THE SHAPE OF THE ELASTIC CURVE OF A WEIGHTLESS ROTATING SHAFT CARRYING ECCENTRICALLY LOCATED POINT MASSES OR DISKS

Docent A. N. Ogurechnikov

The study of the shape of the elastic curve of a rotating shaft which precesses as it undergoes resonance vibrations is of no particular practical significance, since the theoretical basis in calculating the vibration frequency is Wiedler postulate, in which the elastic curve is regarded as plane, and its shape during resonance is assumed to be similar to that of the elastic curve in unrestrained oscillation. However, it is quite frequently claimed that the elastic curve of a vibrating shaft, by virtue of the different directions of eccentricities of its associated masses, is actually a space curve. This assertion, which is entirely correct for forced vibrations of a shaft, becomes invalid when extended to the case of resonance vibrations.

The elastic curve of a shaft undergoing precession during resonance vibrations must be a plane curve, since the shaft vibrates in all axial planes with the same frequency; thus, all the projections of the outline of the elastic curve onto these planes should be similar, which can occur only if the elastic curve is a planar one.

Academician B. S. Stechkin has rigorously proven the validity of Wiedler's postulate for systems executing torsional vibrations. We shall prove the validity of this postulate for systems which execute flexural vibrations, namely:
1) for a shaft with two eccentrically located point masses;
2) for a shaft carrying an eccentrically coupled point mass and an obliquely seated ideal thin disk.

Without detriment to the generality of proof, we shall consider the case of direct synchronous precession, where the eccentricities and the torque transmitted to the shaft by the drive are the sources of the vibrations.

A PERFECTLY ELASTIC WEIGHTLESS SHAFT WITH TWO ECCENTRICALLY COUPLED POINT MASSES

A schematic of the shaft system is shown in Fig. 1. The axis of the system at rest is denoted by point 0. We use the notation: e_1 and e_2 — eccentricities of the coupled point masses; 2τ — angle between the direction of above eccentricities; m_1 and m_2 — point masses coupled to the system; 0_1 and 0_2 — points of the elastic curve of the bent shaft at the sections passing through the sites of coupling of the point masses;

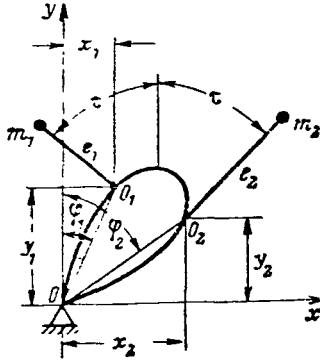*Numbers in the margin indicate pagination in the foreign text.

Figure 1. The Projection of the Elastic Curve of a Shaft with Two Point Masses onto a Plane Perpendicular to its Axis.

y and x - coordinate axes; here y is the axis parallel to the bisector of angle 2τ; φ_1 and φ_2 - direction angles of the shaft deflections.

When the shaft rotates with velocity ω lower than the critical one, a direct synchronous precession ensues. Due to the eccentricities e_1 and e_2 of the coupled masses m_1 and m_2 the shaft axis will be deflected and thus become a spatial curve; the shaft sections, to which these masses are connected, will occupy positions 0_1 and 0_2.

Since

$$\omega \neq \omega_{cr}$$

the shaft deflections are described by

$$y_1 = (y_1 + e_1 \cos \tau)\, m_1 \omega^2 \delta_{11} + (y_2 + e_2 \cos \tau)\, m_2 \omega^2 \delta_{21};$$
$$y_2 = (y_1 + e_1 \cos \tau)\, m_1 \omega^2 \delta_{12} + (y_2 + e_2 \cos \tau)\, m_2 \omega^2 \delta_{22};$$
$$x_1 = (x_1 - e_1 \sin \tau)\, m_1 \omega^2 \delta_{11} + (x_2 + e_2 \sin \tau)\, m_2 \omega^2 \delta_{21};$$
$$x_2 = (x_1 - e_1 \sin \tau)\, m_1 \omega^2 \delta_{12} + (x_2 + e_2 \sin \tau)\, m_2 \omega^2 \delta_{22},$$

where δ_{11}, δ_{12}, δ_{21}, δ_{22} are the compliances of the shaft. If we denote

$$e_1 \cos \tau = \vartheta_1; \quad e_2 \cos \tau = \vartheta_2;$$
$$e_1 \sin \tau = \theta_1; \quad e_2 \sin \tau = \theta_2;$$
$$m_1 \omega^2 \delta_{11} = n_{11}; \quad m_1 \omega^2 \delta_{12} = n_{12};$$
$$m_2 \omega^2 \delta_{22} = n_{22}; \quad m_2 \omega^2 \delta_{21} = n_{21},$$

then this system of equations can be written as

$$y_1 (n_{11} - 1) + n_{21} y_2 = -\vartheta_1 n_{11} - \vartheta_2 n_{21};$$
$$y_1 n_{12} + y_2 (n_{22} - 1) = -\vartheta_1 n_{12} - \vartheta_2 n_{22};$$
$$x_1 (n_{11} - 1) + x_2 n_{21} = \theta_1 n_{11} - \theta_2 n_{21};$$
$$x_1 n_{12} + x_2 (n_{22} - 1) = \theta_1 n_{12} - \theta_2 n_{22}.$$

As can be seen, this system of equations contains two parts: the first two equations contain only y_1 and y_2, while the last two contain only x_1 and x_2.

The determinant composed of coefficients of the unknown y_1 and y_2 (of the first two equations) is

$$D_y = (n_{11} - 1)(n_{22} - 1) - n_{12}n_{21}.$$

The value of y_1 is the fraction

$$g_1 = \frac{\Delta y_1}{D_y},$$

where Δy_1 is a determinant which is obtained from the system's determinant by replacing the first column by the column of free terms

$$\Delta y_1 = -(\vartheta_1 n_{11} + \vartheta_2 n_{21})(n_{22} - 1) + (\vartheta_1 n_{12} + \vartheta_2 n_{22})n_{21},$$

and, consequently,

$$y_1 = \frac{-(\vartheta_1 n_{11} + \vartheta_2 n_{21})(n_{22} - 1) + (\vartheta_1 n_{12} + \vartheta_2 n_{22})n_{21}}{(n_{11} - 1)(n_{22} - 1) - n_{12}n_{21}}.$$

Similarly,

$$\Delta y_2 = -(n_{11} - 1)(\vartheta_1 n_{12} + \vartheta_2 n_{22}) + n_{12}(\vartheta_1 n_{11} + \vartheta_2 n_{21})$$

and

$$y_2 = \frac{-(\vartheta_1 n_{12} + \vartheta_2 n_{22})(n_{11} - 1) + (\vartheta_1 n_{11} + \vartheta_2 n_{21})n_{12}}{(n_{11} - 1)(n_{22} - 1) - n_{12}n_{21}}.$$

The determinant composed of coefficients of unknown x_1 and x_2 (of the two last equations) is

$$D_x = (n_{11} - 1)(n_{22} - 1) - n_{12}n_{21} = D_y;$$
$$\Delta x_1 = (\theta_1 n_{11} - \theta_2 n_{21})(n_{22} - 1) - (\theta_1 n_{12} - \theta_2 n_{22})n_{21},$$

hence we find in the same manner as above

$$x_1 = \frac{(\theta_1 n_{11} - \theta_2 n_{21})(n_{22} - 1) - (\theta_1 n_{12} - \theta_2 n_{22})n_{21}}{(n_{11} - 1)(n_{22} - 1) - n_{12}n_{21}}$$

and, using the expression

$$\Delta x_2 = (n_{11} - 1)(\theta_1 n_{12} - \theta_2 n_{22}) - (\theta_1 n_{11} - \theta_2 n_{21})n_{12},$$

we get

$$x_2 = \frac{(\theta_1 n_{12} - \theta_2 n_{22})(n_{11} - 1) - (\theta_1 n_{11} - \theta_2 n_{21}) n_{12}}{(n_{22} - 1)(n_{11} - 1) - n_{12} n_{21}} .$$

From the deflection coordinates thus found we can obtain the deflections themselves

$$\varrho_1 = \sqrt{x_1^2 + y_1^2} \text{ and } \varrho_2 = \sqrt{x_2^2 + y_2^2},$$

as well as the angles between the direction of deflections and the y axis

$$\tan \varphi_1 = \frac{x_1}{y_1} \text{ and } \tan \varphi_2 = \frac{x_2}{y_2}.$$

Let us now ask two questions: 1) is the elastic curve of the shaft planar? 2) How do angles φ_1 and φ_2 vary with changes in the rpm?

If the elastic curve of the shaft is to be planar, then angles φ_1 and φ_2 must be equal, i.e., the ratios

$$\frac{x_1}{y_1} \text{ and } \frac{x_2}{y_2},$$

must be equal; representing this in terms of previously found values of x and y

$$\frac{x_1}{y_1} = \frac{(\theta_1 n_{11} - \theta_2 n_{21})(n_{22} - 1) - (\theta_1 n_{12} - \theta_2 n_{22}) n_{21}}{-(\vartheta_1 n_{11} + \vartheta_2 n_{21})(n_{22} - 1) + (\vartheta_1 n_{12} + \vartheta_2 n_{22}) n_{21}}$$

and

$$\frac{x_2}{y_2} = \frac{(\theta_1 n_{12} - \theta_2 n_{22})(n_{11} - 1) - (\theta_1 n_{11} - \theta_2 n_{21}) n_{12}}{-(\vartheta_1 n_{12} + \vartheta_2 n_{22})(n_{11} - 1) + (\vartheta_1 n_{11} + \vartheta_2 n_{21}) n_{12}} .$$

We first consider two cases: 1) if $2\tau = 0$ and consequently $\tau = 0$, i.e., if the eccentricities are codirectional away from the shaft, then

$$\vartheta_1 = e_1 \text{ and } \vartheta_2 = e_2,$$

and

$$\theta_1 = 0 \text{ and } \theta_2 = 0.$$

It can be seen from the expressions for $\underline{x}_1$ and $\underline{x}_2$ that the numerators become zero so that $\varphi_1 = 0$ and $\varphi_2 = 0$, since the denominators then are not zero. Then also $y_1 \neq 0$, and $y_2 \neq 0$, since $\vartheta_1 \neq 0$ and $\vartheta_2 \neq 0$. As far as $D_x = D_y \neq 0$ is concerned, the expressions for these terms are finite when $11 \neq 11_{cr}$.

When $\omega = \omega_{cr}$, $\underline{D} = \overline{\underline{D}}_{\underline{x}} = \overline{\underline{D}}_{y} = 0$, or

$$D = (\overline{n}_{11} - 1)(\overline{n}_{22} - 1) - \overline{n}_{12}\overline{n}_{21} =$$
$$= (m_1 \omega_{cr}^2 \delta_{11} - 1)(m_2 \omega_{cr}^2 \delta_{22} - 1) - m_1 \omega_{cr}^2 \delta_{12} m_2 \omega_{cr}^2 \delta_{21} =$$
$$= \omega_{cr}^4 m_1 m_2 (\delta_{11}\delta_{22} - \delta_{12}\delta_{21}) - \omega_{cr}^2 (m_1 \delta_1 + m_2 \delta_{22}) + 1 = 0,$$

since the last expression is a periodic equation when written in the standard form.

In analyzing the first case we see that when the eccentricities are directed in the same direction, the elastic curve of the shaft will be planar if

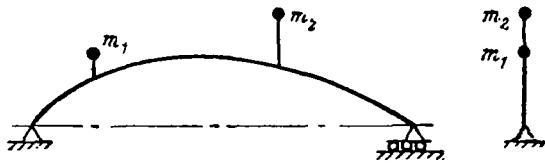$$\omega \neq \omega_{cr} \quad \text{and} \quad \varphi_1 = \varphi_2 = 0.$$

Since $\underline{D} \to 1$ as $\omega \to 0$ and remains greater than zero for all the $\omega < \omega_{cr}$, then the critical angular velocity ω_{cr} is smaller than the partial angular velocity ω_{part}. Since

Figure 2. The Shape of the Elastic Curve of a Shaft Rotating at Subcritical Speed.

$$\omega_{1part} = \sqrt{\frac{1}{m_1 \delta_{11}}}; \quad \omega_{2part} = \sqrt{\frac{1}{m_2 \delta_{22}}}$$

and further $0 < \underline{n}_{22} < 1$ and $0 < \underline{n}_{11} < 1$ when $n_{12} > 0$ and $n_{21} > 0$,

then

$$y_1 > 0 \quad \text{and} \quad y_2 > 0.$$

Consequently, at $\omega < \omega_{cr}$, the elastic curve has a deflection in the direction of the initial eccentricities (Fig. 2).

When the speed is raised by an infinitessimal increment above the critical, the sign of the determinant will change to minus and therefore the deflections y_1 and y_2 will become negative (Fig. 3).

When $\omega \to \infty - e_1$ and $y_2 \to - e_2$ (Fig. 4), since

$$y_2 = \frac{- e_1 n_{11} n_{12} - e_2 n_{11} n_{22} + e_1 n_{11} n_{12} + e_2 n_{21} n_{12}}{n_{11} n_{22} - n_{12} n_{21}} = - e_2$$

and

$$y_1 = \frac{- e_1 n_{11} n_{22} - e_2 n_{22} n_{21} + e_1 n_{12} n_{21} + e_2 n_{22} n_{21}}{n_{11} n_{22} - n_{12} n_{21}} = - e_1.$$

2) If the eccentricities are oppositely directed, so that

$$2\tau = \pi; \quad \tau = \pi/2,$$

from which

$$\vartheta_1 = 0 \text{ and } \vartheta_2 = 0, \text{ and } \theta_1 = e_1 \text{ and } \theta_2 = e_2,$$

then, when $\omega \neq \omega_{cr}$, one obtains

$$x_1 \neq 0; \quad y_1 = 0; \quad x_2 \neq 0; \quad y_2 = 0$$

and

$$|\tan \varphi_1| = |\tan \varphi_2| = |\infty|.$$

Under these conditions the elastic curve will lie in the plane of the x axis.

When $\omega \to \infty$

$$x_1 = \frac{e_1 n_{11} n_{22} - e_2 n_{22} n_{21} - e_1 n_{12} n_{21} + e_2 n_{22} n_{21}}{n_{11} n_{22} - n_{12} n_{21}} = e_1,$$

$$x_2 = \frac{e_1 n_{11} n_{12} - e_2 n_{22} n_{11} - e_1 n_{11} n_{12} + e_2 n_{21} n_{12}}{n_{11} n_{22} - n_{12} n_{21}} = - e_2.$$
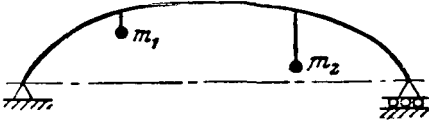
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Figure 3. The Shape of the
Elastic Curve of a Shaft
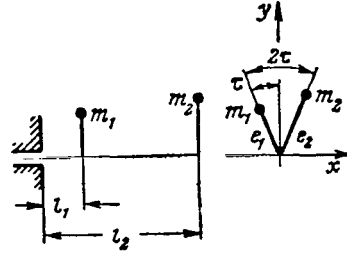Rotating at Supercritical
Speed.

Figure 4. Schematic of a
Cantilevered Weightless
Elastic Shaft Carrying Two
Point Masses.

For small ω (as compared with ω_{cr} and ω_{part}) $D > 0$, $n_{11}n_{22} > n_{12}n_{21}$ and the sign of $\underline{x}_1$ will depend on the sign of the expression

$$\theta_1 n_{11} n_{22} - \theta_2 n_{22} n_{21} - \theta_1 n_{11} + \theta_2 n_{21} - \theta_1 n_{21} n_{12} + \theta_2 n_{22} n_{21}$$

in which, by collecting terms with θ_1 and factoring out θ_1, we get

$$\theta_1 (n_{11} n_{22} - n_{11} - n_{12} n_{21}) + \theta_2 n_{21} =$$
$$= e_1 (m_1 \omega^2 \delta_{11} m_2 \omega^2 \delta_{22} - m_1 \omega^2 \delta_{11} - m_1 \omega^2 \delta_{12} m_2 \omega^2 \delta_{21}) + e_2 m_2 \omega^2 \delta_{21} =$$
$$= e_1 [\omega^4 m_1 m_2 (\delta_{11} \delta_{22} - \delta_{12} \delta_{21})] + e_2 m_2 \omega^2 \delta_{21} - e_1 m_1 \omega^2 \delta_{11}.$$

Analysis of this expression shows that if we neglect ω^4 and divide the expression by ω^2, we will find that the sign of $\underline{x}_1$ depends on the sign of the expression $e_2 m_2 \delta_{21} - e_1 m_1 \delta_{11}$.

Proceeding in the same manner, the sign of $\underline{x}_2$ will depend on the sign of

$$e_2 m_2 \delta_{22} - e_1 m_1 \delta_{12}.$$

If $\omega \to 0$, then

$$\tan \varphi_{1\,(0)} = \frac{-\theta_1 n_{11} + \theta_2 n_{21}}{\vartheta_1 n_{11} + \theta_2 n_{21}} \quad \text{and} \quad \tan \varphi_{2\,(0)} = \frac{-\theta_1 n_{12} + \theta_2 n_{22}}{\vartheta_1 n_{12} + \theta_2 n_{22}}.$$

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Returning to the initial notation, we get

$$\tan \varphi_{1\,(0)} = \tan \tau \frac{e_2 m_2 \delta_{21} - e_1 m_1 \delta_{11}}{e_1 m_1 \delta_{11} + e_2 m_2 \delta_{21}}$$

$$\tan \varphi_{2\,(0)} = \tan \tau \frac{e_2 m_2 \delta_{22} - e_1 m_1 \delta_{12}}{e_1 m_1 \delta_{12} + e_2 m_2 \delta_{22}}$$

We now find angles φ_1 and φ_2 for $\omega \gg \omega_{cr}$ and for any angle 2τ

$$\tan \varphi_1 = \frac{\theta_1 n_{11} n_{22} - \theta_2 n_{22} n_{21} - \theta_1 n_{12} n_{21} + \theta_2 n_{22} n_{21}}{-\vartheta_1 n_{11} n_{22} - \vartheta_2 n_{22} n_{21} + \vartheta_1 n_{12} n_{21} + \vartheta_2 n_{22} n_{21}} =$$

$$= \frac{\theta_1 (n_{11} n_{22} - n_{12} n_{21})}{-\vartheta_1 (n_{11} n_{22} - n_{12} n_{21})} = -\frac{\theta_1}{\vartheta_1} = -\frac{e_1 \sin \tau}{e_1 \cos \tau} = -\tan \tau.$$

Consequently,

$$\varphi_1 = -\tau,$$

$$\tan \varphi_2 = \frac{\theta_1 n_{11} n_{12} - \theta_2 n_{11} n_{22} - \theta_1 n_{11} n_{12} + \theta_2 n_{21} n_{12}}{-\vartheta_1 n_{11} n_{12} - \vartheta_2 n_{11} n_{22} + \vartheta_1 n_{11} n_{12} + \vartheta_2 n_{21} n_{12}} =$$

$$= \frac{-\theta_2 (n_{11} n_{22} - n_{21} n_{12})}{-\vartheta_2 (n_{11} n_{22} - n_{21} n_{12})} = \frac{\theta_2}{\vartheta_2} = \frac{e_2 \sin \tau}{e_2 \cos \tau} = \tan \tau,$$

from which

$$\varphi_2 = \tau.$$

Comparing the expressions just obtained with those for $\tan \varphi_1$ and $\tan \varphi_2$ derived at low rpm, we see that the multiplier of $\tan \varphi$ in the latter equation contains fractions not equal to unity. Consequently, angles φ_1 and φ_2 depend on the shaft rpm.

We shall now consider a system in resonance. If it is assumed that the shaft resonance amplitude is limited by frictional forces and that the resonance vibrations become steady if the system is operating in vacuum, and, furthermore, assuming perfect (i.e., frictionless) bearings we will have to assume that the amplitude will have to be limited by the friction between the fibers of the shaft.

As is known from literature and as can be seen from one case of angle $2\tau = 0$, when the system passes through the critical velocity, the directions of eccentricities relative to the elastic curve of the shaft shift

by the amount π. It is also known that at the critical velocity, the phase shift is $\pi/2$ in the direction of the rotation; i.e., at the instant when the system goes into resonance, the shaft is twisted about the axis of the elastic curve. This also should happen in a system with any value of 2τ. In this case, since even at resonance the angle between the directions of eccentricities should remain 2τ (we shall neglect the torsional deformations of the shaft), the shaft of our system will be twisted so that the directions of the eccentricities will form angles τ with the y axis.

From the expressions derived above for φ_1 and φ_2, we can write equations for $\omega = \omega_{cr}$

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$$n_{ik} = \bar{n}_{ik};$$

$$\tan \bar{\varphi}_1 = \left(\frac{x_1}{y_1}\right)_{cr} = \frac{(\theta_1 \bar{n}_{11} - \theta_2 \bar{n}_{21})(\bar{n}_{22} - 1) - (\theta_1 \bar{n}_{12} - \theta_2 \bar{n}_{22})\bar{n}_{21}}{-(\theta_1 \bar{n}_{11} + \theta_2 \bar{n}_{21})(\bar{n}_{22} - 1) + (\theta_1 \bar{n}_{12} + \theta_2 \bar{n}_{22})\bar{n}_{21}} =$$

$$= \frac{\theta_1(\bar{n}_{11}\bar{n}_{22} - \bar{n}_{12}\bar{n}_{21}) - \theta_2(\bar{n}_{22}\bar{n}_{21} - \bar{n}_{22}\bar{n}_{21}) - \theta_1\bar{n}_{11} + \theta_2\bar{n}_{21}}{-\theta_1(\bar{n}_{11}\bar{n}_{22} - \bar{n}_{12}\bar{n}_{21}) + \theta_2(\bar{n}_{22}\bar{n}_{21} - \bar{n}_{22}\bar{n}_{21}) + \theta_1\bar{n}_{11} + \theta_2\bar{n}_{21}} =$$

$$= \frac{\theta_1(\bar{n}_{11}\bar{n}_{22} - \bar{n}_{12}\bar{n}_{21}) - \theta_1\bar{n}_{11} + \theta_2 n_{21}}{-\theta_1(\bar{n}_{11}\bar{n}_{22} - \bar{n}_{12}\bar{n}_{21}) + \theta_1\bar{n}_{11} + \theta_2\bar{n}_{21}}$$

We collect some terms of the numerator and denominator in parentheses, and reduce the expressions in parentheses to the form

$$D_{cr} = (\bar{n}_{11} - 1)(\bar{n}_{22} - 1) - \bar{n}_{12}\bar{n}_{21} = \bar{n}_{11}\bar{n}_{22} - \bar{n}_{12}\bar{n}_{21} - \bar{n}_{11} - \bar{n}_{22} + 1 = 0:$$

$$\tan \bar{\varphi}_1 = \frac{\theta_1(\bar{n}_{11} + \bar{n}_{22} - 1) - \theta_1\bar{n}_{11} + \theta_2\bar{n}_{21}}{-\theta_1(\bar{n}_{11} + \bar{n}_{22} - 1) + \theta_1\bar{n}_{11} + \theta_2\bar{n}_{21}} =$$

$$= \frac{\theta_1(\bar{n}_{22} - 1) + \theta_2\bar{n}_{21}}{-\theta_1(\bar{n}_{22} - 1) + \theta_2\bar{n}_{21}} = \tan \tau \frac{e_1(\bar{n}_{22} - 1) + e_2\bar{n}_{21}}{-e_1(\bar{n}_{22} - 1) + e_2 n_{21}}$$

or

$$\tan \varphi_{1cr} = -\tan \tau \frac{e_1(\bar{n}_{22} - 1) + e_2\bar{n}_{21}}{e_1(\bar{n}_{22} - 1) - e_2 n_{21}};$$

$$\tan \bar{\varphi}_{2cr} = \left(\frac{x_2}{y_2}\right)_{cr} = \frac{\theta_1\bar{n}_{11}\bar{n}_{12} - \theta_2\bar{n}_{11}\bar{n}_{22} - \theta_1\bar{n}_{12} + \theta_2\bar{n}_{22} - \theta_1\bar{n}_{11}\bar{n}_{12} + \theta_2\bar{n}_{12}\bar{n}_{21}}{-\theta_1\bar{n}_{11}\bar{n}_{12} - \theta_2\bar{n}_{11}\bar{n}_{22} + \theta_1\bar{n}_{12} + \theta_2\bar{n}_{22} + \theta_1\bar{n}_{11}\bar{n}_{12} + \theta_2\bar{n}_{12}\bar{n}_{21}} =$$

$$= \frac{-\theta_2(\bar{n}_{11}\bar{n}_{22} - \bar{n}_{12}\bar{n}_{21}) - \theta_1\bar{n}_{12} + \theta_2\bar{n}_{22}}{-\theta_2(\bar{n}_{11}\bar{n}_{22} - \bar{n}_{12}\bar{n}_{21}) + \theta_1\bar{n}_{12} + \theta_2\bar{n}_{22}} =$$

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$$= \frac{-\theta_2(\bar{n}_{11}+\bar{n}_{22}-1)-\theta_1\bar{n}_{12}+\theta_2\bar{n}_{22}}{-\vartheta_2(\bar{n}_{11}+\bar{n}_{22}-1)+\vartheta_1\bar{n}_{12}+\theta_2\bar{n}_{22}} = \frac{-\theta_2(\bar{n}_{11}-1)-\theta_1\bar{n}_{12}}{-\vartheta_2(\bar{n}_{11}-1)+\vartheta_1\bar{n}_{12}};$$

$$\mathsf{tan}\,\varphi_{2\,\mathrm{cr}} = \tan\tau\,\frac{e_2(\bar{n}_{11}-1)+e_1\bar{n}_{12}}{e_2(\bar{n}_{11}-1)-e_1\bar{n}_{12}}.$$

We find the ratio

$$\left(\frac{y_1}{y_2}\right)_{\mathrm{cr}} = \frac{-\vartheta_1(\bar{n}_{11}\bar{n}_{22}-\bar{n}_{12}\bar{n}_{21})+\vartheta_1\bar{n}_{11}+\vartheta_2\bar{n}_{21}}{-\vartheta_2(\bar{n}_{11}\bar{n}_{22}-\bar{n}_{12}\bar{n}_{21})+\vartheta_1\bar{n}_{12}+\vartheta_2\bar{n}_{22}} =$$

$$= \frac{-\vartheta_1(\bar{n}_{11}+\bar{n}_{22}-1)+\vartheta_1\bar{n}_{11}+\vartheta_2\bar{n}_{21}}{-\vartheta_2(\bar{n}_{11}+\bar{n}_{22}-1)+\vartheta_2\bar{n}_{22}+\vartheta_1\bar{n}_{12}} = \frac{-\vartheta_1(\bar{n}_{22}-1)+\vartheta_2\bar{n}_{21}}{-\vartheta_2(\bar{n}_{11}-1)+\vartheta_1\bar{n}_{12}} =$$

$$= \frac{e_1(\bar{n}_{22}-1)-e_2\bar{n}_{21}}{e_2(\bar{n}_{11}-1)-e_1\bar{n}_{12}}.$$

Similarly,

$$\left(\frac{x_1}{x_2}\right)_{\mathrm{cr}} = \frac{\theta_1(\bar{n}_{22}-1)+\theta_2\bar{n}_{21}}{-\vartheta_2(\bar{n}_{11}-1)-\theta_1\bar{n}_{12}} = -\frac{e_1(\bar{n}_{22}-1)+e_2\bar{n}_{21}}{e_2(\bar{n}_{11}-1)+e_1\bar{n}_{12}}.$$

When $\omega \to 0$ and $\underline{n} \to 0$, and neglecting the product $(\underline{n}_{ik} \cdot \underline{n}_{ps})$, we find

$$\left(\frac{x_1}{x_2}\right)_0 = \frac{-\theta_1\bar{\bar{n}}_{11}+\theta_2\bar{\bar{n}}_{21}}{-\theta_1\bar{\bar{n}}_{12}+\theta_2\bar{\bar{n}}_{22}} = \frac{e_1 m_1\delta_{11}-e_2 m_2\delta_{21}}{e_1 m_1\delta_{12}-e_2 m_2\delta_{22}},$$

$$\left(\frac{y_1}{y_2}\right)_0 = \frac{\vartheta_1\bar{\bar{n}}_{11}+\vartheta_2\bar{\bar{n}}_{21}}{\vartheta_1\bar{\bar{n}}_{12}+\vartheta_2\bar{\bar{n}}_{22}} = \frac{e_1 m_1\delta_{11}+e_2 m_2\delta_{21}}{e_1 m_1\delta_{12}+e_2 m_2\delta_{22}}$$

We shall now prove that the elastic curve of a shaft rotating at critical speed is a planar curve. This is equivalent to the condition that

$$\tan\varphi_{1\mathrm{cr}} = \tan\varphi_{2\mathrm{cr}}$$

or, if we use the previously obtained expressions for the tangents, to

$$\frac{e_1(\bar{n}_{22}-1)+e_2\bar{n}_{21}}{-e_1(\bar{n}_{22}-1)+e_2\bar{n}_{21}} = \frac{e_2(\bar{n}_{11}-1)+e_1\bar{n}_{12}}{e_2(\bar{n}_{11}-1)-e_1\bar{n}_{12}}$$

We shall show that this equality really holds

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$$[e_1(\bar{n}_{22}-1)+e_2\bar{n}_{21}][e_2(\bar{n}_{11}-1)-e_1\bar{n}_{12}]=$$
$$=[-e_1(\bar{n}_{22}-1)+e_2\bar{n}_{21}][e_2(\bar{n}_{11}-1)+e_1\bar{n}_{12}].$$

We remove the parentheses

$$e_1e_2\bar{n}_{11}\bar{n}_{22}-e_1e_2\bar{n}_{11}+e_2^2\bar{n}_{11}\bar{n}_{21}-e_1e_2\bar{n}_{22}+e_1e_2-$$

$$-e_2^2\bar{n}_{21}-e_1^2\bar{n}_{22}\bar{n}_{12}+e_1^2\bar{n}_{12}-e_1e_2\bar{n}_{12}\bar{n}_{21}=-e_1e_2\bar{n}_{11}\bar{n}_{22}+$$

$$+e_1e_2\bar{n}_{11}+e_2^2\bar{n}_{11}\bar{n}_{21}+e_1e_2\bar{n}_{22}-e_1e_2-e_2^2\bar{n}_{21}-$$

$$-e_1^2\bar{n}_{22}\bar{n}_{12}+e_1^2\bar{n}_{12}+e_1e_2\bar{n}_{12}\bar{n}_{21}.$$

Canceling out identical terms [of opposite sign], factoring out and dividing through by e_1e_2, we satisfy ourselves that the remaining terms add up to $D = 0$, and hence the difference in sign does not disprove the identity of the expressions.

It is thus proven that the elastic curve of a shaft carrying two eccentrically coupled point masses with differently directed eccentricities and rotating at critical speed is a planar curve.

A PERFECTLY ELASTIC WEIGHTLESS SHAFT CARRYING AN ECCENTRICALLY COUPLED POINT MASS AND AN OBLIQUELY SEATED PERFECT THIN DISK

Let there be a weightless, elastic cantilevered shaft carrying a point mass and an obliquely seated disk (Fig. 5). The directions of the eccentricity and of the obliqueness form an angle 2τ.

The coordinate axes y and x are associated with the shaft, but in such a manner that on rotation the y axis divides angle 2τ into two equal parts; i.e., the axis is parallel to the bisector of this angle.

We introduce the notation: m_1 - the point mass; e_1 - the eccentricity of this mass; α - the angle of skewness of the disk (a small quantity); θ_2 - the equatorial moment of inertia of the disk mass.

If the skewness of the disk is expressed in terms of its shift (from vertical) at radius r, then
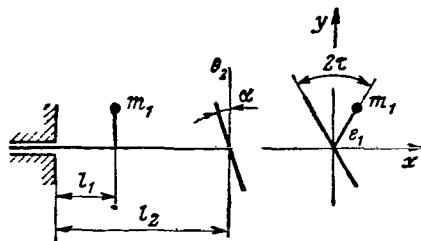
$$\tan\alpha=\frac{a}{r}$$

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Figure 5. Schematic of a Cantilevered Weightless, Elastic Shaft Carrying a Point Mass and an Obliquely Seated Disk.

Making use of the fact that the skewness is small, we write

$$\alpha = \frac{a}{r}.$$

The angles made by the skew disk with axes y and x are

$$\tan \alpha_y = \frac{a \cos \tau}{r} = \tan \alpha \cos \tau;$$

$$\text{and } \alpha_x = \frac{a \sin \tau}{r} = \tan \alpha \sin \tau$$

or

$$\alpha_y = \alpha \cos \tau \text{ and } \alpha_x = \alpha \sin \tau.$$

We now write the equation of shaft deformations when the shaft rotates at any $\omega \neq \omega_{cr}$

$$y_1 = (y_1 m_1 \omega^2 + \omega^2 e m_1 \cos \tau) \delta_{11} + \theta_2 (\alpha_y - \varphi_2) \omega^2 \delta_{21}; \tag{1}$$

$$y_3 = (y_1 m_1 \omega^2 + \omega^2 e m_1 \cos \tau) \delta_{13} + \theta_2 (\alpha_y - \varphi_2) \omega^2 \delta_{23}; \tag{2}$$

$$\varphi_2 = (y_1 m_1 \omega^2 + \omega^2 e m_1 \cos \tau) \delta_{12} + \theta_2 (\alpha_y - \varphi_2) \omega^2 \delta_{22}. \tag{3}$$

We introduce the notation

$$m_1 \omega^2 \delta_{11} = n_{11}; \quad m_1 \omega^2 \delta_{12} = n_{12}; \quad e \sin \tau = \theta_1;$$

$$\theta_2 \omega^2 \delta_{22} = n_{22}; \quad \theta_2 \omega^2 \delta_{21} = n_{21}; \quad e \cos \tau = \vartheta_1.$$

Then, Eqs. (1) and (3) form an independent system

$$y_1 (n_{11} - 1) - \varphi_2 n_{21} = -\vartheta_1 n_{11} - \alpha_y n_{21};$$

$$y_1 n_{12} - \varphi_2 (n_{22} + 1) = -\vartheta_1 n_{12} - \alpha_y n_{22},$$

whose determinants are

$$D = -(n_{11} - 1)(n_{22} + 1) + n_{12}n_{21} = -n_{11}n_{22} + n_{12}n_{21} - n_{11} + n_{22} + 1;$$

$$\Delta y_1 = (n_{22} + 1)(\vartheta_1 n_{11} + \alpha_y n_{21}) - n_{21}(\vartheta_1 n_{12} + \alpha_y n_{22}) =$$
$$= \vartheta_1 (n_{11}n_{22} - n_{12}n_{21} + n_{11}) + \alpha_y (n_{22}n_{21} - n_{22}n_{21} + n_{21}) =$$
$$= \vartheta_1 (n_{11}n_{22} - n_{12}n_{21} + n_{11}) + \alpha_y n_{21}$$

and

$$\Delta\varphi_2 = (n_{11} - 1)(-\vartheta_1 n_{12} - \alpha_y n_{22}) + n_{12}(\vartheta_1 n_{11} + \alpha_y n_{21}) =$$
$$= \vartheta_1 n_{12} + \alpha_y (n_{22} + n_{12}n_{21} - n_{11}n_{22}).$$

Consequently,

$$y_1 = \frac{\Delta y_1}{D} \text{ and } \varphi_2 = \frac{\Delta\varphi_2}{D}.$$

The coordinate y_3 of the second equation is written in terms of y_1 and φ_2 in the form

$$y_3 = y_1 n_{13} + \vartheta_1 n_{13} + \alpha_y n_{23} - \varphi_2 n_{23}.$$

Setting up the equations of deformations in the x direction, we get

$$x_1 = (x_1 - \theta_1) n_{11} + (\alpha_x - \Psi_2) n_{21};$$
$$\Psi_2 = (x_1 - \theta_1) n_{12} + (\alpha_x - \Psi_2) n_{22};$$
$$x_3 = (x_1 - \theta_1) n_{13} + (\alpha_x - \Psi_2) n_{23}.$$

The first two of these equations form the system

$$x_1 (n_{11} - 1) - \Psi_2 n_{21} = \theta n_{11} - \alpha_x n_{21};$$
$$x_1 n_{12} - \Psi_2 (n_{22} + 1) = \theta_1 n_{12} - \alpha_x n_{22},$$

the determinant of which is

$$D = -(n_{11} - 1)(n_{22} + 1) + n_{12}n_{2i} = -n_{11}n_{22} + n_{12}n_{21} - n_{11} + n_{22} + 1.$$

The determinant, as should have been expected, proves to be the same as for the system of equations for deformations (y_1, φ)

$$\Delta x_1 = -\theta_1 (n_{11}n_{22} - n_{12}n_{21} + n_{11}) + \alpha_x n_{21};$$

$$\Delta\Psi_2 = -\alpha_x (n_{11}n_{22} - n_{12}n_{21} - n_{22}) - \theta_1 n_{12};$$
$$x_1 = \frac{\Delta x_1}{D} \text{ and } \Psi_2 = \frac{\Delta\Psi_2}{D},$$

and

$$x_3 = x_1 n_{13} - \Psi_2 n_{23} - \theta_1 n_{13} + \alpha_x n_{23}.$$

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We now seek the tangents of the angles of deflection of the shaft
sections as $\omega \to 0$

$$\tan \varphi_1(0) = \left(\frac{x_1}{y_1}\right)_0 \cong \frac{-\theta_1 n_{11} + a_x n_{21}}{\vartheta_1 n_{11} + a_y n_{21}} =$$

$$= \frac{-e \sin \tau \cdot m_1 \delta_{11} + a\theta_2 \sin \tau \cdot \delta_{21}}{m_1 e \cos \tau \cdot \delta_{11} + \theta_2 a \cos \tau \cdot \delta_{21}} = -\tan \tau \frac{m_1 e \delta_{11} - \theta_2 a \delta_{21}}{m_1 e \delta_{11} + \theta_2 a \delta_{21}},$$

$$\tan \varphi_3(0) \cong \left(\frac{x_3}{y_3}\right) = \frac{-e \sin \tau \cdot m_1 \delta_{13} + \theta_2 a \sin \tau \cdot \delta_{23}}{e \cos \tau \cdot m_1 \delta_{13} + \theta_2 a \cos \tau \cdot \delta_{23}} =$$

$$= -\tan \tau \frac{e m_1 \delta_{13} - \theta_2 a \delta_{23}}{e m_1 \delta_{13} + \theta_2 a \delta_{23}}.$$

We shall find the direction angles of the deflections of the elastic curve as $\omega \to \infty$; here we shall drop all terms containing ω raised to the lowest powers

$$\lim_{\omega \to \infty} D = -n_{11} n_{22} + n_{12} n_{21},$$

$$\lim_{\omega \to \infty} \Delta y_1 = \vartheta_1 (n_{11} n_{22} - n_{12} n_{21}) + a_y (n_{22} n_{21} - n_{22} n_{21}) =$$

$$= \vartheta_1 (n_{11} n_{22} - n_{12} n_{21});$$

$$\lim_{\omega \to \infty} y_1 = \frac{\Delta y_1}{D} = \frac{\vartheta_1 (n_{11} n_{22} - n_{12} n_{21})}{-(n_{11} n_{22} - n_{12} n_{21})} = -\vartheta_1$$

$$y_1 = -\vartheta_1 = -e \cos \tau,$$

$$\lim_{\omega \to \infty} \Delta x_1 = -\theta_1 (n_{11} n_{22} - n_{12} n_{21}),$$

$$\lim_{\omega \to \infty} x_1 = \frac{\Delta x_1}{D} = \frac{-\theta (n_{11} n_{22} - n_{12} n_{21})}{-(n_{11} n_{22} - n_{12} n_{21})} = \theta_1,$$

$$x_1 = \theta_1 = e \sin \tau,$$

$$\tan \varphi_{1(\infty)} = \frac{x_1}{y_1} = -\tan \tau;$$

$$\varphi_{2(\infty)} = \frac{-a_y (n_{11} n_{22} - n_{12} n_{21})}{-(n_{11} n_{22} - n_{12} n_{21})} = a_y.$$

Let us consider the state of the system when it rotates at critical
velocity

$$D = -(n_{11} - 1)(n_{22} + 1) + n_{12} n_{21} = 0;$$

$$D = -n_{11} n_{22} + n_{12} n_{21} - n_{11} + n_{22} + 1 = 0$$

or

$$\tan \varphi_{1\,\text{cr}} = \frac{\Delta x_{1\text{cr}}}{\Delta y_{1\text{cr}}} = \frac{-\theta_1 (n_{11} n_{22} - n_{12} n_{21} + n_{11}) + a_x n_{21}}{\vartheta_1 (n_{11} n_{22} - n_{12} n_{21} + n_{11}) + a_y n_{21}}.$$

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If in the parentheses of the numerator and denominator we reduce the terms to D, then

$$\tan \varphi_{1\,cr} = \frac{-\theta_1(n_{22}+1) + a_x n_{21}}{\vartheta_1(n_{22}+1) + a_y n_{21}},$$

$$\tan \varphi_{3\,cr} = \left(\frac{x_3}{y_3}\right)_{\kappa p} = \frac{x_1 n_{13} - \Psi_2 n_{23} - \theta_1 n_{13} + a_x n_{23}}{y_1 n_{13} - \varphi_2 n_{23} + \vartheta_1 n_{13} + a_y n_{23}}.$$

Multiplying the numerator and denominatorby $\underline{D}_{cr} = 0$, we get

$$\tan \varphi_{3\,cr} = \frac{\Delta x_1 n_{13} - \Delta \Psi_2 n_{23}}{\Delta y_1 n_{13} - \Delta \varphi_2 n_{23}}.$$

In this case, the third and fourth terms of the numerator and denominator become zero. Then

$$\tan \varphi_{3\,cr} = \frac{-\theta_1(n_{22}+1)n_{13} + a_x n_{21} n_{13} - a_x(n_{11}-1)n_{23} + \theta_1 n_{12} n_{23}}{\vartheta_1(n_{22}+1)n_{13} + a_y n_{21} n_{13} - a_y(n_{11}-1)n_{23} - \vartheta_1 n_{12} n_{23}}.$$

If it is assumed that the elastic curve of a shaft rotating at resonance speed is planar, then we must have

$$\tan \varphi_{1\,cr} = \tan \varphi_{3\,cr}.$$

However, this expression will hold only when it will be proven that the expressions in parentheses are proportional to the corresponding expressions outside of parentheses, which means that we must prove

$$\frac{-\theta_1(n_{22}+1) + a_x n_{21}}{\vartheta_1(n_{22}+1) + a_y n_{21}} = \frac{-a_x(n_{11}-1) + \theta_1 n_{12}}{-a_y(n_{11}-1) - \vartheta_1 n_{12}},$$

since

$$\frac{a}{b} = \frac{a+ka}{b+kb} = \frac{a(1+k)}{b(1+k)}$$

We change to a somewhat different expression

$$\frac{-e \sin \tau \cdot (n_{22}+1) + a \sin \tau \cdot n_{21}}{e \cos \tau \cdot (n_{22}+1) + a \cos \tau \cdot n_{21}} = \frac{-a \sin \tau \cdot (n_{11}-1) + e \sin \tau \cdot n_{12}}{-a \cos \tau \cdot (n_{11}-1) - e \cos \tau \cdot n_{12}}$$

Dividing through both sides of the expression by tan τ and, cross multi- plying, we get

$$[-e(n_{22}+1) + an_{21}][-a(n_{11}-1) - en_{12}] =$$
$$= [e(n_{22}+1) + an_{21}][-a(n_{11}-1) + en_{12}].$$

We remove the parentheses

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$$ea(n_{22}+1)(n_{11}-1)+e^2(n_{22}+1)n_{12}-a^2(n_{11}-1)n_{21}-aen_{12}n_{21}=$$

$$=ea(n_{22}+1)(n_{11}-1)+e^2(n_{22}+1)n_{12}-a^2(n_{11}-1)n_{21}+aen_{12}n_{21}.$$

Cancelling out identical terms with opposite sign and dividing through by e, we get

$$a(n_{11}n_{21}+n_{11}-n_{22}-1)-an_{12}n_{21}=$$

$$=-a(n_{11}n_{22}+n_{11}-n_{22}-1)+an_{12}n_{21}.$$

Then, factoring out α and dividing through, we shall satisfy ourselves that the remaining expressions are equal to D = 0, which proves that

$$\tan \varphi_{1\,cr}=\tan\varphi_{3\,cr}$$

as well as the assertion that when the system rotates at resonance rpm, the elastic curve of the shaft is a planar curve.

It is also possible to prove the validity of the Wiedler postulate for systems more complex than those presented above, but this would only involve more complicated calculations without changing the substance of the argument. Thus, Wiedler's postulate, stated as a theorem, has been proven.

In closing, we present two numerical examples which illustrate the fundamental tenets of this article.

Example 1 (see Fig. 4). Given an elastic, weightless bar, carrying two eccentrically coupled point masses m_1 and m_2 (with eccentricities e_1 and e_2) at an angle 2τ from one another, find the critical speeds $\bar{\omega}_1$ and $\bar{\omega}_2$, φ_1 and φ_2, construct the elastic curve at critical rpm, as well as at $\omega = 0$ and $\omega = \infty$. The system data listed above are tabulated below:

Quantity	l_1	l_2	EJ	m_1	m_2	τ	e_1	e_2
Value	2	4	1	3	1	45°	2	4

The matrix of effect coefficients is

$$
\begin{array}{|c|c|}
\hline
\delta_{11} & \delta_{12} \\
\hline
\delta_{21} & \delta_{22} \\
\hline
\end{array}
=
\begin{array}{|c|c|}
\hline
\dfrac{8}{3} & \dfrac{20}{3} \\
\hline
\dfrac{20}{3} & \dfrac{64}{3} \\
\hline
\end{array}
$$

The equation of Ω, the natural frequencies of the system, is

$$\Omega^4 \cdot 3 \cdot 1 \left(\frac{8}{3} \cdot \frac{64}{3} - \frac{400}{9} \right) - \Omega^2 \left(3 \cdot \frac{8}{3} + \frac{64}{3} \right) + 1 = 0$$

or

$$\Omega^4 - \frac{88}{112} \Omega^2 + \frac{3}{112} = 0,$$

from which

$$\Omega_1^2 = \frac{1}{28}; \quad \Omega_1 = 0.19$$

and

$$\Omega_2^2 = \frac{3}{4}; \quad \Omega_2 = 0.865.$$

When $\omega \to 0$

$$\tan \varphi_1(0) = \frac{- \vartheta_1 n_{11} + \vartheta_2 n_{21}}{\vartheta_1 n_{11} + \vartheta_2 n_{21}}.$$

Dividing by $\sin \tau = \cos \tau$ and by $1/3$, we find

$$\tan \varphi_1(0) = \frac{-2 \cdot 3 \cdot 8 + 4 \cdot 1 \cdot 20}{2 \cdot 3 \cdot 8 + 4 \cdot 1 \cdot 20} = \frac{1}{4};$$

$$\tan \varphi_2(0) = \frac{- \vartheta_1 n_{12} + \vartheta_2 n_{22}}{\vartheta_1 n_{12} + \vartheta_2 n_{22}} = \frac{-2 \cdot 3 \cdot 20 + 4 \cdot 1 \cdot 64}{2 \cdot 3 \cdot 20 + 4 \cdot 1 \cdot 64} \simeq 0.361;$$

$$\left(\frac{x_1}{x_2} \right)_{\omega \to 0} = \frac{- \vartheta_1 n_{11} + \vartheta_2 n_{21}}{- \vartheta_1 n_{12} + \vartheta_2 n_{21}} = \frac{-2 \cdot 3 \cdot 8 + 4 \cdot 1 \cdot 20}{-2 \cdot 3 \cdot 20 + 4 \cdot 1 \cdot 64} \simeq 0.235;$$

$$\left(\frac{y_1}{y_2} \right)_{\omega \to 0} = \frac{\vartheta_1 n_{11} + \vartheta_2 n_{21}}{\vartheta_1 n_{12} + \vartheta_2 n_{22}} = \frac{48 + 80}{120 + 256} \approx 0.341.$$

We now determine the angles of deflection of the shaft's sections at the first critical speed

$$\tan \varphi_{1\,cr} = - \tan \tau \, \frac{e_1 (\bar{n}_{22} - 1) + e_2 \bar{n}_{21}}{e_1 (\bar{n}_{22} - 1) - e_2 \bar{n}_{21}} =$$

$$= -1 \, \frac{2(0.762 - 1) + 4 \cdot 0.238}{2(0.762 - 1) - 4 \cdot 0.238} = \frac{1}{-3};$$

$$\bar{n}_{22} = m_2 \delta_{22} \omega_{cr}^2 = 1 \cdot \frac{64}{3} \cdot \frac{1}{28} = 0.762;$$

$$\bar{n}_{21} = m_2 \delta_{21} \omega_{cr}^2 = 1 \cdot \frac{20}{3} \cdot \frac{1}{28} \simeq 0.238;$$

$$\overline{n}_{11} = m_1 \delta_{11} \omega_{cr}^2 = 3 \cdot \frac{8}{3} \cdot \frac{1}{28} = 0.286;$$

$$\overline{n}_{12} = m_1 \delta_{12} \omega_{cr}^2 = 3 \cdot \frac{20}{3} \cdot \frac{1}{28} = 0.714;$$

$$\tan \varphi_{2cr} = \tan \tau \frac{e_2 (\overline{n}_{11} - 1) + e_1 \overline{n}_{12}}{e_2 (\overline{n}_{11} - 1) - e_1 \overline{n}_{12}} =$$

$$= \frac{4 (0.286 - 1) + 2 \cdot 0.714}{4 (0.286 - 1) - 2 \cdot 0.714} = \frac{1}{3}.$$

From the fact that angles φ_1 and φ_2 are equal, we can conclude that the elastic curve of the shaft is planar

$$\left(\frac{y_1}{y_2} \right)_{cr} = \frac{e_1 (\overline{n}_{22} - 1) - e_2 \overline{n}_{21}}{e_2 (\overline{n}_{11} - 1) - e_1 \overline{n}_{12}} = \frac{2 (0.762 - 1) - 4 \cdot 0.238}{4 (0.286 - 1) - 2 \cdot 0.714} = \frac{1}{3};$$

$$\left(\frac{x_1}{x_2} \right)_{cr} = - \frac{e_1 (\overline{n}_{22} - 1) + e_2 \overline{n}_{21}}{e_2 (\overline{n}_{11} - 1) + e_1 \overline{n}_{12}} = - \frac{2 (0.762 - 1) + 4 \cdot 0.238}{4 (0.286 - 1) + 2 \cdot 0.714} = \frac{1}{3}.$$

We now find the projections of the shaft's deflections at $\omega \to \infty$

$$y_{1\infty} = \frac{- \vartheta_1 n_{11} n_{22} - \vartheta_2 n_{21} n_{22} + \vartheta_1 n_{12} n_{21} + \vartheta_2 n_{22} n_{21}}{n_{11} n_{22} - n_{12} n_{21}} =$$

$$= - \vartheta_1 \frac{n_{11} n_{22} - n_{12} n_{21}}{n_{11} n_{22} - n_{12} n_{21}} = - \vartheta_1 = - e \cos \tau = - 2 \frac{\sqrt{2}}{2} = - 1.414;$$

$$x_{1\infty} = \frac{\theta_1 n_{11} n_{22} - \theta_2 n_{22} n_{21} - \theta_1 n_{12} n_{21} + \theta_2 n_{22} n_{21}}{n_{11} n_{22} - n_{12} n_{21}} =$$

$$= \theta_1 = e_1 \sin \tau = 1.414;$$

$$y_2 = \frac{- \vartheta_1 n_{11} n_{12} - \vartheta_2 n_{11} n_{22} + \vartheta_1 n_{11} n_{12} + \vartheta_2 n_{12} n_{21}}{n_{11} n_{22} - n_{12} n_{21}} = - \vartheta_2 =$$

$$= - e_2 \cos \tau = - 2.828;$$

$$x_2 = \frac{\theta_1 n_{11} n_{12} - \theta_2 n_{11} n_{22} - \theta_1 n_{11} n_{12} + \theta_2 n_{12} n_{21}}{n_{11} n_{22} - n_{12} n_{21}} =$$

$$= - \theta_2 = - e_2 \sin \tau = -2.828.$$

Example 2 (see Fig. 5). Given an elastic weightless bar, carrying one eccentrically coupled point mass m_1 with eccentricity e_1 and one ideal disk with moment of inertia θ_2 and skewness angle α, the angle between the direction of the eccentricity and the direction of skewness being 2α, find the critical speed and the coordinates of deflections for ω_{cr}, $\omega = 0$ and $\omega \to \infty$.

Quantity	l_1	l_2	EJ	m_1	θ_2	2τ	α	e_1
Value	2	4	1	2	3	90°	0.1	2

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The matrix of effect coefficients is

$$\left|\begin{array}{c|c} \delta_{11} & \delta_{12} \\ \hline \delta_{21} & \delta_{22} \end{array}\right| = \left|\begin{array}{c|c} 8/3 & 2 \\ \hline 2 \cdot & 4 \end{array}\right|$$

The equation for the frequency of direct synchronous precession is

$$-\,\Omega^4 \cdot 2 \cdot 3 \left(\frac{8}{3} \cdot 4 - 4\right) - \Omega^2 \left(2 \cdot \frac{8}{3} - 3 \cdot 4\right) + 1 = 0$$

or

$$\Omega^4 - \frac{1}{6}\,\Omega^2 - \frac{1}{40} = 0;$$

$$\Omega^2 = \frac{1}{12} \pm \sqrt{\frac{1}{144} + \frac{1}{40}} = \frac{1 \pm 2.145}{12}$$

$$\Omega = 0.512.$$

When $\omega \to 0$

$$\tan \varphi_1(0) = -\tan \tau \frac{m_1 e \delta_{11} - \theta_2 a \delta_{21}}{m_1 e \delta_{11} + \theta_2 a \delta_{21}} =$$

$$= -\frac{2 \cdot 2 \cdot \dfrac{8}{3} - 3 \cdot 0.1 \cdot 2}{2 \cdot 2 \cdot \dfrac{8}{3} + 3 \cdot 0.1 \cdot 2} = -0.895;$$

$$\tan \varphi_3(0) = -\tan \tau \cdot \frac{e m_1 \delta_{13} - \theta_2 a \delta_{23}}{e m_1 \delta_{13} + \theta_2 a \delta_{23}} = -\frac{2 \cdot 2 \cdot \dfrac{20}{3} - 3 \cdot 0.1 \cdot 8}{2 \cdot 2 \cdot \dfrac{20}{3} + 3 \cdot 0.1 \cdot 8} = -0.835.$$

The angle of deflection of the elastic curve when the shaft rotates at critical rpm is

$$\tan \varphi_{cr} = \frac{-\theta_1(n_{22} - 1) + \alpha_x n_{21}}{\vartheta_1(n_{22} - 1) + \alpha_y n_{21}} = -\tan \tau \cdot \frac{e(n_{22} - 1) - a n_{21}}{e(n_{22} - 1) + a n_{21}} =$$

$$= -\frac{2(3.14 - 1) - 0.1 \cdot 1.57}{2(3.14 - 1) + 0.1 \cdot 1.57} = -0.93;$$

$$n_{22} = \theta_2 \Omega^2 \delta_{22} = 3 \cdot 0.262 \cdot 4 = 3.14;$$

$$n_{21} = \theta_2 \Omega^2 \delta_{21} = 3 \cdot 0.262 \cdot 2 = 1.57.$$

The tangent of the angle of displacement of the second angle is not calculated, since the equality of these angles was proven in the general case.

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