The Effect of Damping, Time Delays

and Parameter Variations on the Stability

of a Nonconservative System[1]

by

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Abstract

The stability of the nonconservative double pendulum model of Ziegler is studied with emphasis placed on the effect of changes in the model parameters and on the presence of retardation of the angular position of the follower load. Both the cases of external and internal damping are considered. As in previous studies, the damping is assumed to be small.

Introduction

The stability of nonconservative elastic systems has been the subject of much recent interest. To demonstrate some of the properties of such systems several authors have investigated the behavior of a simple two-dimensional model consisting of an inverted double pendulum subjected to a follower type of load (Figure 1) Ziegler [1][2] discovered that the presence of linear viscous internal damping may tend to destabilize the system by lowering the value of the critical follower load. Herrmann and Jong [2,3] studied the behavior of this model in the case of small internal damping for a follower load and a partial follower load.

The results of these studies indicate that damping and the type of loading may have significant effects on the stability of the system and that considerable care must be exercised in the modeling of the damping mechanism; Herrmann and Jong have recommended further study of other damping models and loadings.

A type of loading which might occur in some systems and which is probably present in most experimental studies is that of a retarded load, that is, a load that acts in a prescribed manner in relation to the system, but which is delayed by the inherent time lag of the mechanism which produces it. Kiusalaas and Davis [4] investigated the stability of the double pendulum model subjected to a retarded follower force with a constant time lag for the case of no damping.

The purpose of this study is to examine the stability of the standard double pendulum model when subjected to a retarded follower load with small

[2] Numbers in brackets designate references at the end of the paper.

time lag but with the inclusion of slight damping, both internal and external. It is found that the presence of damping greatly alters the critical load of the system, an expected result. The effect of the time delay on the follower load is shown to be significant and, in most cases, destabilizing. Yet, for a few cases the time delay raises the critical load above that for the standard follower load, an unexpected result which is significant for the interpretation of experimental data.

A second aspect of this study is the determination of the effects of changes in the model parameters. Previous works have rather consistently studied the double pendulum with mass ratio $m_1/m_2 = 2$, length ratio $\ell_1/\ell_2 = 1$ and spring constant ratio $c_1/c_2 = 1$ (Figure 1). It is of interest, for design and analysis purposes, to have a knowledge of the effects of these ratios on the stability of the system.

The Equations of Motion

Consider the double pendulum of Figure 1, consisting of two rigid, weightless bars of lengths ℓ_1 and ℓ_2 which carry concentrated masses m_1 and m_2, respectively. The generalized coordinates $\phi_1(t)$ and $\phi_2(t)$, with t the time, specify the angles between the vertical and the two bars, and are assumed to remain sufficiently small. The linear elastic restoring moments $c_1\phi_1$ and $c_2(\phi_2-\phi_1)$ and the linear viscous internal damping moments $b_1\dot{\phi}_1$ and $b_2(\dot{\phi}_2-\dot{\phi}_1)$ are assumed to act at the two hinges, respectively, with b_1 and b_2 small but not both zero.

The load P is applied at the free end at an angle $\theta(t)$ with respect to the vertical, and it is assumed that

$$\theta(t) = \phi_2(t-\tau), \tag{1}$$

with $\tau \geqq 0$, a constant time lag; hence, the load acts at time t in such a manner as to be tangential to the position of the upper bar of the pendulum at time $t-\tau$. The special case $\tau = 0$ reduces P to the standard follower load. It is assumed that τ is small.

Consideration of Lagrange's equations in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}_i}\right) + \frac{\partial D}{\partial \dot{\phi}_i} - \frac{\partial T}{\partial \phi_i} + \frac{\partial V}{\partial \phi_i} = Q_i, \; i = 1,2 \tag{2}$$

with the kinetic energy, dissipation function, potential energy and generalized forces given by

$$\begin{aligned}
T &= \frac{1}{2} m_1 \ell_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 (\ell_1 \dot{\phi}_1 + \ell_2 \dot{\phi}_2)^2, \\
D &= \frac{1}{2} b_1 \dot{\phi}_1^2 + \frac{1}{2} b_2 (\dot{\phi}_2 - \dot{\phi}_1)^2, \\
V &= \frac{1}{2} c_1 \phi_1^2 + \frac{1}{2} c_2 (\phi_2 - \phi_1)^2, \\
Q_1 &= P \ell_1 (\phi_1 - \theta), \\
Q_2 &= P \ell_2 (\phi_2 - \theta),
\end{aligned} \tag{3}$$

immediately yields the two linearized equations of motion

$$(m_1+m_2)\ell_1^2\ddot{\phi}_1 + (b_1+b_2)\dot{\phi}_1 + (c_1+c_2-P\ell_1)\phi_1 + m_2\ell_1\ell_2\ddot{\phi}_2 +$$

$$-b_2\dot{\phi}_2 - c_2\phi_2 + P\ell_1\theta = 0,$$

$$(4)$$

$$m_2\ell_1\ell_2\ddot{\phi}_1 - b_2\dot{\phi}_1 - c_2\phi_1 + m_2\ell_2^2\ddot{\phi}_2 + b_2\dot{\phi}_2 +$$

$$+ (c_2-P\ell_2)\phi_2 + P\ell_2\theta = 0.$$

Given the linearity of the equations, assume a solution of the form

$$\phi_i(t) = A_i e^{\omega t}, \quad i = 1,2 \tag{5}$$

and recall that, from Equation (1),

$$\theta(t) = A_2 e^{-\omega\tau} e^{\omega t}. \tag{6}$$

Since the time delay τ is assumed to be small, let

$$\theta(t) \approx A_2(1-\omega\tau)e^{\omega t}. \tag{7}$$

Substitution of Equations (5) and (7) into (4) immediately yields the condition for existence of a nontrivial solution as

$$\begin{vmatrix} \mu^2(\delta+1)\Omega^2+(B_1+B_2)\Omega+\epsilon+1-\mu F & \mu\Omega^2-B_2\Omega-1+\mu F(1-\alpha\Omega) \\ \mu\Omega^2-B_2\Omega-1 & \Omega^2+B_2\Omega+1-F\alpha\Omega \end{vmatrix} = 0, \tag{8}$$

where the parameters of the system have been appropriately nondimensionalized and are defined by

$$F = \frac{P\ell_2}{c_2}; \quad \Omega = \ell_2\left(\frac{m_2}{c_2}\right)^{1/2}\omega; \quad \alpha = \frac{1}{\ell_2}\left(\frac{c_2}{m_2}\right)^{1/2}\tau;$$

$$B_i = \frac{b_i}{\ell_2(c_2 m_2)^{1/2}}, \quad i = 1,2, \quad ; \mu = \frac{\ell_1}{\ell_2}; \quad \delta = \frac{m_1}{m_2}, \quad \epsilon = \frac{c_1}{c_2}. \tag{9}$$

Expansion of the determinant (8) yields the <u>frequency equation</u> in the form

$$p_0\Omega^4 + p_1\Omega^3 + p_2\Omega^2 + p_3\Omega + p_4 = 0, \tag{10}$$

from which, through the familiar Routh-Hurwitz criteria, the conditions for asymptotic stability are obtained as

$$p_i > 0, \quad i = 0,\ldots,4$$
$$X \equiv (p_1 p_2 - p_0 p_3)p_3 - p_4 p_1^2 > 0. \tag{11}$$

<u>The Standard Model</u>

Consider now the case for which

$$\mu = 1, \ \delta = 2, \ \epsilon = 1 \tag{12}$$

which is the "standard" model discussed in References [1-4]. For this particular case the coefficients of the frequency equation take the form

$$p_0 = 2$$

$$p_1 = B_1 + 6B_2 - 2\alpha F$$

$$p_2 = 7 - 2F + B_1B_2 - \alpha B_1F - 2\alpha B_2F \qquad (13)$$

$$p_3 = B_1 + B_2 + \alpha F(F-3)$$

$$p_4 = 1$$

Under the assumption of slight damping, the case of physical interest, and of small time delays the three last terms in p_2 are neglected since they are higher order quantities. By assuming that $B_2 \neq 0$ and defining the ratios

$$\beta = \frac{B_1}{B_2} = \frac{b_1}{b_2}, \quad \gamma = \frac{\alpha}{B_2} = \frac{\tau c_2}{b_2}, \qquad (14)$$

the stability criteria (11) yield, for this standard model, the following four inequalities on the load parameter F:

$$F < \frac{\beta+6}{2\gamma},$$

$$F < \frac{7}{2},$$

$$F(3-F) < \frac{\beta+1}{\gamma}, \qquad (15)$$

$$2\gamma^2F^4 - \gamma(12+2\beta+14\gamma)F^3 + \gamma(78+13\beta+20\gamma)F^2 +$$

$$-[2\beta^2+14\beta+12+104\gamma+19\gamma\beta]F + 4\beta^2+33\beta+4 > 0.$$

These inequalities determine the critical load: the vertical position of the double pendulum is an asymptotically stable equilibrium state for loads below the critical load and an unstable equilibrium state for higher loads.

The values of these critical loads were obtained with the use of a digital computer and are plotted in Figures 2a and 2b. In Figure 2a the critical load is plotted versus the ratio of damping coefficients β for various fixed values of γ, the measure of the time delay. The case $\gamma = 0$ corresponds to the result obtained in [2]. It is clear that for high values of γ an increase in β always leads to an increase in the critical load; for low values of γ, however, the critical load increases up to a certain value of β and then decreases slightly thereafter: the optimum choice of β depends on the magnitude of γ (for $\gamma = 0$, the maximum critical load $F = 2.086$ occurs at $\beta = 11.071$ [2]). It is also noted that for some values of γ a sudden upward jump in the critical load takes place as β is increased beyond a certain value. Mathematically, this occurs when two of the real roots of the quartic (15) become complex conjugates.

The critical load is plotted versus the time delay parameter γ in Figure 2b for various fixed values of the damping ratio β. The presence of a time delay usually produces a destabilizing effect on the system, with a larger time delay producing a lower critical load. Since physically the important problem is that with small damping, it is clear that the delays will have to be small for any degree of stability; hence the reason for the assumption on the smallness of τ. Note, however, the surprising result that in the case $\beta = 5$ a small time delay stabilizes the system; at $\gamma = 2$, for example, the critical load is 4 per cent higher than for the standard follower load ($\gamma = 0$).

Effect of External Damping

Consider now the standard model of Figure 1 but with the internal damping of the previous section replaced by viscous external damping which acts on the masses m_1 and m_2 (but not on the bars) with coefficients b_1 and b_2, respectively. The dissipation function in (3) becomes

$$D = \frac{1}{2} b_1 \dot{\phi}_1^2 + \frac{1}{2} b_2 (\dot{\phi}_2 + \dot{\phi}_1)^2 \tag{16}$$

Following the same procedure as in the previous section, the frequency equation (10) is obtained with coefficients

$$\begin{aligned} p_0 &= 2, \\ p_1 &= B_1 + 2B_2 - 2\alpha F, \\ p_2 &= 7 - 2F + B_1 B_2 - \alpha B_1 F, \\ p_3 &= B_1 + 5B_2 - 2B_2 F + \alpha F(F-3), \\ p_4 &= 1, \end{aligned} \tag{17}$$

from which the stability criteria

$$\begin{aligned} &F < \frac{\beta+2}{2\gamma}, \\ &F < \frac{7}{2}, \\ &F(2+3\gamma-\gamma F) < \beta+5, \\ &2\gamma^2 F^4 - \gamma(4+2\beta+14\gamma)F^3 + (30\gamma+13\gamma\beta+20\gamma^2+4\beta)F^2 + \\ &-(2\beta^2+20\beta+8+44\gamma+19\gamma\beta)F + 4\beta^2 + 25\beta + 16 > 0, \end{aligned} \tag{18}$$

are obtained. The critical loads determined from these equations are depicted in Figure 3a, as plotted against β, and in Figure 3b, as plotted versus γ.

It is to be noted that in the case of external damping the presence of a time delay is always destabilizing: an increase in γ causes a decrease in the critical load (for fixed β). The behavior of the pendulum under changes in the parameter β again depends on γ. For $\gamma > 1/2$ the critical load increases with increasing β, but for $\gamma < 1/2$ the critical load first increases and for values beyond a certain value of β then decreases.

For a standard follower load ($\gamma = 0$), the maximum critical load occurs for $\beta = 2$, where it attains the value $F = 2.086$ (the same maximum value as for the case of internal damping). This value happens to be the value F_e [2] of the critical load of the model for zero damping. Hence, external damping on the double pendulum exhibits a "destabilizing effect" similar to that caused by internal damping: the addition of slight linear viscous external damping lowers the critical load below the value F_e (except for $\beta = 2$, when the critical load equals F_e).[3]

Effect of Parameter Changes

Thus far, only the "standard" model with $\delta = 2$, $\epsilon = 1$ and $\mu = 1$ has been considered. It is desirable to investigate the effect of the mass ratio δ, the spring constant ratio ϵ and the length ratio μ on the stability of the model with internal damping.

The procedure used for this purpose is identical to that used in the previous sections. The results of this analysis are shown in Figures 4 - 6. Figures 4a,b depict the critical load for various values of δ with

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[3] It must be noted that this "destabilizing effect" of external damping does not occur for a continuous cantilevered column subjected to a follower load [5].

$\epsilon = 1$ and $\mu = 1$; the results are shown as a function of β in Figure 4a with a constant time delay parameter $\gamma = 1$, and as a function of γ in Figure 4b for a fixed damping ratio $\beta = 1$. It is noted that the critical load increases with increasing mass ratio δ.

The effect of various spring constant ratios ϵ is shown in Figures 5a,b. It is observed that for $\beta > 1$ the critical load decreases with increasing ϵ, but that the opposite effect occurs for small values of β. Figures 6a,b depict the effect on the critical load of the length ratio μ for fixed values $\delta = 2$ and $\epsilon = 1$.

From these graphs it is noted that this model is extremely sensitive to parameter variations, and that therefore considerable care should be exercised in inferring conclusions about continuous elastic systems from results obtained from a simple two-degree-of-freedom model.

Concluding Remarks

It has been shown that the presence of load retardation, a phenomenon almost inevitable in experimental studies, has a significant effect on the stability of the Ziegler model, the effect being in general destabilizing. The importance of the nature of the damping mechanism, already noted by other authors, and the high sensitivity to parameter variations of the model have been studied and general quantitative results have been obtained.

Acknowledgment

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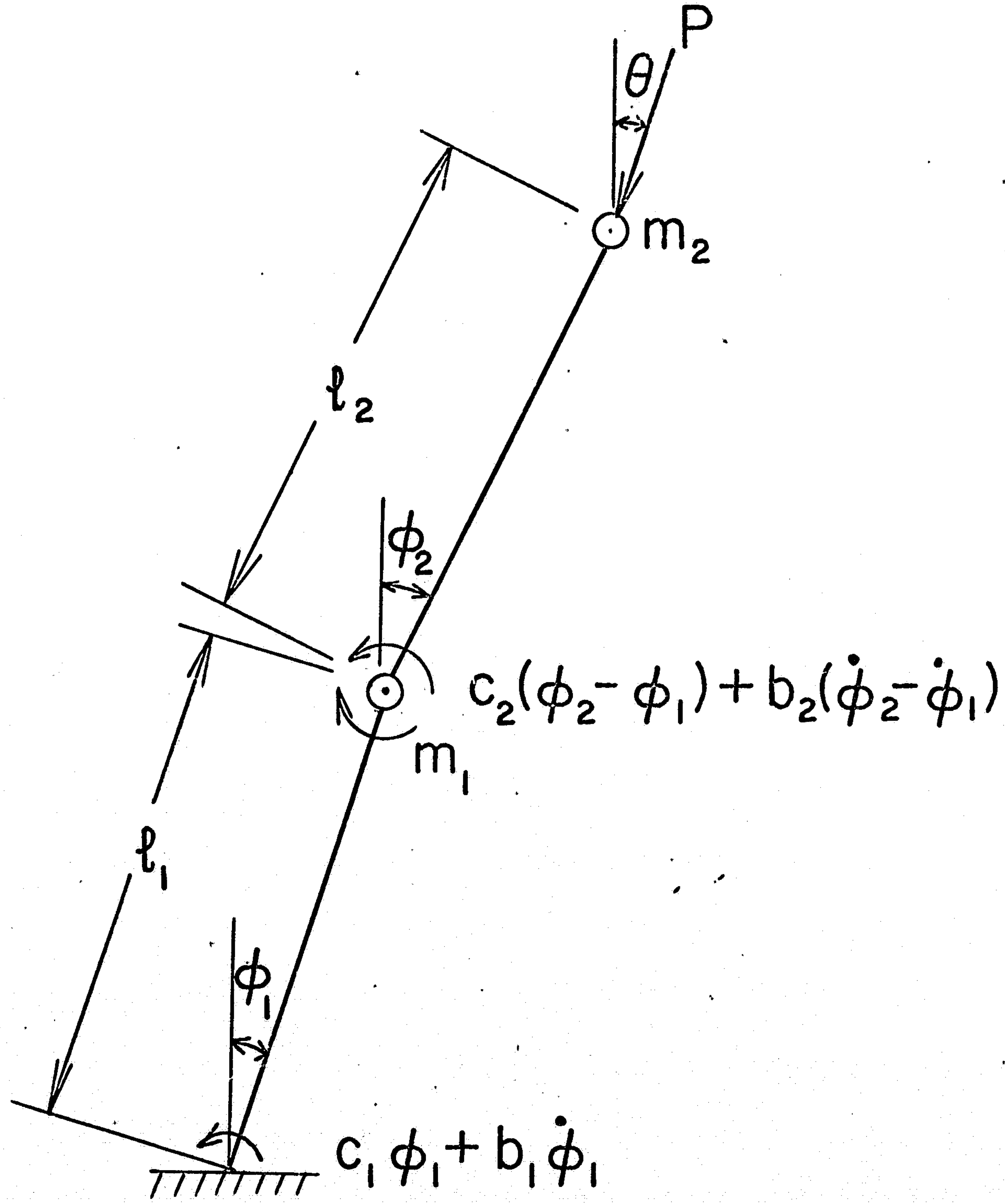
Captions for Figures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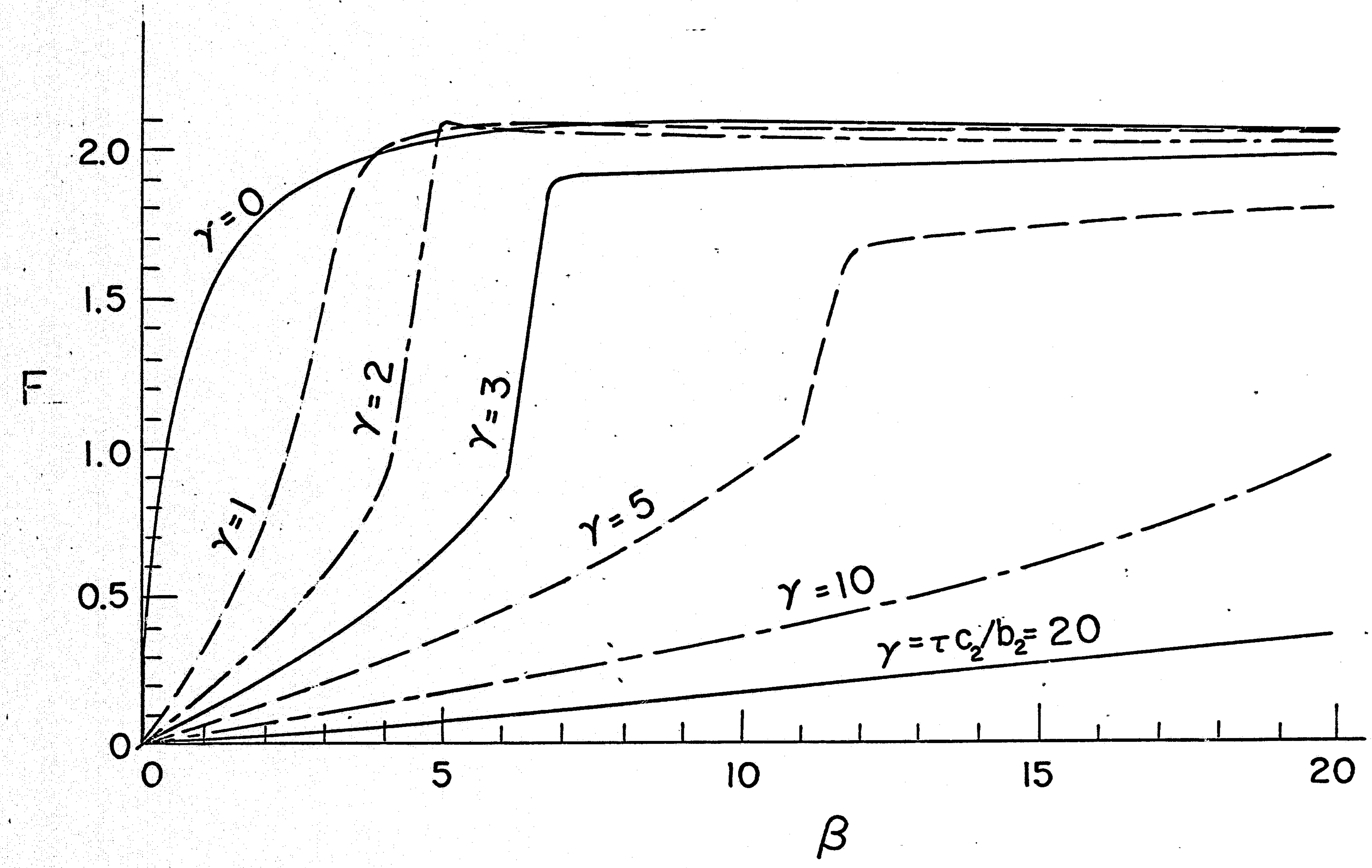
θ
m_2
ℓ_2
ϕ_2
$c_2(\phi_2 - \phi_1) + b_2(\dot{\phi}_2 - \dot{\phi}_1)$
m_1
ℓ_1
ϕ_1
$c_1\phi_1 + b_1\dot{\phi}_1$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2.0
1.5
1.0
0.5
0
5
10
15
20
β
γ = 0
γ = 1
γ = 2
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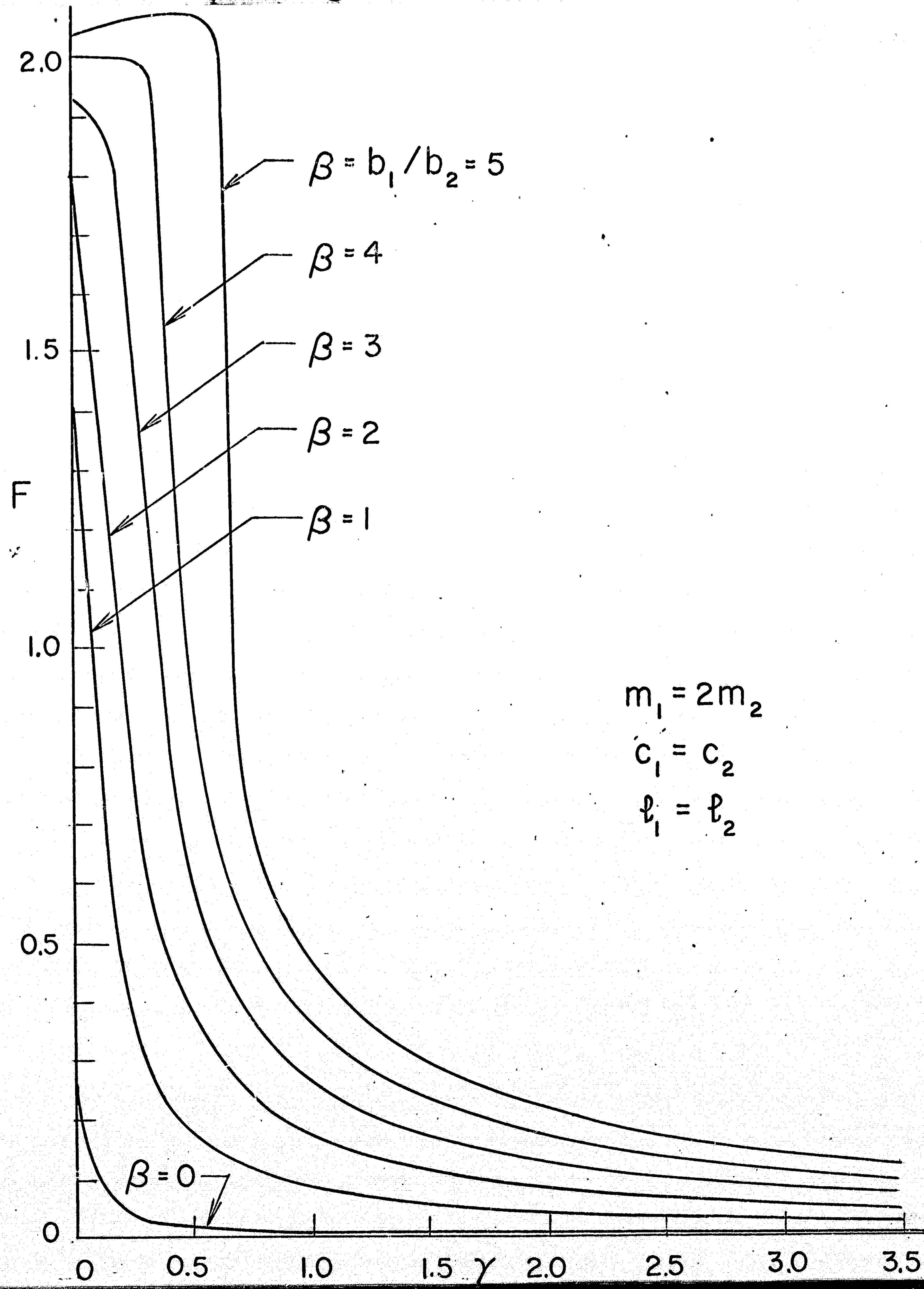
γ = 5
γ = 10
γ = τ c2/b2 = 20



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$\beta = 3$
$\beta = 2$
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$\beta = 0$
F
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1.5
1.0
0.5
0
0
0.5
1.0
1.5
γ
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2.5
3.0
3.5
$m_1 = 2m_2$
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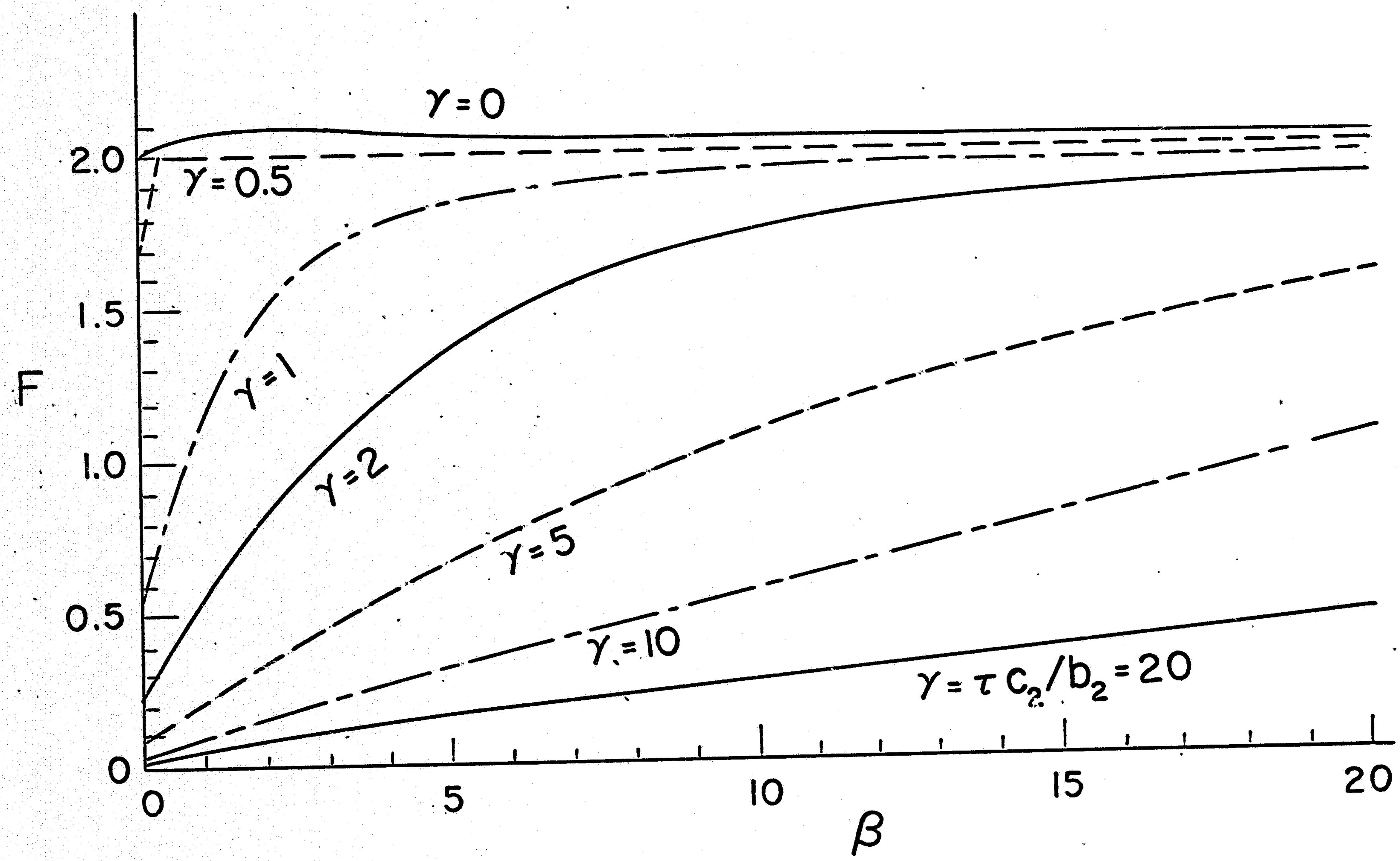

γ = 0
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γ = 1
γ = 2
γ = 5
γ = 10
γ = τ c₂/b₂ = 20
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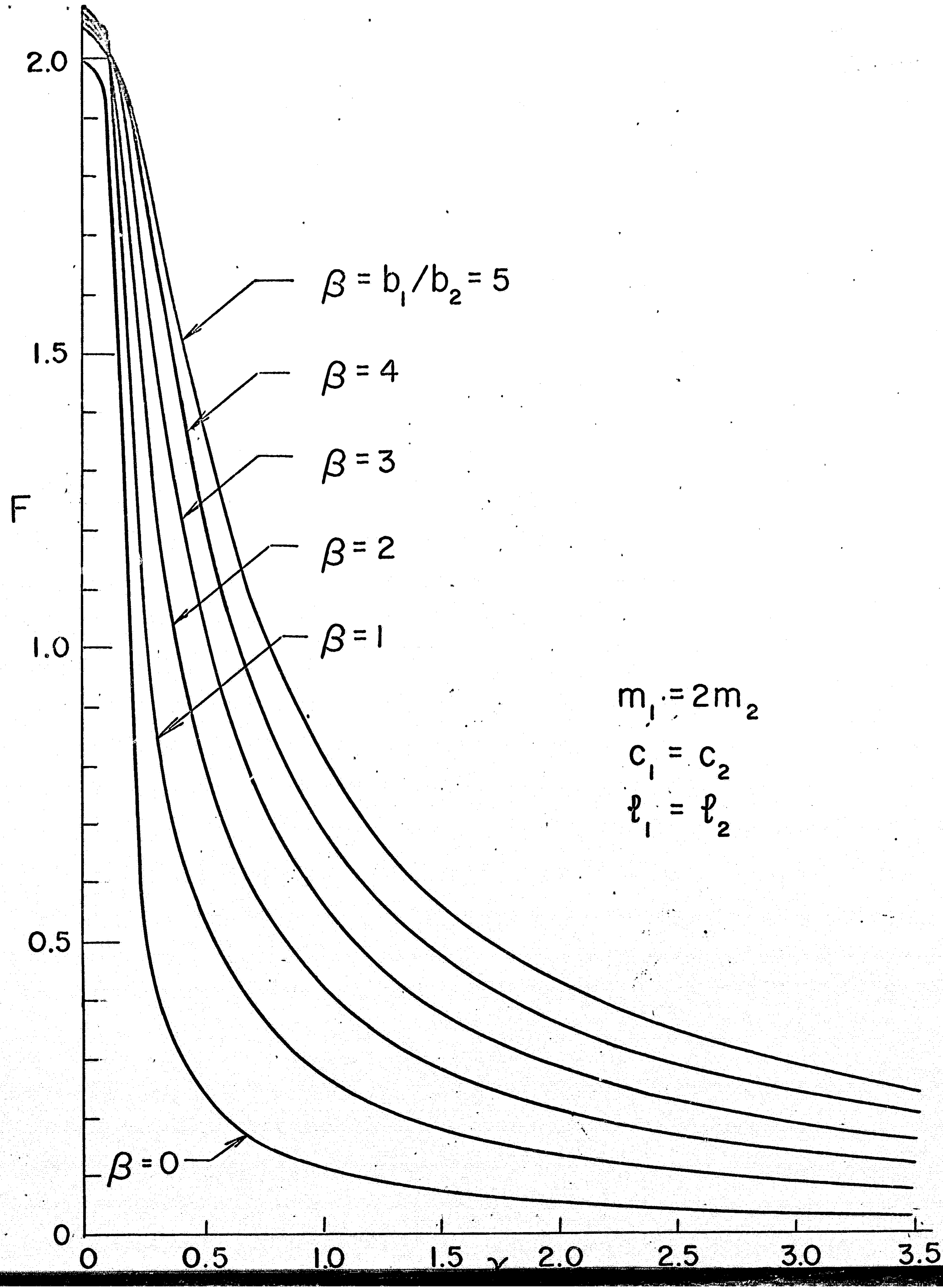
1.0
0.5
0
0
5
10
15
20
β



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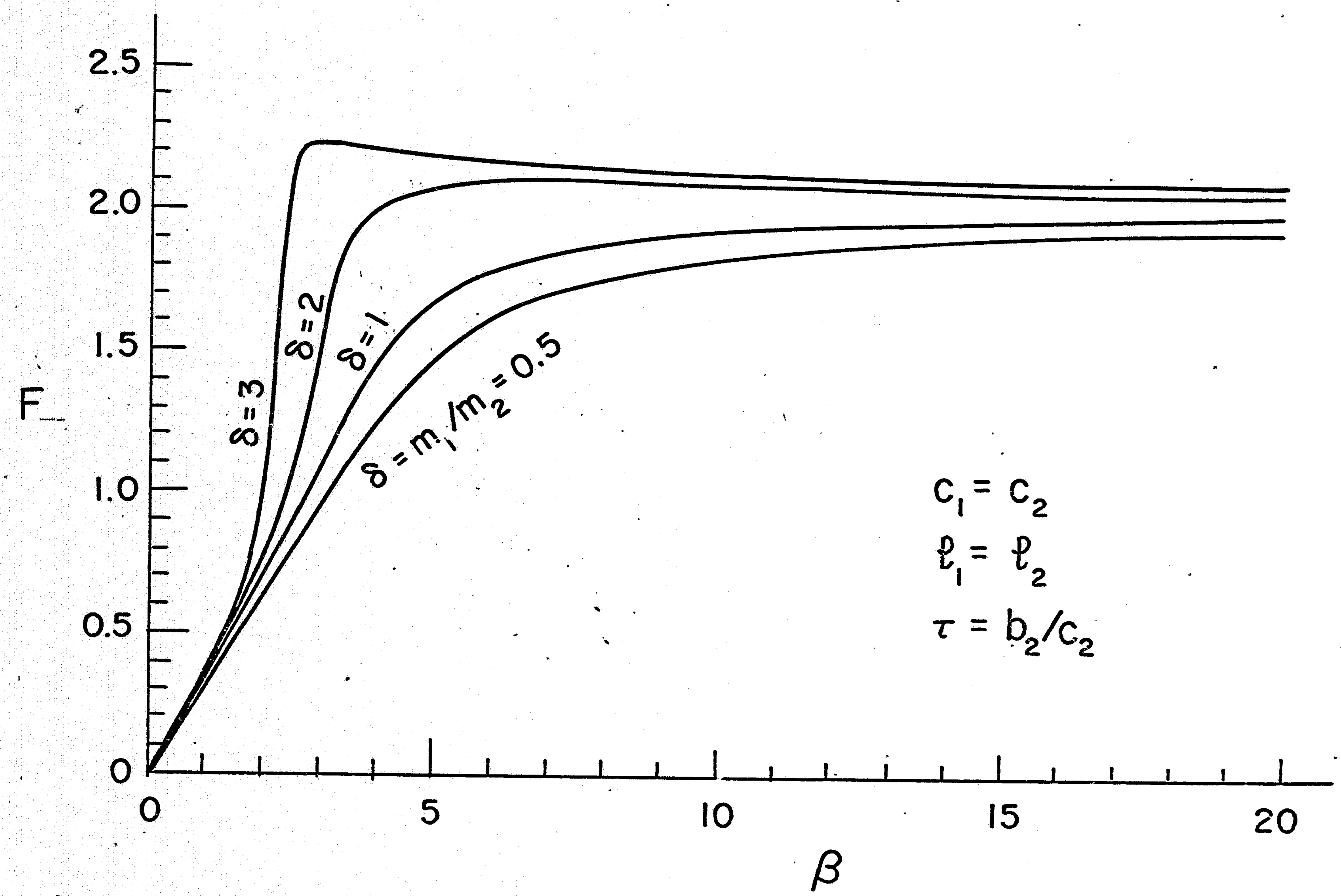
1.5
1.0
0.5
0
F
β = b₁/b₂ = 5
β = 4
β = 3
β = 2
β = 1
β = 0
m₁ = 2m₂
c₁ = c₂
ℓ₁ = ℓ₂
0
0.5
1.0
1.5
γ
2.0
2.5
3.0
3.5



F_
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2.0
1.5
1.0
0.5
0
5
10
15
20
β
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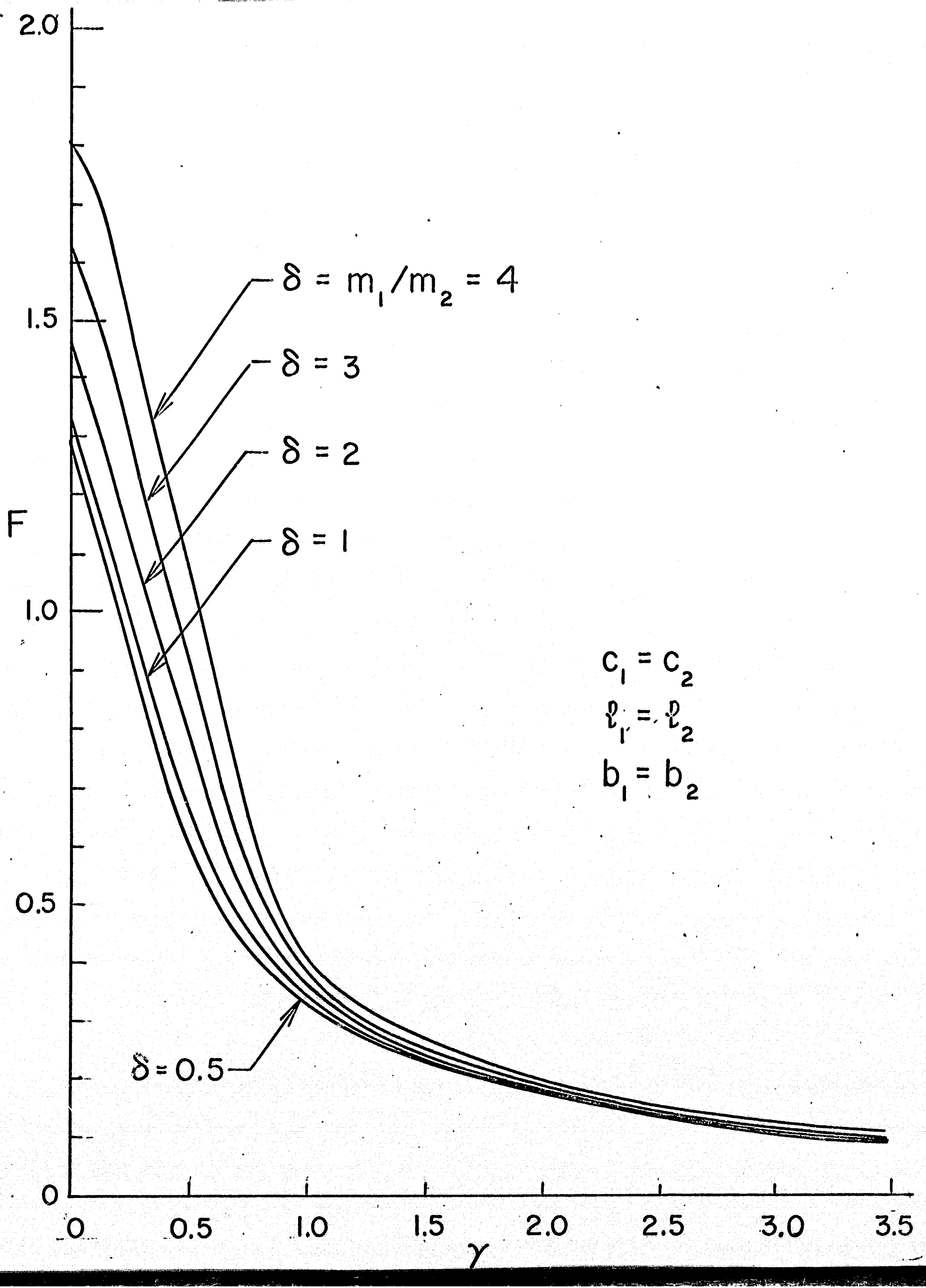
ℓ1 = ℓ2
τ = b2/c2



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0
F
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c1 = c2
l1 = l2
b1 = b2
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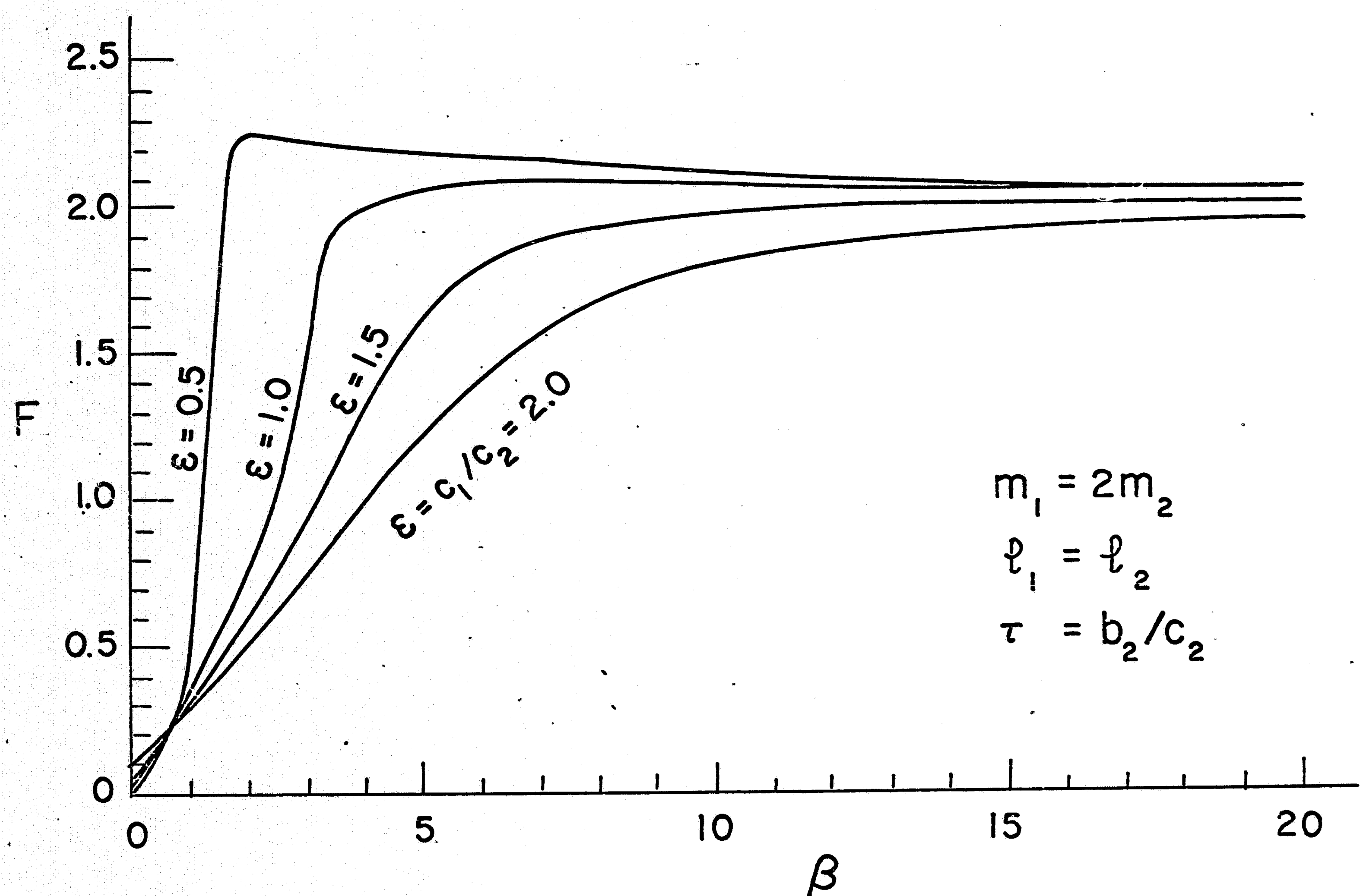
2.5
3.0
3.5
γ



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1.0
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0
F
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ε = 1.5
ε = c₁/c₂ = 2.0
m₁ = 2m₂
ℓ₁ = ℓ₂
τ = b₂/c₂
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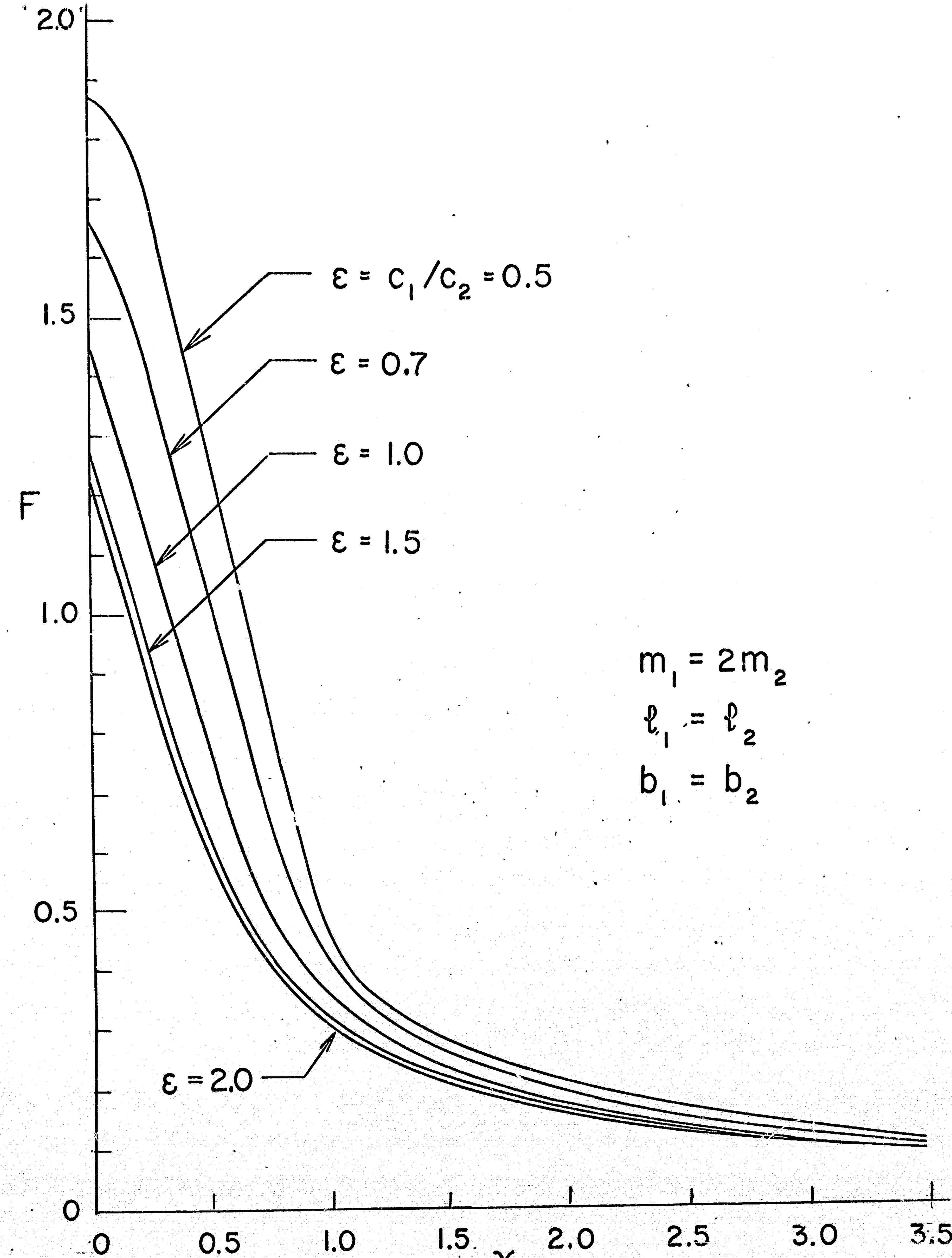
5
10
15
20
β



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0.5
0
F
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m1 = 2m2
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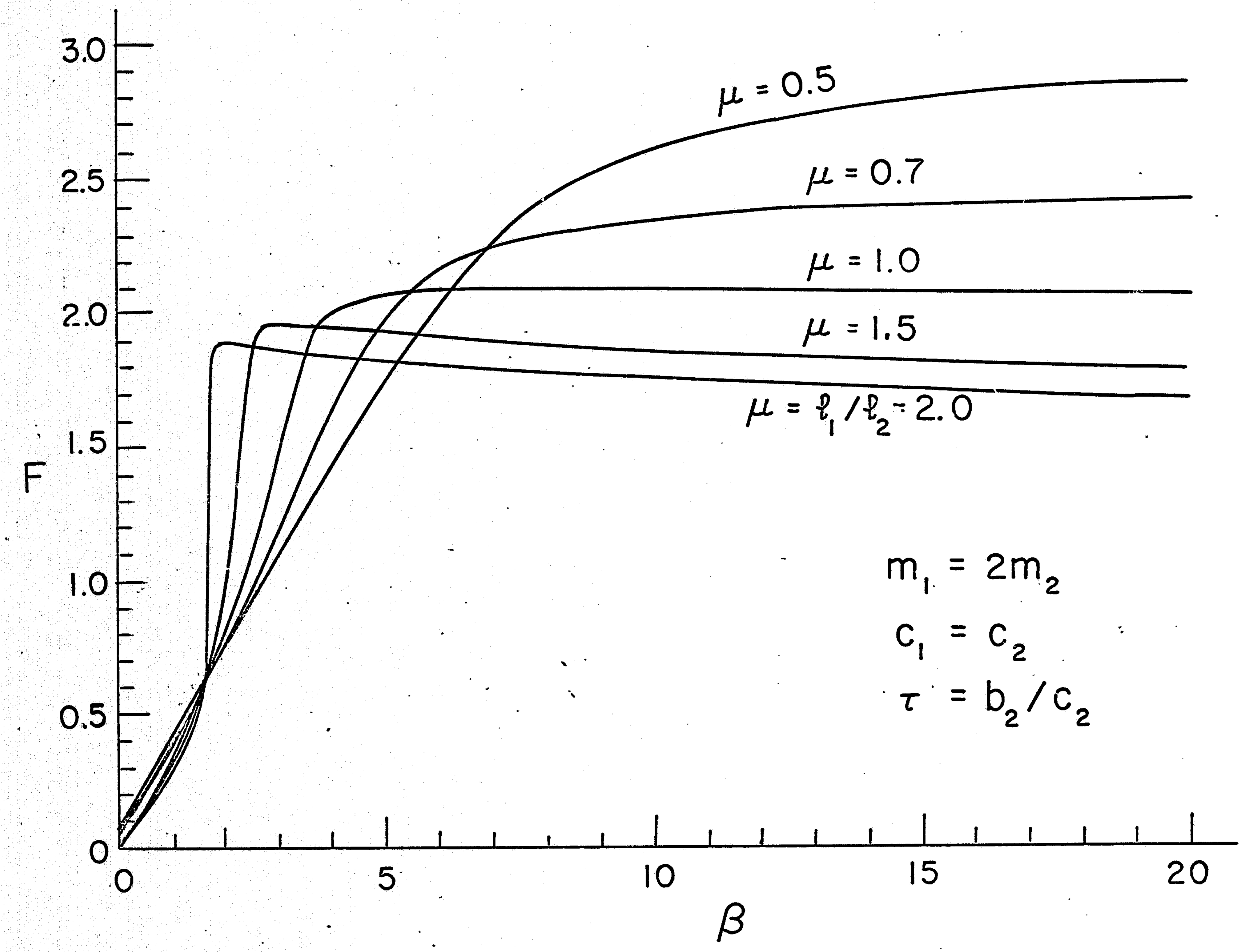
2.0
2.5
3.0
3.5
γ



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2.0
1.5
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0.5
0
F
0
5
10
15
20
β
μ = 0.5
μ = 0.7
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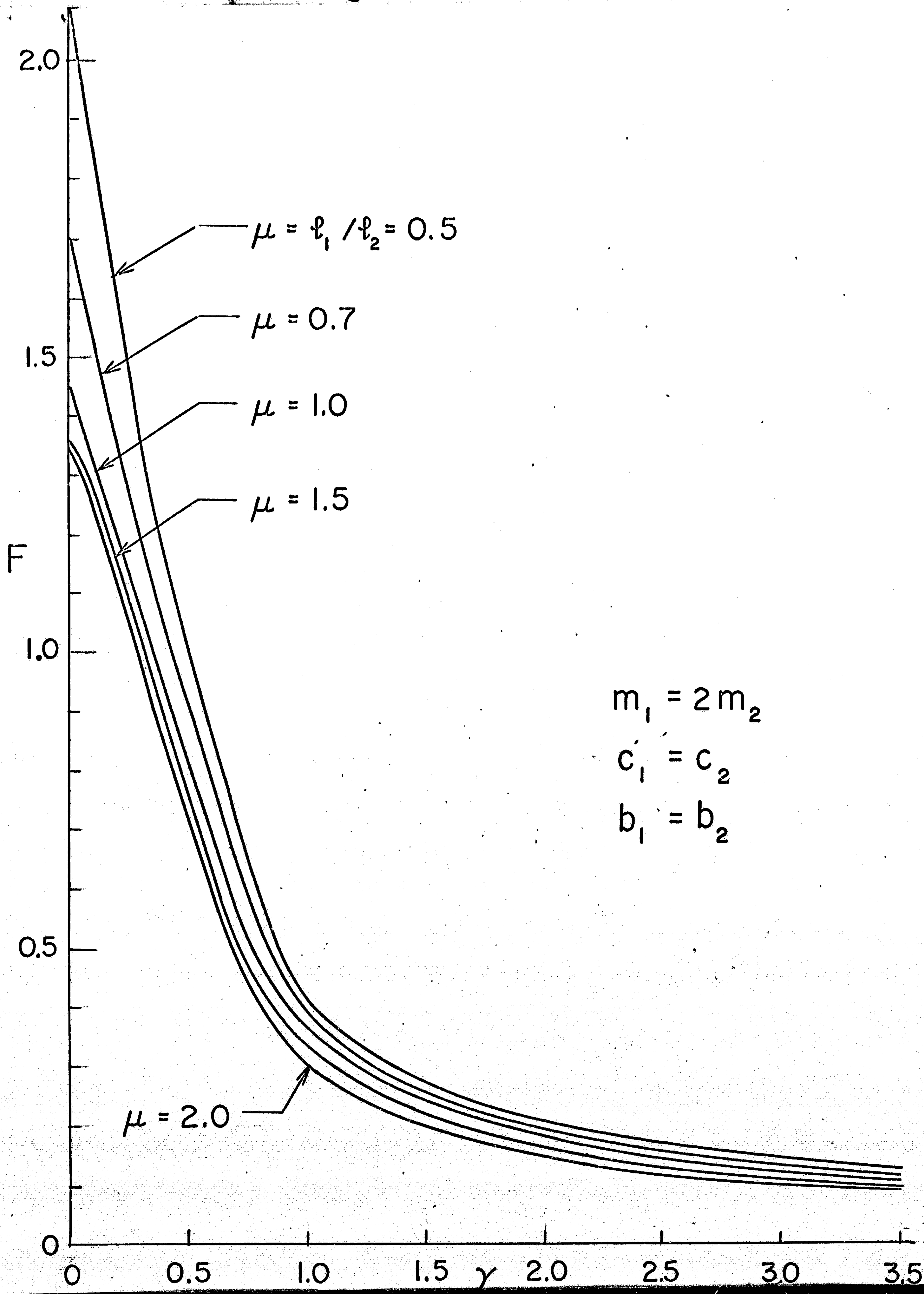
μ = 1.5
μ = ℓ1/ℓ2 = 2.0
m1 = 2m2
c1 = c2
τ = b2/c2



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1.5
1.0
0.5
0
0
0.5
1.0
1.5
γ
2.0
2.5
3.0
3.5
μ = ℓ₁ / ℓ₂ = 0.5
μ = 0.7
μ = 1.0
μ = 1.5
μ = 2.0
m₁ = 2 m₂
c₁ = c₂
b₁ = b₂