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INDIVIDUAL ANGULAR MOMENTUM VECTOR DISTRIBUTION AND ROTATION LAWS FOR THREE DOUBLE-GIMBALED CONTROL MOMENT GYROS

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ABSTRACT

Three double-gimbaled control moment gyros constitute the actuators for one of the attitude control system modes of the Apollo Telescope Mount. A distribution law and a rotation law for the individual angular momentum vectors of the control moment gyros are developed because the attitude control law, in controlling the magnitude and the direction of the total angular momentum vector, uses only three of the available six degrees of freedom. Without a distribution law, a highly undesirable distribution can develop where two of the three individual angular momentum vectors are parallel and the third is antiparallel, reducing the total angular momentum available to one third of that which could be available with a desirable distribution. The desirable distribution is such that, after equilibrium is achieved, the individual angular momentum vectors have equal components along the total. Other results are equal angles between the individual vectors and equal angles between the individual vectors and the total. Because of the equal angles, the desired distribution is called the "isogonal distribution." Isogonal distribution has the following advantages: (1) antiparallel distribution is avoided, (2) the gain available for the attitude control law is maximized, and (3) the cross coupling is minimized. The distribution law uses two degrees of freedom; rotation about the total angular momentum vector still remains free. Therefore, a rotation law is added to minimize the inner gimbal angles with the result that hitting of the inner gimbal stops is avoided as much as possible. Operation of the distribution and rotation laws in case one of the control moment gyros fails is also discussed.

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DEFINITION OF SYMBOLS

i, j, k	cyclic permuted subscripts (1, 2, 3, or 2, 3, 1 or 3, 1, 2)
m, n	general subscripts (range 1, 2, 3; all possible combinations)
$\_$	a bar below a letter indicates a vector quantity
$\cdot$	a dot above a letter indicates a time derivative
A_i	a 2×3 matrix (eq. 9)
B_i	a 3×2 matrix (eq. 11 and 11a)
c	$\equiv \cos$
$\underline{e}_i$	a unit vector in direction of the ith CMG angular momentum
e_{mn}	direction cosines of $\underline{e}_i$ ($i = m$) with respect to the vehicle axis n
e_{ik}'	$\equiv \sqrt{1/(1 - e_{ik}^2)}$
$\underline{e}_T$	$\equiv \underline{e}_1 + \underline{e}_2 + \underline{e}_3$. Normalized total angular momentum of the CMG cluster (maximum value: 3)
e_T	$\equiv \mid \underline{e}_T \mid$
e_{Ti}	components of $\underline{e}_T$ in vehicle space
E_i	intermediate quantity used in the digital program

DEFINITION OF SYMBOLS (Continued)

$\tilde{e}$	$\equiv \begin{bmatrix} 0 & -e_3 & +e_2 \\ +e_3 & 0 & -e_1 \\ -e_2 & +e_1 & 0 \end{bmatrix}$
$\underline{h}_i$	see eq. 5
h_{mn}	components of $\underline{h}_i$ $(i = m)$ in vehicle space
K_D	gain of the isogonal distribution, [1/s]
K_D'	maximum of $\lvert K_D \rvert$, [1/s]
K_R	rotation gain $(K_R > 0)$, [1/s]
q	determinant (see eq. 13)
r_{mn}	intermediate quantities used in the digital program
s	$\equiv$ sin
t	$\equiv$ tan
$\underline{u}_i$	unit vector along the ith vehicle axis
x_V, y_V, z_V	vehicle axes
α_i	angles used in conjunction with the rotation law,
β_i	defined in Figure 4

DEFINITION OF SYMBOLS (Concluded)

$\delta_{1(i)}$	inner gimbal angle of ith CMG
$\delta_{3(i)}$	outer gimbal angle of ith CMG
$\underline{\delta}_i$	$\equiv \begin{bmatrix} \delta_{1(i)} \\ \delta_{3(i)} \end{bmatrix}$
ϵ_i	error of the isogonal distribution, [1/s]
ϵ_R	rotation error, [1/s]
ϵ_{Ri}	part of ϵ_R stemming from the ith CMG, [1/s]
$\epsilon_{Ri}{}'$	$= e_T{}^2 \, \epsilon_{Ri}/K_R$
λ_D	gain modifier for isogonal distribution
$\underline{\omega}_i$	actual angular velocity of $\underline{e}_i$, [1/s]
$\underline{\Omega}$	angular velocity command for rotation, [1/s]

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INDIVIDUAL ANGULAR MOMENTUM VECTOR DISTRIBUTION AND ROTATION LAWS FOR THREE DOUBLE-GIMBALED CONTROL MOMENT GYROS

SUMMARY

A distribution law and a rotation law for the individual angular momentum vectors of the three control moment gyros of the Apollo Telescope Mount are developed. The attitude control law controls the magnitude and the direction of the total angular momentum vector, using only three of the available six degrees of freedom. Without a distribution law, a highly undesirable distribution can develop where two of the three individual angular momentum vectors are parallel and the third is antiparallel, reducing the total angular momentum available by two thirds. The total is always fully available with the desirable distribution as provided by the distribution law, unless gimbal stops interfere. The desirable distribution has the following advantages: (1) antiparallel distribution is avoided, (2) the gain available for the attitude control law is maximized, and (3) the cross coupling is minimized. The rotation law minimizes the inner gimbal angles with the result that hitting of the inner gimbal stops is avoided as much as possible. Operation of the distribution and rotation laws in case one of the CMG's fails is also discussed.

INTRODUCTION

The Apollo Telescope Mount (ATM) carries three double-gimbaled control moment gyros (CMG) [1,2]. The magnitude of the angular momentum of each CMG is fixed, and the arrangement has six degrees of freedom. The attitude control law will command the magnitude and the direction of the total angular momentum; therefore, it utilizes only three degrees of freedom. The distribution of the individual angular momenta is not controlled. This freedom can lead to highly undesirable distributions, such as the antiparallel situation in which two vectors are parallel and the third is antiparallel. To avoid undesirable distributions, a distribution law is developed. The desirable distribution is such that, after equilibrium is achieved, the individual angular momentum vectors have equal components along the total. Other results are equal angles between the individual vectors and equal angles between the individual vectors and the total. Because of these equal angles, the desired distribution is called the "isogonal distribution" (Fig. 1). The isogonal distribution has the following advantages: (1) antiparallel distribution is avoided, (2) the gain available for the attitude control law is maximized, and (3) the cross coupling is minimized. The distribution law uses two degrees of freedom; rotation about the total angular momentum vector still remains free. Therefore, a rotation law is added to minimize the inner gimbal angles with the result that hitting of the inner gimbal stops is avoided as much as possible.

The existing gimbal angles are the inputs to the distribution and rotation (D&R) laws and the gimbal angle rate commands are the outputs. These rate commands are in addition to those from the attitude control law. To develop

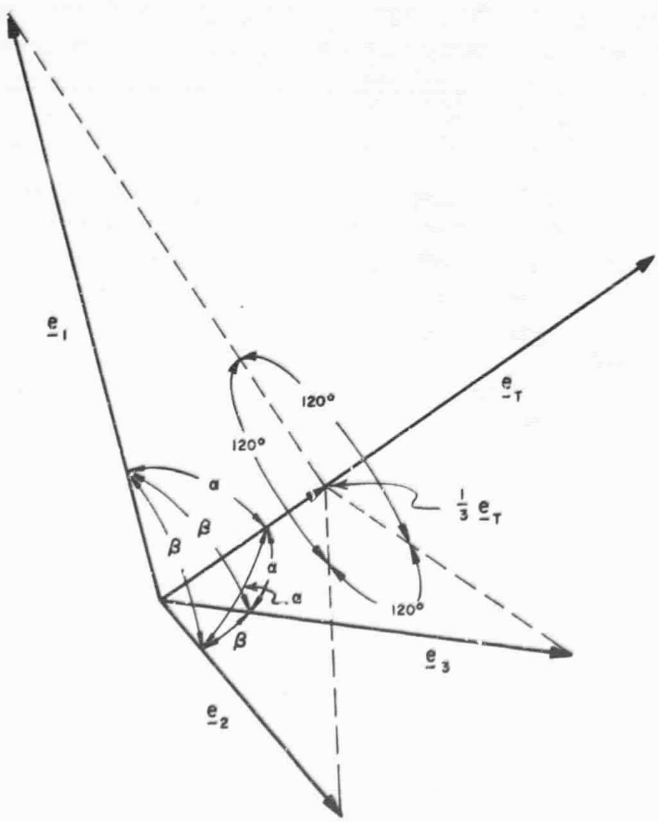

FIGURE 1. ISOGONAL DISTRIBUTION

the D&R laws it is assumed that the angular velocity of the vehicle is negligibly small and that the contribution of the gimbal rates caused by the D&R laws to the total angular momentum is negligible. For simplicity, unit vectors along the individual angular momentum vectors are used and are called $\underline{e}_i$ $(i = 1, 2, 3)$; their sum is $\underline{e}_T$ $(0 \leqq |e_T| \leqq 3)$. To avoid interference of the D&R laws with the attitude control law, it is necessary that these laws do not result in any change of the total angular momentum vector of the CMG cluster (i.e., no acceleration on the vehicle).

The ratio of the angular velocity command to an angular error function may be called the gain for the D&R laws. In addition to a selectable constant, there is an inherent gain which varies with the gimbal angles. This is acceptable if the loop is stable with the highest inherent gain.

DISTRIBUTION LAW

For the sake of argument we can assume that two of the $\underline{e}_i$-vectors are summed first and then the third is added to form $\underline{e}_T$. If we desire to redistribute the first two without disturbing the total ($\dot{\underline{e}}_T = 0$), we can only rotate the first two about their sum. The angular rate $\dot{\underline{e}}_i$ about this sum is made proportional to the scalar function ϵ_i which, for $\underline{e}_1$ and $\underline{e}_2$, is given by $K_D(\underline{e}_1 \cdot \underline{e}_T - \underline{e}_2 \cdot \underline{e}_T)$. This scalar function provides a signal proportional to the deviation from the desired orientation. To assure rotation about the sum of the first two vectors, $\dot{\underline{e}}_i$ is also made proportional to their cross product. Pairing $\underline{e}_1$ and $\underline{e}_2$, for example, we have (Fig. 2)

$$\dot{\underline{e}}_1 = \epsilon_3 (\underline{e}_2 \times \underline{e}_1) \tag{1}$$

$$\dot{\underline{e}}_2 = \epsilon_3 (\underline{e}_1 \times \underline{e}_2) \tag{2}$$

with $$\epsilon_3 = K_D (\underline{e}_1 \cdot \underline{e}_T - \underline{e}_2 \cdot \underline{e}_T) = K_D(\underline{e}_1 \cdot \underline{e}_3 - \underline{e}_2 \cdot \underline{e}_3) \tag{3}$$

By cyclic permutation the equivalent equations for the other two pairs can be developed. The constant K_D can be chosen from other considerations (see discussion at the end of this section). The cyclic permutation allows us

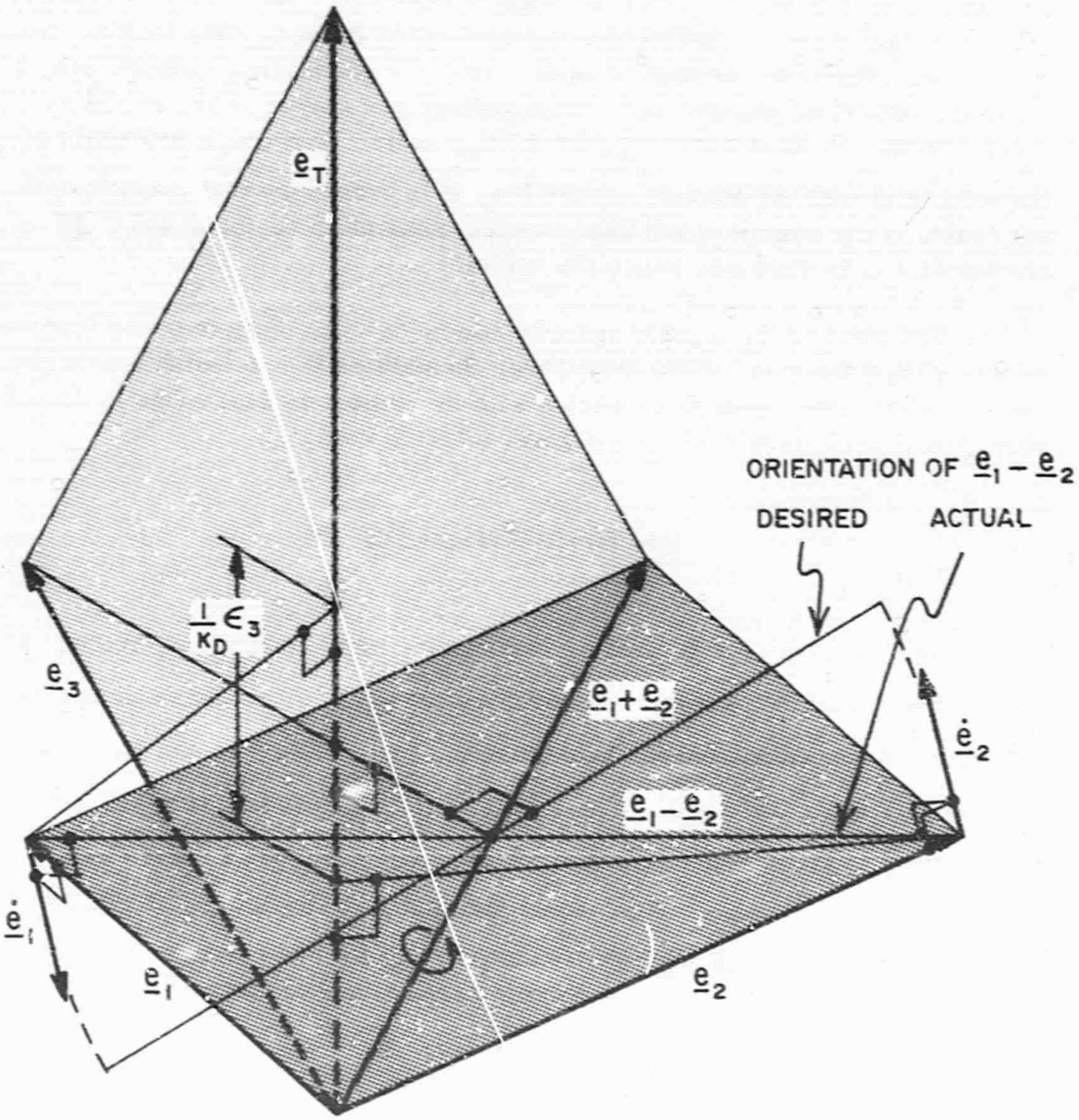

FIGURE 2. DEVELOPMENT OF DISTRIBUTION LAW

to use index notation with the understanding that we have the following three sets of values for i, j, and k:

i	j	k
1	2	3
2	3	1
3	1	2

The vector pairing can be assumed to be simultaneous and the resulting velocities vectorially added to form:

$$\dot{\underline{e}}_i = \underline{h}_i \times \underline{e}_i \tag{4}$$

with $$\underline{h}_i = \epsilon_k \underline{e}_j + \epsilon_j \underline{e}_k \tag{5}$$

It can be seen (after expansion of equations 4 and 5) that no matter what value is chosen for K_D, the sum of the $\dot{\underline{e}}_i$'s (or $\dot{\underline{e}}_T$) is zero (Appendix A).

To develop the equations for the gimbal velocity commands, we must consider the following built-in relationship (Fig. 3):

$$\underline{e}_i = \begin{bmatrix} e_{ii} \\ e_{ij} \\ e_{ik} \end{bmatrix} = c\delta_{1(i)} c\delta_{3(i)} \underline{u}_i - c\delta_{1(i)} s\delta_{3(i)} \underline{u}_j - s\delta_{1(i)} \underline{u}_k \tag{6}$$

where the $\underline{u}_i$'s are unit vectors along the vehicle axes with the indices permuted as previously shown, and the δ's are gimbal angles defined in Figure 3. The second index on the components of $\underline{e}_i$ refers to the vehicle axes; for example, e_{ij} is the direction cosine of $\underline{e}_i$ with respect to vehicle axis j. Therefore we can also write:

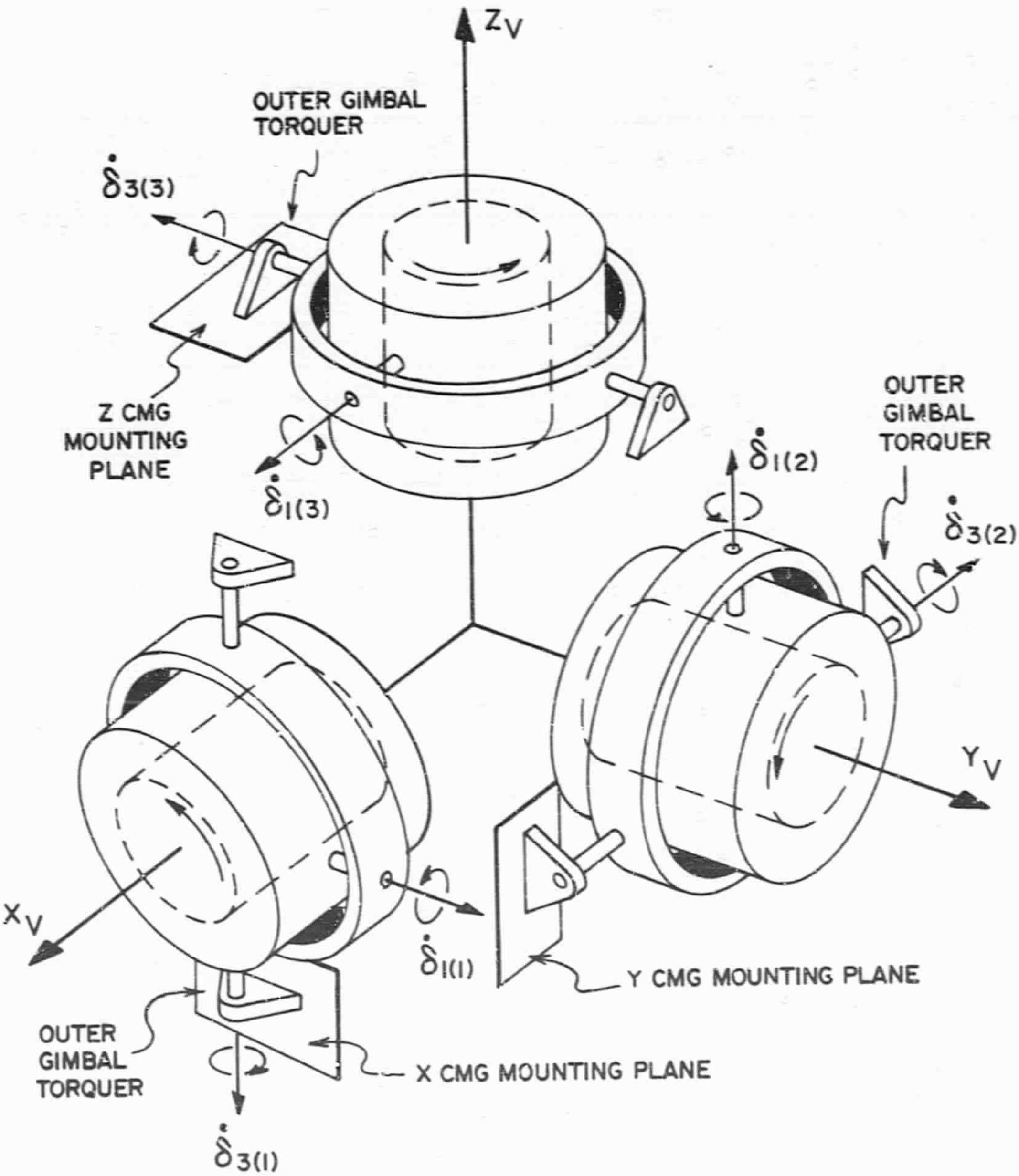

FIGURE 3. CONTROL MOMENT GYRO ORIENTATIONS

$$\begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \underline{u}_3 \end{bmatrix} \tag{7}$$

The generally nonorthogonal direction cosine matrix in equation 7 will be needed later. The e_{ij}'s are given in terms of gimbal angles on page 20.

For an existing gimbal velocity we have

$$\underline{\dot{e}}_i = A_i \underline{\dot{\delta}}_{(i)} \tag{8}$$

with

$$\underline{\dot{e}}_i = \begin{bmatrix} \dot{e}_{ii} \\ \dot{e}_{ij} \\ \dot{e}_{ik} \end{bmatrix} \quad \text{and} \quad \underline{\dot{\delta}}_{(i)} = \begin{bmatrix} \dot{\delta}_{1(i)} \\ \dot{\delta}_{3(i)} \end{bmatrix}$$

$$A_i = \begin{bmatrix} -s\delta_{1(i)} c\delta_{3(i)} & -c\delta_{1(i)} s\delta_{3(i)} \\ +s\delta_{1(i)} s\delta_{3(i)} & -c\delta_{1(i)} c\delta_{3(i)} \\ -c\delta_{1(i)} & 0 \end{bmatrix} \tag{9}$$

As $\underline{\dot{\delta}}_i$-command law we need (see Appendix B and proof below):

$$\underline{\dot{\delta}}_{(i)} = B_i \underline{h}_i \tag{10}$$

with

$$B_i = \begin{bmatrix} +s\delta_{3(i)} & +c\delta_{3(i)} & 0 \\ -t\delta_{1(i)} c\delta_{3(i)} & +t\delta_{1(i)} s\delta_{3(i)} & -1 \end{bmatrix} \tag{11}$$

or in terms of direction cosines (with $e_{ik}' = \sqrt{1/(1 - e_{ik}^2)}$):

$$B_i = \begin{bmatrix} - e_{ik}'e_{ij} & e_{ik}'e_{ii} & 0 \\ e_{ik}(e_{ik}')^2 e_{ii} & - e_{ik}(e_{ik}')^2 e_{ij} & -1 \end{bmatrix} \tag{11a}$$

The existing hardware implementation through resolvers on the gimbals provides the direction cosines of equation 7 whereas the gimbal angles are not available explicitly. Thus, B_i is expressed in terms of the direction cosines (see Appendix C).

By multiplying, it can be shown that

$$\dot{\underline{e}}_i = A_i \, B_i \, \underline{h}_i = - \underline{e}_i \times \underline{h}_i = \underline{h}_i \times \underline{e}_i$$

if it is assumed that the actual and the commanded gimbal rates are identical. Equation 10 is, therefore, the proper solution for the gimbal velocity commands to solve equation 4.

A discussion on the freely selectable isogonal distribution gain K_D (equation 3) is in order. When this gain is positive, we get a right-handed configuration (right-handed means that when we look down $\underline{e}_T$ and identify the individual vectors counterclockwise, we find the sequence 1 - 2 - 3); for a negative K_D, we get a left-handed configuration. Both configurations are stable. Since neither sign has a basic preference, we can dynamically (during operation of the CMG's) select the sign of the gain K_D so that the vectors are driven toward the closer solution. This action implies a knowledge of whether the instantaneous distribution is right- or left-handed. The following determinant affords this knowledge.

$$q = \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix}$$

It can be recognized that this is the determinant of the direction cosines for the angular momentum vectors. If these vectors are mutually perpendicular, we have an orthogonal coordinate system, and it is well known that the value of the determinant q is +1 for a right-handed system and -1 for a left-handed one. For a continuously deformable system as the one at hand, the determinant can be anywhere between these limits and the sign of the determinant can be used as the sign for the distribution gain K_D. A value of zero for q indicates that the three vectors are in a plane. A change in sign for q occurs under certain conditions. Gimbal stops may disallow the isogonal distribution and deform the distribution so much that q changes sign. The distribution gain is much lower than the attitude control gain (as discussed later), and maneuver commands can cause a relatively fast change in the distribution which could result in a sign change of q. Finally, whenever $\underline{e}_T$ passes through zero, q changes sign. Regardless of the reason for the sign change, it is always beneficial for the system behavior. A further improvement in system behavior can be made through the introduction of a multiplier λ_D so that the isogonal distribution gain becomes

$$K_D = \lambda_D K_D' \operatorname{sgn} q$$

where

$$K_D' = |K_D|_{max}$$

and

$$0 \leqq \lambda_D \leqq +1$$

The multiplier λ_D is a function of $e_T = |\underline{e}_T|$. For small e_T we desire $\lambda_D = 0$, thus avoiding the possibility for large angular redistributions when $\underline{e}_T$ passes close to zero. When e_T is in the vicinity of unity (i.e., without a distribution law, antiparallel is a problem), the multiplier λ_D should be at its maximum. For large e_T, a small λ_D is sufficient. The exact function of $\lambda_D(e_T)$ is not critical, and straight line segments may be used (Appendix C). The maximum distribution gain K_D' should be large enough to avoid antiparallel, but not larger than necessary because gimbal stops sometimes disallow the isogonal distribution and the attitude control gain must be higher than the distribution gain so that the attitude control takes over in case of conflict. During normal operation there is no conflict because the distribution and rotation laws do not result in an acceleration of the vehicle.

Temporary deviations from the isogonal distribution can be tolerated to a large extent. For example, for the antiparallel case, $\underline{e}_T = 1$. This case is depicted in Figure 1. The dot product between any of the isogonally distributed vectors and the antiparallel direction is - 1/3, or there is an angle of almost 110 degrees between the desired individual vector direction as demanded by the isogonal distribution law and the antiparallel direction. This indicates that even a deviation of 40 or 50 degrees from the desired direction can be temporarily accepted.

ROTATION LAW

The isogonal distribution law concerns itself only with the relative distribution between the individual and the total angular momentum vectors. A rotation about the total leaves the distribution unaffected. The rotational freedom can therefore be used for some benefit. To reach some desired orientation about $\underline{e}_T$, an angular velocity $\underline{\Omega}$ of the following form must be commanded.

$$\underline{\Omega} = \epsilon_R \underline{e}_T \tag{14}$$

where ϵ_R will be developed later. For the individual angular momentum vectors, the relation exists

$$\dot{\underline{e}}_i = \underline{\Omega} \times \underline{e}_i \tag{15}$$

Equation 15 is identical in form to equation 4; only $\underline{\Omega}$ is substituted for $\underline{h}_i$. Therefore we get analogous to equation 10

$$\dot{\underline{\delta}}_{(i)} = B_i \underline{\Omega} \tag{16}$$

No unique law exists for the rotation contrary to the unique isogonal distribution law. In the application of the three CMG system to ATM, the most benefit can be derived from a rotation law if the inner gimbal angles are minimized. Even under this assumption, minimizing must be first defined, since no unique definition exists. Avoidance of hitting the inner gimbal stops, which are at ±80 degrees, was given priority. It was also desirable to have a continuous function rather than a switching function for the rotation law. Because the three vectors rotate as a unit about $\underline{e}_T$, none of the inner gimbal angles will generally be at an absolute minimum, but a compromise will be reached for all three. The following form was selected for ϵ_R:

$$\epsilon_R = \sum_{i=1}^{3} \epsilon_{Ri} \tag{17}$$

with

$$\epsilon_{Ri} = \frac{K_R}{e_T} t\delta_{1(i)} s\alpha_i c\beta_i \tag{18}$$

Because ϵ_R will become zero at equilibrium and K_R is a freely selectable positive constant, no additional constants are necessary in equation 17.

The reasons for the various quantities in the error function ϵ_{Ri} are given. To make $\underline{\Omega}$ independent of the magnitude of $\underline{e}_T$, we have to divide by e_T. The tangent of the inner gimbal angle (rather than the angle itself) is chosen for two reasons: (1) to give the larger angles more weight and (2) to make use of trigonometric functions which are readily available in the form of the direction cosines (equation 7). Figure 4 defines the two other angles used for $i = 1$. Consider, for the sake of the explanation, that the plane for which $\delta_{1(i)} = 0$ is the equatorial plane. Then the angle α_i indicates the separation of the meridian plane containing $\underline{e}_i$ and the one containing $\underline{e}_T$. The combination of $t\delta_{1(i)} s\alpha_i$ will select the proper sign for the angular velocity and go to zero when the desired conditions are reached. The multiplier $c\beta_i$ takes the effectiveness of a rotation in reducing $\delta_{1(i)}$ into account, eliminating the ϵ_{Ri}-contribution to ϵ_R completely when $\beta_i = \frac{\pi}{2}$ where a rotation does not change $\delta_{1(i)}$.

We can identify several cases; Figure 5 shows an opaque unit sphere, the equator of which indicates $\delta_{1(i)} = 0$. The vertical line is a longitude onto which, for convenience, all intersects for the direction of $\underline{e}_T$ have been placed. Usually $\underline{e}_T$ will not terminate on the unit sphere.

<u>CASE I</u>. — The circle of possible tip locations for $\underline{e}_i$ (assuming a rotation about $\underline{e}_T$) does not intersect the equator, and the inner gimbal angle $\delta_{1(i)}$ is positive. We find two equilibrium conditions: one stable and one unstable.

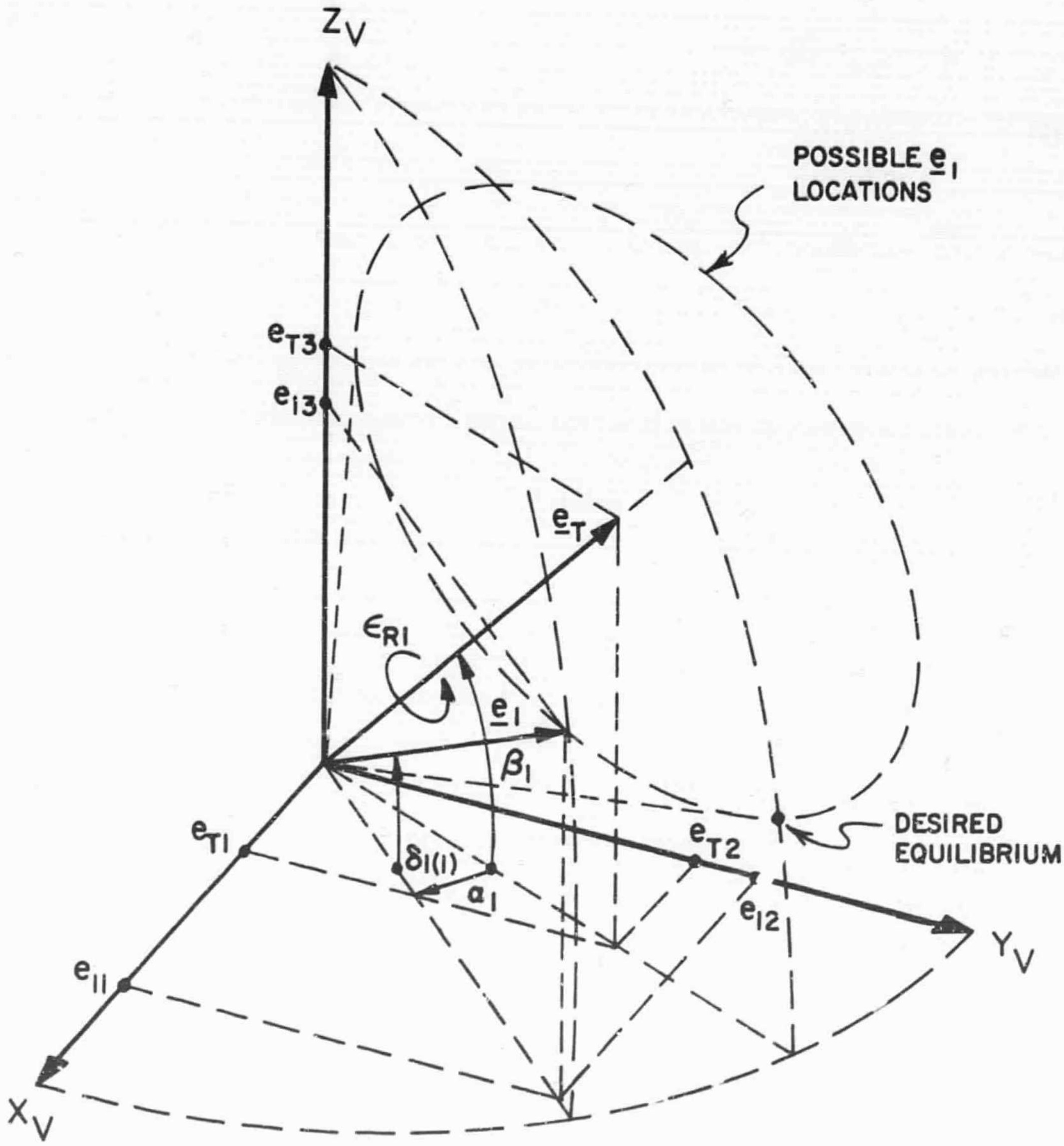

FIGURE 4. ANGLE DEFINITIONS FOR ROTATION LAW

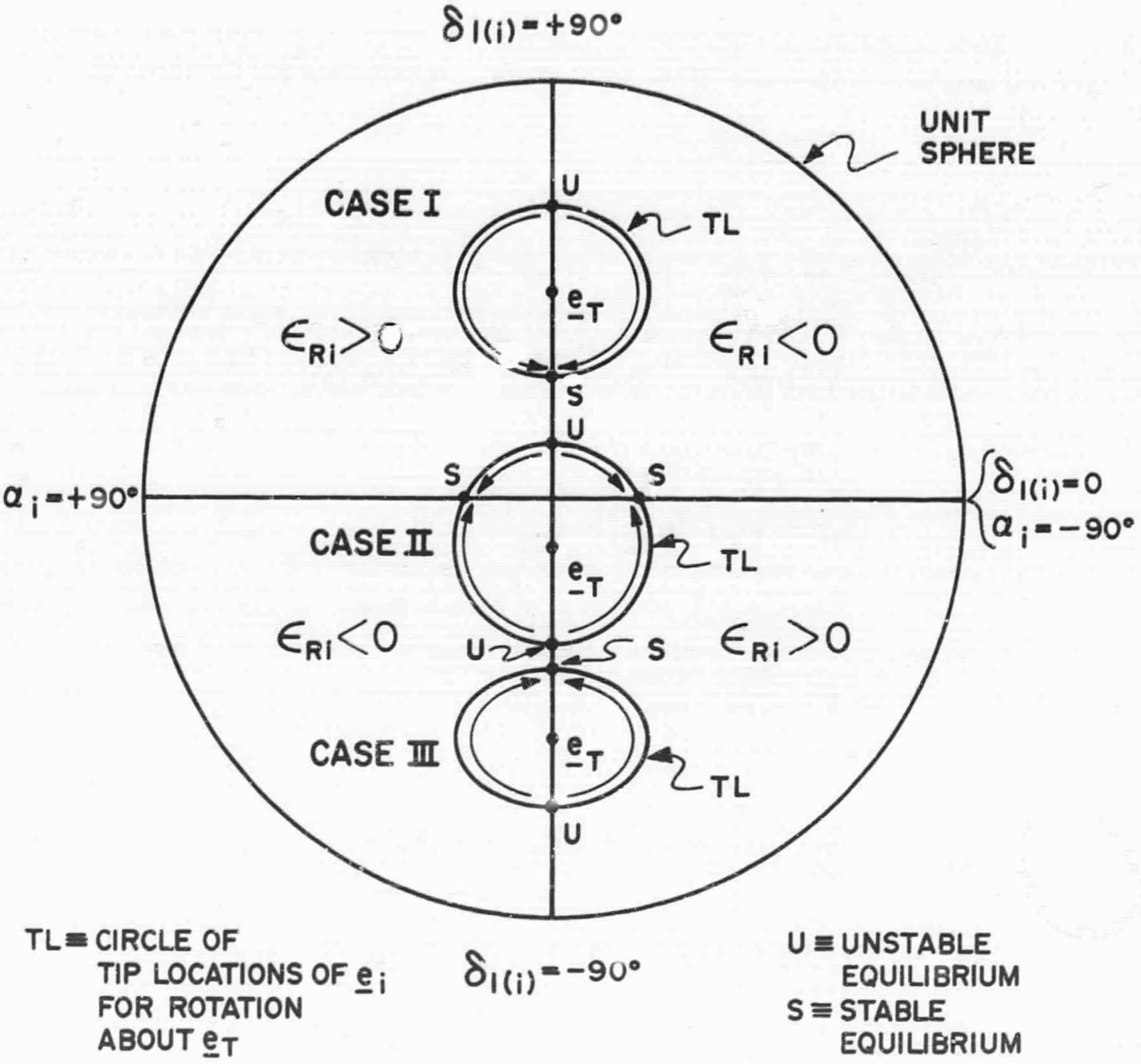

FIGURE 5. EQUILIBRIUM CASES FOR ROTATION

CASE II. — The circle of tip locations intersects the equator. We find four equilibria: two are intersections and are stable, and two are unstable.

CASE III. — This case is equivalent to case I; only the sign on $\delta_{1(i)}$ is reversed.

Determination of equation 18 in terms of the readily available direction cosines and the components of the total angular momentum vector leads to $(i = 1)$.

$$t\delta_{1(1)} = -\, e_{13}e_{13}' \qquad \text{with } e_{13}' = \frac{1}{\sqrt{1 - e_{13}^2}}$$

$$c\beta_1 = \sqrt{1 - \left(\frac{e_{T3}}{e_T}\right)^2} = \frac{1}{e_T}\sqrt{e_T^2 - e_{T3}^2} = \frac{1}{e_T}\sqrt{e_{T1}^2 + e_{T2}^2}$$

$$s\alpha_1 = \frac{e_{11}e_{T2} - e_{12}e_{T1}}{\sqrt{(e_{11}^2 + e_{12}^2)\,(e_{T1}^2 + e_{T2}^2)}}$$

$$= \frac{e_{11}e_{T2} - e_{12}e_{T1}}{\sqrt{(1 - e_{13}^2)\,(e_{T1}^2 + e_{T2}^2)}}$$

$$\epsilon_{R1} = \frac{1}{e_T^2}\left\{-e_{13}\,(e_{13}')^2\,(e_{11}e_{T2} - e_{12}e_{T1})\right\} \tag{19}$$

Cyclic permutation of the indices leads to ϵ_{R2} and ϵ_{R3}:

$$\epsilon_{R2} = \frac{1}{e_T^2}\left\{-\,e_{21}\,(e_{21}')^2\,(e_{22}e_{T3} - e_{23}e_{T2})\right\} \tag{20}$$

$$\epsilon_{R3} = \frac{1}{e_T^2}\left\{-\,e_{32}\,(e_{32}')^2\,(e_{33}e_{T1} - e_{31}e_{T3})\right\} \tag{21}$$

DISTRIBUTION AND ROTATION LAW COMBINATION

Since the desired gimbal velocity commands from the distribution law and from the rotation law are independent of each other, they can be combined into

$$\dot{\underline{\delta}}_{(i)} = B_i (\underline{h}_i + \epsilon_R \underline{e}_T) \tag{22}$$

These gimbal velocity commands should be added to the commands from the attitude control law.

The distribution is unaffected by the rotation as already indicated during the development of the rotation law; i.e., the ϵ_i's of the distribution law are not affected. On the other hand, ϵ_R is affected by a change in the distribution, and it is advisable that K_R (contained in ϵ_R) is made smaller than $K_D{}'$ (contained in $\underline{h}_i$ through ϵ_i). A ratio of $K_R/K_D{}' = 0.1$ was found to be acceptable, but it is not critical either way.

OPERATION WITH TWO CMG'S

A system with three CMG's has the capability to lose one and still be able to control the vehicle, even if the performance is degraded. Two CMG's have only four degrees of freedom, and three are being used for attitude control. There is no need for a distribution law since the two remaining angular momentum vectors have inherently the proper distribution, but the rotation law still can minimize the inner gimbal angles of the operative CMG's. To convert from the three to the two CMG operation, it is only necessary to set all the direction cosines of the inoperative CMG to zero. (Gimbal angle velocity commands issued by the computer to the inoperative CMG should be disregarded.) No distribution command will appear for the operative CMG's. But, as desired, the rotation law is working as usual, having now only two error sources, rather than three (equation 17).

CONCLUSIONS

The isogonal distribution law and the rotation law were implemented in a hybrid simulation, and the behavior was studied with the aid of a three-dimensional display.* Both laws performed as expected. It was also noted that, if gimbal stops disallowed the isogonal distribution, the distribution approximated the desired distribution as close as possible under the imposed restrictions. Analysis of the dynamic performance of the attitude control law showed a marked improvement in response of the desired channel and a strong reduction of the cross coupling into the other channels.

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*The three-dimensional display shows the body axes, the three individual angular momentum vectors, their total, and the total commanded angular momentum vector simultaneously, identified by appropriate coding. This proved an invaluable tool for the investigation of the behavior of the system [3].

APPENDIX A

Sum of the $\dot{\underline{e}}$'s

Expansion and combination of equations 4 and 5 yield

$$\dot{\underline{e}}_1 = (\epsilon_3 \underline{e}_2 + \epsilon_2 \underline{e}_3) \times \underline{e}_1$$

$$\dot{\underline{e}}_2 = (\epsilon_1 \underline{e}_3 + \epsilon_3 \underline{e}_1) \times \underline{e}_2 \qquad \text{(A-1)}$$

$$\dot{\underline{e}}_3 = (\epsilon_2 \underline{e}_1 + \epsilon_1 \underline{e}_2) \times \underline{e}_3$$

For the sum of the $\dot{\underline{e}}_i$'s, we get

$$\begin{aligned} \dot{\underline{e}}_T = &+ (\underline{e}_2 \times \underline{e}_3 + \underline{e}_3 \times \underline{e}_2)\,\epsilon_1 \\ &+ (\underline{e}_3 \times \underline{e}_1 + \underline{e}_1 \times \underline{e}_3)\,\epsilon_2 \qquad \text{(A-2)} \\ &+ (\underline{e}_1 \times \underline{e}_2 + \underline{e}_2 \times \underline{e}_1)\,\epsilon_3 \end{aligned}$$

Because the vectors in parenthesis are all zero, $\dot{\underline{e}}_T$ is zero and it is immaterial what is chosen for the ϵ_i's. The form chosen was (equation 3):

$$\epsilon_1 = K_D(\underline{e}_2 - \underline{e}_3) \cdot \underline{e}_1$$

$$\epsilon_2 = K_D(\underline{e}_3 - \underline{e}_1) \cdot \underline{e}_2$$

$$\epsilon_3 = K_D(\underline{e}_1 - \underline{e}_2) \cdot \underline{e}_3$$

Because the ϵ_i's are independent of each other, the K_D's could be different for each ϵ_i without affecting the condition that $\dot{\underline{e}}_T = 0$. This is not sensible though, because it would not treat the vector pairs alike, and there is no reason for that.

APPENDIX B

Development of Matrix B

We desire to get the gimbal velocities to solve equation 4, which is (subscripts dropped)

$$\dot{\underline{e}} = \underline{h} \times \underline{e} \tag{B-1}$$

The actual angular velocity of $\underline{e}$ can be defined as $\underline{\omega}$ and must be perpendicular to $\underline{e}$ because of the gimbal constraints. Then,

$$\dot{\underline{e}} = \underline{\omega} \times \underline{e} \tag{B-2}$$

This leads to (equations B-1 and B-2)

$$(\underline{h} - \underline{\omega}) \times \underline{e} = 0 \tag{B-3}$$

and this equation can only be solved because $\underline{\omega}$ and $\underline{e}$ are known in terms of the gimbal angles and their velocities. With

$$\underline{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} +c\delta_1 c\delta_3 \\ -c\delta_1 s\delta_3 \\ -s\delta_1 \end{bmatrix} \quad \text{and} \quad \underline{\omega} = \begin{bmatrix} +s\delta_3\dot{\delta}_1 \\ +c\delta_3\dot{\delta}_1 \\ -\dot{\delta}_3 \end{bmatrix}$$

we obtain for equation B-3

$$0 = \begin{bmatrix} 0 & -(h_3 + \dot{\delta}_3) & +(h_2 - c\delta_3\dot{\delta}_1) \\ +(h_3 + \dot{\delta}_3) & 0 & -(h_1 - s\delta_3\dot{\delta}_1) \\ -(h_2 - c\delta_3\dot{\delta}_1) & +(h_1 - s\delta_3\dot{\delta}_1) & 0 \end{bmatrix} \begin{bmatrix} +c\delta_1 c\delta_3 \\ -c\delta_1 s\delta_3 \\ -s\delta_1 \end{bmatrix} \tag{B-4}$$

The third equation in B-4 yields

$$\dot{\delta}_1 = +s\delta_3 h_1 + c\delta_3 h_2 \tag{B-5}$$

With equation B-5 the first or second equation of B-4 can be solved to get

$$\dot{\delta}_3 = +t\delta_1 (s\delta_3 h_2 - c\delta_3 h_1) - h_3 \tag{B-6}$$

or

$$\begin{bmatrix} \dot{\delta}_1 \\ \dot{\delta}_3 \end{bmatrix} = \begin{bmatrix} +s\delta_3 & +c\delta_3 & 0 \\ -t\delta_1 c\delta_3 & +t\delta_1 s\delta_3 & -1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

i.e., $\quad \underline{\dot{\delta}} = B\,\underline{h}$

It might be interesting to note that the cross-product law is

$$(XPR) = B\,A\,B$$

and that

$$\tilde{e} = -\,A\,B$$

where $\tilde{e}$ is defined as

$$\tilde{e} = \begin{bmatrix} 0 & -e_3 & +e_2 \\ +e_3 & 0 & -e_1 \\ -e_2 & +e_1 & 0 \end{bmatrix}$$

APPENDIX C

Suggested Digital Computer Implementation (Sequence of Operations)

Inputs.

$e_{11} = +c\delta_{1(1)} c\delta_{3(1)}$ $\quad K_D'$

$e_{12} = -c\delta_{1(1)} s\delta_{3(1)}$ $\quad K_R$

$e_{13} = -s\delta_{1(1)}$

$e_{21} = -s\delta_{1(2)}$

$e_{22} = +c\delta_{1(2)} c\delta_{3(2)}$

$e_{23} = -c\delta_{1(2)} s\delta_{3(2)}$

$e_{31} = -c\delta_{1(3)} s\delta_{3(3)}$

$e_{32} = -s\delta_{1(3)}$

$e_{33} = +c\delta_{1(3)} c\delta_{3(3)}$

The equivalences of the e_{mn}'s in terms of the gimbal angles are only given for completeness. The e_{mn}'s are provided by resolver chains in the existing hardware and constitute the actual inputs to the digital computer.

Sequence of Computations.

$$(e_{13}')^2 = (1 - e_{13}^2)^{-1}$$

$$e_{13}' = [(e_{13}')^2]^{+\frac{1}{2}}$$

both quantities are needed in the following

$$(e_{21}')^2 = (1 - e_{21}^2)^{-1}$$

$$e_{21}' = [(e_{21}')^2]^{+\frac{1}{2}}$$

$$(e_{32}')^2 = (1 - e_{32}^2)^{-1}$$

$$e_{32}' = [(e_{32}')^2]^{+\frac{1}{2}}$$

$$e_{T1} = e_{11} + e_{21} + e_{31}$$

$$e_{T2} = e_{12} + e_{22} + e_{32}$$

$$e_{T3} = e_{13} + e_{23} + e_{33}$$

$$e_T^2 = e_{T1}^2 + e_{T2}^2 + e_{T3}^2$$

$$e_T = \sqrt{e_T^2}$$

$$q = e_{11}(e_{22}e_{33} - e_{32}e_{23}) + e_{21}(e_{32}e_{13} - e_{12}e_{33}) + e_{31}(e_{12}e_{23} - e_{22}e_{13})$$

$\operatorname{sgn} q = +1$ for $q \geqq 0$

$\operatorname{sgn} q = -1$ for $q < 0$

$$\lambda_D = 0 \qquad \text{for} \qquad e_T \leqq 0.25$$

$$\lambda_D = 2e_T - 0.5 \qquad \text{for } 0.25 < e_T \leqq 0.75$$

$$\lambda_D = +1 \qquad \text{for } 0.75 < e_T \leqq 1.25$$

$$\lambda_D = +3.5 - 2e_T \qquad \text{for } 1.25 < e_T \leqq 1.65$$

$$\lambda_D = +0.2 \qquad \text{for } 1.65 < e_T$$

$$K_D = K_D' \lambda_D \operatorname{sgn} q$$

$$E_1 = e_{21}e_{31} + e_{22}e_{32} + e_{23}e_{33}$$

$$E_2 = e_{11}e_{31} + e_{12}e_{32} + e_{13}e_{33}$$

$$E_3 = e_{11}e_{21} + e_{12}e_{22} + e_{13}e_{23}$$

$$\epsilon_1 = (E_3 - E_2) K_D$$

$$\epsilon_2 = (E_1 - E_3) K_D$$

$$\epsilon_3 = (E_2 - E_1) K_D$$

$$h_{11} = \epsilon_3 e_{21} + \epsilon_2 e_{31}$$

$$h_{12} = \epsilon_3 e_{22} + \epsilon_2 e_{32}$$

$$h_{13} = \epsilon_3 e_{23} + \epsilon_2 e_{33}$$

$$h_{21} = \epsilon_1 e_{31} + \epsilon_3 e_{11}$$

$$h_{22} = \epsilon_1 e_{32} + \epsilon_3 e_{12}$$

$$h_{23} = \epsilon_1 e_{33} + \epsilon_3 e_{13}$$

$$h_{31} = \epsilon_2 e_{11} + \epsilon_1 e_{21}$$

$$h_{32} = \epsilon_2 e_{12} + \epsilon_1 e_{22}$$

$$h_{33} = \epsilon_2 e_{13} + \epsilon_1 e_{23}$$

$$\epsilon_{R1}' = e_{13}(e_{13}')^2(e_{12}e_{T1} - e_{11}e_{T2})$$

$$\epsilon_{R2}' = e_{21}(e_{21}')^2(e_{23}e_{T2} - e_{22}e_{T3})$$

$$\epsilon_{R3}' = e_{32}(e_{32}')^2(e_{31}e_{T3} - e_{33}e_{T1})$$

$$\epsilon_R = \frac{K_R}{e_T^2}(\epsilon_{R1}' + \epsilon_{R2}' + \epsilon_{R3}')$$

$$r_{11} = h_{11} + \epsilon_R e_{T1}$$

$$r_{12} = h_{12} + \epsilon_R e_{T2}$$

$$r_{13} = h_{13} + \epsilon_R e_{T3}$$

$$r_{21} = h_{21} + \epsilon_R e_{T1}$$

$$r_{22} = h_{22} + \epsilon_R e_{T2}$$

$$r_{23} = h_{23} + \epsilon_R e_{T3}$$

$$r_{31} = h_{31} + \epsilon_R e_{T1}$$

$$r_{32} = h_{32} + \epsilon_R e_{T2}$$

$$r_{33} = h_{33} + \epsilon_R e_{T3}$$

Outputs.

$$\dot{\delta}_{1(1)} = -e_{13}'(e_{12}r_{11} - e_{11}r_{12})$$

$$\dot{\delta}_{3(1)} = +(e_{13}')^2 e_{13}(e_{11}r_{11} + e_{12}r_{12}) - r_{13}$$

$$\dot{\delta}_{1(2)} = -e_{21}'(e_{23}r_{22} - e_{22}r_{23})$$

$$\dot{\delta}_{3(2)} = +(e_{21}')^2 e_{21}(e_{22}r_{22} + e_{23}r_{23}) - r_{21}$$

$$\dot{\delta}_{1(3)} = -e_{32}'(e_{31}r_{33} - e_{33}r_{31})$$

$$\dot{\delta}_{3(3)} = +(e_{32}')^2 e_{32}(e_{33}r_{33} + e_{31}r_{31}) - r_{32}$$

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APPROVAL

INDIVIDUAL ANGULAR MOMENTUM VECTOR DISTRIBUTION AND ROTATION LAWS FOR THREE DOUBLE-GIMBALED CONTROL MOMENT GYROS

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The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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