INPUT-OUTPUT PROPERTIES OF MULTIPLE-INPUT MULTIPLE-OUTPUT DISCRETE SYSTEMS: PART II*

by

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ABSTRACT

Part II considers input-output properties of nonlinear time-varying discrete systems. Slightly generalized forms of the Small Gain and the Passivity theorem are derived. Some results of Part I and these theorems are used to derive stability criteria. The memoryless nonlinearities and the multipliers are not required to be noninteracting.

I. Introduction

In Part I of this paper we have derived the best known results concerning the determinateness and the input-output properties of linear discrete feedback systems. In this Part II we are concerned mostly with nonlinear discrete systems. Two fundamental results in stability theory of feedbac' systems are the Small Gain theorem and the Passivity theorem. They are the two basic principles behind most of the stability criteria. These two theorems are not new and they have been used either explicitly or implicitly in many papers. Here we present them in a new, slightly more general form. The corresponding Section IV is essentially tutorial in nature. We hope that these two basic theorems will provide a more unified approach to the stability problem. As applications and illustrations of the power of these two theorems, we present in Section V several stability criteria for certain classes of nonlinear discrete systems. Some features of this paper are as follows: 1) We take the advantage of the simpler analytic properties of the discrete case to obtain simple derivations. 2) We define the stability of feedback systems in terms of their input-output properties. 3) In contrast to most previous results in the multiple-input, multiple-output case we don't require the nonlinearities to be of noninteracting type. 4) By the use of the results of Part I, we are able to include a much broader class of linear subsystems. 5) Using the passivity criterion we obtain a simple derivation of the Tsypkin criterion under less restrictive conditions. 6) The paper is essentially self-contained.

II. Notations

We use the same notations as in Part I. Some new terms are defined below.

The symbol Σ^n and $\Sigma^{n\times n}$ denote the spaces of all sequences in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ respectively; more precisely, $\Sigma^n \triangleq \{\underset{\sim}{x} : J_+ \to \mathbb{R}^n\}$ and $\Sigma^{n\times n} \triangleq \{\underset{\sim}{G} : J_+ \to \mathbb{R}^{n\times n}\}$. If $n = 1$, we simply write Σ.

Let $\underset{\sim}{x} = \{\underset{\sim}{\xi}_i\}_0^\infty \in \Sigma^n$ and let $N \in J_+$. <u>The sequence $\underset{\sim}{x}$ truncated at N</u> is denoted by $\underset{\sim}{x}_N$ and is defined as

$$\underset{\sim}{x}_N = (\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_2, \ldots, \underset{\sim}{\xi}_{N-1}, \underset{\sim}{\xi}_N, \underset{\sim}{0}, \underset{\sim}{0}, \ldots)$$

Let $\|\cdot\|$ denote <u>any</u> norm on Σ^n subject to the condition that for all $\underset{\sim}{x} \in \Sigma^n$ and all $N \in J_+$

$$\|\underset{\sim}{x}_N\| \leq \|\underset{\sim}{x}\| \quad .$$

All ℓ_n^p norms defined in Part I satisfy this condition. The space of all sequences in Σ^n that have finite norm is denoted by $\mathcal{B}$, i.e.

$$\mathcal{B} \triangleq \left\{\underset{\sim}{x} \in \Sigma^n \mid \|\underset{\sim}{x}\| < \infty\right\}$$

Let $\underset{\sim}{x}, \underset{\sim}{y} \in \ell_n^2$. The <u>scalar product of two real sequences</u> $\underset{\sim}{x} \triangleq \{\underset{\sim}{\xi}_i\}_0^\infty$ and $\underset{\sim}{y} \triangleq \{\underset{\sim}{\eta}_i\}_0^\infty$, denoted by $\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle$ is the map of $\Sigma^n \times \Sigma^n$ into $\mathbb{R}_+$ defined by

$$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \sum_{i=0}^{\infty} \underset{\sim}{\xi}_i' \, \underset{\sim}{\eta}_i$$

where $\underset{\sim}{\xi}_i'$ denotes the transpose of $\underset{\sim}{\xi}_i$. Consequently

$$\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = \|\underset{\sim}{x}\|_2^2 \qquad \forall \underset{\sim}{x} \in \ell_n^2$$

Considering truncations at N, we note that

$$\langle \underset{\sim}{x}_N, \underset{\sim}{y} \rangle = \langle \underset{\sim}{x}, \underset{\sim}{y}_N \rangle = \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle_N$$

where we define $\langle x, y \rangle_N$ by $\sum_0^N \underset{\sim}{\xi}_i' \, \underset{\sim}{\eta}_i$.

If z is a complex number, then $\bar{z}$ denotes its complex conjugate. If $\underset{\sim}{e}$ is an n-tuple of complex numbers, then $\underset{\sim}{e}^*$ denotes its conjugate transpose.

III. System Description

We consider the system model shown in Fig. 1. The sequences $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$; $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$; $\underset{\sim}{y}_1$ and $\underset{\sim}{y}_2$ are in Σ^n. $\underset{\sim}{H}_1$, $\underset{\sim}{H}_2$: $\Sigma^n \to \Sigma^n$ are operators which can be linear or/and nonlinear, time-invariant or/and time-varying. Assume that the system $\underset{\sim}{\mathcal{S}}$ is determinate. From Fig. 1 the system $\underset{\sim}{\mathcal{S}}$ is described by the following system equations.

$$\underset{\sim}{e}_1 = \underset{\sim}{u}_1 - \underset{\sim}{y}_2 \tag{1}$$

$$e_2 = u_2 + y_1 \tag{2}$$

$$y_1 = H_1 e_1 \tag{3}$$

$$y_2 = H_2 e_2 \tag{4}$$

Comment:

For simplicity, we consider H_1 and H_2 as operators. In fact H_1 and H_2 can also be allowed to be relations [1].

Definition 1

Let $H : \Sigma^n \to \Sigma^n$ and let $\|\cdot\|$ be any norm on Σ^n. The operator H is said to have finite gain γ_1 if there exists a nonnegative number γ_1 and a constant β_1 (both independent of x) such that

$$\|(Hx)_N\| \leq \gamma_1 \|x_N\| + \beta_1 \qquad \forall x \in \Sigma^n, \ \forall N \in J_+ \tag{5}$$

where $(Hx)_N$ denotes the sequence Hx truncated at N.

Definition 2

Let $H : \Sigma^n \to \Sigma^n$. The operator H is said to be passive if there is a nonnegative function $V : \Sigma^n \times J_+ \to \mathbb{R}_+$ and a constant α such that

$$\langle x, Hx \rangle_N \geq V(x,N) + \alpha \qquad \forall x \in \Sigma^n, \ \forall N \in J_+ \tag{6}$$

In particular, if there is a positive number δ such that

$$V(\underset{\sim}{x},N) \geq \delta\|\underset{\sim}{x}_N\|_2^2 \qquad \forall \underset{\sim}{x} \in \Sigma^n, \ \forall N \in J_+$$

thus

$$\langle \underset{\sim}{x}, \underset{\sim}{H}\underset{\sim}{x} \rangle_N \geq \delta\|\underset{\sim}{x}_N\|_2^2 + \alpha \qquad \forall \underset{\sim}{x} \in \Sigma^n, \ \forall N \in J_+ \tag{7}$$

then $\underset{\sim}{H}$ is said to be <u>strictly passive</u>.

Comments:

1. The definition of gain defined in (5) is more appropriate and more general than that defined by Zames [1] and used by Sandberg [3]. In fact (5) does not require that $\underset{\sim}{H}\underset{\sim}{0} = \underset{\sim}{0}$; this is useful, for example, if $\underset{\sim}{H}$ represents a relay or a hysteresis. As a special case when $\beta_1 = 0$ and $\|\underset{\sim}{x}_N\| \neq 0$, γ_1 can be taken to be

$$\gamma_1 \triangleq \sup_{\substack{N \in J_+ \\ \underset{\sim}{x} \in \Sigma^n}} \frac{\|(\underset{\sim}{H}\underset{\sim}{x})_N\|}{\|\underset{\sim}{x}_N\|}$$

we are then brought back to the definition originally given by Zames [1].

2. The definition of passivity is slightly more general than those used by Zames [1] and Sandberg [3]. Ours is inspired from circuit theory. (See Kuh-Rohrer [2].)

IV. Main Results

In the stability studies of feedback systems in terms of input-output properties, there are two major results, namely, the Small Gain theorem and the Passivity theorem. The Small Gain theorem is applicable to any norm on Σ^n, but with the more restricting condition of requiring that the product of gains of two subsystems be less than 1; while the Passivity theorem is applicable only to ℓ_n^2-norm. It has the advantage that, for the linear time-invariant case, the passivity condition has a frequency domain interpretation. These results have been developed mostly for the continuous systems and are available explicitly or implicitly elsewhere [1, 3, 5, 14, 16]. Here we are concerned only with discrete systems and these two results are generalized and stated in their most general forms.

Theorem 1 (Small Gain Theorem)

Consider the system $\underset{\sim}{\mathcal{S}}$ (Fig. 1) described by (1)-(4), where $\underset{\sim}{H}_1$, $\underset{\sim}{H}_2 : \Sigma^n \to \Sigma^n$. Let $\|\cdot\|$ be any norm on Σ^n and let there be some nonnegative numbers μ_1, μ_2 and some constants ν_1, ν_2 such that

$$\|(\underset{\sim}{H}_1\underset{\sim}{x})_N\| \leq \mu_1\|\underset{\sim}{x}_N\| + \nu_1 \qquad \forall \underset{\sim}{x} \in \Sigma^n, \ \forall N \in J_+ \tag{8}$$

and

$$\|(\underset{\sim}{H}_2\underset{\sim}{x})_N\| \leq \mu_2\|\underset{\sim}{x}_N\| + \nu_2 \qquad \forall \underset{\sim}{x} \in \Sigma^n, \ \forall N \in J_+ \tag{9}$$

Under these conditions, if

$$\mu \triangleq \mu_1\mu_2 < 1 \tag{10}$$

then, for all $N \in J_+$,

$$\|\underset{\sim}{e}_{2N}\| \leq \frac{1}{1-\mu}\left[\|\underset{\sim}{u}_{2N}\| + \mu_1\|\underset{\sim}{u}_{1N}\| + \nu_1 + \mu_1\nu_2\right] \tag{11}$$

Furthermore, if $\underset{\sim}{u}_1, \underset{\sim}{u}_2 \in \mathcal{B}$, then $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{y}_1$ and $\underset{\sim}{y}_2$ are in $\mathcal{B}$.

<u>Theorem 2 (Passivity Theorem)</u>

Consider the system $\underset{\sim}{\mathcal{S}}$ (Fig. 1) described by (1)-(4), where $\underset{\sim}{H}_1$, $\underset{\sim}{H}_2 : \Sigma^n \to \Sigma^n$. Let $\underset{\sim}{H}_1$ satisfy the following conditions:

(i) For some nonnegative number γ_1 and some constant β_1

$$\|(\underset{\sim}{H}_1\underset{\sim}{x})_N\|_2 \leq \gamma_1\|\underset{\sim}{x}_N\|_2 + \beta_1 \qquad \forall \underset{\sim}{x} \in \Sigma^n, \ \forall N \in J_+ \tag{12}$$

(ii) For some constants δ_1 and α_1

$$\langle \underset{\sim}{x}, \underset{\sim}{H}_1\underset{\sim}{x} \rangle_N \geq \delta_1\|\underset{\sim}{x}_N\|_2^2 + \alpha_1 \qquad \forall \underset{\sim}{x} \in \Sigma^n, \ \forall N \in J_+ \tag{13}$$

Let $\underset{\sim}{H}_2$ be such that for some constants ε_2 and α_2

$$\langle \underset{\sim}{x}, \underset{\sim}{H}_2\underset{\sim}{x} \rangle_N \geq \varepsilon_2\|(\underset{\sim}{H}_2\underset{\sim}{x})_N\|_2^2 + \alpha_2 \qquad \forall \underset{\sim}{x} \in \Sigma^n, \ \forall N \in J_+ \tag{14}$$

Under these conditions, if

$$\lambda \triangleq (\delta_1 + \varepsilon_2) > 0 \tag{15}$$

then $\underset{\sim}{u}_1, \underset{\sim}{u}_2 \in \ell_n^2$ implies that $\underset{\sim}{y}_2 \in \ell_n^2$ and consequently $\lim_{i\to\infty} \underset{\sim}{y}_{2i} = \underset{\sim}{0}$. The same results also hold for $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ and $\underset{\sim}{y}_1$.

Comments:

1. In contrast to the continuous case the two preceding theorems need no special assumption concerning the possibility of finite escape time. The assumption of determinateness implies that, for the nonanticipative case, the equations for the successive components of $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ have a unique solution. In the linear case explicit conditions can be given for this to be the case (see Theorem 1, Part I).

2. Many forms of these theorems have appeared in the literature. The best recent ones are due to Zames [1] and to Sandberg [3]. It is interesting to note that our more inclusive definitions do not alter the essential conclusion.

3. With respect to the Passivity theorem, (a) we do not require $\underset{\sim}{H}_1$ to be passive and $\underset{\sim}{H}_2$ to be strictly passive, we need only have $\delta_1 + \varepsilon_2 > 0$. This fact has already been observed by Stern [15] and Cho-Narendra [14]. (b) If $\underset{\sim}{x} \in \ell_n^2$, then $\underset{\sim}{x} \in \ell_n^\infty$ and $\underset{\sim}{x}_i \to \underset{\sim}{0}$ as $i \to \infty$. Therefore the conclusion of the theorem implies that $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{y}_1$ and $\underset{\sim}{y}_2 \in \ell_n^\infty$ and $\to \underset{\sim}{0}$ as $i \to \infty$. (c) If $\underset{\sim}{u}_1 = \underset{\sim}{0}$, the assumption (12) is not required in proving $\underset{\sim}{y}_2, \underset{\sim}{e}_1 \in \ell_n^2$. In other words, if $\underset{\sim}{u}_1 = \underset{\sim}{0}$ and if we are only interested in showing $\underset{\sim}{y}_2$, $\underset{\sim}{e}_1 \in \ell_n^2$, then we don't need the assumption (12), namely, that $\underset{\sim}{H}_1$ has finite gain. However, if we want to have same results for $\underset{\sim}{y}_1$ and $\underset{\sim}{e}_2$, then the assumption (12) is essential.

4. The Passivity theorem and its applications (given in the next section)

can be extended in that, instead of considering only inputs with finite energy, viz.

$$\underset{\sim}{v} = \{\underset{\sim}{v}_i\}_0^\infty \quad \text{with} \quad \|\underset{\sim}{v}\|_2^2 = \sum_{i=0}^{\infty} |v_i|^2 < \infty$$

one may also consider inputs with finite average power, i.e.

$$\underset{\sim}{v} = \{\underset{\sim}{v}_i\}_0^\infty \quad \text{with} \quad \limsup_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} |\underset{\sim}{v}_i|^2 < \infty$$

Under the conditions stated, such finite average power inputs produce finite average power outputs.

The usefulness of Theorem 2 can be greatly enhanced by modifying the system $\underset{\sim}{\mathcal{S}}$ using the multiplier technique. Let $\underset{\sim}{M}$ be a linear map from Σ^n onto Σ^n and suppose that its inverse, $\underset{\sim}{M}^{-1}$, maps Σ^n into Σ^n. The modified system is denoted by $\mathcal{S}_{\sim M}$ and is shown in Fig. 2. It is easy to verify that $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$, $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{y}_1$, $\underset{\sim}{y}_2$ satisfy the system equations of $\underset{\sim}{\mathcal{S}}$ (i.e. (1) to (4)) if and only if $\underset{\sim}{u}_1$, $\hat{\underset{\sim}{u}}_2$, $\underset{\sim}{e}_1$, $\hat{\underset{\sim}{e}}_2$, $\hat{\underset{\sim}{y}}_1$ and $\underset{\sim}{y}_2$ satisfy the system equations of $\mathcal{S}_{\sim M}$. Furthermore $\underset{\sim}{\mathcal{S}}$ is determinate if and only if $\mathcal{S}_{\sim M}$ is determinate.

Theorem 2M below is obtained by transcribing Theorem 2 to the system $\mathcal{S}_{\sim M}$ and using $\hat{\underset{\sim}{e}}_2 = \underset{\sim}{M}\underset{\sim}{e}_2$ and $\hat{\underset{\sim}{u}}_2 = \underset{\sim}{M}\underset{\sim}{u}_2$.

<u>Theorem 2M (Passivity Theorem for the System with Multiplier)</u>

Consider the system $\mathcal{S}_{\sim M}$ shown in Fig. 2, where $\underset{\sim}{H}_1, \underset{\sim}{H}_2, \underset{\sim}{M} : \Sigma^n \to \Sigma^n$. Let $\underset{\sim}{M}\underset{\sim}{H}_1$ satisfy the following conditions:

(i) for some nonnegative constant γ_1' and some constant β_1'

$$\|(\underset{\sim}{M}\underset{\sim}{H}_1\underset{\sim}{x})_N\|_2 \le \gamma_1'\|\underset{\sim}{x}_N\|_2 + \beta_1' \qquad \forall \underset{\sim}{x} \in \Sigma^n, \ \forall N \in J_+ \qquad (12')$$

(ii) for some constants δ_1' and α_1'

$$\langle \underset{\sim}{x}, \underset{\sim}{M}\underset{\sim}{H}_1\underset{\sim}{x}\rangle_N \ge \delta_1'\|\underset{\sim}{x}_N\|_2^2 + \alpha_1' \qquad \forall \underset{\sim}{x} \in \Sigma^n, \ \forall N \in J_+ \qquad (13')$$

Let $\underset{\sim}{H}_2$, $\underset{\sim}{M}$ be such that for some constants ε_2' and α_2'

$$\langle \underset{\sim}{M}\underset{\sim}{x}, \underset{\sim}{H}_2\underset{\sim}{x}\rangle_N \ge \varepsilon_2'\|(\underset{\sim}{H}_2\underset{\sim}{x})_N\|_2^2 + \alpha_2' \qquad \forall \underset{\sim}{x} \in \Sigma^n, \ \forall N \in J_+ \qquad (14')$$

Under these conditions, if

$$\lambda' \triangleq (\delta_1' + \varepsilon_2') > 0 \qquad (15')$$

then for all $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$ with $\underset{\sim}{u}_1 \in \ell_n^2$ and $\underset{\sim}{M}\underset{\sim}{u}_2 \in \ell_n^2$, we have

(a) $\underset{\sim}{e}_1$, $\hat{\underset{\sim}{e}}_2$, $\hat{\underset{\sim}{y}}_1$ and $\underset{\sim}{y}_2$ in ℓ_n^2.

(b) If, in addition, either (i) $\underset{\sim}{H}_1$ has a finite gain
or (ii) $\underset{\sim}{M}^{-1} : \ell_n^2 \to \ell_n^2$
then $\underset{\sim}{y}_1$ is also in ℓ_n^2.

(c) In (b), if (ii) holds, then $\underset{\sim}{e}_2$ is also in ℓ_n^2.

Comment:

It is important to note that in Theorem 2M, we don't require the multiplier $\underset{\sim}{M}$ to be a map of ℓ_n^2 into ℓ_n^2 (a similar comment applies to the continuous case).

V. Applications

We use the theorems above to obtain several stability criteria for some classes of nonlinear discrete time-varying systems. Theorem 1A (below) applies to system $\underset{\sim}{\mathcal{S}}$ (Fig. 1): $\underset{\sim}{H}_1$ is linear time-invariant and specified by its z-transfer function $\underset{\sim}{\tilde{G}}(z)$; it is assumed (ineq. (18)) that $\underset{\sim}{\mathcal{S}}$ is stable under constant linear feedback with gain $\underset{\sim}{K}$. Suppose now that the feedback becomes nonlinear and time-varying; then we use the Small Gain theorem (see (13) below) to ascertain how far it can deviate from the linear gain $\underset{\sim}{K}$ (see (17)). This is essentially a perturbational result. A little thought will show that if (17) is violated only for a finite number of values of m, the boundedness conclusions still hold.

Theorem 1A (Application of Small Gain Theorem)

Consider the system $\underset{\sim}{\mathcal{S}}$ (Fig. 1) with $\underset{\sim}{H} = \underset{\sim}{G}$ being a linear, time-invariant, nonanticipative subsystem and $\underset{\sim}{H}_2 = \underset{\sim}{\Phi}_t$ being a time-varying memoryless nonlinearity. Let the input-output relation of the linear subsystem $\underset{\sim}{G}$ be defined in terms of its impulse response $\underset{\sim}{G}$ by the convolution

$$\underset{\sim}{y}_1 = \underset{\sim}{G} * \underset{\sim}{e}_1 \tag{16}$$

Let the open-loop z-transfer function of $\underset{\sim}{G}$ be of the form

$$\tilde{G}(z) = R\left(1 - z^{-1}\right)^{-1} + \sum_{i=0}^{\infty} G_i z^{-i} \triangleq R\left(1 - z^{-1}\right)^{-1} + \tilde{G}_\ell(z) \qquad (16a)$$

where R is an n×n constant matrix and $G_\ell \triangleq \{G_i\}_0^\infty = \mathcal{Z}^{-1}\{\tilde{G}_\ell(z)\} \in \ell^1_{n\times n}$. Let the time-varying memoryless nonlinearity Φ_t be described by a nonlinear function $\psi_t : \Sigma^n \times J_+ \to \Sigma^n$, which satisfies the condition that for some constant matrix K, some nonnegative number μ_2 and some constant ν_2

$$|\psi_t(\sigma,m) - K\sigma| \le \mu_2|\sigma| + \nu_2 \qquad \forall \sigma \in \Sigma^n, \ \forall m \in J_+ \qquad (17)$$

Under these conditions, if

(a)
$$\inf_{|z|\ge 1} \left|\det\left(I + \tilde{G}(z)K\right)\right| > 0 \qquad (18)$$

and if either $R = 0$ or RK is nonsingular,

(b)
$$\|H\|_1 \mu_2 < 1 \qquad (19)$$

where $H \triangleq \{H_i\}_0^\infty = \mathcal{Z}^{-1}\left\{[I + \tilde{G}(z)K]^{-1}\tilde{G}(z)\right\}$,

then for any fixed $p \in [1,\infty]$ u_1, u_2 in ℓ^p_n implies that e_1, e_2, y_1 and y_2 are also in ℓ^p_n.

Corollary 1A

Consider the single-input, single-output system $\mathcal{S}$ (Fig. 1) with $H_1 = G$ being a linear time-invariant, nonanticipative subsystem and $H_2 = \Phi_t$

being a time-varying memoryless nonlinearity. Let the input-output relation of the linear subsystem G be defined in terms of its impulse response g by the convolution

$$y_1 = g * e_1 \qquad (16')$$

Let the open-loop z-transfer function of G be of the form

$$\tilde{g}(z) = r\left(1 - z^{-1}\right)^{-1} + \sum_{i=0}^{\infty} g_i z^{-i} \triangleq r\left(1 - z^{-1}\right)^{-1} + \tilde{g}_\ell(z) \qquad (16a')$$

where r is a constant and $g_\ell \triangleq \{g_i\}_0^\infty = \mathcal{Z}^{-1}\{\tilde{g}_\ell(z)\} \in \ell^1$. Let the time-varying, memoryless nonlinearity be described by a nonlinear function $\psi_t : \Sigma \times J_+ \to \Sigma$, which satisfies the condition that for some constants k, ν_2' and some nonnegative number μ_2'

$$|\psi_t(\sigma,m) - k\sigma| \leq \mu_2'|\sigma| + \nu_2' \qquad \forall\sigma \in \Sigma,\ \forall m \in J_+ \qquad (17')$$

Under these conditions, if

(a) $$\inf_{|z|\geq 1} |1 + k\tilde{g}(z)| > 0 \qquad (18')$$

and if $r = 0$ or $rk \neq 0$,

(b) $$\|h\|_1\mu_2' < 1 \qquad (19')$$

where $h \triangleq \{h_i\}_0^\infty = \mathcal{Z}^{-1}\{\tilde{g}(z)/1 + h\tilde{g}(z)\}$, then for any fixed $p \in [1,\infty]$, u_1, $u_2 \in \ell^p$ implies that e_1, e_2, y_1 and y_2 are also in ℓ^p.

Theorem 1B

Consider the system $\underset{\sim}{\mathcal{S}}$ (Fig. 1) with $\underset{\sim}{H}_1 = \underset{\sim}{G}$ being a linear, time-invariant, nonanticipative subsystem which is described by (16) and (16a) and $\underset{\sim}{H}_2 = \underset{\sim}{K}$ being a linear, time-varying gain $\underset{\sim}{K}$ which is specified by a sequence of n×n matrices $\{\underset{\sim}{K}_i\}_0^\infty$, where $|\underset{\sim}{K}_i| < \infty \ \forall i \in J_+$. Let the system $\underset{\sim}{\mathcal{S}}$ be determinate, i.e. by Theorem 1, Part I,

$$\det\,[\underset{\sim}{I} + (\underset{\sim}{G}_o + \underset{\sim}{R})\underset{\sim}{K}_i] \neq 0 \qquad \forall i \in J_+$$

Under these conditions, if there is a constant matrix $\underset{\sim}{\overline{K}}$ such that $\underset{\sim}{K}_i \to \underset{\sim}{\overline{K}}$ as $i \to \infty$ and furthermore

$$\inf_{|z|\geq 1} \left|\det\,[\underset{\sim}{I} + \underset{\sim}{\tilde{G}}(z)\underset{\sim}{\overline{K}}]\right| > 0 \tag{18a}$$

then for any fixed $p \in [1,\infty]$, $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$ in ℓ_n^p implies that $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{y}_1$ and $\underset{\sim}{y}_2$ are also in ℓ_n^p.

Roughly speaking Theorem 1B asserts that if a given linear discrete system with time-varying gain tends towards a stable (see (18a)) linear time-invariant system, then the given system is also stable. This result is sharper than that of C. T. Chen [19] in that we do not require that

$$\sum_i |\underset{\sim}{K}_i - \underset{\sim}{\overline{K}}| < \infty .$$

Theorem 2A below is an application of the Passivity theorem. It uses a combination of techniques: some results of Part I of this paper, the multiplier idea and some inequalities of Willems-Brockett [7]. It is worth noting that Theorem 2A applies to the multiple-input multiple-output case and memoryless nonlinearity need <u>not</u> be <u>uncoupled</u>, as was the case, for example, in Refs. [6], and [11].

<u>Theorem 2A (Application of Passivity Theorem)</u>

Consider the same system $\underset{\sim}{\mathcal{S}}$ as in Theorem 1A, where the linear time-invariant nonanticipative subsystem $\underset{\sim}{G}$ is described by (16) and (16a); the memoryless, time-varying nonlinearity $\underset{\sim}{\Phi}_t$ is described by a nonlinear function $\underset{\sim}{\psi}_t : \Sigma^n \times J_+ \to \Sigma^n$ which has the following properties:

N1. for some constant n×n matrix $\underset{\sim}{K}$

$$(\underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2)'\left[\underset{\sim}{\psi}_t(\underset{\sim}{\sigma}_1, m) - \underset{\sim}{\psi}_t(\underset{\sim}{\sigma}_2, m)\right] \geq (\underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2)'\underset{\sim}{K}(\underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2)$$
$$\forall \underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2 \in \Sigma^n, \quad \forall m \in J_+ \tag{20}$$

N2. $$\underset{\sim}{\psi}_t(-\underset{\sim}{\sigma}, m) = -\underset{\sim}{\psi}_t(\underset{\sim}{\sigma}, m) \qquad \forall \underset{\sim}{\sigma} \in \Sigma^n, \ \forall m \in J_+ \tag{21}$$

Let $\underset{\sim}{M}$ be a multiplier whose z-transfer function is of the form

$$\underset{\sim}{\tilde{M}}(z) = \sum_{i=0}^{\infty} \underset{\sim}{M}_i z^{-i} \tag{22}$$

and satisfies the following conditions:

M1. $\underset{\sim}{M} \triangleq \{\underset{\sim}{M}_i\}_0^\infty = \mathcal{Z}^{-1}\{\tilde{\underset{\sim}{M}}(z)\} \in \ell^1_{n\times n}$

M2. $$\inf_{|z|\geq 1} \left|\det \tilde{\underset{\sim}{M}}(z)\right| > 0 \tag{23}$$

M3. for all $i \in J_+$, all elements of $\underset{\sim}{M}_i$ are such that

$$(m_i)_{\alpha\alpha} \geq \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} |(m_i)_{\alpha\beta}| \quad \text{and} \quad (m_i)_{\beta\beta} \geq \sum_{\substack{\alpha=1\\ \alpha\neq\beta}}^{n} |(m_i)_{\alpha\beta}| \tag{24}$$

Under these conditions, if

(i) for the constant matrix $\underset{\sim}{K}$ defined in (20)

$$\inf_{|z|\geq 1} \left|\det [\underset{\sim}{I} + \underset{\sim}{G}(z)\underset{\sim}{K}]\right| > 0 \tag{25}$$

and if $\underset{\sim}{R} = \underset{\sim}{0}$ or $\underset{\sim}{RK}$ is nonsingular,

(ii) for some number $\delta > 0$,

$$\inf_{|z|=1} \lambda\left\{\tilde{\underset{\sim}{M}}(z)[\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\underset{\sim}{K}]^{-1}\tilde{\underset{\sim}{G}}(z) + \hat{\underset{\sim}{G}}'(\bar{z})[\underset{\sim}{I} + \underset{\sim}{K}'\tilde{\underset{\sim}{G}}'(\bar{z})]^{-1}\tilde{\underset{\sim}{M}}'(\bar{z})\right\} \geq \delta > 0 \tag{26}$$

where $\lambda\{\underset{\sim}{W}\}$ denotes the least eigenvalue of the matrix $\underset{\sim}{W}$, then for all $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$ in ℓ^2_n, $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{y}_1$ and $\underset{\sim}{y}_2$ are in ℓ^2_n.

Corollary 2A

Consider the same system $\mathcal{S}$ as in Corollary 1A, where the linear, time-invariant, nonanticipative subsystem G is described by (16') and

(16a') and the memoryless, time-varying nonlinearity Φ_t is described by a nonlinear function $\psi_t : \Sigma \times J_+ \to \Sigma$ which has the following properties:

N1. for some constant k

$$\left[\psi_t(\sigma_1,m) - \psi_t(\sigma_2,m)\right](\sigma_1 - \sigma_2) \geq k(\sigma_1 - \sigma_2)^2 \qquad \forall\sigma_1,\ \sigma_2 \in \Sigma,\ \forall m \in J_+ \tag{20'}$$

N2. $$\psi_t(-\sigma,m) = -\psi_t(\sigma,m) \qquad \forall\sigma \in \Sigma,\ \forall m \in J_+ \tag{21'}$$

Let M be a multiplier whose z-transfer function is of the form

$$\tilde{m}(z) = \sum_{i=0}^{\infty} m_i z^{-i} \tag{22'}$$

and satisfies the following conditions:

M1. $m \triangleq \{m_i\}_0^\infty = \mathcal{Z}^{-1}\{\tilde{m}(z)\} \in \ell^1.$

M2. $$\inf_{|z|\geq 1} |\tilde{m}(z)| > 0 \tag{23'}$$

M3. $$m_i \geq 0 \qquad \forall i \in J_+ \tag{24'}$$

Under these conditions, if

(i) for the constant k defined in (20')

$$\inf_{|z|\geq 1} |1 + k\tilde{g}(z)| > 0 \tag{25'}$$

and if $r = 0$ or $rk \neq 0$

(ii) for some number $\delta' > 0$

$$\inf_{|z|=1} \operatorname{Re}\left\{\tilde{m}(z)\frac{\tilde{g}(z)}{1+k\tilde{g}(z)}\right\} \geq \delta' > 0 \tag{26'}$$

then $u_1, u_2 \in \ell^2$ implies that $e_1, e_2, y_1, y_2 \in \ell^2$.

Theorem 2A is simply an application of usual Passivity theorem [1,3], a special case of Theorem 2, in which one subsystem is passive and the other subsystem is strictly passive and has finite gain. In order to illustrate the application of generalized passivity theorem given in Theorem 2, we present the following theorem.

Theorem 2B

Consider the single-input, single-output system $\mathcal{S}$ (Fig. 1) with $H_1 = G$ being a linear time-invariant, nonanticipative subsystem and $H_2 = \Phi$ being a time-invariant memoryless nonlinearity. Let the open-loop impulse response sequence of G, $g \triangleq \{g_i\}_0^\infty$ be in ℓ^1 and let the the input-output relation of the linear subsystem G be defined in terms of g by

$$y_1 = g * e_1 \tag{27}$$

or equivalently

$$y_{1m} = (g * e_1)_m = \sum_{i=0}^{m} g_{m-i} e_{1i} \tag{28}$$

Let Φ be characterized by a nonlinear function $\psi : \Sigma \to \Sigma$ which satisfies the following assumptions:

N1. for some constants k_1 and k_2,

$$0 < k_1 \le \frac{\psi(\sigma_1) - \psi(\sigma_2)}{\sigma_1 - \sigma_2} \le k_2 \qquad \forall \sigma_1, \sigma_2 \in \Sigma, \quad \sigma_1 \neq \sigma_2 \tag{29}$$

$$\psi(\sigma) = 0 \quad \text{if and only if } \sigma = 0$$

N2. $\psi(-\sigma) = -\psi(\sigma) \quad \forall \sigma \in \Sigma$

Let $M : \Sigma \to \Sigma$ be a multiplier whose z-transfer function is of the form

$$\tilde{m}(z) = \sum_{i=0}^{\infty} m_i z^{-i}$$

where $m \triangleq \{m_i\}_0^{\infty} = \mathcal{Z}^{-1}\{\tilde{m}(z)\} \in \ell^1$. The input-output relation of the multiplier M is defined by the convolution $Mx = m * x$.

Under these conditions, if

$$\lambda \triangleq \inf_{|z|=1} \left\{ \mathrm{Re}[\tilde{m}(z)\tilde{g}(z)] \right\} + \frac{m_o}{k_2} - \frac{1}{k_1} \|m\|_1 > 0 \tag{30}$$

then $u_1, u_2 \in \ell^2$ implies that e_1, e_2, y_1, y_2 are in ℓ^2.

Comments:

1. Assumption N1 implies the following facts:

a. $0 < k_1\sigma^2 \le \sigma\psi(\sigma) \le k_2\sigma^2 \quad \forall\sigma \in \Sigma, \; \sigma \ne 0$ (29a)

b. $0 < \frac{1}{k_2}\psi^2(\sigma) \le \sigma\psi(\sigma) \le \frac{1}{k_1}\psi^2(\sigma) \quad \forall\sigma \in \Sigma, \; \sigma \ne 0$ (29b)

2. The assumptions N1 and N2 above specify an odd monotonically increasing nonlinearity in the sector $[k_1,k_2]$.

3. If, in addition to N1 and N2 defined above, we have additional assumption on the slope of the nonlinearity, e.g. $|d\psi(\sigma)/d\sigma| < k_3$, then a Jury-Lee [10] type of criterion which is in the form of (30) can be obtained easily as an application of Theorem 2.

To illustrate further the power of Theorem 2, we present below a stability criterion which is similar to that of Tsypkin [9]. Our result is more general in that we allow for inputs in ℓ^2 and the conditions on the nonlinearity are slightly less restrictive.

<u>Theorem 2C</u>

Consider the same system $\mathcal{S}$ as in Theorem 2B, where the linear, time-invariant, nonanticipative subsystem G is described by (27) and (28) and the memoryless, time-invariant nonlinearity Φ is described by a nonlinear function $\psi : \Sigma \to \Sigma$ which satisfies the condition that for some constant k

$$0 \le \frac{\psi(\sigma_1) - \psi(\sigma_2)}{\sigma_1 - \sigma_2} \le k \qquad \sigma_1, \sigma_2 \in \Sigma, \;\; \sigma_1 \ne \sigma_2 \tag{31}$$

Let M be the multiplier whose z-transform is $\tilde{m}(z) = 1 + q(1 - z^{-1})$ with

$q \geq 0$. Under these conditions, if

$$\inf_{|z|=1} \mathrm{Re}\left\{\left[1 + q\left(1 - z^{-1}\right)\right] \hat{g}(z)\right\} + \frac{1}{k} > 0 \tag{32}$$

then for all u_1, u_2 in ℓ^2, e_1, e_2, y_1 and y_2 are also in ℓ^2.

VI. Appendix

Proof of Theorem 1

From the system equations (1), (2), (3) and the assumptions (8) and (9), we obtain (using the subscript N to indicate truncation at N),

$$\|e_{2N}\| \le \|u_{2N}\| + \mu_1\|e_{1N}\| + \nu_1 \qquad \forall N \in J_+ \tag{33}$$

and

$$\|e_{1N}\| \le \|u_{1N}\| + \mu_2\|e_{2N}\| + \nu_2 \qquad \forall N \in J_+ \tag{34}$$

Now substituting (34) into (33), we obtain after some manipulations

$$(1 - \mu_1\mu_2)\|e_{2N}\| \le \left[\|u_{2N}\| + \mu_1\|u_{1N}\| + \nu_1 + \mu_1\nu_2\right] \tag{35}$$

Since $1 - \mu_1\mu_2 \triangleq (1 - \mu) > 0$ by (10), inequality (35) yields

$$\|e_{2N}\| \le \frac{1}{1 - \mu}\left[\|u_{2N}\| + \mu_1\|u_{1N}\| + \nu_1 + \mu_1\nu_2\right] \tag{36}$$

Now u_1, $u_2 \in \mathcal{B}$, hence for all $N \in J_+$, $\|u_{1N}\| \le \|u_1\| < \infty$ and $\|u_{2N}\| \le \|u_2\| < \infty$ and as a consequence of (36), $\|e_2\| < \infty$, i.e. $e_2 \in \mathcal{B}$. From the system equations (1)-(4) and the assumptions (8) and (9), we can easily see that e_1, y_1 and y_2 are also in $\mathcal{B}$.

Before we prove Theorem 2, we present first a fundamental lemma which is analogous to Tellegen's Theorem in circuit theory. This lemma is an immediate consequence of the system equations (1)-(4) and the linearity of the scalar product.

Lemma A

Let the system $\underset{\sim}{\mathcal{S}}$ and $\underset{\sim}{\mathcal{S}}_M$ (Fig. 1 and Fig. 2) described by (1)-(4) be determinate. Then for all $N \in J_+$, we have for $\underset{\sim}{\mathcal{S}}$

$$\langle e_1, H_1 e_1 \rangle_N + \langle e_2, H_2 e_2 \rangle_N = \langle u_1, H_1 e_1 \rangle_N + \langle u_2, H_2 e_2 \rangle_N \tag{37}$$

and similarly for $\underset{\sim}{\mathcal{S}}_M$

$$\langle e_1, MH_1 e_1 \rangle_N + \langle Me_2, H_2 e_2 \rangle_N = \langle u_1, MH_1 e_1 \rangle_N + \langle Mu_2, H_2 e_2 \rangle_N \tag{38}$$

Proof of Theorem 2

By Lemma A, we have for any $N \in J_+$

$$\langle e_1, H_1 e_1 \rangle_N + \langle e_2, H_2 e_2 \rangle_N = \langle u_1, H_1 e_1 \rangle_N + \langle u_2, H_2 e_2 \rangle_N \tag{39}$$

Using the assumptions (12)-(14) and Schwarz's inequality, we obtain from (39)

$$\delta_1 \| e_{1N} \|_2^2 + \alpha_1 + \varepsilon_2 \| (H_2 e_2)_N \|_2^2 + \alpha_2 \leq \| (H_1 e_1)_N \|_2 \| u_{1N} \|_2 + \| (H_2 e_2)_N \|_2 \| u_{2N} \|_2 \tag{40}$$

Recalling from the system equations that $y_2 = H_2 e_2$ and $e_1 = u_1 - y_2$, thus we have for any $N \in J_+$,

$$\|\underset{\sim}{e}_{1N}\|_2 \geq \|\underset{\sim}{u}_{1N}\|_2 - \|\underset{\sim}{y}_{2N}\|_2 \quad \text{and} \quad \|\underset{\sim}{e}_{1N}\|_2 \leq \|\underset{\sim}{u}_{1N}\|_2 + \|\underset{\sim}{y}_{2N}\|_2$$

Using these relations and (12) we obtain from (40)

$$\delta_1\left(\|\underset{\sim}{u}_{1N}\|_2 - \|\underset{\sim}{y}_{2N}\|_2\right)^2 + \alpha_1 + \epsilon_2\|\underset{\sim}{y}_{2N}\|_2^2 + \alpha_2$$

$$\leq \left[\gamma_1\left(\|\underset{\sim}{u}_{1N}\|_2 + \|\underset{\sim}{y}_{2N}\|_2\right) + \beta_1\right]\|\underset{\sim}{u}_{1N}\|_2 + \|\underset{\sim}{y}_{2N}\|_2\|\underset{\sim}{u}_{2N}\|_2 \qquad (41)$$

Let $\lambda \triangleq \delta_1 + \epsilon_2$ and use the assumptions $\underset{\sim}{u}_1, \underset{\sim}{u}_2 \in \ell_n^2$; we obtain, after some manipulations, from (41)

$$\lambda\|\underset{\sim}{y}_{2N}\|_2^2 \leq \left[(\gamma_1 + 2\delta_1)\|\underset{\sim}{u}_1\|_2 + \|\underset{\sim}{u}_2\|_2\right]\|\underset{\sim}{y}_{2N}\|_2$$

$$+ \left[(\gamma_1 + |\delta_1|)\|\underset{\sim}{u}_1\|_2^2 + \beta_1\|\underset{\sim}{u}_1\|_2 - \alpha_1 - \alpha_2\right] \qquad (42)$$

or

$$\lambda\|\underset{\sim}{y}_{2N}\|_2^2 \leq k_1\|\underset{\sim}{y}_{2N}\|_2 + k_2 \qquad \forall N \in J_+, \ \forall \underset{\sim}{y}_2 \in \Sigma^n \qquad (43)$$

where

$$k_1 \triangleq \left[(\gamma_1 + 2\delta_1)\|\underset{\sim}{u}_1\|_2 + \|\underset{\sim}{u}_2\|_2\right]$$

and

$$k_2 \triangleq (\gamma_1 + |\delta_1|)\|\underset{\sim}{u}_1\|_2^2 + \beta_1\|\underset{\sim}{u}_1\|_2 - \alpha_1 - \alpha_2$$

are constants independent of $\underset{\sim}{y}_{2N}$ and N. Since $\lambda > 0$ (by assumption), (43) implies that $\underset{\sim}{y}_2 \in \ell_n^2$. Since $\underset{\sim}{e}_1 = \underset{\sim}{u}_1 - \underset{\sim}{y}_2$ and $\underset{\sim}{u}_1 \in \ell_n^2$, we have $\underset{\sim}{e}_1 \in \ell_n^2$. It follows from (12) that $\underset{\sim}{y}_1 \in \ell_n^2$. Finally $\underset{\sim}{e}_2 \in \ell_n^2$ because $\underset{\sim}{e}_2 = \underset{\sim}{u}_1 + \underset{\sim}{y}_1$.

<u>Proof of Theorem 1A</u>

We shall prove the theorem by applying Theorem 1.

By a standard system transformation, we obtain from the system $\underset{\sim}{\mathcal{S}}$ (Fig. 1) the new transformed system $\underset{\sim}{\bar{\mathcal{S}}}$ (Fig. 3), where the linear subsystem $\underset{\sim}{H}$ in the forward path and the nonlinearity $\underset{\sim}{\bar{\Phi}}_t$ in the feedback path become respectively

$$\underset{\sim}{H} = (\underset{\sim}{I} + \underset{\sim}{G}\underset{\sim}{K})^{-1}\underset{\sim}{G} \tag{44}$$

and

$$\underset{\sim}{\bar{\Phi}}_t = \underset{\sim}{\Phi}_t - \underset{\sim}{K}\underset{\sim}{I} \tag{45}$$

The variables $\underset{\sim}{y}_1$, $\underset{\sim}{e}_2$ and $\underset{\sim}{u}_2$ are preserved in the system $\underset{\sim}{\bar{\mathcal{S}}}$ and the new variables $\underset{\sim}{\bar{u}}_1$, $\underset{\sim}{\bar{e}}_1$ and $\underset{\sim}{\bar{y}}_2$ are related to the old variables $\underset{\sim}{u}_1$, $\underset{\sim}{e}_1$ and $\underset{\sim}{y}_2$ by

$$\underset{\sim}{\bar{u}}_1 = \underset{\sim}{u}_1 - \underset{\sim}{K}\underset{\sim}{u}_2 \tag{46}$$

$$\underset{\sim}{\bar{e}}_1 = \underset{\sim}{e}_1 + \underset{\sim}{K}\underset{\sim}{y}_1 \tag{47}$$

$$\underset{\sim}{\bar{y}}_2 = \underset{\sim}{y}_2 - \underset{\sim}{K}\underset{\sim}{e}_2 \tag{48}$$

Since $\underset{\sim}{K}$ is a constant matrix, it is clear from the above relations that $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$, $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{y}_1$ and $\underset{\sim}{y}_2$ are in ℓ_n^p if and only if that $\underset{\sim}{\bar{u}}_1$, $\underset{\sim}{u}_2$, $\underset{\sim}{\bar{e}}_1$, $\underset{\sim}{e}_2$, $\underset{\sim}{y}_1$

and $\bar{y}_2$ are in ℓ_n^p. Therefore the original system $\mathcal{S}$ and the transformed system $\bar{\mathcal{S}}$ are equivalent as far as stability is concerned.

Now by assumptions (16) and (18), it follows from (44) and Theorem 2 of Part I that

$$H \triangleq \{H_i\}_0^\infty = \mathcal{Z}^{-1}\left\{[I + \tilde{G}(z)K]^{-1}\tilde{G}(z)\right\} \in \ell^1_{n\times n}$$

consequently H has a finite norm denoted by $\|H\|_1$. Therefore for any fixed $p \in [1,\infty]$

$$\|(H\bar{e}_1)_N\|_p \le \|H\|_1\|\bar{e}_{1N}\|_p \qquad \forall \bar{e}_1 \in \Sigma^n, \ \forall N \in J_+ \tag{49}$$

This shows that condition (8) of Theorem 1 (Small Gain Theorem) is satisfied with $\mu_1 \triangleq \|H\|_1$ and $\nu_1 = 0$. Relation (45) and assumptions (17) and (19) show that conditions (9) and (10) of Theorem 1 are met. Thus it follows from Theorem 1 that u_1, $u_2 \in \ell_n^p$ implies that $\bar{e}_1$, e_2, y_1 and $\bar{y}_2$ are in ℓ_n^p and by (46)-(48), e_1, y_2 are also in ℓ_n^p.

<u>Proof of Theorem 1B</u>

Perform the system transformation as in the proof of Theorem 1A; we obtain the system $\bar{\mathcal{S}}$ with

$$H = (I + G\bar{K})^{-1}G \tag{44a}$$

and

$$\hat{K} = K - \bar{K} \tag{45a}$$

From the proof of Theorem 4 in Part I and the fact that the system $\underset{\sim}{\mathcal{S}}$ is determinate, we see that the system $\underset{\sim}{\bar{\mathcal{S}}}$ is also determinate. Furthermore because of (18a), $\|\underset{\sim}{H}\|_1 < \infty$. Now, by assumption, $|\underset{\sim}{K}_i| < \infty$ $\forall i \in J_+$ and $\underset{\sim}{K}_i \to \underset{\sim}{\bar{K}}$ as $i \to \infty$; thus for any $\varepsilon \in (0,1)$ there exists an $N(\varepsilon) \in J_+$ such that for all $i \geq N(\varepsilon)$, $\|\underset{\sim}{H}\|_1 |\underset{\sim}{K}_i - \underset{\sim}{\bar{K}}| \leq 1 - \varepsilon$. Therefore the claimed result of the theorem follows immediately from Theorem 1 applied to the system $\underset{\sim}{\bar{\mathcal{S}}}$ for $i \geq N(\varepsilon)$.

Note that we have actually proved that if $|\underset{\sim}{K}_i| < \infty$ $\forall i$, and if for some N, $i > N$ implies that $|\underset{\sim}{K}_i - \underset{\sim}{\bar{K}}| \cdot \|\underset{\sim}{H}\|_1 \leq 1 - \varepsilon$, then the conclusion of Theorem 1B still holds. In other words, it is not necessary for the $\underset{\sim}{K}_i$ to tend to $\underset{\sim}{\bar{K}}$ but only that they eventually get sufficiently close to $\underset{\sim}{\bar{K}}$ and remain there.

Proof of Theorem 2A

We shall prove the theorem by applying Theorem 2.

First we perform the system transformation as in the proof of Theorem 1A to obtain the system $\underset{\sim}{\bar{\mathcal{S}}}$ (Fig. 3). We have noted that system $\underset{\sim}{\mathcal{S}}$ is stable if and only if system $\underset{\sim}{\bar{\mathcal{S}}}$ is stable. Next we introduce the multiplier $\underset{\sim}{M}$ into the system $\underset{\sim}{\bar{\mathcal{S}}}$ to obtain the system $\underset{\sim}{\bar{\mathcal{S}}}_M$ ($\underset{\sim}{\bar{\mathcal{S}}}_M$ can be obtained from Fig. 2 by replacing $\underset{\sim}{u}_1$, $\underset{\sim}{e}_1$, $\underset{\sim}{y}_2$, $\underset{\sim}{H}_1$ and $\underset{\sim}{H}_2$ with $\underset{\sim}{\bar{u}}_1$, $\underset{\sim}{\bar{e}}_1$, $\underset{\sim}{\bar{y}}_2$, $\underset{\sim}{H}$ and $\underset{\sim}{\Phi}_t$ respectively.) Now by assumptions (16) and (18) and the relation (44), it follows from the same reasoning as in the proof of Theorem 1A that $\underset{\sim}{H}$ has finite gain $\|\underset{\sim}{H}\|_1$ as is defined in (49), i.e., $\forall N \in J_+$

$$\|(\underset{\sim}{H}\underset{\sim}{\bar{e}}_1)_N\|_2 \leq \|\underset{\sim}{H}\|_1 \|\underset{\sim}{\bar{e}}_{1N}\|_2 \tag{50}$$

By assumption M1 and (50), we obtain

$$\|(\underset{\sim}{M}\underset{\sim}{H}\overline{\underset{\sim}{e}}_1)_N\|_2 \le \|\underset{\sim}{M}\|_1\|\underset{\sim}{H}\|_1\|\overline{\underset{\sim}{e}}_{1N}\|_2 \qquad \forall \overline{\underset{\sim}{e}}_1 \in \Sigma^n, \ \forall N \in J_+ \tag{51}$$

This shows that condition (12') of Theorem 2M is satisfied with $\gamma_1' \triangleq \|\underset{\sim}{M}\|_1\|\underset{\sim}{H}\|_1$ and $\beta_1' = 0$. Now by assumption (26) and Parseval's theorem, we have

$$\begin{aligned}
\langle \overline{\underset{\sim}{e}}_1, \underset{\sim}{M}\underset{\sim}{H}\overline{\underset{\sim}{e}}_1 \rangle_N &= \frac{1}{2\pi}\oint_{|z|=1} \tilde{\overline{\underset{\sim}{e}}}{}^*_{1N}(z)\left[\tilde{\underset{\sim}{M}}(z)\left(\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\underset{\sim}{K}\right)^{-1}\tilde{\underset{\sim}{G}}(z)\right]\tilde{\overline{\underset{\sim}{e}}}_{1N}(z)\, z^{-1}dz \\
&= \frac{1}{4\pi}\oint_{|z|=1} \tilde{\overline{\underset{\sim}{e}}}{}^*_{1N}(z)\left\{\left[\tilde{\underset{\sim}{M}}(z)\left(\underset{\sim}{I} + \tilde{\underset{\sim}{G}}(z)\underset{\sim}{K}\right)^{-1}\tilde{\underset{\sim}{G}}(z)\right]\right. \\
&\quad + \left.\left[\tilde{\underset{\sim}{G}}{}^*(z)\left(\underset{\sim}{I} + \underset{\sim}{K}'\tilde{\underset{\sim}{G}}{}^*(z)\right)^{-1}\tilde{\underset{\sim}{M}}{}^*(z)\right]\right\}\tilde{\overline{\underset{\sim}{e}}}_{1N}(z)\, z^{-1}\, dz \\
&\ge \delta\|\overline{\underset{\sim}{e}}_{1N}\|_2^2 > 0 \qquad \forall \overline{\underset{\sim}{e}}_1 \in \Sigma^n, \ \forall N \in J_+
\end{aligned} \tag{52}$$

Thus the condition (43') of Theorem 2 is satisfied with $\delta_1' \triangleq \delta > 0$ and $\alpha_1' = 0$. It remains to check the conditions of (14') and (15') of Theorem 2. The assumption N1 and the relation (45) give us

$$(\underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2)'\left[\overline{\underset{\sim}{\psi}}_t(\underset{\sim}{\sigma}_1,m) - \overline{\underset{\sim}{\psi}}_t(\underset{\sim}{\sigma}_2,m)\right] \ge 0 \qquad \forall \underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2 \in \Sigma^n, \ \forall m \in J_+ \tag{53}$$

This coupled with assumption N2 implies that $\overline{\underset{\sim}{\Phi}}_t$ is an odd, monotonically nondecreasing nonlinearity. Since the assumption M3 implies that the matrices $\underset{\sim}{M}_i$'s are doubly dominant [7] for all $i \in J_+$, it follows that [Theorem 4 of Willems and Brockett]

$$\begin{aligned}\langle \hat{\underset{\sim}{e}}_2, \overline{\underset{\sim}{\Phi}}_t \underset{\sim}{M}^{-1} \hat{\underset{\sim}{e}}_2 \rangle_N &= \langle \underset{\sim}{M}\underset{\sim}{e}_2, \overline{\underset{\sim}{\Phi}}_t \underset{\sim}{e}_2 \rangle_N \\ &= \langle \sum_{i=0}^{N} \underset{\sim}{M}_{N-i} \underset{\sim}{e}_{2i}, \overline{\underset{\sim}{\Psi}}_t(\underset{\sim}{e}_{2i}) \rangle \\ &= \sum_{i=0}^{N} \overline{\underset{\sim}{\Psi}}_t'(\underset{\sim}{e}_{2i}) \underset{\sim}{M}_{N-i} \underset{\sim}{e}_{2i} \geq 0 \qquad (54)\end{aligned}$$

where we have used the assumptions M1 and M2 to guarantee the existence of $\underset{\sim}{M}^{-1}$ in the above equation. (In fact M1 and M2 imply that $\underset{\sim}{M}^{-1} \in \ell^1_{n\times n}$; see proof of Theorem 2, Part I). (54) shows that condition (14') of Theorem 2 is satisfied with $\varepsilon_2' = \alpha_2' = 0$. Clearly condition (18') of Theorem 2 is indeed satisfied because $\lambda' \triangleq (\delta_1' + \varepsilon_2') = \delta > 0$. Therefore we have demonstrated that all conditions of Theorem 2 are satisfied. Now $\underset{\sim}{u}_1, \underset{\sim}{u}_2 \in \ell^2_n$ implies that $\overline{\underset{\sim}{u}}_1$ and $\hat{\underset{\sim}{u}}_2$ are in ℓ^2_n because $\overline{\underset{\sim}{u}}_1 = \underset{\sim}{u}_1 - \underset{\sim}{K}\underset{\sim}{u}_2$, $\hat{\underset{\sim}{u}}_2 = \underset{\sim}{M}\underset{\sim}{u}_2$, where $\underset{\sim}{K}$ is an $n\times n$ constant matrix and $\underset{\sim}{M} \in \ell^1_{n\times n}$. Therefore we conclude from Theorem 2 that $\overline{\underset{\sim}{e}}_1$, $\hat{\underset{\sim}{e}}_2$, $\hat{\underset{\sim}{y}}_1$ and $\overline{\underset{\sim}{y}}_2$ are in ℓ^2_n. From Fig. 4, we can easily see that $\underset{\sim}{y}_1$ and $\underset{\sim}{e}_2$ are in ℓ^2_n because $\underset{\sim}{M}$ and $\underset{\sim}{M}^{-1}$ are in $\ell^1_{n\times n}$ which map ℓ^2_n into ℓ^2_n respectively. From the system equations (47) and (48) we obtain easily that $\underset{\sim}{e}_1$ and $\underset{\sim}{y}_2$ are in ℓ^2_n.

Before we prove Theorem 2B, we first quote a lemma [8, 17, 18] which we will use it in our proof later.

Lemma B

Let $f : \Sigma \to \Sigma$ satisfy the following conditions:

$$(\sigma_1 - \sigma_2)\left[f(\sigma_1) - f(\sigma_2)\right] \geq 0 \qquad \forall \sigma_1, \sigma_2 \in \Sigma$$

Then for all $j \in J_+$

$$\sum_{i=0}^{N} \sigma_{i-j} f(\sigma_i) \leq \sum_{i=0}^{N} \sigma_i f(\sigma_i) \qquad \forall \sigma_i \in \Sigma, \ \forall N \in J_+$$

If, in addition, $f(-\sigma) = -f(\sigma)$, then for all $j \in J_+$

$$\left|\sum_{i=0}^{N} \sigma_{i-j} f(\sigma_i)\right| \leq \sum_{i=0}^{N} \sigma_i f(\sigma_i) \qquad \forall \sigma_i \in \Sigma, \ \forall N \in J_+$$

Proof of Theorem 2B

We prove the theorem by means of Theorem 2.

Since, by assumption, g and m are in ℓ^1, we have

$$\|(MGe_1)_N\|_2 = \|(m * g * e_1)_N\|_2 = \|m\|_1 \|g\|_1 \|e_{1N}\|_2 \qquad \forall e_1 \in \Sigma, \ \forall N \in J_+ \tag{55}$$

This shows that (12') of Theorem 2 is satisfied with $\gamma_1' \triangleq \|m\|_1 \|g\|_1$ and $\beta_1' = 0$. Now by Parseval theorem, we get

$$\langle e_1, MGe_1 \rangle_N = \frac{1}{2\pi} \oint_{|z|=1} \tilde{e}^*_{1N}(z)\, \tilde{m}(z)\, \tilde{g}(z)\, \tilde{e}_{1N}(z)\, z^{-1}\, dz$$

$$= \frac{1}{2\pi} \oint_{|z|=1} \mathrm{Re}[\tilde{m}(z)\, \tilde{g}(z)]\, \tilde{e}^*_{1N}(z)\, \tilde{e}_{1N}(z)\, z^{-1}\, dz$$

$$\geq \delta \| e_{1N} \|_2^2 \tag{56}$$

where $\delta \triangleq \inf_{|z|=1} \{\mathrm{Re}[\tilde{m}(z)\, \tilde{g}(z)]\}$.

Clearly (56) is in the form of (13') of Theorem 2 with $\delta_1' = \delta$ and $\alpha_1' = 0$. Before we apply Theorem 2, we need only to check conditions (14') and (15'). Now consider

$$\langle Me_2, \Phi e_2 \rangle_N = \sum_{i=0}^{N} \left(\sum_{j=0}^{i} m_j e_{2(i-j)} \right) \psi(e_{2i})$$

$$= \sum_{i=0}^{N} m_o e_{2i} \psi(e_{2i}) + \sum_{i=1}^{N} \left(\sum_{j=0}^{i-1} m_j e_{2(i-j)} \right) \psi(e_{2i}) \tag{57}$$

and using the assumption N1, N2 and Lemma B, we obtain successively

$$\langle Me_2, \Phi e_2 \rangle_N \geq \frac{m_o}{k_2} \sum_{i=0}^{N} \psi^2(e_{2i}) - \sum_{j=0}^{\infty} |m_j| \sum_{i=1}^{N} |e_{2(i-j)} \psi(e_{2i})|$$

(cont.)

$$\geq \frac{m_o}{k_2} \sum_{i=0}^{N} \psi^2(e_{2i}) - \|m\|_1 \sum_{i=1}^{N} e_{2i}\psi(e_{2i})$$

$$\geq \left(\frac{m_o}{k_2} - \frac{\|m\|_1}{k_1}\right) \|y_{2N}\|_2^2$$

$$\triangleq \varepsilon_2 \|y_{2N}\|_2^2 \tag{58}$$

where we have used (29b) and defined ε_2 by

$$\varepsilon_2 \triangleq \frac{m_o}{k_2} - \frac{\|m\|_1}{k_1}$$

So (58) is in the form of (14') of Theorem 2. By assumption, clearly condition (15') of Theorem 2 is satisfied. Therefore it follows from Theorem 2 that e_1, e_2, y_1 and y_2 are in ℓ^2.

<u>Proof of Theorem 2C</u>

By identical arguments as in the proof of Theorem 2B, we obtain

$$\langle e_1, Me_1 \rangle_N \geq \delta_1' \|e_{1N}\|_2^2 \tag{59}$$

where $\delta_1' \triangleq \inf_{|z|=1} \mathrm{Re}\{[1 + q(1 - z^{-1})]\tilde{g}(z)\}$.

Next we consider

$$\langle Me_2, \Phi e_2 \rangle_N = \sum_{i=0}^{N} (m * e_2)_i \psi(e_{2i}) \tag{60}$$

Denote $\tilde{m}(z) = 1 + q(1 - z^{-1}) = (1 + q) - qz^{-1} \triangleq m_o + m_1 z^{-1}$, then $m_o = 1 + q$ and $m_1 = -q$. Since

$$\begin{aligned}(m * e_2)_i &= m_o e_{2i} + m_1 e_{2(i+1)} \\ &= (1 + q)e_{2i} - qe_{2(i+1)}\end{aligned}$$

We obtain from (60)

$$\langle Me_2, \Phi e_2 \rangle_N = \sum_{i=0}^{N} (1 + q)e_{2i}\psi(e_{2i}) - q \sum_{i=0}^{N} e_{2(i+1)}\psi(e_{2i}) \tag{61}$$

Applying Lemma B to (61) and noting that $q \geq 0$

$$\begin{aligned}\langle Me_2, \Phi e_2 \rangle_N &\geq (1 + q) \sum_{i=0}^{N} e_{2i}\psi(e_{2i}) - q \sum_{i=0}^{N} e_{2(i+1)}\psi(e_{2i}) \\ &\geq (1 + q) \sum_{i=0}^{N} e_{2i}\psi(e_{2i}) - q \sum_{i=0}^{N} e_{2i}\psi(e_{2i}) \\ &= \sum_{i=0}^{N} e_{2i}\psi(e_{2i}) \geq \frac{1}{k} \sum_{i=0}^{N} \psi^2(e_{2i}) = \frac{1}{k} \|\psi(e_{2i})_N\|_2^2 \\ &\triangleq \varepsilon_2 \|\psi(e_{2i})_N\|_2^2\end{aligned}$$

Assumption (32) implies that $\delta_1' + \varepsilon_2 > 0$. So we have shown that all conditions of Theorem 2 are satisfied, conseuqently we conclude from Theorem 2 that $u_1, u_2 \in \ell^2$ implies that e_1, e_2, y_1 and y_2 are in ℓ^2.

Conclusion

Using some results of Part I and slightly generalized versions of the Small Gain and of the Passivity theorems we obtain in a unified manner several general stability criteria for multiple-input, multiple-output discrete systems. We hope further work in this direction will lead to a unified presentation of stability theory of nonlinear feedback systems.

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FIGURE CAPTIONS

Fig. 1. The system $\underset{\sim}{\mathcal{S}}$ under consideration.

Fig. 2. The system $\underset{\sim}{\mathcal{S}}_M$ which is the system $\underset{\sim}{\mathcal{S}}$ with the multiplier $\underset{\sim}{M}$.

Fig. 3. The system $\underset{\sim}{\bar{\mathcal{S}}}$ which is obtained from the system $\underset{\sim}{\mathcal{S}}$ by a standard system transformation.

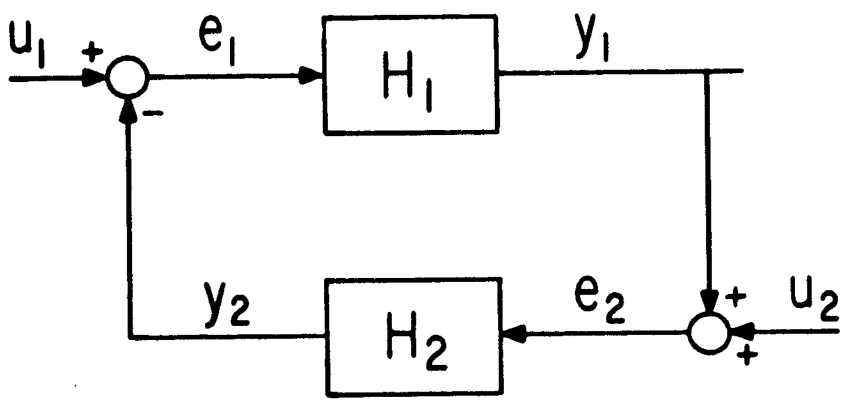

Fig 1

Wu - Desoer

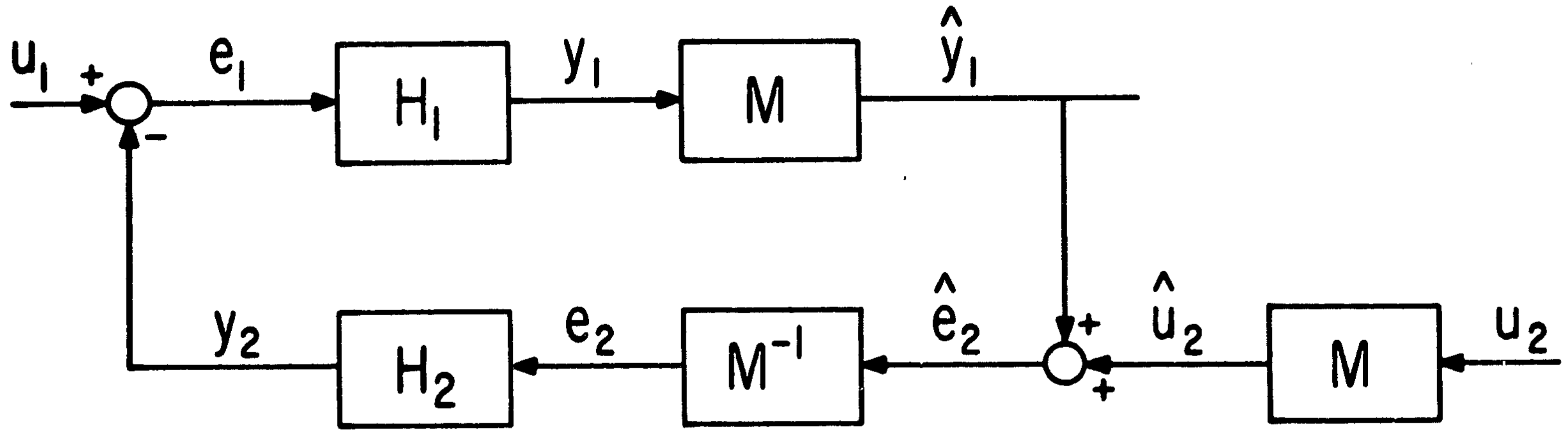

Fig. 2

Wu-Desoer

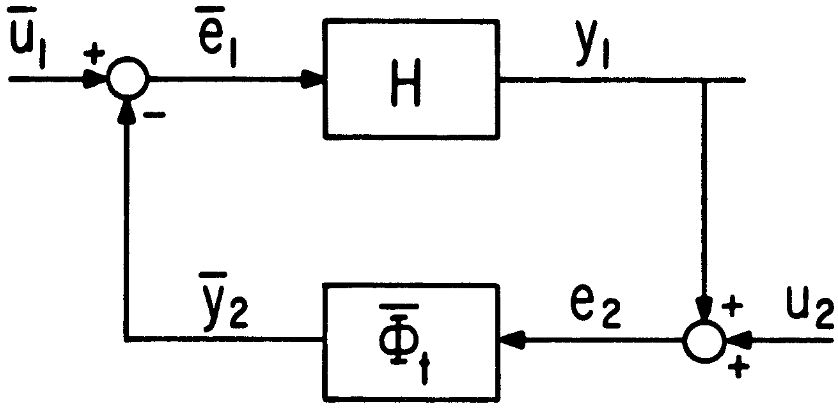

Fig. 3
Wu-Desoer