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STATISTICAL MECHANICS OF SUPERNOVAE

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ABSTRACT

In most treatments of the supernova origin theory of cosmic rays, average values of such observables such as cosmic ray flux, bulk streaming velocity, total energy, etc., are calculated (or estimated) and compared with observation to determine the parameters of the theory. Since supernova explosions are essentially statistical events the question arises as to whether or not the observed values have any simple relationship to average values; in other words are fluctuations important? To investigate this question I have considered that the variables describing the supernova injection event (position, time, etc.) are random variables and that our galaxy is a sample from an ensemble of galaxies. In this manner one can, in principle, calculate all higher moments of the cosmic ray observables. It turns out that an important parameter may be derived for any particular model of cosmic ray transport. If a single supernova event fills an effective volume V_{eff} with cosmic rays for a time τ_c and there are n supernova events per unit time per unit volume on the average, the quantity $\langle N \rangle = n\tau_c V_{eff}$ becomes essentially an inverse "discreteness parameter" in the sense that when it is large fluctuations are relatively unimportant, but when it is small fluctuations can dominate the situation. For currently considered models of our own galaxy the situation appears to be borderline.

I. Introduction

For some time now supernova explosions have been considered a leading contender for the role of primary source of galactic cosmic rays. This is based primarily on the fact that their rate of energy release appears to be sufficient to meet the requirements of the cosmic ray source[1-5] ($\approx 10^{41}$ ergs/sec) and more recently on the possibility that a detailed model of the supernova explosion might predict the energy spectrum of cosmic rays.[6-8]

A second major theme in recent cosmic-ray research has been the question of how cosmic rays propagate through the galaxy once they have been produced by their source. This question has been approached from two different aspects; first what combination of source characteristics and average path length in the galaxy will produce the abundances of the nuclear species observed in the cosmic-rays. There is, of course, a vast literature on this question and I cite here only a few of the most recent articles.[9-13] The second approach to this question is concerned with the fact that the bulk streaming velocity of cosmic rays in the vicinity of the earth appears to be very small ($\lesssim$500 km/sec). A considerable discussion has appeared in the literature concerning how the structure of the galactic magnetic field[14,15,16] and various possible types of plasma instabilities[17,18,19] could be responsible for this fact. A common feature of the theoretical work that has been done so far in these matters is that for any of the various models considered average values of observable quantities are calculated and then compared with observation, in so far as possible. It is the purpose of this paper to show that if cosmic rays do originate in supernova explosions the system we are observing is an essentially noisy one and the values of observable quantities that we measure may not have a great deal to do with the average values. In other words the probability that we are observing a rather large fluctuation away from the average may not be negligible.

II. Statistical Method

We begin by considering the general response function for a supernova explosion. The density of cosmic rays at a point X and at time t due to the i^{th} supernova explosion shall be designated by $\rho_i(X, t)$. Since we shall consider the propagation characteristics of the galaxy to be uniform in time and one spatial direction we have $\rho_i(X, t) = \rho(X - X_i, t - t_i)$ where ρ is some universal function and X_i and t_i are the time and position of the i^{th} supernova explosion. The reason for considering only one spatial coordinate will be made clear in Section III. when we discuss the actual propragation model that we will use.

In a like manner we may discuss the directional streaming velocity of the particles from the i^{th} event $v(X - X_i, t - t_i) \equiv v_i(X, t)$ where the directional flux of particles is given by $j_i(X, t) = v_i(X, t)\rho_i(X, t)$.

At any point in space and time the total density of cosmic ray particles is given by

$$\rho = \sum_i \rho_i$$

where the sum is over all supernova events. For reasons of later mathematical simplicity we shall consider the sum to be over all past and future events and build all causality requirements into the response function $\rho(X, t)$. In a similar fashion we have

$$j = \sum_i j_i \ .$$

The bulk streaming velocity that would be observed is given by

$$j/\rho = \left(\sum_i j_i \Big/ \sum_i \rho_i \right)$$

and since the age of all particles from the i^{th} event is just $t - t_i$ the average of all particles is just

$$\bar{\tau} = \left(\sum_i (t - t_i)\, \rho_i \Big/ \sum_i \rho_i \right)$$

Until now there has been no mention of statistical notions; the coordinates of the supernova events, X_i, t_i have been considered to have determined values and the various observable quantities to be exactly calculable, at least in principle. In reality, of course, we have no idea of the values of the event coordinates (with possibly a few exceptions) that would be required to calculate the values of the observable quantities. Even if they were known to a reasonable degree of accuracy the subsequent computations would be extremely difficult if not impossible.

This situation is reminiscent of the one encountered in statistical mechanics and, in fact, we shall treat this problem from a statistical mechanical point of view. We shall consider our galaxy to be just one sample of an arbitrarily large ensemble of galaxies. In this ensemble the event coordinates X_i, t_i shall be considered to be random variables, distributed with some probability distribution $P(X_1, t_1, X_2, t_2, \cdots X_i, t_i \cdots)$. Our observable quantities ρ, j, v, etc., are now functions of many random variables and while we will not calculate these functions themselves we may in principle calculate average values for any combination of them.

We shall use the method employed by Rice[20] in calculating electrical shot noise. It makes use of the idea of an ensemble average of any function of the random variables; If F is a function of the X_i, t_i then we have

$$\langle F \rangle \equiv \left(\prod_i \int dX_i \int dt_i \right) P(X_i, t_i)\, F(X_i, t_i) \ . \qquad (1)$$

Since we are interested in $-\infty < t_i < +\infty$ there are an infinite number of events and it is not clear what we want as our probability function $P(X_i, t_i)$. Before approaching this question we shall make some simplifying assumptions. First we shall assume that separate events are statistically independent, second we shall assume that the probability of an event occurring in an increment of X_i is $L^{-1}\,dX_i$ if $X_a \leq X_i \leq X_b$ where $L = X_b - X_a$ and is zero otherwise, and third we assume that the probability of an event occurring in an increment of time dt_i is μdt_i where μ is a constant.

We shall now consider the calculation of averages in three stages. We shall first consider the contribution to an observable quantity from only those events occurring in the time interval $-T/2 \leq t_i \leq T/2$ where T is a large but arbitrary finite length of time and further we shall consider only that subset of the ensemble for which exactly n events take place in this interval. For this subset the averaging operation has the form

$$\langle\ \rangle_{nT} = L^{-n}\,T^{-n}\prod_{i=1}^{n}\int_{X_a}^{X_b} dX_i \int_{-T/2}^{T/2} dt_i\ . \tag{2}$$

This operator is well defined and is concerned only with finite quantities. It may therefore be applied directly to any observable quantity in a straightforward manner.

In the next stage we simply note that the quantity n will have a Poisson distribution over the entire ensemble so our average may be extended to the full ensemble by suitably averaging over n. Our results will now be functions of the quantity $\langle n \rangle / T$ the average number of supernova events in the time interval T. Passing to the limit is achieved by simply noting that we have assumed $\langle n \rangle / T$ to be a constant therefore independent of T. We simply replace it with the average number of events per unit time $\bar{n}$. By this method we readily obtain

$$\langle \rho \rangle \equiv \left\langle \sum_i \rho_i \right\rangle = (\bar{n}/L) \int_{X_a}^{X_b} dX_i \int_{-\infty}^{\infty} dt_i \, \rho_i \left(X_i , t_i \right) ,$$

$$\langle \rho^2 \rangle \equiv \left\langle \sum_{i,j} \rho_i \rho_j \right\rangle = \langle \rho \rangle^2 + (\bar{n}/L) \int_{X_a}^{X_b} dX_i \int_{-\infty}^{\infty} dt_i \, \rho_i^2 \left(X_i , t_i \right) \quad (3)$$

etc.

We shall now anticipate a result of the next section. In Section III we will see that our response function $\rho\left(X - X_i , \ t - t_i\right)$ is actually a function of the dimensionless parameters $\left(X - X_i\right)/L_c$ and $\left(t - t_i\right)/\tau_c$ where L_c and τ_c are a characteristic length and time respectively. The integrals may therefore be written as

$$\left(L_c \tau_c\right) \int_{X_a/L_c}^{X_b/L_c} d\left(\frac{X_i}{L_c}\right) \int_{-\infty}^{\infty} d\left(\frac{t_i}{\tau_c}\right) \rho_i \left(\frac{X - X_i}{L_c} , \ \frac{t - t_i}{\tau_c}\right)$$

etc. If we write $\langle N \rangle \equiv \bar{n} L_c \tau_c / L$ and $\langle \rho_i \rangle$, $\langle \rho_i^2 \rangle$, etc., as the integrals over the <u>dimensionless</u> parameters of ρ_i, ρ_i^2, etc., we may write equations (3) as

$$\langle \rho \rangle = \langle N \rangle \langle \rho_i \rangle , \qquad \langle \rho^2 \rangle = \langle N \rangle^2 \langle \rho_i \rangle^2 + \langle N \rangle \langle \rho_i^2 \rangle$$

etc. The parameter $\langle N \rangle$ is the average number of supernova events per unit characteristic length, per unit characteristic time, and since it is essentially the average number of events that are sensibly contributing to the state of cosmic-ray-affairs at any one point in space and time it plays the role of an inverse "discreteness parameter." As such we should expect that fluctuations will be important if $\langle N \rangle$ is small and the contrary to be true if it is large. We shall see that, in fact, this is the case.

One additional point should be discussed before leaving the question of the statistical method. In calculating quantities such as $\langle j/\rho \rangle$ and $\langle \bar{\tau} \rangle$ we face a problem with having the statistical quantities in the denominator. While the operator $\langle \; \rangle_{nT}$, Equation (2), can be applied in principle it would be prohibitively difficult to do in practice. The escape from this difficulty is effected by defining $\delta\rho \equiv \rho - \langle \rho \rangle$ and writing

$$\left\langle \frac{j}{\rho} \right\rangle = \left\langle \frac{j}{\langle \rho \rangle + \delta\rho} \right\rangle = \left\langle \frac{j}{\langle \rho \rangle} \sum_{n=0}^{\infty} \left(- \frac{\delta\rho}{\langle \rho \rangle} \right)^n \right\rangle = \sum_{n=0}^{\infty} (-1)^n \frac{\langle j\,\delta\rho^n \rangle}{\langle \rho \rangle^{n+1}} \; . \tag{4}$$

We would expect this series, though most likely not a convergent one, to be a useful asymptotic one if fluctuations are small, i.e., $\delta\rho/\langle \rho \rangle << 1$. As we might expect this series turns out to be expressible, after some rearranging, as a power series in $1/\langle N \rangle$, our "discreteness parameter."

III. Galactic Propagation Model

We now turn to the question of what particular form the response function $\rho(X - X_i, t - t_i)$ shall have. The form that this function will take is completely dependent on the model one chooses for the propagation of cosmic rays through the galaxy. In principle one could choose as complicated a model as one wished and perform the integrals indicated in Equation (3) numerically.

Since, however, we are interested primarily in determining how the overall results are affected by certain general parameters of a model we shall consider here a model that is relatively simple and amenable to calculation. Furthermore, it is my belief that this model is not without a physical basis. It is in fact suggested by the picture of galactic cosmic ray propagation proposed by Parker[14,15,16] in which the cosmic rays are relatively free to stream along the galactic spiral field all the while leaking out of the surface of the galactic disk. This leakage is accomplished by means of the "bubble blowing" instability also discussed by Parker.[14]

We shall therefore adopt the following picture of cosmic ray propagation in the galaxy. When a supernova explodes it suddenly fills a bubble of volume L_0^3 with a hot cosmic ray gas where L_0 is of the order of the galactic disk thickness. This bubble is threaded by spiral (on the average) magnetic field of the galactic disk and is thus able to expand only along the field lines. (This is why we have considered response functions of only one variable.) This bubble expands in both directions along the field direction behind a "front" that moves with velocity V while behind the front the density and pressure are uniform. While the expansion is taking place the total number of particles is decaying exponentially with a time constant τ_c due to leakage out of the sides. This picture can be thought of as propagation down a leaky pipe. From these considerations we may write the density or response function as

$$\rho\left(X - X_i, t - t_i\right) = \frac{K e^{-(t-t_i)/\tau_c}}{\left[L_0 + V\left(t - t_i\right)\right]} U\left(t - t_i\right) U\left(V\left(t - t_i\right) + L_0 - \left|X - X_i\right|\right) \quad (5)$$

where U(X) is the Heaviside step function and K is a normalizing coefficient. From considerations of continuity we have

$$v_i\,(X, t) = \frac{V\left(X - X_i\right)}{\left[L_0 + V\left(t - t_i\right)\right]} \quad (6)$$

and

$$j_i = v_i \rho_i \, .$$

These formulae hold true for any point in the galaxy that is on a field line that treads the initial bubble, for any other point there will be no effect at all.

There are, of course, many other possibilities for models and response functions; diffusive models, or models with more detailed hydrodynamic expansions for example. However, I believe that the model described above exhibits certain features that would have to be present in any model. The minimum length L_0 has no effect on the average density $\langle \rho \rangle$, however, $\langle \rho^2 \rangle$ and higher order moments diverge as $L_0 \to 0$. This simply means that the probability of a supernova exploding arbitrarily near the point of observation has an overwhelming effect on all fluctuation phenomena. A diffusion model would have to have a built in maximum velocity of propagation otherwise the well known infinite propagation speed at $t = 0$ characteristic of pure diffusion solutions causes $\langle j^2 \rangle$ and higher terms to diverge. Finally, I believe that leakage out of side of the "pipe" is physically called for by the arguments of Parker.

IV. Calculations

I have used Equations (5) and (6) in evaluating* the integrals of Equation (3). The series expansion of Equation (4) has been carried out through all terms of order $1/\langle N \rangle$. In the calculations the free expansion velocity V has been considered a variable parameter and the dependence of $\langle \bar{\tau} \rangle$, $\langle \bar{\tau} \rangle + \sqrt{\langle \delta\tau^2 \rangle}$, $\langle \bar{\tau} \rangle - \sqrt{\langle \delta\tau^2 \rangle}$, $\langle v \rangle$, $\langle v \rangle + \sqrt{\langle \delta v^2 \rangle}$, and $\langle v \rangle - \sqrt{\langle \delta v^2 \rangle}$ upon this parameter are shown in Figures 1 and 2.

If we consider the RMS deviations to be a measure of the uncertainty of any prediction of an observable from a particular model we see that the results for the mean lifetime of cosmic ray particles is not too spectacular. (In this model lifetime is identical with lifetime in the disk since halo storage is not considered.) We see that the best one can do is about 6% accuracy but this is insignificant compared to present uncertainties.

*Data used for the galactic configuration were: Length of spiral flux tube (arm) – 92 Kpc, Earth 36 Kpc from inner end, $L_0 = 0.1$ Kpc, event rate ≈ 0.2 per year for our galaxy, galactic radius 12.5 Kpc, $\tau_c = 10^6$ years.

However, when we come to the bulk streaming velocity we find a different matter entirely. We see that the presently observed limit for the streaming velocity of ≤ 500 km/sec is compatible with a range of inherent streaming velocities from one fourth the speed of light on down. From this we can see that the observations of this streaming velocity may not tell us too much about the relevant parameters of a particular propagation model.

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FIGURE CAPTIONS

Figure 1. Mean particle age $\langle\bar{\tau}\rangle$ and $\langle\bar{\tau}\rangle$ plus and minus the R.M.S. deviation $\sqrt{\langle\delta\bar{\tau}^2\rangle}$ as a function of the expansion velocity V.

Figure 2. Bulk streaming velocity $\langle v\rangle$ and $\langle v\rangle$ plus and minus the R.M.S. deviation $\sqrt{\langle\delta v^2\rangle}$ as a function of the expansion velocity V.

PARTICLE AGE vs. EXPANSION VELOCITY

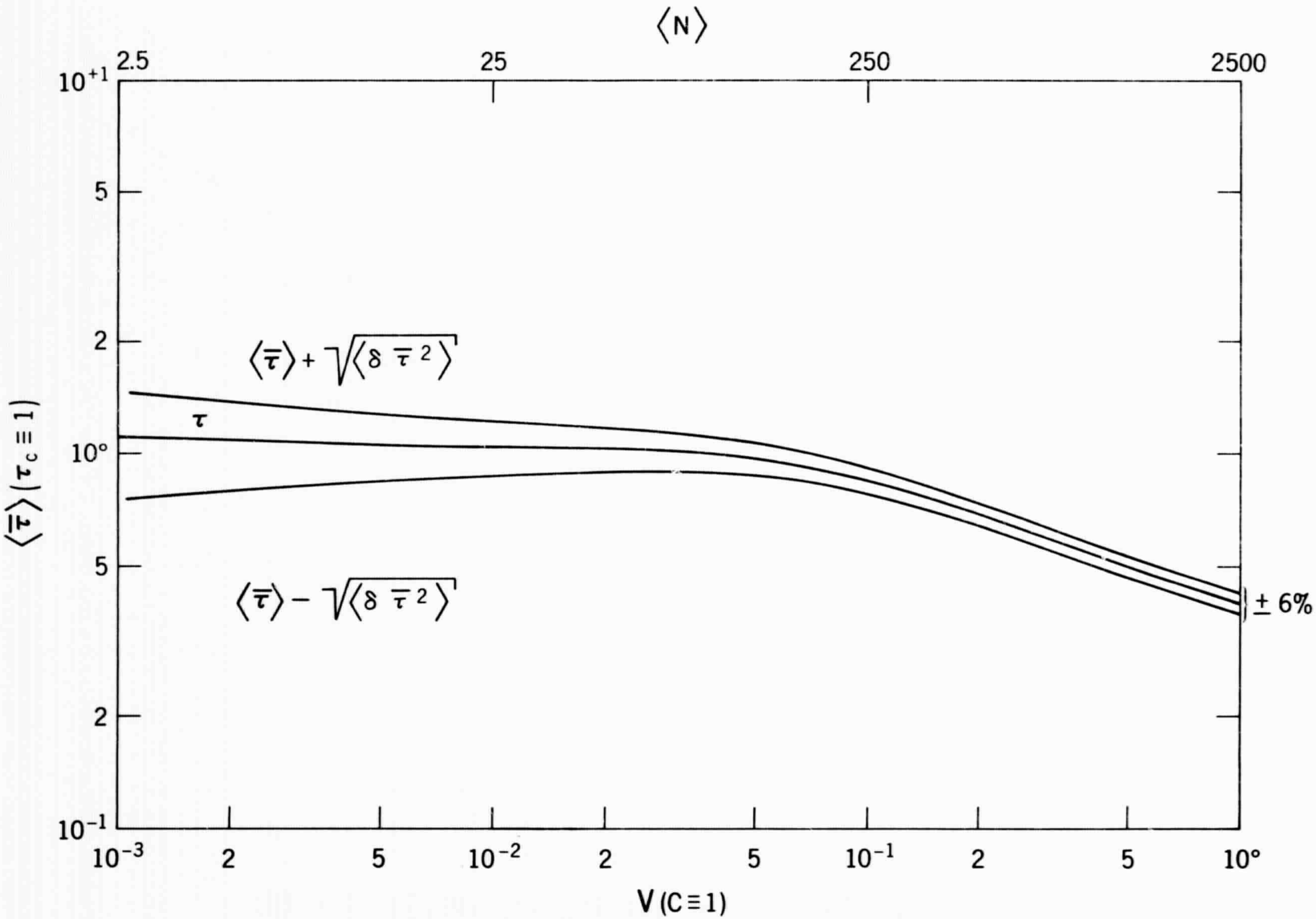

Figure 1.

BULK STREAMING VELOCITY vs. EXPANSION VELOCITY

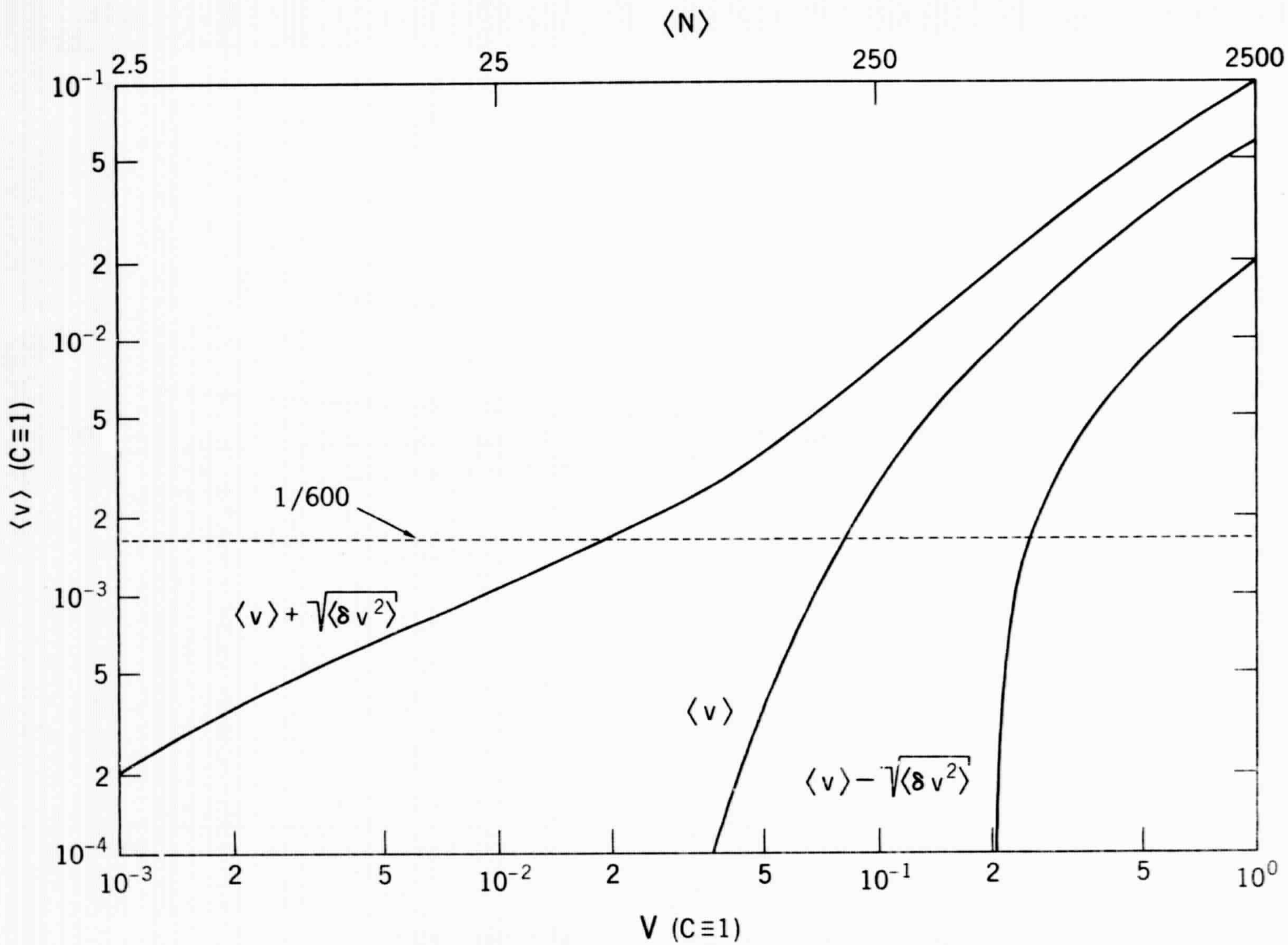

Figure 2.