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Time and Fuel Optimal Controls
In The Presence of Measurement
Uncertainties

**CASE FILE
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By

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$\alpha, \alpha_1, \alpha_2; \alpha_0$	thrust angles ; a ratio
$\underline{I}$	unit diagonal matrix ; an integral
$F, F_1, D, \underline{d}, H$	system dynamics parameters
D	target matrix
$\underline{Z}, Z$	observation vector ; dummy variable
$\underline{v}$	observation noise ; feedback gain vector ;
$\underline{v}$	velocity vector
$\underline{r}, r$	position vector and its magnitude
$\underline{r}$	a correlation vector
$\underline{g}$	gravity vector
$\delta \underline{r}, \delta \underline{v}$	increments of $\underline{r}$ and $\underline{v}$
G, G_0	gravity components
$\underline{f}, \underline{f}_1$	forcing vector
$\underline{f}$	a correlation vector
$M, m; \dot{m}$	masses ; mass rate
Q_W, R_N, C_1	noise covariance matrices
$\delta(t), \delta_{ij}$	Dirac Delta , Kronecker Delta
σ	normalized Wiener Process
C	Kalman Gain
P	error covariance matrix ; a parameter set
μ	a large number
$\underline{e}$	new information random process
E_0	correlation matrix
ϕ	transition matrix
ϕ_N, ϕ_D	numerator and denominator
ρ	a ratio ; a generator for $\underline{b}_e$

$\underline{b}$	stochastic control vector
$\underline{c}$	a constant vector
S	target weighting matrix
Q	observation matrix
ω_0, ω_1	target set size
$\bar{\omega}_0, \omega_W, \omega_W^1, \omega_{\alpha_1}, \omega_{\alpha_2}, \omega_{\alpha}$	coefficients of control
V, V_x, V_{xx}	a function and its first and second partial derivatives
$\mathcal{L}$	weak differential operator
$\Gamma, a_{ijk}, b_{ijk}, \beta_{ij}, \delta_{ijk}$	expansion coefficients
$\beta, \zeta, \xi; \lambda$	random or dummy variables ; a ratio
$\rho(\zeta)$	probability density
$\underline{P}$	a parameter vector
γ, γ_0	stochastic gain functions
ϵ	a scalar bound
$\sigma_{\zeta}, \sigma_{\eta}$	variances
Ω	a set on the real line ; derivative of X
Q_f	an integral
Λ, v, T	Lagrange multipliers or adjoint variables
H	Hamiltonian
$\delta H, \delta_1 H, \delta v, \delta X, \delta X, \delta t_{\mu}$	variations in H, V, X, X, t_{μ}
H_x, H_{pp}	partial derivatives of Hamiltonian
Δx	total variation in x
$\mathcal{X}$	a point set
$\nabla \mathcal{X}$	normal to boundary of $\mathcal{X}$

κ	mass flow constant
Δ	a collective term ; a step size
γ_m	a decision parameter
C_r	a constant
$x, k, a, c, \lambda, \hat{x}$	scalar representation of $X, \underline{k}, F, C, E, C^T, \Lambda, \hat{x}$
$\hat{i}_\omega$	approximation to indicator function i_ω
κ_s	step - size
$\underline{b}_1, \underline{b}_2$	stochastic control variables
$\underline{d}_1, \underline{d}_2$	system parameter vectors
$\underline{v}_1, \underline{v}_2$	ideal feedback control vectors
$\underline{b}_e$	generating vector for $\underline{b}_1, \underline{b}_2$
θ	a cross correlation
$\mu_0, \mu_1, \mu_2 ; \mu_1, \mu_2$	integrals ; scalar coefficients
T_H	a collection of terms
$H_{\alpha_1}, H_{\underline{k}}, H_{\underline{v}}$	partial derivatives of the Hamiltonian
$\eta, \eta_1, \eta_2, \eta_3$	step - size scalars
ψ	a characteristic function
$\underline{s}$	a spacial frequency vector in the transform domain
$\lambda_{ijk\ell}$	a cumulant
W	expected conditional cost
$\underline{l}$	a weighting vector

ABSTRACT

This report considers a class of stochastic optimal control problems associated with the task of controlling a rocket or space vehicle under circumstances of noisy measurements and strictly bounded control inputs. Performance criteria of minimum time, minimum expected fuel consumption and least upper bound fuel consumption are considered.

Optimal feedback gains are precalculated in terms of the desired performance criterion, a priori known statistical information about the measurements and other random quantities, and the known system dynamics and constraints. The closed loop prestructured optimal control consists of a combination of the precalculated feedback gains and the history of noisy measurements on the system.

I. INTRODUCTION

1.1 THE PHYSICAL PROBLEM

This report considers a class of stochastic optimal control problems associated with the task of controlling a rocket or space vehicle under circumstances of noisy measurements and strictly bounded control inputs. Consider a rocket, which is modelled dynamically as a point mass, which is on an approach mission to a near planet. It must execute a continuous terminal guidance maneuver with respect to a nominal trajectory in order to achieve a certain terminal objective to within a prescribed tolerance. The small errors in position and velocity with respect to the nominal trajectory are unknown random quantities. These small errors constitute the state of the system dynamics; for, in the absence of future control or disturbance forces, they suffice to determine exactly the corresponding errors at any future time. The desired terminal objective of the mission is conveniently described in terms of a target set of state variables, i.e. in terms of some combination, statistically averaged over an ensemble of missions, of the terminal values of the state variables.

Control is based on measurements of the state. Typical measurements might be the angle subtended by the planet diameter and the angle between the center of the planet and a distant star. The statistical properties of these measurements are determined by the nominal trajectory and are known. Thus for example the error in measuring the planet diameter depends only on the distance to the planet center. As a result the exact manner in which the observations are coupled into the determination of the state is predetermined to a very good approximation by the nominal trajectory provided only that the deviations from the nominal trajectory, which define the state, are sufficiently small.

For simplicity only motion in a plane with two components of position and two of velocity is considered. A single component of control is exerted by a pair of back-to-back ion engines which are strictly thrust limited. It is desired to find a control law, i.e. a policy for generating the bounded thrust which depends on the history of past measurements and which controls the motion of the rocket to a desired target set with a specified minimum cost.

The stochastic nature of the problem is such that control can only be exerted on that portion of the state which can be measured. The measureable portion of the state components are of course statistical quantities. Therefore, it is reasonable to define the target set also as a statistic in terms of the measureable portion of the state at a specified terminal time.

There are three performance criteria of interest and these turn out to be closely related:

PROBLEM 1: MINIMUM-TIME

The rocket coasts until time t_μ and then exerts control in the interval $[t_\mu, T]$. Find the time t_μ such that the interval $[t_\mu, T]$ is of minimum length and also find the control law.

PROBLEM 2. MINIMUM EXPECTED FUEL-CONSUMPTION

For a control interval $[t_0, T]$ of length greater than $[t_\mu, T]$ as in problem 1, find the control law which minimizes the expected fuel-consumption. If the control signals are only allowed to take the values +1, 0, -1 then fuel consumption is equivalent to the length of the burning interval, in which the signals are +1 or -1. The actual fuel consumption in this problem is a random quantity depending on the measurement history.

PROBLEM 3. LEAST UPPER BOUND FUEL CONSUMPTION

The control signals are constrained to the values +1, 0, -1 over the fixed interval $[t_0, T]$. Find the set of burning intervals, in which the signals are

$+1, -1$, such that the set has minimum length in the sense of Lebesgue measure and find the control law over the burning intervals. The burning intervals do not depend on measurement history but only on the dynamics and statistics which are known *a priori*. Hence the fuel consumption is a deterministic quantity in contrast to the random fuel consumption of Problem 2.

Typical control functions for the three problems are sketched in Fig.1. Note that in problem 1 burning takes place only in the interval $[t_\mu, T]$, in problem 2 burning can take place anywhere on the interval $[t_0, T]$, and in problem 3 burning takes place only in the burning intervals indicated by the heavy lines. The turn-on time t_μ in problem 1, and the length and location of the burning intervals in problem 3 are precalculated as a function of the known system dynamics and measurement statistics. In these two problems burning is continuous in the burning intervals and only the direction of thrust is a random variable, depending on the measurement history. In problem 2 the thrust direction, length, and location of the burning intervals are all random variables, depending on the measurement history. In all three problems the times t_0 and T are assumed to be known. A more complete discussion of the three problems, and several generalizations are presented in section 1.4.

From a practical point of view Problem 3 is the most interesting for it specifies exactly how much fuel should be carried instead of an average amount. It is a reasonable conjecture that the Least Upper Bound (L.U.B.) fuel consumption will be greater than the expected fuel consumption of Problem 2. This is likely to be so because in Problem 3 the burning intervals are fixed and it should be expected that this constraint will impose a cost penalty.

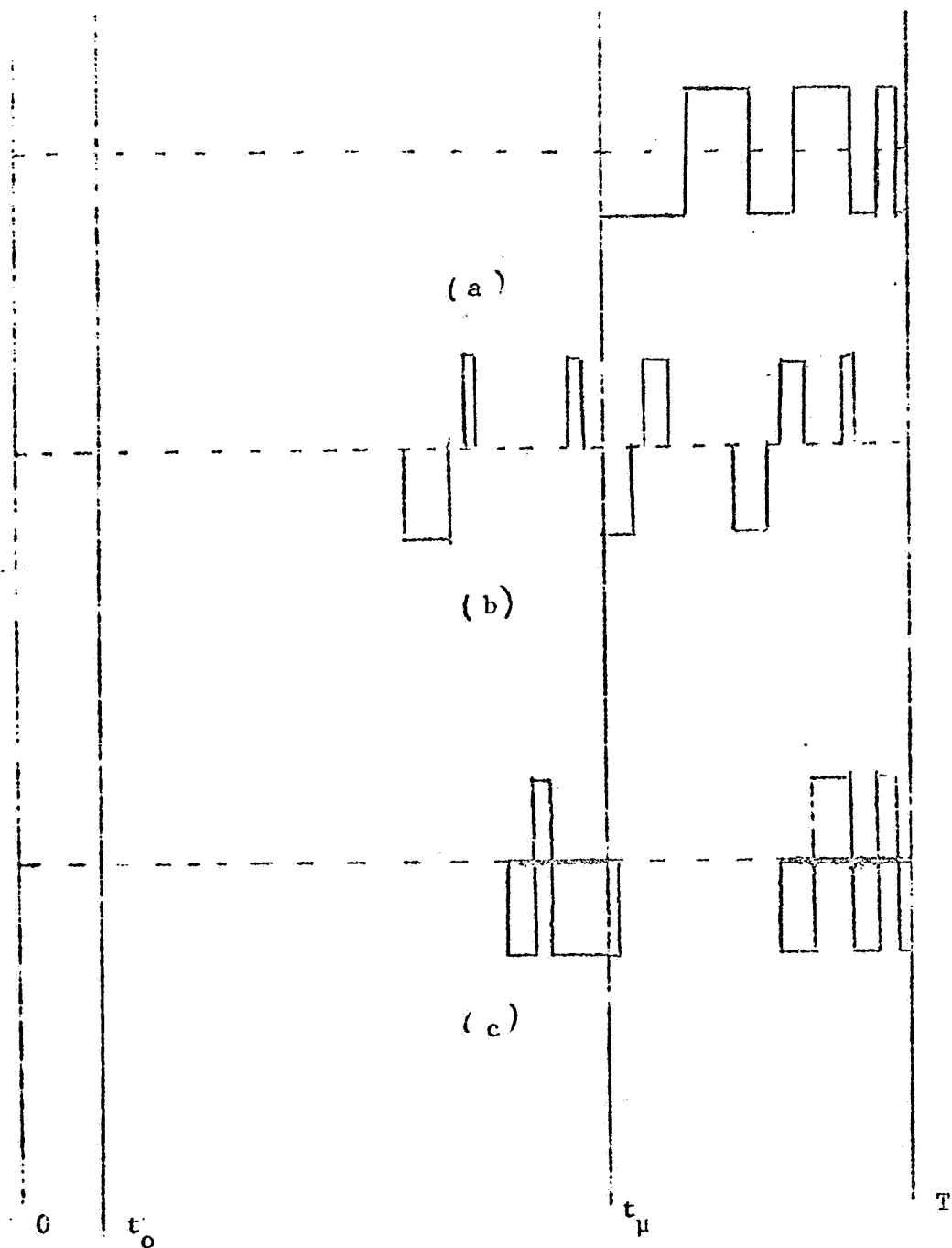


Figure 1: Typical control functions for a) the minimum time problem, b) the minimum expected fuel problem, and c) the least-upper bound fuel consumption problem.

1.2. THE BASIC PHILOSOPHY OF THIS REPORT

In order to appreciate the particular point of view adopted in this report it is first necessary to consider some of the difficulties associated with some other points of view. One such alternative point of view is briefly described below. While we reject it some use will be made of it in Chapter 3 to gain insight into the structure of the problem from our own point of view.

Suppose a system whose continuously evolving dynamics can be described by a state vector, $\underline{x}(t)$, a control, $u(t)$, and possibly random initial conditions, parameters, and disturbing forces is to be optimized by choosing the control $u(t)$ to be a function of the set of past observations $\underline{z}(\sigma); \sigma \leq t$, so as to minimize the performance cost functional

$$J = \min_{u \in [t_0, T]} E[E[\int_{t_0}^T \tilde{L}(\tau, u(\tau), \underline{x}(\tau)) d\tau | \underline{z}(\sigma); \sigma \leq \tau]]] \quad (1.2.1)$$

The above optimization would in addition satisfy two classes of constraints.

Firstly, the random control, u , is strictly bounded. Secondly, the motion of the system at the terminal time, T , is constrained in some sense to a target set region whose dimensions are determined from physical conditions. For example it may be required that $\underline{x}(T) \in \text{Target Set}$ with a probability greater than P .

The only known method which is sufficiently general to encompass the formulation of Eq.(1.2.1) is Dynamic Programming [9],[16],[22]. This requires firstly that an estimator, $\hat{\underline{x}}$, be postulated for the state, $\underline{x}$, which contains all the information about the observations $\underline{z}(\sigma)$ for $\sigma \leq t$. Then Eq.(1.2.1) can be reformulated in terms of $\hat{\underline{x}}$. We require the evaluation of a partial differential equation for the quantity

$$V(\hat{\underline{x}}(t)) = \min_{u \in [t, T]} E[\int_t^{T_0} \tilde{L}_1(\tau, u(\tau), \hat{\underline{x}}(\tau)) d\tau] \quad (1.2.2)$$

then

$$J = E[V] \quad (1.2.3)$$

The major drawback to Eq.(1.2.1) is that it does not include all problems of interest. In particular the least upper bound fuel consumption problem is not described by the formulation of Eq.(1.2.1). On the other hand for the problem of minimum expected fuel consumption the above formulation is quite natural. Where time of flight is involved in the cost to be minimized then Eq.(1.2.1) leads naturally to a first-crossing problem. It is possible by various arguments which involve the simultaneous optimization of parameters and controls to employ the structure of Eq.(1.2.1) for all three problems. However, the Dynamic Programming formalism in the case of the L.U.B. problem especially does not yield a solution algorithm but only some structural properties of the solution. The reader is referred to Section 3.

Even if the formulation of Eq.(1.2.1) is accepted as being appropriate to the particular problem of interest there are still some rather formidable obstacles to its use. The chief obstacle is the difficulty of obtaining a solution to the partial differential equation for V . No general method seems to be available and only special cases have been solved completely. There are in addition certain minor difficulties involving the target set. For example if the probability of first entry into the target region is a constraint then an additional description is required for the evolution of probability transition functions. Such questions are beyond the scope of our interest here.

In summary the general problem formulation of Eq.(1.2.1), where it is appropriate, is of interest primarily in so far as it yields insight into the structure or properties of the optimal control rather than as a means of generating an algorithm. Perhaps the best known result concerning structure is the Separation Theorem. This states that for linear systems and certain convex cost functionals including the quadratic a sufficient statistic for the history of past observations exists and is given by

$$\hat{\underline{x}}(t) = E[\underline{x}(t) | z(\sigma); \sigma \leq t] \quad (1.2.4)$$

Moreover, the optimal control depends on the observations only through $\hat{\underline{x}}$, [4],[53],[23]. While this result has not been shown to apply to our three problems of interest it would appear that the choice of the conditional mean as an estimator is a very reasonable assumption.

The philosophy adopted in this report is to reduce the basic stochastic problem to an equivalent deterministic problem directly in the problem formulation. This requires that an equivalent deterministic state must be found to represent the dynamics of the system. This is based on the notion that the conditional mean which is an a priori zero mean random variable, approximately Gaussian, represents a sufficient statistic. The deterministic state then becomes the set of moments associated with the covariance matrix of the conditional mean. We denote this moment matrix by $X(t)$. It is also necessary in this formulation to find an equivalent deterministic representation for the control $u = u(\hat{\underline{x}}(t), t)$. The corresponding deterministic control quantity, denoted by the vector $\underline{b}$, turns out to be the correlation of the conditional mean $\hat{\underline{x}}(t)$ with the control $u(t)$. This correlation quantity must be evaluated in terms of unknown control system parameters which correspond to feedback gains, thresholds and other such instrumentation quantities. These quantities are denoted by the parameters $\underline{k}, \underline{v}, a, \alpha$. Our deterministic formulation then becomes

$$J = \min_{\underline{k}, \underline{v}, a, \alpha} \int_{t_0}^T L(X(\tau), \tau, \underline{k}, \underline{v}, a, \alpha) d\tau \quad (1.2.5)$$

and the solution of Eq.(1.2.5) involves a set of ordinary differential equations with both initial and terminal boundary conditions. The above approach in a simpler form was followed in [6] and [7] for a specific case.

While Eq.(1.2.5) yields a solution there are some questions as to its exact meaning and validity. For one thing it is only possible to solve Eq.(1.2.5) when the terminal target set is defined in terms of the state matrix, $X(T)$. The relationship of such a target set description to a target set with physical dimensions can only be inferred if one assigns a probability distribution to the conditional mean, $\hat{x}(t)$. Since $\hat{x}(t)$ is approximately Gaussian a rather good interpretation of $X(T)$ is possible. Another difficulty which is potentially more serious is that $\hat{x}(t)$ must be assigned a probability distribution in order to calculate the stochastic control quantity, b , determined by the correlation of $\hat{x}$ and u . The assumption of Gaussian statistics, which is strictly non-valid due to the limiting imposed on u , leads to errors in the stochastic state matrix and consequently to an only approximate evaluation of Eq.(1.2.5). The resultant performance loss is attributed to the fact that all moments in the distribution function of $\hat{x}(t)$ higher than the second one are discarded continuously. A physical argument can however be made that these moments are small.

In summary the philosophy of this report is to adopt the problem formulation of Eq.(1.2.5) rather than that of Eq.(1.2.1) and to try to live with the inadequacies that it represents. When higher order moments in the distribution function of $\hat{x}(t)$ are large then the formalism of Eq.(1.2.5) is incomplete.

The justification of the above approach on physical grounds is that it is only possible to control what can be measured. Consequently, it is very reasonable to attempt to control the moment matrix of the conditional mean since, for example, if the conditional mean itself represents an a priori random error then if the second moments of the conditional mean are small then the random error will also be small. Moreover, if the second moments are small then higher order moments will also be small so that the loss of information due to the non-Gaussian character of the statistics may well be unimportant.

1.3 HISTORICAL BACKGROUND

An excellent introduction to the problem of space guidance and navigation can be found in Battin [1]. Probably the beginnings of the stochastic control problem as applied to space guidance appeared in the important paper by Battin in 1961 [2]. The basic notions of Battin's scheme were as follows: 1) The use of state space notation to describe the propagation of position and velocity deviations; 2) The use of a recursive least-mean square estimate for the state vector conditioned on past observations, which is equivalent to the Kalman Filter [3]; 3) The specification of a velocity to be gained vector, ΔV , as a linear function of the current best estimate of the state which would null the extrapolated terminal best estimate of the state; 4) A stochastic control law based on a threshold concept; namely, the control ΔV is executed at any time if the ratio of the r.m.s. magnitude of ΔV to the r.m.s. uncertainty in ΔV exceeds a prespecified threshold.

The above scheme while not optimal with respect to any performance criterion does enable the designer to cope with some of the basic trade-offs inherent in the problem by adjusting the threshold level. The fundamental consideration is that when control is exerted early it is very effective in that a small velocity correction applied early can correct a large position error. On the other hand, there is a large uncertainty as to just what the velocity correction should be since only a few measurements have been made. Later, after many measurements, the uncertainty in the velocity correction is small; however, the correction is relatively ineffective in that a large velocity correction is required to correct a relatively small position error. As more and more measurements are accumulated, the magnitude of the required r.m.s. correction tends to grow, while the r.m.s. uncertainty in the correction tends to diminish, hence the ratio must increase and eventually cross the threshold.

If the above problem were formulated directly as a stochastic optimal control problem, then the solution would inherently solve the trade-off between control effectiveness and control uncertainty. Early attempts at the stochastic optimal control problem were confined to the case of minimizing an expected quadratic cost functional [4], [5]. Here the estimate of the state is always the Kalman Filter and the best control policy is a linear combination of the current best estimate of the state as in [2]. The control law however bears a close relation to the corresponding deterministic control law.

An early attempt to solve the problem of minimum expected fuel consumption by Breakwell and Tung [6], and see also [7], considered a performance index

of the form $J = \int_0^T \sqrt{E||u||^2} dt$ as an approximation to the desired control law

of the form $J = E \int_0^T ||u|| dt$. The control law was chosen to be a linear

combination of the current best estimate of the state vector as defined by the Kalman Filter. In [7] a version of the stochastic minimum time control with measurement uncertainties was also attempted. Here second-order perturbations to a deterministic control law were obtained to account for small random measurement errors.

Two recent books in the general area are [8] and [9], which deal largely with quadratic cost functions and with stability. Note also [10], and one of the early papers on Stochastic Differential Games [11].

1.4 OPERATION OF THE OPTIMAL CONTROLLER

This section describes in some detail the implementation and operation of the optimum controller for the performance criteria of minimum time, minimum expected fuel consumption and least upper bound fuel consumption. Attention is restricted to the context of the rocket mission. It is hoped that the following discussion will help to clarify the exact definition of these problems and

contrast them with other possibly related problems of interest. At the same time some results from later sections are introduced here for the purpose of familiarizing the reader.

1.4.1 THE MINIMUM TIME PROBLEM

The turn-on time, t_μ , is precomputed in terms of system dynamics and statistics and this specifies the burning interval $[t_\mu, T]$. The thrust policy is based on a precalculated feedback gain vector $\underline{k}(t)$.

During flight measurements are taken, i.e., on celestial bodies, and these are used to calculate on-line the current best estimate of the state, $\hat{\underline{x}}$, from Eq.(2.2.4). The control, u , is then generated by

$$u = \begin{cases} 0 & ; t < t_\mu \\ \text{sgn } \underline{k}^T(t)\hat{\underline{x}}(t) & ; t_\mu \leq t \leq T \end{cases} \quad (3.2.10)$$

The only a priori unknown quantity is the statistic, $\hat{\underline{x}}$. The above problem may be contrasted with the problem, not treated in this report, where the time, t_μ , is also a random variable and it is desired to minimize the expected length of the interval $[t_\mu, T]$. In our problem the minimum interval $[t_\mu, T]$ depends only on the dynamics and measurement statistics which are known a priori rather than on a measurement history.

1.4.2 THE MINIMUM EXPECTED FUEL PROBLEM

Prior to flight we select a suitable time interval $[t_o, T]$ which is greater than $[t_\mu, T]$ for the minimum time problem. The initial time, t_o , is the first time at which a required minimum number of measurements have been taken and it represents the first time burning may occur. The thrust policy is based on a precalculated and stored vector, $\underline{k}(t)$, and a scalar threshold, $a(t)$.

The current best estimate of the state, $\hat{\underline{x}}(t)$, is computed as in the minimum time problem on line from celestial measurements and is given by Eq.(2.2.4). Burning may occur anywhere in the interval $[t_o, T]$ and the control, u , is generated by the policy

$$u = \begin{cases} +1, & \underline{\hat{k}}^T \underline{\hat{x}} > a \sqrt{\underline{k}^T \underline{k}} \\ 0, & -a \sqrt{\underline{k}^T \underline{k}} < \underline{\hat{k}}^T \underline{\hat{x}} < a \sqrt{\underline{k}^T \underline{k}} \\ -1, & \underline{\hat{k}}^T \underline{\hat{x}} < -a \sqrt{\underline{k}^T \underline{k}} \end{cases} \quad (3.2.11)$$

Consequently, the burning times when $u = \pm 1$ are random as well as the estimate $\underline{\hat{x}}$. Thus the actual fuel consumption is random and depends on a specific measurement history but the expected fuel consumption is not and in fact is precalculated in terms of the dynamics and measurement statistics.

1.4.3 THE LEAST UPPER BOUND FUEL PROBLEM.

The significant feature of this problem is that the entire set of burning intervals, $\Omega \subset [t_0, T]$ must be precalculated and stored prior to flight. Hence it is predetermined just exactly when burning will or will not occur on the basis of the dynamics and measurement statistics. This information is stored in the rocket by an indicator function parameter, i_ω , which takes on the value ± 1 during burning and 0 during coasting. By construction of the control, the actual fuel consumption is identical to the length of the burning interval, Ω ; hence the actual fuel consumption is also non-random and is precomputed. Hence the fuel consumption is minimized a priori in terms of the dynamics and measurement statistics. As before we also require the precalculation of a vector $\underline{k}$.

On board, the estimator $\underline{\hat{x}}$ is calculated on-line from celestial measurements from Eq.(2.2.4) as before. The control u is given by

$$u = \left. \begin{array}{ll} \text{sgn } \underline{\hat{k}}^T \underline{\hat{x}} & ; \quad i_\omega = 1 \\ 0 & ; \quad i_\omega = 0 \end{array} \right\} \quad (4.5.10)$$

Thus the direction of the control u is random while the magnitude is precomputed.

The Least Upper Bound Problem may properly be considered to be a generalization of the Minimum Time Problem where we minimize the length of an interval set Ω instead of the length of an interval. The interval $[t_0, T]$ which contains Ω

has the same significance as in the problem of minimum expected fuel consumption.

This problem might also be regarded as a specialization of a more general problem in which the total fuel consumption must be fixed but in addition the burning intervals are allowed to be random. The difficulty here is that it becomes necessary to calculate *a priori* conditions for when the desired fixed amount of fuel will be burned. This is a form of the classical "first crossing problem" solutions for which are generally unknown. Hence it is desirable to avoid this problem.

1.4.4 GENERALIZED L.U.B. PROBLEMS

Up to this point nothing has been said concerning the direction of thrust and the effects on the system dynamics resulting from loss of mass due to burning. Maximum simplification is achieved by simply assuming that thrust takes place in a tangential direction in the orbital plane and moreover that the change in mass is small enough to be negligible. These simplifying assumptions are in fact utilized in most of the following sections.

The L.U.B. problem is however, readily generalized to account for a finite mass flow. This generalization involves a modification of system dynamics; however, the specific form of the instrumentation is unchanged and in fact is identical with that of Section 1.4.3. As a further generalization it is possible to consider thrust vector directions other than tangential. One possibility is to precalculate the direction of thrust in terms of the system dynamics and statistics. The thrust angle $\alpha(t)$ is then stored along with the feedback gain vector $\underline{k}$ and indicator function, i_w . Again the form of the instrumentation is unchanged from that in Section 1.4.3.

Finally we consider the case where the thrust vector is random and depends on the measurement history, i.e. on $\hat{\underline{x}}$ the current best estimate of the state which is calculated on board. The loss of mass is not taken into account.

The thrust angle, α , has the form

$$\alpha = \alpha_0 + \underline{v}^T \underline{\hat{x}} \quad (6.0.1)$$

Here the parameters $\alpha_0(t)$ and $\underline{v}(t)$ are precalculated and stored along with the control parameter $\underline{k}(t)$ and the indicator function i_ω . Actually α_0 , $\underline{k}$, $\underline{v}$ need only be stored for those times for which $i_\omega = 1$. The form of the control u is given as before by Eq.(4.5.10) of Section (1.4.3).

2. Dynamical Equations of Stochastic Control

Dynamical systems which can be described by finite order differential equations can be conveniently represented by a set of first order state equations, see for example [13], as in Eq. (2.0.1).

$$\dot{\underline{x}} = \underline{F}\underline{x} + \underline{D}\underline{u} + \underline{w} \quad (2.0.1)$$

$$\underline{z} = \underline{H}\underline{x} + \underline{v}$$

Here, $\underline{x}$, is an n vector denoting the state, $\underline{u}$, is a p vector denoting the input and $\underline{z}$ is an r vector denoting the observations. The n vector, $\underline{w}$, represents a random forcing term and the r vector, $\underline{v}$, represents a random additive uncertainty to the observations. The canonical form of Eq. (2.0.1) is used generally throughout references [1]-[8] and only with minor modifications in [9] and [10] .

2.1 Motion Of The Vehicle In State Space

It is of interest to relate the general canonical form of Eq. (2.0.1) to the physical parameters involved in the example of the orbiting rocket. Following Battin [1], [2] a very general description of a mass particle in a perhaps complicated gravitational field can be given by

$$\frac{d^2}{dt^2} \underline{r} = \underline{g}(\underline{r}) \quad (2.1.1)$$

where $\underline{r}$ is the radial vector to the point mass from some specified origin and $\underline{g}$ is the gravitational force vector.

Define the velocity vector, $\underline{v}$, by

$$\underline{v} = \frac{d\underline{r}}{dt} \quad (2.1.2)$$

Then a set of first order differential equations is obtained from Eqs. (2.1.1) and (2.1.2)

$$\begin{aligned} \frac{d\underline{r}}{dt} &= \underline{v} \\ \frac{d\underline{v}}{dt} &= \underline{g}(\underline{r}) \end{aligned} \quad (2.1.3)$$

The motion of the mass particle under Eq. (2.1.3) and suitable boundary conditions constitutes the nominal trajectory. Our primary interest will be the small perturbations from the nominal caused by forcing accelerations, $\underline{f}$. The motion of the small perturbations is obtained by applying the perturbation operator, δ , to Eq. (2.1.3) and noting that the operations of perturbation and differentiation can be interchanged.

$$\begin{aligned} \frac{d}{dt} \delta \underline{r} &= \delta \underline{v} \\ \frac{d}{dt} \delta \underline{v} &= G \delta \underline{r} + \underline{f} \end{aligned} \quad (2.1.4)$$

Where G is the Jacobian matrix whose elements are

$$G_{ij} = \begin{Bmatrix} \frac{\partial g_i}{\partial r_j} \end{Bmatrix}$$

Define a state vector

$$\underline{x} = \begin{pmatrix} \delta \underline{r} \\ \delta \underline{v} \end{pmatrix}$$

and a forcing vector

$$\underline{f}_1 = \begin{pmatrix} 0 \\ \underline{f} \end{pmatrix}$$

and a matrix F by

$$F = \left[\begin{array}{c|c} 0 & I \\ \hline G & 0 \end{array} \right]$$

(2.1.5)

Then Eq. (2.1.4) can be written in terms of the state, $\underline{x}$, as $\dot{\underline{x}} = F\underline{x} + \underline{f}_1$.

(2.1.6)

To complete the comparison of Eq. (2.1.6) with the canonical model of Eq. (2.0.1) it is necessary to introduce the random vectors $\underline{w}$ and $\underline{v}$ and the observations. We therefore consider, $\underline{f}_1$, to consist of a control, $D\underline{u}$, where each of the p components of $\underline{u}$ satisfy the constraint

$$|u_i| \leq 1 \quad i=1,2,\dots,p \quad (2.1.6)$$

and in addition an additive random forcing term, $\underline{w}$.

The random acceleration, $\underline{w}$, may be employed as a model of any of the following physical phenomena :

- a) unknown fluctuations in solar winds
- b) uncertainty in the thrust vector, $\underline{D}u$, due to attitude control errors.

In the more general problem of Eq. (2.0.1) where the state, $\underline{x}(t)$, need not refer to the motion of a rocket, the quantity, $\underline{w}$, may refer to a wide variety of random forcing terms.

It is further assumed that noisy observations, $\underline{z}$, are made continuously. These observations are always linear in the state variables, $\underline{x}$, and corrupted with additive white Gaussian noise as in [1], [2], [3], [6], [7]. The general model of Eq. (2.0.1) is now complete.

A simplification that will be adopted throughout this report is that there is only one control variable. With respect to the physical model of the rocket, suppose that the gravitation field, $g(\underline{r})$ of Eq. (2.1.1) is caused by a single body, i.e.

$$g(\underline{r}) = - \frac{(M + m) G_o}{r^3} \underline{r}$$

where M is the mass of the planet, m the mass of the rocket, G_o the universal gravitation constant and $\underline{r}$ is now the distance from the center of gravity of the rocket planet system to the rocket. Then the motion of the rocket is a conic in a plane determined by the angular momentum vector $\underline{h}$. The single control u is directed in the $\hat{r}$ direction, in the plane, i.e. in the direction $\underline{r} \times \underline{h}$. Thus the motion remains in a plane and the effect of control is essentially to modify the magnitude of the angular momentum of the orbit. Note also that planar motion implies that the state has only four components,

With the above simplification of a single control Eq. (2.0.1) becomes

$$\left. \begin{aligned} \dot{\underline{x}}(t) &= F(t)\underline{x}(t) + \underline{d}(t)u(t) + \underline{w}(t) \\ \underline{z}(t) &= H(t)\underline{x}(t) + \underline{v}(t) \\ |u(t)| &\leq 1 \end{aligned} \right\} \quad (2.1.7)$$

Where $F(t)$, $d(t)$, $H(t)$ are all time - varying quantities. With this understanding the argument, t , may subsequently be dropped.

We assume that $\underline{w}$ and $\underline{v}$ are independent white noise processes, jointly Gaussian, which satisfy the conditions

$$\left. \begin{aligned} E [\underline{w}(t)] &= \underline{0} \\ E [\underline{v}(t)] &= \underline{0} \\ E [\underline{w}(t) \underline{v}^T(t+\tau)] &= \underline{0} \text{ for all } t, \tau \end{aligned} \right\} \quad (2.1.8a)$$

Also the "white" property may be expressed as

$$\left. \begin{aligned} E [\underline{w}(t) \underline{w}^T(t+\tau)] &= Q_w(t) \delta(\tau) \\ E [\underline{v}(t) \underline{v}^T(t+\tau)] &= R_n(t) \delta(\tau) \end{aligned} \right\} \quad (2.1.8b)$$

where $\delta(\tau)$ is the Dirac δ function.

The time $t = 0$ is an arbitrary time reference point so chosen such that $T > t_\mu > t_0 > 0$. In the rocket problem, this might correspond to the time of final stage separation. The initial error, $\underline{x}(0)$, is assumed to be a zero mean Gaussian random variable. Note also that $E[\underline{f}_1] = 0$.

The first equation of Eq.(2.1.7) is sometimes written

$$d\underline{x} = \underline{F}\underline{x} dt + \underline{d} \underline{u} dt + \underline{C}_1 d\underline{\sigma}$$

$$\text{where } \underline{C}_1(t) \underline{C}_1^T(t) = Q(t)$$

and $\underline{\sigma}(t)$ is a normalized Wiener process, [9]. Here $d\underline{\sigma}$ is identified with normalized white noise and $\underline{\sigma}(t)$ is defined to satisfy the relation, as in [9]

$$E [(\underline{\sigma}(t) - \underline{\sigma}(s)) (\underline{\sigma}^T(t) - \underline{\sigma}^T(s))] = I | t-s | \quad (2.1.9)$$

2.2 The Optimal Estimator

The question of a simultaneous solution of optimum estimation and control is very challenging and yet not completely solved. The best results to date appear to be given in [23]. Following the examples of [2], [6], and [7], and the justification of [4],[5], and [23] the estimator is specified to be the conditional mean given the entire history of past observations.

$$\hat{\underline{x}}(t) \triangleq E[\underline{x}(t) | z(\sigma), 0 \leq \sigma \leq t] \quad (2.2.1)$$

It is well-known, [3], that the error defined by

$$\tilde{\underline{x}}(t) = \underline{x}(t) - \hat{\underline{x}}(t) \quad (2.2.2)$$

has the important property that

$$E[\tilde{\underline{x}}(t) \underline{z}^T(\sigma), \sigma \leq t] = 0 \quad (2.2.3a)$$

or equivalently

$$E[\tilde{\underline{x}}(t) \hat{\underline{x}}^T(\sigma), \sigma \leq t] = 0 \quad (2.2.3b)$$

It turns out that the estimator given by Eq.(2.2.1) or alternatively by Eq.(2.2.3) will minimize the expected value of any convex loss function of the error, $\tilde{\underline{x}}$, conditioned on the history of past observations. Typical loss functions might be $\tilde{\underline{x}}^T \tilde{\underline{x}}$, $\Sigma(\tilde{\underline{x}})^4$ and so on.

The reader is referred to Kalman's original work, [3], for a fuller discussion. The estimator which satisfies Eq.(2.2.1) for the system of Eq.

(2.1.7) is given by Kalman to satisfy Eq.(2.2.4)

$$\dot{\hat{\underline{x}}} = \underline{F}\hat{\underline{x}} + \underline{d}u + \underline{C}e \quad (2.2.4a)$$

$$\hat{\underline{x}}(0) = \underline{0}$$

$$\underline{e} \triangleq \underline{z} - \underline{H}\hat{\underline{x}} = \underline{H}\tilde{\underline{x}} + \underline{v} \quad (2.2.4b)$$

$$\underline{P} \triangleq \underline{E} \left[\begin{array}{c} \tilde{\underline{x}}\tilde{\underline{x}}^T \\ \underline{xx}^T \end{array} \right]$$

$$\underline{C} = \underline{P} \underline{H}^T \underline{R}_n^{-1} \quad (2.2.4d)$$

$$\dot{\underline{P}} = \underline{F}\underline{P} + \underline{P}\underline{F}^T + \underline{Q} - \underline{P}\underline{H}^T \underline{R}_n^{-1} \underline{H}\underline{P} \quad (2.2.4e)$$

$\underline{P}(0)$ is computed from physical arguments as an independent problem.

The interpretation of the matrix, $\underline{P}$, is that of a measure of the average uncertainty in the state. If we assume no measurement information at $t = 0$, then in theory $\underline{P}(0)$ should be chosen to satisfy a physically determined initial error covariance. On the other hand, in order to obtain a conservative estimator, one can choose $\underline{P}(0) = \mu \underline{I}$, where $\underline{I}$ is an identity matrix and μ is very large.

It follows directly from Eq.(2.2.3a) and Eq.(2.2.4b) that the vector, $\underline{e}(t)$, has the important property of being orthogonal to the history of past observations, $\underline{z}(\sigma)$, $\sigma < t$. The following elementary argument based on Eqs. (2.2.3a and b) demonstrates that the vector, $\underline{e}(t)$, is also orthogonal to past values, $\underline{e}(\sigma)$, so that $\underline{e}(t)$ can be regarded as white noise. For $t > \sigma$, write $\underline{E}[\underline{e}(\sigma)\underline{e}^T(t)]$ from Eq.(2.2.4b) as

$$\begin{aligned} \underline{E}[(\underline{z}(\sigma) - \underline{H}\hat{\underline{x}}(\sigma))(\underline{x}^T(t)\underline{H}^T + \underline{v}^T(t))] &= \underline{E}[\underline{z}(\sigma)\underline{x}^T(t)\underline{H}^T \\ &\quad - \underline{H}\hat{\underline{x}}(\sigma)\underline{x}^T(t)\underline{H}^T + (\underline{z}(\sigma) - \underline{H}\hat{\underline{x}}(\sigma))\underline{v}^T(t)] \end{aligned}$$

The first term in the above expression is null from Eq.(2.2.3a) and the second is null from Eq.(2.2.3b). The remaining two terms are null by the assumption that future observation noise is entirely independent of any past inputs. Q.E.D

The white noise property of $\underline{e}(t)$ was used in [6] and another demonstration was given in [23]. Of course, $\underline{e}(t)$ must be Gaussian since it is a linear combination of other Gaussian random variables and it must satisfy Eq.(2.2.5):

$$E[\underline{e}] = \underline{0} \quad (2.2.5)$$

$$E[\underline{e}(t) \underline{e}^T(t + \tau)] = E_0(t) \delta(\tau)$$

The covariance quantity $E_0(t)$ is calculated directly from Eq. (2.2.4b)

$$E_0 = R_n \quad (2.2.6)$$

Physically, $\underline{e}(t)$, can be interpreted as "new information" in that it constitutes that portion of the observations, $\underline{z}(t)$, which cannot be predicted from past observations. The matrix covariance quantity $\phi E_0 \phi^T$ can be interpreted as the expected rate at which new information is obtained from the observations $\underline{z}(t)$.

2.3 An Integral Formulation

Valuable insight is obtained by expressing Eq. (2.2.4a) in integral form. The integral form uses a transition matrix $\phi(t, \tau)$ which has the properties:

$\phi(t, t) = I$, the unit diagonal matrix

$$\phi(t_1, t_2) = \phi(t_1, t_3) \phi(t_3, t_2) \quad (2.3.1)$$

$$\frac{d}{dt} \phi(t, t_1) = F(t) \phi(t, t_1)$$

A more complete discussion of the integral solution to Eq. (2.2.4b) and its properties may be found in Athans & Falb [13]. Since $\hat{\underline{x}}(0) = \underline{0}$, the integral solution to Eq. (2.2.4b) is

$$\begin{aligned} \hat{\underline{x}}(t) = & \phi(t, 0) \int_0^t \phi^{-1}(\tau, 0) \underline{d}(\tau) u(\tau) d\tau \\ & + \phi(t, 0) \int_0^t \phi^{-1}(\tau, 0) C(\tau) \underline{e}(\tau) d\tau \end{aligned} \quad (2.3.2)$$

NOTATION:

In what follows the second argument of the transition matrix will always be 0. Hence we shall define

$$\phi(t) \triangleq \phi(t, 0) \quad (2.3.3)$$

Note that since the second argument is 0 we do not have full freedom to use the second equation of Eq. (2.3.1). One consequence of this is that the inverse operator must be used. Moreover, the single argument, t , in Eq. (2.3.3) will always be positive.

Note that in Eq. (2.3.2), $\hat{x}(t)$ depends on past values of $\underline{e}(\tau)$ and $u(\tau)$, $\tau \leq t$. The control $u(\tau)$ is non-anticipatory and depends only on preceding observations. In particular $u(\tau)$ can be regarded as a function of $\underline{e}(\tau)$, $0 \leq \tau \leq t$. Eq. (2.3.2) as it stands describes the evolution of, $\hat{x}$, a random quantity depending on a specific measurement history. What is required to eliminate the dependence on specific measurements is an ensemble average quantity which describes the state $\hat{x}$ statistically. Such a quantity is the correlation matrix, X , henceforth called STATE MATRIX, defined by

$$X(t) \triangleq E [\hat{x}(t) \hat{x}^T(t)] \quad (2.3.4a)$$

In order to describe the evolution of the state, $X(t)$, the following correlation quantities are required:

$$r(t, t) \triangleq E [u(t) u(t_1)] \quad (2.3.4b)$$

$$r(t) \triangleq r(t, t) \quad (2.3.4b)$$

$$\underline{b}(t, t_1) \triangleq E [\hat{x}(t) u(t_1)] \quad (2.3.4c)$$

$$\underline{b}(t) \triangleq \underline{b}(t, t) \quad (2.3.4c)$$

$$\underline{f}(t, t_1) \triangleq \begin{cases} E [\underline{e}(t) u(t_1)] & \text{for } t \leq t_1 \\ 0 & t > t_1 \end{cases} \quad (2.3.4d)$$

The vector, $\underline{b}(t)$, in Eq. (2.3.4c) above, will be called the STOCHASTIC CONTROL. Note that a control policy $u(t) = u(\underline{z}(\tau), \tau \leq t)$ determines $\underline{b}(t)$, but $\underline{b}(t)$ alone does not determine a control policy.

We now proceed to evaluate the state matrix, $X(t)$, in terms of the control, $\underline{b}(t)$, and other correlation quantities defined by Eq. (2.3.4).

Postmultiplying Eq. (2.3.2) by $u(t)$ and ensemble averaging yields a relation involving $\underline{b}(t)$.

$$\underline{b}(t) = \phi(t) \int_0^t \phi^{-1}(\tau) \underline{d}(\tau) r(\tau, t) d\tau + \phi(t) \int_0^t \phi^{-1}(\tau) C(\tau) f(\tau, t) d\tau \quad (2.3.5)$$

Also from Eq. (2.3.2) a relation in $X(t)$ is obtained as follows :

$$\begin{aligned} \phi^{-1}(t) E [\hat{\underline{x}}(t) \hat{\underline{x}}(t)^T] \phi^{-1}(t)^T &= \phi^{-1}(t) X(t) \phi^{-1}(t)^T \\ &= E \left[\left(\int_0^t \phi^{-1}(\tau) \underline{d}(\tau) u(\tau) d\tau + \int_0^t \phi^{-1}(\tau) C(\tau) \underline{e}(\tau) d\tau \right) \right. \\ &\quad \left. \left(\int_0^t u(\sigma) \underline{d}^T(\sigma) \phi^{-1}(\sigma)^T d\sigma + \int_0^t \underline{e}^T(\sigma) C^T(\sigma) \phi^{-1}(\sigma)^T d\sigma \right) \right] \\ &= \int_0^t \int_0^t \phi^{-1}(\tau) \underline{d}(\tau) r(\tau, \sigma) \underline{d}^T(\sigma) \phi^{-1}(\sigma)^T d\sigma d\tau \\ &\quad + \int_0^t \int_0^t \phi^{-1}(\sigma) C(\sigma) \underline{f}(\sigma, \tau) \underline{d}^T(\tau) \phi^{-1}(\tau)^T d\sigma d\tau \\ &\quad + \int_0^t \phi^{-1}(\tau) C(\tau) E_0(\tau) C^T(\tau) \phi^{-1}(\tau)^T d\tau \\ &\quad + \int_0^t \int_0^t \phi^{-1}(\tau) \underline{d}(\tau) \underline{f}^T(\sigma, \tau) C^T(\sigma) \phi^{-1}(\sigma)^T d\sigma d\tau \end{aligned} \quad (2.3.6)$$

We wish to solve for $X(t)$ in terms of $\underline{b}(t)$ so as to eliminate $r(\tau, \sigma)$ and $\underline{f}(\sigma, \tau)$. From Eq. (2.3.5) we obtain the identities Eq. (2.3.7a) and Eq. (2.3.7b)

$$\begin{aligned}
& \int_0^t \phi^{-1}(\tau) d(\tau) \int_0^\tau \underline{f}^T(\sigma, \tau) C^T(\sigma) \phi^{-1}(\sigma) d\sigma d\tau \\
&= \int_0^t \phi^{-1}(\tau) d(\tau) \left[\underline{b}^T(\tau) \phi^{-1}(\tau)^T - \int_0^\tau r(\tau, \sigma) \underline{d}^T(\sigma) \phi^{-1}(\sigma)^T d\sigma \right] d\tau \\
&\quad (2.3.7a)
\end{aligned}$$

$$\begin{aligned}
& \int_0^t \int_0^\tau \phi^{-1}(\sigma) C(\sigma) f(\sigma, \tau) \underline{d}^T(\tau) \phi^{-1}(\tau)^T d\sigma d\tau \\
&= \int_0^t \left[\phi^{-1}(\tau) \underline{b}(\tau) - \int_0^\tau \phi^{-1}(\sigma) \underline{d}(\sigma) r(\sigma, \tau) \right] \underline{d}^T(\tau) \phi^{-1}(\tau)^T d\sigma d\tau \\
&\quad (2.3.7b)
\end{aligned}$$

It follows that the sum of the middle two integrals of Eq. (2.3.6) can be written as the combination of Eqs. (2.3.7a) and (2.3.7b)

$$\begin{aligned}
\text{SUM} &= \int_0^t \phi^{-1}(\tau) \left(\underline{d}(\tau) \underline{b}^T(\tau) + \underline{b}(\tau) \underline{d}^T(\tau) \right) \phi^{-1}(\tau)^T d\tau \\
&- \int_0^t \phi^{-1}(\tau) \underline{d}(\tau) \int_0^\tau r(\tau, \sigma) \underline{d}^T(\sigma) \phi^{-1}(\sigma)^T d\sigma d\tau \\
&- \int_0^t \int_0^\tau \phi^{-1}(\sigma) \underline{d}(\sigma) r(\sigma, \tau) \underline{d}^T(\tau) \phi^{-1}(\tau)^T d\sigma d\tau
\end{aligned}$$

But note the identity

$$\begin{aligned}
& \int_0^t \int_0^\tau \phi^{-1}(\sigma) \underline{d}(\sigma) r(\sigma, \tau) \underline{d}^T(\tau) \phi^{-1}(\tau)^T d\sigma d\tau \\
&= \int_0^t \int_\tau^t \phi^{-1}(\tau) \underline{d}(\tau) r(\tau, \sigma) \underline{d}^T(\sigma) \phi^{-1}(\sigma)^T d\tau d\sigma
\end{aligned}$$

obtained by interchanging the order of integration, a permissible operation since the integrand is continuous.

It follows that the two multiple integrals in the SUM combine into a single multiple integral. The quantity SUM can therefore be written

$$\begin{aligned} \text{SUM} &= \int_0^t \phi^{-1}(\tau) [\underline{d}(\tau) \underline{b}^T(\tau) + \underline{b}(\tau) \underline{d}^T(\tau)] \phi^{-1}(\tau) d\tau \\ &\quad - \int_0^t \int_0^t \phi^{-1}(\tau) \underline{d}(\tau) r(\tau, \sigma) \underline{d}^T(\sigma) \phi^{-1}(\sigma)^T d\sigma d\tau \end{aligned}$$

Returning to Eq. (2.3.6) if the quantity SUM is substituted for the middle two integrals the last term in SUM cancels the first term in Eq. (2.3.6) and Eq. (2.3.6) becomes

$$\begin{aligned} \phi^{-1}(t) X(t) \phi^{-1}(t)^T &= \int_0^t \phi^{-1}(\tau) [\underline{d}(\tau) \underline{b}^T(\tau) + \underline{b}(\tau) \underline{d}^T(\tau)] \phi^{-1}(\tau) d\tau \\ &\quad + \int_0^t \phi^{-1}(\tau) C(\tau) E_0(\tau) C^T(\tau) \phi^{-1}(\tau)^T d\tau \end{aligned} \quad (2.3.8)$$

Note that the first term on the right of Eq. (2.3.8) contains the control term, $\underline{b}(\tau)$, while the second term on the right contains the information obtained from observations in the form of $E_0(\tau)$. The second term will henceforth be called the OBSERVATION MATRIX and will be called $Q(t)$.

$$Q(t) \triangleq \int_0^t \phi^{-1}(\tau) C(\tau) E_0(\tau) C^T(\tau) \phi^{-1}(\tau)^T d\tau \quad (2.3.9)$$

It will also be convenient to define a Target Matrix by $D(t)$ according to

$$D(t) \triangleq Q(t) - \phi^{-1}(t) \dot{X}(t) \phi^{-1}(t) \quad (2.3.10)$$

We shall require the observability condition that $Q(t)$ be positive definite for all $t \geq t_0$. This will make $X(t)$ positive definite for $t \geq t_0$. Recall from the problem statements of Section (1.1) that control does not take place prior to $t = t_0$. We shall restrict attention to those systems which have the property that $X(t)$ remains positive definite for $t \geq t_0$ under conditions of control.

On substituting Eqs.(2.3.9) and (2.3.10) into Eq.(2.3.8) the target matrix, $D(t)$, can be directly related to the control vector $\underline{b}(\tau)$.

$$-D(t) = \int_0^t \phi^{-1}(\tau) [\underline{d}(\tau) \underline{b}^T(\tau) + \underline{b}(\tau) \underline{d}^T(\tau)] \phi^{-1}(\tau)^T d\tau \quad (2.3.11)$$

Observe that if the control $u(t) \equiv 0$, so that $\underline{b}(t) \equiv 0$, then $D(t) = 0$. In this event $X(t)$ is related to the observation matrix by $X(t) = \phi(t) Q(t) \phi^T(t)$.

Eq.(2.3.11) is also needed in a differential form for the Hamiltonian formulation of Chapter 4. As a first step, pre multiply Eq.(2.3.8) by $\phi(t)$ and postmultiply it by $\phi^T(t)$ to yield Eq.(2.3.12)

$$\begin{aligned} X(t) = & \phi(t) \int_0^t \phi^{-1}(\tau) [\underline{d}(\tau) \underline{b}^T(\tau) + \underline{b}(\tau) \underline{d}^T(\tau)] \phi^{-1}(\tau)^T d\tau \phi^T(t) \\ & + \phi(t) \int_0^t \phi^{-1}(\tau) C(\tau) E_0(\tau) C^T(\tau) \phi^{-1}(\tau)^T d\tau \phi^T(t) \end{aligned} \quad (2.3.12)$$

Next differentiate $X(t)$ with respect to t , recalling that $\dot{\phi}(t) = F(t)\phi(t)$ and $\phi(t)^T = \phi(t)^T F^T(t)$. This yields

$$\dot{X} = FX + XF^T + \underline{d} \underline{b}^T + \underline{b} \underline{d}^T + C E_0 C^T \quad (2.3.13)$$

Eqs. (2.3.11) and (2.3.13) are the desired results of this section and constitute the basic dynamical relations for the system of Eq. (2.1.7). Note however that Eqs. (2.3.11) and (2.3.13) are incomplete in that we have not given a relation between a control law, $u(t) = u(z(\tau), \tau \leq t)$ and the quantity $\underline{b}(t)$ defined by Eq. (2.3.4c). This relation is the subject of Chapter 3 and will not be discussed further here.

Finally note that there is a correspondence between the dynamics of the deterministic system of Eq. (2.3.14), and Eqs. (2.3.11) and (2.3.13). The deterministic system

$$\dot{\underline{x}}(t) = F(t)\underline{x}(t) + u(t)\underline{d}(t) \quad ; \quad \underline{x}(0) = \underline{c} \quad (2.3.14a)$$

has an integral solution, [13]

$$-(\underline{c} - \phi^{-1}(t)\underline{x}(t)) = \int_0^t \phi^{-1}(\tau)\underline{d}(\tau)u(\tau)d\tau \quad (2.3.14b)$$

Then Eq. (2.3.14a) corresponds to Eq. (2.3.13) and Eq. (2.3.14b) corresponds to Eq. (2.3.11) in the sense described in TABLE 1.

TABLE 1

STOCHASTIC QUANTITY		DETERMINISTIC QUANTITY
$\underline{X}(t)$	STATE	$\underline{x}(t)$
$\underline{b}(t)$	CONTROL	$u(t)$
$Q(t)$	INFORMATION	$\underline{c}$
$D(t)$	TARGET	$\underline{c} - \phi^{-1}(t)\underline{x}(t)$

The target matrix $D(T)$ above implies that control takes an initial state, $\underline{X}(t_0) = 0$ to a terminal state, $\underline{X}(T)$, related to $D(T)$ by Eq. (2.3.10). It is generally unnecessarily restrictive to require that $\underline{X}(T) = 0$, and in fact the

associated cost may be prohibitive or even infinite or otherwise a well-behaved solution may not exist. There is generally no physical reason to directly specify a non-zero $X(T)$. There is, however, physical justification for requiring that $X(T)$ satisfy a target set condition of the form *

$$E [\hat{\underline{x}}^T(T) S(T) \hat{\underline{x}}(T)] = \text{trace} [S(T) X(T)] \leq \omega_0 \quad (2.3.15)$$

Since Eq.(2.3.15) does not uniquely determine $X(T)$ or $D(T)$ the formulation of the optimum control problem must consider Eq.(2.3.15) directly.

The target set condition of Eq.(2.3.15) is a statement about average measure quantities and not about distances and velocities in a physical space that are unrelated to observations. It is thus possible, though highly unlikely, for a "good" target set of Eq.(2.3.15) to be satisfied but yet to have the rocket impact the planet instead of orbiting because the measurement history was a poor non-typical member of the ensemble.

Problems in which the target set is described in terms of the state, $\underline{x}$, of Eq.(2.1.7) rather than in terms of X belong to the class of "first-crossing problems." In such a problem, the issue is to specify a probability of first-crossing into the desired region such that cost is minimized. Such a probability is directly related to the parameter, ω_0 , in Eq.(2.3.15). Hence there is a close relationship between the "first-crossing" problem and the formulation presented here. However, the virtue of our formulation is that it avoids the complexities of this generally insoluble problem.

*

The target matrix $S(T)$ is assumed positive definite for practical convenience. In practice the elements of $S(T)$ can be made arbitrarily small without vanishing if so required.

3. The Parametric Structure Of The Controller

Recall from the discussion following Eq. (2.3.4) that the control $u(t) = u(\underline{z}(\sigma), \sigma \leq t)$ is not yet specified by the stochastic control $\underline{b}(t)$. In order for the stochastic control, $\underline{b}(t)$, to specify a policy it is necessary to make the further assumption of a parametric structure for the control, $u(t)$, which of course satisfies the constraint $|u(t)| \leq 1$. This issue is the subject of this chapter.

While there may exist an optimal structure a full description of it is as yet elusive. However, by employing the concepts of Dynamic Programming as in, [9], much insight can be gained. A structure is postulated which appears to be a suitable first approximation to the optimum. Some preliminary notions are required first.

3.1 The Weak Differential Operator

In the Dynamic Programming formulation of the stochastic optimal control problems 1,2, and 3, a certain limiting operator appears, which has been named the weak differential operator or or the weak differential generator, [9]. The weak differential operator of a function, $V(\underline{x}(t), t)$ is denoted by $\mathcal{L} V(\underline{x}(t), t)$ and is defined by Eq. (3.1.1)

$$\mathcal{L} V(\underline{x}(t), t) = \lim_{\Delta \rightarrow 0} \frac{E [V(\underline{x}(t+\Delta), t+\Delta)] - V(\underline{x}(t), t)}{\Delta} \quad (3.1.1)$$

For the optimal estimator of Eq. (2.2.4a)

$$\dot{\underline{x}} = \underline{F}\underline{x} + \underline{d}u + \underline{C}e \quad (3.1.2)$$

$$E [\underline{e}] = \underline{0}$$

$$E [\underline{e}(t) \underline{e}^T(t + \tau)] = E_0(t) \delta(\tau) \quad (3.1.3)$$

The weak differential operator, $\hat{V}$, is given by, [9], $\hat{V}(\hat{\underline{x}}(t), t) =$

$$\underline{V}_{-\underline{x}}^T (\underline{F}\hat{\underline{x}} + \underline{d} u) + 1/2 \text{ trace } (\hat{\underline{V}}_{\underline{xx}} \underline{C} \underline{E}_o \underline{C}^T) + \frac{\partial V}{\partial t} \quad (3.1.4)$$

where the subscripts denote partial differentiation. Thus

$$\hat{\underline{V}}_{-\underline{x}} = \frac{\partial V}{\partial \underline{x}} = \left\{ \frac{\partial V}{\partial x_1} \right\}$$

$$\hat{\underline{V}}_{\underline{xx}} = \left\{ \frac{\partial V}{\partial x_i \partial x_j} \right\}$$

The derivation of Eq. (3.1.4) from the definition of Eq. (3.1.1) is instructive.

Assume $V(\hat{\underline{x}}, t)$ is smooth in $\hat{\underline{x}}$ and t . Then expanding $V(\hat{\underline{x}}(t + \Delta), t + \Delta)$ in a Taylor Series yields

$$\begin{aligned} V(\hat{\underline{x}}(t + \Delta), t + \Delta) &= V(\hat{\underline{x}}(t), t) + \underline{V}_{-\underline{x}}^T (\hat{\underline{x}}(t + \Delta) - \hat{\underline{x}}(t)) + \underline{V}_t \Delta \\ &\quad + 1/2 \text{ trace } [\hat{\underline{V}}_{\underline{xx}} (\hat{\underline{x}}(t + \Delta) - \hat{\underline{x}}(t)) (\hat{\underline{x}}(t + \Delta) - \hat{\underline{x}}(t))^T] \\ &\quad + \underline{V}_{xt} \Delta (\hat{\underline{x}}(t + \Delta) - \hat{\underline{x}}(t)) + 1/2 \underline{V}_{tt} \Delta^2 + \text{higher order terms.} \end{aligned}$$

Consequently,

$$\begin{aligned} V(\hat{\underline{x}}(t + \Delta), t + \Delta) &= V(\hat{\underline{x}}(t), t) + \underline{V}_{-\underline{x}}^T \int_t^{t+\Delta} (\underline{F}\hat{\underline{x}} + \underline{d}u + \underline{C}\underline{e}) d\tau + \underline{V}_t \Delta \\ &\quad + 1/2 \text{ trace } [\hat{\underline{V}}_{\underline{xx}} \int_t^{t+\Delta} d\tau \int_t^{t+\Delta} d\sigma (\underline{F}\hat{\underline{x}} + \underline{d}u + \underline{C}\underline{e}(\sigma)) \\ &\quad (\underline{F}\hat{\underline{x}}(\tau) + \underline{d}u(\tau) + \underline{C}\underline{e}(\tau))^T \\ &\quad + \text{other terms.} \end{aligned} \quad (3.1.5)$$

Taking the expected value of both sides assuming $\hat{\underline{x}}(t)$ given yields

$$\begin{aligned}
 E [V(\hat{\underline{x}}(t+\Delta), t+\Delta) | \hat{\underline{x}}(t)] &= V(\hat{\underline{x}}(t), t) + V_{\underline{x}}^T \int_t^{t+\Delta} (\underline{F}\hat{\underline{x}} + \underline{du}) d\tau + V_t \Delta \\
 &+ 1/2 \text{ trace } [V_{\underline{x}\underline{x}}^{\wedge\wedge} \int_t^{t+\Delta} d\tau \int_t^{t+\Delta} d\sigma (\underline{F}\hat{\underline{x}}(\sigma) + \underline{du}(\sigma)) \\
 &(\underline{F}\hat{\underline{x}}(\tau) + \underline{du}(\tau)) \\
 &+ 1/2 \text{ trace } [V_{\underline{x}\underline{x}}^{\wedge\wedge} \int_t^{t+\Delta} d\tau \int_t^{t+\Delta} d\sigma C E[\underline{e}(\tau)\underline{e}(\sigma)^T] C^T]
 \end{aligned}
 \tag{3.1.6}$$

From Eq. (3.1.3) the last double integral of Eq.(3.1.6) collapses to a single integral

$$\begin{aligned}
 1/2 \int_t^{t+\Delta} d\sigma \int_t^{t+\Delta} d\tau \text{ trace } [V_{\underline{x}\underline{x}}^{\wedge\wedge} C E\{ \underline{e}(\sigma)\underline{e}^T(\tau) \} C^T] &= \\
 1/2 \int_t^{t+\Delta} \text{ trace } [V_{\underline{x}\underline{x}}^{\wedge\wedge} C E_0 C^T] d\tau
 \end{aligned}
 \tag{3.1.7}$$

On substituting Eq.(3.1.7) back into Eq.(3.1.6) and letting $\Delta \rightarrow 0$ yields

$$\begin{aligned}
E[V(\hat{\underline{x}}(t+\Delta), t+\Delta)] &= V(\hat{\underline{x}}(t), t) + \Delta \left[\hat{V}_{\underline{x}}^T (\hat{F}\hat{\underline{x}} + \hat{d}) + \frac{\partial V}{\partial t} \right. \\
&\quad \left. + \frac{1}{2} \text{trace} (\hat{V}_{\underline{xx}} \hat{C} \hat{C}^T) \right] + \frac{\Delta^2}{2} \text{trace} [\hat{V}_{\underline{xx}} (\hat{F}\hat{\underline{x}} + \hat{d}) (\hat{F}\hat{\underline{x}} + \hat{d})^T] \\
&\quad + \dots +
\end{aligned} \tag{3.1.8}$$

from which Eq. (3.1.4) follows upon neglecting terms of second and higher order in Δ . Note that the expansion of Eq.(3.1.5) is more basic than a Taylor's Series.

One further property is of some interest. Suppose

$$W(t) = E[V(\hat{\underline{x}}(t), t)] \tag{3.1.9}$$

Then for future times it must be true that

$$W(t+\Delta) = E[E[V(\hat{\underline{x}}(t+\Delta), t+\Delta) \mid \hat{\underline{x}}(t)]] \tag{3.1.10}$$

Hence it follows that

$$\begin{aligned}
\frac{dW}{dt}(t) &= \lim_{\Delta \rightarrow 0} \frac{W(t+\Delta) - W(t)}{\Delta} = E \left[\frac{E[V(\hat{\underline{x}}(t+\Delta), t+\Delta) \mid \hat{\underline{x}}(t)]}{\Delta} \right. \\
&\quad \left. - \frac{V(\hat{\underline{x}}(t), t)}{\Delta} \right] = E \left[\frac{\partial V(\hat{\underline{x}}(t), t)}{\partial t} \right]
\end{aligned} \tag{3.1.11}$$

See also Dynkin's Formula [9]

3.2 A Dynamic Programming Evaluation of The Control Law Structure For The Minimum Time, Minimum Expected Fuel and Least Upper Bound Fuel Problems.

In this section the methods of Dynamic Programming are used to evaluate a control law structure for the stated problems of interest. In order to set up the Dynamic Programming arguments in a convenient and yet rigorous fashion the concept of a penalty function will be introduced. Then we shall argue that choosing the penalty function appropriately is equivalent to specifying a boundary condition at the terminal time of the form

$$E[\hat{\underline{x}}(T)^T S \hat{\underline{x}}(T)] = \omega_0 \quad (2.3.15)$$

Recall in Section 2 that the estimator, $\hat{\underline{x}}(t)$, defined by the Kalman Filter was chosen as a sufficient statistic over the set of past observations $\mathcal{Y}(\sigma)$, $\sigma \leq t$. Hence all cost functionals are chosen to depend on the observations only through $\hat{\underline{x}}$. Let us now consider the problem

$$J = \min_{\substack{u \in [t_1, T] \\ p}} E[\hat{\underline{v}}\hat{\underline{x}}^T(T) S \hat{\underline{x}}(T) + \int_{t_1}^T \tilde{L}_1(\tau, p, u(\tau), \hat{\underline{x}}(\tau)) d\tau] \quad (3.2.1)$$

where the interval $[t_1, T]$ is non-random and p is a parameter or set of parameters. The case where the lower limit of integration, t_1 , is a parameter belonging to the parameter set p is also considered.

The solution of Eq.(3.2.1) by Dynamic Programming requires the following general assumptions :

A1) Given $\hat{\underline{x}}(t)$ there exists a scalar function $V_1(\hat{\underline{x}}(t), t)$ whose value equals the minimum remaining cost.

$$V_1(\hat{\underline{x}}(t), t) = \min_{\substack{u \in [t, T] \\ p}} E[(v\hat{\underline{x}}^T(T)S\hat{\underline{x}}(T) + \int_t^T \tilde{L}_1(\tau, p, u(\tau), \hat{\underline{x}}(\tau)) d\tau) | \hat{\underline{x}}(t)] \quad (3.2.2)$$

$$\text{Hence } J = V_1(\hat{\underline{x}}(t_1), t_1)$$

A2) V_1 has a continuous second partial derivative with respect to $\hat{\underline{x}}$, $V_{\hat{\underline{x}}\hat{\underline{x}}}$. Boltyanskii, [20], has shown that these derivatives of V need not exist at every point in the space.

A3) As $\hat{\underline{x}}(t)$ evolves continuously according to Eq.(3.1.2), V defined by Eq.(3.2.2) remains optimum. This is the principle of optimality.

Note that at the terminal time T

$$V(\hat{x}(T), T) = \hat{v}^T \hat{x}(T) S \hat{x}(T) \quad (3.2.3)$$

Let us now consider the problem of minimum time. We are to choose t_1 such that the interval $[t, T]$ is minimum subject to the constraint of Eq. (2.3.15). To put this formulation into that of Eq. (3.2.1) set $t_1 = p$ and $\tilde{L}_1 = 1$. We pick initial values for t_1 and v and solve for $u(\tau)$. When the time interval $T - t_1$ is large then the penalty $E[\hat{v}^T \hat{x}(T) S \hat{x}(T)]$ will be small and vice-versa. Hence there is a value of $t_1 = t_\mu(v)$ such that J is minimized. Now for values of $t_1 = t_\mu(v)$ compare $E[\hat{v}^T \hat{x}(T) S \hat{x}(T)]$ with ω_0 . If v is too large then $E[\hat{v}^T \hat{x}(T) S \hat{x}(T)]$ will be too small since the penalty on terminal error was too large. A contrary argument holds if v is too small. Consequently, there is some value of v such that Eq. (2.3.15) is exactly satisfied.

The problem of minimum expected fuel consumption is even simpler for now $t_1 = t_0$ and $\tilde{L}_1 = |u|$. The performance criterion J is minimized for all values of v and then v is chosen to satisfy Eq. (2.3.15).

For the case of least upper bound fuel consumption we must specify an interval set, Ω , of non-overlapping intervals. Ω is defined by an indicator function, i_ω which is 1 on Ω and 0 elsewhere. Then $t_1 = t_0$ and $\tilde{L}_1 = i_\omega$. We pick v and the set Ω and solve for $u(\tau)$ to minimize J .

When the Lebesgue measure of Ω is large then the penalty will be small and vice-versa. Hence there is some set of intervals Ω depending on v , $\Omega(v)$, such that J is minimized. For all $\Omega(v)$ choose that Ω corresponding to the v such that Eq.(2.3.15) is satisfied. It can be seen that this L.U.B. problem is simply a generalization of the minimum time problem.

We remark that the above arguments are not intended to demonstrate a computational algorithm but only structural properties of the solution.

The principle of Optimality states that V should be optimal for all t . Consequently for all $\Delta > 0$

$$V(\hat{\underline{x}}(t), t) = \min_{u \in [t, T_1]} \{ E[V(\hat{\underline{x}}(t+\Delta), t+\Delta \mid \hat{\underline{x}}(t)) + E \int_t^{t+\Delta} [\tilde{L}_1(t, u(\tau), \underline{x}(\tau) \mid \hat{\underline{x}}(t)) d\tau] \}$$

(3.2.4)

Note that the expectation over future times takes into account only measurements up to and including t which determine $\hat{x}(t)$. Hence $\hat{x}(t+\Delta)$ depends also on the random variables, $e(\tau)$, $t < \tau \leq t + \Delta$. On rearranging terms in Eq. (3.2.4) and dividing through by Δ , we obtain

$$\begin{aligned} \text{Min}_{u \in [t, T]} \quad & \frac{1}{\Delta} \{ E [V(\hat{x}(t+\Delta), t+\Delta | \hat{x}(t))] - V(\hat{x}(t), t) \\ & + E \int_t^{t+\Delta} [\tilde{L}_1(\tau, u(\tau), \hat{x}(\tau) | \hat{x}(t))] d\tau \} = 0 \end{aligned} \quad (3.2.5)$$

Passing to the limit as $\Delta \rightarrow 0$ the first two terms become the weak differential operator, $\mathcal{L}V$, defined by Eq. (3.1.1) and given by Eq. (3.1.4). The last term becomes

$$\lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \int_t^{t+\Delta} [\tilde{L}_1(\tau, u(\tau), \hat{x}(\tau) | \hat{x}(t))] dt \rightarrow \tilde{L}_1(t, u(t), \hat{x}(t))$$

Thus Eq. (3.2.5) becomes in the limit as $\Delta \rightarrow 0$

$$\text{Min}_u \{ \mathcal{L}V + \tilde{L}_1 \} = 0 \quad (3.2.6)$$

On substituting the expression for $\mathcal{L}V$ from Eq. (3.1.4) into Eq. (3.2.6) above the result is

$$\text{Min}_u \{ \underline{v}_x^T (\underline{F}\hat{x} + \underline{d}u) + 1/2 \text{ trace } [\underline{v}_{xx}^T \underline{C} \underline{E}_0 \underline{C}^T] + \partial V / \partial t + \tilde{L}_1 \} = 0 \quad (3.2.7)$$

Eq. (3.2.7) together with the imposed boundary conditions of Eq. (3.2.3) determines the control policy $u = u(\hat{x})$ over the fixed time interval $[t_1, T]$.

General methods for the solution of Eq.(3.2.7) are not available, but as already noted, the equation is useful to indicate general features of the control.

Returning to Eq.(3.2.7) with $\hat{L}_1 = 1$ and the constraint $|u| \leq 1$ the solution for u in terms of V is easily obtained and is given by Eq.(3.2.8)

$$u = -\text{sgn } \underline{V}_x^T \underline{d} \quad (3.2.8)$$

Consequently, the minimum time controller must be bang-bang i.e. take on only the values $+1, -1$.

For the problem of minimum expected fuel consumption where $\hat{L}_1 = |u|$ and again the control is constrained to be bounded by the relation $|u| \leq 1$ the solution for u in terms of V becomes

$$u = \begin{cases} +1 & \underline{V}_x^T \underline{d} \leq -1 \\ 0 & -1 < \underline{V}_x^T \underline{d} < 1 \\ -1 & \underline{V}_x^T \underline{d} \geq 1 \end{cases} \quad (3.2.9)$$

For the problem of least upper bound fuel consumption $|u| \leq 1$ on Ω and $|u| = 0$ elsewhere. Hence on Ω u has the structure of Eq.(3.2.8) and is identically zero elsewhere. In terms of the indicator function, i_ω , we may write

$$u = i_\omega \text{sgn } \underline{V}_x^T \underline{d} \quad (3.2.10)$$

Eqs.(3.2.8), (3.2.9), (3.2.10) do not yet specify the control structure since $\underline{V}_x$ is unknown. Our approach is to attempt a representation for $V(\hat{x}(t), t)$ and then approximate the representation. The resulting structure will not

satisfy Eq. (3.2.7) but it appears to be a reasonable approximation.

Suppose $V(\hat{\underline{x}}(t), t)$ could be represented by an expansion of the

$$V(\hat{\underline{x}}(t), t) = \hat{\underline{x}}^T(t) \Gamma(t) \hat{\underline{x}} + \sum_{i,j,k} a_{ijk} \hat{x}_i \hat{x}_j \hat{x}_k + \sum_{i,j,k,\ell} b_{ijkl} \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_\ell + \dots + \quad (3.2.11)$$

If the first term in the expansion is a suitable approximation then

$$\underline{V}_x^{\wedge} \approx \Gamma(t) \hat{\underline{x}}(t) \quad (3.2.12)$$

from which it follows that the term $\underline{V}_x^{\wedge T} \underline{d}$ in Eqs. (3.2.8), (3.2.9) and (3.2.10) is simply a linear combination of the elements of $\hat{\underline{x}}$.

For the minimum time problem and L.U.B. problems, if we represent the term

$$\underline{V}_x^{\wedge T} \underline{d} \text{ by } -\underline{k}^T \hat{\underline{x}}, \text{ then Eq. (3.2.8) and (3.2.10) become} \quad (3.2.13)$$

$$u(t) = \text{sgn } \underline{k}^T(t) \hat{\underline{x}}(t) \quad (3.2.13)$$

$$u(t) = i_w \text{sgn } \underline{k}^T(t) \hat{\underline{x}}(t) \quad (3.2.14)$$

where $\underline{k}(t)$ is some time-varying vector depending only on the a priori

known dynamics and statistics. Note that the control does not depend on the magnitude of $\underline{k}$.

For the minimum expected fuel problem it is convenient to express the term $\underline{V}_x^{\wedge T} \underline{d}$ by $-\underline{k}^T \hat{\underline{x}} / a \sqrt{\underline{k}^T \underline{k}}$ which is independent of the magnitude of $\underline{k}$. Then Eq. (3.2.9) becomes

$$u = \begin{cases} +1 & \frac{\underline{k}^T \hat{\underline{x}}}{\sqrt{\underline{k}^T \underline{k}}} \geq a \sqrt{\underline{k}^T \underline{k}} \\ 0 & -a \sqrt{\underline{k}^T \underline{k}} < \frac{\underline{k}^T \hat{\underline{x}}}{\sqrt{\underline{k}^T \underline{k}}} < a \sqrt{\underline{k}^T \underline{k}} \\ -1 & \frac{\underline{k}^T \hat{\underline{x}}}{\sqrt{\underline{k}^T \underline{k}}} \leq -a \sqrt{\underline{k}^T \underline{k}} \end{cases} \quad (3.2.15)$$

The parameter a in Eq. (3.2.15) above corresponds to a threshold. In particular when $a = 0$ then Eq. (3.2.15) reduces to Eq.(3.2.13). Thus the proposed parametric structure of the minimum time problem can be regarded as a special case of the parametric structure of the minimum expected fuel problem.

It is of some interest that the structure proposed in Eqs. (3.2.15) and (3.2.13) may actually be more general than that implied by the approximation $V \approx \hat{\underline{x}}^T \Gamma \hat{\underline{x}}$. Thus in Lim [14] where the asymptotic solution is found for a Bellman-Hamilton-Jacobi equation for a quadratic performance with bounded control, a bang-bang control similar to Eq.(3.2.13) is seen to correspond to a V containing both cubic and quadratic terms in $\hat{\underline{x}}$.

More recently, Deyst (16), worked out a second-order sampled minimum expected fuel problem directly with Dynamic Programming in which he solved for the cost function, $V(\hat{\underline{x}}(n))$, recursively starting at the terminal time and working backwards to the n^{th} sample time. This exact solution took the form of Eq.(3.2.15) i.e. a linear function of $\hat{\underline{x}}$ being compared with a symmetric threshold. This result is rather encouraging for the assumptions which lead to Eq.(3.2.15).

3.3 Evaluation of The Stochastic Control Vector, $\underline{b}$.

Recall that the stochastic control vector, $\underline{b}$, appearing in Eqs. (2.3.11) and (2.3.13) and defined by

$$\underline{b} = E[\hat{\underline{x}}u] \quad (2.3.4b)$$

was left undefined in chapter 2. In this section, the control, $\underline{b}$, is determined approximately in terms of the general parametric structure of Eq.(3.2.11). Hence it will then be possible to express the stochastic

state, $\underline{x}$, of Eq. (2.3.13) directly in terms of the parameters, $\underline{k}$ and $\underline{a}$, to the degree of approximation that it is possible to evaluate Eq. (2.3.4b).

If the estimated quantity, $\hat{\underline{x}}$, were truly Gaussian then our evaluation of Eq. (2.3.4b) which follows is exact. The approximation in our evaluation of $\underline{b}$ in Eq. (2.3.4b) comes about by representing, $\hat{\underline{x}}$, as a Gaussian random vector and neglecting 3rd and higher order moments of its distribution function. Recall that the evolution of $\hat{\underline{x}}$ is given by Eq. (2.24a) in terms of the White Gaussian information vector, $\underline{e}$, and non-Gaussian bounded control $\underline{u}$. By assumption, i.e. the small engine, the effect of the control, $\underline{u}$, on $\hat{\underline{x}}$ is small over any small time interval so that only integrated effects over long intervals are significant. But if future controls tend to be independent of present controls the integrated control will tend to a Gaussian by the Central Limit Theorem. Hence the assumption of Gaussian, $\hat{\underline{x}}$, while not strictly true seems to be very reasonable approximation for purposes of calculation.

As a first step in the evaluation of $\underline{b}$ in Eq. (2.3.4b) under the Gaussian hypothesis, consider the simpler problem of calculating the scalar quantity, β , defined in terms of an arbitrary vector, $\underline{c}$, by Eq. (3.3.1)

$$\beta \triangleq E [\underline{c}^T \hat{\underline{x}} \underline{u}] \quad (3.3.1)$$

If β is known for all $\underline{c}$, then $\underline{b}$ can be found from β .

Define the quantities ζ and ξ where

$$\zeta \triangleq \underline{k}^T \hat{\underline{x}} \quad (3.3.2)$$

is the statistic relating the past measurements $\{z(\sigma), \sigma \leq t\}$ to the control, $\underline{u}$, of Eq. (3.2.10) and

$$\xi \triangleq \underline{c}^T \hat{\underline{x}} \quad (3.3.3)$$

On substituting Eq. (3.3.3) into Eq. (3.3.1) the problem now is

to evaluate

$$\beta = E [\xi u] \quad (3.3.4)$$

in terms of the statistic, ζ , given by Eq. (3.3.2). In this regard it is convenient to regard ξ as composed of a sum of two orthogonal components, the first being ζ , the statistic, and the other a component called η orthogonal to ζ .

Thus we write

$$\xi = \alpha \zeta + \eta \quad (3.3.5)$$

Then the condition that $E [\zeta \eta] = 0$ determines α by the following self-evident calculation.

$$E [\xi \zeta] = \alpha E [\zeta^2] + E [\eta \zeta]$$

Consequently

$$\alpha = \frac{E[\xi \zeta]}{E[\zeta^2]} = \frac{\underline{k}^T X \underline{c}}{\underline{k}^T X \underline{k}} \quad (3.3.6)$$

where $X = E [\hat{\underline{x}} \hat{\underline{x}}^T]$

Substituting Eq. (3.3.6) into Eq. (3.3.5) yields

$$\xi = \frac{\underline{k}^T X \underline{c}}{\underline{k}^T X \underline{k}} \zeta + \eta \quad (3.3.7)$$

By construction ζ , and η are uncorrelated zero mean Gaussian random variables, with variances σ_ζ^2 and σ_η^2 respectively. This is because $\hat{\underline{x}}$ is a zero mean random variable as is the true state, $\underline{x}$. See Appendix B.

Recall from Eq. (3.3.6) that

$$\sigma_\zeta^2 = E[\zeta^2] = \underline{k}^T X \underline{k}$$

Thus the joint distribution of ζ and η is

$$p(\zeta, \eta) = p(\zeta)p(\eta) = \frac{1}{2\pi\sigma_\zeta\sigma_\eta} e^{-\frac{\zeta^2}{2\sigma_\zeta^2}} e^{-\frac{\eta^2}{2\sigma_\eta^2}} \quad (3.3.8)$$

$p(\zeta, \eta)$.

Using Eq. (3.2.10)

$$\begin{aligned} \beta = E[u \xi] = E[u \xi(\zeta, \eta)] &= \int_{a\sqrt{\underline{k}^T \underline{k}}}^{\infty} d\zeta \int_{-\infty}^{\infty} d\eta (1 \cdot \xi(\zeta, \eta)) p(\zeta, \eta) \\ &+ \int_{-\infty}^{-a\sqrt{\underline{k}^T \underline{k}}} d\zeta \int_{-\infty}^{\infty} d\eta (-1 \cdot \xi(\zeta, \eta)) p(\zeta, \eta) \end{aligned} \quad (3.3.9)$$

Substituting Eq. (3.3.7) and (3.3.8) into Eq. (3.3.9) yields

$$\begin{aligned} &= \int_{a\sqrt{\underline{k}^T \underline{k}}}^{\infty} d\zeta \int_{-\infty}^{\infty} d\eta \left(\frac{\underline{k}^T \underline{X} \underline{c}}{\underline{k}^T \underline{X} \underline{k}} \zeta + \eta \right) \frac{e^{-\zeta^2/2\sigma_{\zeta}^2} e^{-\eta^2/2\sigma_{\eta}^2}}{2\pi \sigma_{\zeta} \sigma_{\eta}} \\ &- \int_{-\infty}^{-a\sqrt{\underline{k}^T \underline{k}}} d\zeta \int_{-\infty}^{\infty} d\eta \left(\frac{\underline{k}^T \underline{X} \underline{c}}{\underline{k}^T \underline{X} \underline{k}} \zeta + \eta \right) \frac{e^{-\zeta^2/2\sigma_{\zeta}^2} e^{-\eta^2/2\sigma_{\eta}^2}}{2\pi \sigma_{\zeta} \sigma_{\eta}} \end{aligned} \quad (3.3.10)$$

Integrating Eq. (3.3.10) with respect to η yields

$$E[u \xi] = \int_{-\infty}^{\infty} d\zeta \frac{e^{-\zeta^2/2\sigma_\zeta^2}}{a\sqrt{\underline{k}^T \underline{k}}} \frac{\underline{k} \times \underline{c}}{\underline{k}^T \times \underline{k}} \zeta$$

$$- \int_{-\infty}^{-a\sqrt{\underline{k}^T \underline{k}}} d\zeta \frac{e^{-\zeta^2/2\sigma_\zeta^2}}{\sqrt{2\pi} \sigma_\zeta} \frac{\underline{k}^T \times \underline{c}}{\underline{k}^T \times \underline{k}} \zeta$$

On making the change of variable $z = \zeta^2 / 2\sigma_\zeta^2$ on combining terms, the upbove yields

$$E[u \xi] = \frac{2\underline{k}^T \times \underline{c}}{\underline{k}^T \times \underline{k}} \frac{\sigma_\zeta}{\sqrt{2\pi}} \int_0^\infty \frac{a^2 \underline{k}^T \underline{k}}{2\sigma_\zeta^2} e^{-z} dz$$

$$= \sqrt{\frac{2}{\pi}} \frac{\underline{k}^T \times \underline{c}}{\sqrt{\underline{k}^T \times \underline{k}}} e^{-a^2 \underline{k}^T \underline{k} / 2\underline{k}^T \times \underline{k}}$$

(3.3.11)

Define the quantity γ by

$$\gamma = \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{a^2 \underline{k}^T \underline{k}}{2\underline{k}^T \times \underline{k}}}}{\sqrt{\underline{k}^T \times \underline{k}}}$$

(3.3.12)

Then Eq. (3.3.11) becomes

$$\beta = E [\underline{c}^T \hat{\underline{x}} u] = \gamma \underline{k}^T X \underline{c} = \gamma \underline{c}^T X \underline{k}$$

from which it readily follows that

$$\underline{b} = E [\hat{\underline{x}} u] = \gamma X \underline{k} \quad (3.3.13)$$

Eq. (3.3.13) is the main result of this section. It shows explicitly how the parameters $\underline{k}$ and a of the control structure of Eq. (3.2.10) are related to the stochastic control quantity $\underline{b}$.

For the special case of $a \equiv 0$ corresponding to the bang-bang control, define

$$\gamma_0 = \gamma \Big|_{a \equiv 0} = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\underline{k}^T X \underline{k}}} \quad (3.3.14)$$

Finally, on substituting Eq. (3.3.13) into Eq. (2.3.13) we obtain the fundamental dynamical equations for stochastic evolution.

$$\dot{X} = F X + X F^T + \gamma (\underline{d} \underline{k}^T X + X \underline{k} \underline{d}^T) + C E_0 C^T$$

$$X(t_0) = 0$$

$$\text{trace} (S(T)X(T)) = \omega_0 \quad (3.3.15a)$$

An alternate integral formulation based on Eq. (2.3.11) is

$$-D(T) = \int_0^T \gamma(t) \phi^{-1}(t) \{ \underline{d}(t) \underline{k}^T(t) X(t) + X(t) \underline{k}(t) \underline{d}^T(t) \} \phi^{-1}(t)^T dt \quad (3.3.15b)$$

Before concluding this section it is of some interest to compare Eq. (3.3.13) with the linear case. If u is a linear combination of the state variables

$$u = \underline{k}^T \hat{\underline{x}}$$

Then

$$\underline{b} = E[\hat{\underline{x}} u] = X \underline{k} \quad (3.3.16)$$

for any distribution of $\hat{\underline{x}}$. Hence the scalar factor γ represents the main difference between linear control, bang-bang control and threshold control. This implies that a linear control covariance γX yields the same stochastic control $\underline{b}$ as a threshold control with γ given by Eq. (3.3.12) or a saturating control with $\gamma = \gamma_o$ given by Eq. (3.3.14). Since Eq. (3.3.16) is independent of statistics we might conjecture that the structure of Eq. (3.3.15) is a somewhat general result with the parameter γ depending on the statistics and on the control constraints. Hence small deviations from Gaussian statistics would, under this conjecture, amount to nothing more than small deviations in γ . If Eq. (3.3.15a) is stable then errors in X due to errors in γ would at least be bounded and very likely would damp out.

4. The Hamiltonian Formulation (Minimum Principle) For The Minimum Time, Minimum Expected Fuel, And Least Upper Bound Fuel Problems.

The Hamiltonian formulation has become a standard method of attacking optimal control problems, [12], [13], [17], and it has been used in [6], [7], and [14]. A brief description of the method is given in Section 4.2. For a more detailed treatment the interested reader is referred to the above references.

The primary step is to reduce the stochastic problem to an equivalent deterministic problem following the reasoning of Eq.(1.2.4), as was done for example in [6] and [7]. Recall that this is accomplished at the price of continuously discarding all information that is not contained in the covariance matrix, X , of the conditional mean $\hat{x}$. This is achieved by the description of the systems dynamics in terms of X and the stochastic control $\underline{b}$ as in Eq.(3.3.15) and the evaluation of $\underline{b}$ in terms of X as in Eq.(3.3.13). Consequently in order to completely define the equivalent deterministic problem it only remains to find the cost functional of Eq.(1.2.4) and this is done in Section 4.1.

Section 4.2 describes the Minimum Principle. The succeeding Sections apply the Minimum Principle to obtain necessary, and in some cases sufficient, conditions for optimal performance for the problems of minimum time, minimum expected fuel consumption and least upper bound fuel consumption.

4.1 Deterministic Cost

The general formulation for the equivalent deterministic optimal control problem follows the point of view of Eq.(1.2.4) which is rewritten here in terms of the parameters $\underline{k}$, a which were defined in Eq. (3.3.13)

$$J = \min_{\underline{k}, a} \int_{t_0}^T L(\tau, X(\tau), \underline{k}(\tau), a(\tau)) d\tau. \quad (4.1.1)$$

The optimization of Eq.(4.1.1) is to be carried out subject to the constraints of Eq.(3.3.15). By comparison with the point of view of Eq.(1.2.1) we see that the required relation between the random cost rate $\tilde{L}$ and the deterministic cost rate L is

$$L(\tau, X(\tau), \underline{k}(\tau), a(\tau)) = E[E[\tilde{L}(\tau, \underline{x}(\tau), u(\tau) | \mathcal{F}(\sigma); \sigma \leq \tau)]]$$

Needless to say such relation in general does not always hold.

First consider the case of minimum time. Hence $L \equiv 1$. Thus the minimum time problem can be stated as follows:

Find the quantities $\underline{k}(t)$, $a(t)$, t_μ such that

$$J = \int_{t_\mu}^T dt$$

is minimized subject to Eq.(3.3.15).

In the least upper bound fuel problem we take note of the fact, from Eq.(3.5.1) that since control takes on only the values $+1, 0, -1$, the fuel consumption is identical with the burning interval. Hence define the burning interval by the indicator function

$$i_\omega = \begin{cases} 1 & \text{if } u = +1 \\ 0 & \text{if } u = 0 \end{cases} \quad (4.1.2)$$

We assume that i_ω is a non-random function which can be solved a priori. When $i_\omega = 0$ we know a priori that $u = 0$ and hence it follows that the stochastic control $\underline{b} = 0$. Eq.(3.3.15) must be modified to account for this. This modification is most easily accomplished by replacing γ by $i_\omega \gamma$ as indicated by Eq. (3.5.2).

The least upper bound fuel problem may now be stated as follows : Find the quantities, $\underline{k}(t)$, $a(t)$, $i_{\omega}(t)$ over a fixed interval $[t_0, T]$ where $t_0 < t_{\mu}$ such that

$$J_{LUB} = \int_{t_0}^T i_{\omega}(t) dt \quad (4.1.3)$$

is minimized subject to the modified Eq. (3.3.15).

In the problem of minimum expected fuel consumption, which was approximated in [6] and [7], $\tilde{L} = |u|$. Hence it is necessary to calculate the quantity $E|u|$ in order to represent the cost. This calculation now proceeds :

For the control given by Eq. (3.2.10) consider again the statistic given by

$$\zeta = \underline{k}^T \underline{x} \quad (3.3.2)$$

whose mean is 0 and variance $\sigma_{\zeta}^2 = \underline{k}^T \underline{X} \underline{k}$

Then it follows from Eq. (3.2.10) that

$$E |u| = \frac{2}{\sqrt{2\pi} \sigma_{\zeta}} \int_{-\infty}^{\infty} 1 \cdot e^{-\zeta^2 / 2\sigma_{\zeta}^2} d\tau \quad (4.1.4)$$

Define the integral $Q_f(z)$ by

$$Q_f(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^z e^{-t^2} dt \quad (4.1.5)$$

Then from Eq. (4.1.4) it follows that

$$E |u| = 2 \left\{ 1 - Q_f \left(\frac{a\sqrt{k} \sqrt{T_k}}{\sqrt{2} \sigma_\zeta} \right) \right\} \quad (4.1.6)$$

Hence the minimum expected fuel consumption problem can be stated thusly :

Find the quantities $\underline{k}(t)$, $a(t)$ such that

$$J_{\text{Fuel}} = 2 \int_{t_0}^T \left\{ 1 - Q_f \left(\frac{a\sqrt{k} \sqrt{T_k}}{\sqrt{2} \sigma_\zeta} \right) \right\} dt \quad (4.1.7)$$

is minimized for fixed T , t_0 subject to . (3.3.15)

4.2 Optimality Conditions

In this section the Hamiltonian formulation, also known as the Minimum Principle will be stated in terms sufficiently general to encompass a general scalar deterministic cost function, such as Eq.(4.1.7), subject to the constraints of Eq. (3.3.15). A short justification will be given but the reader is referred to already cited references for more complete details.

The problem is as follows :

Minimize the cost functional

$$J_0 = \int_{t_1}^T L(\tau, X(\tau), \underline{p}(\tau)) d\tau \quad (4.2.1)$$

by choosing the piece - wise continuous parameters, $\underline{p}$, which belong to the subset $\mathcal{P}$, to satisfy the constraints

$$\dot{X}(t) = \Omega (X(t), \underline{p}(t)) \quad (4.2.2)$$

so that the problem is autonomous. The boundary condition is

$$\text{trace } \{ S(T) X(T) \} = \omega_0 \quad (4.2.3a)$$

at the fixed terminal time, T . Initially $X(t_1)$ is determined by integrating Eq.(3.3.15a) with $\underline{b} = 0$ from 0 to t_1 . (4.2.3b)

In the above the control parameter, $\underline{p}$, may be identified with the quantities, $\underline{k}$, a , i_ω previously defined in Eq. (3.2.10) and Section 4.1. Although the initial time $t_1 \geq t_0$ is left free in the above formulation as in the minimum time problem it need not be free in other problems of interest. The case where t_1 is fixed is simply a specialization of the more general case where it is free.

The solution to the above problem according to the Minimum Principle is an optimum control $\underline{p}^*$ and an optimum trajectory X^* defined in terms of an ADJOINT MATRIX $\Lambda (X^*, \underline{p}^*)$ and a scalar Hamiltonian, H , given by

$$H = L + \text{trace } \{ \Lambda^T \Omega \} \quad (4.2.4)$$

The adjoint matrix Λ satisfies

$$\dot{\Lambda} = - \frac{\partial H}{\partial X^*} = \left\{ - \frac{\partial H}{\partial X_{ij}^*} \right\} \quad (4.2.5)$$

$$\Lambda(T) = \nu S^T(T) = \nu S(T) ; \nu > 0$$

where v is another positive scalar constant, chosen so that Eq.

(4.2.3a) is satisfied. The optimal control, $p = p^*$ satisfies

$$H^*(\Lambda(t_1), X^*(t_1), p^*(t_1)) = 0 \quad (4.2.7)$$

$$p^* = \arg \min_p H(\Lambda, X^*, p)$$

The motion of the system is given by Eqs. (4.2.5), (4.2.2)

(4.2.6) and (4.2.7) solved simultaneously.

An elementary argument is now presented to justify the matrix version of the Minimum Principle given above.

The assumption is made that a Lagrange multiplier matrix, Λ , and a scalar constant, v , both exist so that the minimization of J_0 in Eq. (4.2.1) subject to the constraints of Eq. (4.2.2) and (4.2.3) is equivalent to the minimization of the single scalar quantity, V , defined by

$$V \triangleq v \left(\text{trace} \{ S(T)X(T) \} - \omega_0 \right) + \int_{t_1}^T L(X, p) \, d\tau$$

$$- \text{trace} \left\{ \int_{t_1}^T \Lambda^T (\dot{X} - \Omega) \, d\tau \right\} \quad (4.2.8)$$

On substituting Eq. (4.2.4) into V , Eq. (4.2.8) becomes

$$V = v \left(\text{trace} \{ S(T)X(T) \} - \omega_0 \right) + \int_{t_1}^T H \, d\tau - \text{trace} \left\{ \int_{t_1}^T \Lambda^T \dot{X} \, d\tau \right\}$$

(4.2.9)

Since V is to be minimum, variations in $V, \delta V$, due to variations in $\underline{p}$, $\delta \underline{p}$, and variations in t_μ , δt_μ , should be non-negative.

Hence

$$\delta V \stackrel{=}{=} 0 \quad (4.2.10)$$

Variations in $H(\Lambda, X^*, \underline{p}^*)$ due to variations in $\underline{p}$ can be written, as for example in [14]

$$\delta H = \text{trace} (H_X^T \delta X) + \text{trace} H_\Lambda^T \delta \Lambda + \delta_1 H$$

where H_X has elements $\left\{ \frac{\partial H}{\partial X_{ij}} \right\}$

(4.2.11)

and $\delta_1 H$ is a general term which contains the first-order variation in H due to $\delta \underline{p}$ as well as higher-order terms in δX and $\delta \underline{p}$.

$$\delta_1 H \approx \frac{\partial H^T}{\partial \underline{p}} \delta \underline{p} + \frac{1}{2} \delta \underline{p}^T H_{pp} \delta \underline{p} + \text{other terms}$$

(4.2.12)

Using Eqs. (4.2.11) and (4.2.12) consider variations in V as follows :

$$\begin{aligned}
\delta V = & \nu \text{ trace } \{ S(T) \delta X(T) \} + \delta \nu \{ \text{trace } (S(T)X(T)) - \omega_0 \} \\
& + \int_{t_1}^T \text{trace } \{ H_X^T \delta X + H^T \delta \Lambda \} dt + \int_{t_1}^T \delta_1 H d\tau \\
& - \text{trace } \left\{ \int_{t_1}^T \Lambda^T \delta \dot{X} d\tau \right\} - \text{trace } \left(\int_{t_1}^T \delta \dot{\Lambda}^T X d\tau \right) - L(t_1) \delta t_1
\end{aligned}
\tag{4.2.13}$$

Note also the following integration by parts

$$\int_{t_1}^T \Lambda^T \delta \dot{X} = \Lambda^T(T) \delta X(T) - \Lambda^T(t_1) \delta X(t_1) - \int_{t_1}^T \dot{\Lambda}^T \delta X d\tau
\tag{4.2.14}$$

Since the initial point, t_1 , is variable write

$$\delta X(t_1) = \Delta X(t_1) - \dot{X}(t_1) \delta t_1
\tag{4.2.15}$$

where $\Delta X(t_1)$ is the total change at t_1 . On substituting Eqs. (4.2.14) and (4.2.15) into Eq. (4.2.13) we obtain

$$\begin{aligned}
\delta V = & \delta \nu [\text{trace } [S(T)X(T)] - \omega_0] + \nu \text{ trace } [S(T) \delta X(T)] \\
& - \text{trace } [\Lambda^T(T) \delta X(T)] + \text{trace } \left[\int_{t_1}^T [H_X^T + \dot{\Lambda}^T] \delta X d\tau \right. \\
& + \text{trace } \left[\int_{t_1}^T (H_\Lambda^T - \dot{X}^T) \delta \Lambda d\tau \right] \\
& + \text{trace } [\Lambda^T(t_1) \Delta X(t_1)] - \text{trace } [\Lambda^T(t_1) \dot{X}(t_1)] \delta t_1 - L(t_1) \delta t_1 \\
& + \int_{t_1}^T \delta_1 H d\tau
\end{aligned}
\tag{4.2.16}$$

On Setting :

$$\left. \begin{aligned} \text{trace} [S(T) X(T)] &= \omega_0 \\ \dot{X} &= H_{\Lambda} = \Omega \\ \dot{\Lambda} &= -H_X \end{aligned} \right\} \quad (4.2.17)$$

$$\Lambda(T) = v S^T(T) = v S(T)$$

$$\Lambda X(t_1) = 0 \quad \text{i.e. } X(t_1) \text{ determined by Eq. (4.2.3b) with } t_0 = t_1$$

$$L(t_1) + \text{trace} [\Lambda^T(t_1) \dot{X}(t_1)] = H(t_1) = 0$$

There Results : T

$$\delta V = \int_{t_1}^T \delta_1 H \, d\tau \quad (4.2.18)$$

so that if $p = p^*$ is such that $\delta_1 H \geq 0$ then $\delta V \geq 0$. Q.E.D

Remark :

It is easily shown that along an optimal trajectory variations in H due to variations in X and Λ cancel out.

Hence

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

When the time is not an explicit argument H is constant. In the particular case when $H(t_1) = 0$ then H is everywhere 0.

Remark :

In the cases of the minimum expected fuel and least upper bound problems the control interval $[t_1, T]$ is fixed. Consequently one merely eliminates the condition $H(t_1) = 0$, and all else is the same.

Remark :

The satisfaction of Eqs. (4.2.5), (4.2.6) and (4.2.7) are necessary conditions for a local minimum for the functional J_0 .

In addition, if there is no conjugate point [26] , then the above conditions may be taken as sufficient for a local minimum. Pontryagin et al , [17] , demonstrated equivalent conditions for a local minimum under conditions more general than that permitted by the usual arguments of the calculus of variations, i.e. allowing $\underline{p}$ to belong to a closed set P rather than an open set.

The satisfaction of a local minimum does not guarantee that a global optimal solution has been found. In fact it must be shown that an optimal solution exists. There are cases, for example, in which J_0 possesses an infimum but not a strict minimum.

The multiplier, Λ , has an interesting geometric interpretation. Consider an optimal path where $\delta H_1 = 0$ propagating backwards from T . Then only the last terms in Eq. (4.2.16) survive at the free left end and hence

$$\delta V(t) = \text{trace} [\Lambda^T(t) \Delta X(t)] - H(t) \delta t \quad (4.2.19)$$

The above leads naturally to the interpretation :

$$\Lambda = \frac{\partial V}{\partial X} \quad (4.2.20a)$$

$$H = - \frac{\partial V}{\partial t} \quad (4.2.20b)$$

The implication of Eq. (4.2.20a) is that Λ is the outward normal to the surface $V(t) = c$ where V is given by Eq. (4.2.9). Note also that Eq.(4.2.20b) is the Hamilton Jacobi Equation which itself can serve as the starting point of the theory. We are led to the following geometrical picture of the Minimum Principle.

First define an augmented scalar state variable, x_0 , by

$$x_0(t) = \int_t^T L(X, p, t) dt \quad (4.2.21)$$

Then consider the set of augmented admissible initial conditions which can be driven to the target set with cost $V(t) = c$ and denote this set by $\mathcal{K}(t)$.

$$\mathcal{K}(t) \triangleq \{ x_0, X \mid X(t) \in \text{target set}, \underline{b}(t) \in B(t), V(t) = c \} \quad (4.2.22)$$

Then the outward normal to the set $\mathcal{K}(t)$ is $\{1, \Lambda(t)\}$ by construction. Next consider the motion of the system i.e. the evolution of the augmented states $\{x_0, X\}$ from a point on the boundary of $\mathcal{K}(t)$. We must have the inner product

$$\begin{aligned} < \nabla \mathcal{K}(t) \cdot (\{x_0(t+\Delta), X(t+\Delta)\} - \{x_0(t), X(t)\}) > \\ &= x_0(t+\Delta) - x_0(t) + \text{trace} \{ \Lambda^T(t) [X(t+\Delta) - X(t)] \} \\ &= H(t) \Delta t \quad \text{as } \Delta \rightarrow 0 \end{aligned} \quad (4.2.23)$$

Consequently, if H is minimized the motion of the system is directed maximally inward which tends to collapse the set of augmented initial conditions to the target set in an optimal fashion.

It is not hard to visualize how the above interpretation leads to nonsensical results if the boundary of $\mathcal{K}$ is irregular. In this case there is no supporting tangent hyperplane whose normal is determined by $\{1, \Lambda\}$. It would appear that a sufficient condition for optimality is that the set $\mathcal{K}$ be closed and convex; for then the path which minimizes the

Hamiltonian is unique.

Sufficient conditions for optimality are given by Lee[19] Mangasarian [27] and Dem'yanov [18]. Unfortunately we cannot simply use any of their results directly because in all cases considered in this report the states, X , and the controls, $\underline{k}$, a , i_ω are coupled. It is therefore necessary in general to make additional tests such as the conjugate point condition [26] or even to submit to search techniques in order to discover all the local minima.

4.3 Solution for A Local Minimum For The Minimum Time Problem

In this section the Minimum Principle of the last section is applied to the problem of finding the minimum time $[t_\mu, T]$ such that the target set condition of Eq.(4.2.3a) is satisfied subject to the dynamics of Eq.(3.3.15).

Recall first of all that the cost rate $L = 1$ and that $\gamma = \gamma_0$ as in Eq.(3.3.14). Consequently, on substituting Eq.(3.3.15) for Eq.(4.2.2) the Hamiltonian becomes

$$H = 1 + \text{trace} \left\{ \dot{\Lambda}^T [FX + XF^T + CE_0 C^T + \frac{\sqrt{2}}{\pi} \frac{1}{\sqrt{\underline{k}^T X \underline{k}}} (\underline{d} \underline{k}^T X + X \underline{k} \underline{d}^T)] \right\} \quad (4.3.1)$$

We are to find $\underline{k}$ to minimize H . Note that since H does not depend on the magnitude of $\underline{k}$, $|\underline{k}|$, we can only choose an optimal direction.

The term in H which contains $\underline{k}$, after some manipulation, becomes

$$\frac{\underline{k}^T X (\Lambda + \Lambda^T) \underline{d}}{\sqrt{\underline{k}^T X \underline{k}}} = \frac{\phi_N}{\phi_D} \quad (4.3.2)$$

To minimize ϕ_N / ϕ_D with respect to $\underline{k}$ it is sufficient to minimize ϕ_N subject to the constraint that ϕ_D (or ϕ_D^2) is constant. This yields

$$\underline{k} = -\eta (\Lambda + \Lambda^T) \underline{d} \quad ; \quad \eta > 0 \quad (4.3.3)$$

The optimal multiplier, Λ , is obtained from

$$\begin{aligned} -\Lambda^T = H_X &= \Lambda^T F + F^T \Lambda^T + \gamma_0 (\Lambda^T \underline{d} \underline{k}^T + \underline{k} \underline{d}^T \Lambda^T) \\ &\quad - \frac{\pi}{4} \gamma_0^3 \underline{k} \underline{k}^T \text{trace} \{ \Lambda^T (\underline{d} \underline{k}^T X X \underline{k} \underline{d}^T) \} \end{aligned} \quad (4.3.4)$$

$$\text{where } \gamma_0 = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\underline{k}^T X \underline{k}}} \quad (4.3.5)$$

The boundary conditions are

$$\left. \begin{aligned} \Lambda(T) &= \gamma S^T(T) = \gamma S(T) \quad ; \quad \gamma > 0 \\ H(t_\mu) &= 0 \end{aligned} \right\} \quad (4.3.6)$$

$X(t_0)$ = determined by Eq. (4.2.3b) with $t_0 = t_1$

$X(t_\mu)$ is determined by

$$\dot{X}(t) = F(t) X(t) + X(t) F(t)^T + C(t) E_0(t) C^T(t) \quad (4.3.7)$$

for times $t_\mu > t_0$

4.4 Solution For A Local Minimum For The Minimum Expected Fuel Problem

Applying the methods of Section (4.2) to the minimum expected fuel consumption problem, the Hamiltonian can be constructed by substituting Eqs. (3.3.15) and (4.1.7) into Eq. (4.2.4)

$$H = 2 \left[1 - Q_f \left(\frac{a \sqrt{\underline{k}^T \underline{k}}}{\sqrt{2} \sqrt{\underline{k}^T X \underline{k}}} \right) \right] + \text{trace} \{ \Lambda^T .$$

$$\left[F X + X F^T + C E_0 C^T + \sqrt{\frac{2}{\pi}} e^{-\frac{a^2 \underline{k}^T \underline{k}}{2 \underline{k}^T X \underline{k}}} \frac{1}{\sqrt{\underline{k}^T X \underline{k}}} (\underline{d} \underline{k}^T X + X \underline{k} \underline{d}^T) \right]$$

(4.4.1)

H is to be simultaneously minimized with respect to $\underline{k}$ and a .

Define the quantity

$$\rho = \frac{a \sqrt{\underline{k}^T \underline{k}}}{\sqrt{2} \sqrt{\underline{k}^T \underline{X} \underline{k}}} \quad (4.4.2)$$

Then the terms involving $\underline{k}$ and a can be written

$$\bar{\omega}_0 = -2 Q_f(\rho) + \sqrt{\frac{2}{\pi}} \frac{e^{-\rho^2}}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} (\underline{k}^T \underline{X} (\underline{\Lambda} + \underline{\Lambda}^T) \underline{d}) \quad (4.4.3)$$

and we are to choose $(\underline{k}, a)$ to minimize $\bar{\omega}_0$. The ranges of ρ and a are both $[0, \infty]$ and for any $\underline{k}$, a determines ρ uniquely, or if a is desired it can be uniquely specified by ρ . Consequently, $\{\underline{k}, \rho\}$ form a sufficient pair of control variables and it is only necessary to minimize $\bar{\omega}_0$ with respect to $\{\underline{k}, \rho\}$.

As in the previous section the asterisks denoting optimal quantities will be dropped where the optimal quantities are clearly understood.

The minimization of $\bar{\omega}_0$ with respect to $\underline{k}$ is exactly the problem of minimizing Eq. (4.3.2) in the minimum time problem and yields

$$\underline{k} = -\eta (\underline{\Lambda} + \underline{\Lambda}^T) \underline{d} ; \quad \eta > 0 \quad (4.4.4)$$

which is exactly the same as Eq. (4.3.3). The resulting $\bar{\omega}_0$ becomes

$$\bar{\omega}_0(\rho) = -2 Q_F(\rho) - \sqrt{\frac{2}{\pi}} e^{-\rho^2} \sqrt{\underline{d}^T (\Lambda + \Lambda^T) \times (\Lambda + \Lambda^T) \underline{d}} \quad (4.4.5)$$

It therefore only remains to minimize $\bar{\omega}_0$ with respect to $\rho > 0$.

Differentiating $\bar{\omega}_0(\rho)$ with respect to ρ yields

$$\begin{aligned} \frac{d\bar{\omega}_0(\rho)}{d\rho} &= \frac{-2}{\sqrt{\pi}} e^{-\rho^2} - \sqrt{\frac{2}{\pi}} 2\rho e^{-\rho^2} \sqrt{\underline{d}^T (\Lambda + \Lambda^T) \times (\Lambda + \Lambda^T) \underline{d}} \\ &= \frac{-2}{\sqrt{\pi}} e^{-\rho^2} \left(1 - \sqrt{2} \rho \sqrt{\underline{d}^T (\Lambda + \Lambda^T) \times (\Lambda + \Lambda^T) \underline{d}} \right) \end{aligned} \quad (4.4.6)$$

On setting $\frac{d\bar{\omega}_0(\rho)}{d\rho} = 0$ there results:

$$\rho = \frac{1}{\sqrt{2} \sqrt{\underline{d}^T (\Lambda + \Lambda^T) \times (\Lambda + \Lambda^T) \underline{d}}}$$

$$\text{or } a = \frac{1}{\sqrt{\underline{d}^T (\Lambda + \Lambda^T)^2 \underline{d}}} \quad (4.4.7)$$

To determine whether ρ is indeed a minimum calculate

$$\frac{d^2 \bar{\omega}_0(\rho)}{d\rho^2} = \frac{-2}{\sqrt{\pi}} e^{-\rho^2} \left(-\sqrt{2} \sqrt{\underline{d}^T (\Lambda + \Lambda^T) \times (\Lambda + \Lambda^T) \underline{d}} \right) > 0 \quad (4.4.8)$$

Which verifies that Eq. (4.4.7) does indeed yield a minimum.

The optimal multiplier, Λ , is obtained from

$$\begin{aligned}
 -\Lambda^T = H_X^T &= \frac{e^{-\rho^2}}{\sqrt{2\pi}} \left(\frac{a \underline{k} \underline{k}^T}{\underline{k}^T X \underline{k}} \right) + \Lambda^T F + F^T \Lambda^T \\
 &+ \gamma \left(\Lambda^T \underline{d} \underline{k}^T + \underline{k} \underline{d}^T \Lambda^T \right) \\
 &+ \sqrt{\frac{2}{\pi}} \frac{e^{-\rho^2} \underline{k} \underline{k}^T}{2 \left(\underline{k}^T X \underline{k} \right)^{3/2}} \left(\frac{a^2 \underline{k}^T \underline{k}}{\underline{k}^T X \underline{k}} - 1 \right) \text{trace} \{ [\Lambda^T (\underline{d} \underline{k}^T X + X \underline{k} \underline{d}^T)] \}
 \end{aligned}
 \tag{4.4.9}$$

$$\text{where } \gamma = \sqrt{\frac{2}{\pi}} \frac{e^{-\rho^2}}{\sqrt{\underline{k}^T X \underline{k}}}$$

The boundary conditions are

$$\left. \begin{aligned}
 \Lambda(T) = v S^T(T) &= v S(T) \quad ; \quad v > 0 \\
 X(t_0) \text{ is determined by Eq. (4.2.3b)} \\
 \text{with } t_0 &= t_1
 \end{aligned} \right\}
 \tag{4.4.10}$$

4.5 Solution For A Local Minimum For The L.U.B. Problem

Before applying the methods of Section 4.2 to the solution of the L.U.B. problem recall the brief discussion of the L.U.B. performance criterion of Eq. (4.1.3). During coasting intervals $\underline{b} = 0$, and this fact can be put into the dynamical equations of motion Eq. (3.3.15) by replacing γ by $i_\omega \gamma_0$ so that $\underline{b}_\omega = i_\omega \gamma_0 \underline{X} \underline{k} = 0$ where $i_\omega = 0$. Eq. (3.3.15a) now becomes

$$\dot{\underline{X}} = \underline{F}\underline{X} + \underline{X}\underline{F}^T + \underline{C}\underline{E}_0^T + i_\omega \gamma_0 (\underline{d} \underline{k}^T \underline{X} + \underline{X} \underline{k} \underline{d}^T) \quad (4.5.1)$$

Consequently, the performance criterion of Eq. (4.1.3) implies that the Hamiltonian should be

$$H = i_\omega + \text{trace} \{ \underline{\Lambda}^T [\underline{F}\underline{X} + \underline{X}\underline{F}^T + \underline{C}\underline{E}_0^T + i_\omega \gamma_0 (\underline{d} \underline{k}^T \underline{X} + \underline{X} \underline{k} \underline{d}^T)] \} \quad (4.5.2)$$

We are to minimize H with respect to the pair $\{ i_\omega, \underline{k} \}$ and the constraint $(i_\omega = 0, 1)$. The terms involving the pair $\{ i_\omega, \underline{k} \}$ can be written

$$\begin{aligned} \omega_W &= i_\omega (1 + \gamma_0 (\underline{k}^T \underline{X} (\underline{\Lambda} + \underline{\Lambda}^T) \underline{d})) \\ &= i_\omega (1 + \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} \underline{k}^T \underline{X} (\underline{\Lambda} + \underline{\Lambda}^T) \underline{d}) \end{aligned} \quad (4.5.3)$$

When $i_\omega = 0$ the solution of $\underline{k}$ is of no consequence and

when $i_\omega = 1$ the minimization of ω_W as a function of $\underline{k}$ is straightforward and in fact is the same as the minimization of Eq. (4.3.2).

The optimal $\underline{k}$ for $i_\omega = 1$ is given by

$$\underline{k} = - \eta (\underline{\Lambda} + \underline{\Lambda}^T) \underline{d} ; \quad \eta > 0 \quad (4.5.4)$$

which is the same as Eq. (4.3.3). Since the above $\underline{k}$ may be presumed to hold when $i_\omega = 0$ i.e. during coasting when it has no physical effect Eq. (4.5.4) can be thought of holding throughout the interval $[t_0, T]$. On substituting Eq. (4.5.4) into Eq. (4.5.3) ω_W is a function of a single variable i_ω

$$\omega_W(i_\omega) = i_\omega \left(1 - \sqrt{\frac{2}{\pi}} \sqrt{\underline{d}^T (\Lambda + \Lambda^T) X (\Lambda + \Lambda^T) \underline{d}} \right) \quad (4.5.6)$$

Clearly the choice of i_ω which minimizes ω_W is

$$i_\omega = \begin{cases} 1 & 1 - \sqrt{\frac{2}{\pi}} \sqrt{\underline{d}^T (\Lambda + \Lambda^T) X (\Lambda + \Lambda^T) \underline{d}} \leq 0 \\ 0 & 1 - \sqrt{\frac{2}{\pi}} \sqrt{\underline{d}^T (\Lambda + \Lambda^T) X (\Lambda + \Lambda^T) \underline{d}} \geq 0 \end{cases} \quad (4.5.7)$$

The optimum multiplier, Λ , is obtained from

$$\begin{aligned} -\dot{\Lambda}^T &= H_X^T = \Lambda^T F + F^T \Lambda^T + i_\omega \gamma_0 (\Lambda^T \underline{d} \underline{k}^T + \underline{k} \underline{d}^T \Lambda^T) \\ &\quad - \frac{\pi}{4} i_\omega \gamma_0 \underline{k} \underline{k}^T \text{trace} \{ \Lambda^T (\underline{d} \underline{k}^T X + X \underline{k} \underline{d}^T) \} \end{aligned} \quad (4.5.8)$$

$$\text{where } \gamma_0 \text{ is given by Eq. (4.3.5) } \Lambda(T) = v S(T) ; v > 0 \quad (4.5.9)$$

$X(t_0)$ is determined by Eq. (4.2.3b) with $t_0 = t_1$

The control is given by

$$u = \begin{cases} \text{sgn } \underline{k}^T \hat{X} & ; i_\omega = 1 \\ 0 & ; i_\omega = 0 \end{cases} \quad (4.5.10)$$

From the discussion in section (4.2) the above conditions are sufficient for the optimum.

4.6 Inclusion Of Mass Flow Rate In The Stochastic Optimization Problems

If we consider the rocket problem then the dynamics of the system must be modified to account for the physical effects of mass flow rate during burning. The additional complexities of a finite mass flow rate are discussed in some detail below.

4.6.0 The Modified F Matrix

It is easy to show that the effect of mass flow is to modify the F matrix of Eq. (3.3.15). This can be demonstrated by modifying Eq. (2.1.3), (2.1.4), and (2.1.5) in the original derivation of F . These equations then should become

$$\frac{d\underline{V}}{dt} + \frac{\dot{m}}{m} \underline{V} = \underline{g}(\underline{x}) \quad (4.6.0.1)$$

$$\frac{d}{dt} \delta \underline{V} + \frac{\dot{m}}{m} \delta \underline{V} = \underline{G} \delta \underline{x} + \underline{f} \quad (4.6.0.2)$$

Hence the pair of equations

$$\frac{d}{dt} \delta \underline{x} = \delta \underline{V}$$

$$\frac{d}{dt} \delta \underline{V} = \underline{G} \delta \underline{x} - \frac{\dot{m}}{m} \delta \underline{V} + \underline{f}$$

implies that

$$\underline{F} = \left[\begin{array}{c|c} \underline{0} & \underline{I} \\ \hline \underline{G} & \underline{0} \end{array} \right]; \underline{F}_1 = - \frac{\dot{m}}{m} \left[\begin{array}{c|c} \underline{0} & \underline{0} \\ \hline \underline{0} & \underline{I} \end{array} \right] \quad (4.6.0.3)$$

so that $\underline{F}$ depends linearly on $\dot{m}$. We shall adopt the notation

$$F = \begin{bmatrix} 0 & I \\ G & 0 \end{bmatrix} \quad F_1 = -\frac{\dot{m}}{m} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \quad (4.6.0.4)$$

so that Eq. (4.6.0.3) implies that F is replaced by $F + F_1$.

4.6.1 The Mass Flow Rate In The Minimum Time Problem.

For the case of the minimum time problem the inclusion of the $\dot{m}$ term is particularly simple. Once t_μ is determined $\dot{m}$ is a fixed constant and m is known from

$$m = m(t_\mu) - \dot{m} t \quad (4.6.1.1)$$

Hence we may replace F by $F + F_1$ in Eq. (3.3.15) and the dynamics of section 4.3 are otherwise unchanged. Of course the computation of $X(t_\mu)$ from Eq. (4.3.7) is unchanged as well.

4.6.2 The Mass Flow Rate In The Minimum Expected Fuel Problem

The minimum expected fuel problem is considerably complicated in principle by the inclusion of the mass flow rate. This is because now $\dot{m}$ is a random quantity and hence the dynamics of both the X equation for the state given by Eq. (3.3.15) and the $C E_0 C^T$ equation for the observation rate given by the Eq. (2.2.4) depend on the random $\dot{m}$ term.

While we do not have a very good way of handling this problem the following approach is at least tractable :

In Eq. (2.1.7) which we now write

$$\dot{\underline{x}} = (F + F_1) \underline{x} + \underline{du} + \underline{w} \quad (4.6.2.1)$$

We calculate F_1 on the basis of a mean value of $\dot{m}$ and then incorporate the deviation from the mean into the forcing term $\underline{w}$. Then

$$\underline{F}_1 = - \frac{E[\dot{m}]}{m} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \quad (4.6.2.2)$$

which is deterministic. If the forcing term, $\underline{w}$, is truly a white noise process then its effect is to add uncertainty to the best estimate, $\hat{\underline{x}}$, or equivalently to reduce the observation rate $C E_o C^T$ which is computed from the Kalman Filter of Eq. (2.2.4). Now the uncertainty in the dynamics due to the random mass rate can be expressed by the matrix

$$\hat{\underline{F}}_1 = \frac{\dot{m} - E[\dot{m}]}{m} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \quad (4.6.2.3)$$

so that the random forcing function, $\underline{w}$, in Eq. (4.6.2.1) in fact is given by

$$\underline{w} = \hat{\underline{F}}_1 \underline{x} \quad (4.6.2.4)$$

where both $\hat{\underline{F}}_1$ and $\underline{x}$ are random. If we assume, and this is not strictly accurate, that $\dot{m}$ is independent of $\underline{x}$ then

$$\text{cov}[\underline{w}] = \frac{E[\dot{m} - E[\dot{m}]]^2}{m^2} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \quad (4.6.2.4)$$

The expression for $E[\dot{m} - E[\dot{m}]]^2$ can be calculated as follows :

Suppose m is normalized so that

$$E[\dot{m}] = E|u| = 2 [1 - Q_f] \left(\frac{a \sqrt{\underline{k}^T \underline{k}}}{\sqrt{2} \sqrt{\underline{k}^T \underline{X} \underline{k}}} \right) \quad (4.6.2.5)$$

Then since $E[u]^2 = E|u|$, a quantity less than unity, since

$u = (\pm 1, 0)$ it follows that

$$E [\dot{m}^2] = E [\dot{m}] \quad (4.6.2.6)$$

Hence it is trivial to show that

$$E [\dot{m} - E [\dot{m}]]^2 = E [\dot{m}] - (E [\dot{m}])^2 \quad (4.6.2.7)$$

The remaining difficulties are associated with the dependence of both $E [\dot{m}]$ and $\text{cov} [\underline{w}]$ on X . This is not reflected in the formulation of the dynamics given by Eq. (3.3.15) or for that matter in the formulation of the Kalman Filter given by Eq. (2.2.4). It is necessary to resort to successive approximations in fact in order to implement the above model.

We therefore suppose a given $E [\dot{m}]$ and a given value for $\text{cov} [\underline{w}]$ and find a complete solution for $X, \underline{k}, a$ from the method of section (4.4). Then using these calculated $x, \underline{k}, a$ we compute $E [\dot{m}]$ and $\text{cov} [\underline{w}]$ and again solve for $X, \underline{k}, a$ from section (4.4). The process is repeated until a stopping condition on $X, \underline{k}, a$ is reached. Convergence of this process is likely if

$$\frac{E [\ddot{m}]}{m} < < 1 \quad (4.6.2.8)$$

because in this case the effect of $\ddot{m}$ is second order anyway and $\underline{b}$ $X, \underline{k}, a$ is not strongly affected by it.

4.6.3 The Mass Flow Rate In The L.U.B. Problem

The effect of mass flow rate in the L.U.B. problem is in some respects the most interesting. We note that $\dot{m}$ is constant when $i_\omega = 1$, and $\dot{m}$ is zero when $i_\omega = 0$. Consequently we replace

$$F \quad \text{by} \quad F + i_\omega F_1 \quad (4.6.3.1)$$

in both Eq. (2.2.4) for the Kalman Filter and Eq. (3.3.15) for the

equations of motion. The equations of motion become

$$\begin{aligned}\dot{\underline{X}} &= \underline{F}\underline{X} + \underline{X}\underline{F}^T + i_w (\underline{F}_1 \underline{X} + \underline{X} \underline{F}_1^T) + \underline{C}\underline{E}_0^T + i_w \gamma_0 (\underline{d} \underline{k}^T \underline{X} + \underline{X} \underline{k} \underline{d}^T) \\ \dot{\underline{P}} &= \underline{F}\underline{P} + \underline{P}\underline{F}^T + i_w (\underline{F}_1 \underline{P} + \underline{P}\underline{F}_1^T) + \underline{Q}_W - \underline{P}\underline{H}^T \underline{R}_n^{-1} \underline{H} \underline{P} \end{aligned} \quad (4.6.3.2)$$

Note that the equations for the Kalman Filter are now included in the equations of motion since they are affected by the control i_w . The $\underline{P}$ equations must be adjoined to the Hamiltonian by a new Lagrange multiplier which we shall designate by $\underline{T}$. Consequently, the Hamiltonian becomes

$$\begin{aligned}H &= i_w + \text{trace} \{ \underline{\Lambda}^T [\underline{F}\underline{X} + \underline{X}\underline{F}^T + \underline{C}\underline{E}_0^T + i_w (\underline{F}_1 \underline{X} + \underline{X}\underline{F}_1^T) \\ &\quad + i_w \gamma_0 (\underline{d} \underline{k}^T \underline{X} + \underline{X} \underline{k} \underline{d}^T)] + \underline{T}^T [\underline{F}\underline{P} + \underline{P}\underline{F}^T + i_w (\underline{F}_1 \underline{P} + \underline{P}\underline{F}_1^T) \\ &\quad + \underline{Q}_W - \underline{P}\underline{H}^T \underline{R}_n^{-1} \underline{H} \underline{P}] \} \end{aligned} \quad (4.6.3.3)$$

As in Section (4.5) we are to minimize H with respect to the pair $(i_w, \underline{k})$ and the constraint $(i_w = 0, 1)$. The terms involving $(i_w, \underline{k})$ are

$$\begin{aligned}\omega'_W &= i_w [1 + \text{trace} \{ \underline{\Lambda}^T (\underline{F}_1 \underline{X} + \underline{X}\underline{F}_1^T) + \underline{T}^T (\underline{F}_1 \underline{P} + \underline{P}\underline{F}_1^T) \\ &\quad + \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} (\underline{k}^T \underline{X} (\underline{\Lambda} + \underline{\Lambda}^T) \underline{d}) \}] \end{aligned} \quad (4.6.3.4)$$

All arguments are now parallel to those of the last section. Hence

$$\underline{k} = - \eta (\underline{\Lambda} + \underline{\Lambda}^T) \underline{d} ; \eta > 0 \quad (4.6.3.5)$$

and

$$i_w = \begin{cases} 1 & \text{if } (1 + \text{trace} \{ \underline{\Lambda}^T (\underline{F}_1 \underline{X} + \underline{X}\underline{F}_1^T) + \underline{T}^T (\underline{F}_1 \underline{P} + \underline{P}\underline{F}_1^T) \\ & \quad - \sqrt{\frac{2}{\pi}} \underline{d}^T (\underline{\Lambda} + \underline{\Lambda}^T) \underline{X} (\underline{\Lambda} + \underline{\Lambda}^T) \underline{d} \} \leq 0) \\ 0 & \text{otherwise} \end{cases}$$

$$(4.6.3.6)$$

Note that $CE_0 C^T = PH^T R_n^{-1} HP$ so that in the calculation of the adjoint equations there will be a contribution to H_P from the $CE_0 C^T$ term.

The adjoint equations are :

$$\begin{aligned} H_X^T &= \Lambda^T F + F^T \Lambda^T + i_\omega (\Lambda^T F_1 + F_1^T \Lambda^T) \\ &\quad + i_\omega \gamma_0 (\Lambda^T \underline{d} \underline{k}^T + \underline{k} \underline{d}^T \Lambda^T) \\ &\quad - \frac{\pi}{4} i_\omega \gamma_0 \underline{k} \underline{k}^T \text{trace} \{ \Lambda^T (\underline{d} \underline{k}^T X + X \underline{k} \underline{d}^T) \} \end{aligned} \quad (4.6.3.7a)$$

$$\begin{aligned} H_P^T &= T^T F + F^T T^T + i_\omega (T^T F_1 + F_1^T T^T) + \Lambda^T PH^T R_n^{-1} H + H^T R_n^{-1} HP \Lambda^T \\ &\quad - T^T PH^T R_n^{-1} H - H^T R_n^{-1} HP T^T \end{aligned} \quad (4.6.3.7b)$$

$$o) = \text{constant}$$

$$i) = vS(T) ; v > 0$$

$$j) = 0 \quad (4.6.3.8)$$

The question of optimality in the case of mass flow is even more uncertain than in the case of no mass flow. However, if a satisfactory solution can be obtained for the zero mass flow case with $F_1=0$ then this solution may be taken as a starting guess for the solution of the mass flow problem.

4.7 Optimization Of Thrust Direction For The L.U.B. Problem

The problems considered thus far have assumed that the rocket is steered by a single control, $u(t)$, and that optimization consists solely of finding a policy $u(t) = u(\hat{\underline{x}}, t)$ for determining u as a function of the current best estimate of the state, $\hat{\underline{x}}$.

With respect to the rocket problem, it must be noted that the thrust of the steering engines must be pointed in a specified direction determined by the vector $\underline{d}$, of Eq.(2.1.7). In the context of the previous problem, the vector, $\underline{d}$, is regarded as fixed, perhaps chosen so that thrust always points in the tangential direction of the orbital plane. However, if the thrust direction is determined by an angle, α , then in fact the complete description of the control is determined by the pair (u, α) . Ideally, both u and α should be regarded as random quantities which depend on $\hat{\underline{x}}$ i.e. on the measurement history. This rather ideal formulation will be reserved for future work. Instead, in this section the steering angle, α , will be regarded as a deterministic quantity which depends only on the system dynamics and on the statistics of the measurements.

Recalling that the forcing function $\underline{du}$ of Eq. (2.1.7) physically can only cause accelerations, the vector $\underline{d}$ can be written in the form

$$\underline{d} = \kappa \begin{pmatrix} 0 \\ 0 \\ \sin \alpha \\ \cos \alpha \end{pmatrix} \quad (4.1.7)$$

where $\alpha = 0$ corresponds to tangential thrusting and where κ is a constant proportional to the mass flow rate and specific impulse of the fuel and inversely proportional to the total mass of the rocket.

Now recalling the results of the last chapter it is necessary to minimize the Hamiltonian H with respect to the triple $(i_\omega, \underline{k}, \alpha)$ or equivalently, as in Eq. (4.6.3.4) to minimize $\bar{\omega}(i_\omega, \underline{k}, \alpha)$ given by

$$\bar{\omega} = i_\omega \left(1 + \text{trace} \{ \Lambda^T [\cdot] + \gamma^T [\cdot] + \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\underline{k}^T X \underline{k}}} (\underline{k}^T X (\Lambda + \Lambda^T) \underline{d}(\alpha)) \right) \quad (4.7.2)$$

With no loss of generality $\bar{\omega}$ can be first minimized with respect to $\underline{k}$, regarding i_ω and α as constant to yield $\bar{\omega} = \bar{\omega}(i_\omega, \underline{k}(i_\omega, \alpha), \alpha)$. The optimal $\underline{k}$ is given by Eq. (4.6.4) for $i_\omega \neq 0$ and as before is undefined for $i_\omega = 0$.

$$\underline{k} = \underline{k}(\alpha) = - \eta (\Lambda + \Lambda^T)^T \underline{d}(\alpha) ; \eta > 0 \quad (4.7.3)$$

On substituting Eq. (4.7.3) into (4.7.2) there results

$$\bar{\omega} = \bar{\omega}(i_\omega, \alpha) = i_\omega \left(1 + \text{trace} \{ \Lambda^T [\cdot] + \gamma^T [\cdot] - \sqrt{\frac{2}{\pi}} \sqrt{\underline{d}^T(\alpha) (\Lambda + \Lambda^T) X (\Lambda + \Lambda^T) \underline{d}(\alpha)} \right) \quad (4.7.4)$$

Next let us minimize $\bar{\omega}$ with respect to α again assuming $i_\omega = 1$ and α undefined for $i_\omega = 0$. This involves finding $\underline{d}(\alpha)$ to maximize the quantity

$$\omega_\alpha = \sqrt{\underline{d}^T(\alpha) (\Lambda + \Lambda^T) X (\Lambda + \Lambda^T) \underline{d}(\alpha)} \quad (4.7.5)$$

To solve Eq. (4.7.5) let us define the quantity

$$M = (\Lambda + \Lambda^T) X (\Lambda + \Lambda^T) = \{ m_{ij} \} \quad (4.7.6)$$

Then ω_α given by Eq. (4.7.5) can be expressed in terms of the

m_{ij} 's of Eq. (4.7.6) as follows :

$$\omega_\alpha = \sin^2 \alpha m_{33} + 2m_{34} \sin \alpha \cos \alpha + m_{44} \cos^2 \alpha \quad (4.7.7)$$

To maximize ω_α we employ the condition

$$\frac{d\omega_\alpha}{d\alpha} = 0$$

The solution yields for α the condition

$$\tan 2\alpha = \frac{2m_{34}}{m_{44} - m_{33}} \quad (4.7.8)$$

Eq. (4.7.8) has two solutions for α . These are for $-\pi/2 < \alpha \leq \pi/2$

$$\left. \begin{aligned} \alpha_1 &= \frac{1}{2} \arctan \frac{2m_{34}}{m_{44} - m_{33}} \\ \alpha_2 &= \frac{1}{2} \arctan \frac{2m_{34}}{m_{44} - m_{33}} + \frac{\pi}{2} \end{aligned} \right\} \quad (4.7.9)$$

We must now find a condition for when α_1 is the maximizing solution to Eq. (4.7.7) and when α_2 is the maximizing solution. The condition for α_1 to be maximizing as opposed to α_2 is that

$$\omega_{\alpha_1} > \omega_{\alpha_2}$$

Since $\alpha_2 = \alpha_1 + \frac{\pi}{2}$ we can express ω_{α_2} in terms of α_1 .

The following identities are useful:

$$\sin \left(\alpha_1 + \frac{\pi}{2} \right) = \cos \alpha_1$$

$$\cos \left(\alpha_1 + \frac{\pi}{2} \right) = -\sin \alpha_1$$

Employing the above identities in Eq. (4.7.7) yields for α_2 :

$$\omega_{\alpha_2} = \cos^2 \alpha_1 m_{33} - 2m_{44} \sin \alpha_1 \cos \alpha_1 + m_{44} \sin^2 \alpha_1$$

It follows that the condition $\omega_{\alpha_1} - \omega_{\alpha_2} \geq 0$ can be written

$$\begin{aligned}
 \omega_{\alpha_1} - \omega_{\alpha_2} &= \sin^2 \alpha_1 m_{33} + 2m_{34} \sin \alpha_1 \cos \alpha_1 + m_{44} \cos^2 \alpha_1 \\
 &\quad - \cos^2 \alpha_1 m_{33} + 2m_{34} \sin \alpha_1 \cos \alpha_1 - m_{44} \sin^2 \alpha_1 \\
 &= (\cos^2 \alpha_1 - \sin^2 \alpha_1)(m_{44} - m_{33}) + 4 \sin \alpha_1 \cos \alpha_1 m_{34} > 0 \\
 &= \cos 2\alpha_1 (m_{44} - m_{33}) + 2 \sin 2\alpha_1 m_{34} > 0
 \end{aligned} \tag{4.7.10}$$

Since Eq.(4.7.8) is satisfied for $\alpha = \alpha_1$, it follows that

$$\cos 2\alpha_1 = \frac{1}{\Delta}$$

$$\sin 2\alpha_1 = \frac{2m_{34}}{\Delta(m_{44} - m_{33})}$$

$$\text{where } \Delta = \sqrt{1 + \left(\frac{2m_{34}}{m_{44} - m_{33}} \right)^2}$$

Eq. (4.7.10) can now be written in the more compact form.

$$\omega_{\alpha_1} - \omega_{\alpha_2} = \frac{1}{\Delta} [(m_{44} - m_{33}) + 4m_{34}^2] \geq 0$$

Finally, α_1 or α_2 is seen to maximize ω_{α} in Eq.(4.7.7) according to whether the quantity γ_m defined by Eq. (4.7.11) positive or negative.

$$\left. \begin{aligned}
 \gamma_m &\stackrel{\Delta}{=} m_{44} - m_{33} + 4m_{34}^2 \\
 \gamma_m > 0 &\rightarrow \alpha = \alpha_1 \\
 \gamma_m < 0 &\rightarrow \alpha = \alpha_2 \\
 \gamma_m = 0 &\rightarrow \alpha = \text{either } \alpha_1 \text{ or } \alpha_2
 \end{aligned} \right\} \tag{4.7.11}$$

Finally let α be optimal, i.e. either α_1 or α_2 as determined by Eqs. (4.7.9) and (4.7.11). The quantity $\underline{d}(\alpha)$ is then determined by Eq. (4.7.1). The quantity $\bar{\omega}$ of Eq. (4.7.4) becomes

$$\bar{\omega}(i_w, \alpha) = i_w \left(1 + \text{trace} \{ \Lambda^T [F_1 X + X F_1^T] + T^T [F_1 P + P F_1] \} - \sqrt{\frac{2}{\pi}} \omega_\alpha(\alpha) \right) \quad (4.7.12)$$

Then by the same arguments as in Eq. (4.6.3.6)

$$i_w = \begin{cases} 1 & \text{if } \left(1 + \text{trace} \{ \Lambda^T [F_1 X + X F_1^T] + T^T [F_1 P + P F_1] \} - \sqrt{\frac{2}{\pi}} \omega_\alpha(\alpha) \right) \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.7.13)$$

where Λ and T are defined by Eq. (4.6.3.7)

Remarks

The optimization of the thrust direction for the case of minimum time and minimum fuel consumption are basically similar to the treatment presented above for the L.U.B. problem. In all cases α is precomputed from dynamics and statistics.

4.8 Discussion

The L.U.B. and minimum time problems as formulated above are seen to be closely mathematically related. In fact the L.U.B. problem can be regarded essentially as a generalized minimum time problem. The major simplification arising from the introduction of the indicator function, i_w , in the L.U.B. problem is the elimination of the many transversality conditions at the transitions between burning and coasting

The above formulation is by no means ideal. It would be very nice to be able to formulate an L.U.B. problem where the total fuel consumption is non - random but the actual burning times are random and depend on the measurement history. This would involve a random indicator function, i_r , which would no doubt be correlated with the burning control u which would satisfy the condition

$$\int_{t_0}^T i_r dt \leq C_r \quad (4.8.1)$$

where C_r is a constant depending only on the system dynamics and measurement statistics which is to be minimized. Since control is to be terminated the first time that Eq. (4.8.1) is satisfied with the equality sign it is clear that this is a type of " first crossing problem ". Our formulation of the L.U.B. problem is deliberately cooked up to avoid the difficulties of " first- crossing problems ".

5. Numerical Techniques And Examples

Numerical methods for the solution of the optimal controls of Chapter 4 are discussed in this chapter. Special reference is made to the L.U.B. problem of section 4.6.3. This problem is of greater interest than the minimum expected fuel problem which is not specifically mentioned; firstly, because the indicator function in the solution is highly discontinuous and secondly because the basic arguments apply also to the random thrust direction L.U.B. problem of Chapter 6.

As a preliminary, two first - order cases are worked out and a numerical graph is obtained to illustrate the performance in terms of system parameters. Section 5.1 treats a simple first order minimum time problem. An analytical expression is obtained for the turn on time and hence the time of flight. Section 5.2 treats the same system as a L.U.B. fuel problem and it turns out that for this special case the minimum time and L.U.B. fuel performance criteria lead to identical control laws. However, the L.U.B. formulation yields additional insight that could not have been obtained directly from the minimum time formulation.

5.1 First Order Minimum Time Problem

The matrix equations for the minimum time problem of section 4.3 are given by Eqs. (3.3.15a), (4.3.1), (4.3.4), (4.3.5), (4.3.6), and (4.3.7). For the first order example the stochastic state matrix becomes a scalar quantity, x , and the adjoint matrix becomes a scalar quantity, λ . We shall take the dynamics matrix, F , to be a scalar, $-a$. Under the assumption that the Kalman Filter has reached a steady state the error covariance matrix, P , is constant and hence $CE_0 C^T$ is constant, and in the scalar case is just a constant, c . The stochastic control, k , is now a scalar, k . Taking $d=1$, and $S(T)=1$ as well we have

$$\gamma_0 = \sqrt{\frac{2}{\pi}} \frac{1}{|k| \sqrt{x}} \quad (5.1.1)$$

$$k = -2\lambda\eta; \quad \eta > 0 \quad (5.1.2)$$

Thus the dynamics which correspond to Eq. (3.3.15a) can be written

$$\begin{aligned} \dot{x} &= -2ax + c; \quad x(0) = 0 \quad 0 \leq t < t_\mu \\ \dot{x} &= -2ax + c - 2\sqrt{\frac{2}{\pi}} \sqrt{x} \operatorname{sgn} \lambda; \quad t_\mu \leq t < T \\ x(T) &= \omega_0 \end{aligned} \quad (5.1.3)$$

The λ equation, from Eq. (4.3.4) is given by

$$\dot{\lambda} = 2a\lambda + \sqrt{\frac{2}{\pi}} \frac{|\lambda|}{\sqrt{x}}; \quad \lambda(T) = v > 0 \quad (5.1.4)$$

Since Eq. (5.1.4) is first order and $\lambda(T)$ is positive it follows that

$\lambda(t) > 0$ for all $t \leq T$. Consequently, the only way

of exerting control on Eq. (5.1.3) is by the specification of turn-on time, t_μ . It is convenient to consider this quantity first.

The formal condition for turn-on time is that the Hamiltonian be 0 from Eq. (4.3.6). Hence

$$\begin{aligned} \lambda \dot{x} \Big|_{t_\mu} &= -1 \quad \text{or} \\ \dot{x}(t_\mu) &= -1/\lambda(t_\mu) \end{aligned} \quad (5.1.5)$$

Since $\lambda > 0$ for all t , $\dot{x}(t_\mu)$ is negative as might be expected. The multiplier $v > 0$ is determined from the condition that $x(T) = \omega_0$.

The physical control is given by the expression

$$u = \operatorname{sgn} k \hat{x} = - \operatorname{sgn} 2\lambda \eta \hat{x} = - \operatorname{sgn} \hat{x} \quad (5.1.6)$$

for $t > t_\mu$

where $\hat{x}$ is related to x by

$$x = E [\hat{x}^2]$$

As shown above the turn-on time is ultimately related to the terminal condition $x(T) = \omega_0$. Consequently it is necessary to solve Eq. (5.1.3) to find t_μ .

Integrating the first equation of Eq. (5.1.3) yields

$$x(t_\mu) = e^{-2at_\mu} x(0) + \frac{c}{2a} (1 - e^{-2at_\mu}) = \frac{c}{2a} (1 - e^{-2at_\mu})$$

when $x(0) = 0$. Taking $at_\mu \gg 1$ this becomes $x(t_\mu) \approx \frac{c}{2a}$ (5.1.7)

The second equation of Eq. (5.1.3) is solved by separation of variables.

First write

$$dt = - \frac{dx}{2ax + 2 \sqrt{\frac{2}{\pi}} \sqrt{x} - c} \quad (5.1.8)$$

Next substitute $y = \sqrt{x}$ yielding

$$T - t_\mu = - \int_{y(t_\mu)}^{y(T)} \frac{2y dy}{2ay^2 + 2\sqrt{\frac{2}{\pi}} y - c} \quad (5.1.9)$$

$$\left. \begin{aligned} \text{where } y(T) &= \sqrt{\omega_0} \\ \text{and } y(t_\mu) &= \sqrt{x(t_\mu)} = \sqrt{\frac{c}{2a}} \end{aligned} \right\} \quad (5.1.10)$$

From Dwight [24] formula 160.11

$y(T)$

$$T - t_\mu = -2 \left[\frac{1}{4a} \log \left| 2ay^2 + 2 \sqrt{\frac{2}{\pi}} y - c \right| - \frac{1}{2a} \sqrt{\frac{2}{\pi}} I \right] \Big|_{y(t_\mu)}$$

where the quantity I defined by the integral

$$I = \int_{y(t_\mu)}^{y(T)} \frac{dx}{2ax + 2\sqrt{\frac{2}{\pi}} x - c}$$

is given by formula 160.01 in [24] as

$$I = \frac{1}{\sqrt{\frac{8}{\pi} + 8ac}} \log \left| \frac{4ay + 2\sqrt{\frac{2}{\pi}} - 2\sqrt{\frac{2}{\pi} + 2ac}}{4ay + 2\sqrt{\frac{2}{\pi}} + 2\sqrt{\frac{2}{\pi} + 2ac}} \right|$$

$y(T)$
 $y(t_\mu)$

Writing out the complete expression for $T - t_\mu$ in terms of $y(T)$ and $y(t_\mu)$ yields the following

$$a(T - t_\mu) = -\frac{1}{2} \log \left| \frac{2a^2 y^2(T) + 2\sqrt{\frac{2}{\pi}} a y(T) - ac}{2a^2 y^2(t_\mu) + 2\sqrt{\frac{2}{\pi}} a y(t_\mu) - ac} \right|$$

$$+ \frac{1}{2\sqrt{1 + \pi ac}} \left\{ \log \left| \frac{\sqrt{2\pi a y(T) + 1} - \sqrt{1 + \pi ac}}{\sqrt{2\pi a y(T) + 1} + \sqrt{1 + \pi ac}} \right| \right.$$

$$\left. - \log \left| \frac{\sqrt{2\pi a y(t_\mu) + 1} - \sqrt{1 + \pi ac}}{\sqrt{2\pi a y(t_\mu) + 1} + \sqrt{1 + \pi ac}} \right| \right\} \quad (5.1.11)$$

On substituting Eq. (5.1.10) into Eq. (5.1.11) above there results

$$\begin{aligned}
 a(T - t_\mu) = & -\frac{1}{2} \log \left| \frac{2\sqrt{\pi} a^2 \omega_0 + 2\sqrt{2} a \sqrt{\omega_0} - \sqrt{\pi} ac}{2\sqrt{ac}} \right| \\
 & + \frac{1}{2\sqrt{1 + \pi ac}} \left\{ \log \left| \frac{\sqrt{2\pi} a \sqrt{\omega_0} + 1 - \sqrt{1 + \pi ac}}{\sqrt{2\pi} a \sqrt{\omega_0} + 1 + \sqrt{1 + \pi ac}} \right| \right. \\
 & \left. - \log \left| \frac{\sqrt{\pi ac} + 1 - \sqrt{1 + \pi ac}}{\sqrt{\pi ac} + 1 + \sqrt{1 + \pi ac}} \right| \right\}
 \end{aligned}
 \tag{5.1.12}$$

It is interesting that the terms $2\sqrt{\pi} a^2 \omega_0 + 2\sqrt{2} a \sqrt{\omega_0} - \sqrt{\pi} ac$ and $\sqrt{2\pi} a \sqrt{\omega_0} + 1 - \sqrt{1 + \pi ac}$ both vanish when

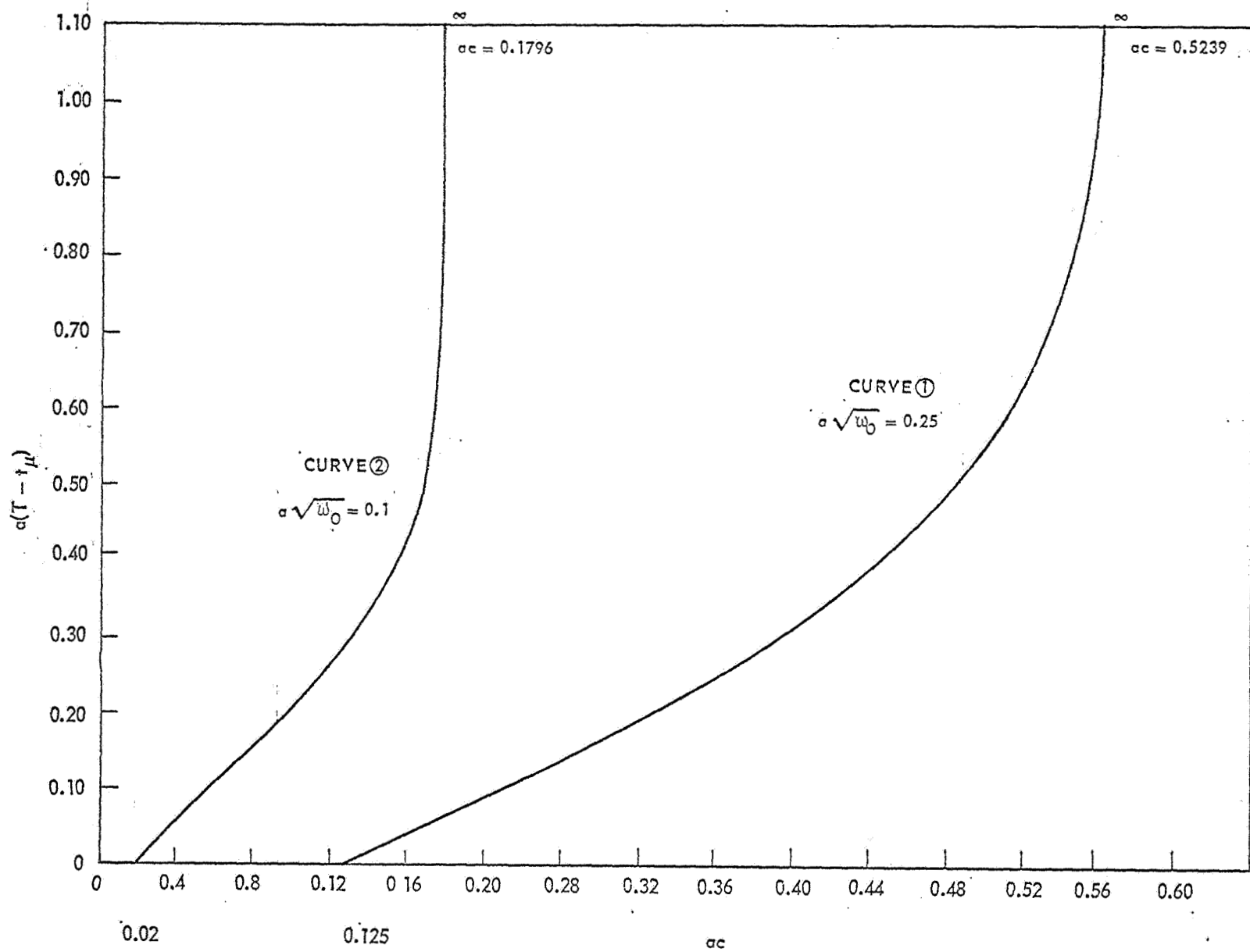
$$a \sqrt{\omega_0} = \frac{\sqrt{1 + \pi ac} - 1}{\sqrt{2\pi}}
 \tag{5.1.13}$$

When Eq. (5.1.13) is satisfied the log terms in Eq. (5.1.12) blow up.

The dimensionless time quantity $a(T - t_\mu)$ given by Eq. (5.1.12) is plotted in Fig. 2 as a function of the parameter ac for fixed values of the parameter $a \sqrt{\omega_0}$. The range of ac is determined by Eqs. (5.1.10) and (5.1.13). Thus the condition

$$y(t_\mu) > y(T) \rightarrow ac > 2 (a \sqrt{\omega_0})^2
 \tag{5.1.14a}$$

Also the stopping condition of Eq. (5.1.13) implies



Curves For The Minimum Time Example

$$ac < \frac{1}{\pi} \left(\left(\sqrt{2\pi} a \sqrt{\omega_0} + 1 \right)^2 - 1 \right) \quad (5.1.14)$$

As might be expected from Eqs. (5.1.14a & b), curve 1 where $a \sqrt{\omega_0} = .25$ starts at a greater value of ac and blows up at a larger value of ac than curve 2 where $a \sqrt{\omega_0} = 1$. There is an interesting partial physical explanation of this phenomenon. First of all Eq. (5.1.7) indicates that there is some minimum value of c , which determines the rate at which new information is fed into the system, so that the variance, x , can reach a specified value without any control action. This is because the system of Eq. (5.1.3) is stable and hence x tends to run down. Next under control action the system is more stable and tends to run down faster. The natural stability of the system tends to decrease the variance, x , while the new information coming into the system which is determined by c tends to increase the variance, x . At a balancing condition of these opposing forces the variance, x is in a steady state.

From Eq. (5.1.3) the condition of balance between the forces of control and information should occur when

$$\dot{x} = 0 = c - 2ax - 2\sqrt{\frac{2}{\pi}} \sqrt{x} \quad (5.1.15)$$

$$\text{or } a \sqrt{x} = \frac{\sqrt{1 + \pi ac} - 1}{\sqrt{2\pi}} \quad (5.1.16)$$

Hence when $x = \omega_0$ as specified by the target condition at time T in Eq. (5.1.3) then Eqs. (5.1.13) and (5.1.15) are identical. Hence for a given information rate it is simply not possible to regulate to an arbitrarily small target region.

We should therefore regard Eq. (5.1.13) or (5.1.16) as stochastic controllability conditions.

5.2 First Order L.U.B. Problem

The matrix equations for the L.U.B. problem are given by Eqs. (4.5.1), (4.5.4), (4.5.7), (4.5.8), (4.5.9). Employing the same notation as in the first order minimum time problem we arrive at the following first order dynamical equations

$$\dot{x} = -2ax + c - 2i_w \sqrt{\frac{2}{\pi}} \sqrt{x} \quad (5.2.1)$$

The adjoint equation from Eq. (5.1.4), since λ is positive is

$$\dot{\lambda} = 2a\lambda + i_w \sqrt{\frac{2}{\pi}} \frac{\lambda}{\sqrt{x}} \quad (5.2.2)$$

The control k is given by

$$k = -2\eta\lambda; \quad \eta > 0 \quad (5.2.3)$$

and the indicator function is given by the condition

$$i_w = \begin{cases} 1 & \text{if } 1 - 2\sqrt{\frac{2}{\pi}} \lambda \sqrt{x} \leq 0 \\ 0 & \text{if } 1 - 2\sqrt{\frac{2}{\pi}} \lambda \sqrt{x} > 0 \end{cases} \quad (5.2.4)$$

The boundary conditions are as before

$$x(t_0) = 0 \quad x(T) = \omega_0 \quad \lambda(T) = v > 0 \quad (5.2.5)$$

The control is given by

$$u = -\text{sgn } k \hat{x} \quad (5.2.6)$$

The numerical solution of Eqs. (5.2.1), (5.2.2), and (5.2.4) may proceed on the assumption that an initial value $\lambda, \lambda(t_0)$ is given. Then it is possible to integrate Eqs. (5.2.1), and (5.2.2) forward simultaneously. This "shooting method" is shown to have numerical instabilities since Eq. (5.2.2) is unstable in the forward direction. A more practical numerical method is suggested in Section 5.3. Note that when $x = 0, i_\omega = 0$ so that the λ equation never has an infinite derivative.

At time $t=T$ we compare $X(T)$ with ω_0 . If $X(T) > \omega_0$ then this can be corrected by decreasing $\lambda(t_0)$. This decreases $\lambda(t)$ for all t and makes i_ω take on the value zero more often. Conversely, if $x(T) < \omega_0$, $\lambda(t_0)$ should be increased. A very simple linear updating scheme for $\lambda(t_0)$ is proposed in Eq. (5.2.7) where $\Delta\lambda(t_0)$ is the change in $\lambda(t_0)$.

$$\Delta\lambda(t_0) = -\frac{\lambda(t_0)}{\omega_0} (x(T) - \omega_0) \quad (5.2.7)$$

For the numerical integration of Eqs. (5.2.1) and (5.2.2) a simple first-order Euler method may suffice. Thus a general equation of the type

$$\dot{z} = g(z, i_\omega) \quad (5.2.8a)$$

might be integrated with the algorithm

$$z(k+1) = z(k) + g(z(k), i_\omega(k)) \cdot h \quad (5.2.8b)$$

where h is the step-size increment. Perhaps the most insight can be obtained from a phase plane sketch in the x, λ plane. From Eqs. (5.2.1) and (5.2.2)

$$\frac{dx}{d\lambda} = \frac{-2ax + c - 2i_{\omega} \sqrt{\frac{2}{\pi}} \sqrt{x}}{2a\lambda + i_{\omega} \sqrt{\frac{2}{\pi}} \frac{\lambda}{\sqrt{x}}} \quad (5.2.9)$$

Eq. (5.2.9) yields the equation for the isoclines. Initially when $x=0$ it follows from Eq. (5.2.4) that $i_{\omega} = 0$ so that

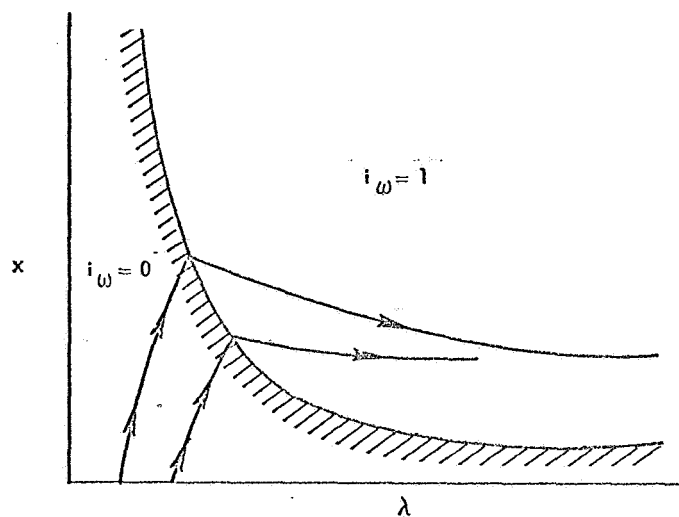
$$\left. \frac{dx}{d\lambda} \right|_{x=0} = \frac{c}{2a\lambda} \quad (5.2.10)$$

There are two critical regions in the phase plane corresponding to $i_{\omega} = 0$ and $i_{\omega} = 1$ as shown in Fig. 3a, b. Eq. (5.2.4) gives the equation for the separating boundary, shown in Fig. 3a, b which is written as Eq. (5.2.11)

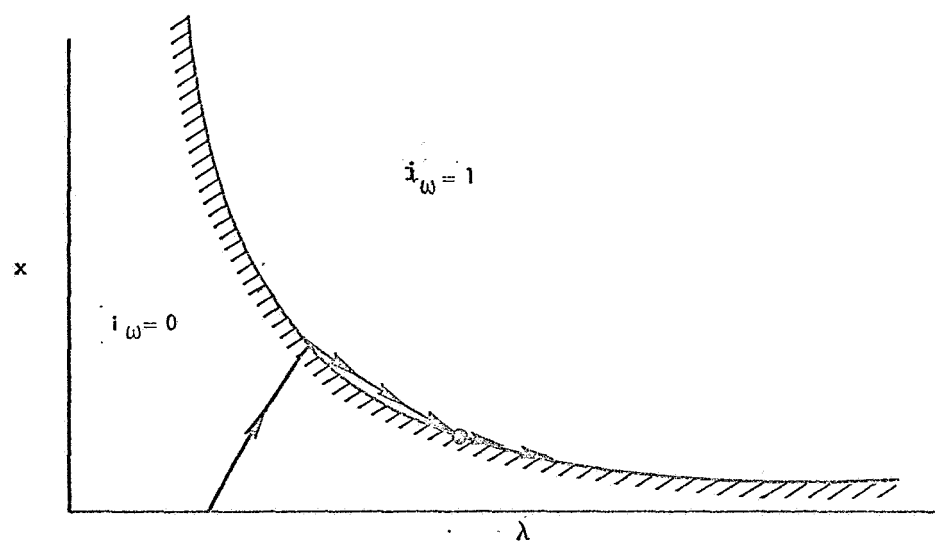
$$\lambda^2 x = \frac{\pi}{8} \quad (5.2.11)$$

Note that Eq. (5.2.2) implies that λ increases monotonically with time. Consequently, the x, λ contours sketched in Fig. 3a, b are equivalent to x, t contours on a different scale for the abscissa. As λ (or t) increases Eq. (5.2.11) implies that x must decrease so that the switching boundary must approach $x = 0$ for large λ or large T . However, the actual variance x never drops below the value determined by the target value, ω_0 . We argue that once the region $i_{\omega} = 1$ is entered, motion never again enters the $i_{\omega} = 0$ region. Consider the two possible cases illustrated in Fig. 3a and Fig. 3b. Fig. 3a illustrates the circumstance in which motion once having entered the interior of the region $i_{\omega} = 1$ remains in the interior. This corresponds exactly to the case of minimum time performance discussed in section 5.1

PHASE PLANE SKETCHES OF x, λ CONTOURS. TIME INCREASES IN THE DIRECTION OF THE ARROWS



a



b

because as soon as the $i_\omega = 1$ region has been entered the system behaves as if it were under minimum time control. Fig. 3b illustrates a possible circumstance of chattering along the switching boundary; a circumstance which we shall show is impossible. It is clear in any case that once motion has entered the $i_\omega = 1$ and then decayed down to the switching boundary it cannot actually re-enter the $i_\omega = 0$ region for any distance. This is because as soon as control terminates the action of the forcing term c makes x increase so that it again re-enters the $i_\omega = 1$ boundary. We now argue that in fact once the switching boundary is reached and control is applied motion does not chatter down the boundary but does in fact penetrate into the $i_\omega = 1$ region as in Fig. 3a

If chattering did occur on the boundary then the slope $\frac{dx}{d\lambda}$ given by Eq. (5.2.9) for $i_\omega = 1$ would be more negative than the slope of the boundary itself as given from Eq. (5.2.11). That is for chattering we must have

$$\frac{-2ax + c - 2\sqrt{\frac{2}{\pi}}\sqrt{x}}{2a\lambda + \sqrt{\frac{2}{\pi}}\frac{\lambda}{x}} < -\frac{2x}{\lambda} \quad (5.2.12)$$

where x and λ are related by Eq. (5.2.11). On eliminating x in favor of λ and substituting into Eq. (5.2.12), the chatter condition reduces to

$$-c > 2ax \quad (5.2.13)$$

which is impossible since all quantities in Eq. (5.2.13) are positive. Hence chatter does not occur.

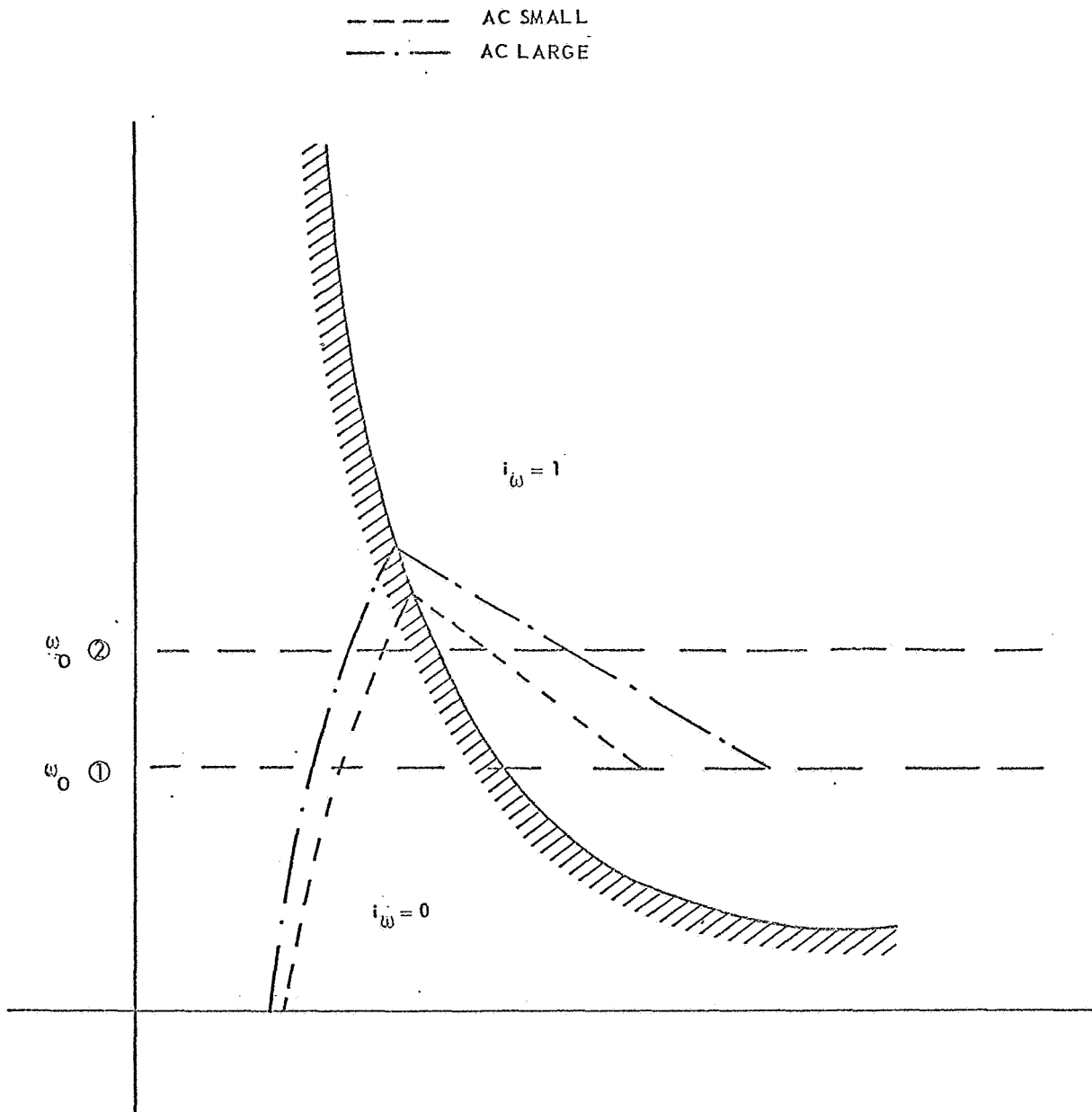
Since it has now been established that the first order L.U.B. problem and the first order minimum time problem are identical in the region $i_\omega = 1$ it is of interest to interpret the curves of Fig. 2 in terms of the x, λ phase plane. Fig. 4 illustrates the phase plane contours for curves 1 and 2 which correspond to Fig. 2. Recall also that since λ increases monotonically with time the λ axis is equivalent to a time axis with different scale. Each dashed curve in Fig. 4 corresponds to both curve 1 and curve 2 in Fig. 2. However, curve 1 of Fig. 2 corresponds to motion of either dashed curve in Fig. 2 over the range from the switching curve to the lower ω_0 line while curve 2 of Fig. 2 corresponds to motion from the switching curve to the upper ω_0 line. Thus it is clear why time must increase as ω_0 decreases. Fig. 4 also illustrates why time tends to increase with increasing α . Note that the curves, if continued, would level off at a value of x given by Eq. (5.1.16).

5.3 Iterative Solution Of The Tangential Thrusting L.U.B.

Problem Of Section 4.6.3

Given a solution of Eq. (2.1.3) nominal values of $\underline{r}$ and $\underline{v}$ of the trajectory are determined. Based on these nominal values it is possible to generate the quantities F , F_1 , and $\underline{d}$. A good discussion of these perturbation quantities may be found in Battin [1],[2].

For the remainder of this chapter only the perturbed motion will be considered and it will be assumed that F, F_1 , and $\underline{d}$ are known. Consequently, the basic stochastic equations of motion that must be solved are given by Eq. (4.6.3.2) together with other previously given defining relations which are here rewritten as Eq. (5.3.1) for the interval



Phase Plane Contours of Figure 2

$$[t_0, T]$$

$$\begin{aligned}\dot{X} &= FX + XF^T + i_\omega (F_1 X + X F_1^T) + C E_0 C^T + i_\omega \gamma_0 (\underline{dk}^T X + X \underline{kd}^T) \\ \dot{P} &= FP + PF^T + i_\omega (F_1 P + P F_1^T) + Q_W - PH^T R_n^{-1} HP \\ X(t_0) &= \text{constant} \\ P(t_0) &= \text{constant} \\ C &= PH^T R_n^{-1} \\ E_0 &= R_n\end{aligned}\tag{5.3.1}$$

Note that the P equation generates the matrix quantity CEC^T .

The time t_0 is the first time that $X(t_0)$ is positive definite.

$X(t_0)$ and $P(t_0)$ are determined by Eq.(5.3.2) over the interval

$$[0, t_0]$$

$$\begin{aligned}\dot{X} &= FX + XF^T + C E_0 C^T \\ \dot{P} &= FP + PF^T + Q_W - PH^T R_n^{-1} HP \\ X(0) &= 0 \\ P(0) &= \mu I \quad \mu = \text{a very large number} \\ C &= PH^T R_n^{-1} \\ E_0 &= R_n\end{aligned}\tag{5.3.2}$$

$X(t_0)$ is positive definite

Having determined $X(t_0)$ and $P(t_0)$ from Eq.(5.3.2) we may now employ Eq. (5.3.1) to define the motion of the system in terms of the control variables i_ω , $\underline{k}$. Recall that the reason for determining a t_0 such that $X(t_0)$ is positive definite was to satisfy the sufficient conditions for optimality. The remaining description of the L.U.B. problem is the adjoint equations and the determination of the

control variables.

The adjoint equations are given by Eq. (4.6.3.7) and are rewritten here as Eq. (5.3.3)

$$\dot{\Lambda} = -F^T \Lambda - \Lambda F - i_{\omega} (F_1^T \Lambda + \Lambda F_1) - i_{\omega} \gamma_0 (\underline{k} d^T \Lambda + \Lambda \underline{d} k^T) \\ + \frac{\pi}{4} i_{\omega} \gamma_0 \underline{k} \underline{k}^T \text{trace} \{ \Lambda^T (\underline{d} k^T X + X \underline{k} d^T) \} \quad (5.3.3a)$$

$$\dot{T} = -F^T T - T F - i_{\omega} (F_1^T T + T F_1) \\ - H^T R_n^{-1} H P T - T P H^T R_n^{-1} H - \Lambda^T P H^T R_n^{-1} H - H^T R_n^{-1} H P \Lambda^T \quad (5.3.3b)$$

$$\Lambda(T) = v S(T)$$

$$T(T) = 0 \quad (5.3.3c)$$

The control quantities from Eqs. (4.6.3.5) and (4.6.3.6) are given by

$$\underline{k} = - \eta (\Lambda + \Lambda^T) \underline{d} \quad ; \quad \eta > 0 \quad (5.3.4)$$

$$i_{\omega} = \begin{cases} 1 & \text{if } (1 + \text{trace} \{ \Lambda^T (F_1 X + X F_1^T) + T^T (F_1 P + P F_1) \} \\ & - \sqrt{\frac{2}{\pi}} \sqrt{\underline{d}^T (\Lambda + \Lambda^T) X (\Lambda + \Lambda^T) \underline{d}}) \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.3.5)$$

Note also the target set condition which determines the multiplier v in Eq. (5.3.3)

$$\text{trace} [S(T) X(T)] = \omega_0 \quad (5.3.6)$$

The basic iterative procedure that is proposed has elements common to currently in vogue gradient or sweep methods and it is expected that it will share most of the difficulties commonly associated with these methods, which basically involve numerical instabilities. A list of

such methods, all of them having much in common, and none of them guaranteeing convergence is given in [21]. Numerical computation remains an art rather than a science.

It is an axiom of numerical computation that continuity is a highly desirable attribute. The indicator function defined by Eq. (5.3.5) does not have continuous derivatives and consequently small changes in X and Λ and the other variables are likely to lead to large changes in the indicator function. Continuity can however be introduced with the following artifice: The indicator function

$$i_w(x) \triangleq \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$$

can be approximated by the continuous function

$$\hat{i}_w(x, \epsilon) \triangleq \frac{\sqrt{x^2 + \epsilon^2} - x}{2\sqrt{x^2 + \epsilon^2}} \quad (5.3.7a)$$

which has the property that

$$\lim_{\epsilon \rightarrow 0} \hat{i}_w(x, \epsilon) = \begin{cases} 1 & x < 0 \\ 1/2 & x = 0 \\ 0 & x > 0 \end{cases} \quad (5.3.7b)$$

Letting $x = I + \text{trace} \{ \Lambda^T (F_1 X + X F_1^T) + T^T (F_1 P + P F_1) \}$

$$= \sqrt{\frac{2}{\pi}} \sqrt{\underline{d}^T (\Lambda + \Lambda^T) X (\Lambda + \Lambda^T) \underline{d}}$$

the first step in our computational procedure is to replace $i_\omega(x)$ by $\hat{i}_\omega(x, \epsilon)$ in Eq. (5.3.5).

Next since it can be assumed that mass flow will be a small perturbing effect on the final calculation we shall solve the constant mass case which is simpler and use the resulting solution as a starting point for the variable mass problem. Thus set

$$\begin{aligned} F_1 &= 0 \\ T &= 0 \end{aligned} \quad (5.3.8)$$

as a first approximation

The P equations can not be integrated ahead of the actual iterative process and the values for CEC^T retained in storage.

Proceeding to the actual iterative process we choose a value of ϵ in Eq. (5.3.7) and a step-size for the integration as well as an integration algorithm. One of the more popular integration algorithms is the 4th order Runge - Kutta procedure which has the desirable feature of being self-starting. The procedure is as follows :

- 1) Choose an initial guess for the control variables $\underline{k}(t), i_\omega(t)$ over the interval $[t_0, T]$.
- 2) Integrate Eq. (5.3.1) forward from t_0 to T to determine X (and P) which is (are) retained in storage.
- 3) Test the target condition of Eq. (5.3.6). If the inequality

$$\| \text{trace} [S(T)X(T)] - \omega_0 \| \leq \delta \quad (5.3.9)$$

is satisfied then the iteration terminates except on the first pass. If not then we must find a value of $\Lambda(T)$. According to

Eq. (5.3.3c), $\Lambda(T)$ is determined solely by the undetermined multiplier ν .

On the first pass ν is to be guessed at. However, during the iteration procedure ν is to be updated based on the previous value of $\nu = \nu_0$.

Specifically, suppose $\text{trace} [S(T)x(T)] > \omega_0$ so that motion is outside the target set. Increasing ν gives more weight to the positive term $\nu (\text{trace} [S(T)x(T)] - \omega_0)$ in Eq. (4.2.8). Since the term is weighted more heavily the minimization tends to reduce it.

Conversely when $\text{trace} [S(T)x(T)] \leq \omega_0$ so that motion is inside the target region, then making ν smaller puts less weight on the term which will tend to increase it. An updating scheme for ν in terms of the past value of ν_0 which is based on the above idea is

$$\nu - \nu_0 = \kappa_s [\text{trace} \{ S(T)x(T) \} - \omega_0]$$

$$1 > \kappa_s > 0 \quad (5.3.10)$$

The step-size κ_s must be chosen to facilitate convergence.

- 4) Integrate backwards in time Eqs. (5.3.3), (5.3.4) and (5.3.5) simultaneously. This determines new values for the control variables $\underline{k}(t)$ and $\underline{i}_\omega(t)$ in terms of the stored values of $X(c)$.
- 5) Return to step 1.

When convergence has been completed put in the F_1 terms and the equation for T and repeat the process for the changing mass system based on the starting values of $\underline{k}$ and $\underline{i}_\omega$ which resulted from the constant mass system.

The last step is to reduce the parameter ϵ to as small a value as possible. In the actual control of course ϵ will be set to zero.

Convergence of the method depends on the type of computer used, the skill of the programming and the time allotted for computation. Double precision, small step-size, large ϵ and small κ_s add to the

expense but tend to make convergence more likely. The trade-off between cost and numerical accuracy is a matter of skill and judgement. One of the possible difficulties in the computation might be the occurrence of chatter such as is depicted in Fig. 3b of Section 5.1. Chatter occurs when

$$1 + \text{trace} \{ \Lambda^T (F_1 X + X F_1^T) + T^T (F_1 P + P F_1) \} - \sqrt{\frac{2}{\pi}} \sqrt{d^T (\Lambda + \Lambda^T) X (\Lambda + \Lambda^T) d} = 0 \quad (5.3.11)$$

over a finite interval. Then the indicator function i_ω goes from 1 to 0 and back to 1 at an infinite rate. The replacement of i_ω by $\hat{i}_\omega$ as in Eq. (5.3.7a) would lead to the condition $\hat{i}_\omega = 1/2$ on the interval in question.

There probably is no really good way of handling the problem of chatter from a computational point of view. Heuristically, one might suppose based on experience with deterministic controls that chatter is most likely to occur as the motion approaches a very small target set. Hence chatter might be avoided if the target set parameter, ω_0 , is not too small.

5.4 Iterative Solution For The Extended L.U.B. Problem

Of Section 4.7

The extended L.U.B. problem of section 4.7 introduces the additional control variable, α , representing the precomputed direction of thrust. This parameter is determined by Eqs. (4.7.9) and (4.7.11). Recall that

Eq. (4.7.11) is equivalent to

$$\alpha = \begin{cases} \alpha_1 & \text{if } \gamma_m > 0 \\ \alpha_2 & \text{if } \gamma_m < 0 \\ \alpha_1 \text{ or } \alpha_2 & \text{if } \gamma_m = 0 \end{cases} \quad (5.4.1)$$

where γ_m is some function of X and Λ . The updating of $\alpha(t)$ takes place simultaneously with that of $\underline{k}(t)$ and $i_\omega(t)$ as described in step 4 of the last section 5.3. If the quantity γ_m of Eq. (5.4.1) is sensitive to step-size errors, round off errors or truncation errors, the discontinuity of α given by Eq. (5.4.1) may possibly destabilize the numerical calculations. Therefore it might be of some use to introduce the same trick of forcing continuity on α much as continuity was forced onto the parameter i_ω as in Eq. (5.3.7a). We may thus define a new parameter $\hat{\alpha}$ to approximate α which is given by

$$\hat{\alpha} = 1/2 (\alpha_1 + \alpha_2) + \frac{\gamma_m}{\gamma_m^2 + \epsilon^2} (\alpha_1 - \alpha_2) \quad (5.4.2a)$$

which has the property that

$$\lim_{\epsilon \rightarrow 0} \hat{\alpha} = \begin{cases} \alpha_1 & \text{if } \gamma_m > 0 \\ \alpha_2 & \text{if } \gamma_m < 0 \\ 1/2 (\alpha_1 + \alpha_2) & \text{if } \gamma_m = 0 \end{cases} \quad (5.4.2b)$$

On replacing α by $\hat{\alpha}$ and calculating $\hat{\alpha}$ together with $\underline{k}$ and i_ω the procedure is the same as in section 5.3.

6. L.U.B. Optimal Control When The Thrust Direction Depends On The Observation History

In the work of the previous chapter the controller has been designed on the basis that the pair of back - to - back rocket motors always point in a pre-assigned direction. In Section (4.7) it was shown how this thrust direction could be precomputed as a function of the $\hat{a}$ priori known statistics and system dynamics. In this chapter, the problem of the least upper bound fuel control is further generalized so that the thrust direction is a random quantity depending on on-board physical measurements. For purposes of simplicity it is assumed that the rocket has a constant mass. The analysis can be extended to cover the realistic case of a finite mass flow rate exactly in the same way that Section 4.6.3 extends the analysis of Section 4.5.

Recall that the thrust angle α appears in the definition of the vector, $\underline{d}$, in Eq.(4.7.1) as

$$\underline{d} = \kappa \begin{bmatrix} 0 \\ 0 \\ \sin \alpha \\ \cos \alpha \end{bmatrix} \quad (4.7.1)$$

The basic assumption that will be made in this chapter, for which some justification is given in Section 6.2 is that α takes on the structure

$$\alpha \stackrel{\Delta}{=} \underline{v}^T \hat{\underline{x}} + \alpha_0 \quad (6.0.1)$$

The parameter, $\alpha_0(t)$ in Eq.(6.0.1) will closely approximate the value given by Eq.(4.7.11) if $\underline{v}$ has only small elements. A complete solution to the optimal control problem consists of the calculation of the quantities $\underline{k}$, $\underline{v}$, α_0 , and i_ω .

6.1 Dynamical Equations of Motion

Let us return to the equations of motion for the best estimator given by Eq.(2.2.4) which we modify using the arguments of Section (4.6) for the indicator function description of burning as follows :

$$\dot{\hat{\underline{x}}} = F \hat{\underline{x}} + C \underline{e} + i_\omega \underline{u} \underline{d} \quad (6.1.1)$$

Eq.(6.1.1) as it stands does not adequately describe the influence of the random variable α on the estimate of the state, $\hat{\underline{x}}$. A clearer description of the interrelation of α and $\underline{u}$ can be gained by defining the vectors

$$\underline{d}_1 = \kappa \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} ; \quad \underline{d}_2 = \kappa \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (6.1.2)$$

and then rewriting Eq.(6.1.1) in the form

$$\dot{\hat{\underline{x}}} = F \hat{\underline{x}} + C \underline{e} + i_\omega (\underline{u} \sin \alpha \underline{d}_1 + \underline{u} \cos \alpha \underline{d}_2) \quad (6.1.3)$$

In the above, we have made explicit use of the identity

$$\underline{d}_1 \underline{u} \sin \alpha + \underline{d}_2 \underline{u} \cos \alpha = \underline{d} \underline{u} \quad (6.1.4)$$

Eq.(6.1.4) can now be used to find the evolution of the stochastic

state X . Recall first the definition of the stochastic control $\underline{b}$ by Eq. (2.3.4c).

$$\underline{b} \underline{d}^T \triangleq E [\hat{\underline{x}} u] \underline{d}^T = E [\hat{\underline{x}} \underline{d}^T u]$$

On substituting the identity of Eq. (6.1.4) for $\underline{d}^T u$ the above equations can be written as

$$E [\hat{\underline{x}} u \sin \alpha] \underline{d}_1^T + E [\hat{\underline{x}} u \cos \alpha] \underline{d}_2^T = \underline{b}_1 \underline{d}_1^T + \underline{b}_2 \underline{d}_2^T \quad (6.1.5)$$

where

$$\begin{aligned} \underline{b}_1 &\triangleq E [\hat{\underline{x}} u \sin \alpha] \\ \underline{b}_2 &\triangleq E [\hat{\underline{x}} u \cos \alpha] \end{aligned} \quad (6.1.6)$$

Eq. (2.3.13) then can be written as

$$\dot{X} = FX + XF^T + CE_0 C^T + i_\omega (\underline{b}_1 \underline{d}_1^T + \underline{b}_2 \underline{d}_2^T + \underline{d}_1 \underline{b}_1^T + \underline{b}_2 \underline{d}_2^T) \quad (6.1.7)$$

$X(t_0)$ determined by Eq. (5.3.2) with $t_0 = t_1$

The evolution of Eq. (6.1.7) is thus determined by the pair of stochastic control vectors $\underline{b}_1$ and $\underline{b}_2$ which are to be calculated in terms of the parametric structure of Eqs. (3.2.13) and (6.0.1). The optimization consists of forming a Hamiltonian which depends on i_ω , $\underline{k}$, $\underline{v}$, and α_0 and then proceeding as in section 4. Before proceeding with these basic arguments section 6.2 which follows presents a discussion of the structure of Eq. (6.0.1) and a comparison based on Dynamic Programming of the form of a more desired structure.

6.2 Structure Of The Controller

In Eq.(6.0.1) we assumed that the thrust angle can be represented by a simple linear structure and that it depends only on a single vector $\underline{y}(t)$. This assumption in fact may well be poor in that the α which it yields is a poor approximation to the optimal α . In this section the arguments of Dynamic Programming of Section (3.2) are updated for Eq.(6.1.3) and are used to find the "true" optimal structure for the thrust angle, α .

On making the correspondence of Eq.(6.1.4) the Bellman - Hamilton - Jacobi equation of Eq.(3.2.5) becomes

$$\begin{aligned} \text{Min}_{u, \alpha} \left\{ \frac{V^T}{x} \left[\hat{F} \underline{x} + i_{\omega} \underline{d}_1 u \sin \alpha + i_{\omega} \underline{d}_2 u \cos \alpha \right] \right. \\ \left. + 1/2 \text{ trace } [\hat{V}_{xx} C E_0 C^T] + \frac{\partial V}{\partial t} + \hat{L}_1 \right\} = 0 \end{aligned} \quad (6.2.1)$$

Thus when $\hat{L}_1 = 1$ and $|u| \leq 1$ and $i_{\omega} = 1$ we must find

$$\text{Min}_{u, \alpha} u \left[\hat{V}_x^T \underline{d}_1 \sin \alpha + \hat{V}_x^T \underline{d}_2 \cos \alpha \right] \quad (6.2.2)$$

Clearly α should be chosen such that

$$\left| \hat{V}_x^T \underline{d}_1 \sin \alpha + \hat{V}_x^T \underline{d}_2 \cos \alpha \right| \text{ is maximum} \quad (6.2.3)$$

Our solution for α is as follows :

First assume $\hat{V}_x^T \underline{d}_1$ and $\hat{V}_x^T \underline{d}_2$ are both positive. Then differentiating $\hat{V}_x^T \underline{d}_1 \sin \alpha + \hat{V}_x^T \underline{d}_2 \cos \alpha$ with respect α and setting the resulting derivative to zero yields

$$\tan \alpha = \frac{\underline{\hat{v}}_1^T \underline{\hat{d}}_1}{\underline{\hat{v}}_2^T \underline{\hat{d}}_2} \quad (6.2.4)$$

so that $\sin \alpha$ and $\cos \alpha$ are both positive and Eq.(6.2.3) is indeed maximized. Next if $\underline{\hat{v}}_1^T \underline{\hat{d}}_1$ is negative and $\underline{\hat{v}}_2^T \underline{\hat{d}}_2$ positive then from Eq.(6.2.4) if $\sin \alpha$ is negative and $\cos \alpha$ positive : then both terms in Eq.(6.2.3) are positive ; while if $\sin \alpha$ is positive and $\cos \alpha$ is negative then both terms in Eq.(6.2.3) are negative. Similar arguments hold if $\underline{\hat{v}}_1^T \underline{\hat{d}}_1$ is positive and $\underline{\hat{v}}_2^T \underline{\hat{d}}_2$ negative or if both terms are negative. We conclude that Eq.(6.2.4) yields the optimum α .

Then from Eq.(6.2.2) the optimum u is given in terms of the optimum α by

$$u = - \operatorname{sgn} \left[\underline{\hat{v}}_1^T \underline{\hat{d}}_1 \sin \alpha + \underline{\hat{v}}_2^T \underline{\hat{d}}_2 \cos \alpha \right] \quad (6.2.5)$$

If we are willing to accept the representation of the V function as in Eq.(3.2.8) and then to accept the approximation of Eq.(3.2.9) that $\underline{\hat{V}}_x$ can be represented as a linear function of the $\underline{\hat{x}}$, then Eqs. (6.2.4) and (6.2.5) can be written

$$\left. \begin{aligned} \tan \alpha &= \frac{\underline{\hat{v}}_1^T \underline{\hat{x}}}{\underline{\hat{v}}_2^T \underline{\hat{x}}} \\ u &= \operatorname{sgn} \underline{\hat{k}}^T \underline{\hat{x}} \end{aligned} \right\} \quad (6.2.6)$$

where $\underline{\hat{v}}_1$ and $\underline{\hat{v}}_2$ are vectors determined by the partial derivative of

V. The above structure for α is more complex than our analytical techniques can assimilate. We therefore simplify Eq.(6.2.6) at the expense of representational accuracy by expanding in a Taylor's series about $\alpha = \alpha_0$.

$$\begin{aligned} \alpha &= \tan^{-1} \frac{\underline{\underline{v}}_1^T \hat{\underline{\underline{x}}}}{\underline{\underline{v}}_2^T \hat{\underline{\underline{x}}}} \approx \frac{\underline{\underline{v}}_1^T \hat{\underline{\underline{x}}}}{\underline{\underline{v}}_2^T \hat{\underline{\underline{x}}}} - 1/3 \left(\frac{\underline{\underline{v}}_1^T \hat{\underline{\underline{x}}}}{\underline{\underline{v}}_2^T \hat{\underline{\underline{x}}}} \right)^3 + 1/5 \left(\frac{\underline{\underline{v}}_1^T \hat{\underline{\underline{x}}}}{\underline{\underline{v}}_2^T \hat{\underline{\underline{x}}}} \right)^5 \\ &= \alpha_0 + \underline{\underline{v}}^T \hat{\underline{\underline{x}}} + \sum_{i,j} \beta_{ij} \hat{x}_i \hat{x}_j + \sum_{ijk} \delta_{ijk} \hat{x}_i \hat{x}_j \hat{x}_k \end{aligned} \quad (6.2.7)$$

Eq.(6.0.1) results when the higher order terms in Eq.(6.2.7) are discarded. The resulting approximation is best when $\hat{\underline{\underline{x}}}$ is small which occurs near the beginning and near the end of the trajectory of motion. Again recall that the scalar, $\alpha_0(t)$ in Eq.(6.2.7) approximates the value of $\alpha(t)$ given by Eq.(4.7.11) when $\underline{\underline{v}}$ has only small elements.

6.3 Evaluation of the Stochastic Control Vectors, $\underline{b}_1$, and, $\underline{b}_2$.

Before attempting to optimize the L.U.B. fuel consumption using the deterministic dynamical equation Eq.(6.1.7) it is first necessary to evaluate $\underline{b}_1$ and $\underline{b}_2$, which are defined by Eq.(6.1.5), in terms of the parameters $\underline{v}, \underline{k}, \alpha_0$ which were defined in Eqs.(6.0.1) and (6.2.6).

As in Section 3.3 it is only possible to make this evaluation under the approximation that, $\hat{\underline{x}}$, is a Gaussian random vector. The justification of this approximation here is the same as given in Section 3.3.

As a first step in the evaluation of Eq.(6.1.5) under the Gaussian approximation consider the simpler problem of finding

$$\underline{b}_e = E[e^{j\alpha \hat{\underline{x}}u}] \quad (6.3.1)$$

Then clearly

$$\underline{b}_1 = \text{Im}\{\underline{b}_e\} \quad \text{and} \quad \underline{b}_2 = \text{Re}\{\underline{b}_e\} \quad (6.3.2)$$

On substituting for α from Eq.(6.0.1) into Eq.(6.3.1)

$$\underline{b}_e = E[e^{j(\alpha_0 + \underline{v}^T \hat{\underline{x}})u}] = e^{j\alpha_0 u} E[e^{j\underline{v}^T \hat{\underline{x}} u}] \quad (6.3.3)$$

since α_0 is constant. Thus it can be seen that the major problem in the evaluation of Eq.(6.3.3) is the calculation of the term

$$\rho_1 = E[e^{j\underline{v}^T \hat{\underline{x}} u}] \quad (6.3.4)$$

We can still simplify the necessary calculations one step further by considering the quantity

$$\rho = E[e^{j\underline{v}^T \hat{\underline{x}} u}] \quad (6.3.5)$$

and noting the identity

$$E[e^{j \underline{v}^T \underline{x}}] = \frac{1}{j} \frac{\partial}{\partial \underline{v}} \rho \quad (6.3.6)$$

Thus the major effort of this Section will be devoted to calculating the scalar quantity, ρ , in Eq.(6.3.5) for u given by

$$u = \text{sgn } \underline{k}^T \underline{x} \quad (6.3.7)$$

The procedure which will be used to evaluate Eqs.(6.3.5) and (6.3.6) is somewhat parallel to the procedure of Section 3.3. We define random variables

$$\left. \begin{aligned} \zeta &= \underline{k}^T \underline{x} \\ \xi &= \underline{v}^T \underline{x} \end{aligned} \right\} \quad (6.3.8)$$

and attempt to express ξ in terms of ζ . Thus we write

$$\xi = \lambda \zeta + \eta \quad (6.3.9)$$

where λ is a constant representing the projection of ξ onto ζ and η is orthogonal to ζ i.e. $E[\zeta\eta] = 0$. Then λ is calculated from

$$\begin{aligned} E[\xi\zeta] &= \lambda E[\zeta^2] + E[\eta\zeta] = \lambda E[\zeta^2] \\ \lambda &= \frac{E[\xi\zeta]}{E[\zeta^2]} = \frac{\underline{k}^T \underline{X} \underline{v}}{\underline{k}^T \underline{X} \underline{k}} \end{aligned} \quad (6.3.10)$$

By construction (ζ, η) are uncorrelated zero mean Gaussian random variables with variances σ_ζ^2 and σ_η^2 defined by

$$\left. \begin{aligned} \sigma_\zeta^2 &= E[\zeta^2] = \underline{k}^T \underline{X} \underline{k} \\ \sigma_\eta^2 &= E[\eta^2] = E[\xi - \lambda\zeta]^2 = \underline{v}^T \underline{X} \underline{v} - \frac{(\underline{k}^T \underline{X} \underline{v})^2}{\underline{k}^T \underline{X} \underline{k}} \end{aligned} \right\} \quad (6.3.11)$$

The quantity ρ of Eq.(6.3.5) can now be defined in terms of the joint distribution for (ζ, η) which is

$$p(\zeta, \eta) = p(\zeta)p(\eta) = \frac{1}{2\pi\sigma_\zeta\sigma_\eta} e^{-\frac{\zeta^2}{2\sigma_\zeta^2}} e^{-\frac{\eta^2}{2\sigma_\eta^2}} \quad (6.3.12)$$

Then ρ is given by

$$\begin{aligned} \rho &= \int_{-\infty}^{\infty} d\eta \int_0^{\infty} d\zeta \cdot 1. e^{j(\lambda\zeta + \eta)} p(\zeta)p(\eta) \\ &\quad - \int_{-\infty}^{\infty} d\eta \int_{-\infty}^0 d\zeta \cdot 1. e^{j(\lambda\zeta + \eta)} p(\zeta)p(\eta) \end{aligned} \quad (6.3.13)$$

Making the substitution $\zeta_1 = -\zeta$ and noting that $p(\zeta_1) = p(\zeta)$ the second term in Eq.(6.3.13) can be written

$$- \int_{-\infty}^{\infty} d\eta \int_0^{\infty} d\zeta e^{j(-\lambda\zeta + \eta)} p(\zeta)p(\eta)$$

Hence Eq.(6.3.13) becomes

$$\begin{aligned} \rho &= \int_{-\infty}^{\infty} d\eta e^{j\eta} p(\eta) \int_0^{\infty} d\zeta p(\zeta) (e^{j\lambda\zeta} - e^{-j\lambda\zeta}) \\ &= 2j \int_{-\infty}^{\infty} d\eta e^{j\eta} p(\eta) \int_0^{\infty} d\zeta p(\zeta) \sin \lambda\zeta \end{aligned} \quad (6.3.14)$$

Consider now the first integral in Eq.(6.3.14). The real part containing $\cos \eta$ is even while the imaginary part containing $\sin \eta$ is odd and hence integrates out to zero. Thus

$$\int_{-\infty}^{\infty} \frac{d\eta}{\sqrt{2\pi} \sigma_{\eta}} e^{j\eta} e^{-\frac{\eta^2}{2\sigma_{\eta}^2}} = \frac{1}{\sqrt{2\pi} \sigma_v} \int_{-\infty}^{\infty} \cos \eta e^{-\frac{\eta^2}{2\sigma_{\eta}^2}} d\eta \quad (6.3.15)$$

To evaluate Eq.(6.3.15) we use formula 863.3 from Dwight [24]

$$\int_0^{\infty} e^{-a^2 x^2} \cos 2px \, dx = \frac{\sqrt{\pi}}{2a} e^{-\frac{p^2}{a^2}} \quad (6.3.16)$$

with $a = \frac{1}{\sqrt{2} \sigma_{\eta}}$ and $p = \frac{1}{2}$

which yields

$$\int_{-\infty}^{\infty} d\eta e^{j\eta} \underline{p}(\eta) = e^{-\frac{\sigma_{\eta}^2}{2}} = e^{-\frac{\underline{v}^T \underline{X} \underline{v}}{2} + \frac{(\underline{k}^T \underline{X} \underline{v})^2}{2 \underline{k}^T \underline{X} \underline{k}}} \quad (6.3.17)$$

The second integral in Eq.(6.3.14) is given only in series form.

Bierens de Haan [25] gives the form

$$\int_0^{\infty} e^{-px^2} \sin qx \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+2)(n+3)\dots(2n+2)} \frac{q^{2n+1}}{\rho^{n+1}} \quad (6.3.18)$$

In order to represent the quantity, ρ , and also $\underline{b}_1$ and $\underline{b}_2$ in terms of the vector, $\underline{v}$, define the quantity

$$\theta \triangleq \lambda \sigma_{\zeta} = \frac{\underline{k}^T \underline{X} \underline{v}}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} \quad (6.3.19)$$

and the integrals

$$\begin{aligned} \mu_o(\theta) &\triangleq \frac{1}{\sqrt{2\pi} \sigma_{\eta 0}} \int_0^{\infty} d\zeta e^{-\frac{\zeta^2}{2\sigma_{\zeta}^2}} \sin \lambda \zeta = \frac{1}{\sqrt{\pi}} \int_0^{\infty} dx e^{-x^2} \sin(\sqrt{2} \theta x) \\ &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+2)(n+3)\dots(2n+2)} (\sqrt{2}\theta)^{2n+1} \end{aligned} \quad (6.3.20a)$$

$$\mu_1(\theta) \triangleq \frac{d\mu_0}{d\theta} = \sqrt{\frac{2}{\pi}} \int_0^\infty x \, dx \, e^{-x^2} \cos(\sqrt{2} \theta x) \quad (6.3.20b)$$

$$\mu_2(\theta) = \frac{d\mu_1}{d\theta} = -\frac{2}{\sqrt{\pi}} \int_0^\infty x^2 \, dx \, e^{-x^2} \sin(\sqrt{2} \theta x) \quad (6.3.20c)$$

Then Eq. (6.3.14) becomes

$$\rho = 2j \, e^{-\frac{\underline{v}^T \underline{X} \underline{v}}{2} - \frac{\theta^2}{2}} \mu_0(\theta) \quad (6.3.21)$$

The vector, $\underline{b}_e$, given by Eq. (6.3.3) is given in terms of ρ by

$$\underline{b}_e = \frac{e^{j\alpha_0}}{j} \frac{\partial}{\partial \underline{v}} \rho \quad (6.3.22)$$

On noting that

$$\frac{\partial \theta}{\partial \underline{v}} = \frac{\underline{X} \underline{k}}{\sqrt{\underline{k}^T \underline{X} \underline{k}}}$$

we substitute Eq. (6.3.21) into Eq. (6.3.22) and obtain $\underline{b}_e = 2 \, e^{j\alpha_0} \underline{b}_v$ (6.3.23a)

$$\text{where } \underline{b}_v = e^{-\frac{\underline{v}^T \underline{X} \underline{v}}{2} - \frac{\theta^2}{2}} \mu_0(\theta) \left[\underline{X} \underline{v} + \frac{\underline{k}^T \underline{X} \underline{v}}{\underline{k}^T \underline{X} \underline{k}} \underline{X} \underline{k} + \frac{\mu_1(\theta) \underline{X} \underline{k}}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} \right] \quad (6.3.23b)$$

Hence it follows that

$$\underline{b}_1 = 2 \sin \alpha_0 \underline{b}_v \quad (6.3.24a)$$

$$\underline{b}_2 = 2 \cos \alpha_0 \underline{b}_v \quad (6.3.24b)$$

Note that when $\underline{v} = 0$, $\theta = 0$, $\mu_0(0) = 0$ and

$$\mu_1(0) = \frac{2}{\pi} \int_0^\infty x \, dx \, e^{-x^2} = \frac{1}{2} \sqrt{\frac{2}{\pi}}$$

Therefore in this case

$$\underline{b}_1 = \sin \alpha_0 \gamma_0 \underline{X} \underline{k} \quad (6.3.25a)$$

$$\underline{b}_2 = \cos \alpha_0 \gamma_0 \underline{X} \underline{k} \quad (6.3.25b)$$

which exactly reduces to the case treated in Section (4.7).

6.4 The Hamiltonian Formulation of the L.U.B. Optimal Control with Random Thrust Direction.

In this section the methods of Section 4.2 are applied to the L.U.B. optimal control problem with random thrust direction. For purposes of simplicity the finite mass flow rate will be ignored so that we are treating the constant mass case. The basic idea again is to construct a Hamiltonian that can be simultaneously minimized by the choice of parameters i_ω , $\underline{k}$, $\underline{v}$, α_o which determines the control strategy. This procedure yields a local minimum of the cost functional. Arguments are given in Section 6.5 that the local minimum is likely to be a global minimum.

The evolution of the state matrix, X , is obtained by the substitution of Eqs. (6.3.24a) and (6.3.24b) into Eq. (6.1.7). This yields

$$\begin{aligned} \dot{X} = & FX + XF^T + CE_o C^T + 2 i_\omega \left(\frac{\underline{v}^T X \underline{v}}{2} + \frac{\theta^2}{2} \mu_o(\theta) \right) \\ & + \left[\left(\underline{X} \underline{v} + \frac{\underline{k}^T X \underline{v}}{\underline{k}^T X \underline{k}} \underline{X} \underline{k} + \mu_1(\theta) \underline{X} \underline{k} \right) (\sin \alpha_o \underline{d}_1^T + \cos \alpha_o \underline{d}_2^T) + \right. \\ & \left. (\sin \alpha_o \underline{d}_1 + \cos \alpha_o \underline{d}_2) \left(\underline{v}^T X + \frac{\underline{k}^T X \underline{v}}{\underline{k}^T X \underline{k}} \underline{k}^T X + \frac{\mu_1(\theta) \underline{k}^T X}{\sqrt{\underline{k}^T X \underline{k}}} \right) \right] \end{aligned} \quad (6.4.1)$$

$X(t_o)$ given by Eq. (5.3.2).

An important physical assumption for the random thrust direction optimal control problem is that the cost of obtaining the desired thrust direction using the attitude control system is negligible in terms of fuel consumption.

This assumption might be valid if attitude control were accomplished with reaction wheels and a clutching mechanism that transfers angular momentum. Of course it is also very implicitly assumed that thrusting takes place at the center of mass of the vehicle so that there are no attitude errors introduced by the control rockets themselves.

With the above assumptions the cost functional is the same as that given by Eq. (4.1.3)

$$J = \int_{t_0}^T i_{\omega}(t) dt \quad (4.1.3)$$

Following the same reasoning as in Section (4.2) we now construct a Hamiltonian to be minimized simultaneously by i_{ω} , $\underline{k}$, $\underline{v}$, α_o . It is convenient though to first define the quantity

$$T_H \triangleq \text{trace} \left\{ \Lambda^T \left(\underline{X} \underline{v} + \frac{\underline{k}^T \underline{X} \underline{v}}{\underline{k}^T \underline{X} \underline{k}} \underline{X} \underline{k} + \frac{\mu_1(\theta) \underline{X} \underline{k}}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} \right) \underline{b}_2^T \right\} (\sin \alpha_o \underline{d}_1^T + \cos \alpha_o \underline{d}_2^T) +$$

$$\text{trace} \left\{ \Lambda^T (\sin \alpha_o \underline{d}_1 + \cos \alpha_o \underline{d}_2) \left(\underline{v}^T \underline{X} + \frac{\underline{k}^T \underline{X} \underline{v}}{\underline{k}^T \underline{X} \underline{k}} \underline{k}^T \underline{X} + \frac{\mu_1(\theta)}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} \underline{k}^T \underline{X} \right) \right\} \quad (6.4.2)$$

The Hamiltonian is given by

$$H = i_{\omega} + \text{trace} \left\{ \Lambda^T \left[\underline{F} \underline{X} + \underline{X} \underline{F}^T + \underline{C} \underline{E}_o \underline{C}^T \right] \right\}$$

$$+ 2i_{\omega} e^{-\frac{\underline{v}^T \underline{X} \underline{v}}{2}} e^{\frac{\theta^2}{2}} \mu_o(\theta) T_H \quad (6.4.3)$$

where

θ is given by Eq. (6.3.19).

The minimization of Eq.(6.4.3) with respect to $\underline{k}$ and $\underline{v}$ can only be accomplished iteratively at each stage of the computation and will be discussed later in this chapter. On the other hand the minimization of H with respect to i_ω is quite straightforward. Define the quantity

$$\psi_W = 1 + 2e^{-\frac{\underline{v}^T X \underline{v}}{2}} e^{\frac{\theta^2}{2}} \mu_o(\theta) T_H \quad (6.4.4)$$

Then

$$i_\omega = \begin{cases} 1 & \text{when } \psi_W \leq 0 \\ 0 & \text{when } \psi_W > 0. \end{cases} \quad (6.4.5)$$

Of course in the above we understand $\psi_W = \psi_W(\Lambda^*, \underline{k}^*, \underline{v}^*)$ where the starred quantities are optimum.

Next consider the computation of the adjoint equation given by

$$\left. \begin{aligned} \dot{\Lambda} &= -\frac{\partial H}{\partial X} = -H_X \\ \Lambda(T) &= vS(T) \end{aligned} \right\} \quad (4.2.17)$$

As a logical first step in the computation of H_X we first compute the partial derivative with respect to X of various components of H . Thus

$$\frac{\partial}{\partial X} e^{-\frac{\underline{v}^T X \underline{v}}{2}} = -\frac{1}{2} e^{-\frac{\underline{v}^T X \underline{v}}{2}} \underline{v} \underline{v}^T \quad (6.4.6a)$$

$$\frac{\partial}{\partial X} \theta = \frac{\partial}{\partial X} \frac{\underline{k}^T X \underline{v}}{\sqrt{\underline{k}^T X \underline{k}}} = \frac{\underline{k} \underline{v}^T}{\sqrt{\underline{k}^T X \underline{k}}} - \frac{1}{2} \frac{\underline{k}^T X \underline{v}}{(\underline{k}^T X \underline{k})^{3/2}} \underline{k} \underline{k}^T \quad (6.4.6b)$$

$$\frac{\partial}{\partial X} \mu_o(\theta) = \mu_1(\theta) \frac{\partial \theta}{\partial X} \quad (6.4.6c)$$

$$\frac{\partial}{\partial X} \mu_1(\theta) = \mu_2(\theta) \frac{\partial \theta}{\partial X} \quad (6.4.6d)$$

$$\frac{\partial}{\partial X} \left(\frac{\underline{k}^T \underline{X} \underline{v}}{\underline{k}^T \underline{X} \underline{k}} \right) = \frac{\underline{k} \underline{v}^T}{\underline{k}^T \underline{X} \underline{k}} - \frac{2 \underline{k}^T \underline{X} \underline{v}}{(\underline{k}^T \underline{X} \underline{k})^2} \underline{k} \underline{k}^T \quad (6.4.6e)$$

$$\frac{\partial}{\partial X} \frac{1}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} = -\frac{1}{2} \frac{\underline{k} \underline{k}^T}{2(\underline{k}^T \underline{X} \underline{k})^{3/2}} \quad (6.4.6f)$$

$$\frac{\partial}{\partial X} T_H = \frac{\Lambda(\sin \alpha_o \underline{d}_1 + \cos \alpha_o \underline{d}_2)(\underline{v}^T + \frac{\underline{k}^T \underline{X} \underline{v}}{\underline{k}^T \underline{X} \underline{k}} \underline{k}^T + \mu_1(\theta) \frac{\underline{k}^T}{\sqrt{\underline{k}^T \underline{X} \underline{k}}})}{\sqrt{\underline{k}^T \underline{X} \underline{k}}}$$

$$+ \frac{(\underline{v} + \frac{\underline{k}^T \underline{X} \underline{v}}{\underline{k}^T \underline{X} \underline{k}} \underline{k} + \mu_1(\theta) \underline{k})}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} (\sin \alpha_o \underline{d}_1^T + \cos \alpha_o \underline{d}_2^T) \Lambda$$

$$+ [\text{trace} \{ \Lambda^T \underline{X} \underline{k} (\sin \alpha_o \underline{d}_1^T + \cos \alpha_o \underline{d}_2^T) \} \\ + \text{trace} \{ \Lambda^T (\sin \alpha_o \underline{d}_1 + \cos \alpha_o \underline{d}_2) \underline{k} \underline{X} \}] \quad .$$

$$\left(\frac{\partial}{\partial X} \left(\frac{\underline{k}^T \underline{X} \underline{v}}{\underline{k}^T \underline{X} \underline{k}} \right) + \mu_1(\theta) \frac{\partial}{\partial X} \frac{1}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} + \frac{1}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} \frac{\partial}{\partial X} \mu_1(\theta) \right)$$

(6.4.6g)

Using Eq.(6.4.6) above we compute

$$\begin{aligned}
-\dot{\Lambda} = H_X = & F^T \Lambda + \Lambda F + 2i \mu_o(\theta) T_H e^{\theta^2/2 - \frac{\underline{v}^T X \underline{v}}{2}} \left[\frac{\partial \theta \theta}{\partial X} - \frac{\underline{v} \underline{v}^T}{2} \right] \\
& + 2i \frac{\underline{v}^T X \underline{v}}{2} e^{\theta^2/2 - \frac{\underline{v}^T X \underline{v}}{2}} \left[T_H \frac{\partial \mu_o(\theta)}{\partial \Lambda} + \mu_o(\theta) \frac{\partial T_H}{\partial X} \right]
\end{aligned}$$

(6.4.7)

$$\Lambda(T) = \underline{v}^T S(T)$$

The optimal control is established from the condition that H be minimized with respect to α_o , $\underline{k}$, $\underline{v}$. Due to the complexity of Eq.(6.4.3) for H it is necessary to perform the minimization iteratively.

The iterative calculation of the optimal control requires the conditions

$$\left. \begin{aligned} H_{\alpha_o} &= 0 \\ H_{\underline{k}} &= 0 \\ H_{\underline{v}} &= 0 \end{aligned} \right\} \quad (6.4.8)$$

where the subscripts denote partial differentiation. A description of a typical steepest-descent type algorithm [21] is reserved for Section (6.5). However, an essential calculation is the evaluation of the partial derivatives of the Hamiltonian which appear in Eq.(6.4.8). We therefore conclude this section by deriving expressions for

$$H_{\alpha_o}, H_{\underline{k}} \text{ and } H_{\underline{v}}.$$

In order to find H_{α_o} it is first necessary to obtain the derivative of T_H with respect to α_o . This may readily be obtained from Eq.(6.4.2).

$$\begin{aligned}
\frac{\partial T_H}{\partial \alpha_0} &= \text{trace} \left\{ \Lambda^T (X \underline{v} + \frac{k^T X \underline{v} X \underline{k}}{k^T X k} + \frac{\mu_1(\theta) X \underline{k}}{k^T X k}) (\cos \alpha_0 \underline{d}_1^T - \sin \alpha_0 \underline{d}_2^T) \right\} \\
&+ \text{trace} \left\{ \Lambda^T (\cos \alpha_0 \underline{d}_1 - \sin \alpha_0 \underline{d}_2) (\underline{v}^T X + \frac{k^T X \underline{v} k^T X}{k^T X k} + \frac{\mu_1(\theta) k^T X}{k^T X k}) \right\} \\
&\quad \underline{v}^T X \underline{v} / 2 \quad \theta^2 / 2 \quad \frac{\partial T_H}{\partial \alpha_0} \quad \frac{k^T X k}{k^T X k} \quad \frac{k^T X k}{k^T X k} \quad (6.4.9)
\end{aligned}$$

$$H_{\alpha_0} = 2i \omega e \quad e \quad \mu_0(\theta) \frac{\partial T_H}{\partial \alpha_0} \quad (6.4.10)$$

The calculation of $H_{\underline{k}}$ and $H_{\underline{v}}$ follows. As in Eq.(6.4.6) define

$$\frac{\partial}{\partial \underline{k}} \mu_0(\theta) = \mu_1(\theta) \frac{\partial \theta}{\partial \underline{k}} \quad (6.4.11a)$$

$$\frac{\partial}{\partial \underline{k}} \mu_1(\theta) = \mu_2(\theta) \frac{\partial \theta}{\partial \underline{k}} \quad (6.4.11b)$$

$$\frac{\partial}{\partial \underline{k}} \left(\frac{k^T X \underline{v}}{k^T X k} \right) = \frac{X \underline{v}}{k^T X k} - \frac{k^T X \underline{v}}{(k^T X k)^2} X \underline{k} \quad (6.4.11c)$$

$$\frac{\partial}{\partial \underline{k}} \frac{1}{\sqrt{k^T X k}} = - \frac{X \underline{k}}{(k^T X k)^{3/2}} \quad (6.4.11d)$$

$$\frac{\partial \theta}{\partial \underline{k}} = \frac{\partial}{\partial \underline{k}} \frac{k^T X \underline{v}}{\sqrt{k^T X k}} = \frac{X \underline{v}}{\sqrt{k^T X k}} - \frac{k^T X \underline{v}}{(k^T X k)^{3/2}} X \underline{k} \quad (6.4.11e)$$

$$\begin{aligned}
\frac{\partial T_H}{\partial \underline{k}} &= X(\Lambda + \Lambda^T) (\sin \alpha_0 \underline{d}_1 + \cos \alpha_0 \underline{d}_2) \left(\frac{k^T X \underline{v}}{k^T X k} + \frac{\mu_1(\theta)}{k^T X k} \right) \\
&+ \left[\frac{\partial}{\partial \underline{k}} \frac{k^T X \underline{v}}{k^T X k} + \mu_1(\theta) \frac{\partial}{\partial \underline{k}} \frac{1}{\sqrt{k^T X k}} + \frac{1}{\sqrt{k^T X k}} \frac{\partial}{\partial \underline{k}} \mu_1(\theta) \right] \cdot
\end{aligned}$$

$$\begin{aligned}
&\left[\text{trace} \left\{ \Lambda^T X \underline{k} (\sin \alpha_0 \underline{d}_1^T + \cos \alpha_0 \underline{d}_2^T) \right\} + \right. \\
&\quad \left. \text{trace} \left\{ \Lambda^T (\sin \alpha_0 \underline{d}_1^T + \cos \alpha_0 \underline{d}_2^T) \underline{k}^T X \right\} \right] \quad (6.4.11f)
\end{aligned}$$

Then the quantity, $H_{\underline{k}}$, is given from Eq. (6.4.3) as

$$\begin{aligned}
 H_{\underline{k}} = & 2i_{\omega} e^{-\frac{\underline{v}^T \underline{\Lambda} \underline{v}}{2}} \left\{ \mu_0(\theta) T_H e^{\frac{\theta^2}{2}} \theta \frac{\partial \theta}{\partial \underline{k}} + T_1 e^{\frac{\theta^2}{2}} \frac{\partial \mu_0}{\partial \underline{k}}(\theta) \right. \\
 & \left. + \mu_0(\theta) e^{\frac{\theta^2}{2}} \frac{\partial T_H}{\partial \underline{k}} \right\} \quad (6.4.12)
 \end{aligned}$$

A similar calculation to obtain $H_{\underline{v}}$ is as follows :

$$\frac{\partial}{\partial \underline{v}} \theta = \frac{\partial}{\partial \underline{v}} \left(\frac{\underline{k}^T \underline{X} \underline{v}}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} \right) = \frac{\underline{X} \underline{k}}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} \quad (6.4.13a)$$

$$\frac{\partial}{\partial \underline{v}} \mu_0(\theta) = \mu_1(\theta) \frac{\partial}{\partial \underline{v}} \theta \quad (6.4.13b)$$

$$\frac{\partial}{\partial \underline{v}} \mu_1(\theta) = \mu_2(\theta) \frac{\partial}{\partial \underline{v}} \theta \quad (6.4.13c)$$

$$\frac{\partial T_H}{\partial \underline{v}} = \underline{X}(\underline{\Lambda}^T + \underline{\Lambda})(\sin \alpha_0 \underline{d}_1 + \cos \alpha_0 \underline{d}_2) + \left[\frac{\underline{X} \underline{k}}{\underline{k}^T \underline{X} \underline{k}} + \frac{1}{\sqrt{\underline{k}^T \underline{X} \underline{k}}} \frac{\partial \mu_1(\theta)}{\partial \underline{v}} \right]$$

$$\left\{ \text{trace} \{ \underline{\Lambda}^T \underline{X} \underline{k} (\sin \alpha_0 \underline{d}_1^T + \cos \alpha_0 \underline{d}_2^T) \} + \text{trace} \{ \underline{\Lambda}^T (\sin \alpha_0 \underline{d}_1 + \cos \alpha_0 \underline{d}_2) \underline{k}^T \underline{X} \} \right\} \quad (6.4.13d)$$

Then the quantity $H_{\underline{v}}$ is given by

$$H_{\underline{v}} = 2i_{\omega} e^{-\frac{\underline{v}^T X \underline{v}}{2}} e^{\frac{\theta^2}{2}} \mu_o(\theta) T_H \left(\theta \frac{\partial \theta}{\partial \underline{v}} \cdot X \underline{v} \right) + T_H \frac{\partial \mu_o(\theta)}{\partial \underline{v}} + \mu_o(\theta) \frac{\partial T_H}{\partial \underline{v}} \quad (6.4.14)$$

In summary, necessary conditions have been presented for the optimal control of the L.U.B. optimal fuel problem with random thrust angle and constant mass. These conditions are the simultaneous satisfaction of Eqs.(6.4.1), (6.4.7), (4.2.3a), (4.2.3b) and (6.4.8). The question of sufficiency cannot be answered with any certainty. Convergence to a global optimal solution depends on the correct choice of initial conditions. If a starting solution is taken as the optimal solution for the predetermined thrust direction then this corresponds to choosing $\underline{v} \equiv 0$ in Eq.(6.0.1) and α is given by Eq.(4.7.11). Presumably iterations from this starting solution will ultimately converge on the correct values of $i_{\omega}(t)$, $\underline{k}(t)$, $\underline{v}(t)$ and $\alpha_o(t)$.

6.5 Iterative Solution For The L.U.B. Problem With Random Thrust Direction.

In Section 6.4 necessary conditions for a local minimum were established for the L.U.B. problem with random thrust direction. These conditions ultimately depend on an adjoint matrix $\Lambda(t)$. The numerical solution consists of solving a two-point boundary value problem in which, X , the stochastic state is given at time t_0 and in which Λ , the adjoint matrix is given at the terminal time T .

A similar problem, the case of L.U.B. optimal control with predetermined thrust direction, is discussed from a computational point of view in Section 5.3. The predetermined thrust problem treated in Section 5.3 is simpler than the random thrust problem in the sense that the determination of all the control variables i_ω , $\underline{k}$, α_0 follows explicitly from the minimum Hamiltonian condition. On the other hand the random thrust L.U.B. problem implies the direct determination of only one control variable, i_ω , by Eq.(6.4.5) while the others are determined indirectly from Eq.(6.4.8). Classical gradient methods and their relatives, [21], generally treat two-point boundary value problems in which the control variables are determined indirectly from a condition such as Eq.(6.4.8). These classical methods require that the control variables

$$\left. \begin{aligned} \Delta \alpha_0 &= -\eta_1 H_{\alpha_0} \\ \Delta \underline{k} &= -\eta_2 H_{\underline{k}} \\ \Delta \underline{v} &= -\eta_3 H_{\underline{v}} \end{aligned} \right\} \quad (6.5.1)$$

where the quantities η_1 , η_2 , η_3 are positive scalars. The changes in the control variables α_0 , $\underline{k}$, $\underline{v}$ are zero in theory when Eq.(6.4.8) is satisfied.

The iterative procedure recommended in this section is a combination relaxation and modified gradient method. The control variable, i_ω , is to be determined separately from the other control variables α_0 , $\underline{k}$, $\underline{v}$. The procedure for finding i_ω is exactly that of Section 5.3. The first step is to

introduce smoothness by replacing i_ω by $\hat{i}_\omega$ which is defined by

$$\hat{i}_\omega = \frac{\sqrt{\psi_W^2 + \epsilon^2} - \psi_W}{2\sqrt{\psi_W^2 + \epsilon^2}} \quad (6.5.2)$$

where ψ_W is defined by Eq.(6.4.4). This step is exactly parallel to that of Eq.(5.3.7). For a given ϵ , X and Λ , $\hat{i}_\omega$ is exactly determined by Eq.(6.5.2).

Remark: Since α_0 is determined by Eq.(6.4.8) there is no analogy to Section 5.4 for the L.U.B. problem with random thrust direction.

The modified gradient method is basically a procedure for determining the scalar gains η_1, η_2, η_3 so that small changes in the control variables lead to desired small changes in the state matrix at the terminal time T . Recall from the procedure of Section 5.3 that the specified changes in the target condition according to Eq.(5.3.10) served to specify changes in the multiplier, Λ , which then specified the changes in the control variables. Our modified gradient method is an attempt to similarly restrict the step-sizes η_1, η_2, η_3 so as to limit possible changes in the target condition.

If the Lagrange multipliers v and Λ are fixed but the controls $\alpha_0, \underline{k}, \underline{v}$ are varied then there will be a variation of the state matrix at time T , $X(T)$ and the corresponding variation in the cost according to Eq.(4.2.16) is

$$\delta J = \text{trace} \left[(vS(T) - \Lambda^T(T)) \delta X(T) \right] + \int_{t_0}^T \delta_1 H \, db \quad (6.5.3)$$

where to first order terms

$$\delta_1 H = \frac{\partial H}{\partial \alpha_0} \delta \alpha_0 + \frac{\partial H^T}{\partial \underline{k}} \delta \underline{k} + \frac{\partial H}{\partial \underline{v}} \delta \underline{v} \quad (6.5.4)$$

The boundary condition relating Λ and v of Eq.(4.2.17) can always be imposed at the time T whether or not Λ and v are themselves optimally chosen.

Consequently the first term of Eq.(6.5.3) vanishes. On substituting in Eq.(6.5.1) into Eqs.(6.5.4) and (6.5.3) there results

$$\delta J = - \int_{t_0}^T \left[\eta_1 \left(\frac{\partial H}{\partial \alpha_0} \right)^2 + \eta_2 \left(\frac{\partial H^T}{\partial \underline{k}} \cdot \frac{\partial H}{\partial \underline{k}} \right) + \eta_3 \left(\frac{\partial H^T}{\partial \underline{v}} \cdot \frac{\partial H}{\partial \underline{v}} \right) \right] dt. \quad (6.5.5)$$

Eq.(6.5.5) is a measure of the indicated improvement of the cost in terms of the step sizes η_1, η_2, η_3 . If this improvement is fixed at a certain realistic rate and the steps are made equal i.e. $\eta = \eta_1 = \eta_2 = \eta_3$ then η may be determined from

$$\eta = \frac{|\delta J|}{\int_{t_0}^T \left[\left(\frac{\partial H}{\partial \alpha_0} \right)^2 + \left(\frac{\partial H^T}{\partial \underline{k}} \cdot \frac{\partial H}{\partial \underline{k}} \right) + \left(\frac{\partial H^T}{\partial \underline{v}} \cdot \frac{\partial H}{\partial \underline{v}} \right) \right] dt} \quad (6.5.6)$$

The complete iterative procedure is now as follows

- 1) Choose an initial guess for the control variables $\alpha_0, \underline{k}, \underline{v}, \hat{i}_\omega$ over the interval $[t_0, T]$
- 2) Integrate Eq.(6.4.1) forward from t_0 to T to determine X which is retained in storage
- 3) Test the target condition of Eq.(5.3.6). If the inequalities

$$|| \text{trace} \left\{ S(T)X(T) \right\} - \omega_0 || \leq \delta_1 \quad (6.5.7a)$$

$$\int_{t_0}^T \left[\left(\frac{\partial H}{\partial \alpha_0} \right)^2 + \left(\frac{\partial H^T}{\partial \underline{k}} \cdot \frac{\partial H}{\partial \underline{k}} \right) + \left(\frac{\partial H^T}{\partial \underline{v}} \cdot \frac{\partial H}{\partial \underline{v}} \right) \right] dt \leq \delta_2 \quad (6.5.7b)$$

are satisfied then the iteration terminates except the first pass.

If the inequality Eq.(6.5.7a) is not satisfied then a new $\Lambda(T)$ must be obtained. Following the same argument presented in Step 3 of Section 5.3 determine a multiplier v by

$$v - v_o = \kappa_s [\text{trace} \{S(T)X(T)\} - \omega_o] \quad 1 > \kappa_s > 0 \quad (5.3.10)$$

where v_o is the previous value. Then $\Lambda(T) = vS(T)$.

4) Integrate Eq.(6.4.7) backwards in time determining $\Lambda(t)$. Then either

a) determine $\hat{i}_\omega$ from Eq.(6.5.2) or

b) determine $\Delta\alpha_o$, Δk , Δv from Eqs.(6.5.1) and (6.5.6). Then

$$\alpha_o = \alpha_o(\text{OLD}) + \Delta\alpha_o$$

$$\underline{k} = \underline{k}(\text{OLD}) + \Delta\underline{k}$$

$$\underline{v} = \underline{v}(\text{OLD}) + \Delta\underline{v}$$

Any control variable not updated is of course unchanged from before.

5) Return to step 1.

After the convergence conditions of Step 3 have been satisfied the parameter ϵ appearing in the determination of $\hat{i}_\omega$ in Eq.(6.5.2) is reduced and the above process is repeated.

7. CONCLUSIONS AND RECOMMENDATIONS

In this report a class of stochastic optimal control problems which are characterized by a small strictly bounded input have been considered for performance criteria of minimum time, minimum expected fuel consumption and least upper bound fuel consumption. Solutions have direct application to the task of controlling a rocket or space vehicle with a low thrust engine. In addition the techniques developed herein may well find use in a variety of problems.

The entire basis for the class of solutions that has been developed hinges on the calculation of the stochastic control vector, $\underline{b}$, given by Eq.(3.3.13). The calculation of $\underline{b}$ is founded on two critical assumptions: First, that the control law has the general structure of Eqs.(3.2.13) or (3.2.14), and second, that the estimator $\hat{\underline{x}}$ of the state is a zero mean Gaussian random variable. It is clear that while the estimator, $\hat{\underline{x}}(t)$, is zero mean if $\underline{x}$ itself is zero mean as is shown in Appendix B, it can not be a Gaussian random variable since the motion of the system is forced by a strictly bounded control. Arguments based on the Central Limit Theorem are given in Chapter 3 to support the proposition that $\hat{\underline{x}}(t)$ is at least approximately Gaussian. However these arguments are not very rigorous, and the assumption can probably be justified only for the case of vanishingly small control magnitudes. In this case $\hat{\underline{x}}(t)$ will be Gaussian because the measurement noise is assumed to be Gaussian, and except for the control the system is linear. For control magnitudes of practical interest it appears that $\hat{\underline{x}}(t)$ is definitely nongaussian, and therefore the results that have been obtained here must be regarded as being in the nature of a first-order approximation.

Attempts have been made to determine the evolution of the probability density function of $\hat{\underline{x}}(t)$. In general this involves nonlinear Fokker-Planck equations, solutions to which are not known. In order to circumvent this difficulty one may attempt to compute the evolution of moments or cumulants ; however so far these attempts have been stymied by the fact that stability of the resulting equations can not be guaranteed.

It is clear that the present state of knowledge concerning the solution of nonlinear stochastic differential equations is such that exact solutions can generally not be expected, and even good approximate solutions are difficult to obtain. Under these conditions the rather crude approximations made in this analysis can perhaps be justified from the point of view that a rough approximation is better than no solution at all.

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APPENDIX A.

Derivation of Eq.(6.1.7) From The Weak Differential Operator.

Consider the stochastic differential equation

$$\dot{\underline{\hat{x}}} = \underline{\hat{F}}\underline{\hat{x}} + \underline{C}\underline{e} + \underline{h} \quad (\text{A.1})$$

where $E[\underline{e}] = 0$, $E[\underline{e}(t)\underline{e}(\tau)^T] = E_0 \delta(t-\tau)$

and

$\underline{h}$ represents a random quantity correlated with $\underline{x}$ by

$$E[\underline{\hat{x}} \underline{h}^T] = B. \quad (\text{A.2})$$

For a scalar function of $\underline{\hat{x}}$, $V(\underline{\hat{x}})$, the weak differential operator is defined by

$$\mathcal{L}V(\underline{\hat{x}}(t)) \triangleq \lim_{\Delta \rightarrow 0} \frac{E[V(\underline{\hat{x}}(t+\Delta))] - V(\underline{\hat{x}}(t))}{\Delta} \quad (\text{3.1.1})$$

and if $\frac{\partial V}{\partial t} = 0$, is given by the following version of Eq.(3.1.4)

$$\mathcal{L}V = \underline{V}_{\underline{x}}^T (\underline{\hat{F}}\underline{\hat{x}} + \underline{h}) + \frac{1}{2} \text{tr}(\underline{V}_{\underline{x}} \underline{\hat{C}} E_0 \underline{C}^T) \quad (\text{A.3})$$

We are interested in the evolution of the quantity

$X = E[\underline{\hat{x}} \underline{\hat{x}}^T]$ as a function of time.

For any arbitrary matrix, A, we can relate the scalar V to the desired quantity X as follows: First define $V(\underline{\hat{x}})$ by

$$V(\underline{\hat{x}}) = \frac{1}{2} \underline{\hat{x}}^T (A + A^T) \underline{\hat{x}} = \frac{1}{2} \text{trace} \{ (A+A^T) \underline{\hat{x}} \underline{\hat{x}}^T \} \quad (\text{A.4})$$

$$\text{Hence } E[V(\underline{\hat{x}})] = \frac{1}{2} \text{trace} \{ (A+A^T) X \} = \text{trace} \{ AX \} \quad (\text{A.5})$$

Now the evolution of the quantity $E[V(\underline{\hat{x}})]$ is given by

$$\frac{d}{dt} E[V(\underline{\hat{x}})] = \lim_{\Delta \rightarrow 0} E \left[\frac{E[V(\underline{\hat{x}}(t+\Delta))] - V(\underline{\hat{x}}(t))}{\Delta} \right] = E[\mathcal{L}V] \quad (\text{A.6})$$

Hence from Eqs.(A.5) and (A.6) it follows that

$$\begin{aligned}
\text{trace } \{\dot{A}\dot{X}\} &= E[\dot{\mathcal{L}} V] \\
&= E[\hat{\underline{x}}^T (A+A^T) (\underline{F}\underline{x} + \underline{h})] + \frac{1}{2} \text{tr}\{(A+A^T) C E_o C^T\} \\
&= \text{trace } \{(A+A^T) \underline{F}\underline{x} + (A+A^T) \overline{\hat{\underline{x}}^T \underline{h}}\} + \frac{1}{2} \text{tr}\{(A+A^T)^T C E_o C^T\} \\
&= \text{trace } \{A(\underline{F}\underline{x} + \underline{x}\underline{F}^T) + A(\underline{B} + \underline{B}^T) + A(C E_o C^T)\} \\
&= \text{trace } \{A[\underline{F}\underline{x} + \underline{x}\underline{F}^T + \underline{B} + \underline{B}^T + C E_o C^T]\} \tag{A.7}
\end{aligned}$$

Thus

$$\dot{\underline{X}} = \underline{F}\underline{x} + \underline{x}\underline{F}^T + \underline{B} + \underline{B}^T + C E_o C^T \tag{A.8}$$

In Eq. (6.1.7)

$$\underline{h} = i_\omega u \sin \alpha \underline{d}_1 + i_\omega u \cos \alpha \underline{d}_2 \tag{A.9}$$

$$\underline{B} = i_\omega E[\hat{\underline{x}} u \sin \alpha] \underline{d}_1^T + i_\omega E[\hat{\underline{x}} u \cos \alpha] \underline{d}_2^T \tag{A.10}$$

or from Eq. (6.1.5)

$$\underline{B} = i_\omega (\underline{b}_1 \underline{d}_1^T + \underline{b}_2 \underline{d}_2^T) \tag{A.11}$$

Hence Eq. (6.1.7) follows from (A.11).

APPENDIX B.

DEMONSTRATION THAT $\hat{\underline{x}}$ IS UNBIASED

Consider again Eq. (A.1)

$$\dot{\hat{\underline{x}}} = F \hat{\underline{x}} + C \underline{e} + \underline{h} \quad (\text{A.1})$$

where $E[\hat{\underline{x}}(0)] = 0$ and where

$$E[\underline{h}] = 0 \quad \text{if} \quad E[\hat{\underline{x}}] = 0 \quad (\text{B.1})$$

Recall that typically $\underline{h} = \underline{d} \underline{u} = \underline{d} \operatorname{sgn} \hat{\underline{x}}$ so that condition (B.1) will always be satisfied. Let V be the scalar function

$$V = \underline{\hat{\underline{x}}}^T \hat{\underline{x}} \quad (\text{B.2})$$

and let W be defined by

$$W = E[V] = E[\underline{\hat{\underline{x}}}^T \hat{\underline{x}}] \quad \text{so that} \quad W(0) = 0 \quad \text{by assumption.} \quad (\text{B.3})$$

We now show that $W \equiv 0$. From the weak differential operator of Eq.

(A.3)

$$\dot{V} = \dot{\underline{\hat{\underline{x}}}}^T (F \hat{\underline{x}} + \underline{h}) \quad (\text{B.4})$$

Consequently ,

$$\frac{dW}{dt} = E[\dot{V}] = \underline{\hat{\underline{x}}}^T E[F \hat{\underline{x}} + \underline{h}] \quad (\text{B.5})$$

From Eq. (B.1) $\frac{dW}{dt}(0) = 0$. Since $W(0) = 0$ also and since

W satisfies a first order equation it follows that $W \equiv 0$.

Consequently, since $\underline{\hat{\underline{x}}}$ is arbitrary $E[\hat{\underline{x}}] \equiv 0$.

REFERENCES

1. Richard H. Batten , ASTRONAUTICAL GUIDANCE M.I.T. Press 1964
2. Richard H. Batten , A STATISTICAL OPTIMIZATION PROCEDURE FOR SPACE FLIGHT:
 ARS SPACE FLIGHT REPORT TO THE NATION
 New York Oct. 9-15, 1961
3. R.E. Kalman , NEW METHODS AND RESULTS IN LINEAR PREDICTION AND
 FILTERING THEORY. RIAS TECHNICAL REPORT 61-1
4. P.D. Joseph and J.T. Tou On A Linear Control Theory
 TRANS. AMER. INST. ELECT. ENGINEERS
 80 PT II Sept. 61
5. T.L. Gunkel III and G.F. Franklin , A GENERAL SOLUTION FOR LINEAR
 SAMPLED DATA CONTROL. TRANS. Ser D J.Basic Eng.85 1963
6. J.V. Breakwell and F. Tung, MINIMUM EFFORT CONTROL OF SEVERAL
 TERMINAL COMPONENTS. J. SIAM CONTROL
 Ser A Vol. 2#3, 1964
7. J.V. Breakwell, F.L. Rauch and F. Tung, THEORY OF MINIMUM EFFORT CONTROL
 NASA CP - 378 Jan 1966.
8. M. Aoki , OPTIMIZATION OF STOCHASTIC SYSTEMS-ACADEMIC PRESS 1967
9. H.J. Kushner, STOCHASTIC STABILITY AND CONTROL. ACADEMIC PRESS 1967
10. W.M. Wonham, LINEAR SYSTEMS WITH STATE DEPENDENT NOISE
 SIAM J. CONTROL Ser.A Vol.5 #3 1967
11. J.L. Speyer, A STOCHASTIC DIFFERENTIAL GAME WITH CONTROLLABLE
 STATISTICAL PARAMETERS. IEEE TRANS on SYSTEMS SCIENCE
 AND CYBERNETICS, Vol ssc -3 #1 Jan 1967
12. C. Lietmann , AN INTRODUCTION TO OPTIMAL CONTROL . McGrawhall 1967

13. Athans and Falb, OPTIMAL CONTROL McGraw Hill 1966
14. Yen S. Lim, ASYMPTOTIC SOLUTIONS OF AN OPTIMAL SERVO PROBLEM.
IEEE XTIONS AUTOMATIC CONTROL Vol. AC - 13 #1 Feb. 1968
5. Athens and Tse , A DIRECT DERIVATION OF THE OPTIMUM LINEAR FILTER
USING THE MAXIMUM PRINCIPLE IEEE XTIONS
AUTOMATIC CONTROL Vol. AC-12 #6 Dec.1967
16. J.J. Deyst Jr. OPTIMAL CONTROL OF A RE-ENTRY VEHICLE IN THE
PRESENCE OF MEASUREMENT UNCERTAINTIES. IEEE
TRANS. ON AUTOMATIC CONTROL Vol. AC -13 #2 April 1968
17. L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. MISCHENCO
THE MATHEMATICAL THEORY OF OPTIMAL PROCESSES, INTERSCIENCE
(J.Wiley) 1962
18. V.F. Dem'yenov THE SOLUTION OF SOME OPTIMAL CONTROL PROBLEMS
J. SIAM Ser.A CONTROL Vol.6 # Feb 1968
19. E.B. Lee A SUFFICIENT CONDITION IN THE THEORY OF OPTIMAL CONTROL
J.SIAM Ser.A CONTROL Vol.1.1 #3 1963
20. V.G. Boltyanskii , SUFFICIENT CONDITIONS FOR OPTIMALITY AND
THE JUSTIFICATION OF DYNAMIC PROGRAMMING
J.SIAM Ser.A CONTROL Vol.4 #2 1966
21. H.J. Kelly METHOD OF GRADIENTS CHAPTER 6 IN OPTIMIZATION TECHNIQUES
G. Leitmann ed. ACADEMIC PRESS 1962
- A.E. Bryson , W.F. Denham A STEEPEST ASCENT METHOD FOR SOLVING
OPTIMAL PROGRAMMING PROBLEMS. J. OF APPLIED MECHANICS
VOL.29 June 1962
- S.R. McReynolds, A.E. Bryson A SUCCESSIVE SWEEP METHOD FOR
SOLVING OPTIMAL PROGRAMMING PROBLEMS - presented
at the 6th JACC Troy N.Y. June 1965

22. Harold J. Kushner SUFFICIENT CONDITIONS FOR THE OPTIMALITY
OF A STOCHASTIC CONTROL J. SIAM Ser.A CONTROL
vol. 3 #3 1965
23. W.M. Wonham , ON THE SEPARATION THEOREM OF STOCHASTIC CONTROL
J. SIAM Ser.A CONTROL Vol. 6 #2 May 1968
24. H.B. Dwight TABLES OF INTEGRALS AND OTHER MATHEMATICAL DATA
MacMillan 1955
25. D. Bierens de Haan table 263 Nonvelles Tables, Hafner 1957
26. J.V.Breakwell , Yu Chi Ho , ON THE CONJUGATE POINT CONDITION
FOR THE CONTROL PROBLEM TR 441 CRUFT LAB
Harvard U. March 1964
27. O.L. Mangasarian , SUFFICIENT CONDITIONS FOR THE OPTIMAL CONTROL
OF NON- LINEAR SYSTEMS. J. SIAM Ser.A CONTROL
Vol. 4 #1 1966.
28. P.Dorato, C.M. Hsieh, P.N.Robinson, OPTIMAL BANG__BANG CONTROL of
LINEAR STOCHASTIC SYSTEMS with a SMALL NOISE
PARAMETER
IEEE PG-AC VOL. AC - 12 #6 Dec. 1967