CERTAIN COMPUTATIONS CONCERNING A SPECIAL CASE OF THE THREE BODY PROBLEM

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There is widespread opinion that the general problem of the three bodies is today at the same point where Lagrange left it. However, this opinion must be shown to be no longer pertinent. Even when attention is focused on the great works limited to the special cases of the planetary system, and even when it is taken into account that no new integrals can be included in the functions previously employed in the analysis (propositions by Bruns and Poincaré), there are still a number of works done on the three-body problem which are in part purely analytical, in part aided by numerous computations (Hill, Gyldén, Poincaré, Thiele, v. Haerdtl, Burrau, Darwin, Schwarzschild, Charlier, Levi-Civita, Plummer, Strömgren *et al*), which not only provide true advances, but also warrant the highest hopes for even greater progress in the future.

This area previously indicated that progress was up in the air and at the same time would come from various directions (see for example my paper on Darwin's work, *Vierteljahrsschrift* of the A.G. 1898, page 22). Since I resumed my investigation and calculations in this area some time ago, it seems appropriate that I now say a few words about my work plans, for I consider it quite possible that similar investigations and calculations will be carried out elsewhere in the world, and because it would therefore be quite desirable to establish contact with the scientists who are doing similar work.

My investigations will concern themselves for the time being mainly with the prime issue in the identical special case as before (A. N. 3230 and 3251, see also 3289), i.e. two volumes of equal weight with a circular motion and the third volume infinitely small. (I would like to mention in passing that one does not impose as large a limitation of generality as it may seem. That is to say, it is possible to give form to the similarities of the general problem which causes the worst singularities in the special case to stand out. I hope to return to this elsewhere.)

In A. N. 3251 I have shown how the pure librations, that is, those in periodic solutions, which have their natural beginning in the Lagrangian libration point L, finish with an ejection path which is the same both in drawing and in numbers. As I see it, the case must arise very frequently in which a class of periodic solvents has an ejection path as a border path. That is to say, taking any periodic solvent whatsoever one must be able to transfer to the associated solvent (with the help of the "variation equations") and continue this procedure until approaching the important point of discontinuity, the mass point itself, thus, until the class finishes with an ejection path. The great importance of the ejection path has brought me to investigate the series expansion of all ejection paths, and--for the great Darwinian work was published in the meantime, accepting the mass ratio as 1:10--with any relationship between the large masses. I will write the similarities of the problem with the Darwinian unit designators:

$$E'' - \tfrac{1}{4}n(\cos 2iF - \cos 2E).F' = \frac{n^2}{64}\sin 4E + b.\sin 2E +$$
$$+ \frac{n^2-2}{64}\cdot 3\sin 3E\cos iF - \frac{n^2-2}{64}\sin E\cos 3iF + \frac{n^2-2}{2}\sin E.$$
$$F'' + \tfrac{1}{4}n(\cos 2iF - \cos 2E).E' = \frac{n^2}{64i}\sin 4iF + \frac{b}{i}\sin 2iF +$$
$$+ \frac{n^2-2}{64i}\cdot 3\cos E.\sin 3iF - \frac{n^2-2}{64i}\cos 3E\sin iF + \frac{n^2}{2i}\sin iF.$$

E and F are the coordinates of the infinitely small mass, the coordinates which arise from the relative coordinate (x and y of Darwin, ξ and η in the A. N.-dissertations of Thiele and Burrau) through the transformation:

$$x + iy = \cos^2 \tfrac{1}{2}(E + iF) - \frac{1}{n^2}, \qquad (i = \sqrt{-1})$$

The independent variable τ arises from the time t through the transformation

$$dt = r_1 r_2 d\tau = \tfrac{1}{4}\sin(E + iF)\sin(E - iF).d\tau$$

Through this transformation presented by Thiele in A. N. 3289 the singularities of the mass point are neutralized. The apexes of the ejection paths are transformed into paths which pass uninterrupted through the mass points. The class of periodic solvents which were previously separated by an ejection path are unified into a single class following the transformation. For the ejection paths themselves, a series expansion for E and F progressing from the powers of τ, which converge at least adequately in an extensive proximity to the mass points.

The constant n denotes the relationship of the two large masses. For example, should masses of the same size be desired, $n = \sqrt{2}$ should be used. The constant b is closely related to the Jacobian constant. I prefer to use the constant $B = 2b + \frac{11n^2 - 12}{16}$ in the series expansion. Besides, the constant a arises in the series expansion which denotes the initial angle of ejection. $a = \sqrt{2}$ yields, for example, the ejection in the direction of the other mass; $a = 0$ yields the ejection in the opposite direction.

The series expansion becomes:

$$E = a.\tau + a\{B - 1/2\} \cdot \frac{\tau^3}{3!} + 2n\sqrt{2-a^2} \cdot \frac{\tau^4}{4!} + e_5 \cdot \frac{\tau^5}{5!} + \\ + e_6 \cdot \frac{\tau^6}{6!} + \cdots e_9 \cdot \frac{\tau^9}{9!}$$

$$F = \sqrt{2-a^2}.\tau + \sqrt{2-a^2}\{B + 1/2\} \cdot \frac{\tau^3}{3!} - 2na \cdot \frac{\tau^4}{4!} + \\ + f_5 \cdot \frac{\tau^5}{5!} + f_6 \cdot \frac{\tau^6}{6!} + \cdots f_9 \cdot \frac{\tau^9}{9!}$$

$$e_5 = a\{(B - 1/2)^2 - a^2(4B - 1/2)\}$$

$$f_5 = \sqrt{2-a^2}\{(B + 1/2)^2 + (2 - a^2)(4B + 1/2)\}$$

$$e_6 = n\sqrt{2-a^2}\{22B + 20(2 - a^2) - 15\}$$

$$f_6 = na\{-22B + 20a^2 - 15\}$$

$$e_7 = a\{(B - 1/2)^3 - 11/2\, a^2(B - 1/2)(8B - 1) + \\ + a^4(16B + 45/2\, n^2 - 23) + 45 - 125/2\, n^2\}$$

$$f_7 = \sqrt{2-a^2}\{(B + 1/2)^3 + 11/2\,(2 - a^2)(B + 1/2)(8B + 1) + \\ + (2 - a^2)^2(16B + 45/2\, n^2 - 22) + 45 - 125/2\, n^2\}$$

$$e_8 = n\sqrt{2-a^2}\{182(2 - a^2)^2 + (2 - a^2)(1008B - 224) + \\ + 204B^2 - 924B + 203\}$$

$$f_8 = -na\{182a^4 - a^2(1008B + 224) + 204B^2 + \\ + 924B + 203\}$$

$$
\begin{aligned}
e_9 = a\{ & -(2334B+3985)n^2+1620B+1710+ \\
& +(B-{}^1/_2)^4+a^2(4060n^2-2520-51(B-{}^1/_2)^2(8B-1)) \\
& +a^4(912B^2+810n^2B-1200B-405n^2+{}^{1677}/_4)+ \\
& +a^6(473-{}^{945}/_2n^2-64B)\} \\
f_9 = \sqrt{2-a^2}\{ & -(2334B-3985)n^2+1620B-1710+ \\
& +(B+{}^1/_2)^4+ \\
& +(2-a^2)(-4060n^2+2520+51(B+{}^1/_2)^2(8B+1))+ \\
& +(2-a^2)^2(912B^2+810n^2B-420B+405n^2-{}^{1563}/_4)+ \\
& +(2-a^2)^3(-472+{}^{945}/_2n^2+64B)\}
\end{aligned}
$$

The coefficients e and f have been calculated independently by me and my friend N. P. Bertelsen. I have tried to ascertain the validity of these coefficients through this double calculation. We agreed on the expressions up through e_9 and f_9. But the e_{10} and f_{10} gave us such difficulty that we gave up trying to go further. The coefficients which we had already calculated were more than sufficient to be able to begin any ejection path and carry it far enough so that the usual numerical integration was quite reliable.

While the series developed in this manner seem relatively equally complicated, no matter what mass relationship we assume, the similarities are significantly simplified by assuming two masses of identical size. Let us assume $n = \sqrt{2}$, thus:

$$
\begin{aligned}
E'' - \frac{\sqrt{2}}{4}(\cos 2iF - \cos 2E)F' &= \frac{1}{32}\sin 4E + b\sin 2E \\
F'' + \frac{\sqrt{2}}{4}(\cos 2iF - \cos 2E)E' &= \frac{1}{32 \cdot i}\sin 4iF + \\
&\quad + \frac{b}{i}\sin 2iF + \frac{1}{i}\sin iF,
\end{aligned}
$$

in whose similarities we only have to change the independent variable τ by $2\sqrt{2}.\psi$ in order to arrive at the units used in the previous A. N. dissertation. I plan to undertake various projects in the future using these similarities. First, I have undertaken further interesting ejection paths[1] but I have also given myself the task of working directly with Fourier's series, that is, to establish the coefficients of Fourier's series expansion of a periodic solvent directly from the similarities through the method of undetermined coefficients. In order to be able to properly evaluate a periodic solvent, it must be placed in the form of a Fourier series, and it would be excellent if the solvent could be derived in this form directly from the similarities. I have begun to present the pertinent large formula equipment, but am not yet finished with it and mention it now only to come in contact with other scientists doing similar work. The formula is nevertheless complete, in which

$$e^{a_0 + a_1 e^{iv} + a_{-1} e^{-iv} + a_2 e^{2iv} + a_{-2} e^{-2iv} + \ldots\ldots a_7 e^{7iv} + a_{-7} e^{-7iv}}$$

is presented as the power series of e^{iv} and from hereon will serve as easily as sin as it does as cos of a Fourier series.

I have stopped with a_7 and a_{-7} because on one hand, the formula is scarcely manageable with further sections, and on the other side, the ejection border path of the pure librations transformed into E and F were as well

1 I have lacked good tables of the hyperbolic functions as well as sine tables with natural numbers in my numerous calculations therefore I have set about to publish such tables. These will be printed in the near future by the Reimerschen Verlag, Berlin.

presented by seven sect. ons, and it can be expected that this very border path is even more difficult to present than the associated paths (even associated pure librations as well as associated periodic solvents which form a loop around the large mass in ξ and η).

Mr. Plummer developed the first sections of the Fourier series for oscillating satellites in "Monthly Notices" Volume LXIV, 1903. Because he stopped at the usual relative coordinates he encountered difficulties which he discusses as follows: "A series of orbits of oscillating satellites has been found by M. Burrau to be limited by a'trajectory of ejection.' The real obstacle in the way of following up to this limit the continuation of the class of solutions considered in this paper is the limited range over which the force function can be expanded in ascending powers of the relative coordinates. Whether this difficulty can be surmounted is a question which must be left for further consideration."

Plummer's difficulties were completely taken care of by the transformation in E and F. The ejection border path has already been developed, as indicated above, into a Fourier series, and I hope soon to be able to present details about this in another article.