NASA TECHNICAL NOTE

NASA TN D-5381

c. 1

NASA TN D-5381

APPROXIMATION TO THE STATISTICS OF MIDCOURSE VELOCITY CORRECTIONS

by Lawrence H. Hoffman and George R. Young

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • SEPTEMBER 1969

APPROXIMATION TO THE STATISTICS OF MIDCOURSE VELOCITY CORRECTIONS

By Lawrence H. Hoffman and George R. Young

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

APPROXIMATION TO THE STATISTICS OF MIDCOURSE VELOCITY CORRECTIONS

By Lawrence H. Hoffman and George R. Young
Langley Research Center

SUMMARY

A simple approximation for the mean and standard deviation of the magnitude of a statistical velocity correction vector is derived. Approximate methods are presented for making probability statements about the velocity correction vector. These methods are shown to be adequate for calculating the fuel budget for a velocity correction. Comparisons are made with exact results obtained by numerical integration. Results obtained by the root-sum-square method of determining a fuel budget are also compared with the exact numerically integrated results.

The data presented indicate that the probability density function of the magnitude of a velocity correction vector can be fitted accurately with a Gamma distribution having an integer shape parameter, and thus probability statements about the magnitude can be made. The accuracy of the Gamma distribution fit is examined for a number of representative covariance matrices.

INTRODUCTION

Estimating the amount of fuel required for a midcourse velocity correction maneuver is a problem of considerable importance for many types of space missions. Overestimation of the required fuel budget may result in overly large fuel systems and thereby reduce usable payload and mission utility. Conversely, underestimation may result in compromising mission reliability.

In this paper an approximation to the statistical magnitude of a velocity correction is obtained. It is assumed herein that velocity corrections are impulsive and can be represented by a trivariate, normally distributed, random vector with zero component means and with a specified covariance matrix. Justification of the normal-distribution assumption may be inferred from the following observations:

(1) Injection-error sources are usually distributed normally.

(2) The nonstandard trajectories differ only slightly from the nominal; hence, linear methods may be used for error propagation.

(3) The velocity correction is usually the result of a linear function of the state at midcourse time.

(4) Linear mapping of normally distributed variables results in normal variables.

The reader at this point may ask why it is necessary to approximate at all. Why not write the equation for the variable of interest (the magnitude of the velocity correction vector) and use the standard methods of transforming random variables to obtain the probability density and cumulative distribution functions? The answer is that the equations are intractable and have only been solved symbolically for the general case (ref. 1).

Several methods have been used to make probability statements about the magnitude of a velocity correction vector. Two of these, the Monte Carlo method (refs. 2 and 3) and numerical integration, can be made as accurate as desired if a computer is employed. A third method is to use the distribution of the magnitude of a random variable that is normally distributed with zero mean and with a standard deviation which is the root-sum-square value of the three standard deviations for the three-dimensional distribution. This distribution, alternately known as Chi, half-normal, or generalized Rayleigh, is referred to herein as Rayleigh and is compared with the results of the present paper.

SYMBOLS

a	constant
a_1,a_2,a_3	coefficients of polynomial
b	constant representing upper limit of integration
$E(\ )$	expected value of ( )
e	base of natural logarithm
f	function of V_x^*,V_y^*,V_z^*
$g(\overline{V})$	normal density function of V_x,V_y,V_z
$g(V_x^*,V_y^*,V_z^*)$	normal density function of transformed variables
$h(\lvert\overline{V}\rvert)$	Gamma density function
i,j,k,n	integers

$K(\lambda)$	polynomial in λ
$P(\ )$	probability of ( )
r	radius
T	trace of midcourse maneuver matrix
V_x, V_y, V_z	components of $\overline{V}$
V'_x, V'_y, V'_z	components of $\overline{V}'$
V^*_x, V^*_y, V^*_z	transformed variables
$\overline{V}$	midcourse maneuver vector (velocity to be gained)
$\overline{V}'$	midcourse maneuver vector before diagonalization of covariance matrix
α, β	parameters in Gamma density
$\Gamma(\ )$	Gamma function of ( )
θ, ϕ	polar coordinates
$\lambda, \lambda_1, \lambda_2, \lambda_3$	eigenvalues
$\mu_{\lvert\overline{V}\rvert}$	mean of $\lvert\overline{V}\rvert$
μ_x, μ_y, μ_z	means of components of $\overline{V}$
ν_{ijk}	ijkth moment of transformed variables about the origin
$\sigma_{\lvert\overline{V}\rvert}$	standard deviation of $\lvert\overline{V}\rvert$
$(\ )^T$	transpose of ( )
$\lvert\ \rvert$	magnitude

ANALYSIS

Calculation of the Moments of $\left|\overline{V}\right|$

In the formulation of most fuel budget problems, one normally has been given or has calculated a velocity-to-be-gained covariance matrix of the form

$$E\left[\overline{V}'\left(\overline{V}'\right)^T\right] = \begin{bmatrix} E\left(V_x'V_x'\right) & E\left(V_x'V_y'\right) & E\left(V_x'V_z'\right) \\ E\left(V_y'V_x'\right) & E\left(V_y'V_y'\right) & E\left(V_y'V_z'\right) \\ E\left(V_z'V_x'\right) & E\left(V_z'V_y'\right) & E\left(V_z'V_z'\right) \end{bmatrix} \tag{1}$$

In order to get the mean and variance of the magnitude of $\overline{V}'$, it is necessary to obtain the first and second moments about the origin. Thus, the problem herein is to determine the moments about the origin of $\left|\overline{V}'\right|$. One can, without loss of generality (see app.), apply a similarity transformation to equation (1) and work with the diagonalized matrix

$$E\left(\overline{V}\,\overline{V}^T\right) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2}$$

where the λ's are eigenvalues of the matrix in equation (1). The magnitude of the vector $\overline{V}$ can be written as

$$\left|\overline{V}\right| = \left(V_x^2 + V_y^2 + V_z^2\right)^{1/2} \tag{3}$$

The moments of $\left|\overline{V}\right|$ are required. The vector $\overline{V}$ has a three-dimensional normal distribution which can be written as

$$g\left(\overline{V}\right) = \frac{(2\pi)^{-3/2}}{\sqrt{\lambda_1\lambda_2\lambda_3}}\, e^{-\frac{1}{2}\left[\frac{\left(V_x-\mu_x\right)^2}{\lambda_1} + \frac{\left(V_y-\mu_y\right)^2}{\lambda_2} + \frac{\left(V_z-\mu_z\right)^2}{\lambda_3}\right]} \tag{4}$$

where

$$\overline{V} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} \qquad \left(-\infty < V_x, V_y, V_z < \infty\right)$$

and where μ_x, μ_y, and μ_z are the means of the components of $\overline{V}$.

The most direct way to find the nth moment of $|\overline{V}|$ is to multiply $g(\overline{V})$ from equation (4) by $|\overline{V}|^n$ and integrate from $-\infty$ to ∞ in all three variables. After finding the first and second moments about the origin, the standard deviation can be found from

$$\sigma^2_{|\overline{V}|} = E\left(|\overline{V}|^2\right) - E\left(|\overline{V}|\right)^2 \tag{5}$$

Unfortunately, the form of equation (3) prohibits the integration of the product of equations (3) and (4) in general, and recourse must be made to numerical integration (see ref. 1).

The approach adopted here is to expand $|\overline{V}|^n$ in a Taylor series about some point, multiply the resulting series by $g(\overline{V})$ from equation (4), and perform the triple integration. From the form of a Taylor series, it is obvious that this procedure is equivalent to finding the mixed moments of V_x, V_y, and V_z about the origin. Since $g(\overline{V})$ is symmetric about the origin, the most obvious expansion is about the point $(V_x,V_y,V_z) = (0,0,0)$. However, the derivatives of $|\overline{V}|^n$ are not defined at the origin. If the expansion were about some point other than the origin, the resulting Taylor series radius of convergence would stop at the origin and the expansion would not predict across the origin. To avoid this problem, a transformation of coordinates was used which would preserve the statistical properties of $\left(V_x^2 + V_y^2 + V_z^2\right)^{n/2}$ but which would allow the integration to take place within the series radius of convergence. The following transformation was assumed:

$$\left.\begin{aligned} V_x^* &= |V_x| \\ V_y^* &= |V_y| \\ V_z^* &= |V_z| \end{aligned}\right\} \tag{6}$$

Thus $\left[\left(V_x^*\right)^2 + \left(V_y^*\right)^2 + \left(V_z^*\right)^2\right]^{n/2} = \left(V_x^2 + V_y^2 + V_z^2\right)^{n/2}$ and the functions of interest have not been changed. The transformed density function would then be the sum of the densities in the eight octants (see ref. 4, p. 222). For the transformed density function, the following equation can be written:

$$g\left(V_x^*,V_y^*,V_z^*\right) = \frac{(2\pi)^{-3/2}}{\sqrt{\lambda_1\lambda_2\lambda_3}}\left\{ e^{-\frac{1}{2}\left[\frac{\left(V_x^*-\mu_x\right)^2}{\lambda_1}+\frac{\left(V_y^*-\mu_y\right)^2}{\lambda_2}+\frac{\left(V_z^*-\mu_z\right)^2}{\lambda_3}\right]} + e^{-\frac{1}{2}\left[\frac{\left(V_x^*-\mu_x\right)^2}{\lambda_1}+\frac{\left(V_y^*-\mu_y\right)^2}{\lambda_2}+\frac{\left(V_z^*+\mu_z\right)^2}{\lambda_3}\right]} + e^{-\frac{1}{2}\left[\frac{\left(V_x^*-\mu_x\right)^2}{\lambda_1}+\frac{\left(V_y^*+\mu_y\right)^2}{\lambda_2}+\frac{\left(V_z^*-\mu_z\right)^2}{\lambda_3}\right]} + \ldots + e^{-\frac{1}{2}\left[\frac{\left(V_x^*+\mu_x\right)^2}{\lambda_1}+\frac{\left(V_y^*+\mu_y\right)^2}{\lambda_2}+\frac{\left(V_z^*+\mu_z\right)^2}{\lambda_3}\right]}\right\} \qquad \left(V_x^*,V_y^*,V_z^* \geqq 0\right) \qquad (7)$$

After setting $\mu_x = \mu_y = \mu_z = 0$,

$$g\left(V_x^*,V_y^*,V_z^*\right) = \frac{8(2\pi)^{-3/2}}{\sqrt{\lambda_1\lambda_2\lambda_3}}\, e^{-\frac{1}{2}\left[\frac{\left(V_x^*\right)^2}{\lambda_1}+\frac{\left(V_y^*\right)^2}{\lambda_2}+\frac{\left(V_z^*\right)^2}{\lambda_3}\right]} \qquad \left(V_x^*,V_y^*,V_z^* \geqq 0\right) \qquad (8)$$

The intention now is to make the Taylor series expansion about a point in the domain of the new distribution. The analysis that follows requires the calculation of the ijkth moment of the transformed variables. Let

$$\nu_{ijk} = \int_0^\infty \int_0^\infty \int_0^\infty \left(V_x^*\right)^i \left(V_y^*\right)^j \left(V_z^*\right)^k g\left(V_x^*,V_y^*,V_z^*\right) dV_x^*\, dV_y^*\, dV_z^*$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \left(V_x^*\right)^i \left(V_y^*\right)^j \left(V_z^*\right)^k \frac{8(2\pi)^{-3/2}}{\sqrt{\lambda_1\lambda_2\lambda_3}}\, e^{-\frac{1}{2}\left[\frac{\left(V_x^*\right)^2}{\lambda_1}+\frac{\left(V_y^*\right)^2}{\lambda_2}+\frac{\left(V_z^*\right)^2}{\lambda_3}\right]} dV_x^*\, dV_y^*\, dV_z^* \qquad (9)$$

which can be written in the form,

$$\nu_{ijk} = \int_0^\infty \int_0^\infty \int_0^\infty \left[\frac{\left(V_x^*\right)^i 2e^{-\frac{1}{2}\frac{\left(V_x^*\right)^2}{\lambda_1}}}{\sqrt{2\pi\lambda_1}}\right]\left[\frac{\left(V_y^*\right)^j 2e^{-\frac{1}{2}\frac{\left(V_y^*\right)^2}{\lambda_2}}}{\sqrt{2\pi\lambda_2}}\right]\left[\frac{\left(V_z^*\right)^k 2e^{-\frac{1}{2}\frac{\left(V_z^*\right)^2}{\lambda_3}}}{\sqrt{2\pi\lambda_3}}\right] dV_x^*\, dV_y^*\, dV_z^* \qquad (10)$$

Then equation (10) can be integrated three times to yield

$$\nu_{ijk} = \frac{\left(2\lambda_1\right)^{i/2}\left(2\lambda_2\right)^{j/2}\left(2\lambda_3\right)^{k/2}\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(\frac{j+1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\pi^{3/2}} \tag{11}$$

Equation (11) represents the product moments about the origin of the magnitudes of the components of $\overline{V}$. Substituting various values of i, j, and k into equation (11) yields

$$\left.\begin{array}{lll} \nu_{100} = \sqrt{\frac{2\lambda_1}{\pi}} & \nu_{010} = \sqrt{\frac{2\lambda_2}{\pi}} & \nu_{001} = \sqrt{\frac{2\lambda_3}{\pi}} \\ \nu_{200} = \lambda_1 & \nu_{020} = \lambda_2 & \nu_{002} = \lambda_3 \\ \nu_{110} = \frac{2}{\pi}\sqrt{\lambda_1\lambda_2} & \nu_{101} = \frac{2}{\pi}\sqrt{\lambda_1\lambda_3} & \nu_{011} = \frac{2}{\pi}\sqrt{\lambda_2\lambda_3} \end{array}\right\} \tag{12}$$

Now, $E\left(\left|\overline{V}\right|\right)$ can be found by the Taylor series expansion. The series expansion will be made about the point $\left(\sqrt{\frac{a\lambda_1}{\pi}},\sqrt{\frac{a\lambda_2}{\pi}},\sqrt{\frac{a\lambda_3}{\pi}}\right)$ and the maximum percentage error in $\sigma^2_{\left|\overline{V}\right|}$ minimized. This form is taken because when $a = 2$, the expansion would be about the means of the components. Expansion of equation (3) yields

$$\begin{aligned}
\left|\overline{V}\right| = {} & f\left(\sqrt{\frac{a\lambda_1}{\pi}},\sqrt{\frac{a\lambda_2}{\pi}},\sqrt{\frac{a\lambda_3}{\pi}}\right) + f_{V_x^*}\left(\sqrt{\frac{a\lambda_1}{\pi}},\sqrt{\frac{a\lambda_2}{\pi}},\sqrt{\frac{a\lambda_3}{\pi}}\right)\left(V_x^* - \sqrt{\frac{a\lambda_1}{\pi}}\right) + f_{V_y^*}\left(\sqrt{\frac{a\lambda_1}{\pi}},\sqrt{\frac{a\lambda_2}{\pi}},\sqrt{\frac{a\lambda_3}{\pi}}\right)\left(V_y^* - \sqrt{\frac{a\lambda_2}{\pi}}\right) \\
& + f_{V_z^*}\left(\sqrt{\frac{a\lambda_1}{\pi}},\sqrt{\frac{a\lambda_2}{\pi}},\sqrt{\frac{a\lambda_3}{\pi}}\right)\left(V_z^* - \sqrt{\frac{a\lambda_3}{\pi}}\right) + \frac{1}{2} f_{V_x^*V_x^*}\left(\sqrt{\frac{a\lambda_1}{\pi}},\sqrt{\frac{a\lambda_2}{\pi}},\sqrt{\frac{a\lambda_3}{\pi}}\right)\left(V_x^* - \sqrt{\frac{a\lambda_1}{\pi}}\right)^2 \\
& + \frac{1}{2} f_{V_y^*V_y^*}\left(\sqrt{\frac{a\lambda_1}{\pi}},\sqrt{\frac{a\lambda_2}{\pi}},\sqrt{\frac{a\lambda_3}{\pi}}\right)\left(V_y^* - \sqrt{\frac{a\lambda_2}{\pi}}\right)^2 + \frac{1}{2} f_{V_z^*V_z^*}\left(\sqrt{\frac{a\lambda_1}{\pi}},\sqrt{\frac{a\lambda_2}{\pi}},\sqrt{\frac{a\lambda_3}{\pi}}\right)\left(V_z^* - \sqrt{\frac{a\lambda_3}{\pi}}\right)^2 \\
& + f_{V_x^*V_y^*}\left(\sqrt{\frac{a\lambda_1}{\pi}},\sqrt{\frac{a\lambda_2}{\pi}},\sqrt{\frac{a\lambda_3}{\pi}}\right)\left(V_x^* - \sqrt{\frac{a\lambda_1}{\pi}}\right)\left(V_y^* - \sqrt{\frac{a\lambda_2}{\pi}}\right) \\
& + f_{V_y^*V_z^*}\left(\sqrt{\frac{a\lambda_1}{\pi}},\sqrt{\frac{a\lambda_2}{\pi}},\sqrt{\frac{a\lambda_3}{\pi}}\right)\left(V_y^* - \sqrt{\frac{a\lambda_2}{\pi}}\right)\left(V_z^* - \sqrt{\frac{a\lambda_3}{\pi}}\right) \\
& + f_{V_x^*V_z^*}\left(\sqrt{\frac{a\lambda_1}{\pi}},\sqrt{\frac{a\lambda_2}{\pi}},\sqrt{\frac{a\lambda_3}{\pi}}\right)\left(V_x^* - \sqrt{\frac{a\lambda_1}{\pi}}\right)\left(V_z^* - \sqrt{\frac{a\lambda_3}{\pi}}\right) + \ldots
\end{aligned} \tag{13}$$

where the subscripts on the f denote partial differentiation. By taking the expected value of $|\overline{V}|$ from equation (13) and substituting the partial derivatives, it is possible to obtain, after some manipulation, the second-order equation

$$E\left(|\overline{V}|\right) \approx \sqrt{\frac{2T}{\pi}}\left[1 + \frac{\pi - 2}{\sqrt{2a}T^2}\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right)\right] \tag{14}$$

where T is the trace of the matrix in equation (2) and where expressions from equations (12) have been used for the moments. The second moment of $|\overline{V}|$ about the origin can be evaluated without expanding in a Taylor series because

$$E\left(|\overline{V}|^2\right) = E\left[\left(V_x^*\right)^2 + \left(V_y^*\right)^2 + \left(V_z^*\right)^2\right] = E\left[\left(V_x^*\right)^2\right] + E\left[\left(V_y^*\right)^2\right] + E\left[\left(V_z^*\right)^2\right] \tag{15}$$

which from equations (9) to (12) can be written as

$$E\left(|\overline{V}|^2\right) = \nu_{200} + \nu_{020} + \nu_{002} = \lambda_1 + \lambda_2 + \lambda_3 \tag{16}$$

The variance can then be written as

$$\sigma^2_{|\overline{V}|} = E\left(|\overline{V}|^2\right) - E\left(|\overline{V}|\right)^2 \tag{17}$$

or, to second order, as

$$\sigma^2_{|\overline{V}|} \approx \lambda_1 + \lambda_2 + \lambda_3 - \left\{\sqrt{\frac{2T}{\pi}}\left[1 + \frac{\pi - 2}{\sqrt{2a}T^2}\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right)\right]\right\}^2 \tag{18}$$

Higher order approximations to equations (14) and (18) can be obtained at will with a corresponding increase in complexity. However, in the next section, in an attempt to make probability statements about $|\overline{V}|$, an approximation will be needed which will negate to a large extent the usefulness of a third-order fit to the moments.

Association of Probability Statements With $|\overline{V}|$

Given the mean and standard deviation of the magnitude of the velocity to be gained, it is of interest to determine what probability of having enough velocity capability can be associated with the mean plus 1σ, the mean plus 2σ, and so forth. This problem is difficult at best, because it requires the integration of the density function of the magnitude of $\overline{V}$ between arbitrary limits. In fact, as stated previously, the density function has only been written symbolically in reference 1. Probability statements can be made, however, if one is willing to accept an approximate fit to the density function with a Gamma distribution. The Gamma distribution is a logical choice because it is a standard distribution defined on the interval $(0,\infty)$ as is $|\overline{V}|$. The Gamma distribution is characterized by the scale and shape parameters which can be related to the mean and variance.

A variable with a Gamma distribution has a density function which can be written as

$$h\left(|\overline{V}|\right) = \frac{1}{\Gamma(\alpha+1)\beta^{\alpha+1}}|\overline{V}|^{\alpha} e^{-\frac{|\overline{V}|}{\beta}} \qquad \left(0 \leqq |\overline{V}| < \infty\right) \tag{19}$$

The Gamma density has a mean of $\beta(\alpha+1)$ and a second moment about the origin of $\beta^2(\alpha+1)(\alpha+2)$. From the first and second moments about the origin, the variance can be calculated from equation (5) to be

$$\sigma^2_{|\overline{V}|} = \beta^2(\alpha+1) \tag{20}$$

The Gamma distribution can be fitted to the first and second moments. The fit is

$$E\left(|\overline{V}|\right) = \sqrt{\frac{2T}{\pi}}\left[1 + \frac{\pi - 2}{\sqrt{2a}T^2}\left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right)\right] = \beta(\alpha+1) = \mu_{|\overline{V}|} \tag{21a}$$

and

$$E\left(|\overline{V}|^2\right) = \lambda_1 + \lambda_2 + \lambda_3 = \beta^2(\alpha+1)(\alpha+2) = \sigma^2_{|\overline{V}|} + \mu^2_{|\overline{V}|} \tag{21b}$$

These two equations can be solved simultaneously to yield

$$\alpha = \frac{\mu^2_{|\overline{V}|}}{\sigma^2_{|\overline{V}|}} - 1 \tag{22a}$$

and

$$\beta = \frac{\sigma^2_{|\overline{V}|}}{\mu_{|\overline{V}|}} \tag{22b}$$

Given equations (19), (22a), and (22b), probability statements can be made by

$$P\left(0 \leqq |\overline{V}| \leqq b\right) = \int_0^b h\left(|\overline{V}|\right) d|\overline{V}| \tag{23}$$

Equation (23) must be integrated numerically unless α is a positive integer, in which case it can be integrated successively by parts to yield (see ref. 4, p. 128)

$$P\left(0 \leqq |\overline{V}| \leqq b\right) = 1 - \sum_{k=0}^{k=\alpha} \frac{1}{k!}\left(\frac{b}{\beta}\right)^k e^{-b/\beta} \tag{24}$$

An attempt should be made to fit the distribution with the integer nearest to the resulting α and thereby simplify the calculations.

NUMERICAL RESULTS AND DISCUSSION

In order to check the range of validity of the approximation included in the analysis section, 17 covariance matrices were chosen to be representative of general midcourse maneuver covariance matrices. Each of the matrices was chosen such that the trace $\left(\lambda_1 + \lambda_2 + \lambda_3\right)$ was equal to 1 and the eigenvalues had various ratios. (See fig. 1.)

In figure 1(a) is presented a plane in three-dimensional eigenvalue space equivalent to $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Then, every choice of three eigenvalues (with $T = 1$) must be a point on that plane. Figure 1(b) is a small sketch indicating that the eigenvalue plane has been sectioned, and section B is presented in figure 1(c). The 17 points are indicated in figure 1(c) along with the corresponding eigenvalue ratios. Because of symmetry of the mathematical formulation, each point of section B is equivalent to a point on each of the remaining five sections. For example, the point (0.5,0.5,0) would give the same mean and standard deviation of the magnitude as the point (0.5,0,0.5) and the point (0,0.5,0.5). Investigating the results for these 17 cases is equivalent to considering 76 points in figure 1(a), which should be more than adequate to represent the plane. The vertical line in figure 1(c) corresponds to the three-dimensional cases with two eigenvalues the same. The inclined line also corresponds to three-dimensional cases. The horizontal line corresponds to the two-dimensional cases. Along this line the largest eigenvalue λ_1 becomes greater as it approaches the point (1,0,0) from the right. In the vertical direction the smallest eigenvalue λ_3 becomes smaller as it approaches the point (1,1,0) from above.

For each of these 17 cases the mean and standard deviation were calculated from equations (14) and (18), respectively. It was not known, a priori, which value of a in equations (14) and (18) would yield the most accurate values for the mean and standard deviation and therefore a was varied from 2 to 2.8 by increments of 0.1 and the percentage absolute error in the standard deviation plotted in figure 2 as a function of a. The approximate values were compared with exact values obtained by multiplying $g\left(\overline{V}\right)$ from equation (4) by $|\overline{V}|$ and performing the numerical integration. The actual integration was performed by transforming to polar coordinates, integrating out the r coordinate analytically, and integrating from 0° to 360° in ϕ and θ numerically by a 40-point Gauss integration technique. The integrated values were accurate to 12 places.

The error in the mean was not plotted because it is directly related to the error in the standard deviation. From figure 2 it can be seen that a value of $a = 2.7$ gives the best fit. This value was chosen and will be used for the remainder of the present paper. The results of the approximations for $a = 2.7$ for each of the 17 cases are presented in table I. In the second, third, and fourth columns are shown the actual eigenvalues used for each of the cases, and the corresponding ratios are given in the fifth column. The next three columns show the exact mean, the approximate mean, and the percent error in mean. For all cases the percent error in mean was 1.65 or less. The error for case 5 is zero because the function considered in the expansion was $|\overline{V}| = |V_x| = \sqrt{V_x^2}$ and the Taylor series had only one term. In the next three columns the same information is shown for the standard deviation. The standard-deviation errors are much larger than the errors in mean because the standard deviation is calculated from equation (18) as a function of the square of the mean. In no case, however, was the error found to be greater than 4.52 percent; the accuracy is probably adequate for fuel budget calculations.

Next the Gamma function was fitted to the mean and the standard deviation for the 17 cases. Equation (22a) was used to find the parameter α and equation (22b) was used to find β. As stated previously, if α is a positive integer, equation (23) can be integrated by parts to obtain equation (24). Therefore, in the Gamma function fit investigated herein, the nearest integer was chosen for α and the corresponding β was calculated as $\mu_{|\overline{V}|}/\alpha + 1$. Figures 3(a), 3(b), 3(c), and 3(d) show a comparison of this approximate density function (eq. (19)) and the exact density function for four eigenvalue ratios. Included in each figure is a plot of the right half of a normal density function with a mean of 0 and a standard deviation of 1. Also included is the Rayleigh density function as discussed in the introduction. It is easily seen that for all these ratios except 1-0-0, the approximation fits better than the Rayleigh density. For the 1-0-0 ratio, the Rayleigh is exact. However, the important curve in terms of probability statements is the cumulative distribution function (eq. (23)), and therefore only four of the worst density functions have been presented as a matter of interest.

The cumulative functions tell a much better story of the fit (see figs. 4(a) to 4(q)). In figure 4 the magnitude of the velocity to be gained $|\overline{V}|$, normalized to the square root of the trace, has been plotted as a function of probability on probability paper. For reference the normal cumulative curve has been included. Probability paper has the characteristic of making the cumulative normal distribution appear as a straight line. The curvature, then, is a measure of the deviation from a normal curve.

It can be seen from figures 4(a) to 4(q) that the approximation is consistently better than the Rayleigh results (except perhaps where the Rayleigh curve crosses the exact curve). One other exception occurs for the eigenvalue ratio 1-0-0 (fig. 4(e)); as explained previously, the Rayleigh and exact results are equivalent for this ratio. It has commonly been assumed that the Rayleigh assumption is conservative (see, for instance, ref. 2). Figure 4 shows that at $|\overline{V}|/\sqrt{\overline{T}} = 3$ the Rayleigh assumption is conservative (i.e., the real probability is higher than predicted) for all ratios. However, for the lower values of $|\overline{V}|/\sqrt{\overline{T}}$ (less than 1.2), or probability less than 75 percent, the Rayleigh assumption is not generally conservative. For this reason, care should be used in the application of the Rayleigh assumption.

CONCLUDING REMARKS

A method for calculating the first two moments of the distribution of the magnitude of a midcourse maneuver has been presented. A Gamma distribution with an integer shape parameter has been fitted to these two moments and the approximation results compared with exact integrated results for representative eigenvalue ratios. In addition, a comparison is made with results of the Rayleigh assumption commonly used. Approximations for the mean, standard deviation, and cumulative distribution function are shown to be reasonable. The Rayleigh assumption is shown to be conservative for values of probability greater than 75 percent or normalized velocity-to-be-gained magnitude greater than 1.2.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., April 23, 1969,
194-82-01-05-23.

APPENDIX

USING THE RESULTS WITHOUT DIAGONALIZATION

In the text the assumption was made that a matrix of the form

$$E\left(\overline{V}\overline{V}^T\right) = \begin{bmatrix} E\left(V_xV_x\right) & 0 & 0 \\ 0 & E\left(V_yV_y\right) & 0 \\ 0 & 0 & E\left(V_zV_z\right) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{A1}$$

was to be given and the analysis could be performed on this matrix. However, in the more general case, a matrix of the form

$$E\left[\overline{V}'\left(\overline{V}'\right)^T\right] = \begin{bmatrix} E\left(V_x'V_x'\right) & E\left(V_x'V_y'\right) & E\left(V_x'V_z'\right) \\ E\left(V_y'V_x'\right) & E\left(V_y'V_y'\right) & E\left(V_y'V_z'\right) \\ E\left(V_z'V_x'\right) & E\left(V_z'V_y'\right) & E\left(V_z'V_z'\right) \end{bmatrix} \tag{A2}$$

will be available.

The problem of finding λ_1, λ_2, and λ_3 in terms of elements of the matrix in equation (A2) requires solution of a cubic. However, for the second-order solutions (eqs. (14) and (18)), it is only necessary to know $\lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$. If these two expressions are known, it would not be necessary to diagonalize the matrix in equation (A2). In order to find these two quantities, the characteristic equation of the matrix in equation (A2) can be written as (see ref. 5)

$$K(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{A3}$$

and the characteristic equation for the matrix in equation (A1) can be written as

$$K(\lambda) = \lambda^3 - \left(\lambda_1 + \lambda_2 + \lambda_3\right)\lambda^2 + \left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right)\lambda - \lambda_1\lambda_2\lambda_3 = 0 \tag{A4}$$

If the coefficients of like powers of λ are equated, the result is

$$\left.\begin{aligned} a_1 &= -\left(\lambda_1 + \lambda_2 + \lambda_3\right) \\ a_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\ a_3 &= -\lambda_1\lambda_2\lambda_3 \end{aligned}\right\} \tag{A5}$$

APPENDIX

If the characteristic equation is written for the matrix in equation (A2), the following expressions are obtained for a_1, a_2, and a_3:

$$\left.\begin{aligned}
a_1 &= -\left[E\left(V_x'V_x'\right) + E\left(V_y'V_y'\right) + E\left(V_z'V_z'\right)\right] \\
a_2 &= E\left(V_x'V_x'\right)E\left(V_y'V_y'\right) + E\left(V_x'V_x'\right)E\left(V_z'V_z'\right) + E\left(V_y'V_y'\right)E\left(V_z'V_z'\right) \\
&\quad - E^2\left(V_y'V_z'\right) - E^2\left(V_x'V_y'\right) - E^2\left(V_x'V_z'\right) \\
a_3 &= -\left[E\left(V_x'V_x'\right)E\left(V_y'V_y'\right)E\left(V_z'V_z'\right) - E\left(V_x'V_x'\right)E^2\left(V_y'V_z'\right) - E^2\left(V_x'V_y'\right)E\left(V_z'V_z'\right)\right. \\
&\quad \left. - E^2\left(V_x'V_z'\right)E\left(V_y'V_y'\right) + 2E\left(V_x'V_y'\right)E\left(V_x'V_z'\right)E\left(V_y'V_z'\right)\right]
\end{aligned}\right\} \tag{A6}$$

Combining equations (A5) and (A6) yields

$$\left.\begin{aligned}
\lambda_1 + \lambda_2 + \lambda_3 &= E\left(V_x'V_x'\right) + E\left(V_y'V_y'\right) + E\left(V_z'V_z'\right) \\
\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 &= E\left(V_x'V_x'\right)E\left(V_y'V_y'\right) + E\left(V_x'V_x'\right)E\left(V_z'V_z'\right) + E\left(V_y'V_y'\right)E\left(V_z'V_z'\right) \\
&\quad - E^2\left(V_y'V_z'\right) - E^2\left(V_x'V_y'\right) - E^2\left(V_x'V_z'\right)
\end{aligned}\right\} \tag{A7}$$

Thus, $\lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ can be found in any coordinate system without diagonalization. The results of the text can then be applied to any covariance matrix, diagonal or not.

REFERENCES

1. Blumenson, L. E.; and Miller, K. S.: Properties of Generalized Rayleigh Distributions. Ann. Math. Statist., vol. 34, 1963, pp. 903-910.

2. Smith, Gerald L.; and Gadeberg, Burnett L.: Monte Carlo Computation of the Statistics of the Midcourse Velocity Corrections for a Lunar Mission. NASA TN D-2296, 1964.

3. Skidmore, Lionel J.; and Penzo, Paul A.: Monte Carlo Simulation of the Midcourse Guidance for Lunar Flights. AIAA J., vol. 1, no. 4, Apr. 1963, pp. 820-831.

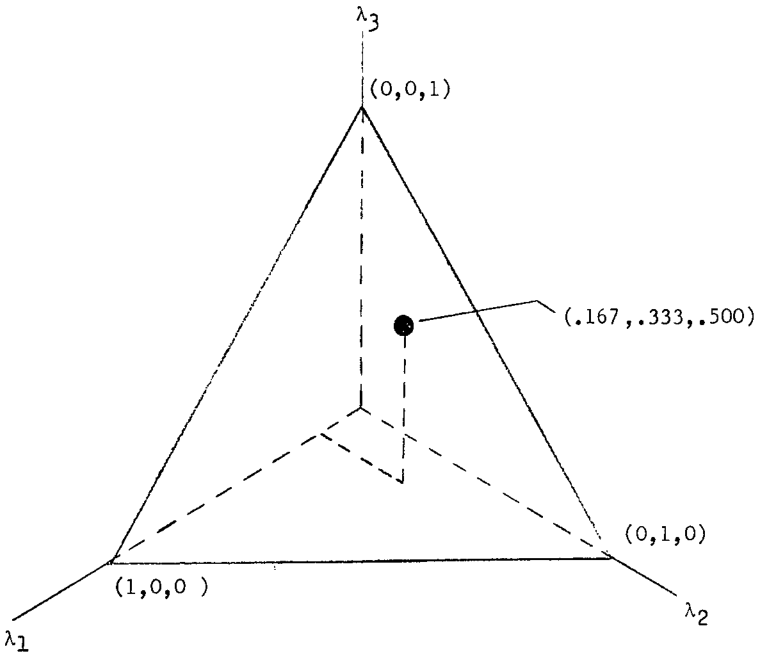
4. Mood, Alexander M.; and Graybill, Franklin A.: Introduction to the Theory of Statistics. McGraw-Hill Book Co., Inc., c.1963.

5. Margenau, Henry; and Murphy, George Moseley: The Mathematics of Physics and Chemistry. D. Van Nostrand Co., Inc., c.1956.

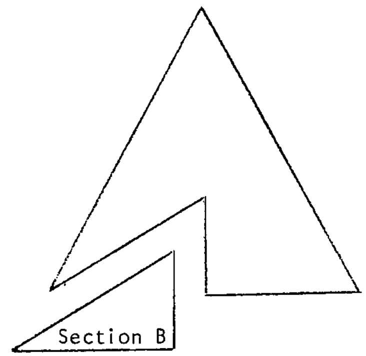
TABLE I.- ACCURACY OF CALCULATION OF THE MEANS AND STANDARD DEVIATIONS OF $|V|$ FOR VARIOUS EIGENVALUE RATIOS

$[a = 2.7]$

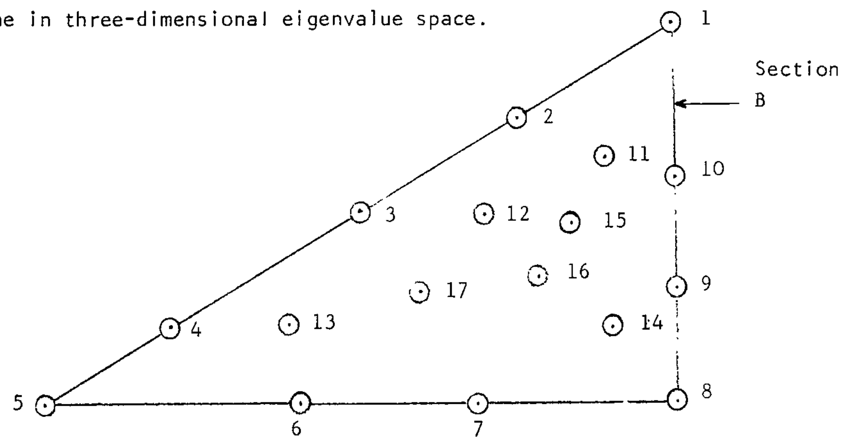
Case	Eigenvalues			Ratio of eigenvalues	Exact mean	Approximate mean	Percent error in mean	Exact standard deviation	Approximate standard deviation	Percent error in standard deviation	α	β	β for integer nearest to α
	λ_1	λ_2	λ_3										
1	0.333	0.333	0.333	1-1-1	0.9213	0.9285	+0.78	0.3888	0.3712	-4.52	5.256	0.1484	0.1547
2	.250	.250	.500	1-1-2	.9158	.9204	+.49	.4016	.3910	-2.63	4.539	.1661	.1534
3	.666	.166	.166	1-1-4	.8991	.8958	-.36	.437	.444	+1.52	3.065	.2203	.2239
4	.866	.066	.066	1-1-13	.8591	.8449	-1.65	.5118	.5349	+4.51	1.495	.3386	.4224
5	1.00	0	0	1-0-0	.7978	.7978	0	.6028	.6028	0	.7519	.4554	.3989
6	.800	.200	0	1-4-0	.8643	.8606	-.43	.5030	.5093	+1.24	1.855	.3014	.2868
7	.650	.350	0	7-13-0	.8811	.8870	+.67	.4728	.4616	-2.37	2.692	.2401	.2217
8	.500	.500	0	1-1-0	.8862	.8958	+1.09	.4632	.4443	-4.09	3.066	.2203	.2239
9	.450	.450	.100	2-9-9	.9077	.9125	+.531	.4195	.4089	-2.52	3.978	.1833	.1825
10	.400	.400	.200	1-2-2	.9173	.9233	+.653	.3981	.3840	-3.54	4.780	.1597	.1539
11	.443	.333	.224	443-333-224	.9179	.9238	+.643	.3967	.3827	-3.515	4.825	.1586	.1539
12	.570	.270	.160	57-27-16	.9089	.9109	+.2110	.4168	.4126	-1.009	3.873	.1869	.1822
13	.769	.154	.0769	1-2-10	.8809	.8721	-1.01	.4730	.489	+3.42	2.176	.2745	.2907
14	.520	.410	.070	52-41-7	.9025	.9069	+.495	.4306	.4212	-2.203	3.637	.1956	.1814
15	.500	.333	.166	1-2-3	.9133	.9176	+.481	.4074	.3974	-2.45	4.333	.1721	.1835
16	.555	.333	.111	1-3-5	.9063	.9092	+.3121	.4225	.4164	-1.449	3.767	.1907	.1818
17	.650	.250	.100	2-5-13	.8984	.8968	-.171	.4392	.4423	+.714	3.111	.2181	.2242



(a) Plane in three-dimensional eigenvalue space.



(b) Sectioning of the plane.



(c) Section B, with 17 points and corresponding eigenvalue ratios indicated.

Case	Eigenvalue ratio	Case	Eigenvalue ratio
1	1-1-1	10	1-2-2
2	1-1-2	11	443-333-224
3	1-1-4	12	57-27-16
4	1-1-13	13	1-2-10
5	1-0-0	14	52-41-7
6	1-4-0	15	1-2-3
7	7-13-0	16	1-3-5
8	1-1-0	17	2-5-13
9	2-9-9		

Figure 1.- Definition of eigenvalue ratios.

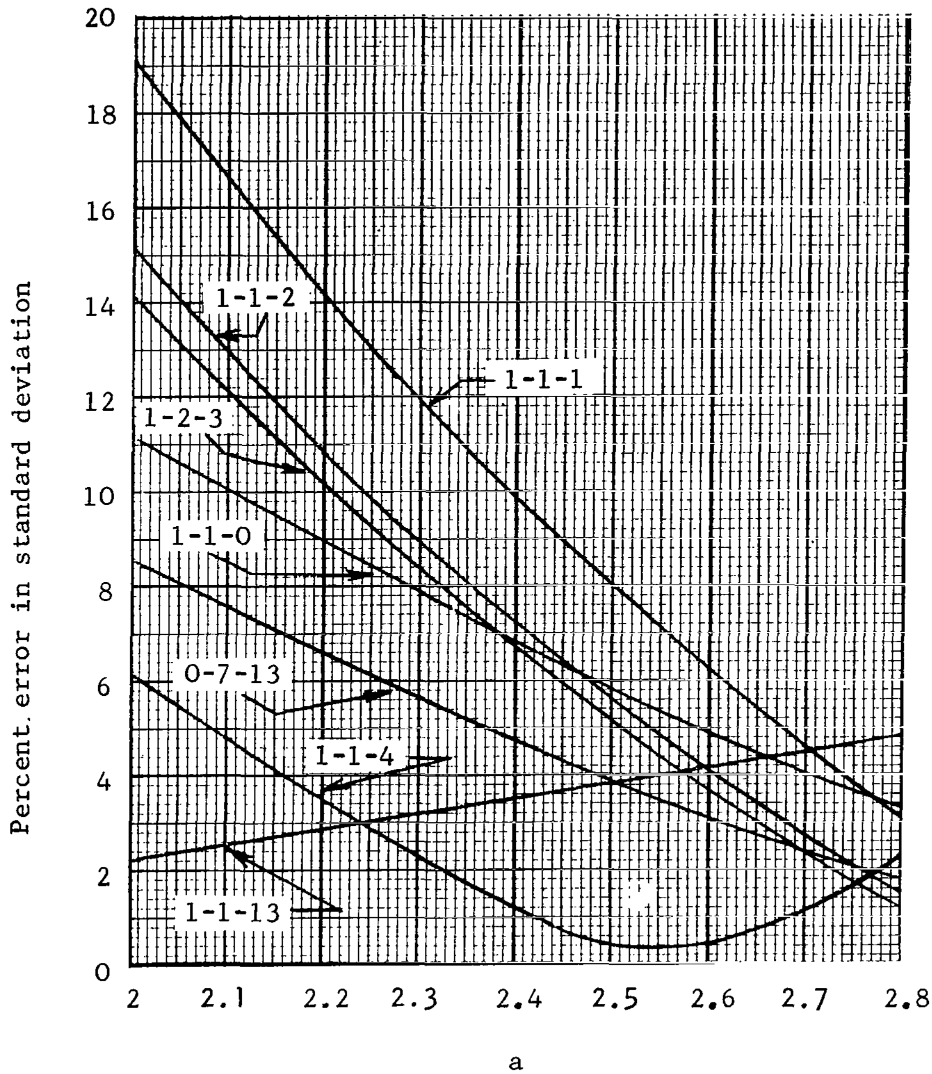
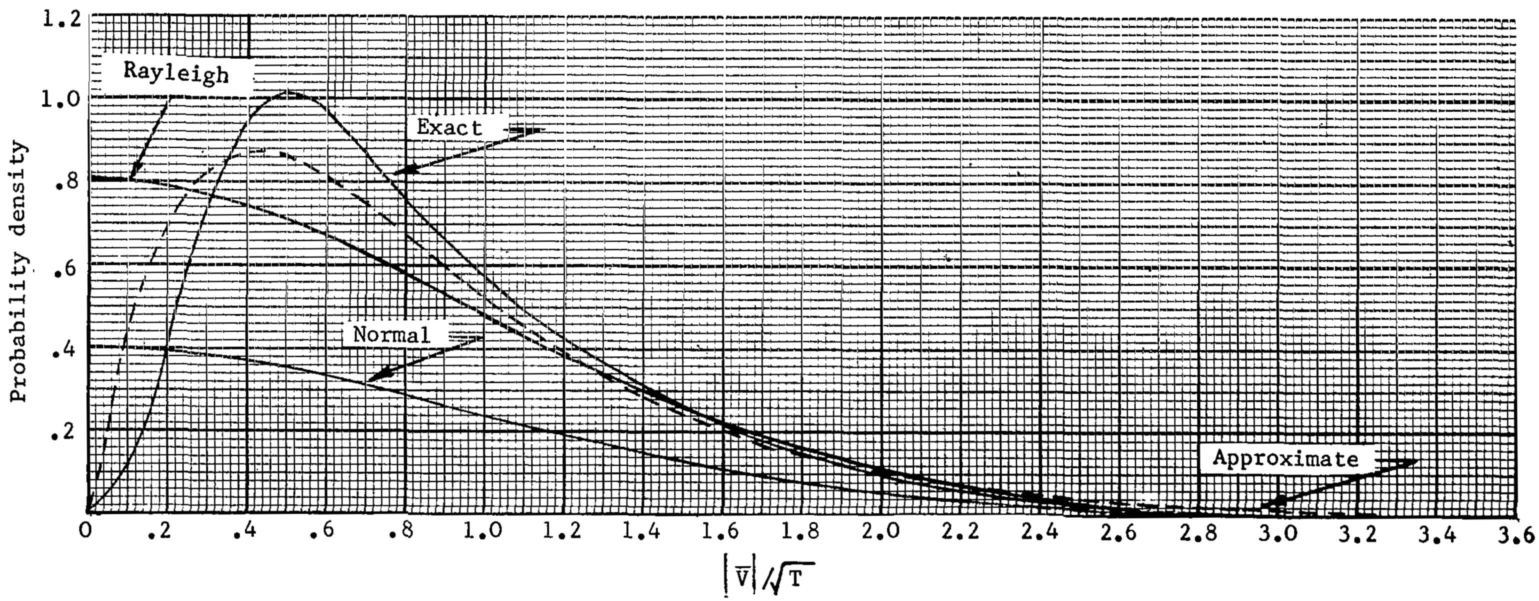
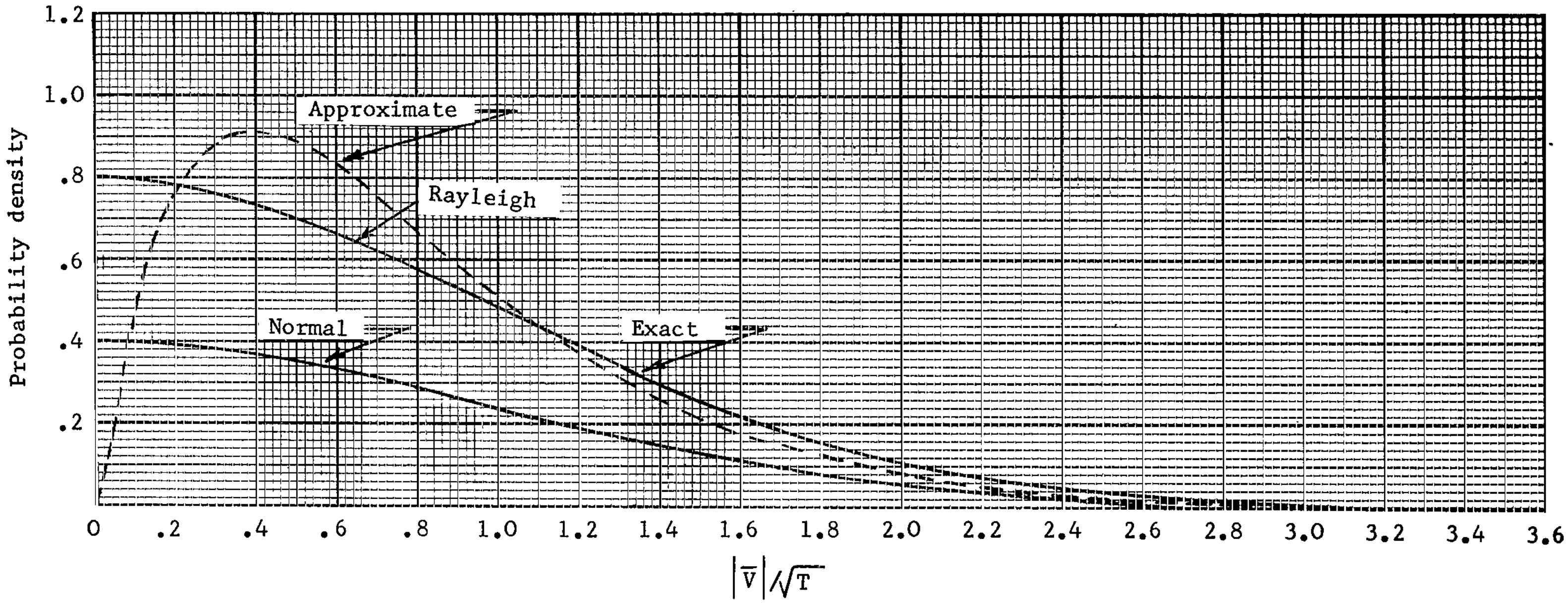


Figure 2.- Variation of percent error in standard deviation as a function of the parameter a for various eigenvalue ratios, with the point of expansion being $\left(\sqrt{\frac{a\lambda_1}{\pi}}, \sqrt{\frac{a\lambda_2}{\pi}}, \sqrt{\frac{a\lambda_3}{\pi}}\right)$.



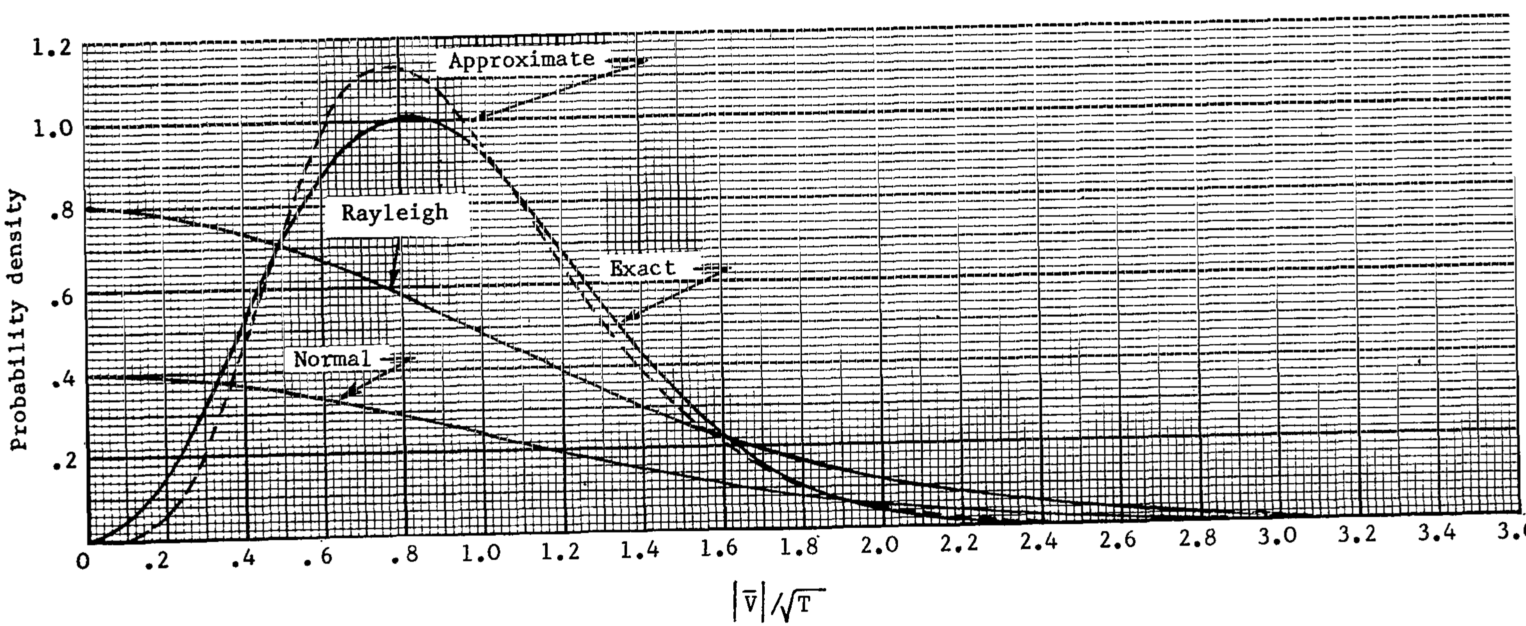
(a) Eigenvalue ratio of 1-1-13.

Figure 3.- Comparison of approximate, normal, exact, and Rayleigh density functions.



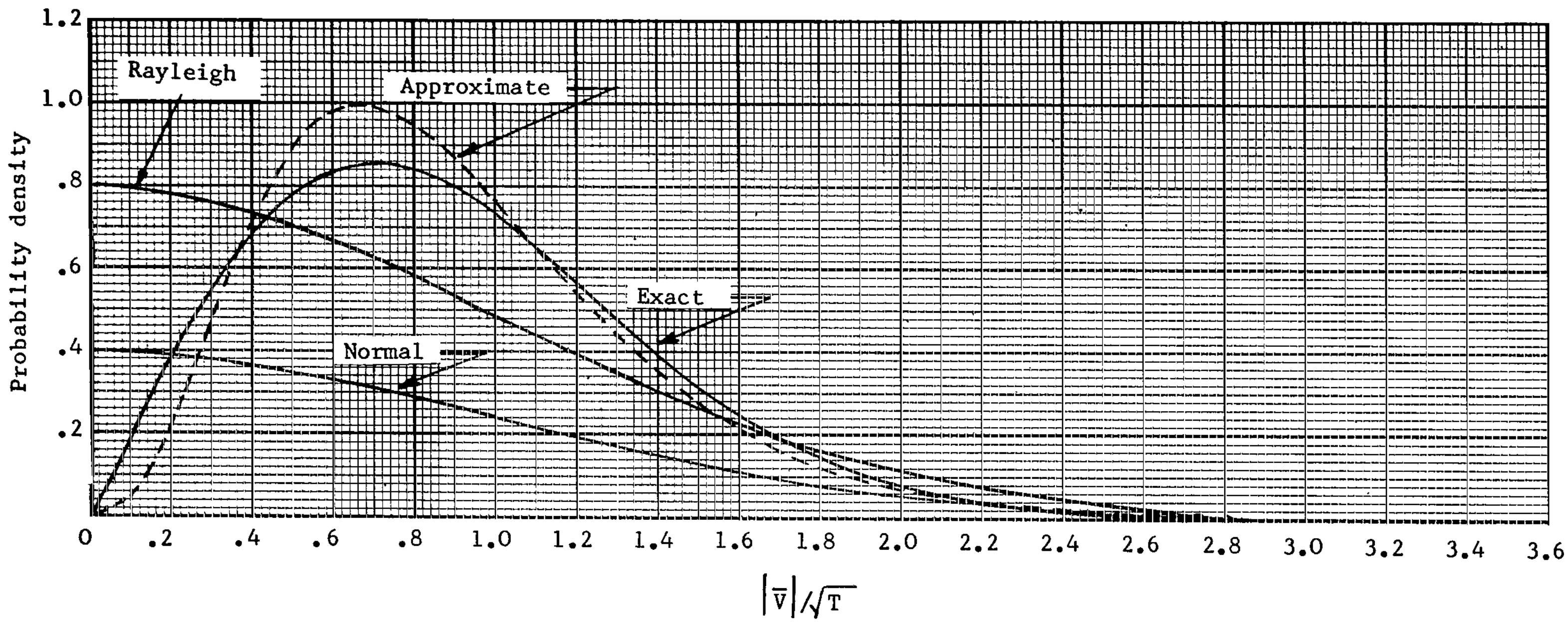
(b) Eigenvalue ratio of 1-0-0.

Figure 3.- Continued.



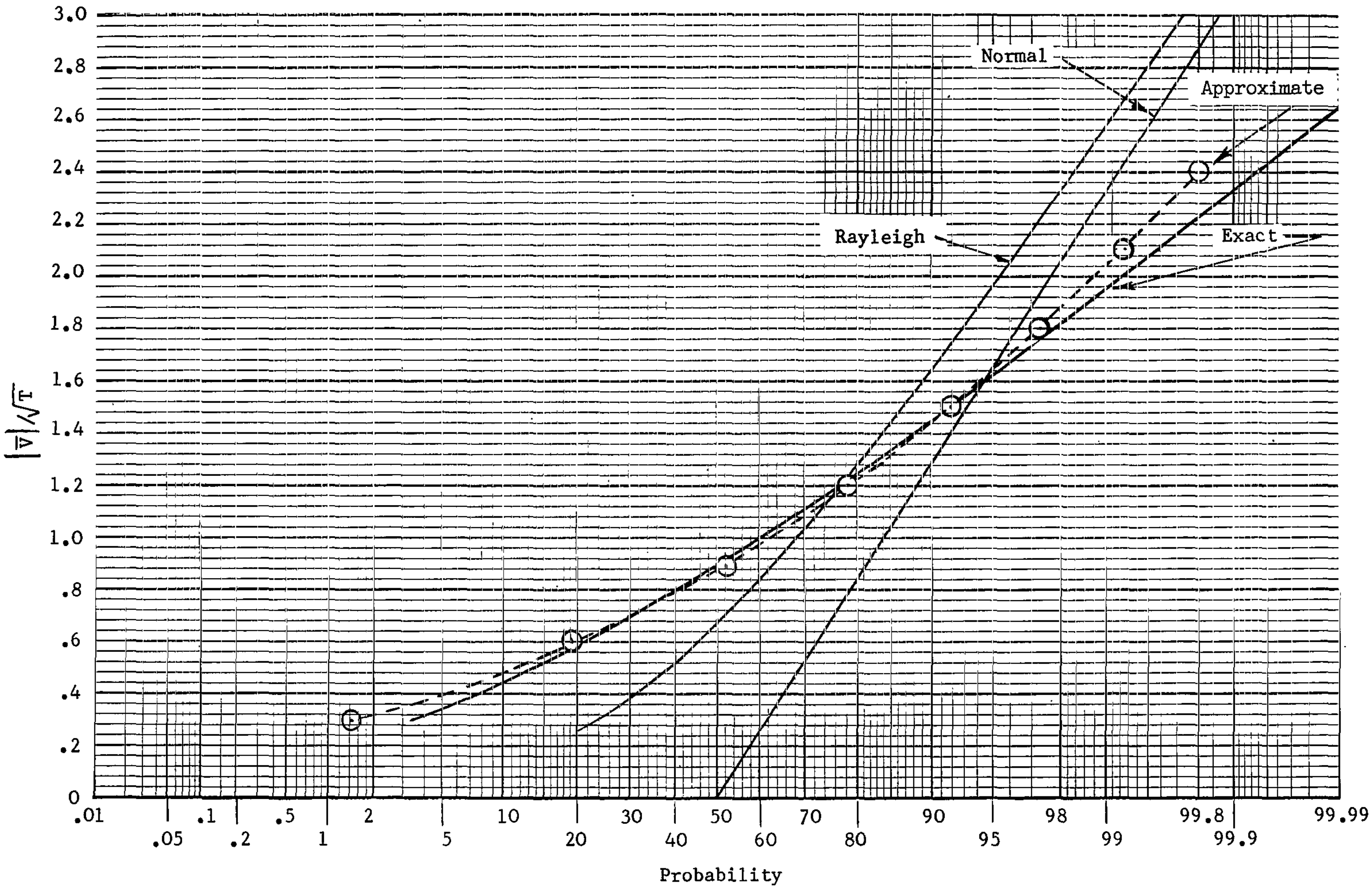
(c) Eigenvalue ratio of 1-1-1.

Figure 3.- Continued.



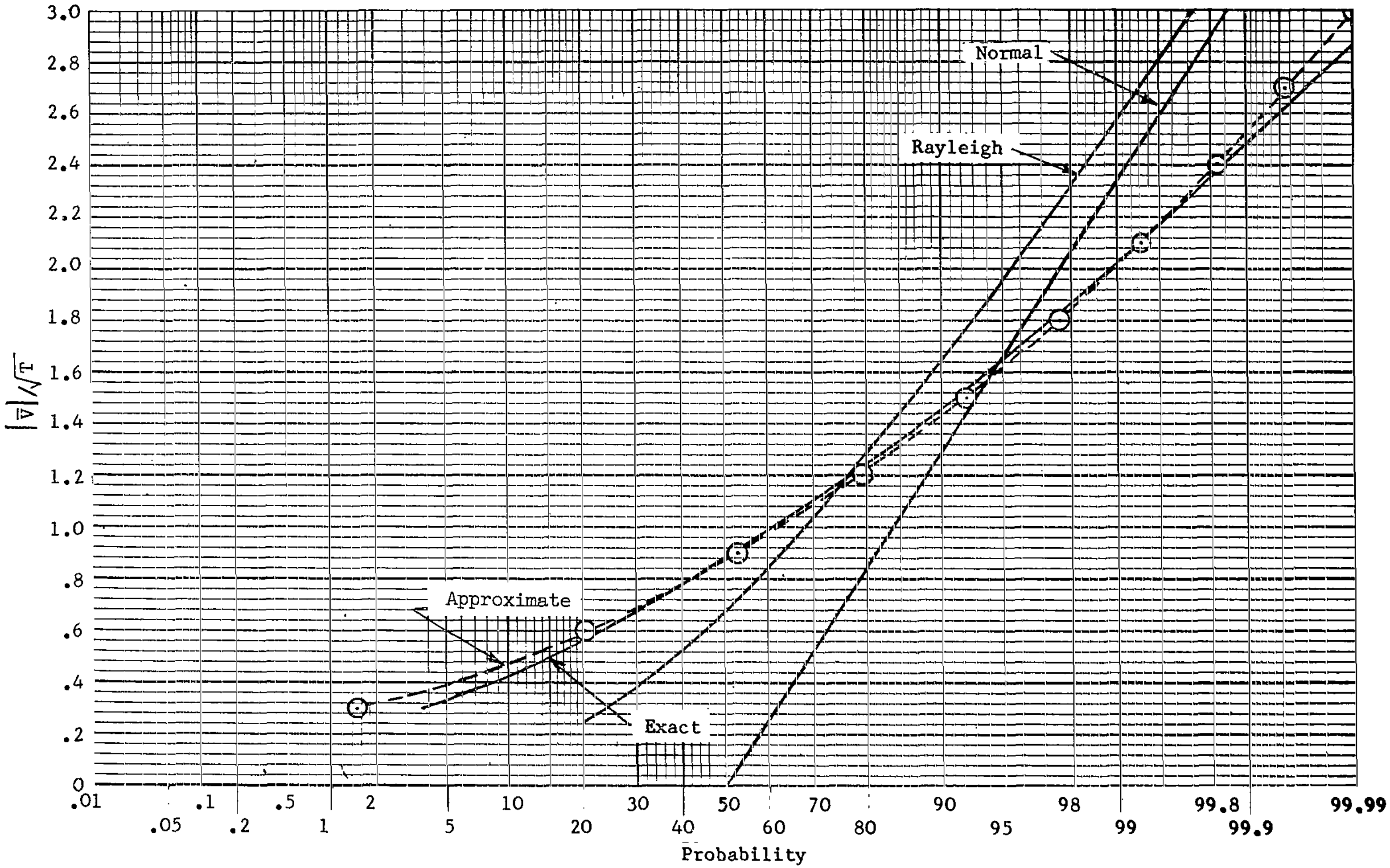
(d) Eigenvalue ratio of 1-1-0.

Figure 3.- Concluded.



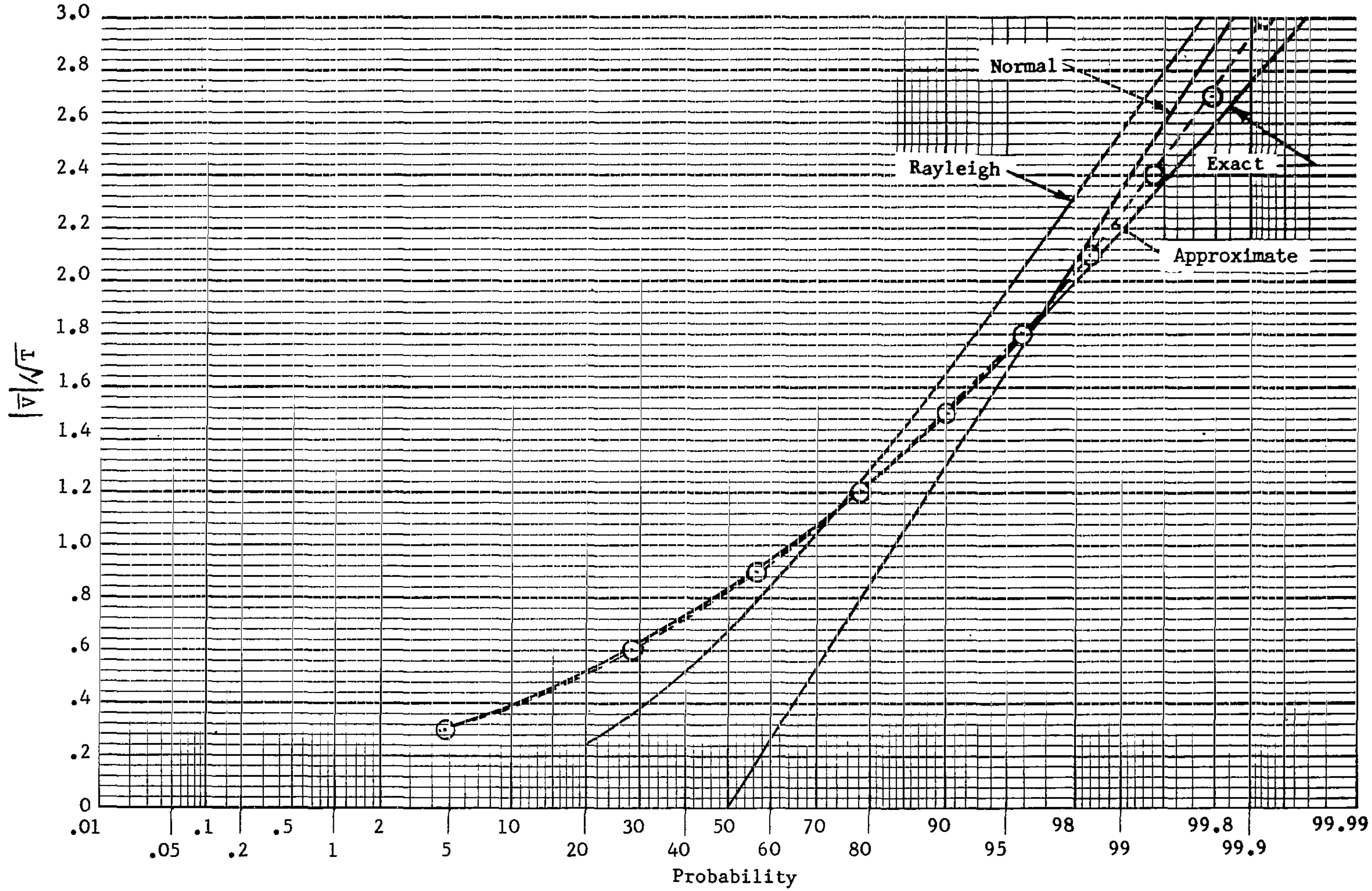
(a) Eigenvalue ratio of 1-1-1.

Figure 4.- Comparison of approximate, normal, exact, and Rayleigh cumulative distribution functions of $|\overline{V}|$. a = 2.7.



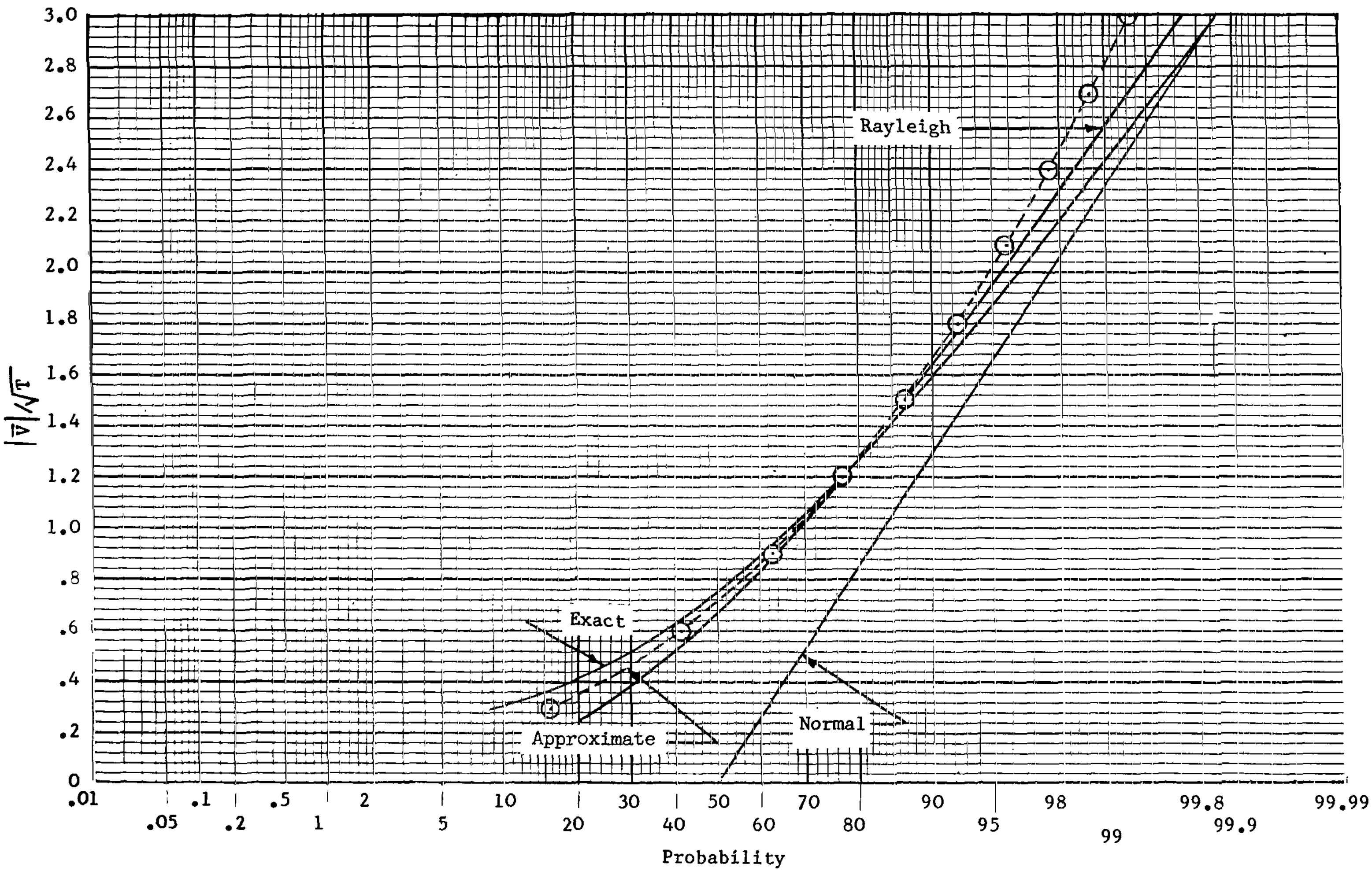
(b) Eigenvalue ratio of 1-1-2.

Figure 4.- Continued.



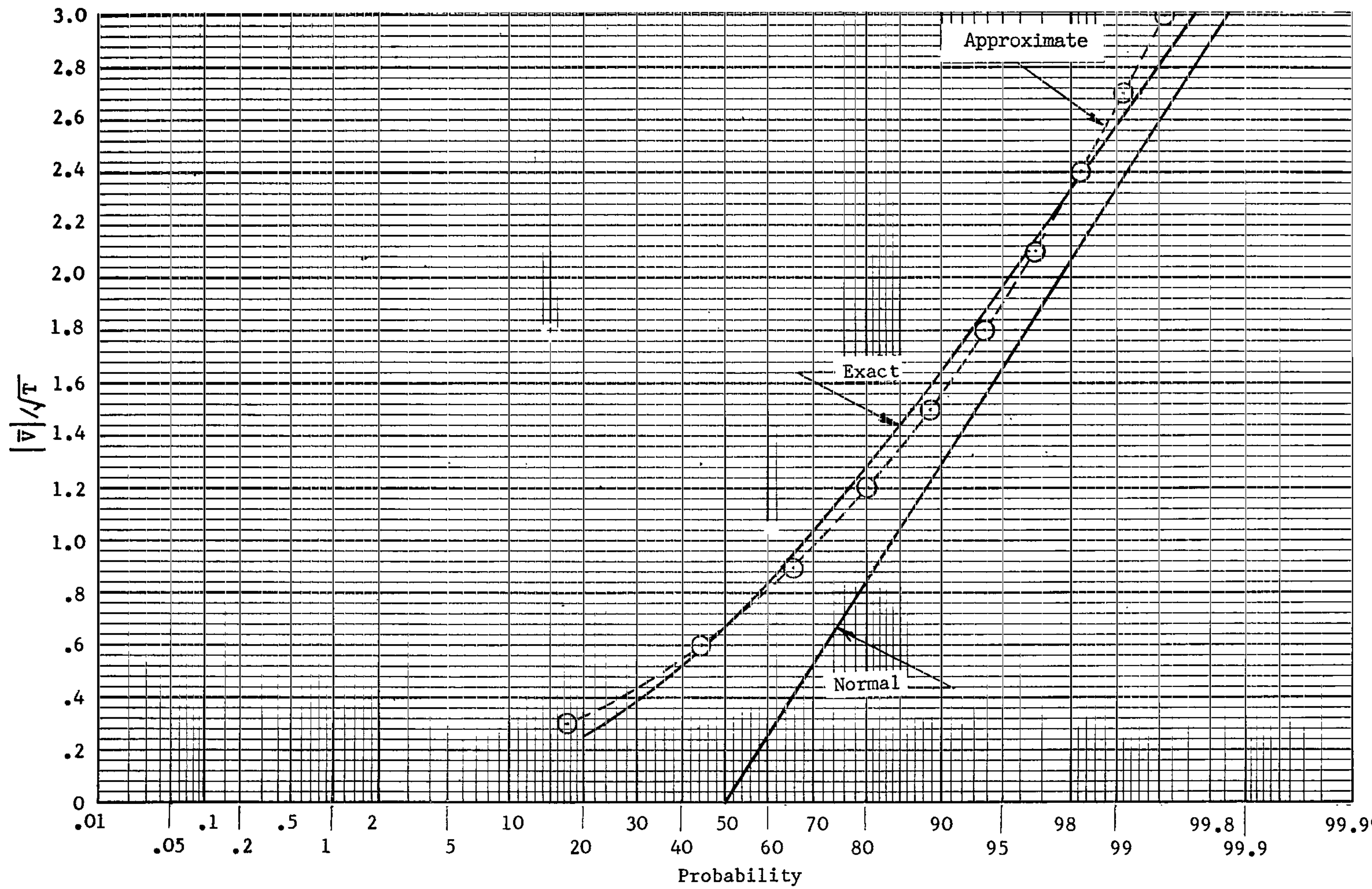
(c) Eigenvalue ratio of 1-1-4.

Figure 4.- Continued.



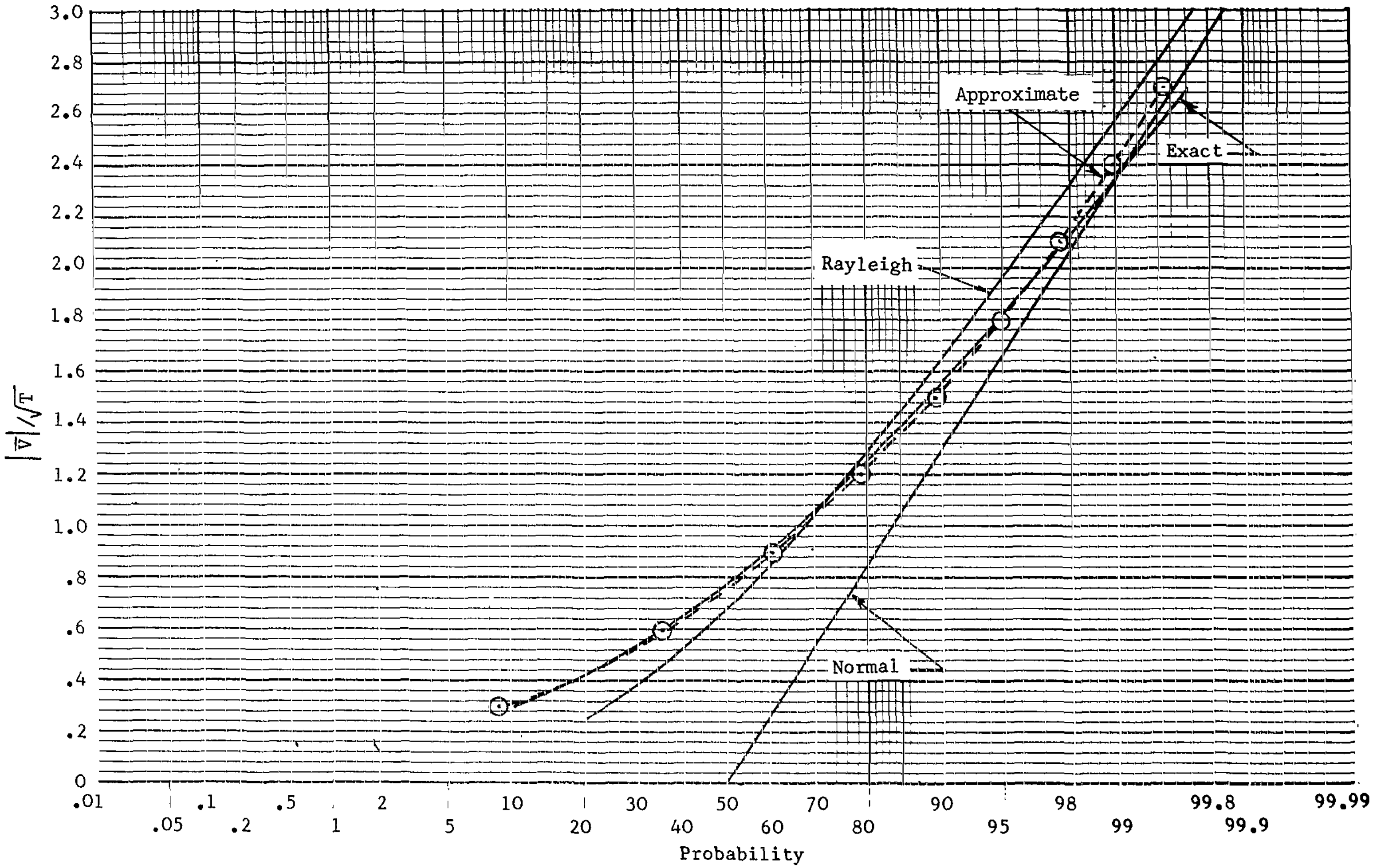
(d) Eigenvalue ratio of 1-1-13.

Figure 4.- Continued.



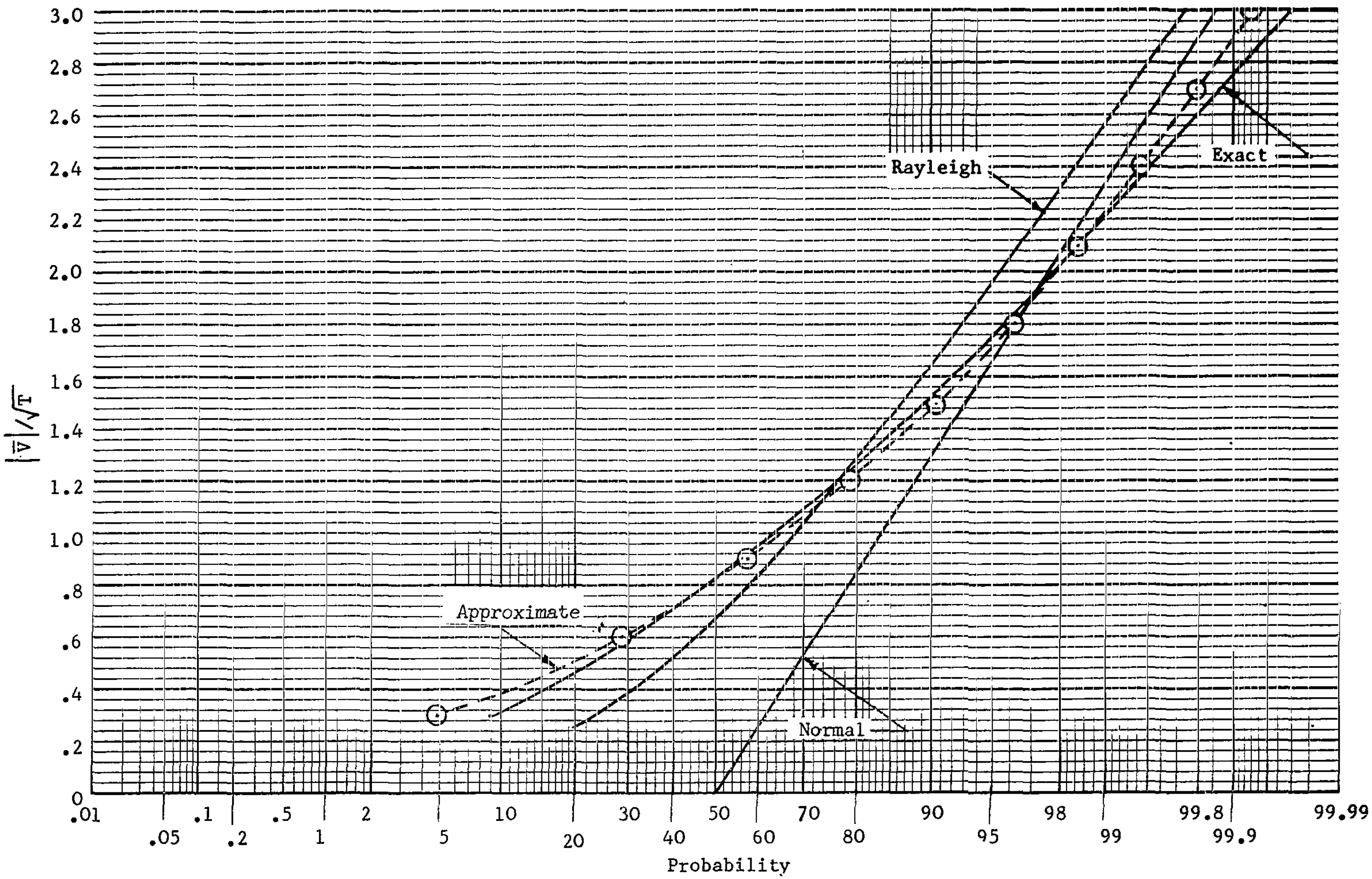
(e) Eigenvalue ratio of 1-0-0.

Figure 4.- Continued.



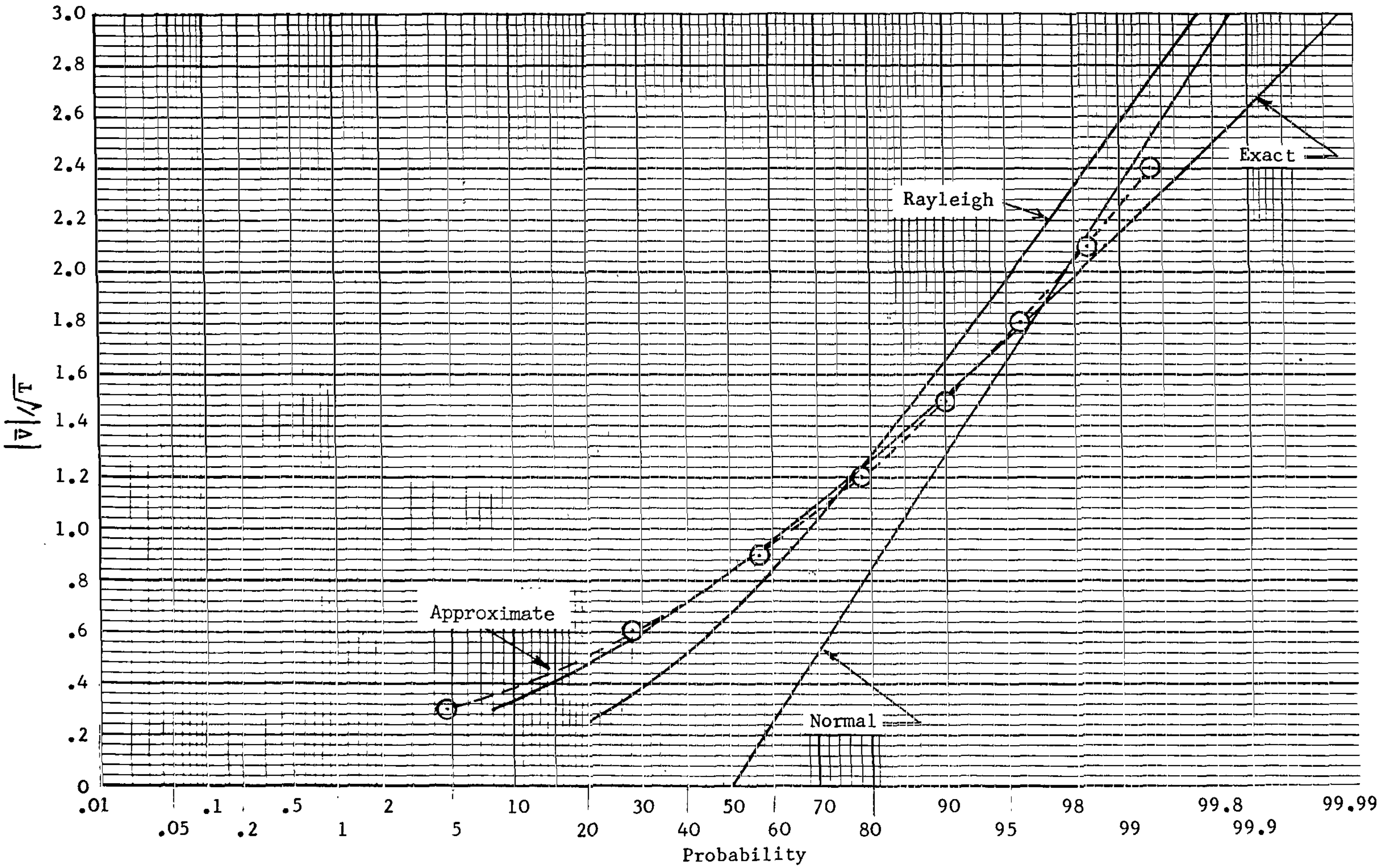
(f) Eigenvalue ratio of 1-4-0.

Figure 4.- Continued.



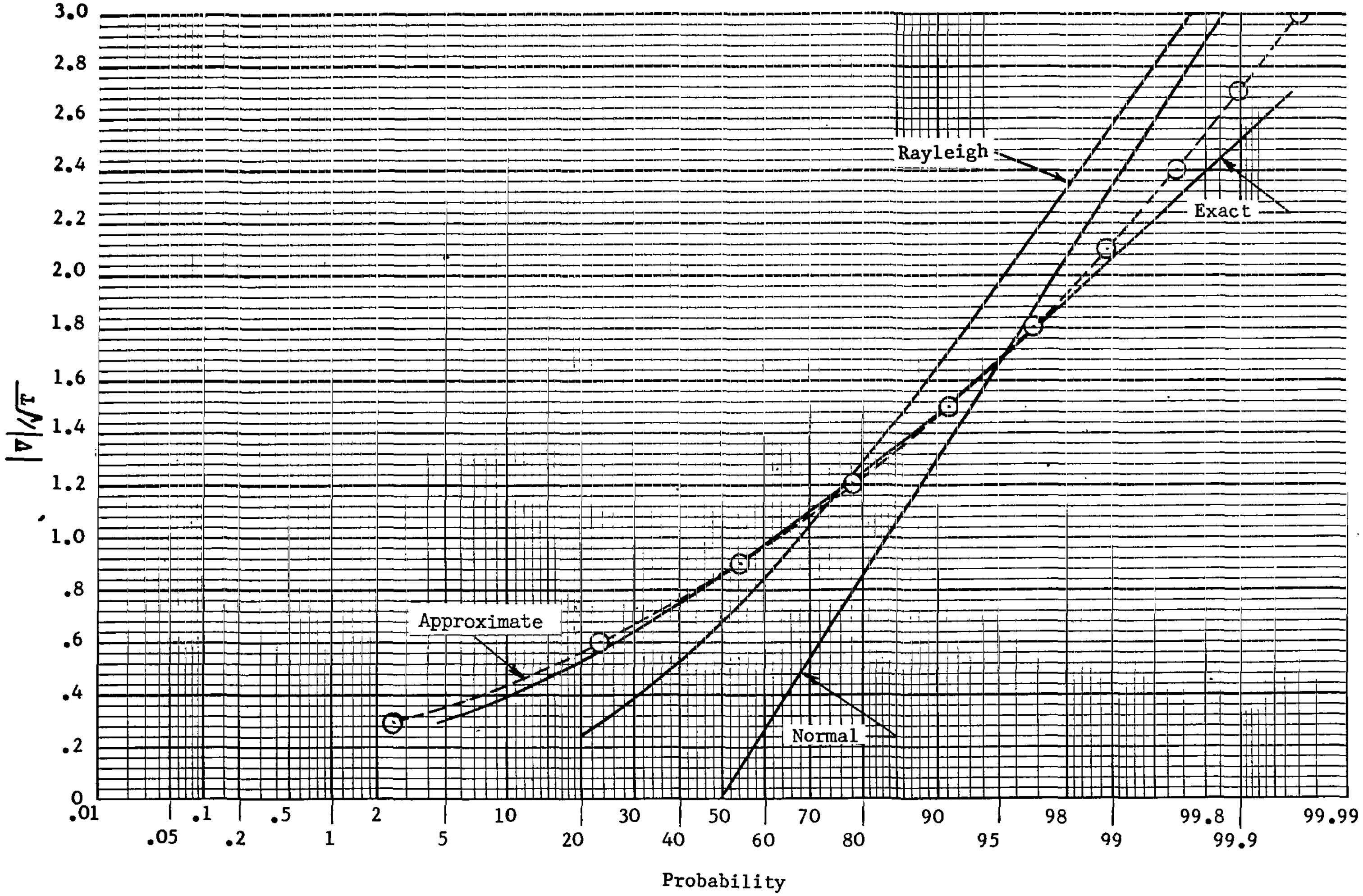
(g) Eigenvalue ratio of 7-13-0.

Figure 4.- Continued.



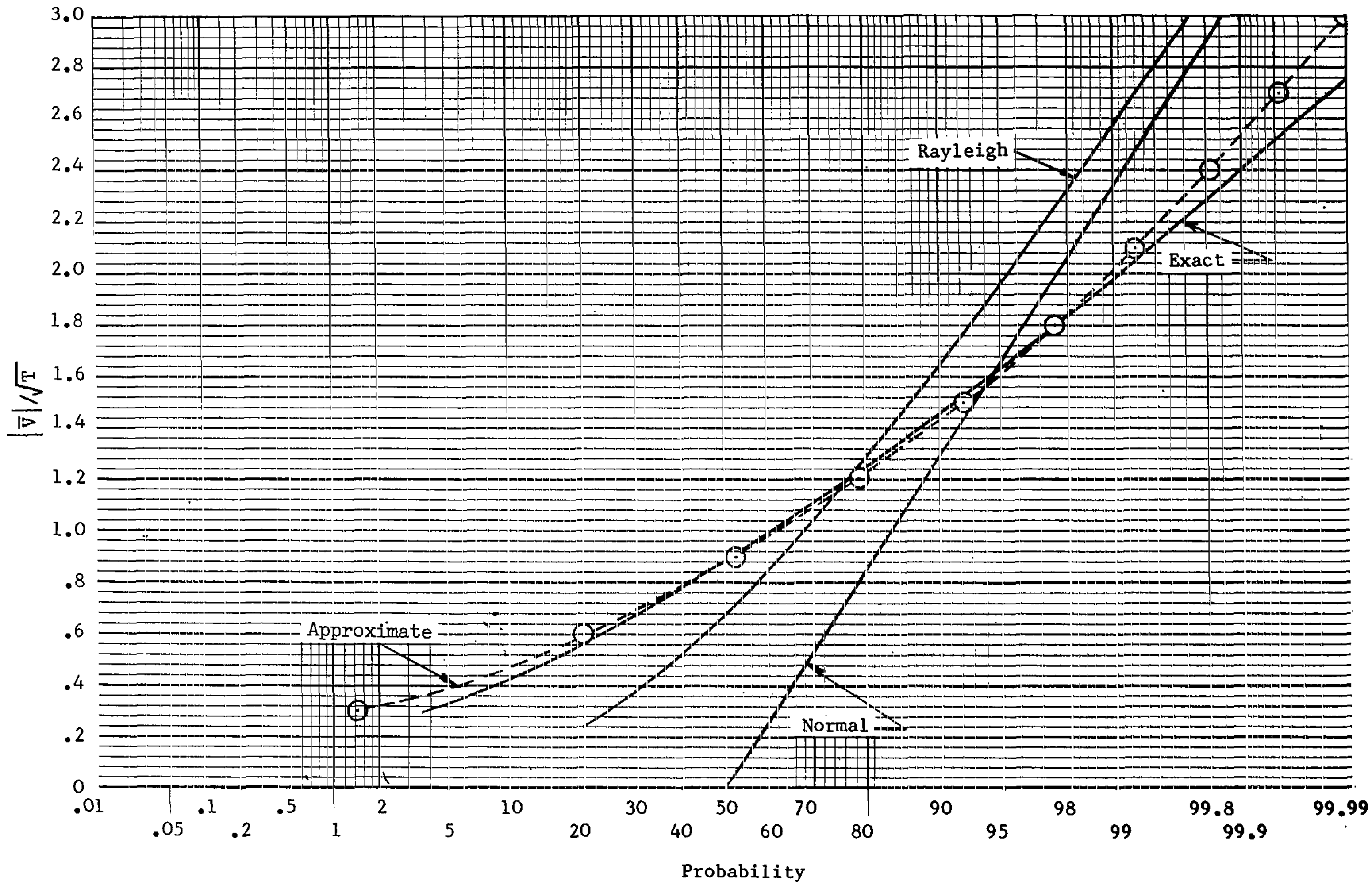
(h) Eigenvalue ratio of 1-1-0.

Figure 4.- Continued.



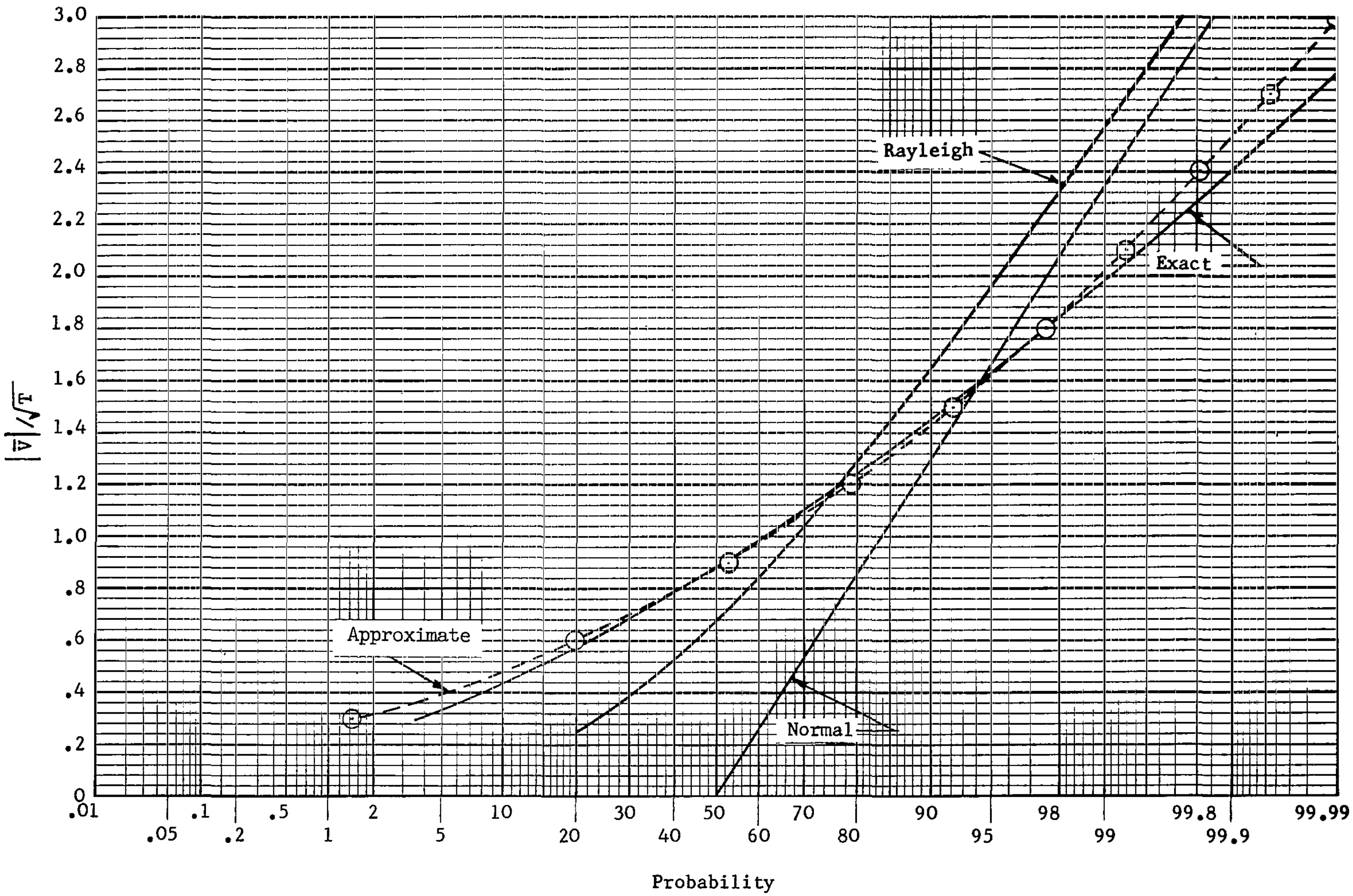
(i) Eigenvalue ratio of 2-9-9.

Figure 4.- Continued.



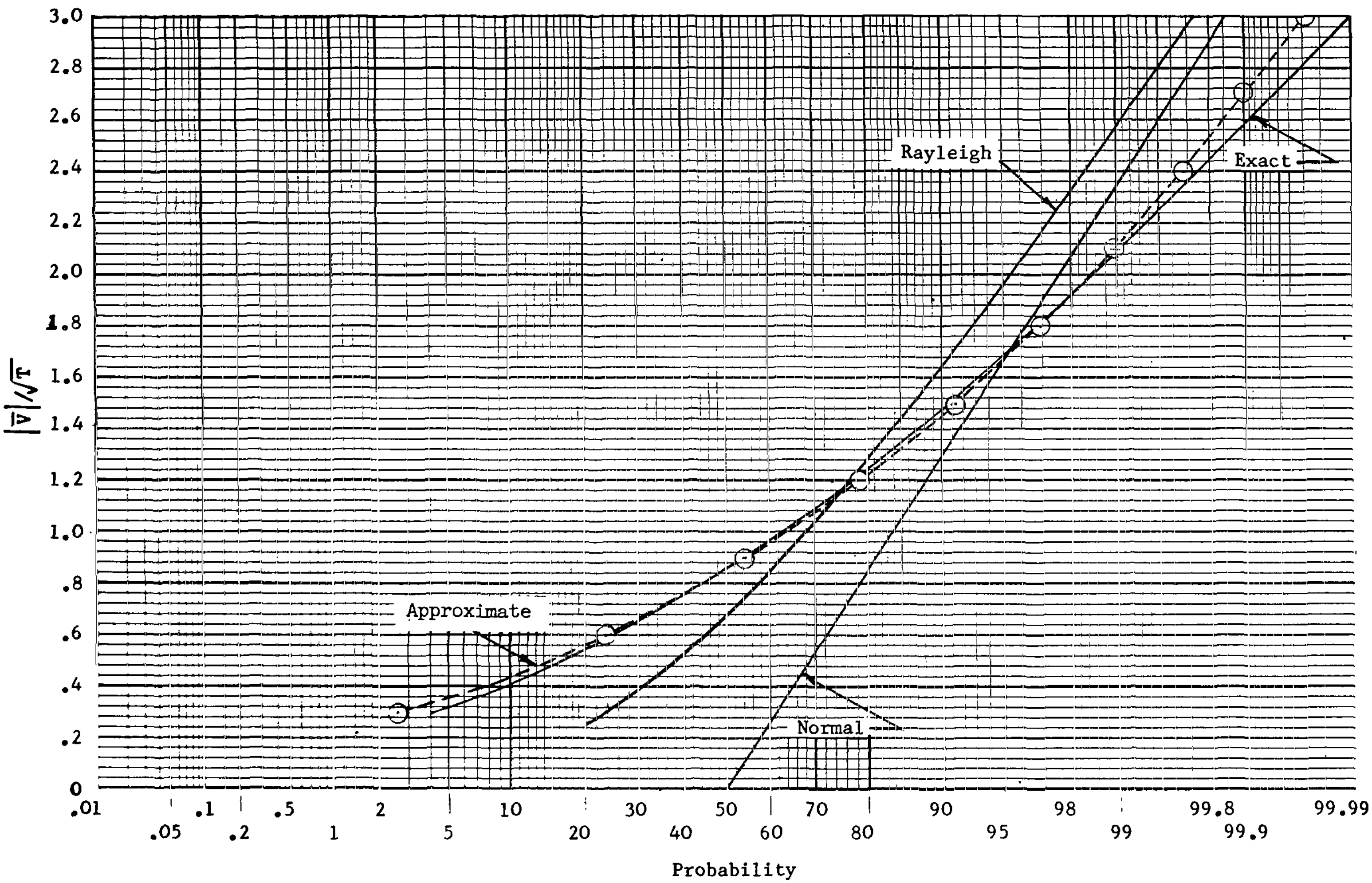
(j) Eigenvalue ratio of 1-2-2.

Figure 4.- Continued.



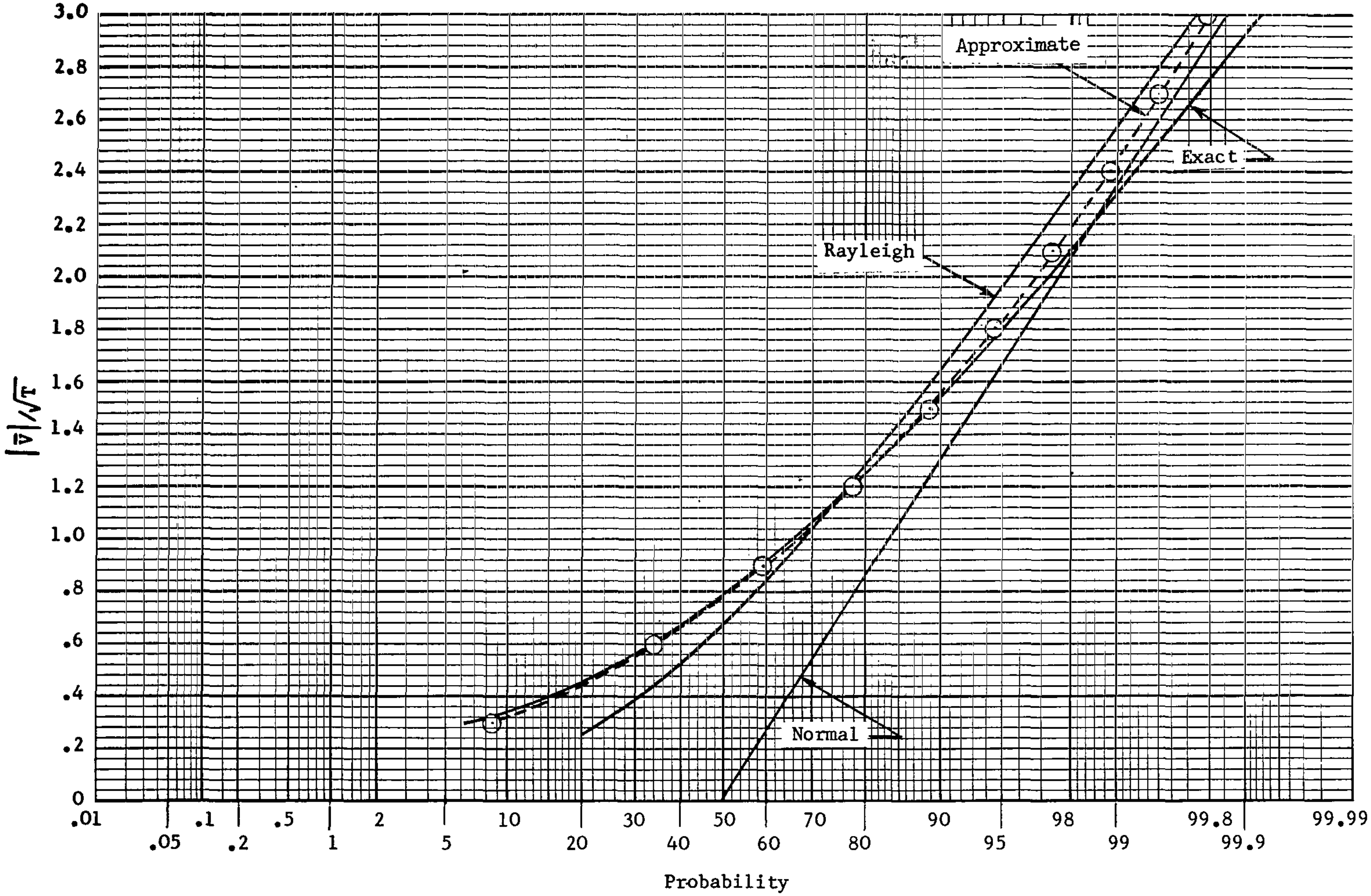
(k) Eigenvalue ratio of 443-333-224.

Figure 4.- Continued.



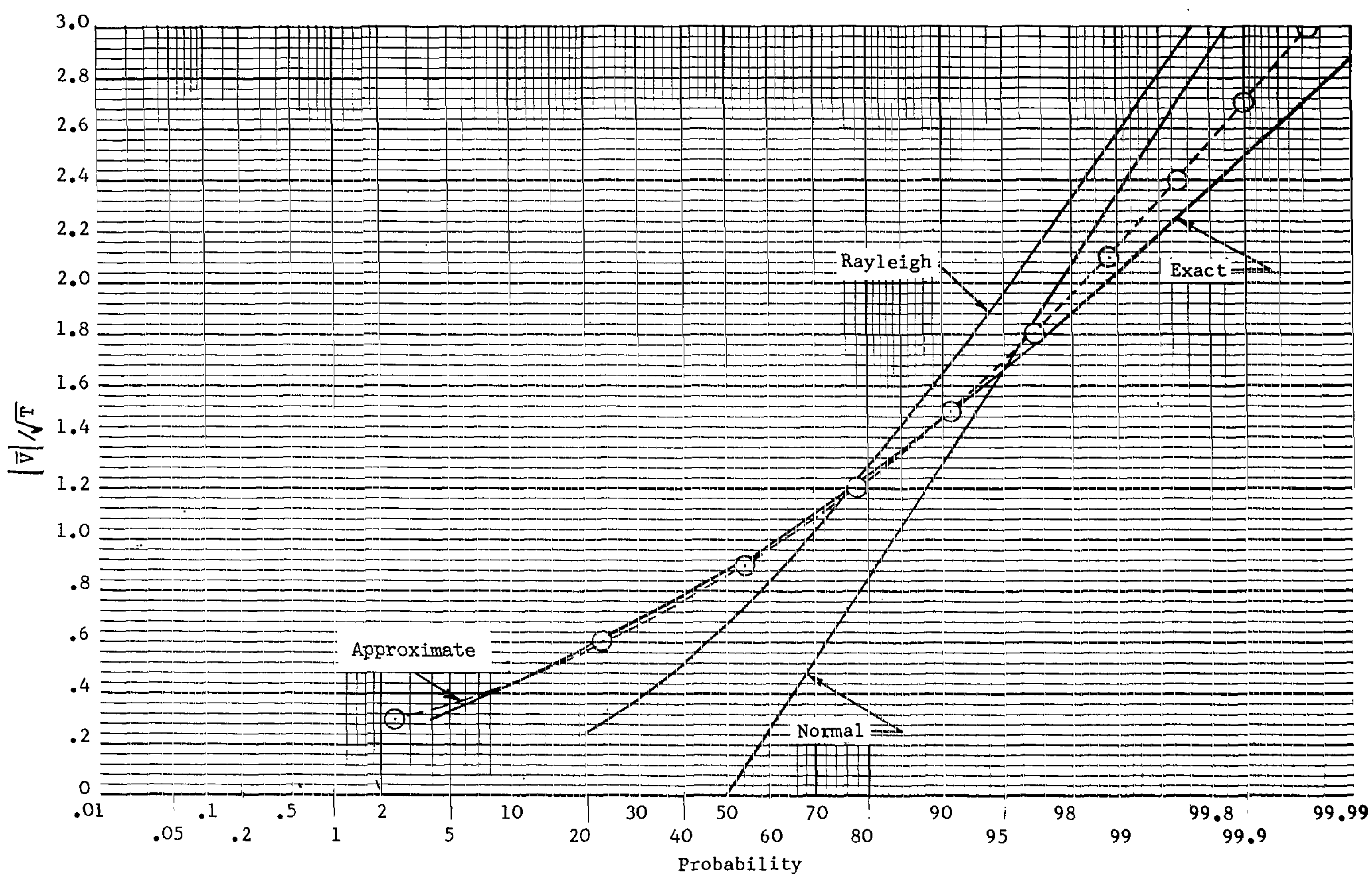
(l) Eigenvalue ratio of 57-27-16.

Figure 4.- Continued.



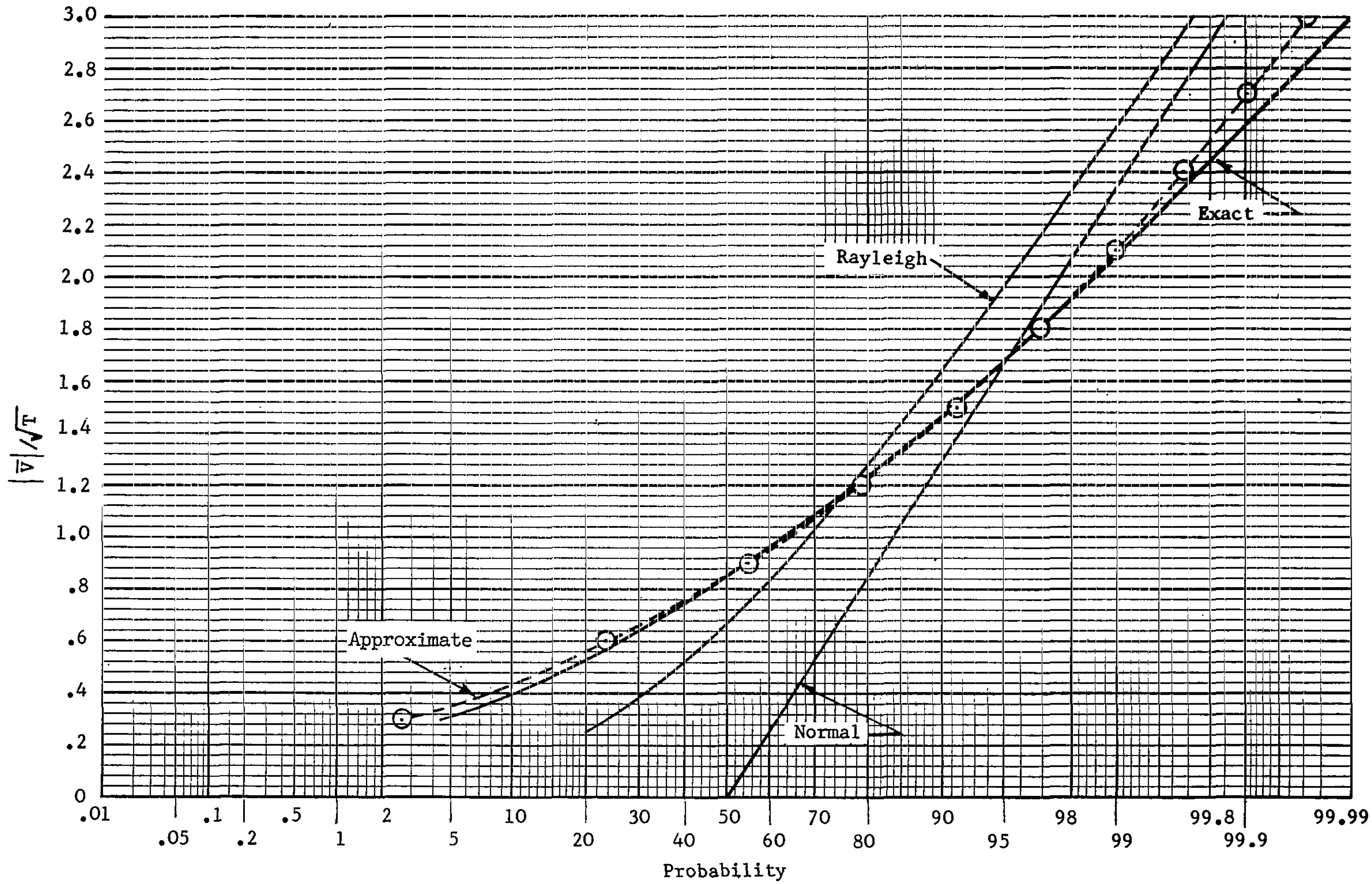
(m) Eigenvalue ratio of 1-2-10.

Figure 4.- Continued.



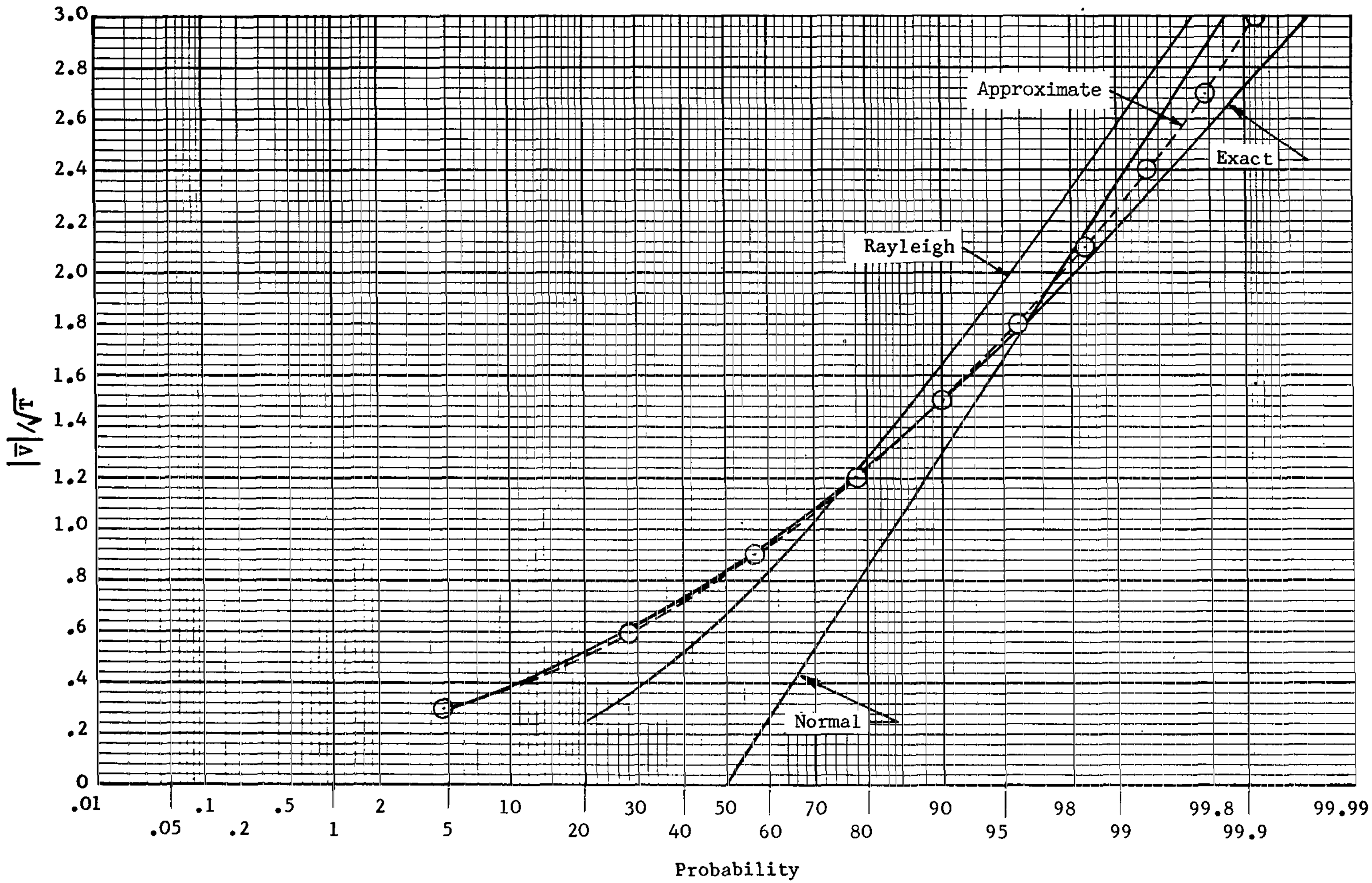
(n) Eigenvalue ratio of 1-2-3.

Figure 4.- Continued.



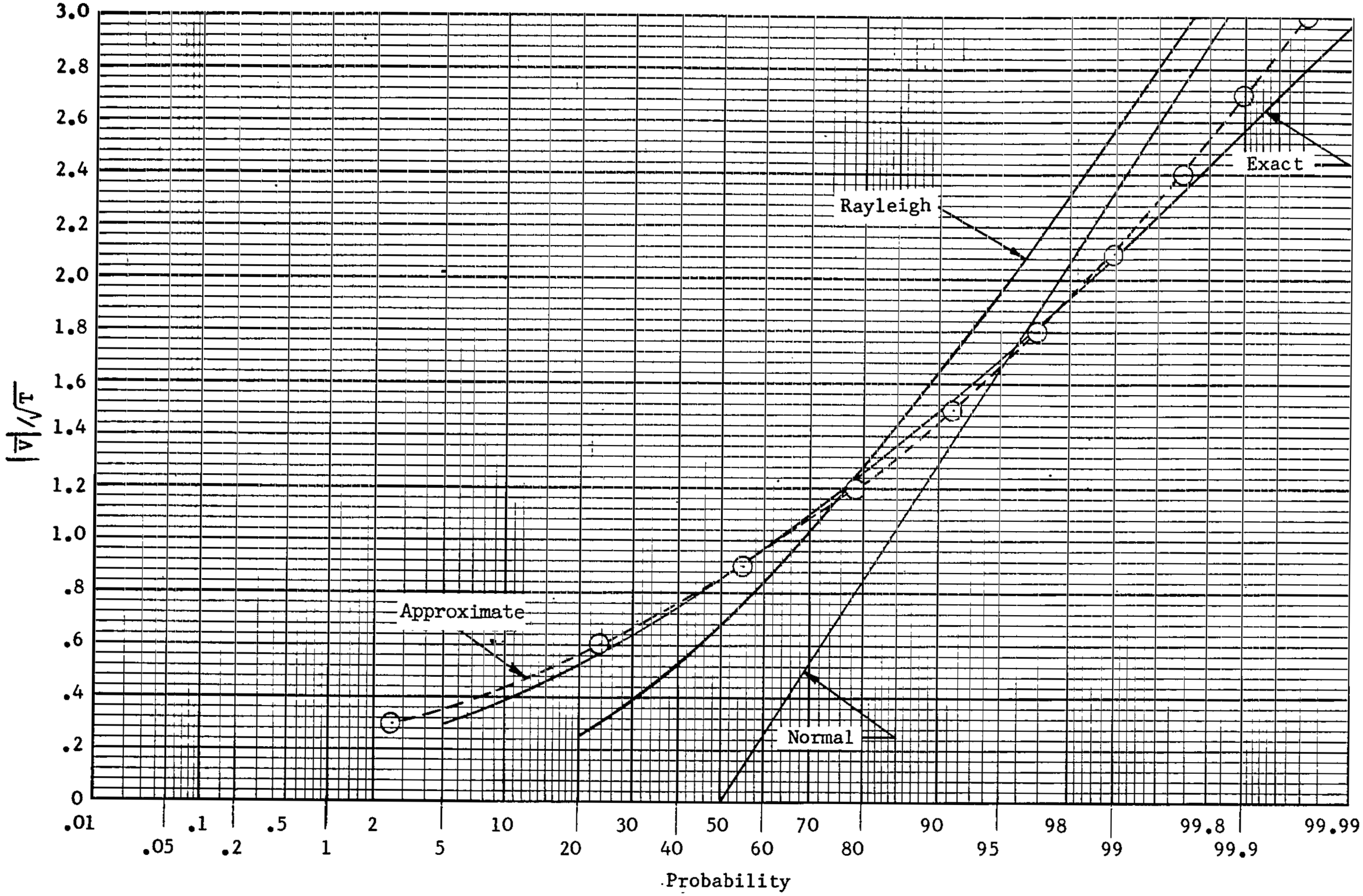
(o) Eigenvalue ratio of 1-3-5.

Figure 4.- Continued.



(p) Eigenvalue ratio of 2-5-13.

Figure 4.- Continued.



(q) Eigenvalue ratio of 52-41-7.

Figure 4.- Concluded.

NASA-Langley, 1969 — 30 L-6166

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D. C. 20546

OFFICIAL BUSINESS

FIRST CLASS MAIL

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS AN
SPACE ADMINISTRATION

POSTMASTER: If Undeliverable (Section 158 Postal Manual) Do Not Return

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

— NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546