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The Inversion Theorem and Plancherel's
Theorem in Infinite Dimensions

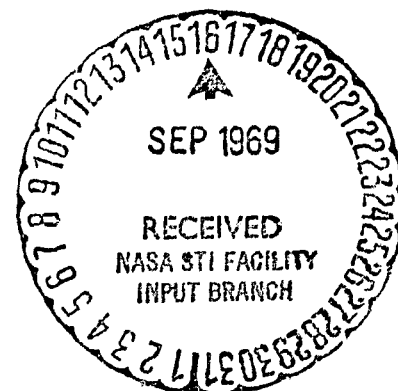
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The Inversion Theorem and Plancherel's

Theorem in a Banach Space

1. Introduction

Let G be a locally compact abelian group with Haar measure μ , and let X be a Banach space and $\mathbb{C}$ be the set of complex numbers. A classic theorem due to Plancherel ([1], [2]) states that the Fourier transform maps $L_1(G, \mathbb{C}) \cap L_2(G, \mathbb{C})^+$ onto a dense subset of $L_2(\hat{G}, \mathbb{C})$ ($\hat{G}$ is the dual group of G and has Haar measure m) in such a way that $\int_G \alpha(g) \overline{\beta(g)} \mu(dg) = \int_{\hat{G}} \hat{\alpha}(\gamma) \overline{\hat{\beta}(\gamma)} m(d\gamma)$ for all α, β in $L_1(G, \mathbb{C}) \cap L_2(G, \mathbb{C})$ where $\hat{\alpha}$ is the Fourier transform of α , given by $\hat{\alpha}(\gamma) = \int_G \overline{\gamma(g)} \alpha(g) \mu(dg)$ for all γ in $\hat{G}$. Here (g, γ) denotes the action of the character γ on g in G . In this paper we extend this result to functions taking values in an inner product subspace of a Banach algebra.

Another well-known theorem ([1], [2]) states that if α is a positive definite element of $L_1(G, \mathbb{C}) \cap L_\infty(G, \mathbb{C})$ then $\hat{\alpha}$ is in $L_1(\hat{G}, \mathbb{C})$ and

$$(1.1) \quad \alpha(g) = \int_{\hat{G}} \gamma(g) \hat{\alpha}(\gamma) m(d\gamma)$$

⁺For $1 \leq p \leq \infty$ $L_p(G, X)$ is the space of μ -measurable functions f mapping G into X . For $1 \leq p < \infty$ we use the norm $\|\cdot\|_p$, where $\|f\|_p = \{\int_G \|f(g)\|^p \mu(dg)\}^{1/p}$, and for $p = \infty$ we use the norm $\|f\|_\infty$ which is the (μ) essential supremum of $\|f(g)\|$ on G . $\|\cdot\|$ denotes the norm in X .

for (almost) all g in G . This inversion theorem is also generalized to functions assuming values in certain admissible Banach spaces.

Our work relies heavily on an extension of Bochner's theorem established in [3]. We show that if p is in $L_1(G, X) \cap L_\infty(G, X)$, if p is positive definite (positivity is defined with respect to a particular cone in X), and if $p(0)$ satisfies a certain finiteness condition, then $\hat{p}$, the Fourier transform of p , is in $L_1(\hat{G}, X)$ and the inversion formula 1.1 given for α holds for p . A sharper theorem states that if p is in $L_1(G, X) \cap L_\infty(G, X)$, if p is positive definite, and if there is a real, finite, regular Borel measure λ such that $\|\int_G \alpha(g)p(g)\mu(dg)\| \leq \int_{\hat{G}} |\hat{\alpha}(r)| \lambda(dr)$ for all α in $L_1(G, \mathbb{C})$, then $\hat{p}$ is in $L_1(\hat{G}, X)$ and 1.1 is satisfied by p .

Using this theory we give new proofs of some results due to Hewitt and Wigner ([4]).

Now assume G is compact and $\mathcal{N}$ is the set of Hilbert-Schmidt operators on a separable Hilbert space H . Then we show that the closed maximal ideals of the algebra $L_2(G, \mathcal{N})$ are in a one to one correspondence with $\hat{G}$. The same result holds for $L_2(G, A)$ where A is any separable simple H^* -algebra.

Finally we prove existence and uniqueness theorems for equations of the form

$$(1.2) \quad q(g) = r(g) + \int_G p(g-g')q(g')\mu(dg')$$

where r is in $L_2(G, \mathcal{N})$, p is in $L_1(G, \mathcal{L}(H, H))$, H is a separable Hilbert space and $\mathcal{L}(H, H)$ is the space of continuous linear operators mapping H into H (so $\mathcal{N} \subset \mathcal{L}(H, H)$). Solutions q are to be elements of $L_2(G, \mathcal{N})$.

2. Bochner's Theorem and Dominated Functions

Let X be a Banach space, X^* the dual of X and X^{**} the dual of X^* . For ϕ in X^* we denote the action of ϕ on $x \in X$ by (x, ϕ) . Given a subset of X^* we can define a cone of "positive" elements in X .

DEFINITION 2.1. Let Φ be a subset of X^* . The subset K_Φ of X given by

$$(2.2) \quad K_\Phi = \{x \in X: (x, \phi) \geq 0 \text{ for all } \phi \in \Phi\}$$

is called the cone determined by Φ .

Sometimes we write simply K if Φ is fixed by the context.

K_Φ is the set of "positive" elements.

Let G be a σ -finite locally compact abelian group with Haar measure μ and let $\hat{G}$ be its dual group with Haar measure m .

DEFINITION 2.3. Let p be a measurable map of G into X . Then p is Φ -positive definite if

$$(2.4) \quad \sum_{n=1}^N \sum_{m=1}^N c_n \bar{c}_m (p(g_n - g_m), \varphi) \geq 0$$

for any integer N , any $c_1, \dots, c_N$ in C , any $g_1, \dots, g_N$ in G ,
and all φ in Φ . If p is in $L_\infty(G, X)$ then p is integrally
 Φ -positive definite if

$$(2.5) \quad \left(\int_G \int_G \alpha(g) \bar{\alpha}(g') p(g - g') d\mu d\mu, \varphi \right) \geq 0$$

for all α in $L_1(G, C)$ and all φ in Φ .

Next we impose a condition which relates Φ to the topology of X .

DEFINITION 2.6. The family Φ is full if there is a $\rho > 0$ such
that

$$(2.7) \quad \|x\| \leq \rho \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{ |(x, \varphi)| / \|\varphi\| \}$$

for all x in X .

The following two propositions examine the relationship between the two notions of positive-definiteness.

PROPOSITION 2.8. If Φ is full and p is Φ -positive definite then
 p is in $L_\infty(G, X)$ and $p(0)$ is in K_Φ .

Proof: It is readily shown that for g in G , φ in Φ , $|(p(g), \varphi)| \leq (p(0), \varphi)$ so that $\|p(g)\| \leq \rho \|p(0)\|$.

PROPOSITION 2.9. Let p be in $L_\infty(G, X)$ such that one version of p is ωX -continuous⁺. Then p is Φ -positive definite if and only if p is integrally Φ -positive definite.

Proof: See [3] or [7].

We shall see shortly (corollary 2.15) that all those elements of $L_\infty(G, X)$ of interest to us have the continuity required in proposition 2.9.

Next we recall some results from measure theory. Let S be a locally compact topological space and let $\Sigma(S)$ be the Borel field of S (i.e. the smallest σ -field containing the closed sets of S).

DEFINITION 2.10. A vector measure ν is a weakly countably additive set function defined on $\Sigma(S)$ and taking values in X . ν is weakly regular if the scalar measures $(\nu(\cdot), \phi)$ are regular⁺⁺ for all ϕ in X^* . ν is Φ -positive if $(\nu(E), \phi) \geq 0$ for all ϕ in Φ and E in $\Sigma(S)$.

DEFINITION 2.11. A set function ν^{**} mapping $\Sigma(S)$ into X^{**} is weak-*regular if $(\phi, \nu^{**}(\cdot))$ is a regular scalar measure for all ϕ in X^* . ν^{**} is Φ -positive if $(\phi, \nu^{**}(E)) \geq 0$ for all ϕ in Φ , E in $\Sigma(S)$.

⁺The mapping f of G into X is ωX -continuous if it is continuous when the weak topology is imposed on X . G retains its usual topology.

⁺⁺A scalar measure λ is regular if, given $\epsilon > 0$ and $E \in \Sigma(S)$ with $\|\lambda\|(E) < \infty$ (i.e. λ has finite variation on E), then there is a compact $K \subset E$ and an open $O \supset E$ such that $\|\lambda\|(O-K) < \epsilon$.

If ν is a vector measure we denote its variation on a measurable set E by $\|\nu\|(E)$ and its semi-variation by $|\nu|(E)$ ([5], [6]). The following theorem, an extension of Bochner's theorem, is essential to our work. The proof is given in [3]. We assume Φ is full.

THEOREM 2.12. (A) If ν is a weakly regular Φ -positive vector measure defined on $\Sigma(\hat{G})$ and if

$$(2.13) \quad p(g) = \int_{\hat{G}} (g, \gamma) \nu(d\gamma)$$

then p is an integrally Φ -positive definite element of $L_{\infty}(G, X)$.

(B) If p is an integrally Φ -positive definite element of $L_{\infty}(G, X)$, then there is a set function ν^{**} mapping $\Sigma(\hat{G})$ into X^{**} such that (i) ν^{**} is weak-*-regular, Φ -positive with finite semi-variation and (ii)

$$(2.14) \quad (p(g), \varphi) = \int_{\hat{G}} (g, \gamma) (\varphi, \nu^{**}(d\gamma))$$

for all φ in X^* and almost all g in G .

COROLLARY 2.15. If p is an integrally Φ -positive definite element of $L_{\infty}(G, X)$, then one version of p is ωX -continuous. If p is given by 2.13, where ν is a weakly regular Φ -positive vector measure defined on $\Sigma(\hat{G})$, then p is a continuous map of G into X .

Proof. This follows from the relevant regularity. See also [7].

With the aid of theorem 2.12 we can prove a useful inversion theorem. However, a different version of Bochner's theorem will allow us to establish a sharper theorem. We require first the following

DEFINITION 2.16. p in $L_\infty(G, X)$ is dominated if there exists a finite, regular, positive Borel measure λ , such that

$$(2.17) \quad \left\| \int_G \alpha(g) p(g) \mu(dg) \right\| \leq \int_G |\hat{\alpha}(\gamma)| \lambda(d\gamma)$$

for all α in $L_1(G, \mathbb{C})$, where $\hat{\alpha}$ is the Fourier transform of α , i.e. $\hat{\alpha}(\gamma) = \int_G \overline{(\xi, \gamma)} \alpha(\xi) \mu(d\xi)$.

DEFINITION 2.18. Let Φ be a subset of X . Assume there is a function ϕ_0 mapping K_Φ into $\mathbb{R}^+ \cup \{\infty\}$ in a linear manner such that ϕ_0 is uniformly positive on K_Φ , i.e. there exists $k > 0$ such that $k(x, \phi_0) \geq \|x\|$ for all x in K_Φ . Furthermore assume there is an at most countable sequence $\{\phi_i\}$ in Φ and a sequence $\{c_i\}$ in $\mathbb{R}^+$ such that $(x, \phi_0) = \sum_{i=1}^{\infty} c_i (x, \phi_i)$ for all x in K_Φ . Then we say that the pair (Φ, X) is admissible. We let $K_0 = \{x \in K_\Phi : (x, \phi_0) < \infty\}$.

LEMMA 2.19. If (Φ, X) is admissible, if Φ is full, and if $p \in L_\infty(G, X)$ is integrally Φ -positive definite with $p(0)$ in K_0 , then

$\mathbb{R}^+$ is the set of non-negative real numbers.

p is dominated.

Proof: We note first that $p(0)$ is well defined by corollary 2.15.

Let $\psi(\alpha) = \int_G \alpha(g) p(g) \mu(dg)$ for all α in $L_1(G, C)$, then $(\psi(\alpha), \varphi) = \int_{\hat{G}} \hat{\alpha}(\gamma) (\varphi, \nu^{**}(d\gamma))$ for some weak-* regular, Φ -positive set function ν^{**} given by theorem 2.12. We can actually define $\hat{\psi}(f)$ mapping $C_0(\hat{G})^+$ into X by $(\hat{\psi}(f), \varphi) = \int_{\hat{G}} f(\gamma) (\varphi, \nu^{**}(d\gamma))$. Then $\hat{\psi}$ is a bounded linear map, $\|\hat{\psi}(f)\| \leq \|f\|_{\infty} |\nu^{**}|(\hat{G})$.

If f is in $C_0(\hat{G})$ then $f = f_1 - f_2 + if_3 - if_4$ where f_1 is in $C_0(\hat{G})$, $f_1(\gamma) \geq 0$, and each pair of functions (f_1, f_2) , (f_3, f_4) has disjoint support. Hence $f_1(\gamma) \leq |f(\gamma)|$, and $\hat{\psi}(f_1)$ is in K_{Φ} so that $\|\hat{\psi}(f_1)\| \leq k(\hat{\psi}(f_1), \varphi_0) = k \sum_{j=1}^{\infty} c_j (\hat{\psi}(f_1), \varphi_j) = k \sum_j c_j \int_{\hat{G}} f_1(\gamma) (\varphi_j, \nu^{**}(d\gamma))$. Consider now the set function λ given by $\lambda(E) = \sum_{i=1}^{\infty} c_i (\varphi_i, \nu^{**}(E))$, $E \in \Sigma(\hat{G})$. Then $\lambda(E) \geq 0$ for all E in $\Sigma(\hat{G})$, and also λ is additive. Moreover $\lambda(E) \leq (p(0), \varphi_0) < \infty$ as $p(0)$ is in K_0 .

λ is countably additive because $\lambda(\bigcup_j E_j) = \sum_i \sum_j c_i (\varphi_i, \nu^{**}(E_j)) = \sum_j \sum_i c_i (\varphi_i, \nu^{**}(E_j)) = \sum_j \lambda(E_j)$, if the E_j are disjoint (note that $c_i (\varphi_i, \nu^{**}(E_j)) \geq 0$ for all i, j). Also λ is regular, for given $\epsilon > 0$ and E in $\Sigma(\hat{G})$, there is a number N such that $\sum_{i=N+1}^{\infty} c_i (\varphi_i, \nu^{**}(\hat{G})) < \epsilon/2$ and there is a compact $K \subset E$ and an open $O \supset E$ such that $(\varphi_i, \nu^{**}(O-K)) < \epsilon/2Nc_i$, $i = 1, 2, \dots, N$. Hence $\lambda(O-K) < \epsilon$.

$^+C_0(\hat{G})$ is the space of continuous functions mapping $\hat{G}$ into C , which vanish at ∞ if $\hat{G}$ is only locally compact.

$$\text{Then } \|\hat{\psi}(f)\| \leq \sum_{i=1}^4 \|\hat{\psi}(f_i)\| \leq k \sum_i \int_G \hat{f}_i(\gamma) d\lambda \leq 4k \int_G |\hat{f}(\gamma)| d\lambda.$$

It follows that if $\lambda' = 4k\lambda$ then $\|\psi(\alpha)\| \leq \int_G |\hat{\alpha}(\gamma)| d\lambda'$. This establishes the lemma.

We can now state the alternate version of Bochner's theorem. Assume Φ is full and countable.

THEOREM 2.20. p is a dominated, integrally Φ -positive definite element of $L_\infty(G, X)$ if and only if there is a weakly regular Φ -positive vector measure ν mapping $\Sigma(\hat{G})$ into X such that ν has finite variation, i.e. $\|\nu\|(\hat{G}) < \infty$, and such that

$$(2.21) \quad p(g) = \int_G \hat{g}(\gamma) \nu(d\gamma).$$

For the proof see [3]. In this case, of course, p is continuous by corollary 2.15.

3. Inversion Theorems

If $p \in L_1(G, X)$ we recall that the Fourier transform of p is given by

$$(3.1) \quad \hat{p}(\gamma) = \int_G \hat{g}(\gamma) p(g) \mu(dg).$$

For convenience we let $\mathcal{P} = \{p \in L_\infty(G, X) : p \text{ is integrally } \Phi\text{-positive definite}\}$ and $\mathcal{P}_d = \{p \in \mathcal{P} : p \text{ is dominated}\}$. We recall

that if $p \in \mathcal{P}$ then p is ωX -continuous (corollary 2.15). If (Φ, X) is admissible then $\mathcal{T}_0$ is the set of functions p mapping G into X such that p is ωX -continuous and such that $p(0)$ is in K_0 where K_0 is defined in 2.18.

PROPOSITION 3.2. (A) If $p \in \text{span}\{L_1(G, X) \cap \mathcal{P}\}$ and if $\varphi \in \text{span}\{\Phi\}$ then $(\hat{p}(\cdot), \varphi) \in L_1(\hat{G}, C)$ and (B) if the Haar measure of G is fixed then the Haar measure of $\hat{G}$ can be so normalized that

$$(3.3) \quad (p(g), \varphi) = \int_{\hat{G}} \hat{g}(g, r) (\hat{p}(r), \varphi) m(dr)$$

is valid for all $p \in \text{span}\{L_1(G, X) \cap \mathcal{P}\}$ and all $\varphi \in \text{span}\{\Phi\}$.

Proof: It is evident the results need only hold for $p \in L_1(G, X) \cap \mathcal{P}$, $\varphi \in \Phi$. But this follows from the scalar inversion theorem ([2], p. 22).

A better result is the following.

THEOREM 3.4. Assume Φ is full and countable and (Φ, X) is admissible. (A) If $p \in \text{span}\{L_1(G, X) \cap \mathcal{P} \cap \mathcal{T}_0\}$ then $\hat{p} \in L_1(\hat{G}, X)$, and (B) if μ is fixed then m can be so normalized that

$$(3.5) \quad p(g) = \int_{\hat{G}} \hat{g}(g, r) \hat{p}(r) m(dr)$$

for all p in $\text{span}\{L_1(G, X) \cap \mathcal{P} \cap \mathcal{T}_0\}$ and all g in G .

Proof: Again we need only prove the results for p in $L_1(G, X) \cap \mathcal{P} \cap \mathcal{T}_0$. For such a p and for φ in Φ we have from 3.3 and 2.14

$$(3.6) \quad (p(g), \varphi) = \int_{\hat{G}} (g, r) (\hat{p}(r), \varphi) m(dr) = \int_{\hat{G}} (g, r) (\varphi, \nu^{**}(dr))$$

so that for any $E \in \Sigma(\hat{G})$, $\varphi \in \Phi$ $\int_E (\hat{p}(r), \varphi) m(dr) = (\varphi, \nu^{**}(E)) \geq 0$.

So, in fact, for any $p \in L_1(G, X) \cap \mathcal{P}$ we have

$$(3.7) \quad (\hat{p}(r), \varphi) \geq 0, \varphi \in \Phi, r \in \hat{G}.$$

Now $\infty > (p(0), \varphi_0) = \sum_{i=1}^{\infty} c_i (p(0), \varphi_i) = \sum_{i=1}^{\infty} c_i (\varphi_i, \nu^{**}(\hat{G})) = \sum_{i=1}^{\infty} c_i \int_{\hat{G}} (\hat{p}(r), \varphi_i) m(dr) = \int_{\hat{G}} (\hat{p}(r), \varphi_0) m(dr) \geq \int_{\hat{G}} \|\hat{p}(r)\| m(dr)$ using the monotone convergence theorem, the fact that $\hat{p}(r) \in K_{\Phi}$ for all $r \in \hat{G}$, and the fact that $\hat{p}$ is continuous so $\|\hat{p}(\cdot)\|$ is measurable. As p is measurable and G is σ -finite, then p is essentially separably valued and so is $\hat{p}$. As $\hat{p}$ is also continuous it is measurable. Hence (A) is established.

Now 3.6 yields $(p(g), \varphi) = (\int_{\hat{G}} (g, r) \hat{p}(r) m(dr), \varphi)$ for any $\varphi \in \Phi$ and almost all $g \in G$. As Φ is full and countable we have $p(g) = \int_{\hat{G}} (g, r) \hat{p}(r) m(dr)$ for almost all g . This proves the theorem.

We give now the sharper theorem which does, however, require $\hat{G}$ to be σ -finite.

THEOREM 3.8. Assume Φ is full and countable and $\hat{G}$ is σ -finite.

(A) if $p \in \text{span}\{L_1(G, X) \cap \mathcal{P}_d\}$ then $\hat{p} \in L_1(\hat{G}, X)$, (B) if μ is fixed then m can be so normalized that 3.5 holds for all $p \in \text{span}\{L_1(G, X) \cap \mathcal{P}_d\}$.

Proof: If $p \in L_1(G, X)$ then $\hat{p} \in L_\infty(\hat{G}, X)$. If $p \in \mathcal{P}_d$ also and $p(g) = \int_{\hat{G}} \hat{p}(g, r) \nu(dr)$ as in 2.21, then $\nu(E) = \int_E \hat{p}(r) m(dr)$ for $E \in \Sigma(\hat{G})$ such that $m(E) < \infty$. Then $\|\nu\|(E) = \int_E \|\hat{p}(r)\| m(dr)$ for $m(E) < \infty$, or, for any such E , $\int_E \|\hat{p}(r)\| m(dr) \leq \|\nu\|(\hat{G}) < \infty$ as ν has finite variation. Now $\hat{G}$ is σ -finite so if $\{\hat{G}_n\}$ is a sequence in $\Sigma(\hat{G})$ increasing to $\hat{G}$ then $\int_{\hat{G}} \|\hat{p}(r)\| m(dr) =$

$\lim_{n \rightarrow \infty} \int_{\hat{G}_n} \|\hat{p}(r)\| m(dr) \leq \|\nu\|(\hat{G}) < \infty$. It follows by the monotone convergence theorem that $\hat{p} \in L_1(\hat{G}, X)$. (B) follows readily.

We note that lemma 2.19 and theorem 3.8 give an immediate proof of theorem 3.4 if $\hat{G}$ is σ -finite. Actually theorem 3.4 is the more useful theorem although theorem 3.8 is sharper.

COROLLARY 3.9. If p is given by

$$(3.10) \quad p(g) = \int_{\hat{G}} \hat{p}(g, r) \nu(dr)$$

where ν is a weakly regular Φ -positive vector measure with finite variation and if p is in $L_1(G, X)$, then 3.5 holds.

4. The Plancherel Theorem

As usual this theorem is set in a Hilbert space and so

we must first develop the necessary structure. Assume now that X is a Banach algebra with continuous involution $x \rightarrow x^*$.

DEFINITION 2.14. The triplet (ϕ, X, X_0) is strongly admissible if
 (i) (ϕ, X) is admissible, (ii) X_0 is a non-trivial subspace of X
such that xx^* is in K_0^+ for all x in X , and (iii) there exists
 $k_0 > 0$ such that if $x \in X_0$ then

$$(4.2) \quad k_0 \|xx^*\| \geq \|x\|^2.$$

We note that 4.2 is satisfied if X is a C^* -algebra.

Now we have

PROPOSITION 4.3. If X is a Banach algebra and if (ϕ, X, X_0) is
strongly admissible then X_0 is a Hilbert space under the norm
 $\|\cdot\|_0$ where $\|x\|_0^2 = \langle x, x \rangle_0$ and $\langle x, y \rangle_0 = (xy^*, \phi_0)$.

Proof. ϕ_0 is only defined on K and we do not know that if $x, y \in X_0$ then $xy^* \in K$. However we can extend ϕ_0 by setting $(xy^*, \phi_0) = \sum_{i=1}^{\infty} c_i(xy^*, \phi_i)$ where $\{c_i\}, \{\phi_i\}$ define ϕ_0 on K . Then $|\langle x, y \rangle_0| = |(xy^*, \phi_0)| = \left| \sum_{i=1}^{\infty} c_i(xy^*, \phi_i) \right| \leq \sum_{i=1}^{\infty} c_i(xy^*, \phi_i)^{1/2} (yy^*, \phi_i)^{1/2}$ where the last inequality follows because ϕ_i is a positive functional. Hence we can define $\langle x, y \rangle_0$ for $x, y \in X_0$ and $|\langle x, y \rangle_0| \leq \|x\|_0 \|y\|_0$. It follows from 2.18 and 4.2 that $k_0 \|x\|_0^2 \geq \|x\|^2$ and that $\|\cdot\|_0$ is a norm.

$^+K_0$ is defined in 2.18.

If $\{x_n\}$ is Cauchy in $\|\cdot\|_0$ then it is Cauchy in $\|\cdot\|$, so $x_n \rightarrow x \in X$. As K is closed then $xx^* \in K$. Also $\{x_n\}$ is bounded in $\|\cdot\|_0$ because it is Cauchy, so $\sum_{i=1}^{\infty} c_i(x_n x_n^*, \varphi_i) \leq M$, hence $\sum_{i=1}^{\infty} c_i(xx^*, \varphi_i) \leq M$ or $x \in K_0$. Choose $m(\epsilon)$ such that if $n, m > m(\epsilon)$ then $\|x_n - x_m\|_0 < \epsilon$. Then $\sum_{i=1}^N c_i([x - x_m][x - x_m]^*, \varphi_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^N c_i([x_n - x_m][x_n - x_m]^*, \varphi_i) \leq \limsup_{n \rightarrow \infty} \sum_{i=1}^{\infty} c_i([x_n - x_m][x_n - x_m]^*, \varphi_i) < \epsilon^2$ so that for $m > m(\epsilon)$ $\|x - x_m\|_0 < \epsilon$, or X_0 is a Hilbert space.

If X is a Banach algebra and G is σ -finite, then $L_1(G, X)$ is also a Banach algebra ([8]). If X has the involution $x \rightarrow x^*$, then we can define an involution on $L_1(G, X)$ as $p \rightarrow p^*$ where $p^*(g) = p(-g)^*$.

THEOREM 4.4. If G is σ -finite, X is a Banach algebra with continuous involution, Φ is a full and countable subset of X^* and (Φ, X, X_0) is strongly admissible, then (i) if $\{e_\alpha\}$ is an orthonormal basis for X_0 and there exists k_1 such that $|\langle x, e_\alpha \rangle| \leq k_1 \|x\|$ for $x \in X_0$ and all α , then the Fourier transform maps $L_1(G, X) \cap L_2(G, X_0)$ onto a dense subset of $L_2(\hat{G}, X_0)$, (ii) for $q, r \in L_1(G, X) \cap L_2(G, X_0)$

$$(4.5) \quad \int_G q(g) r(g) \mu(dg) = \int_{\hat{G}} \hat{q}(r) \hat{r}(r) m(dr),$$

(iii) for $q, r \in L_1(G, X) \cap L_2(G, X_0)$

$$(4.6) \quad \langle q, r \rangle = \langle \hat{q}, \hat{r} \rangle,$$

where $\langle q, r \rangle = \int_G \langle q(g), r(g) \rangle \mu(dg)$ and $\langle \hat{q}, \hat{r} \rangle = \int_{\hat{G}} \langle \hat{q}(r), \hat{r}(r) \rangle m(dr)$.

Proof: We shall put $\|q\|_1 = \int_G \|q(g)\| \mu(dg)$ and $\|q\|_2 = \{\int_G \|q(g)\|_0^2 \mu(dg)\}^{1/2}$ for $q \in L_1(G, X) \cap L_2(G, X_0)$. Let $p(g) = (q * q^*)(g)$. As $q \in L_1(G, X)$ so is p with $\|p\|_1 \leq \|q\|_1^2$. It can also be shown that $p \in C_0(G, X_0)^+$ as $q \in L_2(G, X_0)$. Now $p(0) = \int_G q(g)q(g)^* \mu(dg) \in K$ so $(p(0), \varphi_0) = (\int_G q(g)q(g)^* \mu(dg), \varphi_0) = \sum_{i=1}^{\infty} c_i \int_G (q(g)q(g)^*, \varphi_i) \mu(dg) = \int_G (q(g)q(g)^*, \varphi_0) \mu(dg) = \int_G \|q(g)\|_0^2 \mu(dg) = \|q\|_2^2 < \infty$ using the monotone convergence theorem. Hence $p \in L_1(G, X) \cap \mathcal{T}_0$.

Now $C_0(G, X_0) \subset C_0(G, X)$ so $p \in L_{\infty}(G, X)$. Also $\int_G \int_G \alpha(g) \overline{\alpha(g')} p(g-g') \mu(dg) \mu(dg') = \int_G [\int_G \alpha(g) q(g-g'') \mu(dg)] [\int_G \alpha(g') q(g'-g'') \mu(dg')] \mu(dg'') = \int_G q'(g) q'(g)^* \mu(dg)$ using the Fubini and Tonelli theorems with $\alpha \in L_1(G, \mathbb{C})$. $q' = \alpha * q \in L_2(G, X_0)$ ([8]) so $q'(g) \in X_0$ a.e. or $q'(g)q'(g)^* \in K_0$ a.e. Hence if $\varphi \in \mathcal{P}$ then $(\int_G q'(g)q'(g)^* \mu(dg), \varphi) = \int_G (q'(g), q'(g)^*, \varphi) \mu(dg) \geq 0$ or $p \in \mathcal{P}$.

Consequently theorem 3.4 yields $p(g) = \int_{\hat{G}} \langle g, r \rangle \hat{p}(r) m(dr)$. Then $\infty > \|q\|_2^2 = \langle q, q \rangle = \sum_{i=1}^{\infty} c_i (p(0), \varphi_i) = \sum_i c_i \int_{\hat{G}} \langle \hat{p}(r), \varphi_i \rangle m(dr) = \int_{\hat{G}} \langle \hat{p}(r), \varphi_0 \rangle m(dr) = \langle \hat{q}, \hat{q} \rangle$. We have used the monotone convergence theorem again. Hence the Fourier transform maps into $L_2(\hat{G}, X_0)$.

By the usual expansion $\langle q, r \rangle = \langle \hat{q}, \hat{r} \rangle$. This establishes (iii).

⁺If Y is a Banach space then $C_0(G, Y)$ is the space of continuous functions mapping G into Y , which vanish at infinity if G is only locally compact rather than compact.

Moreover $\int_G q(g)q(g)^*\mu(dg) = p(0) = \int_{\hat{G}} \hat{p}(\gamma)m(d\gamma) = \int_{\hat{G}} \hat{q}(\gamma)\hat{q}(\gamma)^*m(d\gamma)$. Also if x, y are elements of a Banach algebra with involution then

$$(4.7) \quad 4xy^* = (x+y)(x+y)^* - (x-y)(x-y)^* + i(x+iy)(x+iy)^* - i(x-iy)(x-iy)^*$$

so that (ii) is also proved.

We need only show that $Q = \{\hat{q} \in L_2(\hat{G}, X_0) : q \text{ in } L_1(G, X) \cap L_2(G, X_0)\}$ is dense in $L_2(\hat{G}, X_0)$. As μ is translation invariant so is $L_1(G, X) \cap L_2(G, X_0)$ and hence Q is invariant under multiplication by $(g, \cdot)$ for any $g \in G$. If $r \in L_2(\hat{G}, X_0)$ and $\langle q, r \rangle = 0$ for all $q \in Q$, then $\int_{\hat{G}} (q(\gamma)r(\gamma)^*, \varphi_0)(g, \gamma)m(d\gamma) = 0$ for all $q \in Q$ and $g \in G$. As $(q(\cdot)r(\cdot)^*, \varphi_0) \in L_1(\hat{G}, C)$ it follows that $(q(\gamma)r(\gamma)^*, \varphi_0) = 0$ a.e. for every $q \in Q$, or $\langle q(\gamma), r(\gamma) \rangle_0 = 0$ a.e. As $L_1(G, X) \cap L_2(G, X_0)$ is invariant under multiplication by $(\cdot, r)$, $r \in \hat{G}$, then Q is invariant under translation.⁺ Hence to every $r_0 \in \hat{G}$ there corresponds $q_0 \in Q$ such that $q_0(r_0) \neq 0$ so $q_0(\gamma) \neq 0$ in a neighborhood of r_0 as q_0 is continuous. If $\{e_\alpha\}$ is the basis of X_0 mentioned in the statement of part (i), then $q_0(\cdot) = \sum_{\alpha} q_{\alpha}(\cdot)e_{\alpha}$ so there exists α_0 such that $q_{\alpha_0}(\gamma) \neq 0$ in a neighborhood of r_0 . If $q_0(\cdot) = \hat{p}(\cdot)$ then $p = \sum_{\alpha} p_{\alpha}e_{\alpha}$ and as $p \in L_2(G, X_0)$, $p_{\alpha} \in L_2(G, C)$. By hypothesis $|\langle x, e_{\alpha} \rangle_0| \leq k_1 \|x\|$ so $p_{\alpha} \in L_1(G, C)$ and $\hat{p}_{\alpha}(\gamma) = q_{\alpha}(\gamma)$. Hence

⁺By this we mean that f_{r_0} is in Q for any r_0 in $\hat{G}$ if f is in Q and $f_{r_0}(\gamma) = f(\gamma + r_0)$.

$p_{\alpha_0}(\cdot)e_\alpha \in L_1(G, X) \cap L_2(G, X_0)$ for any α and $p_{\alpha_0}e_\alpha(\cdot) = q_{\alpha_0}(\cdot)e_\alpha \in Q$. Since for each γ in a neighborhood of γ_0 $\{q_{\alpha_0}(\gamma)e_\alpha\}_\alpha$ forms a complete set in X_0 , and since $0 = \langle q_{\alpha_0}(\gamma)e_\alpha, r(\gamma) \rangle$, then $r(\gamma) = 0$ in a neighborhood of γ_0 . But γ_0 was arbitrary so $r = 0$, or Q is orthogonal only to 0 in $L_2(\hat{G}, X_0)$, a Hilbert space. Hence Q is dense in $L_2(\hat{G}, X_0)$. This completes the proof.

COROLLARY 4.8. Under the assumptions of the theorem the Fourier transform can be extended in a unique manner to an isometry of $L_2(G, X_0)$ onto $L_2(\hat{G}, X_0)$.

Proof: We need only show $L_1(G, X) \cap L_2(G, X_0)$ is dense in $L_2(G, X_0)$. But $C_c(G, X_0)^+$ is dense in $L_2(G, X_0)$ ([7]). Hence if $f \in L_2(G, X_0)$ then there exists $\{f_n\}_1^\infty \subset C_c(G, X_0) \cap L_2(G, X_0)$ such that $\|f_n - f\|_2 \rightarrow 0$. Then $f_n \in C_c(G, X)$ and f_n is measurable so $f_n \in L_1(G, X)$.

Remark: The equality (4.5) holds for all $q, r \in L_2(G, X_0)$.

5. Examples

Here we give some examples of admissible pairs and strongly admissible triplets.

EXAMPLE 5.1. Let $X = L_1([0, 1], C)$ so X is weakly complete, and let Φ consist of elements ϕ_i such that

$C_c(G, X_0)$ denotes the set of functions in $C_0(G, X_0)$ having compact support.

$$(5.2) \quad (x, \varphi_i) = \int_0^1 \chi_i(t) x(t) dt \quad x \in X$$

where $\chi_i(\cdot)$ is the indicator function of one of a countable collection of sets $\{E_i\}$ dense in $\Sigma([0,1])$ under the usual Hausdorff metric. Assume $E_1 = [0,1]$. Then it can be shown ([3], [7]) that Φ is full and that K is the cone of non-negative (a.e.) functions. Let $(x, \varphi_0) = (x, \varphi_1) = \int_0^1 x(s) ds = \|x\|_1$ for $x \in K$. Hence (Φ, X) is admissible and $K_0 = K$.

If p is in $\mathcal{P}$ then $p(0)$ is in $K = K_0$ by propositions 2.8 and 2.9 and by corollary 2.15. So $p \in \mathcal{T}_0$ and the inversion theorem states that if $p \in \text{sp}[L_1(G, L_1([0,1], C)) \cap \mathcal{P}]$ then $\hat{p} \in L_1(\hat{G}, L_1([0,1], C))$ and $p(g) = \int \hat{G}(g, r) \hat{p}(r) m(dr)$.

The author does not know of any non-trivial subspace X_0 which would make (Φ, X, X_0) strongly admissible.

EXAMPLE 5.3. Let $X = H$, a separable Hilbert space with a fixed orthonormal basis $\{e_i\}_1^\infty$. Let H_0 be the set of elements of H with all but a finite number of components zero, with non-zero components being real, rational non-negative, and with norm less than or equal to one. Then $\Phi = H_0$ is full ([3], [7]) and countable and $K_\Phi = \{h \in H: h_i \geq 0\}$.⁺ Let $(h, \varphi_i) = \langle h, e_i \rangle$, $i = 1, 2, \dots$ and $\varphi_0 = \sum_1^\infty \varphi_i$. Then φ_0 maps K into $[0, \infty]$, and for h in K

⁺ $h_i = \langle h, e_i \rangle$

$$(h, \varphi_0)^2 = (\sum h_i)^2 \geq \sum h_i^2 = \|h\|^2$$

so that (Φ, H) is admissible and $K_0 = \{h \in K; \sum_1^\infty h_i < \infty\}$.

H becomes a Banach algebra if we define $hk = \sum_1^\infty h_i k_i e_i$.

Let $h^* = \sum \bar{h}_i e_i$. For h in H hh^* is in K and $(hh^*, \varphi_0) = \sum_i h_i \bar{h}_i = \|h\|^2$. We do not have $k\|hh^*\| \geq \|h\|^2$ for some $k > 0$, but we do have $\|h\|_0 = \|h\|$ which is sufficient to show that $X_0 = H$.

Hence (Φ, H, H) is strongly admissible, and the Plancherel theorem applies. Note that the condition $|\langle h, e_i \rangle| \leq \|h\|$ also holds.

EXAMPLE 5.4. Let $X = \mathcal{L}(H, H)$, the linear bounded operators mapping the separable Hilbert space H into itself. Let H_0 be a countable dense subset of the unit ball in H and let $\Phi = \{\varphi \in X^*: (T, \varphi) = \langle Th, h \rangle, T \in \mathcal{L}(H, H), h \in H_0\}$. Let $\{e_i\}$ also be in H_0 for some orthonormal basis $\{e_i\}$. Then Φ is full and countable and K_Φ is the cone of positive operators ([3] or [7]). Let $(T, \varphi_0) = \sum_1^\infty \langle Te_i, e_i \rangle$. So $\varphi_0 = \sum_1^\infty \varphi_i$ is the trace, where $(T, \varphi_i) = \langle Te_i, e_i \rangle$. Then $\varphi_0: K \rightarrow [0, \infty]$, $(T, \varphi_0) = \text{tr } T \geq \|1\|$ if T is positive. Hence $(\Phi, \mathcal{L}(H, H))$ is admissible and K_Φ is the cone of positive operators of finite trace and so a subset of the trace class.

We can see that in one case the condition $p \in \mathcal{T}_0$ is necessary for the inversion theorem to hold. Let G be the circle group so that $\hat{G}$ is countable. Label its elements $\gamma_1, \gamma_2, \dots$, and let the set function ν be given by

$$(5.5) \quad \langle \nu(\{r_n\})e_i, e_j \rangle = p_n \delta_{ni} \delta_{nj}^+, \quad i, j, n = 1, 2, \dots$$

A

where $\infty > M \geq p_n \geq 0$. ν can be extended to a countably additive measure of finite semi-variation in the obvious way. Let p be given by

$$(5.6) \quad p(t) = \sum_{n=1}^{\infty} e^{it r_n} \nu(\{r_n\})$$

Then p is in $\mathcal{P}$ (theorem 2.12 (A)) and p is in $L_1(G, X)$ because G is compact and $\|p(t)\| \leq M$. If $\hat{p}$ is to be in $L_1(\hat{G}, X)$ then $\|\nu\|(\hat{G})$ must be finite or $\sum_{n=1}^{\infty} p_n = \text{tr } p(0) < \infty$.

Finally let $X_0 = \mathcal{N}$, the Hilbert-Schmidt operators ([5a]).

Then for T in $\mathcal{N}$, TT^* is in the trace class and is positive so that TT^* is in K_0 . Also $\mathcal{L}(H, H)$ is a C^* -algebra so $(\Phi, \mathcal{L}(H, H), \mathcal{N})$ is strongly admissible. A basis for $\mathcal{N}$ is given by $\{T_{ij}\}$ where $\langle T_{ij}e_k, e_\ell \rangle = \delta_{ik} \delta_{j\ell}$, $k, \ell = 1, 2, \dots$. Then $|\langle T, T_{ij} \rangle_0| = |\langle Te_i, e_j \rangle| \leq \|T\|$, and the condition in (i) of theorem 4.4 also holds.

6. On a Theorem of Magnus

We use the preceding theory to deduce a result of Hewitt and Wigner's ([4]). Let $U(\cdot)$ be a continuous n -dimensional unitary representation of G i.e. $U(g+g') = U(g)U(g')$, $U(0) = I$ and U is a continuous mapping of G into $\mathcal{L}(C^n, C^n)$. Then there is a unitary matrix V and characters $r_1, \dots, r_n$ such that

$^+ \delta_{ni}$ is the Kronecker delta.

$$(6.1) \quad U(g) = V^{-1} \begin{bmatrix} (g, r_1) & & 0 \\ & \ddots & \\ 0 & & (g, r_n) \end{bmatrix} V$$

for all g in G ([9]). Hence $U(\cdot)$ is given as a function of n characters. Let A be a symmetric compact neighborhood of 0 in $\hat{G}$ having finite positive measure, and let $E(r_1, \dots, r_n)$ be the function on $\hat{G}^n$ which equals 1 if $r_j - r_k$ is in A for all j, k , and equals zero otherwise. Let p be in $L_1(G, \mathcal{L}(C^n, C^n))$ and let

$$(6.2) \quad \hat{p}(U) = \int_G p(g) U(g) \mu(dg)$$

Theorem 6.3. If p is in $L_1(G, \mathcal{L}(C^n, C^n)) \cap L_\infty(G, \mathcal{L}(C^n, C^n))$ and if for any α in $L_1(G, C)$ $\int_G \int_G \alpha(g) \overline{\alpha(g')} p(g-g') \mu(dg) \mu(dg')$ is positive semidefinite, then there is a constant κ , $0 < \kappa < \infty$ such that

$$(6.4) \quad \kappa p(g) = \int_{\hat{G}^n} \hat{p}(U) U(-g) E(r_1, \dots, r_n) m(dr_1) \dots m(dr_n).$$

Proof: Using the setting of example 5.4 with $H = C^n$ we have $p \in \mathcal{P}$. Then p is continuous and trace $p(0) < \infty$ so that p is in $\mathcal{T}_0$. Hence $p(g) = \int_{\hat{G}} (g, r) \hat{p}(r) m(dr)$ by theorem 3.4. But $U(g) = \sum_{i=1}^n \overline{(g, r_i)} \pi_i$ where the π_i are projections onto mutually orthogonal

one-dimensional subspaces ([9]). Then $\hat{p}(U) = \int_G p(g) U(g) \mu(dg) = \sum \hat{p}(r_i) \pi_i$ and $\hat{p}(U) U(-g) = \sum_{i=1}^n \hat{p}(r_i) \pi_i \sum_{j=1}^n (g, r_j) \pi_j = \sum_{i=1}^n \hat{p}(r_i) (g, r_i) \pi_i$.

Now let $\kappa = \int_{\hat{G}^{n-1}} E(r_1, \dots, r_n) m(dr_1) \dots m(dr_n)$ where we omit integration with respect to r_i . Then κ is independent of i and of the value of r_i by the choice of E , and $0 < \kappa \leq m(A)^{n-1} < \infty$. Now we have

$$\begin{aligned} & \int_{\hat{G}^n} \hat{p}(U) U(-g) E(r_1, \dots, r_n) m(dr_1) \dots m(dr_n) \\ &= \sum_{i=1}^n \int_{\hat{G}^n} \hat{p}(r_i) (g, r_i) E(r_1, \dots, r_n) m(dr_1) \dots m(dr_n) \pi_i \\ &= \sum_{i=1}^n \kappa \int_{\hat{G}} \hat{p}(r_i) (g, r_i) m(dr_i) \pi_i \\ &= \kappa p(g) \end{aligned}$$

and the theorem is established.

Note that p could actually be a finite linear combination of functions satisfying the requirements of the theorem. The extension of this theorem to infinite dimensions will be treated elsewhere.

The other result of [4] is

THEOREM 6.5. If p is in $L_1(G, \mathcal{L}(C^n, C^n)) \cap L_2(G, \mathcal{L}(C^n, C^n))$, then

$$(6.6) \quad \kappa \text{ trace } \int_G p(g) p(g)^* \mu(dg) =$$

$$\text{trace } \int_{\hat{G}^n} \hat{p}(U) \hat{p}(U)^* E(r_1, \dots, r_n) m(dr_1) \dots m(dr_n)$$

Proof: The method of proof is similar to the one given above but uses Plancherel's theorem which yields

$$\int_G p(g) p(g)^* \mu(dg) = \int_{\hat{G}} \hat{p}(\gamma) \hat{p}(\gamma)^* m(d\gamma).$$

7. The Maximal Ideals of $L_2(G, \mathcal{N})$

Assume G is a compact abelian group and $\mathcal{N}$ is the space of Hilbert-Schmidt operators in $\mathcal{L}(H, H)$, where H is a separable Hilbert space. We show that the (closed) maximal ideals of $L_2(G, \mathcal{N})$ correspond to $\hat{G}$.

LEMMA 7.1. If h is a continuous $*$ -homomorphism⁺ of $L_2(G, \mathcal{N})$ onto $\mathcal{N}$, then $M = \text{kernel}(h)$ is a maximal closed self-adjoint ideal such that $M^\perp$ is isometrically $*$ -isomorphic to $\mathcal{N}$.

Proof: h is continuous so M is a closed 2-sided ideal. M is self adjoint as h is a $*$ -homomorphism. $\mathcal{N}$ is a full matrix algebra so it is a simple H^* -algebra. Also $\mathcal{N}$ and $L_2(G, \mathcal{N})/M$ i.e. $M^\perp$ are homeomorphically $*$ -isomorphic ([9], p. 181), so $M^\perp$ is a minimal closed ideal and M is a maximal ideal.

Note that if H is infinite dimensional, then M is not regular for if it were there would exist $p \in L_2(G, \mathcal{N})$ such that for all $q \in L_2(G, \mathcal{N})$ $p * q - q \in M$ or $h(p)h(q) = h(q)$. This means $h(p)$ would be an identity in $\mathcal{N}$, but the identity $I \in \mathcal{L}(H, H)$ is not in

⁺If the isomorphism takes $x \rightarrow T$ then $x^* \rightarrow T^*$ the adjoint of T .

$\mathcal{N}$. If H is finite dimensional then such a p exists as I is in $\mathcal{N}$ and h is onto.

THEOREM 7.2. To every γ in $\hat{G}$ there corresponds a closed maximal ideal M of $L_2(G, \mathcal{N})$ given by $M = \text{kernel}(h)$ and

$$(7.3) \quad h(p) = \hat{p}(\gamma).$$

Proof. Fix γ in $\hat{G}$ and define h by 7.3. As G is compact then $L_2(G, \mathcal{N}) \subset L_1(G, \mathcal{N})$ and so $\hat{p}$ is given by the usual integral if p is in $L_2(G, \mathcal{N})$. Direct computation shows h is a $*$ -homomorphism with norm less than or equal to 1. We need only show h is onto, then the result follows by the preceding lemma.

Given U in $\mathcal{N}$ let $q(\lambda) = \begin{cases} U & \lambda = \gamma \\ 0 & \lambda \neq \gamma \end{cases}$. Then q is in $L_2(\hat{G}, \mathcal{N})$. (N.B. $\hat{G}$ is discrete, $m(\{\gamma\}) = 1$). By the Plancherel theorem there exists p in $L_2(G, \mathcal{N})$ such that $\hat{p} = q$. Then $\hat{p}(\gamma) = U$ or h is onto.

Observe that $M = \ker(h) = \{p: \hat{p}(\gamma) = 0\}$ or $M = \{p \in L_2(G, \mathcal{N}): \langle p(\cdot)e_j, e_i \rangle = p_{ij}(\cdot) \in M_\gamma, i, j = 1, \dots\}$ where M_γ is the ideal in $L_2(G, \mathbb{C})$ corresponding to γ in $\hat{G}$. (N.B. There is a 1-1 correspondence between $\hat{G}$ and the maximal ideals of $L_2(G, \mathbb{C})$ ([1]).) As $\|p\|_2^2 = \int_G \sum_{ij} |p_{ij}(g)|^2 \mu(dg) = \sum_{ij} \|p_{ij}\|_2^2$, we can say that $M = M_\gamma \times M_\gamma \times \dots$

LEMMA 7.4. Let P_{ij} be the projection of $\mathcal{N}$ onto the ij^{th} basis element. If M is an ideal in $L_2(G, \mathcal{N})$ then $P_{ij}M$ is an ideal in $L_2(G, \mathbb{C})$.

Proof: The basis elements b_{ij} of $\mathcal{N}$ are determined by $\langle b_{ij}e_r, e_s \rangle = \delta_{is}\delta_{jr}$ where $\{e_r\}$ is a fixed basis of H . Then $P_{ij}p = \langle p, b_{ij} \rangle_0 = \sum_{\ell} \langle pb_{ij}^* e_{\ell}, e_{\ell} \rangle = \sum_{rs} p_{rs} (b_{ij})_{rs} = p_{ij}$. Let α be an element of $L_2(G, \mathbb{C})$ and set $q(\cdot) = \alpha(\cdot)b_{jj}$. Then $p*q$ is in M if p is in M as M is an ideal. Hence

$$P_{ij}(p*q) = \sum_{\ell} p_{i\ell} * q_{\ell j} = \sum_{\ell} p_{i\ell} * \alpha \delta_{\ell j} = p_{ij} * \alpha$$

or $p_{ij} * \alpha$ is in $P_{ij}M$.

Let us write $P_{ij}M = M_{ij}$.

LEMMA 7.5. If M is a closed ideal in $L_2(G, \mathcal{N})$ then the M_{ij} are all identical and closed.

Proof: First we show M_{ij} is closed. Assume $\alpha_n \in M_{ij}$, $\alpha_n \rightarrow \alpha$. Then $\alpha_n b_{ij} \rightarrow \alpha b_{ij}$. Let $\alpha_n b_{ij} = p_n \in L_2(G, \mathcal{N})$. Then $P_{rs}(p_n) = 0$ for $r \neq i$, $s \neq j$, and $0 \in M_{rs}$, so $P_{rs}(p_n) \in M_{rs}$ for all r, s . Hence p_n is in M . Now $\|p_n - p_m\|_2 = \|\alpha_n - \alpha_m\|_2$ and so p_n is Cauchy, hence converges to an element p of M as M is closed. Also $P_{ij}(p_n) \rightarrow P_{ij}(p)$ by continuity of projections. Hence $\alpha = P_{ij}(p) \in M_{ij}$, or M_{ij} is closed.

Now if $p \in M$, $q \in L_2(G, \mathcal{N})$ then $p * q \in M$ and $P_{ij}(p * q) \in M_{ij}$ or $\sum_{\ell} q_{i\ell} * p_{\ell j} \in M_{ij}$. But $q_{i\ell} * p_{\ell j} \in M_{\ell j}$ as $p_{\ell j} \in M_{\ell j}$, so if φ_{ϵ} in $L_2(G, C)$ is an approximate identity for $L_2(G, C)$, ([1]), and if $q = \varphi_{\epsilon}(\cdot) b_{ik}$, then the dense subset $\{\varphi_{\epsilon} * p_{kj} : p \in M, \epsilon = \frac{1}{n}, n = 1, 2, \dots\}$ of M_{kj} is also a subset of M_{ij} . As M_{ij} is closed then $M_{kj} \subset M_{ij}$ for any i, j, k . So $M_{ij} = M_{kj}$ for any i, k, j . Now using $q * p$ we obtain $M_{ij} = M_{ik}$ for any i, k, j , so the M_{ij} are all identical.

THEOREM 7.6. There is a 1-1 correspondence between the closed maximal ideals of $L_2(G, \mathcal{N})$ and $\hat{G}$, i.e. the regular maximal ideals of $L_2(G, C)$ or $L_1(G, C)$. This correspondence is given by

$$(7.7) \quad M_{\gamma} = \{p \in L_2(G, \mathcal{N}) : \hat{p}(\gamma) = 0\}.$$

Proof: By Theorem 7.4 we know every γ corresponds to a closed maximal ideal in $L_2(G, \mathcal{N})$ and 7.7 describes this correspondence. Conversely if $M \in L_2(G, \mathcal{N})$ is closed, maximal then there exists M_0 , a closed ideal in $L_2(G, C)$ and $M = \{p \in L_2(G, \mathcal{N}) : p_{ij} \in M_0\}$. As M_0 is a closed ideal it can be written as

$$M_0 = \sum_{i \in I} \oplus N_i = \bigcap_{i \notin I} N_i^{\perp}$$

where $\{N_i\}_1^{\infty}$ are the minimal ideals of $L_2(G, C)$ and $N_i \subset M_0$ for

$i \in I$ ([1]) and $N_i^\perp$, the orthogonal complement of N_i , is a regular maximal ideal. If M_0 is not a regular maximal ideal, then $M_0 \subset M_1$, $M_0 \neq M_1$ where M_1 is a regular maximal ideal. But then γ corresponding to M_1 gives rise to a closed maximal ideal $\tilde{M} \in L_2(G, \mathcal{N})$ and $\tilde{M} \supset M$, $\tilde{M} \neq M$. This contradicts the maximality of M . Hence M_0 is a regular maximal ideal, and moreover, 7.7 holds. This proves the theorem.

We note that if H is infinite dimensional then $\mathcal{N}$ is, and none of the closed maximal ideals are regular, whereas if H is finite dimensional all are. We also note that if H has dimension $n < \infty$ then by an argument similar to the one in [1], page 161, the closed ideals of $L_1(G, \mathcal{L}(H, H)) = L_1(G, \mathcal{N})$ correspond in a one to one fashion to the closed ideals of $L_2(G, \mathcal{N})$ and so the maximal ideals of $L_1(G, \mathcal{L}(H, H))$ can be studied through the transform on $\hat{G}$. Unfortunately we cannot prove this for non-compact groups.

8. Convolution Equations for Operators

The above theory can be used to solve operator integral equations much as in the scalar case. Let G be a locally compact, abelian, σ -finite group, H be a separable Hilbert space, and $\mathcal{L}(H, H)$, $\mathcal{N}$ be as before.

PROPOSITION 8.1. If $q \in L_2(G, \mathcal{N})$, $p \in L_1(G, \mathcal{L}(H, H))$ then $p * q \in L_2(G, \mathcal{N})$ and $\|p * q\|_2 \leq \|p\|_1 \|q\|_2$.

Proof: This is straightforward and will be omitted. See also [7].

Consider now

$\mathcal{A}$

$$(8.2) \quad q(g) = \int_G p(g^{-1}g') q(g') \mu(dg') + r(g)$$

or equivalently

$$(8.3) \quad q = p * q + r$$

where $p \in L_1(G, \mathcal{L}(H, H))$, $r \in L_2(G, \mathcal{N})$. We are looking for solutions q of 8.3 in $L_2(G, \mathcal{N})$.

THEOREM 8.4. If r is in $L_2(G, \mathcal{N})$, p is in $L_1(G, \mathcal{L}(H, H))$ and if $\sup_{r \in \hat{G}} \|\hat{p}(r)\| < 1$ then 8.3 has a solution in $L_2(G, \mathcal{N})$.

Note that $\|p\|_1 \geq \|\hat{p}(r)\|$, $r \in \hat{G}$.

Proof: Consider $I - \hat{p}(r)$. As $\|\hat{p}(r)\| < 1$ we know that $(I - \hat{p}(r))^{-1}$ exists for each $r \in \hat{G}$ and $\|(I - \hat{p}(r))^{-1}\| \leq (1 - \|\hat{p}(r)\|)^{-1}$. It follows $(I - \hat{p}(\cdot))^{-1} \in L_\infty(\hat{G}, \mathcal{L}(H, H))$ and so $\|(I - \hat{p}(\cdot))^{-1} \hat{r}(\cdot)\|_2 \leq \|(I - \hat{p}(\cdot))^{-1}\|_\infty \|\hat{r}(\cdot)\|_2$. Hence there exists $q \in L_2(G, \mathcal{N})$ such that $\hat{q}(\cdot) = (I - \hat{p}(\cdot))^{-1} \hat{r}(\cdot)$ by the Plancherel theorem. Let $w(g) = (p * q)(g)$ so $w \in L_2(G, \mathcal{N})$ by proposition 8.1. It can be shown by an approximation argument that $p * q(r) = \hat{p}(r) \hat{q}(r)$. Then $\widehat{r+w} = \hat{r} + \hat{w} = \hat{r} + \hat{p} \hat{q} = (I + \hat{p}(\cdot))(I - \hat{p}(\cdot))^{-1} \hat{r} = (I - \hat{p})^{-1} \hat{r} = \hat{q}$. Hence q satisfies 8.3.

COROLLARY 8.5. The above solution is unique in $L_2(G, \mathcal{N})$.

Proof: If q_0 is any other solution of (8.3) in $L_2(G, \mathcal{N})$ then $\hat{q}_0 = \hat{r} + \hat{p}\hat{q}_0$ so $\hat{q}_0 = (I - \hat{p})^{-1}\hat{r} = \hat{q}$ or $q_0 = q$.

We wish to extend the above theorem to cases where $\|p\|_1 \geq 1$. This can be done by utilizing some results due to Falt and Freedman ([8]). Let W be the set of all continuous linear operators Z mapping $L_2(G, \mathcal{N})$ into itself such that there is a uniformly continuous function $z(\cdot)$ mapping $\hat{G}$ into $\mathcal{L}(\mathcal{N}, \mathcal{N})$ with $Zp(r) = z(r)\hat{p}(r)$ for all r in $\hat{G}$, all p in $L_2(G, \mathcal{N})$. We use the norm

$$(8.6) \quad \|Z\|_W = \sup_{r \in \hat{G}} \|z(r)\|_{\mathcal{L}(\mathcal{N}, \mathcal{N})}$$

where $\|x\|_{\mathcal{N}}^2 = \sum_i \|x e_i\|^2$ for x in $\mathcal{N}$. For p in $L_2(G, \mathcal{N})$ $p(g) = \hat{p}(-g)$ for almost all g in G . Also $\mathcal{N}$ is a B-algebra so W is a B-algebra by the same proof as in [8].

Let B be given by

$$(8.7) \quad B = \{T \in \mathcal{L}(L_2(G, H), L_2(G, H)) : Tx(g) = \int_G p(g-g')x(g')dg' + \lambda x(g) \text{ for some } p \in L_1(G, \mathcal{L}(H, H)) \text{ and } \lambda \in \mathbb{C}\}.$$

We see that under the norm $\|\cdot\|_B$, given by $\|T\|_B = \|p\|_1 + |\lambda|$, B becomes a Banach space, in fact a B-algebra isometrically isomorphic

to $L_1(G, \mathcal{L}(H, H)) \oplus \mathbb{C}$. Also if $T = (p, \lambda)$ then $\hat{T}(\gamma) = \hat{p}(\gamma) + \lambda I$.

We shall now identify B with $\tilde{B}$, a B -algebra of linear operators of $L_2(G, \mathcal{H})$ into itself. For h in H , p in $L_2(G, \mathcal{H})$, g in G , and T in B let $\tilde{T}$ be defined by

$$(8.8) \quad (\tilde{T}p)(g)h = T(p(\cdot)h)(g),$$

so if $T = (q, \lambda)$ then $\tilde{T}p = q * p + \lambda p$ and $\|\tilde{T}\| = \|T\|_B$. Hence $\tilde{B}$ and B are isometrically isomorphic (in the algebra sense). As $\tilde{T}p = \hat{q}\hat{p} + \lambda\hat{p}$ by proposition 8.1, we have $\tilde{B} \subset W$, although the norms are different.

Let $\mathcal{M}$ be the maximal ideal space of $L_1(G, \mathbb{C}) \oplus \mathbb{C}$ (or just $L_1(G, \mathbb{C})$ if G is discrete), so we can put $\mathcal{M} \cong \hat{G} \cup \{\infty\}$, the one point compactification of $\hat{G}$. Then define $\sigma(\hat{T}(\gamma)) = \{\lambda: \hat{T}(\gamma) - \lambda I \text{ does not have an inverse in } \mathcal{L}(H, H)\}$. Also $\Sigma_{\tilde{B}}(\tilde{T}) = \{\lambda: \tilde{T} - \lambda \text{ does not have an inverse in } \tilde{B}\}$, $\Sigma_W(Z) = \{\lambda: Z - \lambda \text{ does not have an inverse in } W\}$ and $\Sigma_B(T) = \{\lambda: T - \lambda \text{ does not have an inverse in } B\}$. Evidently $\Sigma_B(T) = \Sigma_{\tilde{B}}(\tilde{T})$. As B , W and $\tilde{B}$ have identities then $T - \lambda$, $Z - \lambda$ and $\tilde{T} - \lambda$ are defined for λ in $\mathbb{C}$.

DEFINITION 8.9. Let T be in B and let $\{e_i\}$ be an orthonormal basis of H . Let $H_n = \text{span}\{e_1, \dots, e_n\}$ and let E_n be the projection of H onto H_n . Then $T_n = E_n T E_n$ is in B and T is approximable if $\hat{T}_n(\gamma)$ converges to $\hat{T}(\gamma)$ uniformly on $\hat{G} \cup \{\infty\}$.

PROPOSITION 8.10. T in B is approximable if and only if each $\hat{T}(\gamma)$ is a completely continuous element of $\mathcal{L}(H, H)$ for each γ in $\hat{G} \cup \{\infty\}$, and the map $\gamma \rightarrow \hat{T}(\gamma)$ is continuous on $\hat{G} \cup \{\infty\}$.

Proof: See [8].

Now we have

THEOREM 8.11. If T in B is approximable, then $\Sigma_W(\tilde{T}) \subset \bigcup_{\gamma \in \hat{G} \cup \{\infty\}} \sigma(\hat{T}(\gamma)) \subset \Sigma_B(T) = \Sigma_B(\tilde{T})$.

Proof: The proof is the same as that given in [8] for $L_2(G, H)$ rather than $L_2(G, \mathcal{N})$. We need only note if $x \in \mathcal{N}$, $A \in \mathcal{L}(H, H)$ then $\|Ax\|_{\mathcal{N}} \leq \|A\| \|x\|_{\mathcal{N}}$ so $A \in \mathcal{L}(\mathcal{N}, \mathcal{N})$ and in fact $\|A\|_{\mathcal{L}(H, H)} = \|A\|_{\mathcal{L}(\mathcal{N}, \mathcal{N})}$ so that $\sup_{\gamma \in \hat{G}} \|\hat{T}(\gamma)\|_{\mathcal{L}(H, H)} = \sup_{\gamma \in \hat{G}} \|\hat{T}(\gamma)\|_{\mathcal{L}(\mathcal{N}, \mathcal{N})}$. For more details see [7] and [8].

We say p in $L_1(G, \mathcal{L}(H, H))$ is approximable if the corresponding element $(p, 0)$ in B is.

THEOREM 8.12. Let p in $L_1(G, \mathcal{L}(H, H))$ be approximable, let r be in $L_2(G, \mathcal{N})$, and let $1 \notin D$, a domain containing $\bigcup_{\gamma \in \hat{G}} \sigma(\hat{p}(\gamma))$ in its interior. Then 8.3 has a unique solution in $L_2(G, \mathcal{N})$.

REMARK. We note first that if $p \in L_1(G, \mathcal{L}(H, H))$ then $\hat{p}$ is in $C_0(\hat{G}, \mathcal{L}(H, H))$ so $\hat{p}(\infty) = 0$. Hence p is approximable if and only if $\hat{p}(\gamma)$ is a completely continuous element of $\mathcal{L}(H, H)$ for every $\gamma \in \hat{G}$.

Proof: $\bigcup_{\gamma \in \hat{G} \cup \{\infty\}} \sigma(\hat{p}(\gamma)) = \bigcup_{\gamma \in \hat{G}} \sigma(\hat{p}(\gamma)) \cup \{0\}$ so we can extend D to D' such that $1 \notin D'$ and D' contains $\bigcup_{\gamma \in \hat{G} \cup \{\infty\}} \sigma(\hat{p}(\gamma))$ in its interior. Now we can define $F(p) \in W$ where $F(t) = (1-t)^{-1}$ is analytic on D' , a domain containing $\Sigma_W(p)$. If Δ is the identity in W then $F(p) = (\Delta - p)^{-1} \in W$ and $F(\hat{p}(\gamma)) = (I - \hat{p}(\gamma))^{-1}$, $\gamma \in \hat{G}$, ([9], page 203). If Γ is a simple closed rectifiable curve enclosing $\bigcup_{\gamma \in \hat{G} \cup \{\infty\}} \sigma(\hat{p}(\gamma))$ in D' then we have for $x \in L_2(G, \mathcal{N})$

$$\begin{aligned}
 (8.13) \quad \widehat{F(p)x}(\gamma) &= \frac{1}{2\pi i} \int_{\Gamma} F(t) (t\Delta - p)^{-1} dt \, x(\gamma) \\
 &= \frac{1}{2\pi i} \int_{\Gamma} F(t) (t\Delta - p)^{-1} \widehat{x}(\gamma) dt \\
 &= \frac{1}{2\pi i} \int_{\Gamma} F(t) (t - \hat{p}(\gamma))^{-1} dt \, \hat{x}(\gamma) \\
 &= F(\hat{p}(\gamma)) \hat{x}(\gamma).
 \end{aligned}$$

Hence if $r \in L_2(G, \mathcal{N})$ and $q = F(p)r \in L_2(G, \mathcal{N})$ then $\hat{q}(r) = F(\hat{p}(r))\hat{r}(\gamma) = (I - \hat{p}(\gamma))^{-1}\hat{r}(\gamma)$. Consequently $r + p^*q = \hat{r} + \hat{p}\hat{q} = \hat{r} + \hat{p}(I - \hat{p})^{-1}\hat{r} = (I - \hat{p})^{-1}\hat{r} = \hat{q}$ so by the Plancherel theorem q is a solution of 8.3.

Uniqueness can be proved by the method of corollary 8.5.

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