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BOCHNER'S THEOREM IN INFINITE DIMENSIONS

by

P. L. Falb¹

and

U. Haussmann²

Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, R. I.

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BOCHNER'S THEOREM IN INFINITE DIMENSIONS

1. Introduction

Let G be a locally compact abelian group. A well-known theorem of Bochner ([1], [2]) states that a mapping ψ of G into \mathbb{C} is positive definite and continuous if and only if there is a unique nonnegative finite regular Borel measure m_ψ on \hat{G} (the dual group of G) such that $\psi(g) = \int_{\hat{G}} (\gamma, g) dm_\psi$ where (γ, g) denotes the action of the character γ on g . An alternate version of the theorem ([3]) states that if A is a semi-simple, self-adjoint, commutative Banach algebra and ψ is a linear functional on A , then ψ is positive and extendable if and only if there is a finite positive Baire measure ν_ψ on \mathcal{M} (the maximal ideal space of A) such that $\psi(\alpha) = \int_{\mathcal{M}} \hat{\alpha}(M) d\nu_\psi$ where $\hat{\alpha}$ is the Gelfand transform of $\alpha \in A$. Here we shall extend these theorems to mappings taking values in a Banach space X . Our results generalize the extension of Bochner's theorem made in [4].

We shall, in fact, first prove that if A is a self-adjoint, commutative Banach algebra and ψ is a linear map of A into the Banach space X , then ψ is positive⁺ and "almost" extendable if and only if there is a weak-*⁻regular, finite, positive set function ν_ψ^{**} mapping $\Sigma(\mathcal{M})$ (the Borel field of \mathcal{M}) into X^{**} such that $\psi(\alpha) = \int_{\mathcal{M}} \hat{\alpha}(M) d\nu_\psi^{**}$ (where $\psi(\alpha)$ is viewed as an element of X^{**}). We next show that if the mapping $\hat{\psi}$ of \hat{A} into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$ is weakly compact⁺⁺, then ν_ψ^{**} can be viewed as a weakly regular positive vector measure ν_ψ .

⁺Positivity is with respect to a suitable cone in X .

⁺⁺This means that $\hat{\psi}$ maps bounded sets in A into weakly compact sets in X .

mapping $\Sigma(\mathcal{M})$ into X and, conversely, if $\psi(\alpha) = \int_{\mathcal{M}} \hat{\alpha}(M) d\nu_{\psi}$ where ν_{ψ} is a weakly regular positive vector measure on $\Sigma(\mathcal{M})$ to X , then ψ is positive and "almost" extendable and $\hat{\psi}$ is weakly compact. In the case where $A = L_1(G, C)$, these results lead to a representation of ψ by an element p_{ψ} of $L_{\infty}(G, X)$ i.e. $\psi(\alpha) = \int_G \alpha(g) p_{\psi}(g) d\mu$ where μ is the Haar measure on G . We then develop an extended Bochner's theorem for maps p in $L_{\infty}(G, X)$. Finally, we use some particular Banach spaces to illustrate the theory.

The general results obtained here are combined with the transform theory on $L_1(G, X)$ to develop an inversion theorem and a Plancherel theorem in [5]. These theorems are also applied to the solution of convolution equations in Hilbert spaces in [5]. The convolution equations arise in the study of problems relating to the stability and control of systems described by parabolic partial differential equations.

2. Positive Functions

Let X be a Banach space and let X^* and X^{**} be the dual spaces of X and X^* , respectively. If ϕ is an element of X^* , then the operation of ϕ on x is denoted by (x, ϕ) . The notion of positivity that we use is based on a cone of "positive" elements contained in X . We assume that the cone is defined by a family of elements of X^* . More precisely, we have

DEFINITION 2.1. Let Φ be a subset of X^* . The subset K_{Φ} (or simply K when Φ is fixed by the context) of X given by

$$(2.2) \quad K_{\Phi} = \{x \in X: (x, \phi) \geq 0 \text{ for all } \phi \text{ in } \Phi\}$$

is called the cone determined by Φ .

Now let A be a Banach algebra with an involution given by $\alpha \rightarrow \alpha^*$, $\alpha \in A$, and let ψ be a linear mapping of A into X . We then have

DEFINITION 2.3. The mapping ψ is positive with respect to the cone K_Φ (or Φ -positive) if $\psi(\alpha\alpha^*) \in K_\Phi$ for all α in A .

We observe that ψ is Φ -positive if and only if the mappings $(\psi(\cdot), \varphi)$ of A into C are positive functionals for all φ in Φ . Note also that if ψ is Φ -positive, then, for any φ in Φ , the functional $B_\varphi(\alpha, \beta)$ given by

$$(2.4) \quad B_\varphi(\alpha, \beta) = (\psi(\alpha\beta^*), \varphi)$$

is a symmetric bilinear form satisfying the Cauchy inequality

$$(2.5) \quad |B_\varphi(\alpha, \beta)|^2 \leq B_\varphi(\alpha, \alpha)B_\varphi(\beta, \beta)$$

for α, β in A .

DEFINITION 2.6. The mapping ψ is symmetric with respect to Φ (or simply symmetric) if $(\psi(\alpha), \varphi) = \overline{(\psi(\alpha^*), \varphi)}$ for all φ in Φ and α in A .

If A has a unit e , then every Φ -positive mapping is symmetric since $(\psi(\alpha), \varphi) = (\psi(\alpha e), \varphi) = B_\varphi(\alpha, e) = \overline{B_\varphi(e, \alpha)} = \overline{(\psi(\alpha^*), \varphi)}$ for all φ . If A does not have a unit, then A can be imbedded in an algebra $\tilde{A} = A \oplus C$ with a unit in a natural way. Letting e be the unit in \tilde{A} , we can extend

ψ to a linear mapping $\tilde{\psi}_{x_0}$ of \tilde{A} into X by setting $\tilde{\psi}_{x_0}(\alpha + ce) = \psi(\alpha) + cx_0$ for a given x_0 in X . Clearly ψ is symmetric if and only if $\tilde{\psi}_{x_0}$ is. We now have

DEFINITION 2.7. A Φ -positive mapping ψ is almost extendable if (i) ψ is symmetric, (ii) ψ is continuous, and (iii) $|(\psi(\alpha), \varphi)|^2 \leq d \|\psi\| \|\varphi\| (\psi(\alpha\alpha^*), \varphi)$ for all φ in Φ and α in A where d is a constant with $d \geq 1$.

DEFINITION 2.8. A Φ -positive mapping ψ is extendable if ψ is symmetric and if there is an x_0 in X such that $|(\psi(\alpha), \varphi)|^2 \leq (x_0, \varphi)(\psi(\alpha\alpha^*), \varphi)$ for all φ in Φ and α in A .

If A has a unit e , then any Φ -positive mapping is extendable.

If A does not have a unit, then we have

PROPOSITION 2.9. A Φ -positive mapping ψ is extendable if and only if there is an extension $\tilde{\psi}$ of ψ to \tilde{A} which is Φ -positive.

Proof: If $\tilde{\psi}$ is a Φ -positive extension of ψ and e is the unit in \tilde{A} , then, letting $x_0 = \tilde{\psi}(e)$, we deduce immediately that $|(\psi(\alpha), \varphi)|^2 = |(\tilde{\psi}(\alpha), \varphi)|^2 \leq (x_0, \varphi)(\tilde{\psi}(\alpha\alpha^*), \varphi) = (x_0, \varphi)(\psi(\alpha\alpha^*), \varphi)$ (by 2.5) and that ψ is symmetric.

On the other hand, if ψ is extendable, then let $\tilde{\psi}(\alpha + ce) = \tilde{\psi}_{x_0}(\alpha + ce) = \psi(\alpha) + cx_0$. Since $(\tilde{\psi}([\alpha + ce][\alpha + ce]^*), \varphi) = (\psi(\alpha\alpha^*), \varphi) + 2\operatorname{Re} \bar{c}(\psi(\alpha), \varphi) + |c|^2(x_0, \varphi)$, we have $(\tilde{\psi}([\alpha + ce][\alpha + ce]^*), \varphi) \geq (\psi(\alpha\alpha^*), \varphi) - 2|c| |(\psi(\alpha), \varphi)| + |c|^2(x_0, \varphi) \geq [(\psi(\alpha\alpha^*), \varphi)^{1/2} - |c|(x_0, \varphi)^{1/2}]^2 \geq 0$ (as ψ is extendable). Thus, $\tilde{\psi}$ is Φ -positive.

PROPOSITION 2.10. If there is an approximate identity $\{e_n\}$ in A , then a continuous Φ -positive mapping ψ is almost extendable.

Proof: Since $(\psi(\alpha^*), \varphi) = \lim_{n \rightarrow \infty} (\psi(e_n \alpha^*), \varphi) = \lim_{n \rightarrow \infty} \overline{(\psi(\alpha e_n^*), \varphi)} = \overline{(\psi(\alpha), \varphi)}$, ψ is symmetric and, since $|B_\varphi(e_n, \alpha)|^2 \leq B_\varphi(e_n, e_n) B_\varphi(\alpha, \alpha) \leq \|\psi\| \|\varphi\| B_\varphi(\alpha, \alpha) = \|\psi\| \|\varphi\| (\psi(\alpha\alpha^*), \varphi)$, ψ is almost extendable.

In order to prove the extension of Bochner's theorem, we require a condition on the family Φ defining the cone of "positive" elements. As we shall see, the essential point is to deduce an estimate of the form $\|\psi(\alpha)\|^2 \leq k \|\psi(\alpha\alpha^*)\|$ from estimates of the form $|(\psi(\alpha), \varphi)|^2 \leq d \|\varphi\|^2 \|\psi\| \|\psi(\alpha\alpha^*)\|$ (ψ almost extendable) or $|(\psi(\alpha), \varphi)|^2 \leq \|\varphi\|^2 x_0 \|\psi(\alpha\alpha^*)\|$ (ψ extendable). The following definition allows us to do this.

DEFINITION 2.11. The family Φ is full if there is a $\rho > 0$ such that

$$(2.12) \quad \|x\| \leq \rho \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{ |(\psi(x), \varphi)| / \|\varphi\| \}$$

for all x in X^+

We now have

LEMMA 2.13. If A has a unit e , if the involution on A is continuous, and if Φ is full, then every Φ -positive mapping ψ is continuous and almost extendable.

Proof: Suppose first that α is a Hermitian element of A with $\|\alpha\| \leq 1$. The binomial series $(1-t)^{1/2} = 1 - \frac{t}{2} - \frac{t^2}{2^2 2!} - \dots$ converges absolutely for

⁺This could be replaced by the following: Φ is full relative to ψ if there is a $\rho > 0$ such that $\|\psi(\alpha)\| \leq \rho \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{ |(\psi(\alpha), \varphi)| / \|\varphi\| \}$ for all α in A .

$|t| \leq 1$ and so the series $e - \frac{\alpha}{2} - \frac{\alpha^2}{2^2 2!} - \dots$ converges absolutely in A .

Since the involution is continuous, the sum β of this series is a Hermitian element of A with $\beta\beta^* = \beta^2 = e - \alpha$. It follows that $(\psi(e - \alpha), \phi) = (\psi(\beta\beta^*), \phi) \geq 0$ and hence, that $(\psi(e), \phi) \geq (\psi(\alpha), \phi)$. Replacing α by $-\alpha$, we have $(\psi(e), \phi) \geq (\psi(-\alpha), \phi)$. But $(\psi(\alpha), \phi)$ is real (since α is Hermitian) and so $|(\psi(\alpha), \phi)| \leq \|\phi\| \|\psi(e)\|$. Since ϕ is full, $\|\psi(\alpha)\| \leq \rho \|\psi(e)\|$.

Now, if α is any element of A , then $\alpha = \frac{1}{2}(\alpha + \alpha^*) - \frac{i}{2}(i(\alpha - \alpha^*))$. Since the involution is continuous, there is a $c > 0$ such that $\|\alpha^*\| \leq c\|\alpha\|$ and so, if $\|\alpha\| \leq 2/c+1$, then $\|(\alpha + \alpha^*)/2\| \leq 1$ and $\|i(\alpha - \alpha^*)/2\| \leq 1$. It follows that $\|\psi(\alpha)\|^2 \leq 2\rho \|\psi(e)\|$ for all α in A with $\|\alpha\| \leq 2/c+1$. Thus, ψ is bounded and therefore continuous.

Since $|(\psi(\alpha), \phi)|^2 \leq (\psi(e), \phi)(\psi(\alpha\alpha^*), \phi) \leq \|\psi\| \|\phi\| (\psi(\alpha\alpha^*), \phi)$, ψ is almost extendable.

COROLLARY 2.14. If the involution on A is continuous, if ϕ is full, and if ψ is ϕ -positive and extendable, then ψ is continuous and almost extendable.

Proof: Apply proposition 2.9 and the lemma.

Let G be a σ -finite locally compact abelian group and let $A = L_1(G, \mathbb{C})$. The involution on $L_1(G, \mathbb{C})$ is given by $\alpha^*(g) = \overline{\alpha(-g)}$ and is continuous since $L_1(G, \mathbb{C})$ is semi-simple. Observe that if ϕ is full and ψ is a ϕ -positive mapping of $L_1(G, \mathbb{C})$ into X , then ψ is continuous and almost extendable if ψ is extendable (corollary 2.14) and conversely, ψ is almost extendable if ψ is continuous (proposition 2.10).

Now let us introduce the following

DEFINITION 2.15 Let p be an element of $L_\infty(G, X)$. The mapping p is Φ -positive definite if

$$(2.16) \quad \sum_{n=1}^N \sum_{m=1}^N c_n \overline{c_m} (p(g_n - g_m), \varphi) \geq 0$$

for any integer N , any c_1, \dots, c_N in C , any g_1, \dots, g_N in G , and all φ in Φ . The mapping p is integrally Φ -positive definite if

$$(2.17) \quad \left(\int_G \int_G \alpha(g) \overline{\alpha(g')} p(g - g') d\mu d\mu, \varphi \right) \geq 0$$

for all α in $L_1(G, C)$ and all φ in Φ .

We then have

PROPOSITION 2.18. Let p be a continuous element of $L_\infty(G, X)$. Then p is Φ -positive definite if and only if p is integrally Φ -positive definite.

Proof: If p is Φ -positive definite, then p is integrally Φ -positive definite by a result of Naimark ([6], p. 397). Conversely, if p is integrally Φ -positive definite, then there is a continuous positive definite function f_φ mapping G into C such that $f_\varphi(g) = (p(g), \varphi)$ locally almost everywhere on G ([6], p. 397) for each φ in Φ . Since $(p(\cdot), \varphi)$ is continuous, $f_\varphi(\cdot) = (p(\cdot), \varphi)$ everywhere and hence, p is Φ -positive definite.

Now it is a fact that ψ is a bounded weakly compact linear map of $L_1(G, C)$ into X with separable range if and only if there is a p with (essentially) weakly compact range in $L_\infty(G, X)$ such that

$$(2.19) \quad \psi(\alpha) = \int_G \alpha(g) p(g) d\mu$$

for all α in $L_1(G, C)$ ([9], p. 279, or [7], p. 507). Moreover, $\|\psi\| = \|p\|_\infty$. The fact that the weakly compact maps in $\mathcal{L}(L_1(G, C), X)$ are essentially the same as the functions with (essentially) weakly compact range in $L_\infty(G, X)$ will allow us to relate the notion of Φ -positivity to the notions of Φ -positive definiteness and integral Φ -positive definiteness.

LEMMA 2.20. Let Φ be full. If ψ is a weakly compact linear mapping of $L_1(G, C)$ into X which is Φ -positive and extendable, then there is an (essentially unique) integrally Φ -positive p in $L_\infty(G, X)$ such that

$$(2.21) \quad \psi(\alpha) = \int_G \alpha(g) p(g) d\mu$$

for all α in $L_1(G, C)$. Conversely, if p is an integrally Φ -positive definite element of $L_\infty(G, X)$ and ψ is given by 2.21, then ψ is Φ -positive and almost extendable.

Proof: Assume that ψ is given. In view of [9], p. 279, the mapping p exists and we need only show that p is integrally Φ -positive definite.

But

$$(2.22) \quad \psi(\alpha\alpha^*) = \int_G \int_G \alpha(g-g') \overline{\alpha(-g')} p(g) d\mu d\mu = \int_G \int_G \alpha(g) \overline{\alpha(g')} p(g-g') d\mu d\mu$$

by virtue of the Fubini and Tonelli theorems and the invariance of Haar measure. Conversely, given p , we simply note that $\psi(\alpha\alpha^*)$ is determined

by 2.22 in order to prove that ψ is Φ -positive. Moreover, since ψ is continuous, ψ is almost extendable by proposition 2.10.

3. Bochner's Theorem for Algebras

Before proving the generalization of Bochner's theorem to maps of A into X , we recall the following

DEFINITION 3.1. Let S be a locally compact topological space and let $\Sigma(S)$ be the Borel field of S . A vector measure ν is a weakly countably additive set function taking values in X . The vector measure ν is weakly regular if the scalar measures $(\nu(\cdot), \phi)$ are regular⁺ for all ϕ in X^* . The vector measure ν is Φ -positive if $(\nu(E), \phi) \geq 0$ for all ϕ in Φ and E in $\Sigma(S)$. A set function ν^{**} mapping $\Sigma(S)$ into X^{**} is weak-* -regular if $(\phi, \nu^{**}(\cdot))$ is a regular scalar measure for all ϕ in X^* . The set function ν^{**} is Φ -positive if $(\phi, \nu^{**}(E)) \geq 0$ for all ϕ in Φ and E in $\Sigma(S)$.

We now have

THEOREM 3.2. Let A be a self-adjoint commutative Banach algebra whose involution satisfies the condition $(\hat{\alpha}^*) = \bar{\alpha}$ (e.g. A semi-simple) and let Φ be a full family. If ψ is a mapping of A into X , then ψ is Φ -positive and almost extendable if and only if there is a set function ν^{**} mapping $\Sigma(\mathcal{A})$ into X^{**} such that (i) ν^{**} is weak-* -regular, (ii) ν^{**} is Φ -positive, (iii) ν^{**} is finite i.e. $\|\nu^{**}\|(\mathcal{A}) < \infty$, (iv) the mapping $T_{\nu^{**}}$

⁺ A scalar measure μ is regular if given $\epsilon > 0$ and $E \in \Sigma(S)$ with $\|\mu\|(E) < \infty$, then there is a compact $K \subseteq E$ and an open $O \supseteq E$ such that $\|\mu\|(O-K) < \epsilon$.

of X^* into the scalar measures on \mathcal{M} given by $T_{v^{**}}(\varphi) = (v^{**}(\cdot), \varphi)$ is continuous in the X and $C_0(\mathcal{M})^+$ topologies in these spaces respectively, and (v)

$$(3.3) \quad (\psi(\alpha), \varphi) = \int_{\mathcal{M}} \hat{\alpha}(M) d(v^{**}, \varphi)$$

for all α in A and all φ in X^* .

Proof: Suppose first that ψ is Φ -positive and almost extendable. Then ψ is continuous. Let $\hat{\psi}$ be the map of \hat{A} into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$. Then $\|\hat{\psi}(\hat{\alpha})\| = \|\psi(\alpha)\|$ and $|(\psi(\alpha), \varphi)|^2 \leq d\|\psi\|\|\varphi\|(\psi(\alpha\alpha^*), \varphi) \leq d\|\psi\|\|\varphi\|^2\|\psi(\alpha\alpha^*)\|$ for all φ in Φ (since ψ is almost extendable). Since Φ is full, there is a $\rho > 0$ such that $\|\psi(\alpha)\| \leq \rho \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{ |(\psi(\alpha), \varphi)| / \|\varphi\| \}$. Thus, there is a

constant $k (= \rho^2 d\|\psi\|)$ such that

$$(3.4) \quad \|\psi(\alpha)\|^2 \leq k\|\psi(\alpha\alpha^*)\|$$

for all α in A . It follows that $\|\psi(\alpha)\|^2 \leq k\|\psi(\alpha\alpha^*)\| \leq k^{1+1/2}\|\psi([\alpha\alpha^*]^2)\|^{1/2} \dots \leq k^2\|\psi\|_0^2\|\hat{\alpha}\|_\infty^2$ and hence, that $\hat{\psi}$ is a bounded linear map.

Since A is self-adjoint and commutative, \hat{A} is dense in $C_0(\mathcal{M})$ and $\hat{\psi}$ can, therefore, be extended to $C_0(\mathcal{M})$. Let $\hat{\psi}_e$ denote the extension of $\hat{\psi}$ to $C_0(\mathcal{M})$. We claim that there is a weak- $*$ -regular set function v^{**} on $\Sigma(\mathcal{M})$ such that

⁺If \mathcal{M} is compact, then $C_0(\mathcal{M})$ is the set of all continuous complex valued functions on \mathcal{M} . If \mathcal{M} is locally compact but not compact, then $C_0(\mathcal{M})$ is the set of all continuous complex valued functions on \mathcal{M} which "vanish at infinity".

$$(3.5) \quad (\hat{\psi}_e(f), \varphi) = \int_{\mathcal{M}} f(M) d(\nu^{**}, \varphi)$$

for all f in $C_0(\mathcal{M})$ and φ in X^* .

To verify this claim, we let $M(\mathcal{M})$ be the space of all complex valued regular measures μ on \mathcal{M} for which $\|\mu\|$ is finite ([2]). Note that $C_0(\mathcal{M})^* = M(\mathcal{M})$ by the Riesz representation theorem. If $E \in \Sigma(\mathcal{M})$, then let T_E be the element of $C_0(\mathcal{M})^{**}$ defined by

$$(3.6) \quad T_E(\mu) = \mu(E), \quad \mu \in M(\mathcal{M})$$

Now define a set function ν^{**} of $\Sigma(\mathcal{M})$ into X^{**} by setting

$$(3.7) \quad \nu^{**}(E) = \hat{\psi}_e^{**}(T_E)$$

for E in $\Sigma(\mathcal{M})$. We show that ν^{**} is weak- $*$ -regular. If φ is an element of X^* , then $\hat{\psi}_e^*(\varphi)$ is, by the Riesz representation theorem, a measure μ_φ in $M(\mathcal{M})$. But

$$(3.8) \quad \mu_\varphi(E) = T_E(\mu_\varphi) = T_E(\hat{\psi}_e^*(\varphi)) = \hat{\psi}_e^{**}(T_E)(\varphi) = (\nu^{**}(E), \varphi)$$

and so, the set function ν^{**} is weak- $*$ -regular. Moreover, since $\hat{\psi}_e^*(\varphi) = (\nu^{**}(\cdot), \varphi)$ by 3.8, the mapping $T_{\nu^{**}}$ satisfies (iv). Also, $(\hat{\psi}_e(f), \varphi) = \hat{\psi}_e^*(\varphi)(f) = \int_{\mathcal{M}} f(M) d\mu_\varphi = \int_{\mathcal{M}} f(M) d(\nu^{**}, \varphi)$ for f in $C_0(\mathcal{M})$ so that 3.3 is satisfied. It is easy to check that $\|\nu^{**}\|(\mathcal{M}) = \|\hat{\psi}_e\|$ ([7], p. 492) and so, (iii) is satisfied.

All that remains to establish the first half of the theorem is to prove that ν^{**} is Φ -positive. If f is an element of $C_0(\mathcal{M})$ with $f(M) \geq 0$ for all M , then $f^{1/2}$ is in $C_0(\mathcal{M})$ and there is a sequence $\{\alpha_n\}$ in A such that $\lim_{n \rightarrow \infty} \hat{\alpha}_n = f^{1/2}$. Since $(\hat{\psi}(\alpha_n \alpha_n^*), \varphi) = \int_{\mathcal{M}} |\hat{\alpha}_n(M)|^2 d(\nu^{**}, \varphi)$, it follows that if φ is an element of Φ , then $0 \leq (\psi(\alpha_n \alpha_n^*), \varphi) = \int_{\mathcal{M}} |\hat{\alpha}_n(M)|^2 d(\nu^{**}, \varphi)$ and hence, by taking limits, that

$$(7.9) \quad \int_{\mathcal{M}} f(M) d(\nu^{**}, \varphi) \geq 0$$

for all φ in Φ and all f in $C_0(\mathcal{M})$ with $f(\cdot) \geq 0$. But $(\nu^{**}(\cdot), \varphi)$ when restricted to the Baire sets in \mathcal{M} is a Baire measure, and as such, is positive. The Baire measure can be extended to a unique regular Borel measure ([9]) which must (by uniqueness) be $(\nu^{**}(\cdot), \varphi)$. It follows that ν^{**} is Φ -positive.

Now suppose that ν^{**} is given. Since the mapping $T_{\nu^{**}}$ is continuous in the X and $C_0(\mathcal{M})$ topologies, the linear mapping $\varphi \rightarrow \int_{\mathcal{M}} f(M) d(\nu^{**}, \varphi)$ is, for each fixed f in $C_0(\mathcal{M})$, continuous in the X -topology of X^* and is, therefore, generated by an element x_f of X . Thus, the mapping $\hat{\psi}_e$ of $C_0(\mathcal{M})$ into X given by $\hat{\psi}_e(f) = x_f$ is a bounded linear map of $C_0(\mathcal{M})$ into X . If α is an element of A , then let $\psi(\alpha) = \hat{\psi}_e(\hat{\alpha})$. Since $\|\psi(\alpha)\| = \|\hat{\psi}_e(\hat{\alpha})\| \leq \|\hat{\psi}_e\| \|\hat{\alpha}\|_{\infty} \leq \|\hat{\psi}_e\| \|\alpha\|$, ψ is a continuous linear map. If φ is an element of Φ , then $(\psi(\alpha \alpha^*), \varphi) = \int_{\mathcal{M}} |\hat{\alpha}(M)|^2 d(\nu^{**}, \varphi) \geq 0$ and $(\psi(\alpha^*), \varphi) = \int_{\mathcal{M}} \overline{\hat{\alpha}(M)} d(\nu^{**}, \varphi) = \overline{\int_{\mathcal{M}} \hat{\alpha}(M) d(\nu^{**}, \varphi)} = \overline{(\psi(\alpha), \varphi)}$ so that ψ is Φ -positive. Also, $|(\psi(\alpha), \varphi)|^2 \leq [\int_{\mathcal{M}} |\hat{\alpha}(M)|^2 d(\nu^{**}, \varphi)]$

$[\int_{\mathcal{M}} 1^2 d(v^{**}, \varphi)] \leq (\psi(\alpha\alpha^*), \varphi)(v^{**}(\mathcal{M}), \varphi) \leq \|v^{**}\|(\mathcal{M})\|\varphi\|(\psi(\alpha\alpha^*), \varphi) \leq$
 $\max\{1, \|v^{**}\|(\mathcal{M})/\|\psi\|\}\|\psi\|\|\varphi\|(\psi(\alpha\alpha^*), \varphi)$ so that ψ is almost extendable.

COROLLARY 3.10. Let A be a self-adjoint commutative Banach algebra with $(\hat{\alpha}^*) = \bar{\hat{\alpha}}$ and let Φ be a full family. If ψ is Φ -positive and almost extendable and if $\hat{\psi}$ is weakly compact, then there is a weakly regular Φ -positive vector measure ν on $\Sigma(\mathcal{M})$ such that

$$(3.11) \quad \psi(\alpha) = \int_{\mathcal{M}} \hat{\alpha}(M) d\nu$$

for all α in A . Conversely, if ν is a weakly regular Φ -positive vector measure and ψ is given by (3.11), then ψ is Φ -positive and almost extendable and $\hat{\psi}$ is weakly compact.

Proof: Suppose that ψ is given. Since $\hat{\psi}$ is weakly compact, $\hat{\psi}_e^+$ is weakly compact and so, $\hat{\psi}_e^{**}(C_0(\mathcal{M})^{**})$ is contained in the natural imbedding of X in X^{**} . Thus, the set function v^{**} given by 3.7 may be identified with a mapping ν of $\Sigma(\mathcal{M})$ into X . In that case, $(\nu(\cdot), \varphi)$ is an element of $M(\mathcal{M})$ for all φ in X^* . It follows that $\nu(\cdot)$ is a weakly regular vector measure (as $\nu(\cdot)$ is weakly countably additive). Clearly ν is Φ -positive. Moreover, since $(\psi(\alpha), \varphi) = \int_{\mathcal{M}} \hat{\alpha}(M) d(\nu, \varphi) = (\int_{\mathcal{M}} \hat{\alpha}(M) d\nu, \varphi)$ for all φ in X^* , 3.11 is satisfied.

On the other hand, if ν is given and ψ is defined by 3.11 (note that $\hat{\alpha}(\cdot)$ is bounded and continuous), then ψ is Φ -positive and

⁺We use the notation of the proof of the theorem.

almost extendable. In fact, $\|\psi(\alpha)\| \leq \|\hat{\alpha}\|_{\infty} \|v\|(\mathcal{M}) \leq \|\alpha\| \|v\|(\mathcal{M})$ and $|(\psi(\alpha), \varphi)|^2 \leq (\psi(\alpha\alpha^*), \varphi)(v(\mathcal{M}), \varphi)$ so that ψ is extendable ($x_0 = v(\mathcal{M}) \in X$). Thus, to complete the proof we need only show that $\hat{\psi}$ is weakly compact.

Now, $\hat{\psi}$ is clearly linear and, since $\|\hat{\psi}(\hat{\alpha})\| = \|\psi(\alpha)\| \leq \|v\|(\mathcal{M}) \|\hat{\alpha}\|_{\infty}$, $\hat{\psi}$ is continuous. Let $\hat{\psi}_e$ be the mapping of $C_0(\mathcal{M})$ into X defined by $\hat{\psi}_e(f) = \int_{\mathcal{M}} f(M) dv$. Thus, it will be enough to prove that $\hat{\psi}_e$ is weakly compact.

If φ is an element of X^* , then $\hat{\psi}_e^*(\varphi) = (v(\cdot), \varphi)$ is an element of $M(\mathcal{M})$. Since the set $\{(v(\cdot), \varphi) : \varphi \in X^*, \|\varphi\| \leq 1\}$ is weakly sequentially compact as a subset of the space of scalar measures and since v is weakly regular, $\hat{\psi}_e^*$ is a weakly compact mapping. It follows that $\hat{\psi}_e$ is weakly compact and the corollary is established.

COROLLARY 3.12. If v satisfies the conditions of corollary 3.10 and ψ is given by 3.11, then ψ is extendable. Conversely, if ψ is extendable (rather than almost extendable) and if the involution on A is continuous (e.g. A semi-simple), then a v satisfying the conditions of corollary 3.10 exists (the other hypotheses of corollary 3.10 are, of course, assumed).

Proof: The first assertion was established in the course of the proof of corollary 3.10. The second assertion is an immediate consequence of corollary 2.14.

COROLLARY 3.13. If X is weakly complete, if A and Φ satisfy the conditions of corollary 3.10, and if ψ is Φ -positive and almost extendable, then $\hat{\psi}$ is weakly compact.

Proof: By the argument given in the proof of theorem 3.2, $\hat{\psi}$ is a continuous linear map. If A has a unite, then \mathcal{M} is compact. Since \hat{A} is dense in $C_0(\mathcal{M})$, we may extend $\hat{\psi}$ to a continuous linear map $\hat{\psi}_e$ of $C_0(\mathcal{M})$ into X . As X is weakly complete, $\hat{\psi}_e$ is weakly compact ([7], p. 494) and a fortiori so is $\hat{\psi}$. If A does not have a unit, then we extend A to $\tilde{A} = A \oplus C$. Letting x_0 be an element of X , we extend ψ to a mapping $\tilde{\psi}$ of \tilde{A} into X by setting $\tilde{\psi}(\alpha + \lambda e) = \psi(\alpha) + \lambda x_0$. Then $\hat{\tilde{\psi}}(\hat{\alpha} + \lambda \hat{e}) = \hat{\psi}(\hat{\alpha}) + \lambda x_0$ is a bounded linear map of $\hat{\tilde{A}}$ into X . It follows that $\hat{\tilde{\psi}}$ is weakly compact and hence, that $\hat{\psi}$ is weakly compact.

4. Bochner's Theorem on a Group

Let G be a σ -finite locally compact abelian group and let $A = L_1(G, C)$. The involution on A is given by $\alpha^*(g) = \overline{\alpha(-g)}$ and is continuous. Let X be a Banach space and let Φ be a full family. We shall prove a generalization of Bochner's theorem for integrally Φ -positive definite mappings p in $L_\infty(G, X)$ by combining lemma 2.20 with theorem 3.2 and its corollaries. We have

THEOREM 4.1. (A) If ν is a weakly regular Φ -positive vector measure defined on $\Sigma(\hat{G})$ (the Borel field of the dual group \hat{G}) and if

$$(4.2) \quad p(g) = \int_{\hat{G}} \overline{\gamma(g)} d\nu$$

then p is an integrally Φ -positive definite element of $L_\infty(G, X)$.

(B) If p is an integrally Φ -positive definite element of $L_\infty(G, X)$, then there is a set function ν^{**} mapping $\Sigma(\hat{G})$ into X^{**} such

that (i) ν^{**} is weak*-regular, Φ -positive, and finite, (ii) the map $T_{\nu^{**}}$ given by $T_{\nu^{**}}(\varphi) = (\nu^{**}(\cdot), \varphi)$ is continuous in the X topology of X^* and the $C_0(\hat{G})$ topology of $M(\hat{G})$, and (iii)

$$(4.3) \quad (p(g), \varphi) = \int_{\hat{G}} \overline{(\gamma, g)} d(\nu^{**}, \varphi)$$

for all φ in X^* and (almost) all g in G .

Proof: (A) Let $p(\cdot)$ be given by 4.2. Suppose, for the moment, that $p(\cdot)$ is measurable. Then p is in $L_\infty(G, X)$ since $\|p(g)\| \leq \|\nu\|(\hat{G})$ for all g . Let $\psi(\alpha) = \int_G \alpha(g) p(g) d\mu$ for α in $L_1(G, \mathbb{C})$. Then

$$\begin{aligned} (\psi(\alpha), \varphi) &= \int_G \alpha(g) \int_{\hat{G}} \overline{(\gamma, g)} d(\nu, \varphi) d\mu \\ &= \int_{\hat{G}} \int_G \alpha(g) \overline{(\gamma, g)} d\mu d(\nu, \varphi) \\ &= \int_{\hat{G}} \hat{\alpha}(\gamma) d(\nu, \varphi) = (\int_{\hat{G}} \hat{\alpha}(\gamma) d\nu, \varphi) \end{aligned}$$

for all φ in X^* by the Fubini and Tonelli theorems. Since \hat{G} and \mathcal{M} can be identified ([2] or [3]), we have $\psi(\alpha) = \int_{\mathcal{M}} \hat{\alpha}(M) d\nu$ (as ν may be viewed as a measure on \mathcal{M}). But then (corollary 3.10) ψ is Φ -positive and extendable (corollary 3.12). The result follows immediately from 2.22 of lemma 2.20.

Thus, to complete the proof of (A), we need only show that p is measurable. To do this it will be sufficient to show that, for any set $F \subset G$ with $\mu(F) < \infty$, $P_F(\cdot) = \chi_F(\cdot) p(\cdot)$ is the limit in measure of a sequence of simple functions where χ_F is the characteristic function of F .

Since ν is weakly regular, there is a finite, positive, regular scalar measure λ such that $\|\nu\|(E) \rightarrow 0$ if and only if $\lambda(E) \rightarrow 0$ where $\|\nu\|(E)$ is the semi-variation of ν on E ([7]). Therefore, given $\eta > 0$, there is a $\xi > 0$ such that if $\lambda(\hat{G}-K)^+ < \xi$, then $\|\nu\|(\hat{G}-K) < \eta/4$ for K compact in \hat{G} . Since λ is finite and regular, there is a compact set $K \subset \hat{G}$ for which $\lambda(\hat{G}-K) < \xi$ and hence for which $\|\nu\|(\hat{G}-K) < \eta/4$. Let $\eta_1 = \eta/2\|\nu\|(\hat{G})$ and let $N(g; K, \eta_1) = \{g' \in G: |1 - (\gamma, g')| < \eta_1, \gamma \in K\} + g$. Then $N(g; K, \eta_1)$ is an open neighborhood of g in G .

Now G is σ -finite and so there is an increasing sequence of sets G_n with $\mu(G_n) < \infty$ and $\bigcup G_n = G$. Moreover, since Haar measure is regular, given $\epsilon > 0$ there is a compact set $L_n \subseteq G_n$ such that $\mu(G_n - L_n) < \epsilon$. The sets $N(g; K, \eta_1)$, $g \in L_n$, form an open cover of L_n . Thus there are g_1, \dots, g_{M_n} in L_n such that $L_n \subseteq \bigcup_{i=1}^{M_n} N(g_i; K, \eta_1)$. Let $N_1^n = N(g_1; K, \eta_1)$ and $N_{i+1}^n = N(g_{i+1}; K, \eta_1) - (N(g_1; K, \eta_1) \cup \dots \cup N(g_i; K, \eta_1))$. Then $L_n \subseteq \bigcup_{i=1}^{M_n} N_i^n$ and the union is disjoint. Let p_0 be defined on L_n by $p_0(g) = p(g_i)$ if $g \in N_i^n$ and let $p_n^{\epsilon, \eta}(\cdot)$ be given by

$$(4.4) \quad p_n^{\epsilon, \eta}(g) = \begin{cases} p_0(g) & g \in L_n \\ 0 & g \notin L_n \end{cases}$$

Then $p_n^{\epsilon, \eta}(\cdot)$ is a simple function and we claim that

$$(4.5) \quad \mu^*({g \in G_n: \|p(g) - p_n^{\epsilon, \eta}(g)\| > \eta}) < \epsilon$$

where $\mu^*(E) = \inf_{E_1 \supseteq E} \mu(E_1)$. For if g is in L_n , then $\|p(g) - p_n^{\epsilon, \eta}(g)\| =$

*Here $\hat{G}-K$ is the complement of K .

$\|p(g) - p_0(g)\| = \|\int_{\hat{G}} \overline{(\gamma, g)} [1 - \overline{(\gamma, g_1 - g)}] d\nu\| \leq \|\int_{\hat{G}-K} \overline{(\gamma, g)} [1 - \overline{(\gamma, g_1 - g)}] d\nu\| +$
 $\|\int_K \overline{(\gamma, g)} [1 - \overline{(\gamma, g_1 - g)}] d\nu\| \leq \frac{\eta}{2} + \eta_1 \|v\|(\hat{G}) = \eta$ (for some i) so that $\{g \in G_n:$
 $\|p(g) - p_n^{\epsilon, \eta}(g)\| > \eta\} \subseteq G_n - L_n$. It follows that 4.5 holds. Let $p_n(g) =$
 $p_n^{1/n, 1/n}(g)$ so that p_n is a simple function.

Now suppose that a is any positive number. We show that

$$(4.6) \quad \lim_{n \rightarrow \infty} \mu^* (\{g \in F: \|p(g) - p_n(g)\| > a\}) = 0$$

for any $F \subset G$ with $\mu(F) < \infty$. So let $\epsilon > 0$ be given. Then there is an $n_0 \geq \max(1/a, 2/\epsilon)$ such that $\mu(F \cap (G - G_n)) < \epsilon/2$ for $n \geq n_0$. It follows that

$$\begin{aligned}
 \mu^* (\{g \in F: \|p(g) - p_n(g)\| > a\}) &\leq \mu^* (\{g \in F \cap G_n: \|p(g) - p_n(g)\| > a\}) + \epsilon/2 \\
 &\leq \mu^* (\{g \in F \cap G_n: \|p(g) - p_n(g)\| > 1/n\}) + \epsilon/2 \\
 &\leq 1/n + \epsilon/2 \leq \epsilon
 \end{aligned}$$

for $n \geq n_0$. In other words, p_n converges to p in measure on F . The proof of (A) is now complete.

(B) Let $\psi(\alpha) = \int_G \alpha(g) p(g) d\mu$. Then ψ is Φ -positive and almost extendable by lemma 2.20. It follows from theorem 3.2 that there is a set function ν^{**} on $\Sigma(\mathcal{M})$ such that (i) and (ii) are satisfied and

$$(4.7) \quad (\psi(\alpha), \varphi) = \int_{\mathcal{M}} \hat{\alpha}(M) d(\nu^{**}, \varphi)$$

for all φ in X^* . Since \hat{G} and \mathcal{M} can be identified, ν^{**} may be viewed as a set function on $\Sigma(\hat{G})$ and

$$(4.8) \quad (\psi(\alpha), \varphi) = \int_{\hat{G}} [\int_G \alpha(g) \overline{(\gamma, g)} d\mu] d(\nu^{**}, \varphi)$$

for all φ in X^* . Application of the Fubini and Tonelli theorems then yields

$$(4.9) \quad \int_G \alpha(g) (p(g), \varphi) d\mu = (\psi(\alpha), \varphi) = \int_G \alpha(g) q_{\varphi}(g) d\mu$$

where $q_{\varphi}(g) = \int_{\hat{G}} \overline{(\gamma, g)} d(\nu^{**}, \varphi)$. Since $(p(\cdot), \varphi)$ and $q_{\varphi}(\cdot)$ are in $L_{\infty}(G, \mathbb{C})$, we have $\|(p(\cdot), \varphi) - q_{\varphi}(\cdot)\|_{\infty} = 0$ for all φ in X^* . In other words, 4.3 holds. The proof of (B) is now complete.

REMARK 4.10. Since $\overline{(\gamma, g)} = (-\gamma, g)$ and since the measure ν_1 (or the set function ν_1^{**}) given by $\nu_1(E) = \nu(-E)$ (or $\nu_1^{**}(E) = \nu^{**}(-E)$) has the same properties as ν (or ν^{**}), $p(g)$ is given by $p(g) = \int_{\hat{G}} (\gamma, g) d\nu_1$ (or satisfies $(p(g), \varphi) = \int_{\hat{G}} (\gamma, g) d(\nu_1^{**}, \varphi)$). [This agrees with convention in the scalar case.]

We observe that if the hypotheses of (A) are satisfied and $\psi(\alpha) = \int_G \alpha(g) p(g) d\mu$, then the mapping $\hat{\psi}$ of \hat{A} into X given by $\hat{\psi}(\hat{\alpha}) = \psi(\alpha)$ is weakly compact (corollary 3.10). Note also that if X is weakly complete and $p(\cdot)$ is an integrally Φ -positive definite element of $L_{\infty}(G, X)$, then $\hat{\psi}$ is weakly compact. This leads to

COROLLARY 4.11. If X is weakly complete and if p is an integrally Φ -

positive definite element of $L_\infty(G, X)$, then there is a weakly regular Φ -positive vector measure ν on $\Sigma(\hat{G})$ such that

$$(4.12) \quad (p(g), \varphi) = \left(\int_{\hat{G}} \hat{\alpha}(r, g) d\nu, \varphi \right)$$

for all φ in X^* and (almost) all g in G . If, in addition, Φ is countable, then

$$(4.13) \quad p(g) = \int_{\hat{G}} \hat{\alpha}(r, g) d\nu$$

for (almost) all g in G .

Proof. The first assertion follows from corollary 3.12. On the other hand, if $\Phi = \{\varphi_i\}$ is countable, then there is a μ -null set N such that

$$(p(g) - q(g), \varphi) = 0 \text{ for all } \varphi \text{ in } \Phi \text{ and } g \notin N \text{ where } q(g) = \int_{\hat{G}} \hat{\alpha}(r, g) d\nu.$$

But then $\|p(g) - q(g)\| \leq \rho \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(p(g) - q(g), \varphi)| / \|\varphi\|\} = 0$ for $g \notin N$. It

follows immediately that $\|p(\cdot) - q(\cdot)\|_\infty = 0$, i.e. that 4.13 holds.

In order to state our final corollary we require

DEFINITION 4.14. The element p of $L_\infty(G, X)$ is dominated if there exists a finite regular positive Borel measure λ such that

$$(4.15) \quad \left\| \int_{\hat{G}} \alpha(g) p(g) d\mu \right\| \leq \int_{\hat{G}} |\hat{\alpha}(r)| d\lambda$$

for all α in $L_1(G, C)$, where $\hat{\alpha}$ is the Fourier transform of α .

COROLLARY 4.16. Assume Φ is countable. Then p is a dominated integrally Φ -positive definite element of $L_\infty(G, X)$ if and only if there exists a weakly regular Φ -positive vector measure ν of finite variation mapping $\Sigma(\hat{G})$ into X such that

$$(4.17) \quad p(g) = \int_{\hat{G}} (r, g) d\nu.$$

Proof. We have only to note that there exists an isomorphism between the set of weakly regular vector measures $\nu: \Sigma(\hat{G}) \rightarrow X$ with finite variation and the set of bounded linear operators $T: C_0(\hat{G}) \rightarrow X$ for which there exists a finite regular positive Borel measure λ such that $\|T(f)\| \leq \int_{\hat{G}} |f(r)| d\lambda$. This isomorphism is given by $T(f) = \int_{\hat{G}} f(r) d\nu$, ([9], p. 380, or [11]). Now using theorem 4.1 (B) we have, if we assume the existence of p , that $(\int_{\hat{G}} f(r) d\nu, \phi) = \int_{\hat{G}} f(r) d(\nu^{**}, \phi)$ for any f in $C_0(\hat{G})$, ϕ in X^* . But $C_0(\hat{G})^* = M(\hat{G})$, the space of regular complex valued measures defined on $\Sigma(\hat{G})$ of finite variation, and $(\phi, \nu^{**}), (\nu, \phi)$ are in $M(\hat{G})$. Thus, for any E in $\Sigma(\hat{G})$, $(\nu(E), \phi) = (\phi, \nu^{**}(E))$. Consider $\nu(E)$ as an element of X^{**} , then $\nu(E) = \nu^{**}(E)$ and so ν^{**} is actually a measure. From the countability of Φ we derive (4.17).

The converse follows immediately from theorem 4.1 (A).

5. Some Examples

We now give several examples of spaces to which the theory applies.

EXAMPLE 5.1. Let $X = L_1([0,1], \mathbb{C})$. Note that X is weakly complete. If $\Sigma = \Sigma([0,1])$ is the Borel field on $[0,1]$, then Σ is a separable metric space with respect to the usual metric $d(E, E') = \mu(E \Delta E')$ where $E \Delta E' =$

$(E-E') \cup (E'-E)$ is the symmetric difference of E and E' . Let $\{E_i\}$ be a countable dense set in Σ with $E_1 = [0,1]$. Let χ_i be the characteristic function of E_i and let ϕ_i be the element of X^* given by

$$(5.2) \quad (x(\cdot), \phi_i) = \int_0^1 \chi_i(s) x(s) ds$$

If $\Phi = \{\phi_i\}$, then K_Φ is the cone of (essentially) nonnegative functions. Note also that $\|\phi_i\| \leq 1$.

Now we claim that Φ is full. Set $x^+(s) = \max\{0, x(s)\}$ and $x^-(s) = \max\{0, -x(s)\}$ for real x in X . Then $x(s) = x^+(s) - x^-(s)$ and $|x(s)| = x^+(s) + x^-(s)$. Moreover, x^+ and x^- are nonnegative. Letting $\int_0^1 x_0(s) ds = \max\{\int_0^1 x^+(s) ds, \int_0^1 x^-(s) ds\}$ (i.e. $x_0 = x^+$ or x^- according to which integral is greater), we see that

$$(5.3) \quad \|x(\cdot)\|_1 \leq 2 \int_0^1 x_0(s) ds$$

for real x in X . Now, suppose, for example, that $x_0 = x^+$. Since x^+ is measurable, $(x^+)^{-1}([0, \infty)) = E$ is in Σ and $\int_0^1 x_0(s) ds = \int_E x^+(s) ds = \int_E x(s) ds$. As $\{E_i\}$ is dense in Σ , there is a sequence $\{E_{i,n}\}$ such that $d(E_{i,n}, E) \rightarrow 0$ as $n \rightarrow \infty$. But $|\int_{E_{i,n}} x(s) ds - \int_E x(s) ds| \leq \int_{E_{i,n} \Delta E} |x(s)| ds$ and

$\lim_{n \rightarrow \infty} \int_{E_{i,n} \Delta E} |x(s)| ds = 0$ as $\mu(E_{i,n} \Delta E) \rightarrow 0$ as $n \rightarrow \infty$ and x is in

$L_1([0,1], \mathbb{R})$. It follows that there is a sequence $\{\phi_{i,n}\}$ such that $\lim_{n \rightarrow \infty} (x, \phi_{i,n}) = \int_0^1 x_0(s) ds$ and hence, that $\int_0^1 x_0(s) ds \leq \sup_{\phi \in \Phi} |(x, \phi)|$. Now choose any x in

X . Then $x = x_1 + ix_2$ where $x_1(\cdot), x_2(\cdot)$ are real valued. But $\|\phi\| \leq 1$ for ϕ in Φ and so,

$$(5.4) \quad \|x\|_1 \leq \|x_1\|_1 + \|x_2\|_1 \leq 4 \sup_{\varphi \in \Phi} |(x, \varphi)| \leq 4 \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{ |(x, \varphi)| / \|\varphi\| \}$$

for all x in X .

EXAMPLE 5.5. Let $X = H$ be a separable Hilbert space. Fix an orthonormal basis $\{e_i\}$ in H . If $h \in H$, then $h = h_1 + ih_2$ where $h_1 = \sum \operatorname{Re}[\langle h, e_i \rangle] e_i$ and $h_2 = \sum \operatorname{Im}[\langle h, e_i \rangle] e_i$. An element h is real if $h = h_1$. Let H_0 be the set of all real elements h such that (i) $\|h\| \leq 1$, (ii) h is positive i.e. $\langle h, e_i \rangle \geq 0$ for all i , (iii) h is rational i.e. $\langle h, e_i \rangle$ is rational for all i , and (iv) h is finite i.e. only a finite number of components $\langle h, e_i \rangle$ of h are not zero. Since H^* can be identified with H , we let $\Phi = H_0$. In other words, if $\varphi \in \Phi$, then $(h, \varphi) = \langle h, k \rangle$ for some k in H_0 . The cone K_Φ is the set of all positive real elements of H .

We claim that Φ is full. Suppose first that $h = h_1$ is real. Then $h_1 = h_1^+ - h_1^-$ where $\langle h_1^+, e_i \rangle = \max\{0, \langle h_1, e_i \rangle\}$ and $\langle h_1^-, e_i \rangle = \max\{0, -\langle h_1, e_i \rangle\}$ for all i . Note that $\|h_1\|^2 = \|h_1^+\|^2 + \|h_1^-\|^2$. Let k_n^+ be the element of H_0 with components $\langle k_n^+, e_i \rangle$ given by

$$(5.6) \quad \langle k_n^+, e_i \rangle = \begin{cases} r_i & i \leq N \\ 0 & i > N \end{cases}$$

where N is chosen so that

$$(5.7) \quad \sum_{N+1}^{\infty} |\langle h_1^+, e_i \rangle|^2 < \frac{1}{2n^2} \|h_1^+\|^2$$

and r_i is a nonnegative rational such that

$$(5.8) \quad \langle h_1^+, e_i \rangle \geq r_i \|h_1^+\rangle > \left\{ \langle h_1^+, e_i \rangle - \frac{\|h_1^+\|}{n\sqrt{2N}} \right\}$$

Clearly $\|k_n^+ - \frac{h_1^+}{\|h_1^+\|}\| < 1/n$. It follows that $(h_1^+, k_n^+) \rightarrow \|h_1^+\|$ as $n \rightarrow \infty$.

Similarly, there is a sequence k_n^- in H_0 such that $(h_1^-, k_n^-) \rightarrow \|h_1^-\|$ as $n \rightarrow \infty$.

Noting that for any $h = h_1 + ih_2$ in H , $|(h, \varphi)| \geq \max\{|(h_1, \varphi)|, |(h_2, \varphi)|\}$, we have $\|h_1\|^2 = \|h_1^+\|^2 + \|h_1^-\|^2 = \lim_{n \rightarrow \infty} (h_1^+, k_n^+)^2 + \lim_{n \rightarrow \infty} (h_1^-, k_n^-)^2 \leq$

$\overline{\lim} |(h_1^+, k_n^+)|^2 + \overline{\lim} |(h_1^-, k_n^-)|^2 \leq 2 \sup_{\varphi \in \Phi} |(h_1, \varphi)|^2$. Now, if $h = h_1 + ih_2$ is

any element of H , then $\|h\|^2 = \|h_1\|^2 + \|h_2\|^2 \leq 2 \sup_{\varphi \in \Phi} |(h_1, \varphi)|^2 +$

$2 \sup_{\varphi \in \Phi} |(h_2, \varphi)|^2 \leq 4 \sup_{\varphi \in \Phi} |(h, \varphi)|^2$. Since $\|\varphi\| \leq 1$ if $\varphi \in \Phi = H_0$, we deduce

that $\|h\| \leq 2 \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(h, \varphi)| / \|\varphi\|\}$ for all h in H . Thus, Φ is full.

EXAMPLE 5.9. Let H be a separable Hilbert space and let $X = \mathcal{L}(H, H)$ be the space of bounded linear maps of H into itself. Let H_0 be a countable dense subset of the closed unit ball in H and let $\Phi = \{\varphi \in X^* :$

$(T, \varphi) = \langle Tk, k \rangle$, for some k in $H_0\}$. The cone K_Φ is the set of positive operators in $\mathcal{L}(H, H)$.

Since $\|T\| \leq 2 \sup_{\|h\| \leq 1} |\langle Th, h \rangle|$ for T in

$\mathcal{L}(H, H)$ and since $\|\varphi\| \leq \|k\|^2 \leq 1$ for k in H_0 , we have $\|T\| \leq$

$2 \sup_{\substack{\varphi \in \Phi \\ \varphi \neq 0}} \{|(T, \varphi)| / \|\varphi\|\}$. In other words, Φ is full.

EXAMPLE 5.10. Let \mathcal{D} be a bounded domain in \mathbb{R}^n and let $X = L_p(\mathcal{D}, \mathbb{C})$

where $1 < p < \infty$. Let Σ be the Borel field of \mathcal{D} . Then Σ is a separable metric space with respect to the usual metric $d(E, E') = \mu(E \Delta E')$.

Let $\Sigma_0 = \{E_i\}$ be a countable dense set in Σ which include all hyper-

cubes with rational vertices contained in \mathcal{D} and let $Q = \{a+bi \in \mathbb{C} :$

a, b rational $\}$. Let \mathcal{S} be the set of simple functions of the form

$\sum_{i=1}^n q_i \chi_{E_i}$ where the q_i are in Q and the E_i are disjoint elements of Σ_0 .

Note that \mathcal{S} is a countable subset of $L_q(\mathcal{D}, C)$ where $1/p + 1/q = 1$. It is easy to check that \mathcal{S} is dense in $L_q(\mathcal{D}, C)$. An element $\sum_{i=1}^n q_i \chi_{E_i}$ of \mathcal{S} is positive real if q_i is a nonnegative real number for $i = 1, \dots, n$.

Let Φ be the subset of \mathcal{S} consisting of all the positive real elements.

Since $X^* = L_p(\mathcal{D}, C)^* = L_q(\mathcal{D}, C)$, $\Phi \subset X^*$ and the cone K_Φ is simply the set of nonnegative functions in $L_p(\mathcal{D}, C)$. The proof that Φ is full is straightforward and is, therefore, left to the reader. Theorem 4.1, when interpreted in this context, becomes:

COROLLARY 5.11. If p is an element of $L_\infty(G, L_p(\mathcal{D}, C))$ such that $\int_G \int_G \xi(g) \bar{\xi}(g') p(g-g') d\mu d\mu$ is a nonnegative function in $L_p(\mathcal{D}, C)$ for all $\xi(\cdot)$ in $L_1(G, C)$, then $p(g) = \int_{\hat{G}} \gamma(g) d\nu$ where ν is a weakly regular measure on \hat{G} such that $\nu(F)$ is a nonnegative function in $L_p(\mathcal{D}, C)$ for all F in $\Sigma(\hat{G})$, and conversely.

This corollary plays a role in the study of positive solutions of certain partial differential equations.

EXAMPLE 5.12. Let H be a separable Hilbert space and let $\mathcal{L} = \mathcal{L}(H, H)$ be the closed ideal of compact operators in $\mathcal{L}(H, H)$. It is well-known ([8]) that $\mathcal{L}(H, H)^* = \mathcal{L}_1 \oplus \mathcal{L}^\perp$ where \mathcal{L}^\perp is the annihilator of \mathcal{L} and \mathcal{L}_1 is the trace class. Moreover, \mathcal{L}_1 is isometrically isomorphic with \mathcal{L}^* and \mathcal{L}^{**} is isometrically isomorphic with $\mathcal{L}_1^* = \mathcal{L}(H, H)$. Now let H_0 be a countable dense subset of the closed unit ball in H and let $\Phi = \{\varphi \in \mathcal{L}^*: (T, \varphi) = \langle Tk, k \rangle \text{ some } k \text{ in } H_0\}$. The cone K_Φ is the set of positive compact operators and Φ is a countable full family.

References

- [1] S. Bochner, Monotone funktionen, Stieltjessche integrale, und harmonische analyse, Math. Ann. 108, 378-410 (1933).
- [2] W. Rudin, "Fourier Analysis on Groups", Interscience, New York, 1962.
- [3] L. Loomis, "An Introduction to Abstract Harmonic Analysis", Van Nostrand, New York, 1953.
- [4] P. Falb, On a theorem of Bochner, Institut Haute Etudes Scientifique Publications Mathematiques, 36, (1969).
- [5] U. Haussmann, The inversion and Plancherel theorems in a Banach space, (to appear).
- [6] M. Naimark, "Normed Rings", P. Noordhoff, Groningen, 1959.
- [7] N. Dunford and J. Schwartz, "Linear Operators, Part I: General Theory", Interscience, New York, 1958.
- [8] J. Dixmier, "Les Algebres d'Operateurs dans l'Espace Hilbertien", Gauthiers-Villars, Paris, 1957.
- [9] N. Dinculeanu, "Vector Measures", Pergamon Press, New York, 1967.
- [10] P. Halmos, "Measure Theory", Van Nostrand, New York, 1950.
- [11] U. Haussmann, "Harmonic Analysis in Banach Space", Ph.D. Dissertation, June 1970, Brown University.