dynamic system described by the following mathematical model[8]:

$$F(\ddot{x},\dot{x},x,t,q_0) = 0 \qquad (4\text{-}1)$$

where q_o is the parameter of interest. The sensitivity coefficient is defined by

$$u(t,q_o) = \lim_{\Delta q \to o} \frac{x(t,q_o+\Delta q) - x(t,q_o)}{\Delta q} \qquad (4\text{-}2)$$

or

$$u(t,q_o) = \frac{\partial}{\partial q_o} x(t,q_o)$$

where $x(t,q_o+\Delta q)$ is the perturbed solution that results when $(q_o+\Delta q)$ is substituted for q_o in the original mathematical model. The perturbed value of x can be approximated at time t by

$$x(t,q_o+\Delta q) = x(t,q_o) + u(t,q_o)\Delta q \qquad (4\text{-}3)$$

Therefore the error in x at time t can be estimated by knowing $u(t,q_o)$ and Δq. A sensitivity equation may be obtained by taking the partial derivative of the system model equation with respect to q_o. The result is

$$\frac{\partial F}{\partial \ddot{x}} \ddot{u} + \frac{\partial F}{\partial \dot{x}} \dot{u} + \frac{\partial F}{\partial x} u = - \frac{\partial F}{\partial q_o} \qquad (4\text{-}4)$$

Through simultaneous numerical integration of equations (4-1) and (4-4), $u(t,q_o)$ can be found, and the perturbed value $x(t,q_o+\Delta q)$ may be evaluated.

Now consider the dynamical equations for planetary entry (equation A-22). For purposes of writing these equations, the following notation is used:

$$\underline{z}^* = [z_1, z_2, z_3] = [\Theta, v, \Omega] \qquad (4\text{-}5a)$$

and

$$\underline{\eta}^* = [\eta_1, \eta_2, \eta_3, \eta_4] = [a, b, c, \frac{A}{A_o}] \qquad (4\text{-}5b)$$

where * indicates the transposed vector.

Then we obtain the state equation as:

$$\frac{dz_i}{dx} = F(z, \underline{\eta}, x) \qquad i = 1, 2, 3 \qquad (4\text{-}5c)$$

Before proceeding to the mathematical definitions of the parameter sensitivity function, it is necessary to describe the reference and perturbed trajectories used in that definition. The reference state variables will be given as $\hat{z}_i(x) = z_i(x, \hat{\underline{\eta}}, x_E)$ where the normalized altitude x is the independent variable and x_E is the entry altitude. This function results from solving the trajectory dynamics with the reference values of the system parameters $\hat{\underline{\eta}}$, and constant initial conditions $\hat{z}_i(x_E)$. The perturbed trajectory is given by $z_i(x) = z_i(x, \hat{\underline{\eta}}_\ell, \hat{\eta}_j + \Delta\eta_j, x_E)$ where $\Delta\eta_j$ is the perturbed parameter and $\hat{\underline{\eta}}_\ell$ is a vector of the reference parameters without the $\hat{\eta}_j$ term. This perturbed trajectory can be obtained by substituting $\hat{\eta}_j + \Delta\hat{\eta}_j$ into the equations of motion and using the definition of the initial conditions

$$z_i(x_E) = \hat{z}_i(x_E) \qquad (4\text{-}6)$$

for all $\Delta\eta_j$. Physically this means that reference and perturbed state variables are equal at the entry altitude (before any effects of the atmosphere are felt).

Therefore, we can now define the sensitivity function by:

$$u_{ij}(x,x_E) = \lim_{\Delta\eta_j \to 0} \frac{z_i(x,\underline{\hat{\eta}}_\ell,\hat{\eta}_j+\Delta\eta_j,\ x_E) - z_i(x,\underline{\hat{\eta}},x_E)}{\Delta\eta_j}$$

$$= \frac{\partial \hat{z}_i\ (x,\underline{\hat{\eta}},x_E)}{\partial\eta_j} \qquad \begin{matrix} i = 1,2,3 \\ j = 1,2,3,4 \end{matrix} \tag{4-7}$$

The perturbed state variable may be then approximated at altitude x by:

$$z_i(x) = \hat{z}_i(x) + \hat{u}_{ij}(x,x_E)\Delta\eta_j(x_E) \tag{4-8}$$

The quantity $\Delta\eta_j(x_E)$ is a step function applied at $x=x_E$. Equation (4-8) shows that the effects of parameter deviations on the state variables at altitude x can be estimated (at x_E) once the sensitivity coefficients are known.

The sensitivity equations are obtained by taking the partial derivative of equation (4-5c) with respect to $\hat{\eta}_j$. This results in

$$\frac{d}{dx}u_{1j}(x,x_E) = \frac{\partial F_1}{\partial z_1}u_{1j}(x,x_E) + \frac{\partial F_1}{\partial z_2}u_{2j}(x,x_E) \tag{4-9}$$

$$\frac{d}{dx}\hat{u}_{2j}(x,x_E) = \frac{\partial F_2}{\partial z_1}u_{1j}(x,x_E) + \frac{\partial F_2}{\partial z_2}u_{2j}(x,x_E) + \frac{\partial F_2}{\partial\hat{\eta}_j}$$

$$\frac{d}{dx}u_{3j}(x,x_E) = \frac{\partial F_3}{\partial z_1}\hat{u}_{1j}(x,x_E)$$

where $j = 1,2,3,4$. F_1, F_2, and F_3 are given by equation (A-22), and the partial derivatives $\frac{\partial F_k}{\partial F_j}$ z_i are evaluated along the reference trajectory. The initial conditions for $u_{ij}(x,\underline{\hat{n}},x_E)$ come directly from equations (4-6 and (4-7).

$$u_{ij}(x_E,x_E) = \frac{\partial z_i(x_E)}{\partial \hat{n}_j} = 0 \tag{4-10}$$

Using vector and matrix notation, the sensitivity equations can be written as

$$\frac{d}{dx}\underline{\hat{u}}_j(x,x_E) = A(x)\,\underline{\hat{u}}_j(x,x_E) + \underline{h}_j(x) \tag{4-11}$$

$$\underline{\hat{u}}_j(x_E,x_E) = [0]$$

where

$$\underline{\hat{u}}_j = \begin{bmatrix} \hat{u}_{1j} \\ \hat{u}_{2j} \\ \hat{u}_{3j} \end{bmatrix} \qquad \underline{h}_j = \begin{bmatrix} 0 \\ \frac{\partial F}{\partial \hat{n}_j} \\ 0 \end{bmatrix} \tag{4-12}$$

and

$$A(x) = \begin{bmatrix} \frac{\partial F_1}{\partial z_1} & \frac{\partial F_1}{\partial z_2} & 0 \\ \frac{\partial F_2}{\partial z_1} & \frac{\partial F_2}{\partial z_2} & 0 \\ \frac{\partial F_3}{\partial z_1} & 0 & 0 \end{bmatrix}$$

If the index j is allowed to take the values $1,2,3,4$ in equation (4-11), there are 12 sensitivity equations. These together with the three equations of motion are solved simultaneously. Then the resulting sensitivity functions will describe all possible effects of system parameter changes on the state variables. Values of the sensitivity functions at the terminal altitude x_T, $\hat{u}_{ij}(x_T, x_E)$ can be used to estimate the terminal error as a function of the entry conditions.

However, we would like to find the sensitivity coefficients as a function of present altitude rather than at the entry altitude. Suppose we define the present normalized altitude as s where $(x_E \leq s \leq x \leq x_T)$. Suppose we replace x_E by s in equation (4-11). This resets the sensitivities to zero at altitude s. Mathematically it means we are dealing with a new perturbed solution which coincides with the reference trajectory at altitude s (instead of at altitude x_E). The sensitivity functions describing the new perturbed solution are given by

$$\frac{d}{dx}\underline{\hat{u}}_j(x,s) = A(x)\,\underline{\hat{u}}_j(x,s) + \underline{h}_j(x)$$

$$\underline{\hat{u}}(s,s) = 0 \tag{4-13}$$

It is noted that $\underline{h}_j(x)$ is not a function of s[9]. The method of predicting the state variables at x_T is similar to that method already shown with one important exception. Before the assumption was made that the reference and perturbed trajectories were identical. However at a lower altitude s, the effects of state variable deviations at that altitude must also be included. This will be discussed later.

Now our sensitivity coefficients are being solved as a function of x. Clearly, we are only interested in the values at the terminal altitude x_T. The variable of interest is the present altitude s. We would like new sensitivity equations whose solution give analytical descriptions of the sensitivities at the terminal altitude as a function of the present altitude s $[\underline{\hat{u}}_j(x_T,s)]$. To do this, equation (4-13) is solved using the state transition matrix $\phi(x,s)$. This matrix is governed [10] by

$$\frac{d}{dx}\,\phi(x,s) = A(x)\;\phi(x,s)$$
$$\phi(s,s) = I \tag{4-14}$$

We also need the adjoint transition matrix described by

$$\frac{d}{dx}\,\psi^*(x_T,x) = -\,\psi^*(x_T,x)\;A(x)$$
$$\psi^*(x_T,x_T) = I \tag{4-15}$$

From these, the following identity is obtained[9]

$$\psi^*(x_T,s) = \phi(x_T,s) \tag{4-16}$$

Now the solution has the following integral representation:

$$\underline{\hat{u}}_j(x,s) = \int_s^x \phi(x,\lambda)\;\underline{h}_j(\lambda)\;d\lambda \tag{4-17}$$

since $\quad \underline{\hat{u}}_j(s,s) = [0]$

setting $x = x_T$ in (4-17) yields

$$\underline{u}_j(x_T,s) = \int_s^{x_T} \phi(x_T,\lambda)\;\underline{h}_j(\lambda)\;d\lambda \tag{4-18}$$

Substituting equation (4-16) into (4-18) and differentiating the results with respect to s gives

$$\frac{d}{ds}\underline{\hat{u}}_j(x_T,s) = -\psi^*(x_T,s)\ \underline{h}_j(s) \qquad (4\text{-}19)$$

$$x_E \leq s \leq x_T$$

where $\underline{\hat{u}}_j(x_T,x_E)$ is the initial conditions for (4-19). Thus we may pre-compute and store $\phi(x_T,x_E)$ and $\underline{\hat{u}}_j(x_T,x_E)$ and then solve the following (similar to 4-15)

$$\frac{d}{ds}\psi^*(x_T,s) = -\psi^*(x_T,s)\ A(s) \qquad (4\text{-}20)$$

$$\psi(x_T,x_E) = \phi(x_T,x_E)$$

Therefore, equations (4-19)and (4-20) are solved simultaneously with the equations of motion to obtain the proper sensitivity coefficients for updated parameter information at altitude s.

Now to completely predict the terminal state variable deviation, the effects of present state variable deviation must also be included. The state variable sensitivity coefficients are defined as

$$\hat{w}_{ij}(x,s) = \frac{\partial z_i(x,s)}{\partial z_j(s)} \qquad (4\text{-}21)$$

ν

Again using vector notation

$$\underline{\hat{w}}_j(x,s) = [\hat{w}_{1j}(x,s),\ \hat{w}_{2j}(x,s),\ \hat{w}_{3j}(x,s)] \qquad (4\text{-}22)$$

The state variable sensitivity equations

$$\frac{d}{dx}\underline{\hat{w}}_j(x,s) = A(x)\ \underline{\hat{w}}_j(x,s) \qquad (4\text{-}23)$$

are obtained by the same process used to obtain the parameter sensitivity equations. The system matrix $A(x)$ is the same as that given in equation (4-12). The initial conditions are given by

$$\underline{\hat{w}}_j(s,s) = \underline{d}_j \tag{4-24}$$

where

$$\underline{d}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{d}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \underline{d}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4-25}$$

the solution of equation (4-23) subject to (4-24) is

$$\underline{\hat{w}}_j(x,s) = \phi(x,s)\ \underline{d}_j \tag{4-26}$$

where $\phi(x,s)$ is given by (4-14). Using equation (4-16) and setting $x = x_T$ we obtain

$$\underline{\hat{w}}_j(x_T,s) = \psi^*(x_T,s)\ \underline{d}_j \tag{4-27}$$

We now have all the sensitivity coefficients necessary to completely describe the terminal state variable deviations subject to the flight conditions at the present altitude.

C. Computation of Step Controls in the Flight Angle

The terminal range error based on the flight conditions at the present altitude may be estimated by

$$\begin{aligned} \Omega(x_T) - \hat{\Omega}(x_T) = {} & \hat{u}_{31}(x_T,s)[a(s)-\hat{a}] + \hat{u}_{32}(x_T,s)[b(s)-\hat{b}] \\ & + \hat{u}_{33}(x_T,s)[c(s)-\hat{c}] + \hat{u}_{34}(x_T,s)[\tfrac{A}{A_o}(s)-1] \\ & + \hat{w}_{32}(x_T,s)[v(s)-\hat{v}(s)] + \hat{w}_{33}(x_T,s)[\Omega(s)-\hat{\Omega}(s)] \\ & + \hat{w}_{31}(x_T,s)\ \Delta\Theta(s) \end{aligned} \tag{4-28}$$

The $\hat{u}_{3_j}$ and the $\hat{w}_{3_j}$ are the parameter and the state variable sensitivity coefficients, respectively, as defined in the previous section. The a(s), b(s), and c(s) are the updated values of the atmospheric parameters at altitude s^4. $\frac{A}{A_o}(s)$ corresponds to the ballistic coefficient change. $\Omega(s)$ and $v(s)$ are the measured values of the actual range angle and normalized velocity, while $\hat{\Omega}(s)$ and $\hat{v}(s)$ are the reference values.

Our problem is concerned with the values of the variable $\Delta\Theta(s)$. If $\Delta\Theta(s)$ is defined as the difference between the actual and the reference values of the flight path angle, equation (4-28) will give the uncontrolled terminal range error. However, consider $\Delta\Theta(s)$ defined as

$$\Delta\Theta(s) = \Theta(s) + \Delta\Theta_c(s) - \hat{\Theta}(s) \tag{4-29}$$

where $\Theta(s)$ is the measured actual value of the flight path angle, and $\hat{\Theta}(s)$ is the known reference value. We define $\Delta\Theta_c(s)$ as the discrete change necessary to reduce the terminal error (see fig. 4-1). Therefore, $[\Theta(s) + \Delta\Theta_c(s)]$ represents what the value of the flight angle in the actual atmosphere should be so that the range error is zero. The value of $\Delta\Theta_c(s)$ which results in zero range error can now be found by substituting equation (4-29) into (4-28) and by setting equation (4-28) equal to zero.

The increment of velocity, which must be added perpendicular to the vehicle's velocity vector for a path change $\Delta\Theta_c(s)$, can be found through geometrical considerations. It is related to the vehicle's velocity by the tangent of the angular change (see fig. 4-1).

$$\tan [\Delta\Theta_c(s)] = \frac{V_c(s)}{V(s)} \tag{4-30}$$

By making a small angle approximation we get

$$V_c(s) = V(s)\ \Delta\Theta_c(s) \tag{4-31}$$

A discussion of the impulsive force and weight of fuel necessary for a discrete change $\Delta\Theta_c(s)$ in the flight path angle is included in Appendix C.

D. Numerical Example

We now consider a numerical example to illustrate the usefulness of the discrete change in the flight path angle as a trim to decrease the terminal range error. In this example, a VM-8 model is taken as the reference atmosphere and a VM-4 model is taken as the actual atmosphere.

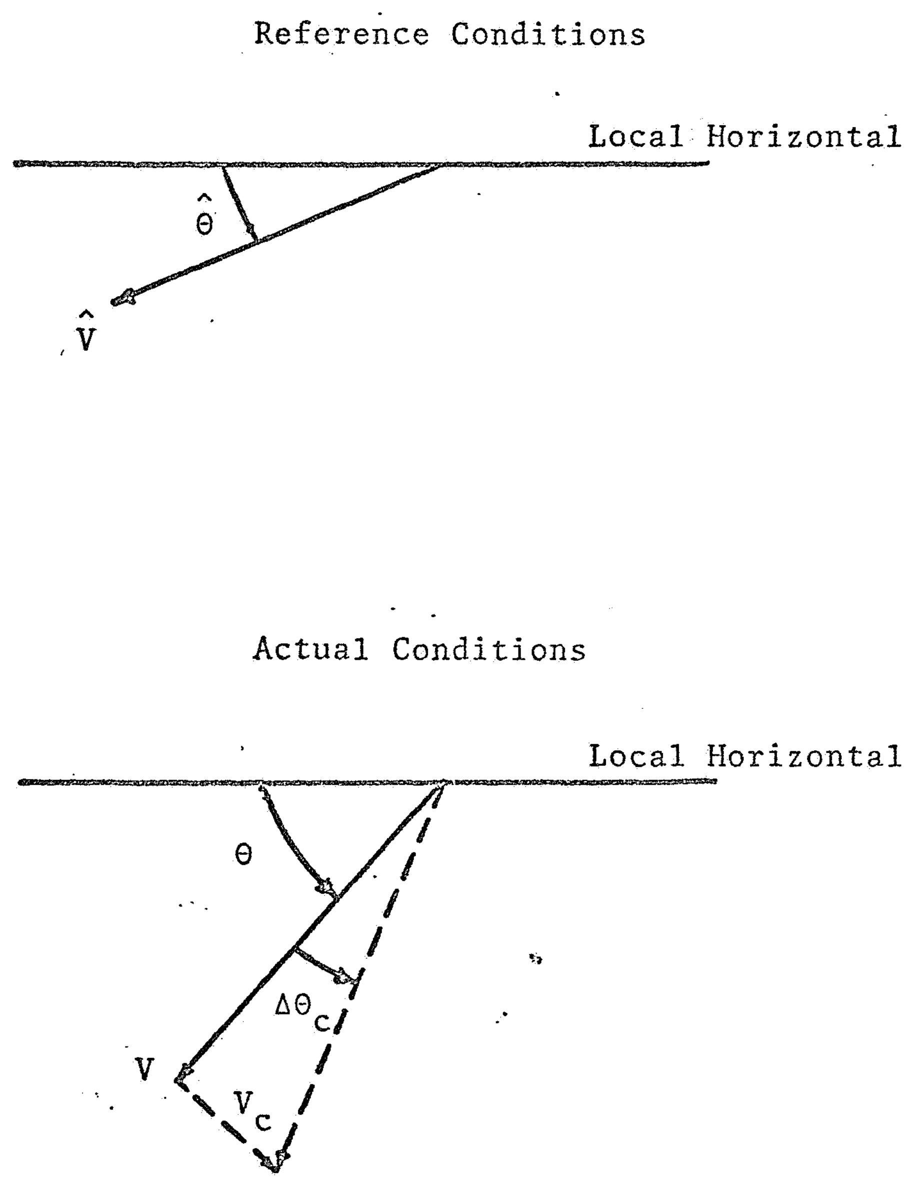

Reference Conditions
Local Horizontal
$\hat{\Theta}$
$\hat{V}$
Actual Conditions
Local Horizontal
Θ
$\Delta\Theta_c$
V
V_c

Figure 4-1

The ballistic coefficient for this example is taken to be .15 slugs/ft^2 (half the NASA reference value). The drag surface area must then be halved at an altitude of approximately ninty thousand feet if the range error is to be made zero (see fig. 3-13). For this simulation, we assumed exact traking of the actual density parameters. However, the change in the ballistic coefficient is made at a slightly lower altitude to induce an error in the terminal flight range.

The numerical data for the reference trajectory was taken as

$$
\begin{aligned}
Y_E &= 89{,}400 \text{ ft.} \\
Y_T &= 20{,}000 \text{ ft.} \\
\hat{\theta}_E &= .15287 \text{ radians} \\
\hat{V}_E &= 12{,}300 \text{ ft/sec} \\
\hat{\Omega}_E &= 0.0 \text{ radians} \\
\hat{a} &= -10.5729 \\
\hat{b} &= 2.7027 \\
\hat{c} &= -.501556 \\
N &= 1.0 \\
h &= 61{,}000 \text{ ft.} \\
g_o &= 12.3 \text{ ft/sec}^2 \\
R_o &= 10.86 \times 10^6 \text{ ft.} \\
\frac{m}{C_D A_o} &= .15 \text{ slugs/ft}^2
\end{aligned}
$$

Using these values, the quantities $\hat{\underline{u}}_j(x_T,x_E)$ and $\phi(x_T,x_E)$

were found by simultaneously solving the differential equations (3-2), (4-13), and (4-14). The following values were obtained

$$\hat{\underline{u}}_1(x_T,x_E) = \begin{bmatrix} 0.6709 \\ -0.02587 \\ -0.009218 \end{bmatrix}$$

$$\hat{\underline{u}}_2(x_T,x_E) = \begin{bmatrix} -0.3166 \\ 0.006299 \\ 0.007126 \end{bmatrix}$$

$$\hat{\underline{u}}_3(x_T,x_E) = \begin{bmatrix} 2.297 \\ -0.03674 \\ -0.06105 \end{bmatrix}$$

$$\hat{\underline{u}}_4(x_T,x_E) = \begin{bmatrix} 0.6709 \\ -0.02587 \\ -0.009215 \end{bmatrix}$$

$$\phi\ (x_T,x_E) = \begin{bmatrix} -1.032 & -0.03100 & 0.0 \\ -0.0009976 & -0.0008936 & 0.0 \\ -0.08010 & 0.005639 & 1.0 \end{bmatrix}$$

Now using these values as the initial conditions in the updating equations, we solve for $\hat{\underline{u}}_j(x_T,s)$ and $\psi^*(x_T,s)$ as functions of the present altitude s. The actual controlled VM-4 trajectory was simulated with the following data

Y_E = 89,400 ft.

Y_T = 20,000 ft.

Θ_E = .15287 radians

V_E = 12,300 ft/sec

Ω_E = 0.0 radians

$$a = -9.9075$$

$$b = 2.3255$$

$$c = -.54392$$

$$N = 1.0$$

$$h = 61{,}000 \text{ ft.}$$

$$g_o = 12.3 \text{ ft/sec}^2$$

$$R_o = 10.86 \times 10^6 \text{ ft.}$$

$$\frac{m}{C_D A_o} = .30 \text{ slugs/ft}^2$$

and the actual state variables were from obtained from this simulation.

The flight angle was discretely changed four times along the actual trajectory. Table 4-1 shows the results for the updated control. The case of both bounded and unbounded values of $\Delta\Theta_c$ was considered. Table 4-2 shows the terminal range error for the actual trajectories with no control, with ballistic coefficient change only, and with bounded and unbounded trim.

V. DISCUSSION OF RESULTS AND CONCLUSIONS

It has been shown that a guidance scheme which uses a discrete change in the ballistic coefficient and a sequence of small discrete changes in the flight path angle as the controls is able to compensate for a certain range of atmospheric parameter deviations. From a good estimate of the atmospheric composition, a reference trajectory can be chosen which has the required terminal conditions. Then based upon small deviations in the reference parameters which describe the atmosphere, the ballistic coefficient can be altered so that the terminal constraints are still satisfied. Then if inaccurate estimates of the actual atmospheric parameters exist at the altitude of change in the drag surface, the terminal errors which result may be corrected by small step changes in the flight path angle of the vehicle.

If we consider the final velocity constraint as an inequality, it is only necessary to keep the ballistic coefficient less than .30 slugs/ft^2 to insure that the constraint is satisfied. The changing of the drag surface area need only be employed to make the capsule reach a reference range independent of the actual atmosphere which exists. The size of the change necessary is based on the estimates of the possible atmospheric parameter deviations before the mission. Then with a knowledge of the actual atmosphere obtained during entry, it is possible to find an altitude at which changing the drag surface will bring the craft to the reference range at twenty thousand feet.

If the composition of the actual atmosphere is similar to that of the reference atmosphere,there exists an altitude at which changing the drag surface by a factor of two will result in zero range error. However if the actual atmospheric composition differs greatly from the reference atmosphere, the reference range can only be reached by a change in the ballistic coefficient which is much greater than a factor of two. The range for unpowered aerodynamic entry is extremely sensitive to the composition of the atmosphere. Therefore, a drag control scheme seems advisable only if a reasonable estimate of the atmospheric composition is know before the mission.

It has also been shown that the sequence of step changes in the flight path angle is very effective in reducing small errors due to changing the ballistic coefficient at the wrong altitude. However, it seems important to compare the case for unbounded values of $\Delta\Theta_c$ and then for bounded values. The range error using four corrections of unbounded magnitude is reduced from -6000 yards to +90 yards. However the velocity changes $[V'_c(s)]$ necessary at each altitude are quite large for a vehicle which will weigh approximately five thousand pounds. They range from 282 feet per second for the first correction to 110 feet per second for the fourth correction (see table 4-1). The magnitude of a thrust of one second duration required to achieve these velocities [$V_c(s)$] range from fifty thousand pounds for the first correction to seventeen thousand pounds for the fourth correction. These forces are extremely large and the entire scheme requires approximately three hundred pounds of fuel for an average chemical engine (see Appendix C).

A bounded step change seems to be a much better solution from both the numerical example and physical limitations. The range error for bounded control is larger than that for unbounded control. However, the error for a bound of one hundredth of a radian is still less than one mile (1300 yards). More important is the fact that the magnitude of the velocity changes [$V_c(s)$] are very small, especially when compared to the V_c's necessary for unbounded control. For a bound of .01 radians, the velocities range from 85 feet per second to 8 feet per second. The thrust of one second duration required for a bound of .01 radians ranges only from fourteen thousand pounds for the first correction to eleven hundred pounds for the fourth correction. These forces are much more reasonable and using this bound only requires sixty pounds of fuel for all four corrections.

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APPENDIX A

Equations of Motion:

Consider two dimensional entry into a non-rotating atmosphere of a spherical planet. The geometry is given by figure A-1. The unit vector parallel to position vector $\underline{e}_r$ can be described in the velocity coordinate frame by

$$\underline{e}_r = \cos\Theta\ \underline{e}_N - \sin\Theta\ e_T \qquad \text{(A-1)}$$

the vehicle position vector $\underline{r}$ can be expressed as

$$\underline{r} = r\ \underline{e}_r \qquad \text{(A-2)}$$

or substituting (A-1) into (A-2)

$$\underline{r} = r\cos\Theta\ \underline{e}_N - r\sin\Theta\ \underline{e}_T \qquad \text{(A-3)}$$

an expression for the velocity in the $\underline{e}_T$, $\underline{e}_N$ system is found by differentiating (A-3) with respect to time:

$$\begin{aligned}\dot{\underline{r}} = \frac{d}{dt}(r\cos\Theta)\ \underline{e}_N + r\cos\Theta\ \dot{\underline{e}}_N - \frac{d}{dt}(r\sin\Theta)\ \underline{e}_T \\ - r\sin\Theta\ \dot{\underline{e}}_T\end{aligned} \qquad \text{(A-4)}$$

But the angular motion of the $\underline{e}_T$, $\underline{e}_N$ system with respect to the fixed reference plane is given by

$$\begin{aligned}\dot{\underline{e}}_T &= -\ (\dot{\Omega}+\dot{\Theta})\ \underline{e}_N \\ \dot{\underline{e}}_N &= (\dot{\Omega}+\dot{\Theta})\ \underline{e}_T\end{aligned} \qquad \text{(A-5)}$$

substituting (A-5) into (A-4) gives

$$\dot{\underline{r}} = [r\dot{\Omega}\sin\Theta + \dot{r}\cos\Theta]\,\underline{e}_N + [r\dot{\Omega}\cos\Theta - \dot{r}\sin\Theta]\,\underline{e}_T \qquad \text{(A-6)}$$

However since $\underline{e}_T$ is defined to be in the direction of the velocity vector

$$\dot{\underline{r}} = \underline{V} = V\,\underline{e}_T \qquad \text{(A-7)}$$

By combining (A-6) and (A-7), the following two scalar equations are obtained:

$$\begin{aligned} V &= r\dot{\Omega}\cos\Theta - \dot{r}\sin\Theta \\ 0 &= r\dot{\Omega}\sin\Theta + \dot{r}\cos\Theta \end{aligned} \qquad \text{(A-8)}$$

The solution to (A-8) for $\dot{\Omega}$ and $\dot{r}$ gives the following geometric results:

$$\dot{\Omega} = \frac{V\cos\Theta}{r} \qquad \text{(A-9)}$$

$$\dot{r} = -V\sin\Theta \qquad \text{(A-10)}$$

For the equations of motion, the vehicle acceleration is needed. By differentiating equation (A-7) with respect to time, the expression for the acceleration is

$$\ddot{\underline{r}} = \dot{V}\,\underline{e}_T + V\,\dot{\underline{e}}_T \qquad \text{(A-11)}$$

Substitution of (A-5) and (A-9) into (A-11) gives

$$\ddot{\underline{r}} = \dot{V}\,\underline{e}_T - V[\dot{\Theta} + \frac{V\cos\Theta}{r}]\,\underline{e}_N \qquad \text{(A-12)}$$

Next, the external forces acting on the entry vehicle are considered. These are drag, lift and gravity.

The external force may be expressed by

$$F = - mg \, \underline{e}_r + L \, \underline{e}_N - D \, \underline{e}_T \tag{A-13}$$

where m = the mass of the entry vehicle

g = gravitational acceleration

L = lift force

D = drag force

By substituting equation (A-1) into (A-13), we get

$$F = (L - mg \cos\Theta) \, \underline{e}_N + (mg \sin\Theta - D) \, \underline{e}_T \tag{A-14}$$

Now by use of Newtons 2nd Law, we obtain the dynamical equation of motion:

$$\dot{V} \, \underline{e}_T - V(\dot{\Theta} + \frac{V \cos\Theta}{r}) \underline{e}_N = (g \sin\Theta - \frac{D}{m}) \, \underline{e}_T + (\frac{L}{m} - g \cos\Theta) \, \underline{e}_N \tag{A-15}$$

the resulting scalar equations are

$$\dot{V} = g \sin\Theta - \frac{D}{m} \tag{A-16}$$

and

$$V\dot{\Theta} = (g - \frac{V}{r}) \cos\Theta - \frac{L}{m} \tag{A-17}$$

If R_o is the planet's radius and y is the altitude of the entry capsule above the planets surface, the relation between r and y is

$$r = R_o + y \tag{A-18}$$

the drag can be approximated by

$$D = \frac{1}{2} \rho(y) \, V^2 (C_D A_o) \, (\frac{A}{A_o}) \tag{A-19}$$

where ρ = atmospheric density

C_D = drag coefficient

A_o = reference area

A = actual area

substituting (A-18) and (A-19) into (A-9), (A-10), (A-16), and (A-17) leads to the time domain equations of motion:

$$\dot{y} = -V \sin\theta$$

$$\dot{\theta} = \frac{1}{V}\left(g - \frac{V^2}{R_o+y}\right)\cos\theta - \frac{1}{2}\frac{C_L}{C_D}(y)\ V\left(\frac{C_D A_o}{m}\right)\left(\frac{A}{A_o}\right)$$

$$\dot{V} = g\sin\theta - \frac{1}{2}\rho(y)\ V^2\left(\frac{C_D A_o}{m}\right)\left(\frac{A}{A_o}\right) \qquad \text{(A-20)}$$

$$\dot{\Omega} = \frac{V\cos\theta}{R_o+y}$$

Now to make the equations dimensionless we make the following change of variables

$$x = N\left(\frac{h-y}{h}\right)$$

and

$$v = \frac{V}{\sqrt{g_o R_o}} \qquad \text{(A-21)}$$

where x = normalized altitude

v = " velocity

h = reference altitude

N = scale factor

g_o = surface gravitational acceleration

For purposes of analysis, ballistic entry without lift is assumed. Now equation (A-20) expresses the dynamics of entry with time as the independent variable. However, the variable of interest is altitude. Therefore making the change of variables (A-21) and expressing altitude as the independent variable, the equations of motion become

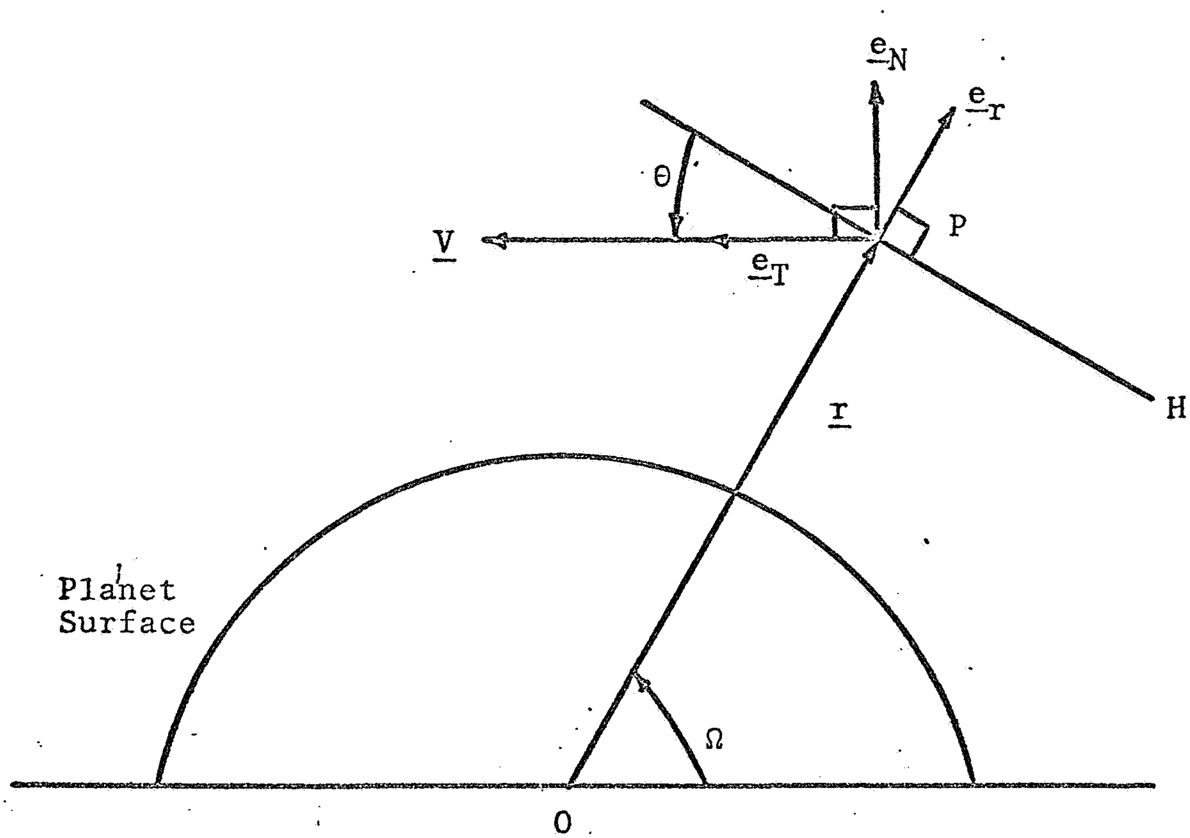
$$\frac{d\Theta}{dx} = \frac{h}{N}\frac{1}{g_oR_o}\frac{1}{v^2\sin\Theta}\left[g(x) - \frac{v^2 g_o R_o}{R_o + h(1-x/N)}\right]\cos\Theta$$

$$= F_1(x,\Theta,v)$$

$$\frac{dv}{dx} = \frac{h}{N}\frac{1}{g_oR_o}\frac{1}{v^2\sin\Theta}\left[g(x)\sin\Theta - \frac{1}{2}\rho(x)g_oR_ov^2\left(\frac{C_DA_o}{m}\right)\left(\frac{A}{A_o}\right)\right]$$

$$= F_2\left(x,\Theta,v,\rho,\frac{A}{A_o}\right)$$

$$\frac{d\Omega}{dx} = \frac{h}{N}\frac{\cot\Theta}{R_o + h(1-x/N)} = F_3(x,\Theta) \qquad \text{(A-22)}$$



where O = Center of Planet

P = Position of Vehicle

PH= Local Horizontal

Ω = Range Angle

$\underline{V}$ = Velocity

θ = Flight Path Angle

$\underline{r}$ = Position Vector

$\underline{e}_T$= Unit Vector in the Dirction of Velocity

$\underline{e}_N$= Unit Vector Perpendicular to $\underline{e}_T$

Figure A-1

APPENDIX B

Martian Atmospheric Density Models

At present there are two density models for Mars being used by NASA[2]. These are[2] an isothermal density model

$$\rho = \rho_{ref} e^{-\beta y} \qquad y > \text{height of tropopause} \qquad \text{(B-1)}$$

and an adiabatic density model

$$\rho = \rho_o \left(1 + \frac{\Gamma}{T_o} y\right)^{\frac{1}{\gamma - 1}} \qquad y \quad \text{height of tropopause} \qquad \text{(B-2)}$$

where ρ_o = surface density (slugs/ft³)

T_o = surface temperatures (^{o}R)

β = inverse scale height (1/ft)

Γ = a diabatic lapse rate (^{o}R/ft)

γ = ratio of specific heats

y = height above surface (ft)

If the density is assumed to be a continuous function of altitude, then it can be shown that

$$\rho_{ref} = \rho_o e^{\beta H_{trop}} \left(1 + \frac{\Gamma}{T_o} H_{trop}\right)^{\frac{1}{\gamma - 1}} \qquad \text{(B-3)}$$

where H_{trop} equals the altitude of the tropopause

The adiabatic density model defined the atmosphere over the main part of aerodynamic breaking. The uncertainty in the present definition of the atmosphere is indicated by the allowable range of the adiabatic parameters

$$1.32(10)^{-5} < \rho_o < 4.98(10)^{-5}$$
$$-.00321 < \Gamma < -.00213$$
$$360 < T_o < 495$$
$$1.37 < \gamma < 1.43$$
$$.0043 < \frac{\Gamma}{T_o} < .0089$$

To obtain the modified adiabatic density equation expressed by three parameters, let us use x as our independent variable where x is defined as

$$x = \left(\frac{h-y}{h}\right) N \qquad \text{(B-4)}$$

and h = reference altitude

N = scale factor

Solving equation (B-4) for y and substituting into (B-2) gives

$$\rho(x) = \rho_o\left[1+\frac{\Gamma}{T_o} h\left(1-\frac{x}{N}\right)\right]^{\frac{1}{\gamma-1}} \qquad \text{(B-5)}$$

Now letting

$$a = \ln\rho_o \qquad \text{(B-6)}$$

$$b = \frac{1}{\gamma-1} \qquad \text{(B-7)}$$

$$c = \frac{\Gamma h}{T_o} \qquad \text{(B-8)}$$

equation (B-5) becomes

$$\rho(x) = e^a\left[1 + c\left(1-\frac{x}{N}\right)\right]^b \qquad \text{(B-9)}$$

Therefore the atmospheric models used for this report are expressed as

$$\rho=\rho_{ref}e^{-\beta y} \qquad y > \text{height of tropopause} \tag{B-10}$$

and

$$\rho=e^{a}[1+c(1-\frac{x}{N})]^{b} \qquad x < \text{normalize height of tropopause}$$

The values of the atmospheric parameters for the models used in this report are given in the following table:

	Mars Atmospheric Parameters h=61,000 ft.,N=1.0				
VM	a	b	c	β	ρ_{ref}
2	-10.2348	2.7027	-.50156	6.07×10^{-5}	2.23×10^{-4}
4	- 9.9075	2.3256	-.54392	5.89×10^{-5}	2.70×10^{-4}
8	-10.5729	2.7027	-.50156	6.07×10^{-5}	1.58×10^{-4}
1	-10.8977	2.6316	-.26248	2.15×10^{-5}	3.12×10^{-5}
3	-10.5384	2.6316	-.26248	2.15×10^{-5}	4.48×10^{-5}
7	-11.2353	2.6316	-.26248	2.15×10^{-5}	2.23×10^{-5}

TABLE B-1

APPENDIX C

Characteristics of the Impulsive Force

To achieve a small discrete change in the flight path angle, an impulsive force will have to be applied to the vehicle. This force will be applied for a small interval of time, and its magnitude will be determined by $\Delta\Theta_c(s)$.

The magnitude of the velocity component applied perpendicular to the actual vehicle velocity to change the flight angle by $\Delta\Theta_c$ is given by equation (4-31)

$$V_c(s) = V(s)\,\Delta\Theta_c \tag{C-1}$$

From Newton's second law, we can relate velocity to the time integral of force.

$$\int F\,dt = mV_c \tag{C-2}$$

However, because of the impulsive nature of the force, the following approximation may be made:

$$\int F\,dt \approx F\Delta t \tag{C-3}$$

where Δt is the time interval over which the force F is applied. Hence an estimate of the magnitude of the force necessary to change the flight direction is given by

$$F = \frac{m_v\,V(s)\,\Delta\Theta_c(s)}{\Delta t} \tag{C-4}$$

where m_v is the mass of the vehicle.

It is also possible to obtain an estimate of the fuel needed to achieve an impulsive force of magnitude F. The specific impulse of a rocket is defined as the thrust per unit weight rate of flow[13].

$$I_{sp} = \frac{F}{\dot{m}_{av} g} \tag{C-5}$$

Since F is applied for time Δt, where Δt is small, we can approximate $\dot{m}_{av}$ by

$$\dot{m}_{av} \approx \frac{m_p}{\Delta t} \tag{C-6}$$

where m_p is the fuel used in time Δt. Substituting equation (C-5 and (C-6) into (C-4) we get

$$m_p g = \frac{F \Delta t}{I_{sp}} \tag{C-7}$$

Substituting equation (C-4) into (C-7) we get the weight of fuel necessary for flight path angle change $\Delta\Theta_c(s)$ as

$$m_p g = \frac{m_v V(s) \ \Delta\Theta_c(s)}{I_{sp}} \tag{C-8}$$

Table C-1 gives the magnitude of force and the fuel consumption for a typical chemical rocket that are needed for the bound and unbound values of $\Delta\Theta_c(s)$ given in table 4-1.

		ALTITUDE OF UPDATE $\Delta t = 1$ sec, $I_{sp} = 400$ sec			
TYPE OF CONTROL		76,000	62,000	48,000	34,000 ft
UNBOUNDED	F (lbs)	-50000	35000	30000	17000
	$m_p g$ (lbs)	110	85	75	40
BOUNDED	F (lbs)	-14000	-6500	2500	1100
$\lvert\Delta\Theta_c\rvert<.01$	$m_p g$ (lbs)	35	16	6	3
BOUNDED	F (lbs)	-20000	-9400	3500	1700
$\lvert\Delta\Theta_c\rvert<.015$	$m_p g$ (lbs)	50	25	9	4

TABLE C-1