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PHENOMENOLOGICAL THEORY OF LOUDNESS

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ABSTRACT

Alternative time- and frequency-domain equations are presented for predicting the loudness of a wide variety of statistically stationary and nonstationary sounds, either continuous or discontinuous. Zwislocki's theory of temporal summation and S. S. Stevens' psychoacoustic conversion law are incorporated in the present mathematical theory. Frequency domain formulas of Zepler and Harel for impulsive sonic booms and Jones for steady noise represent specializations of the present formulas. For sinusoidal inputs, modified Fletcher-Munson auditory response curves are predicted. For an impulsive input the measured response is also predicted.

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SUMMARY

A unified theory is derived which should permit the loudness of most sounds, continuous and discontinuous, to be predicted from known time or frequency characteristics of the sound. It is assumed that the input sound intensity averaged over a finite time is uniquely related to loudness. This relation is modified to include operational processes by which the human auditory system converts intensity into loudness. The processes by which the input pressure signal is transmitted to the brain are assumed to be linear. However, the conversion from a physical (neurological) signal into psychological response (loudness), which occurs in the brain, is nonlinear.

Physically, the input sound pressure wave is linearly converted in successive steps into an electrical wave, which reproduces the original waveform, by hair cells within the organ of Corti. Next, the auditory nerve endings respond to the time rate of change of this current rather than to the current itself. The resulting information is transmitted to the brain along the auditory nerve. This information is evaluated in the brain and subjectively interpreted as loudness.

Mathematically, the time domain representation of the current output of the hair cells is a Fourier convolution of the impulse response of the entire preceding system with the original input sound. The response of the auditory nerve endings corresponds to time differentiation of the Fourier convolution. The signal transmitted to the brain contains information regarding "electric power," which is assumed to be uniquely related to loudness. The loudness is a function of a finite-time integral of the power. The conversion from physical output to psychoacoustic response is accomplished by using S. S. Stevens' psychoacoustic conversion law. The frequency domain representations of these processes are derived by using Fourier series and transforms. The fact that the loudness is a function taken over finite times implies that the frequency representations can be written in terms of "running" Fourier transforms. The complete history of the loudness is predicted.

The complete auditory system must act as a nonideal, band-pass, filter. The response of a selected filter characterizes the system. Part of the selected response function may be attributed to the time differentiation process. The rest of the function is a generalization of that obtained by Zwislocki in his theory of temporal auditory summation. Thus, Zwislocki's theory is implicit in the present one.

The present theory was tested using two fundamental inputs, sine waves and impulses. For sine wave inputs the theory predicts the Fletcher-Munson frequency response curves minus a diffraction correction for the disturbance created by the human head. For impulsive inputs the theory predicts loudness proportional to the intensity, as measured.

Frequency domain formulas of Zepler and Harel for impulsive sonic booms and Jones for steady noise represent specializations of the present formulas.

INTRODUCTION

Well-known empirical methods exist for predicting the loudness of certain statistically stationary sounds (refs. 1 to 3), that is, of certain sounds whose statistics are independent of time. Methods for predicting the loudness of certain statistically nonstationary sounds may be less well known (refs. 4 to 6). No means exists for predicting the loudness of all sounds of either statistical class. Moreover, no single scheme has been shown to predict correctly the loudness of some sounds in both classes. The main deterrent in developing a unified theory of loudness appears to be an impression that the complete auditory system is so complex that a concise mathematical representation of the entire system is not feasible (ref. 6), and that, because the psychoacoustic response to an acoustic input is highly nonlinear, Fourier analysis is not readily applicable to the entire system (refs. 6 and 7). Thus, there would seem to be an inherent difficulty in relating time and frequency representations of psychoacoustic response to a given acoustic input. The main purpose of this report is to show that a practical unified theory of loudness based on Fourier methods is possible and that the theory proposed herein leads to predictions in good agreement with experiment.

The theory to be described resulted from a desire to obtain alternative time and frequency descriptions of the loudness of sonic booms produced by supersonic aircraft. Such a theory might be useful in determining the extent to which undesirable human response to sonic booms could be minimized by controlling the boom pressure signature (ref. 8). The present theory appears to have much broader validity than originally intended.

There are at least three forms of theory which might be developed, namely, one based on the physics and psychophysiology of the ear, nervous system and brain, a phenomenological theory in which the major elements of the complete auditory system are represented by simplified mathematical models, or a completely empirical theory in which each input is directly associated with an ultimate output response as determined experimentally. The first form of theory is likely to get bogged down by physical and mathematical complexities. The last form of theory (empirical) is likely to be impractical because new response tests would have to be performed for each new waveform. It is not likely that such a theory would improve understanding of the hearing process. However, in cases where very rapid pressure changes are the overwhelming determinant of response (as in the case of sonic booms), the empirical approach may still prove completely satisfactory for engineering calculations. Herein the phenomenological approach has been adopted with the hope that it will lead to reasonably accurate estimates of response to a variety of input signatures based on simple mathematical representations of the operational characteristics of the human auditory system.

The complete auditory system consists of three principal elements: the ear, the nervous system, and the brain (fig. 1). The primary operational functions of the ear,

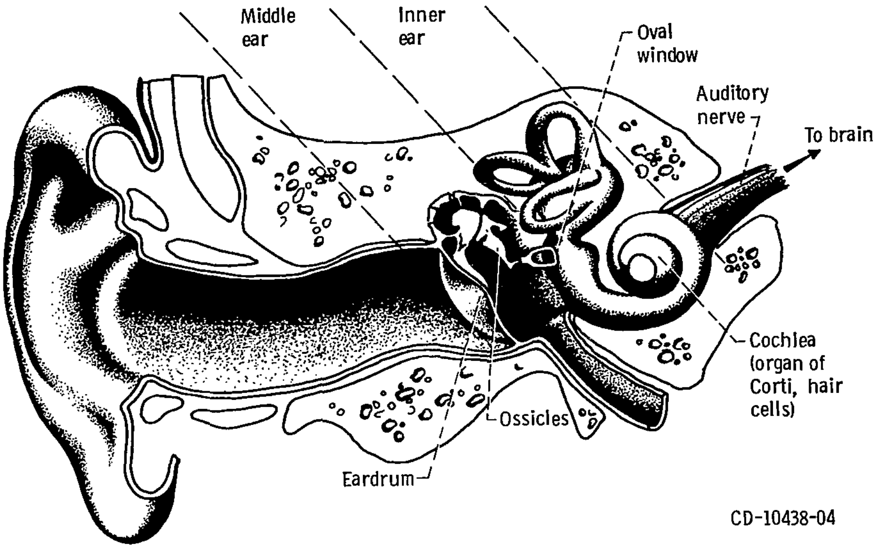


Figure 1. - Auditory system.

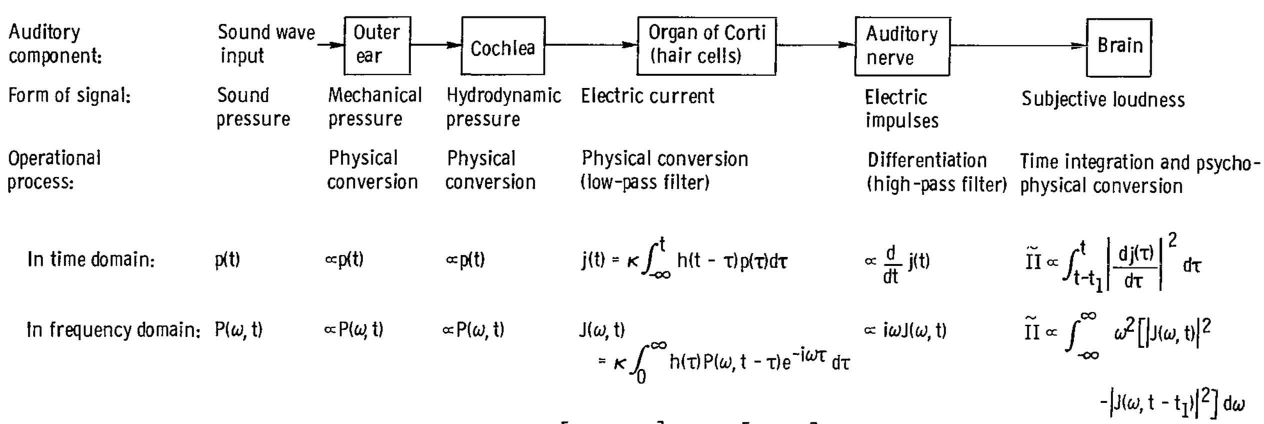
nervous system, and brain are assumed to be, respectively, pressure amplification; physical conversion, filtering and time differentiation; and autocorrelation and "psychophysical conversion" (fig. 2). The "physical conversion" is from a sound wave to mechanical pressure to a hydrodynamic wave and, finally, to an electrical wave. "Filtering" simply implies that all the energy of the incident sound waves is not transmitted to the brain. "Autocorrelation" concisely describes the mathematical process of integrating the "power" with respect to time. The term "psychophysical conversion" is intended to imply the conversion of a signal magnitude from objective, physical measure to subjective, psychological measure; that is, from physical intensity to loudness in the case of statistically stationary sounds.

In the ear (refs. 3 and 9 to 11) (see fig. 1) the sound pressure fluctuations in the atmosphere are mechanically amplified by the eardrum and ossicles into hydrodynamic pressure waves within the cochlea. The conversion into hydraulic waves occurs at the oval window. Within the cochlea the hydraulic pressure waves are further converted into electrical waves by hair cells in the organ of Corti. These waves are reproductions of the original sound pressure waveform (ref. 3, p. 109f). At the auditory nerve endings, the electrical waves are then encoded as electrical impulses of uniform amplitude which are transmitted to the brain through a bundle of nerve fibers comprising the auditory nerve. It appears from the uniform amplitude pulse code signals that the auditory nerve can be regarded as a lossless transmission line. The amplitude of the electrical wave must exceed a certain threshold value in order to produce an impulse in the auditory nerve. But most importantly the time rate of change of the electrical signal determines the number of nerve fibers along which the impulses will be transmitted (ref. 3, p. 112).

The continued change of wave amplitude produces successive impulses in each nerve fiber. The number of fibers which transmit impulses to the brain determines the loudness of the original sound, as subjectively interpreted in the brain. It seems reasonable to expect that the concept of "electric power" (output from the hair cells) can be associated with one aspect of the information transmitted to the brain and that this power integrated over a finite time duration is uniquely related to loudness. The well-known time integration of the signal probably occurs in the brain.

The preceding paragraphs outline the processes that will serve as the basis for a theory of loudness. Although the various mathematical operations will be associated with specific elements of the auditory system, possible incorrect associations (ref. 11) are not likely to affect the theory as long as the assumed operations do occur essentially in the order described.

The theory will be developed according to the following procedure. The loudness is assumed to be uniquely related to the sound intensity integrated over a finite time, the auditory integration time. This average sound intensity is expressed as a function of the sound pressure history (time domain), or, alternatively, as a function of the sound pressure spectrum (frequency domain). (The time and frequency domain analyses will be presented consecutively, rather than in parallel.) Next, the operational characteristics (pressure amplification, physical conversion, filtering, and time differentiation) of the auditory system are introduced. Information regarding the original sound intensity ultimately appears in the brain as information regarding the finite-time-average electric power reaching the auditory nerve. This power is expressed mathematically in both time- and frequency-domain representations. The power formula is made more explicit by specifying the filter characteristics of the auditory system in analytical form. Finally, the psychoacoustic response called loudness is related to the power by applying Stevens' law (ref. 12). This completes the process of relating the sound pressure to



Loudness level, $L = l_p\left[\log G(p)/G_0\right] + l_\omega \log\left[Q(\omega)/Q_0\right] - L_m(\omega).$

Figure 2. - Proposed model of auditory system.

loudness through a chain of operations which presumably occur in the auditory system and brain.

The succession of auditory components, the operational processes they perform, and the corresponding physical and mathematical representations of the processes are diagramed in figure 2.

PHYSICAL QUANTITIES RELATED TO LOUDNESS

It is assumed that the subjective psychoacoustic quality called loudness is a single-valued function of the finite-time-averaged intensity of the sound input at the ear. S. Lifshitz (ref. 13) appears to have been the first to propose this relation. Its validity is well established (refs. 1, 3, 12, 14, and 15). Specifically, the average acoustic intensity over all time (average energy flux over all time),

$$\overline{\Psi} = \overline{pv_n} \tag{1}$$

usually serves as a physical measure uniquely associated with loudness. In equation (1), p is the acoustic pressure, v_n is the normal component of acoustic particle velocity through a control surface having unit area, and the overbars denote infinite time averages. (All symbols are defined in appendix A.) At distances from the sound source which are large in comparison with the extent of the source, equation (1) is approximated by the well-known plane-wave relation

$$\overline{\Psi} = \frac{\overline{p^2}}{\rho c} \tag{2}$$

which in more detailed form is written as

$$\overline{\Psi} = \lim_{\mathscr{T}\to\infty}\left(\frac{1}{2\rho c\mathscr{T}}\right)\int_{-\mathscr{T}}^{\mathscr{T}} |p(t)|^2 \, dt \tag{3}$$

where t is time, ρ is the atmospheric density, and c is the speed of sound. For stationary sounds, equation (3) determines an adequate physical measure of loudness. However, for momentary sounds, such as sonic booms, the intensity Ψ averaged over all time is an unsatisfactory physical measure of loudness because $\overline{\Psi}$ may vanish. Even for statistically stationary sounds, practical necessity requires that the averaging time be finite. In audition a close approximation to $\overline{\Psi}$, both physically and psychologically, is obtained for averaging times less than a second. Let $\widetilde{\Psi}$ denote this average, where the tilde indicates that the average is taken over a finite time duration t_1. Then the average intensity

$$\tilde{\Psi}(t) = \left(\frac{1}{\rho c t_1}\right)\int_{t-t_1}^{t} |p(\tau)|^2 \, d\tau \tag{4}$$

is a practical physical measure of loudness for continuous sounds, regardless of their time dependence (i.e., statistics). For statistically stationary sounds, $\tilde{\Psi}$ is independent of t.

Equation (4) as it stands cannot be correct, or at least complete. To illustrate this, consider the following example. Suppose that throughout an arbitrary auditory integration interval t_1, $p(t) = \text{Constant} \neq 0$. Equation (4) indicates that the auditory response would be nonvanishing and, hence, that auditory response occurs. In fact, auditory response does not occur in this circumstance. Thus, equation (4) must be incomplete, or incorrect. This difficulty will be eliminated when the operational characteristics of the human auditory system are considered.

PHYSICAL INPUT-OUTPUT RELATIONS

Equation (4) is incomplete because it does not include operational characteristics of the auditory system; namely, the pressure amplification induced in the middle ear, the physical conversion from a pressure signal to an electrical signal by the hair cells in the inner ear, and the response to time rates of change of electrical current by the auditory nerve endings. By assuming that these processes are linear, they can be readily treated analytically. (Linearity was previously assumed in the loudness theory of Bürck, Kotowski, and Lichte (ref. 5) and is indicated by measurements (ref. 3, p. 110).)

The successive operations performed by the auditory system involve transfer functions which relate the input and output signal amplitudes. Because the auditory system does not constitute an all-pass filter, part of the energy of the input signal does not reach the brain. The complete auditory system acts essentially as a quasi-linear band-pass filter. For humans the pass band extends roughly from 20 to 20 000 hertz. Any linear filter is characterized by two alternative quantities; namely, the frequency response function $H(\omega)$ and its Fourier transform, the impulse response function $h(t)$. Specifically,

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} \, dt \tag{5a}$$

$$h(t) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} \, d\omega \tag{5b}$$

which may be denoted by

$$h(t) \longleftrightarrow H(\omega) \tag{6}$$

where ω is the angular frequency. The frequency response function $H(\omega)$ describes the filter output for a sinusoidal input; the impulse response function $h(t)$ describes the filter output for an impulsive input (delta function).

Time-Domain Analysis

The hydrodynamic pressure fluctuations in the cochlea are assumed to be proportional to the input atmospheric pressure fluctuations. Next in succession, the electrical output of the hair cells in the inner ear is assumed to be proportional to the hydrodynamic pressure fluctuations. These proportionality constants can be lumped into a single constant κ. Hence, for an arbitrary sound pressure input $p(t)$ to this linear system, the resulting electric current output $j(t)$ from the hair cells is simply given (ref. 16, p. 83) in the time domain by

$$j(t) = \kappa \int_{-\infty}^{\infty} h(\tau) p(t - \tau) d\tau \tag{7a}$$

$$j(t) = \kappa \int_{-\infty}^{\infty} h(t - \tau) p(\tau) d\tau \tag{7b}$$

which are alternative expressions. The current $j(t)$ reaches the auditory nerve endings. But the auditory nerve endings respond to the time rate of change of this current (ref. 3, pp. 112, 259), rather than to the current itself. Thus, the information transmitted to the brain along the auditory nerve concerns

$$\frac{d}{dt} j(t) = \kappa \int_{-\infty}^{\infty} h(\tau) \frac{\partial}{\partial t} p(t - \tau) d\tau \tag{8a}$$

$$\frac{d}{dt} j(t) = \kappa \int_{-\infty}^{\infty} \frac{\partial}{\partial t} [h(t - \tau)] p(\tau) d\tau \tag{8b}$$

rather than $j(t)$.

At the outset it was emphasized that loudness is uniquely related to the intensity of the sound input. After the hydrodynamic wave is converted into an electrical wave, electric power Π, which is proportional to the acoustic intensity Ψ, replaces intensity as the appropriate physical measure of loudness. The output of the hair cells is a reproduction of the original sound waveform. Thus, in parallelism with equation (4), the finite-time-average electric power output of the hair cells $\tilde{\Pi}_c(t)$ is given by

$$\tilde{\Pi}_c(t) = \left(\frac{R}{t_1}\right) \int_{t-t_1}^{t} |j(\tau)|^2 \, d\tau \tag{9}$$

where an electrical resistance R has been introduced to give the equation dimensions of power. When the response characteristics of the auditory nerve endings are included, equation (9) must be replaced by

$$\tilde{\Pi}(t) = Rt_1 \int_{t-t_1}^{t} \left| \frac{d}{d\tau} j(\tau) \right|^2 d\tau \tag{10}$$

where $\frac{d}{d\tau} j(\tau)$ is given by equations (8). Information regarding this power is transmitted to the brain along the auditory nerve. The mode of transmission is essentially lossless and need not be specified in the present theory.

In summary, equations (8) and (10) relate the input sound pressure to the electric power reaching the auditory nerve. Information regarding this power is transmitted to the brain, wherein the information is interpreted as possessing the subjective quality called loudness.

The preceding theory is specified in the time domain. The corresponding results will now be derived for the frequency domain.

Frequency-Domain Analysis

In the frequency-domain representation, the electric power is to be expressed as a function of the sound pressure spectrum $P(\omega)$, which is the Fourier transform of the sound pressure history, or signature $p(t)$. In other words,

$$P(\omega) \longleftrightarrow p(t) \tag{11}$$

Because the auditory integration period t_1 is finite, the nature of the sound pressure spectrum after passage of a finite time may sometimes be of interest. This so-called "running" spectrum $P(\omega, t)$ is given by (ref. 16, p. 148f)

$$P(\omega, t) = \int_{-\infty}^{t} p(\tau)e^{-i\omega\tau}\, d\tau \tag{12a}$$

or

$$P(\omega, t) \longleftrightarrow \theta(t - \tau)p(\tau) \tag{12b}$$

where

$$\theta(t - \tau) \equiv \begin{cases} 1 & (\tau < t) \\ 0 & (\tau > t) \end{cases} \tag{13}$$

Also,

$$\left|\theta(t - \tau)\right|^2 = \theta(t - \tau) \tag{14}$$

The running pressure spectrum $P(\omega, t)$ is a function of $P(\omega)$ (ref. 16, p. 149). Specifically,

$$P(\omega, t) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} \left[\pi\delta(y - \omega) - i(y - \omega)^{-1}\right] P(y)e^{i(y-\omega)t}\, dy \tag{15}$$

relates the two spectra, where δ is the unit impulse function. In particular,

$$P(\omega, t = \infty) \equiv P(\omega) \tag{16}$$

Similar relations apply for the current. Thus, the current possesses a spectrum $J(\omega)$ given by

$$J(\omega) \longleftrightarrow j(t) \tag{17}$$

as well as a running spectrum,

$$J(\omega, t) \longleftrightarrow \theta(t - \tau)j(\tau) \tag{18}$$

Equations (15) and (16), with P replaced by J, also apply.

In the time-domain representation, the current is related to the pressure according to equation (7). The corresponding frequency-domain representation is (ref. 16, p. 86)

$$J(\omega) = \kappa H(\omega)P(\omega) \tag{19}$$

As shown in appendix B, the running current spectrum may be expressed in terms of the response functions and pressure input by

$$J(\omega, t) = \kappa \int_{-\infty}^{t} H(\omega, t - \tau)p(\tau)e^{-i\omega\tau}\, d\tau \tag{20a}$$

$$J(\omega, t) = \kappa \int_{0}^{\infty} h(\tau)P(\omega, t - \tau)e^{-i\omega\tau}\, d\tau \tag{20b}$$

where

$$H(\omega, t) \longleftarrow \theta(t - \tau)h(\tau) \tag{21}$$

and $H(\omega, t)$ is the running frequency response function. In parallel with equation (16), $H(\omega, t = \infty) \equiv H(\omega)$. More generally, equation (15) applies with P replaced by H.

It is noteworthy that the integrands in equations (20) consist of products of time and frequency functions. If $t \rightarrow \infty$, and $h(\tau)$ and $p(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$, equations (20) reduce to equation (19). Even more importantly from the standpoint of steady noise, if $H(\omega, t - \tau)$ is effectively constant throughout the interval $(-\infty, t)$, except for a short time $t - \tau << t_1$ (where, as before, t_1 is the auditory integration period), equations (20) reduce to

$$J(\omega, t) \approx \kappa H(\omega)P(\omega, t) \tag{22}$$

which involves the running spectrum $P(\omega, t)$, rather than the ordinary spectrum $P(\omega)$ contained in equation (19).

In determining the electric power, the time derivative of the current, rather than the current itself, is most significant. The frequency-domain representation of this derivative is given by the Fourier transform

$$\frac{d}{dt} j(t) \longleftarrow i\omega J(\omega) \tag{23}$$

However, the loudness is determined by the electric power integrated over a finite time interval t_1 (cf. eq. (10)). Hence, the running transform of the time derivative applies. The running transform is given by

$$\frac{d}{d\tau}[\theta(t-\tau)j(\tau)] \longleftrightarrow i\omega J(\omega, t) \tag{24}$$

In expanded form, namely

$$\frac{d}{d\tau}[\theta(t-\tau)j(\tau)] = \theta(t-\tau)\frac{d}{d\tau}j(\tau) - \delta(t-\tau)j(\tau) \tag{25}$$

the left-hand side of equation (24) is seen to include an impulsive transient associated with switching off the integration. The existence of this transient is independent of the specific time dependence of $j(t)$. Hence, the transient is not determined by the input. The transient is not associated with the human auditory system because the human mind detects no subjective loudness of an impulsive nature associated with initiation or termination of the auditory integration process. The auditory system remains in the "on" state all the time. Thus, this "switching" transient is unphysical and should be omitted in computing the power. The switching transient is purely a consequence of the mathematics of the Fourier transform and does not appear if $\frac{d}{dt}j(t)$ is represented by a Fourier series expansion over the period t_1. Finally, equation (10), the original time-domain equation for the power, does not contain switching transients.

With the switching transients eliminated it is shown in appendix C that, in the frequency-domain representation, the electric power is given by

$$\tilde{\Pi}(t) = \frac{Rt_1}{2\pi}\int_{-\infty}^{\infty}\left[\left|J(\omega,t)\right|^2 - \left|J(\omega,t-t_1)\right|^2\right]\omega^2\,d\omega \tag{26}$$

which, when accompanied by equations (20), expresses the average power as a function of the input pressure history or its spectrum. Equation (26) exhibits a high-frequency weighting ($\propto \omega^2$) of the average power by virtue of the nerve endings' response to the time derivative of the current. Thus, the auditory nerve endings act as high-pass filters.

The integral in equation (26) may be divergent. To avoid this difficulty, $\frac{d}{dt}j(t)$ in equation (10) may be expanded as a Fourier series over the integration interval. The power is ultimately expressed by the sum of the squares of the Fourier coefficients. This approach is especially useful when $\frac{d}{dt}j(t)$ is periodic. The case where $p(t)$ is a pure tone is considered in appendix D. If $p(t)$ is unknown and information is available regarding the spectrum, but the integral in equation (26) is divergent, then methods described by Bennett (refs. 17 and 18) may be used to evaluate the power.

The resulting power formulas may now be summarized. In the time-domain representation,

$$\tilde{\Pi}(t) = \kappa^2 R t_1 \int_{t-t_1}^{t} \left| \int_{-\infty}^{\infty} h(\hat{\tau}) \frac{\partial}{\partial \tau} p(\tau - \hat{\tau}) d\hat{\tau} \right|^2 d\tau \qquad (27a)$$

$$\tilde{\Pi}(t) = \kappa^2 R t_1 \int_{t-t_1}^{t} \left| \int_{-\infty}^{\infty} \frac{\partial}{\partial \tau} [h(\tau - \hat{\tau})] p(\hat{\tau}) d\hat{\tau} \right|^2 d\tau \qquad (27b)$$

If h(t) and p(t) are both real, the absolute value signs in equation (27) can, of course, be removed. In the frequency-domain representation,

$$\tilde{\Pi}(t) = \frac{\kappa^2 R t_1}{2\pi} \int_{-\infty}^{\infty} \left[\left| \int_{-\infty}^{t} H(\omega, t - \tau) p(\tau) e^{-i\omega\tau} d\tau \right|^2 - \left| \int_{-\infty}^{t-t_1} H(\omega, t - t_1 - \tau) p(\tau) e^{-i\omega\tau} d\tau \right|^2 \right] \omega^2 d\omega \qquad (28a)$$

$$\tilde{\Pi}(t) = \frac{\kappa^2 R t_1}{2\pi} \int_{-\infty}^{\infty} \left[\left| \int_{0}^{\infty} h(\tau) P(\omega, t - \tau) e^{-i\omega\tau} d\tau \right|^2 - \left| \int_{0}^{\infty} h(\tau) P(\omega, t - t_1 - \tau) e^{-i\omega\tau} d\tau \right|^2 \right] \omega^2 d\omega \qquad (28b)$$

If the sound input is steady noise and $H(\omega, t - \tau)$ is effectively constant, except during the short time interval $t - \tau << t_1$, then equation (22) is valid. Therefore,

$$\tilde{\Pi}(t) \approx \frac{\kappa^2 R t_1}{2\pi} \int_{-\infty}^{\infty} \left[|P(\omega, t)|^2 - |P(\omega, t - t_1)|^2 \right] |H(\omega)|^2 \omega^2 d\omega \qquad (29a)$$

or

$$\tilde{\Pi}(t) \approx \frac{\kappa^2 R t_1}{2\pi} \int_{-\infty}^{\infty} \left[|P(\omega, t)|^2 - |P(\omega, t - t_1)|^2 \right] |H_c(\omega)|^2 \, d\omega \tag{29b}$$

follows from equation (28a), where, by definition,

$$H_c(\omega) = i\omega H(\omega) \tag{30}$$

and

$$h_c(t) = \frac{d}{dt} h(t) \tag{31}$$

$$h_c(t) \longleftrightarrow H_c(\omega) \tag{32}$$

If $h(t)$ and $p(t)$ are quasi-impulsive, that is, if $h(t)$ and $p(t)$ are nonvanishing only over a time period shorter than the integration time t_1, then equations (28) simplify. Suppose that $p(t)$ is initiated at some time t_o and is quasi-impulsive. Assume that t_o occurs within the auditory integration interval, that is, $t - t_1 \leq t_o < t$. Then, the second time integral in equations (28) vanishes. Assume that $p(t)$ effectively vanishes permanently again at some later time t_o' less than the upper limit of integration, that is, $t_o < t_o' < t$. Then, if the running frequency response $H(\omega, t)$ effectively becomes independent of time for time durations $t - t_o'$ or larger, that is, if $H(\omega, t - t_o') \approx H(\omega)$, then $H(\omega, t - \tau)$ may be extracted from the integrand of the first integral in equation (28a). The condition $H(\omega, t - t_o') \approx H(\omega)$ corresponds to $h(\tau \geq t - t_o') \approx 0$, and in conjunction with causality (to be discussed) implies that $h(t)$ is quasi-impulsive. The remaining integral equals $P(\omega, t)$. But, because $p(t)$ effectively vanishes for $t > t_o'$, it follows that $P(\omega, t) \approx P(\omega)$. Hence, equation (28a) reduces to

$$\tilde{\Pi} \approx \frac{\kappa^2 R t_1}{2\pi} \int_{-\infty}^{\infty} |H(\omega) P(\omega)|^2 \omega^2 \, d\omega \tag{33a}$$

or

$$\tilde{\Pi} \approx \frac{\kappa^2 R t_1}{2\pi} \int_{-\infty}^{\infty} |H_c(\omega) P(\omega)|^2 \, d\omega \tag{33b}$$

Equation (28b) can be reduced to equations (33) by a similar argument.

In general, the response characteristics of the complete auditory system are mathematically entangled with the input pressure. However, in the special case of equations (29) and (33), the response characteristics are separable from the input. Thus, the quantities $h_c(t)$ and $H_c(\omega)$ represent the response functions for the complete auditory system preceding the auditory nerve.

CAUSALITY AND THE PALEY-WIENER THEOREM

In the preceding section the electrical power output was related to the sound pressure input by introducing response characteristics of the auditory system. The next step is to specify the response characteristics explicitly. However, before doing so it is important to consider the consequences of causality and the Paley-Wiener theorem in this regard. The Paley-Wiener theorem not only has a bearing on the response characteristics but also provides an important result regarding the spectra of short duration sounds.

First, consider causality. Cause precedes effect. Thus, if $p(t)$ is initiated at some time $t = t_o$, then $j(t) = 0$ for $t < t_o$. Without loss of generality, one can set $t_o = 0$.

A function which vanishes for $t < 0$ is called causal (ref. 16, p. 13). The impulse response $h(t)$ is causal (ref. 16, p. 85). Hence, $H(\omega, t)$ is also causal. It follows from equations (7) and (20), respectively, that $j(t)$ and $J(\omega, t)$ must be causal.

Causality in conjunction with the Paley-Wiener theorem (ref. 19, p. 16ff or ref. 16, pp. 215-217 and 222) leads to important consequences regarding the auditory response characteristics as well as the spectra of finite duration sounds. The Paley-Wiener theorem states that, if

$$\int_{-\infty}^{\infty} \frac{\left|\ln\left|F(\omega)\right|\right|}{1+\omega^2}\, d\omega < \infty \tag{34}$$

where $F(\omega)$ is square integrable, that is,

$$\int_{-\infty}^{\infty} \left|F(\omega)\right|^2 d\omega < \infty \tag{35}$$

then $f(t)$, which is the Fourier transform of $F(\omega)$, is causal. Note that, if $F(\omega)$ vanishes over any nonvanishing interval $\Delta\omega$, the inequality (eq. (34)) is violated. Therefore, the spectrum of any causal function must be nonvanishing at all, except possible discrete, frequencies if the spectrum satisfies the inequality (eq. (35)).

Let the auditory impulse response $h(t)$ correspond to $f(t)$ and the frequency response $H(\omega)$ correspond to $F(\omega)$. From the Paley-Wiener theorem it follows that, because $h(t)$ is causal, the auditory frequency-response function $H(\omega)$ must be nonvanishing for all, except possible discrete, frequencies. This means that the auditory response cannot be represented by a simple ideal band-pass filter (ref. 20).

Let the electric current $j(t)$ correspond to $f(t)$ and the current spectrum $J(\omega)$ correspond to $F(\omega)$. Then, because $j(t)$ is causal and $J(\omega)$ is square integrable for finite signal amplitudes, the electrical output of the hair cells associated with any sound input must include all, except possible discrete, frequencies in the audible range.

Finally, let the sound pressure $p(t)$ correspond to $f(t)$ and the pressure spectrum $P(\omega)$ correspond to $F(\omega)$. If $p(t)$ is causal and $P(\omega)$ is square integrable, $P(\omega)$ must be nonvanishing for all, except possible discrete, frequencies. The sonic boom represents an important sound satisfying the Paley-Wiener conditions. Hence, the production of an inaudible sonic boom is impossible. This conclusion is reached without even taking account of the auditory system.

In the preceding applications of the Paley-Wiener theorem the results are valid in principle. However, the amplitudes of the response functions or signals have not been considered. By definition, the response functions vanish outside the audible frequency range. Inside the audible range the signals may be too weak to produce any response.

EXPLICIT FILTER CHARACTERISTICS OF THE AUDITORY SYSTEM

The filter characteristics of the auditory system can now be specified explicitly. In equations (30) and (31) the response characteristics of the complete auditory system, namely $h_c(t)$ and $H_c(\omega)$, have been specified as functions of $h(t)$ and $H(\omega)$. Because the auditory system is assumed to perform linear filtering, it represents a stable system in the sense that its response to any bounded input is bounded. This implies that $h_c(t)$ is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |h_c(t)|\,dt < \infty \tag{36}$$

or $h_c(t) \to 0$ faster than $1/t$ as $t \to \infty$. If $h_c(t)$ is absolutely integrable, $H_c(\omega) \to 0$ as $|\omega| \to \infty$ by virtue of the Riemann-Lebesgue Lemma (ref. 21), namely

$$\lim_{|\omega| \to \infty} \int_{-\infty}^{\infty} h_c(t) e^{-i\omega t}\, dt = \lim_{|\omega| \to \infty} H_c(\omega) = 0 \tag{37}$$

Because the human sensory system is excited only by time-dependent inputs, it follows that $H_c(0) = 0$. The quantity $H_c(\omega)$ must, therefore, describe a band-pass filter. The human auditory system as a passive filter is causal. The causality condition in conjunction with the Paley-Wiener theorem indicates that the auditory response cannot be characterized by an ideal band-pass filter because the impulse response of such a filter is acausal. The impulse response at time $t = 0$ is given by

$$h_c(0) = \frac{1}{\pi} \mathcal{Re} \int_0^{\infty} H_c(\omega) d\omega \tag{38}$$

which follows from equation (32) and causality and where $\mathcal{Re}$ denotes the real part. Because the auditory response is nonvanishing in the audible frequency pass-band, it follows that $\mathcal{Re} |H_c(\omega)| > 0$ for $0 < \omega < \infty$ because $\mathcal{Re} H_c(\omega)$ cannot change sign. Hence,

$$|h_c(0)| > 0 \tag{39}$$

To summarize: In attempting to relate its time- and frequency-response characteristics, the complete auditory system as a linear system corresponds to a nonideal, band-pass, filter. The associated response to an infinite impulse is necessarily nonvanishing at the instant the impulse is applied, but vanishes faster than $1/t$ as $t \to \infty$.

The preceding arguments apply with respect to the running frequency response $H_c(\omega, t)$, as well as with respect to $H_c(\omega)$. Thus, for example, in terms of $H_c(\omega, t)$,

$$\theta(t - \tau) h_c(\tau) \longleftrightarrow H_c(\omega, t) \tag{40}$$

leads to

$$h_c(t) \longleftrightarrow H_c(\omega, \infty) \tag{41}$$

Hence,

$$h_c(t) \longleftrightarrow 2\, \mathcal{Re}\, H_c(\omega, \infty) \equiv 2\, \mathcal{Re}\, H_c(\omega) \tag{42}$$

and equation (38) follows by setting $t = 0$.

Response functions of the type described above are commonly associated with multistage amplifiers (ref. 22) wherein the frequency-response function is represented by the ratio of polynomial functions of frequency so that the impulse response is, then, necessarily described by decaying exponential functions of time. An appropriate Fourier transform pair is

$$H_c(\omega) = iA\omega(\omega_1 + i\omega)^{-1}(\omega_2 + i\omega)^{-1} \tag{43}$$

in the frequency domain, which corresponds to (ref. 23)

$$h_c(t) = \theta(t)A(\omega_2 - \omega_1)^{-1}\left(\omega_2\, e^{-\omega_2 t} - \omega_1\, e^{-\omega_1 t}\right) \tag{44}$$

in the time domain, where A is a constant and ω_1 and ω_2 are the pass-band cutoff frequencies. These cutoff frequencies are properties of the auditory system. Predicting the values of these frequencies requires a more detailed physical analysis of the auditory system than that provided herein. Moreover, the cutoff frequencies, by definition, are not directly measurable but can only be estimated from auditory transmittance curves. From equations (30) and (43) it follows that

$$H(\omega) = A(\omega_1 + i\omega)^{-1}(\omega_2 + i\omega)^{-1} \tag{45}$$

which is the transform of

$$h(t) = \theta(t)A(\omega_2 - \omega_1)^{-1}\left(e^{-\omega_1 t} - e^{-\omega_2 t}\right) \tag{46}$$

When ω_1 and ω_2 are evaluated, it is found, in fact, that the impulse response $h(t)$ is quasi-impulsive. Equation (46) is identical to the impulse response formula for nerves, as derived from physical arguments by Zwislocki in his "Theory of Temporal Auditory Summation" and confirmed by prior experiments of Galambos on medullary nerves (ref. 7). Equations (45) and (46) describe a low-pass filter. This in conjunction with the high-pass weighting - due to the auditory nerve endings' response to pressure rates of change, rather than pressures - in equation (26) causes the complete auditory system to perform band-pass filtering, as indicated by equations (43) and (44).

The explicit equation (46) for $h(t)$ may now be introduced in equations (27) to provide the final time-domain formulation for the electric power. Thus,

$$\tilde{\Pi}(t) = \kappa^2 A^2 (\omega_2 - \omega_1)^{-2} R t_1 \int_{t-t_1}^{t} \left| \int_0^{\infty} \left(e^{-\omega_1 \hat{\tau}} - e^{-\omega_2 \hat{\tau}}\right) \frac{\partial}{\partial \tau} p(\tau - \hat{\tau}) d\hat{\tau} \right|^2 d\tau \quad (47a)$$

$$\tilde{\Pi}(t) = \kappa^2 A^2 (\omega_2 - \omega_1)^{-2} R t_1 \int_{t-t_1}^{t} \left| \int_{-\infty}^{\tau} \left[\omega_2 \, e^{-\omega_2(\tau-\hat{\tau})} - \omega_1 \, e^{-\omega_1(\tau-\hat{\tau})}\right] p(\hat{\tau}) d\hat{\tau} \right|^2 d\tau$$

(47b)

Only the sound pressure history and values of ω_1 and ω_2 need to be specified in order to complete the calculation of $\tilde{\Pi}$.

To complete the frequency-domain formulation for $\tilde{\Pi}$, the running response function $H(\omega, t)$ must be written explicitly for potential introduction in equation (28a). By virtue of equations (21) and (46),

$$H(\omega, t) = \frac{A\theta(t)}{\omega_2 - \omega_1}\left[\frac{\omega_2 - \omega_1}{(\omega_2 + i\omega)(\omega_1 + i\omega)} - \frac{1}{\omega_1 + i\omega} e^{-(\omega_1+i\omega)t} + \frac{1}{\omega_2 + i\omega} e^{-(\omega_2+i\omega)t}\right]$$

(48)

However,

$$\omega_1 + i\omega = \omega_1\left(1 + \frac{i\omega}{\omega_1}\right)$$

$$= \omega_1\left[1 + \left(\frac{\omega}{\omega_1}\right)^2\right]^{1/2} e^{i\theta_1} \quad (49)$$

where

$$\theta_1 = \tan^{-1} \frac{\omega}{\omega_1} \quad (50)$$

and similarly for $\omega_2 + i\omega$. Then, $H(\omega, t)$ may be written in a form in which its real and imaginary parts are easily recognized, namely

$$H(\omega, t) = \frac{A\theta(t)}{\omega_2 - \omega_1}\left\{\frac{\omega_2 - \omega_1}{\omega_1\omega_2}\left[1 + \left(\frac{\omega}{\omega_1}\right)^2\right]^{-1/2}\left[1 + \left(\frac{\omega}{\omega_2}\right)^2\right]^{-1/2} e^{-i(\theta_1+\theta_2)}\right.$$

$$\left. - \frac{1}{\omega_1}\left[1 + \left(\frac{\omega}{\omega_1}\right)^2\right]^{-1/2} e^{-(\omega_1+i\omega)t-i\theta_1} + \frac{1}{\omega_2}\left[1 + \left(\frac{\omega}{\omega_2}\right)^2\right]^{-1/2} e^{-(\omega_2+i\omega)t-i\theta_2}\right\} \tag{51}$$

By comparing the time-dependent terms in equation (51) with the time-independent term it can be shown that the time-dependent terms are generally significant over a time less than 10 percent of the integration period t_1. Hence, when considering steady noise, the effect of the time-dependent terms in $H(\omega, t)$ can be neglected, so that equations (29) are valid.

More generally, a frequency-domain formulation for $\widetilde{\Pi}$ is obtained by substituting the preceding expression for $H(\omega, t)$ in equation (28a). The result is lengthy and will not be written down. A simpler-looking alternative is obtained by substituting the expression for $h(t)$ given by equation (46) into equation (28b). The result is

$$\widetilde{\Pi}(t) = \frac{Rt_1}{2\pi}(\kappa A)^2(\omega_2 - \omega_1)^{-2}\int_{-\infty}^{\infty}\left[\left|\int_0^{\infty}\left(e^{-\omega_1\tau} - e^{-\omega_2\tau}\right)P(\omega, t - \tau)e^{-i\omega\tau}\,d\tau\right|^2\right.$$

$$\left. - \left|\int_0^{\infty}\left(e^{-\omega_1\tau} - e^{-\omega_2\tau}\right)P(\omega, t - t_1 - \tau)e^{-i\omega\tau}\,d\tau\right|^2\right]\omega^2\,d\omega \tag{52}$$

In the special case where $h(t)$ and $p(t)$ are quasi-impulsive, equations (33b) and (43) apply. Then,

$$\widetilde{\Pi} \approx \frac{Rt_1}{2\pi}\left(\frac{\kappa A}{\omega_1\omega_2}\right)^2\int_{-\infty}^{\infty}|P(\omega)|^2 \frac{\omega^2\,d\omega}{\left[1 + (\omega/\omega_1)^2\right]\left[1 + (\omega/\omega_2)^2\right]} \tag{53}$$

PSYCHOACOUSTICAL RELATIONS

Expressions have been obtained for the electric power information presumably received by the auditory nerve as a function of the input pressure signature or spectrum and the physical characteristics of the auditory system. In the brain the information regarding this objective physical power is converted into subjective responses, such as loudness and annoyance, as well as into other objective and subjective responses, some of which may be classified as startle responses. Only loudness will be evaluated herein, but the other subjective responses may be evaluated in a similar fashion (ref. 24).

S. S. Stevens has shown (refs, 24 and 25) for a wide variety of psychophysical phenomena that, if φ is the magnitude of a physical stimulus and ψ is a psychological magnitude (determined by subjective judgments), then

$$\psi = k\varphi^{m} \tag{54}$$

where k and m are constants dependent on the phenomenon and often on the individual as well. Equation (54) may be aptly called Stevens' law. It supplants the well-known Fechner law,

$$\psi = k_f \ln \varphi \tag{55}$$

which is experimentally invalid (ref. 26).

If Stevens' law is applied to loudness (ref. 12), then

$$\mathscr{L} = k_o \tilde{\Pi}^{l} \tag{56}$$

where $\mathscr{L}$ is the loudness (sones) and k_o and l are constants. A loudness level L (phons) was defined by Stevens (ref. 12) as

$$L = 33.3 \log \mathscr{L} + 40 \tag{57}$$

for an input frequency of 1000 hertz. For other frequencies the coefficients may be different. By considering equation (56) and noting that $\tilde{\Pi}$ is a function of p and ω, a more general equation for the loudness level is

$$L(p, \omega) = l_p \log \frac{G(p)}{G_o} + l_\omega \log \frac{Q(\omega)}{Q_o} - L_m(\omega) \tag{58}$$

where $G(p)$ and $Q(\omega)$ are, respectively, functions of p and ω to be determined from $\tilde{\Pi}$; G_0 and Q_0 are reference values; the constant l_p determines the loudness level rise rate as a function of sound pressure; the constant l_ω determines the rise rate as a function of ω; and $L_m(\omega)$ is a function of frequency which accounts for the fact that the detection threshold occurs at a nonvanishing sound pressure. (Note that log signifies logarithm to the base 10.) The condition $l_\omega \neq l_p$ would imply that the psychoacoustic conversion is frequency dependent and, hence, that the brain introduces an additional filtering effect. Similar formulas could be given for noisiness or annoyance and other psychoacoustic phenomena. Equation (58) finally quantitatively relates subjective loudness judgments to the physical sound-pressure input.

For loudness levels L greater than 40 phons (up to at least 110 phons) at 1000 hertz, doubling the loudness $\mathscr{L}$ corresponds to a 10-phon increase in the loudness level (ref. 3, p. 193; ref. 12), a 10-decibel increase in intensity level Υ (ref. 3, p. 186f; ref. 12), or tripling the sound pressure p. For loudness levels less than 40 phons, the relation between loudness and loudness level is not so simple. In this range the loudness varies to a much greater degree as a function of loudness level (ref. 3, p. 193; or ref. 4).

The minimum detectable intensity level change is generally between 0.25 and 1.0 decibel (3- to 12-percent change in sound pressure ratio) for sine waves at intensity levels greater than 30 decibels, and considerably greater at lower intensity levels (ref. 3, p. 146). For impulsive sounds the minimum detectable intensity level change appears to be somewhat greater. For example, for sonic boom N waves, changes less than 2 decibels, or 25 percent in sound pressure, are apparently undetectable (ref. 27).

PARTICULAR SOLUTIONS

In calculating the loudness of various inputs the choice of a time- or frequency-domain calculation is decided simply on the basis of ease of calculation. At the very least, if the preceding theory is valid, it must be capable of predicting the loudness levels of pure tones and impulsive inputs. Thus, these examples will serve to illustrate the initial applications of the theory.

Pure Tone Input

Assume that

$$p(t) = p_0 \cos(\omega_0 t - \Delta) \tag{59}$$

where p_o is the pressure amplitude at the ear, ω_0 is the driving frequency, and Δ is a phase shift. As shown in appendix D,

$$\tilde{\Pi}(t) = \frac{\kappa^2 R t_1 p_o^2 \omega_0^2}{2} \left[|H(\omega_0)|^2 t_1 - \mathcal{R}e \frac{1}{2i\omega_0} H^2(\omega_0) e^{2i(\omega_0 t - \Delta)} \left(1 - e^{-2i\omega_0 t_1}\right) \right] \tag{60}$$

If

$$t_1 >> \frac{1}{\omega_0} \tag{61}$$

equation (60) reduces to

$$\tilde{\Pi} = \frac{1}{2} \kappa^2 R t_1^2 p_o^2 \omega_0^2 |H(\omega_0)|^2 \tag{62a}$$

$$\tilde{\Pi} = \frac{1}{2} \kappa^2 R t_1^2 p_o^2 |H_c(\omega_0)|^2 \tag{62b}$$

that is, the time-average power is proportional to the pressure intensity, the transmittance of the complete auditory system, the square of the averaging time, and is independent of t. Inequality (eq. (61)) is, in fact, satisfied by the auditory system. According to Steudel (ref. 4; ref. 14, p. 158), $t_1 \approx 3\times10^{-4}$ second. The integration time t_1 determined by Steudel is incorrect because the input signature, an impulse, was not of sufficient duration to determine the true integration time. More recent measurements (refs. 6 and 28 to 30) indicate that the integration time is more nearly 0.1 second, with possible dependence on intensity. Von Békésy adopted (ref. 11) the value $t_1 = 0.2$ second. Using this value, equations (62) are valid if $\omega_0 >> 5$, that is, if the driving frequency is much greater than 1 hertz, say 10 hertz or greater.

In order to introduce the result given by equation (62) in equation (58), let

$$G(p) = \frac{1}{2} \kappa^2 R p_o^2 \tag{63}$$

$$G_o = \frac{1}{2} \kappa^2 R p_r^2 \tag{64}$$

$$Q(\omega) = \left|H_c(\omega_0)\right|^2 \tag{65}$$

$$Q_o = \left|H_c(\omega_m)\right|^2 \tag{66}$$

where p_r is a reference sound pressure ($p_r = 2\times10^{-4}$ dyne cm^{-2}), and the frequency-response function $H(\omega)$ peaks at the frequency ω_m. By introducing the preceding relations in equation (58) the loudness level of a pure tone is, therefore, given by

$$L_o(p_o, \omega_0) = l_p \log\left(\frac{p_o}{p_r}\right)^2 + l_\omega \log \frac{\left|H_c(\omega_0)\right|^2}{\left|H_c(\omega_m)\right|^2} - L_m(\omega_m) \tag{67a}$$

Finally, by virtue of equation (43),

$$L_o(p_o, \omega_0) = l_p \log\left(\frac{p_o}{p_r}\right)^2 + l_\omega \log\left\{\frac{(\omega_0/\omega_1)^2}{(\omega_m/\omega_1)^2} \frac{\left[1 + (\omega_m/\omega_1)^2\right]\left[1 + (\omega_m/\omega_2)^2\right]}{\left[1 + (\omega_0/\omega_1)^2\right]\left[1 + (\omega_0/\omega_2)^2\right]}\right\} - L_m(\omega_m) \tag{67b}$$

Note that the second term on the right-hand side of equations (67) vanishes if $\omega_0 = \omega_m$.

The dependence of the loudness level L_o obtained from equation (67b) on sound pressure and frequency should correspond with Stevens and Davis' response curves (ref. 14, p. 124; or ref. 15, p. 201). These curves, derived from data by Fletcher and Munson (ref. 1; ref. 3, p. 188), were obtained by using pure tones supplied by earphones with sound pressures measured near the eardrum. Intensity levels are presented as a function of frequency with loudness level as a parameter, where the intensity level Υ is defined by

$$\Upsilon = 10 \log\left(\frac{p_o}{p_r}\right)^2 \tag{68}$$

The curves of Stevens and Davis differ from the better known Fletcher-Munson curves (ref. 1; ref. 3, p. 188; or ref. 15, p. 200). The Fletcher-Munson curves correspond to introducing the observer into the sound field facing the source of sound after the free-field sound pressure has been measured. As a result the Fletcher-Munson curves include diffraction of the sound by the human head. The present theory does not attempt to account for this diffraction phenomenon, but rather, corresponds closely to the experimental situation represented by the curves of Stevens and Davis. The theoretical and experimental results will be compared following discussion of the theoretical loudness of an impulsive input.

Impulsive Input

Let the sound pressure be represented by

$$p(t) = \pi_o \delta(t) \tag{69}$$

where π_o is the impulse. With the impulse response given by equation (46), it is shown in appendix D that the average power is given by

$$\begin{aligned}\tilde{\Pi}(t) = \frac{1}{2} Rt_1 \left(\frac{\kappa A \pi_o}{\omega_2 - \omega_1}\right)^2 \Bigg\{ &\omega_2 \left[\theta(t)\left(1 - e^{-2\omega_2 t}\right) + \theta(t - t_1)\left(e^{-2\omega_2(t-t_1)} - 1\right)\right] \\ &+ \omega_1 \left[\theta(t)\left(1 - e^{-2\omega_1 t}\right) + \theta(t - t_1)\left(e^{-2\omega_1(t-t_1)} - 1\right)\right] \\ &- \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} \left[\theta(t)\left(1 - e^{-(\omega_1+\omega_2)t}\right) + \theta(t - t_1)\left(e^{-(\omega_1+\omega_2)(t-t_1)} - 1\right)\right]\Bigg\}\end{aligned} \tag{70}$$

When $t = t_1$ the result is especially interesting because it represents the situation where the integration period begins with the impulse, and, hence, is the period associated with maximum loudness. For the human auditory system ω_1, $\omega_2 >> 1/t_1$, so that

$$\tilde{\Pi}(t_1) \approx \frac{1}{2} Rt_1 \frac{(\kappa A \pi_o)^2}{\omega_1 + \omega_2} \tag{71}$$

By applying equation (58), the maximum loudness level is found to be

$$L_{\delta}(\pi_o) = l_p \log\left(\frac{\pi_o}{\pi_r}\right)^2 + l_\omega \log \frac{\left(\omega_1 + \omega_2\right)_r}{\omega_1 + \omega_2} - L_{\delta_o} \tag{72}$$

where π_r is a reference impulse, $\left(\omega_1 + \omega_2\right)_r$ is the sum of the cutoff frequencies at the reference pressure, and L_{δ_o} is a constant loudness level which accounts for the fact that the linear relation between loudness level and intensity level does not include the coordinate origin.

COMPARISON WITH EXPERIMENT

Theoretical and experimental curves of the loudness level spectra for pure tones, with intensity level Υ as parameter, are compared in figure 3. The test curves are cross-plots of Stevens and Davis' equal-loudness curves. In the form displayed in figure 3 the similarity of the curves to ordinary filter curves is evident. Theoretical points computed for 1/3-octave intervals using equation (67b) for sinusoidal inputs have been superposed for comparison purposes.

In order to fit Stevens and Davis' response curves, the constants l_ω, ω_1, and ω_2 were adjusted to obtain a best fit at each intensity level. The "peak" frequency ω_m was not chosen independently despite the fact that it represents an independent constant. Rather, ω_m was assumed to be the geometric mean of the cutoff frequencies ω_1 and ω_2; that is, $\omega_m = \sqrt{\omega_1 \omega_2}$. This assumption causes the frequency-response curves to be symmetrical about ω_m on the log-log scale in figure 3. Despite this unnecessary restriction the theoretical and experimental curves are found to be in very good agreement. The two sets of curves generally differ by less than 2 phons over the entire intensity level range of 10 to 130 decibels and audible frequency range. The greatest disparity of the results is 10 phons at the highest frequencies in the midintensity-level (70 to 90 db) range. The 2-phon difference is certainly no greater than the errors induced by the instrumentation, by averaging results (loudness level probable error of the order of 6 db according to ref. 31) from a large number of tests (297 observations on 11 observers), and by cross-plotting the response curves from published equal-loudness contours. In fact, the largest disparity might result primarily from the experiment, rather than from inadequacy of the theory.

By fitting equation (67b) to the test data for each input intensity level a set of values of the constants l_ω, ω_1, ω_2, and ω_m are obtained for each intensity level. Values of $l_p \log(p_o/p_r)^2 - L_m(\omega_m)$ and l_ω as functions of intensity level have been plotted in figure 4 with $l_p = 10$ phons. A similar plot for ω_1, ω_2, and ω_m is given in figure 5.

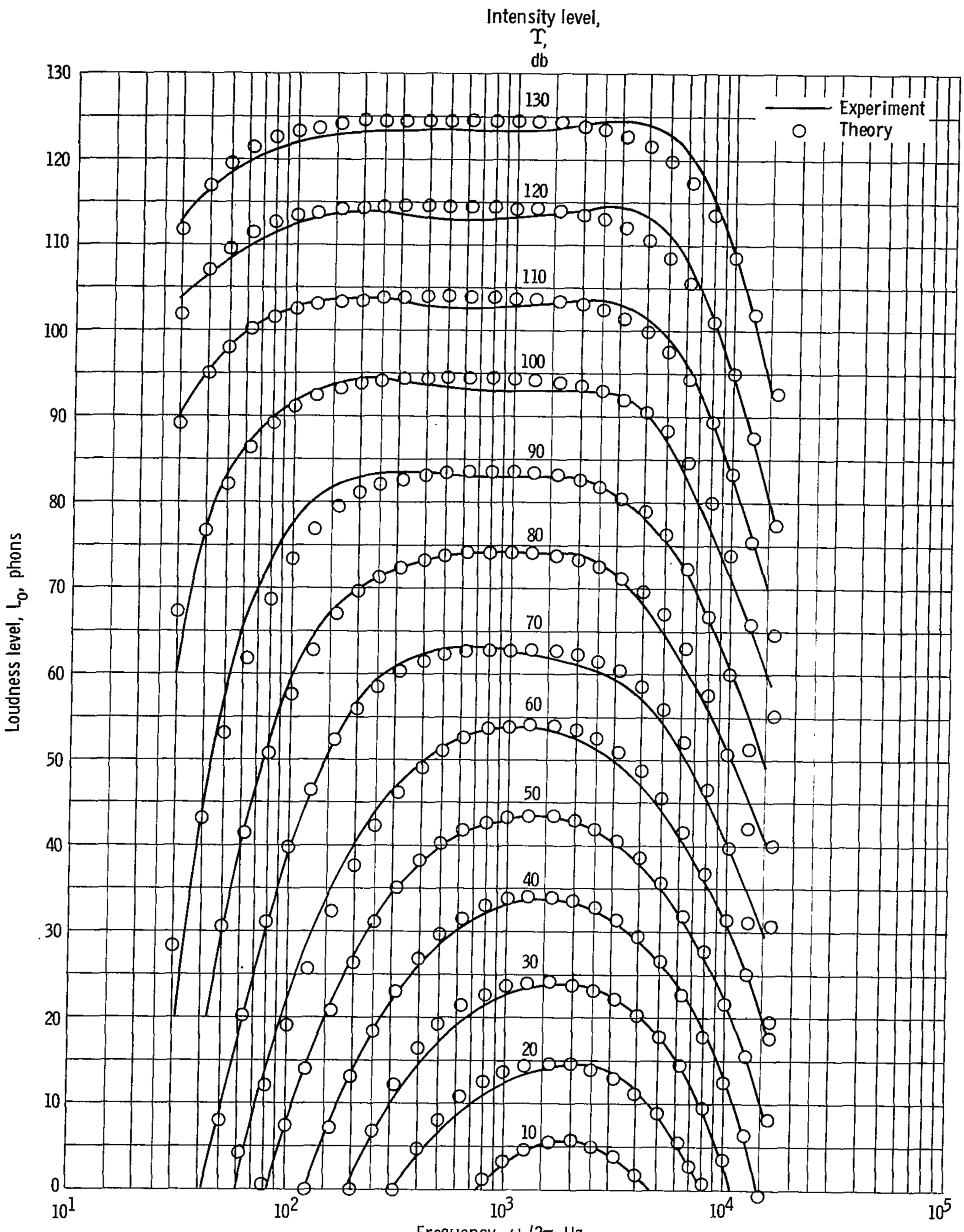


Figure 3. - Psychoacoustic filter curves (individually fit).

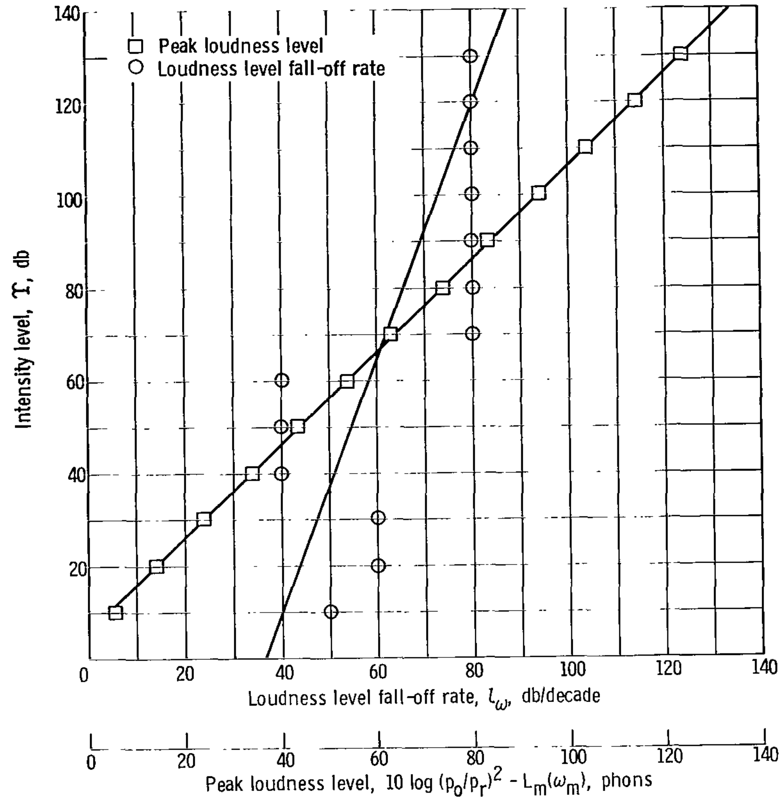


Figure 4. - Loudness level coefficients as functions of intensity level.

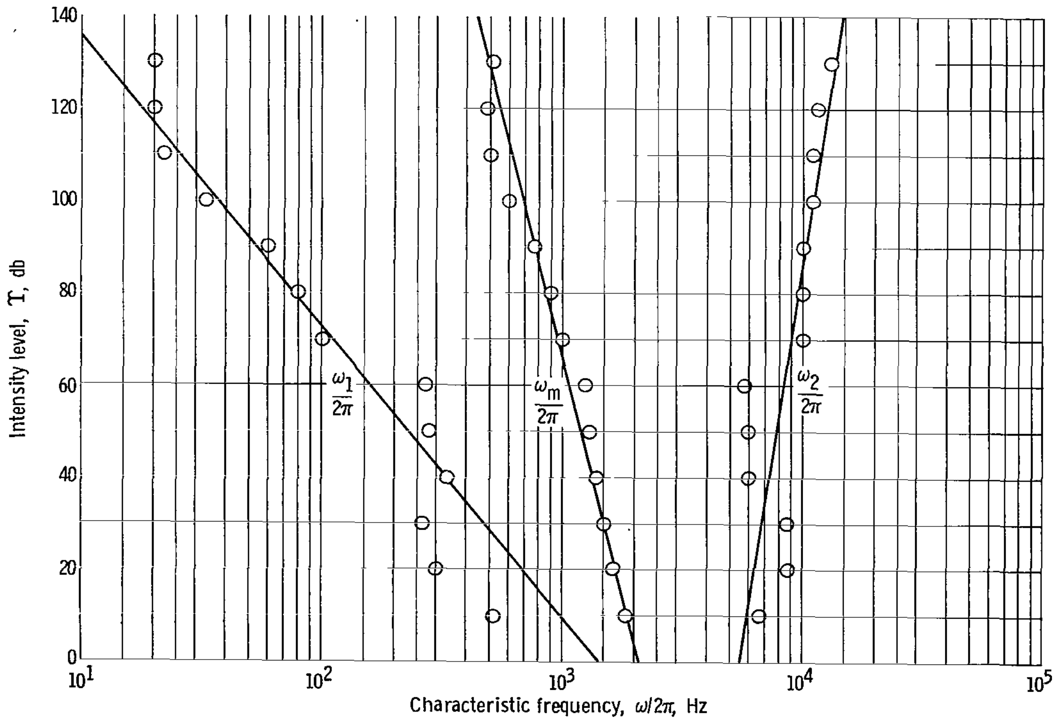


Figure 5. - Characteristic frequencies as function of intensity level.

Note that in most cases the values of the constants can be fit fairly well by straight lines on the log-log plots, although the scatter of the values for l_ω is excessive. In figure 5 it is apparent that the auditory bandwidth increases markedly as the input intensity is increased. However, the geometric mean frequency ω_m remains in the frequency interval 500 to 2000 hertz throughout the entire intensity level range of 10 to 130 decibels, the higher geometric mean frequency occurring at the lower intensity level.

The straight-line fits to the data in figures 4 and 5 can be formulated to provide a set of equations which determine the frequency response over the entire intensity and frequency ranges covered by the data. Of course, the resulting formulas cannot be expected to fit the experimental data nearly as well as the individual fitting process used to obtain figure 3. The formulas for the coefficients determined by the straight lines in figures 4 and 5 are as follows:

$$10 \log(p_o/p_r)^2 = 1.0030\ \Upsilon \text{ phons} \tag{73}$$

$$L_m(\omega_m) = 6 \text{ phons} \tag{74}$$

$$l_\omega = 0.3636\ \Upsilon + 36.6, \text{ phons} \tag{75}$$

$$\log(\omega_1/2\pi) = -0.0158\ \Upsilon + 3.1461 \tag{76}$$

$$\log(\omega_2/2\pi) = 0.003066\ \Upsilon + 3.7404 \tag{77}$$

$$\log(\omega_m/2\pi) = -0.004816\ \Upsilon + 3.3222 \tag{78}$$

These formulas in conjunction with equation (67b) determine the loudness level L_o as a function of the input sound intensity level Υ and frequency ω_0. Note that the values of the auditory constants are actually functions of the intensity and that $l_\omega \neq l_p = 10$. From this inequality it follows that the psychoacoustic conversion involves additional filtering. The frequency-response curves predicted by using the formulas above are compared in figure 6 with the experimental curves. The disagreement between theory and experiment is generally much less than 10 phons, which corresponds to loudnesses differing by a factor of 2. The disagreement for any particular intensity level may, of course, be reduced by slightly manipulating the values of the constants in equations (73) to (78).

It is important to be able to relate the loudness of various pressure signatures to a single reference input so that the loudness of different signatures can be compared on a single loudness scale. It is most desirable to relate the loudness of sine waves and impulses in this manner. By virtue of Stevens' law (eq. (56)) equally loud sine waves and

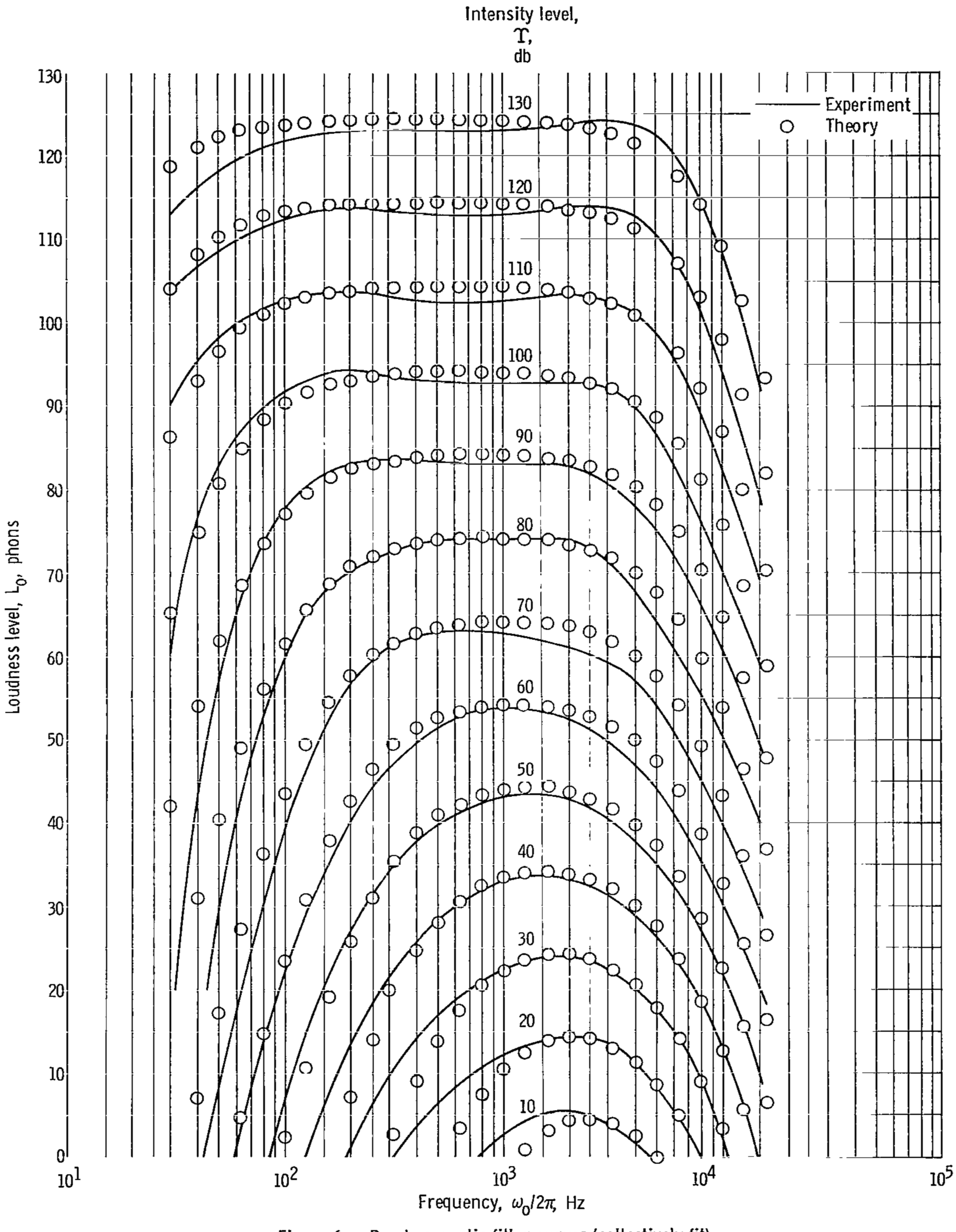


Figure 6. - Psychoacoustic filter curves (collectively fit).

impulses should correspond to a constant intensity level difference (which might be zero) over the entire range of intensities for which the law is valid. Unfortunately, the production of perfect impulses is impossible. Alternatively, input signatures resembling impulses can be produced which will serve to illustrate how the sine wave and impulsive intensity levels corresponding to equal loudness levels are related. For example, Steudel (ref. 4) compared the amplitude for equal loudness of an exponentially decaying finite-amplitude impulse with that of a 1000-hertz sine wave. The initial rise time of the impulse was unspecified but very short in comparison with the 1-millisecond time constant of the exponential decay. For equal loudness levels the intensity level of the impulse was found to be approximately 10 decibels greater than that of the 1000-hertz sine wave over a wide range of intensities. Alternatively, for equal input intensities the loudness level of the 1000-hertz sine wave was 10 phons greater than that of the impulse. Steudel's results (minus the data points, for which the scatter is ±10 phons) are effectively exhibited in figure 7. By replacing his results for the sine wave by those of Stevens and Davis (adopted from Fletcher and Munson's tests) the curves in figure 7 have been based on scales with known reference values, whereas Steudel's original curves were not. Steudel's measurements extended only up to loudness levels of 100 phons.

In figure 7, Steudel's curve for the exponentially decaying impulse is also compared

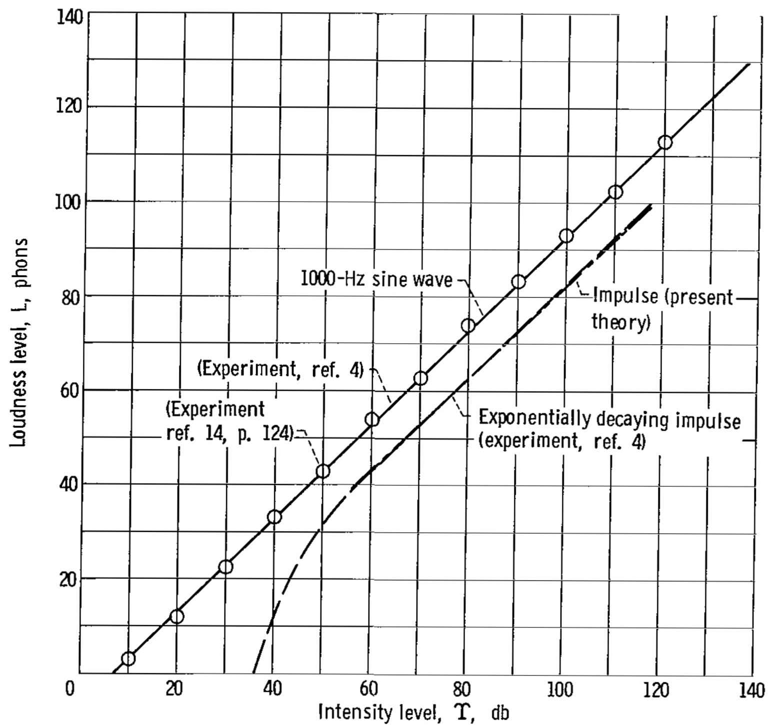


Figure 7. - Relation between loudness level and intensity level for 1000-hertz sine wave and impulses.

with the predicted loudness of a true impulse according to the present theory. The relation between intensity and loudness level for the impulse is given by equation (72), where

$$\Upsilon = 10 \log(\pi_o/\pi_r)^2 \tag{79}$$

because $\pi_o \propto p_o$ and $\pi_r \propto p_r$. The loudness level L_δ is not quite proportional to the intensity level Υ because of the appearance of the ratio of sums $\omega_1 + \omega_2$ in equation (72). As noted previously ω_1 and ω_2 are functions of Υ. The curve shown is for

$$l_p = l_\omega = 10 \text{ phons} \tag{80}$$

$$L_{\delta_o} = 16 \text{ phons} \tag{81}$$

$$\left(\omega_1 + \omega_2\right)_r = 2\pi \cdot 6900 \tag{82}$$

where the selected reference used to determine $\left(\omega_1 + \omega_2\right)_r$ is $\Upsilon = 0$ decibels (cf. fig. 4). The agreement between the theoretical and experimental curves is excellent over the intensity level interval 55 to 115 decibels, where the upper value corresponds to the limit for which Steudel presented data.

COMPARISON WITH OTHER THEORIES

The present theory encompasses some existing theories and formulas for loudness and deviates from others. For example, among the latter category, Steudel (ref. 4) proposed a loudness equation based on the square of the impulse integral; that is,

$$\left[\int_{t_o}^{t_o+t} \left|p(\tau) - p_o\right| d\tau\right]^2_{max}$$

where $p = p_o$ when $t = t_o$, and the integral is to be maximized. This integral correctly accounts for the loudness when $p = p_o$ throughout the time interval t; specifically, it indicates no response. It also correctly accounts for the loudness of a step function as being proportional to the square of the pressure amplitude of the step. However, more generally, it predicts that all pressure signatures which possess the same amplitude and impulse over the integration period will be equally loud. Vast experience with sonic booms indicates that this result is incorrect. Moreover, the loudness of continuous, statistically stationary, sounds is known to be proportional to the acoustic intensity, not to the impulse. Thus, Steudel's equation must be incorrect. Bürck, Kotowski, and Lichte (ref. 5) criticized Steudel's formula on other grounds.

Bürck, Kotowski, and Lichte frequency analyzed impulsive sounds in a manner bearing some similarity to, but cruder than, that adopted herein. They assumed that the mathematical physics was linear and applied Fourier methods. The input signature was transformed into the frequency domain and mathematically filtered by an ideal broad-band filter whose cutoff frequencies were a function of the expected loudness. The loudness was effectively defined as the square root of the transmitted energy in the filter bandwidth. Zepler and Harel (ref. 6) independently repeated Bürck, Kotowski, and Lichte's frequency-domain approach but performed nonideal filtering numerically using experimental frequency response curves for humans, and defined loudness as herein. Errors, debated elsewhere (refs. 32 and 33), in Zepler and Harel's experimental procedure led to ideas which culminated in the present theory.

Zepler and Harel's frequency-domain formulation was intended to apply to sonic booms, which are quasi-impulsive if the body producing the boom is sufficiently short. Their filtering process is numerical, but in the theory proposed herein it is expressed by a simple analytical function. Their formula for the "weighted energy density" is equivalent to equation (33b). Equation (33b) is valid if $p(t)$ is initiated and effectively vanishes within the period t_1. On the contrary, the amplitude-limited ramp function selected by Zepler and Harel possesses maximum amplitude, equal to the shock overpressure, at time t_1. Because the effective loudness-producing portion of the signature is concentrated in the time interval ($\ll t_1$) occupied by the shock, equation (33b) might conceivably be valid in this case also. However, no attempt will be made herein to test the validity of Zepler and Harel's application of equation (33b).

A frequency-domain formulation for the annoyance of statistically stationary noise has been proposed by Jones (ref. 34). Jones' formula is basically equivalent to that of Zepler and Harel for loudness and, hence, to equation (33b). Presumably equation (29b), rather than equation (33b), is the correct equation for steady noise. The two equations differ in that equation (33b) involves the ordinary pressure spectrum whereas equation (29b) involves the running pressure spectra associated with the initiation and termination of the auditory integration period. If the integration process is extended from the finite interval t_1 (≈ 0.2 sec) to the infinite interval $(-\infty, \infty)$ equation (29b) reduces to equation (33b). The assumption that the sound intensity spectrum $|P(\omega)|^2$ over all time is equivalent to the difference of running intensity spectra $|P(\omega, t)|^2$ over the period t_1 is tacit in all noise tests, but apparently has never been verified.

Unlike the present analysis, those of Zepler and Harel (ref. 6) and Jones (ref. 34) do not attempt to link the mathematics to operations which occur within the auditory system.

CONCLUDING REMARKS

A few general remarks should be made regarding implications of present loudness theories.

The claimed successes of the frequency-domain theories of Zepler and Harel for impulsive sonic booms and Jones for continuous noise indicate that the validity of the present theory may be quite general. In particular, Jones' results indicate that the existing multitude of schemes and units for evaluating the subjective aspects of noise such as loudness and annoyance may be irrelevant. It is suggested from the present theory that Jones' introduction of still another noise unit, ''perceived sound level,'' may also be unnecessary. Stevens' loudness and loudness level units, sones and phons, respectively, appear more than sufficient.

An important question in auditory studies concerns the extent to which the auditory system is nonlinear. The present theory, in conjunction with the results of Zepler and Harel and particularly those of Jones, substantiates the conclusion of Bürck, Kotowski, and Lichte that the auditory system, at least that part preceding the brain, is effectively linear with regard to loudness. The nonlinear part of the system is involved in the psychoacoustic conversion which occurs in the brain. This does not mean that the action of the ear and nervous system is completely linear, but only that possible nonlinearities have a negligible effect on loudness.

Although Zwislocki's theory (ref. 7) of temporal auditory summation is incorporated in the present theory, the theoretical constants ω_1 and ω_2 (in the present notation) are shown herein to be response cutoff frequencies and have values (fig. 5) which differ greatly from those given by Zwislocki. The difference in the values of the constants may result simply from the fact that Zwislocki's values were calculated from Galambos' measurements of stapedius muscle contractions in response to electric shocks to the medulla rather than auditory hair cell electrical outputs from pressure disturbances.

Finally, it should be recognized that the present theory obviously does not include all known effects on loudness. For example, auditory fatigue (ref. 3, p. 272f) and the ''cocktail party'' effect are two phenomena not incorporated in the present theory.

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National Aeronautics and Space Administration
Cleveland, Ohio, June 12, 1969,
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APPENDIX A

SYMBOLS

A	coefficient
c	speed of sound
$F(\omega)$	function of frequency
$f(t)$	function of time
G_o	reference sound pressure function
$G(p)$	sound pressure function
$H(\omega)$	frequency response
$H(\omega, t)$	"running" frequency response
$h(t)$	impulse response
I	$\int_{-\infty}^{\infty} h(\tau) \frac{\partial}{\partial t} p(t - \tau) d\tau = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} [h(t - \tau)] p(\tau) d\tau$
i	$\sqrt{-1}$
$J(\omega)$	electric current spectrum
$J(\omega, t)$	"running" electric current spectrum
$j(t)$	time-dependent electric current
k	coefficient in Stevens' psychophysical law
k_f	coefficient in Fechner's psychophysical law
k_o	loudness coefficient
L	loudness level, phons (ref. 1000-Hz sine wave at intensity 10^{-16} W cm^{-2})
L_m	loudness level corresponding to vanishing intensity level
$L_m(\omega_m)$	loudness level at peak response frequency corresponding to vanishing intensity level
L_o	loudness level of sine wave pressure input
L_{δ_o}	impulse loudness level corresponding to vanishing intensity level
$\mathscr{L}$	loudness, sone (loudness of 1000-Hz sine wave 40 db above threshold reference intensity)

ln	natural logarithm
log	common logarithm
l_p	loudness level pressure coefficient, phons
l_ω	loudness level frequency coefficient, phons/decade
m	exponent in Stevens' psychophysical law
n	integer
$P(\omega)$	pressure spectrum
$P(\omega, t)$	"running" pressure spectrum
p	acoustic pressure
p_o	input pressure amplitude at the ear
p(t)	pressure time history, or signature
Q_o	reference frequency function
$Q(\omega)$	frequency function
R	electrical resistance
$\mathcal{Re}$	real part
T	duration of impulse
$\mathcal{T}$	time duration
$t, \hat{t}$	time
t_o	input initiation time
t_o'	effective termination time of input
t_1	integration period, auditory integration period
v_n	velocity component normal to control surface
y	angular frequency (dummy variable)
α_n	Fourier coefficient
Δ	phase angle
$\Delta\omega$	radian frequency increment
δ	unit impulse function
η	$t - \hat{\tau}$
θ	$\omega_o t_1/2\pi$
$\theta(t)$	unit step function

θ_1	$\tan^{-1}(\omega/\omega_1)$
θ_2	$\tan^{-1}(\omega/\omega_2)$
κ	units conversion factor relating acoustic pressure and electric current
ξ	$\tau - \hat{\tau}$
Π	electrical power
Π_c	electrical power output from hair cells
π_o	input impulse
π_r	reference impulse
ρ	mass density
$\tau, \hat{\tau}$	time (dummy variables)
Υ	acoustic intensity level, db (ref. 10^{-16} W cm^{-2})
φ	magnitude of physical stimulus in psychophysical law
Ψ	acoustic intensity
ψ	psychological magnitude in psychophysical law
ω, ω'	angular frequency
ω_m	angular frequency at which auditory frequency response peaks for a given input amplitude
ω_0	auditory input frequency
ω_1	auditory lower cutoff frequency
ω_2	auditory upper cutoff frequency
$\longleftrightarrow$	Fourier transform

Subscripts:

c	complete auditory system
r	reference

Superscripts:

*	complex conjugate
$\overline{}$	infinite time average
$\sim$	finite time average

APPENDIX B

RUNNING CURRENT SPECTRUM

In order to express the running current spectrum $J(\omega, t)$ as a function of the input sound pressure $p(t)$, or its spectrum $P(\omega, t)$, note that

$$J(\omega, t) \longleftrightarrow \theta(t - \tau)j(\tau) \tag{18}$$

Because $h(t)$ is causal (ref. 16, p. 85), equations (7) may be written as

$$j(\tau) = \kappa \int_{-\infty}^{\infty} \theta(\hat{\tau})h(\hat{\tau})p(\tau - \hat{\tau})d\hat{\tau} \tag{7a'}$$

$$j(\tau) = \kappa \int_{-\infty}^{\infty} \theta(\tau - \hat{\tau})h(\tau - \hat{\tau})p(\hat{\tau})d\hat{\tau} \tag{7b'}$$

Using equation (7b'),

$$J(\omega, t) = \kappa \int_{-\infty}^{\infty} \theta(t - \tau)e^{-i\omega\tau}\, d\tau \int_{-\infty}^{\infty} \theta(\tau - \hat{\tau})h(\tau - \hat{\tau})p(\hat{\tau})d\hat{\tau}$$

$$= \kappa \int_{-\infty}^{\infty} p(\hat{\tau})d\hat{\tau} \int_{-\infty}^{\infty} \theta(t - \tau)\theta(\tau - \hat{\tau})h(\tau - \hat{\tau})e^{-i\omega\tau}\, d\tau$$

Let

$$\xi = \tau - \hat{\tau} \tag{B1}$$

Then,

$$J(\omega, t) = \kappa \int_{-\infty}^{\infty} p(\hat{\tau})e^{-i\omega\hat{\tau}}\, d\tau \int_{-\infty}^{\infty} \theta(t - \hat{\tau} - \xi)\theta(\xi)h(\xi)e^{-i\omega\xi}\, d\xi$$

$$= \kappa \int_{-\infty}^{\infty} \theta(t - \hat{\tau})H(\omega, t - \hat{\tau})p(\hat{\tau})e^{-i\omega\hat{\tau}}\, d\hat{\tau} \tag{20a'}$$

An alternative expression for $J(\omega, t)$ is obtained using equation (7a'). Thus,

$$J(\omega, t) = \kappa \int_{-\infty}^{\infty} \theta(t - \tau)e^{-i\omega\tau} d\tau \int_{-\infty}^{\infty} \theta(\hat{\tau})h(\hat{\tau})p(\tau - \hat{\tau})d\hat{\tau}$$

$$= \kappa \int_{-\infty}^{\infty} \theta(\hat{\tau})h(\hat{\tau})d\hat{\tau} \int_{-\infty}^{\infty} \theta(t - \tau)p(\tau - \hat{\tau})e^{-i\omega\tau} d\tau$$

$$= \kappa \int_{-\infty}^{\infty} \theta(\hat{\tau})h(\hat{\tau})d\hat{\tau} \int_{-\infty}^{\infty} \theta(\eta - \xi)p(\xi)e^{-i\omega(\xi+\hat{\tau})} d\xi$$

where

$$\eta = t - \hat{\tau} \tag{B2}$$

Finally,

$$J(\omega, t) = \kappa \int_{-\infty}^{\infty} \theta(\hat{\tau})h(\hat{\tau})P(\omega, t - \hat{\tau})e^{-i\omega\hat{\tau}} d\hat{\tau} \tag{20b'}$$

APPENDIX C

FREQUENCY-DOMAIN REPRESENTATION OF ELECTRIC POWER

In the time domain the electric power corresponding to the information output of the auditory nerve endings is given by

$$\widetilde{\Pi}(t) = Rt_1 \int_{t-t_1}^{t} \left| \frac{d}{d\tau} j(\tau) \right|^2 d\tau \tag{10}$$

or

$$\widetilde{\Pi}(t) = Rt_1 \int_{-\infty}^{\infty} \left\{ \left| \frac{d}{d\tau} [\theta(t - \tau)j(\tau)] \right|^2 - \left| \frac{d}{d\tau} [\theta(t - t_1 - \tau)j(\tau)] \right|^2 \right\} d\tau \tag{10'}$$

where the impulsive transients contained in equation (10'), but not in equation (10), are unphysical and are to be neglected. The Fourier transform of the first term in the integrand in equation (10') is given by

$$\frac{d}{d\tau} [\theta(t - \tau)j(\tau)] \longrightarrow i\omega J(\omega, t) \tag{24}$$

Hence, the frequency-domain representation of the integral of the first term in the integrand in equation (10') is

$$\int_{-\infty}^{\infty} \left| \frac{d}{d\tau} [\theta(t - \tau)j(\tau)] \right|^2 d\tau$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\omega \cdot \omega J(\omega, t)e^{i\omega\tau} \int_{-\infty}^{\infty} d\omega' \cdot \omega' J^*(\omega', t)e^{-i\omega'\tau}$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\omega \cdot \omega J(\omega, t) \int_{-\infty}^{\infty} d\omega' \cdot \omega' J^*(\omega', t) \int_{-\infty}^{\infty} d\tau \cdot e^{-i(\omega' - \omega)\tau}$$

$$= \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} d\omega \cdot \omega J(\omega, t) \int_{-\infty}^{\infty} d\omega' \cdot \omega' J^*(\omega', t)\delta(\omega' - \omega)$$

$$= \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} \omega^2 J(\omega, t)J^*(\omega, t)d\omega$$

or

$$\int_{-\infty}^{\infty} \left| \frac{d}{dt} [\theta(t - \tau)j(\tau)] \right|^2 d\tau = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} \omega^2 |J(\omega, t)|^2 \, d\omega \tag{C1}$$

By repeating this process with respect to the second term in the integrand in equation (10') and then introducing the two results into equation (10'), it follows that

$$\widetilde{\Pi}(t) = \frac{Rt_1}{2\pi} \int_{-\infty}^{\infty} \left[|J(\omega, t)|^2 - |J(\omega, t - t_1)|^2 \right] \omega^2 \, d\omega \tag{26}$$

which is the desired frequency-domain representation for $\widetilde{\Pi}(t)$.

APPENDIX D

POWER OUTPUT FOR VARIOUS INPUTS

Pure Tone

Time-domain analysis. - Assume that

$$p(t) = p_o \cos(\omega_0 t - \Delta) \tag{59}$$

$$p(t) = \frac{1}{2} p_o \left[e^{i(\omega_0 t - \Delta)} + e^{-i(\omega_0 t - \Delta)} \right] \tag{D1}$$

For abbreviation, let

$$I \equiv \int_{-\infty}^{\infty} h(\hat{\tau}) \frac{\partial}{\partial \tau} p(\tau - \hat{\tau}) d\hat{\tau} \tag{D2}$$

so that equation (27a) for the power becomes

$$\tilde{\Pi}(t) = \kappa^2 R t_1 \int_{t-t_1}^{t} |I|^2 \, d\tau \tag{D3}$$

By introducing the right-hand side of equation (D1) in equation (D2), it results that

$$I = \frac{i\omega_0 p_o}{2} \left[H(\omega_0) e^{i(\omega_0 \tau - \Delta)} - H(-\omega_0) e^{-i(\omega_0 \tau - \Delta)} \right] \tag{D4}$$

so that

$$|I|^2 = \frac{\omega_0^2 p_o^2}{2} \left[|H(\omega_0)|^2 - \mathcal{R}e \, H^2(\omega_0) e^{2i(\omega_0 \tau - \Delta)} \right] \tag{D5}$$

because

$$|H(-\omega_0)|^2 = |H(\omega_0)|^2 \tag{D6}$$

$$H^*(\omega_0) = H(-\omega_0) \tag{D7}$$

$$H^*(-\omega_0) = H(\omega_0) \tag{D8}$$

Equation (D6) is valid for a symmetrical filter. The last two equations are valid if h(t) is real. When equation (D5) is introduced in equation (D3), it follows that

$$\tilde{\Pi}(t) = \frac{\kappa^2 R t_1 p_0^2 \omega_0^2}{2} \left[|H(\omega_0)|^2 t_1 - \mathcal{R}e \, \frac{1}{2i\omega_0} H^2(\omega_0) e^{2i(\omega_0 t - \Delta)} \left(1 - e^{-2i\omega_0 t_1}\right) \right] \tag{60}$$

Frequency-domain analysis. - With p(t) given by equation (59) and $I = \frac{d}{d\tau} j(\tau)$ given by equation (D4), let I be expanded as a Fourier series,

$$I = \sum_{n=-\infty}^{\infty} \alpha_n e^{2\pi i n \tau / t_1} \tag{D9}$$

where

$$\alpha_n = \frac{1}{t_1} \int_{t-t_1}^{t} I e^{-2\pi i n \tau / t_1} \, d\tau \tag{D10}$$

From equation (D9) it follows that

$$\int_{t-t_1}^{t} |I|^2 \, d\tau = t_1 \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \tag{D11}$$

because α_n is independent of τ. Hence, considering equation (D3), the power is given by

$$\tilde{\Pi}(t) = \kappa^2 R t_1 \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \tag{D12}$$

From equations (D4) and (D10),

$$\alpha_n = \frac{1}{2}\,\omega_0 p_o\left[H(\omega_0)e^{i(\omega_0 t-\Delta)}\left(1 - e^{-i\omega_0 t_1}\right)\cdot(\omega_0 t_1 - 2\pi n)^{-1}\,e^{-2\pi i n t/t_1}\right.$$

$$\left. + H(-\omega_0)e^{-i(\omega_0 t-\Delta)}\left(1 - e^{i\omega_0 t_1}\right)\cdot(\omega_0 t_1 + 2\pi n)^{-1}\,e^{-2\pi i n t/t_1}\right] \tag{D13}$$

Hence,

$$\sum_{n=-\infty}^{\infty}|\alpha_n|^2 = \left(\frac{1}{2}\,\omega_0 p_o\right)^2\left\{|H(\omega_0)|^2\left(1 - e^{-i\omega_0 t_1}\right)\left(1 - e^{i\omega_0 t_1}\right)\right.$$

$$\cdot\sum_{n=-\infty}^{\infty}\left[(\omega_0 t_1 - 2\pi n)^{-2} + (\omega_0 t_1 + 2\pi n)^{-2}\right]$$

$$+\left[H^2(\omega_0)e^{2i(\omega_0 t-\Delta)}\left(1 - e^{-i\omega_0 t_1}\right)^2\right.$$

$$\left. + H^2(-\omega_0)e^{-2i(\omega_0 t-\Delta)}\left(1 - e^{i\omega_0 t_1}\right)^2\right]$$

$$\left.\cdot\sum_{n=-\infty}^{\infty}\left[(\omega_0 t_1)^2 - (2\pi n)^2\right]^{-1}\right\} \tag{D14}$$

Applying the formula (ref. 35)

$$2\sum_{n=1}^{\infty}\frac{\theta^2 + n^2}{\left(\theta^2 - n^2\right)^2} = -\theta^{-2} + \pi^2\csc^2\pi\theta \tag{D15}$$

with

$$\theta \equiv \frac{\omega_0 t_1}{2\pi} \tag{D16}$$

there results

$$\sum_{n=-\infty}^{\infty}\left[(\omega_0 t_1 - 2\pi n)^{-2} + (\omega_0 t_1 + 2\pi n)^{-2}\right] = 2\left(1 - e^{i\omega_0 t_1}\right)^{-1}\left(1 - e^{-i\omega_0 t_1}\right)^{-1} \tag{D17}$$

Also, using the formula (ref. 36)

$$\sum_{n=1}^{\infty}\left(\theta^2 - n^2\right)^{-1} = \frac{1}{2}\,\theta^{-2}(\pi\theta \cot \pi\theta - 1) \tag{D18}$$

it follows that

$$\sum_{n=-\infty}^{\infty}\left[(\omega_0 t_1)^2 - (2\pi n)^2\right]^{-1} = \frac{1}{2i\omega_0 t_1}\left(1 + e^{i\omega_0 t_1}\right)\left(1 - e^{i\omega_0 t_1}\right)^{-1} \tag{D19}$$

By introducing the sums given by equations (D17) and (D19) in equation (D14) and then introducing the result in equation (D12), equation (60) is obtained.

Impulse

Time-domain analysis. - Assume that

$$p(t) = \pi_o \delta(t) \tag{69}$$

where π_o is the impulse. The electric power is given by equation (D3), where

$$I = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[h(t - \tau)\right] p(\tau) d\tau \tag{D20}$$

By introducing equation (69) in equation (D20), it follows that

$$I = \pi_o \frac{d}{dt} h(t) = \pi_o h_c(t) \tag{D21}$$

where $h(t)$ is given by equation (46). Hence,

$$\tilde{\Pi}(t) = \kappa^2 R t_1 \int_{t-t_1}^{t} |I|^2 \, d\tau \tag{D22}$$

$$\tilde{\Pi}(t) = \kappa^2 R t_1 \left(\int_0^t |I|^2 \, d\tau + \int_{t-t_1}^{0} |I|^2 \, d\tau \right) \tag{D23}$$

where the second integral in the last expression must vanish if $t \leq t_1$ because $h(t)$ is causal. By performing the indicated integrations the results are

$$\tilde{\Pi}(t \leq 0) = 0 \tag{D24a}$$

$$\tilde{\Pi}(0 \leq t \leq t_1) = \frac{1}{2} R t_1 \left(\frac{\kappa A \pi_o}{\omega_2 - \omega_1} \right)^2$$

$$\left[\omega_2 \left(1 - e^{-2\omega_2 t} \right) + \omega_1 \left(1 - e^{-2\omega_1 t} \right) - \frac{4\omega_1 \omega_2}{\omega_1 + \omega_2} \left(1 - e^{-(\omega_1 + \omega_2) t} \right) \right] \tag{D24b}$$

$$\tilde{\Pi}(t_1 \leq t) = \frac{1}{2} R t_1 \left(\frac{\kappa A \pi_o}{\omega_2 - \omega_1} \right)^2 \left\{ \omega_2 e^{-2\omega_2 t} \left(e^{2\omega_2 t_1} - 1 \right) + \omega_1 e^{-2\omega_1 t} \left(e^{2\omega_1 t_1} - 1 \right) \right.$$

$$\left. - \frac{4\omega_1 \omega_2}{\omega_1 + \omega_2} e^{-(\omega_1 + \omega_2) t} \left[e^{(\omega_1 + \omega_2) t_1} - 1 \right] \right\} \tag{D24c}$$

These three equations may be combined to yield equation (70).

Because ω_1, $\omega_2 >> 1/t_1$, it follows from equation (D24b), or (D24c), that, when $t = t_1$,

$$\tilde{\Pi}(t_1) \approx \frac{1}{2} R t_1 \left(\frac{\kappa A \pi_o}{\omega_2 - \omega_1} \right)^2 \left(\omega_2 + \omega_1 - \frac{4\omega_1 \omega_2}{\omega_1 + \omega_2} \right) \approx \frac{1}{2} R t_1 \frac{(\kappa A \pi_o)^2}{\omega_1 + \omega_2} \tag{71}$$

Frequency-domain analysis. - Because $p(t)$ is impulsive and $h(t)$ is quasi-impulsive, equation (53) may be used to evaluate $\tilde{\Pi}$. Thus, by virtue of equations (11) and (69),

$$P(\omega) = \pi_o \tag{D25}$$

Hence, equation (53) becomes

$$\tilde{\Pi}(t_1) \approx \frac{1}{2\pi} Rt_1 (\kappa A \pi_o)^2 \int_{-\infty}^{\infty} \frac{\omega^2 \, d\omega}{\left(\omega^2 + \omega_1^2\right)\left(\omega^2 + \omega_2^2\right)}$$

which results in (ref. 37)

$$\tilde{\Pi}(t_1) \approx \frac{1}{2} Rt_1 \frac{(\kappa A \pi_o)^2}{\omega_1 + \omega_2} \tag{71}$$

in agreement with the result from the time-domain analysis.

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