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SOLUTION OF LAMBERT'S PROBLEM FOR SHORT ARCS

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ABSTRACT

Approximation formulas are found for $\dot{x}(0)$ and $\dot{x}(1)$, where $x(t)$ satisfies $\ddot{x} = f(x, t)$, $x(0) = x_0$, $x(1) = x_1$. The results are applied to an example of two-body motion.

CONTENTS

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1. INTRODUCTION

Consider the following boundary-value problem:

$$\ddot{x} = f(x, t), \qquad x(0) = x_0, \qquad x(1) = x_1, \tag{1}$$

where t is a scalar, x and $f(x, t)$ are column matrices, and the over-dots indicate differentiation with respect to t.

Formulas will be found for $\dot{x}_0$ and $\dot{x}_1$ such that

$$\dot{x}_0 = \dot{\phi}(0), \qquad \dot{x}_1 = \dot{\phi}(1),$$

where $\phi(t)$ satisfies

$$\phi(0) = x(0), \qquad \ddot{\phi}(0) = \ddot{x}(0), \qquad \dddot{\phi}(0) = \dddot{x}(0),$$

$$\phi(1) = x(1), \qquad \ddot{\phi}(1) = \ddot{x}(1), \qquad \dddot{\phi}(1) = \dddot{x}(1).$$

It will not be necessary to find $\phi(t)$ but only to make certain assumptions as to its form.

The problem defined by (1) has an interesting history in celestial mechanics, going back to Euler, Lambert, Lagrange, and Gauss. Recent books by Battin [1] and Escobal [2] describe a total of eight methods for its solution in the case of an inverse-square, central force field. All the methods, however, are iterative, requiring considerable computation to obtain a solution. The method developed in this paper, while limited to cases of moderate time-span, offers a concise, explicit formula with no need for iteration. Within this time-span limitation, the method has the further advantage of allowing for completely general velocity-independent force functions.

The author has found the method useful in the calculation of short arc intercept trajectories for space vehicles, in preliminary orbit determination from observations of the position of a spacecraft at two times, and for ephemeris interpolation.

2. AN APPROXIMATE SOLUTION

Let α be any component of x, $\alpha(0) = \alpha_0$, $\alpha(1) = \alpha_1$, and assume approximations of the form

$$\dot{\alpha}_0 = a_0 \alpha_0 + b_0 \alpha_1 + a_2 \ddot{\alpha}_0 + b_2 \ddot{\alpha}_1 + a_3 \dddot{\alpha}_0 + b_3 \dddot{\alpha}_1 , \qquad (2)$$

$$\dot{\alpha}_1 = c_0 \alpha_0 + d_0 \alpha_1 + c_2 \ddot{\alpha}_0 + d_2 \ddot{\alpha}_1 + c_3 \dddot{\alpha}_0 + d_3 \dddot{\alpha}_1 . \qquad (3)$$

The scalars a_0, b_0, a_2, b_2, a_3, and b_3 are determined by assuming (2) to be exact when $\alpha = \phi_i\,(t)$, $i = 0, \ldots, 5$, where the ϕ_i's are linearly independent over the interval [0,1] with derivatives through the third order at $t = 0$ and $t = 1$. The coefficients in (3) are determined in a similar way. If we use the same set of ϕ_i's for each component of x, we can write

$$\dot{x}_0 = a_0 x_0 + b_0 x_1 + a_2 \ddot{x}_0 + b_2 \ddot{x}_1 + a_3 \dddot{x}_0 + b_3 \dddot{x}_1 , \qquad (4)$$

$$\dot{x}_1 = c_0 x_0 + d_0 x_1 + c_2 \ddot{x}_0 + d_2 \ddot{x}_1 + c_3 \dddot{x}_0 + d_3 \dddot{x}_1 . \qquad (5)$$

We eliminate the third derivatives in (4) and (5) by using

$$\dddot{x} = P\dot{x} + w , \qquad (6)$$

obtained from (1), where P is a matrix with element in the ith row and jth column equal to the value of $\partial f^i/\partial x^j$, and w is a column matrix with ith element equal to the value of $\partial f^i/\partial t$, f^i and x^j being respectively the ith component of f and the jth component of x. Substituting (6) into (4) and (5), we obtain

$$\left(I - a_3 P_0\right) \dot{x}_0 - b_3 P_1 \dot{x}_1 = \beta , \qquad (7)$$

$$-c_3 P_0 \dot{x}_0 + \left(I - d_3 P_1\right) \dot{x}_1 = \gamma , \tag{8}$$

where I is the unit matrix and

$$\beta = a_0 x_0 + b_0 x_1 + a_2 \ddot{x}_0 + b_2 \ddot{x}_1 + a_3 w_0 + b_3 w_1 ,$$

$$\gamma = c_0 x_0 + d_0 x_1 + c_2 \ddot{x}_0 + d_2 \ddot{x}_1 + c_3 w_0 + d_3 w_1 ,$$

$\ddot{x}_0$ and $\ddot{x}_1$ being computed from $\ddot{x} = f(x, t)$. Solving (7) and (8) we obtain

$$\left(B + b P_1 P_0\right) \dot{x}_0 = \beta + P_1 \left(b_3 \gamma - d_3 \beta\right) , \tag{9}$$

$$\left(B + b P_0 P_1\right) \dot{x}_1 = \gamma + P_0 \left(c_3 \beta - a_3 \gamma\right) . \tag{10}$$

$$B = I - a_3 P_0 - d_3 P_1 ,$$

$$b = a_3 d_3 - b_3 c_3 .$$

Note that $B + b P_0 P_1$ is the transpose of $B + b P_1 P_0$ if P is symmetric, as is the case for a central force field.

3. A POLYNOMIAL APPROXIMATION

Substituting successively

$$\alpha = \phi_i (t) = t^i , \qquad i = 0, 1, 2, 3, 4, 5$$

into Equation (2), we obtain

$$a_0 + b_0 = 0 , \qquad b_0 = 1 ,$$

$$b_0 + 2a_2 + 2b_2 = 0 ,$$

$$b_0 + 6b_2 + 6a_3 + 6b_3 = 0 ,$$

$$b_0 + 12b_2 + 24b_3 = 0 ,$$

$$b_0 + 20b_2 + 60b_3 = 0 .$$

The solution of this set of equations is

$$a_0 = -1 , \quad b_0 = 1 , \quad a_2 = -\frac{7}{20} , \quad b_2 = -\frac{3}{20} , \quad a_3 = -\frac{1}{20} , \quad b_3 = \frac{1}{30} . \tag{11}$$

In a similar way we find

$$c_0 = -1 , \quad d_0 = 1 , \quad c_2 = \frac{3}{20} , \quad d_2 = \frac{7}{20} , \quad c_3 = \frac{1}{30} , \quad d_3 = -\frac{1}{20} . \tag{12}$$

4. NUMERICAL EXAMPLE

For a particle moving in an inverse-square central force field, (1) becomes

$$\ddot{x} = -\mu(\tau_1 - \tau_0)^2 \frac{x}{|x|^3} , \qquad x(0) = x_0 , \qquad x(1) = x_1 , \tag{13}$$

where x is the position vector of the particle, μ is a constant, the real time τ and the pseudo-time t being related by $t(\tau_1 - \tau_0) = \tau - \tau_0$ with $x(\tau_0) = x_0$, $x(\tau_1) = x_1$.

As a numerical example take two vectors from a circular orbit separated by an angle of 15°. We can let $\tau_0 = 0$, $\tau_1 = \pi/12\sqrt{\mu}$, and

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \qquad x_1 = \begin{bmatrix} \cos 15^\circ \\ \sin 15^\circ \end{bmatrix} .$$

Substituting these values and (11), (12), (13) into (9) and (10) gives

$$\dot{x}_0 = \begin{bmatrix} -.00000022 \\ .261799360 \end{bmatrix}, \qquad \dot{x}_1 = \begin{bmatrix} -.06775845 \\ .252878819 \end{bmatrix}.$$

The true values are

$$\dot{x}_0 = \begin{bmatrix} 0 \\ .261799388 \end{bmatrix}, \qquad \dot{x}_1 = \begin{bmatrix} -.06775867 \\ .252878790 \end{bmatrix}.$$

5. REMARKS

The method developed above can be carried through when (1) has the more general form

$$\ddot{x} = f(x, t) + A(t)\dot{x}, \qquad x(0) = x_0, \qquad x(1) = x_1,$$

where $A(t)$ is a matrix function of t.

The time-span restriction may be less severe when x in (1) represents a small deviation from a reference trajectory, where the solution of the boundary problem for the reference trajectory has been found by other means. For such problems we have $x_0 = x_1 = 0$. For example, we might obtain a reference trajectory by solving a two-body problem with the given boundary conditions by one of the standard methods [1,2]. Then we could apply the method developed above to Encke's formulation [1] of the differential equation for the deviation of the position vector from the reference trajectory.

If we attempt to approximate $\dot{x}_0$ and $\dot{x}_1$ in such a way that $\dot{x}_0 = \dot{\phi}(0)$ and $\dot{x}_1 = \dot{\phi}(1)$, where $\phi(t)$ satisfies $\phi^{(4)}(0) = x^{(4)}(0)$ and $\phi^{(4)}(1) = x^{(4)}(1)$ as well as the other relations specified in the introduction, we are forced to solve a set of nonlinear equations unless (1) is linear in x.

REFERENCES

1. Battin, Richard H.: 1964, Astronautical Guidance, McGraw-Hill Book Co., New York, pp. 58-87.

2. Escobal, P. R.: 1965, Methods of Orbit Determination, John Wiley and Sons, Inc., pp. 187-235.