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SECOND-ORDER WAVE STRUCTURE IN SUPERSONIC FLOWS

by David A. Caughey

Prepared by
PRINCETON UNIVERSITY
Princeton, N. J.

for

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Abstract

Second-order effects upon the wave systems associated with bodies in uniform supersonic flight are considered. An appropriate scaling of the independent variables is introduced to take into account the locally small, but cumulative, nonlinearities which undermine the validity of the classical perturbation theories at large distances from the body.

For planar flows, the solution valid at large distances from the body (the wave structure solution) is uniformly valid to the body surface, and the boundary conditions may be applied directly. For this case, the simple wave results of Friedrichs as corrected and extended by Lighthill are confirmed, with a minor correction noted for the location of the rear shock at very great distances from the body. It is noted that effects locally of third order must be taken into account behind the trailing shocks to give proper global integrals relating to the lift and drag of the airfoil section and to determine the positions of the trailing shocks to second order.

For flows about finite bodies, the wave structure solution must be matched with a quasi-cylindrical, local solution to provide the inner boundary condition. This procedure gives rise to an intermediate (or one-and-one-half-order) solution which corresponds to the first-order solution over a slightly modified body. In these two wave structure solutions, the azimuthal angle enters only as a parameter, and the flow in each azimuthal plane may be thought of as that about some equivalent body of revolution. Dependence upon azimuthal angle does arise in the true second-order wave structure solution. The complete general solution has been found for this second-order equation, but the third-order local solution is required to effect the matching which determines the inner boundary condition. For flows over bodies whose surfaces lie everywhere near an axis aligned to the flow, an additional matching is required with a solution valid near the body surface to determine the local solution. This slender-body solution is also required to third order to determine the wave structure solution to second order. The lack of a local second-order theory of sufficient generality is the major hurdle in determining the wave structure solution. Local third-order solutions can presently be obtained for only the simplest conical geometries.

Predicted shock angles for the flow over slender wedges (planar flow) and cones (axisymmetric flow) are compared with exact inviscid calculations to show the improvement afforded by inclusion of second-order effects. Inclusion of the full second-order solution is required to achieve appreciable improvement over the first-order theory for the conical case.

Table of Contents

Nomenclature

English alphabet

$\mathcal{A}$	source coefficient for first-order cone flow
a	$=\sqrt{(\partial p/\partial \rho)_s}$, acoustic velocity
$\mathcal{B}$	source coefficient for second-order cone flow
$\mathcal{C}$	source coefficient for third-order cone flow
c	body chord
$F(\xi_b)$	first-order wave structure function
$\mathcal{F}(\lambda)$	first-order local source function
$\mathcal{F}_1(\lambda)$, $\mathcal{F}_2(\lambda)$	first-order local source and dipole functions
$\mathcal{F}_l(x_1,x_2)$	local planar horseshoe vortex distribution
$\mathcal{F}_t(x_1,x_2)$	local planar source distribution
$\mathcal{F}_t(\lambda)$, $\mathcal{F}_l(\lambda)$	equivalent axisymmetric source and vortex distributions
$f(\xi)$	body shape function
$f_1(\xi)$, $f_2(\xi)$	quasi-cylindrical local source and dipole body shape functions
$G(\xi_b)$	second-order wave structure function
$\mathcal{G}(\lambda)$	second-order local source function
$\mathcal{G}_1(\lambda)$, $\mathcal{G}_2(\lambda)$	second-order local source and dipole distributions (quasi-cylindrical)
H	total enthalpy
$H(\xi_b)$	third-order wave structure function
$\mathcal{H}(\lambda)$	third-order local source function
$h(\xi_b)$	second-order wave structure function (planar)
$\hat{i}, \hat{j}, \hat{k}$	unit vectors in (x,y,z) space

K	$= \dfrac{\Gamma M^4 \tau}{\beta^3}$, basic scaling parameter
K_{sl}	$= \dfrac{\Gamma M^4 \tau^2}{\beta^2}$; basic scaling parameter for slender body flows
M	$= U/a_\infty$, Mach number
Δm	mass flux
$O(\varepsilon)$	signifies quantity which approaches (const)$\cdot\varepsilon$ as ε approaches zero
p	fluid pressure
Δp	momentum flux
$\vec{q}$	vector fluid velocity
q	magnitude of fluid velocity
q_n	component of $\vec{q}$ normal to shock surface
r	$=\sqrt{y^2+z^2}$, radial coordinate in plane perpendicular to flow in physical space
$\bar{r}$	$=\sqrt{\bar{\eta}^2+\bar{\zeta}^2}$, radial coordinate in plane perpendicular to flow in Prandtl-Glauert system
r_0	radius of mean cylindrical body surface
S	specific entropy of fluid
$s(\eta)$	function of intergation
T	fluid temperature
t	$= \bar{r}/\bar{\xi}$, conical variable
U	freestream velocity
u	x- component of perturbation velocity
v	y- or r- component of perturbation velocity
(x,y,z)	cartesian system with x-axis parallel to freestream
(x,y)	local coordinates for shock origin problem
x_1, x_2	dummy variables of integration

Greek alphabet

β	$=\sqrt{M^2-1}$, cotangent of Mach angle
Γ	$=\frac{1}{a}\left(\frac{\partial \rho a}{\partial \rho}\right)_s$, thermodynamic parameter
γ	specific heat ratio for calorically perfect gas
$\boldsymbol{\nabla}$	nabla operator in physical space
∇	nabla operator in plane perpendicular to freestream in Prandtl-Glauert system
δ	departure of quantity from its freestream value
ζ	$=\lvert\boldsymbol{\nabla}\times\mathbf{q}\rvert$, magnitude of vorticity
$\vec{\zeta}$	$=\boldsymbol{\nabla}\times\mathbf{q}$, vorticity
$\bar{\zeta}, \bar{\eta}$	$=\beta z/c$, $\beta y/c$
η	$=K\bar{\eta}$ in planar flows; $=2K\sqrt{\bar{r}}$ in finite-body flows — "age" variable
$\tilde{\eta}$	$=K^2\eta$ in planar flows; $=16K^4\eta$ in finite-body flows — "age" variable in Lighthill region
θ	$=\tan^{-1}(\bar{\zeta}/\bar{\eta})$, polar angle in cross-plane in Prandtl-Glauert system
Θ	streamline inclination angle
K	$=\left(6+\frac{3}{M^2}-2\frac{\Gamma M^2}{\beta^2}+2\mu\right)$.
λ	dummy variable of integration
λ'	$=x_1-x_2\cos\Theta$, dummy variable of integration
μ	$=\frac{4\Gamma-\sigma-7}{2}$, thermodynamic parameter
ν	$=-\frac{a^2}{\rho T}\left(\frac{\partial \rho}{\partial S}\right)_p$, thermodynamic parameter
Ξ	$=\bar{\xi}-\bar{\eta}$ in planar flows; $=\bar{\xi}-\bar{r}$ in finite body flows — "phase" variable
$\bar{\xi}$	$=x/c$.

$\tilde{\xi}$	$= K\xi$ in planar flows; $= 4K^2\xi$ in finite-body flows	"phase" variable in Lighthill region
ξ_0	first-order shock position	
ξ_{1_v}	shift in virtual origin of shock, accounting for local structure	
ρ	fluid density	
σ	$= \frac{\rho}{\Gamma}\left(\frac{\partial \Gamma}{\partial \rho}\right)_S$, thermodynamic parameter	
τ	thickness parameter, generally corresponding to maximum body surface inclination	
Φ	$= \int dq$, velocity potential	
φ	$= \frac{1}{\tau\beta}\phi$, slender body perturbation potential	
ϕ	$= \frac{\beta}{\tau c}\left(\frac{1}{U}\Phi - x\right)$, perturbation potential	
ψ	$= \sqrt{\bar{r}}\,\phi$, reduced perturbation potential	
Ω	region of integration for planar bodies; area on mean body plane upstream of intersection of forward Mach cone from point of interest	

Subscripts

0	first-order quantity
$\frac{1}{2}$	one-and-one-half-order quantity
1	second-order quantity
∞	quantity evaluated in freestream
b	quantity evaluated on body surface
ch	quantity pertaining to characteristic
i	inner expansion of quantity, accurate to order included in parentheses. (Parentheses indicate complete expansion, including lower-order terms).
l	quantity pertaining to leading shock
o	outer expansion of quantity, accurate to order included in parentheses. (See note above pertaining to inner expansion).

sh	quantity pertaining to shock
t	quantity pertaining to trailing shock
x, y	x- and y- components of vector quantity

Additional symbols

$[\ ]$	signifies jump in quantity across discontinuity, with jump evaluated at $\xi = \xi_0$.
$\overline{(\ )}$	signifies mean of quantity on either side of discontinuity, evaluated at $\xi = \xi_0$.

Chapter One

The Introduction

1.A. General Introduction

The present treatise is concerned with the mathematical description of the wave systems which emanate from bodies in supersonic flight, and, in particular, the behavior of these systems at large distances from the body. A perturbation theory, which includes account of essential cumulative nonlinearities, is considered, accurate to terms of second order.

Such a problem is of practical interest as it relates to the calculation of sonic boom overpressures, where we are interested in predicting the effects of a high flying, supersonic aircraft in the vicinity of the ground. It is also of theoretical interest as it illustrates techniques useful in solving problems of weakly nonlinear wave propagation. For, while the classical linearized theories of supersonic aerodynamics are perfectly adequate for the prediction of disturbances near the body (and especially surface pressures), the failure of these theories at large distances from the body has long been recognized, and a correct theory must in some way account for cumulative nonlinearity within the wave system.

The equations of steady, compressible fluid flow are inherently nonlinear, even when the simplifications of negligible viscosity and irrotationality are used. The mathematical treatment of nonlinear equations is still largely restricted to especially simple cases, and the complexity of the fluid flow equations precludes treatment in any generality without additional assumptions to provide further simplification.

Perhaps the most useful simplification -- at least for the case of supersonic aerodynamics -- is to consider that the flow is everywhere a small perturbation upon some known flow field (usually uniform). The nonlinear problem is then broken up into a hierarchy of simpler (often linear) problems, and the solution may be found as an asymptotic series, successive terms of which contain factors of higher order in the perturbation parameter.

The classical approach to perturbation theory for steady supersonic aerodynamics yields the scalar wave equation for the velocity potential, to be solved subject to appropriate boundary conditions, as a first approximation. Higher approximations also satisfy the wave equation, but with additional nonhomogeneous terms which are known functions of the lower-order solutions. Thus, a hierarchy of linear problems is set up and may be solved to any degree of accuracy desired. Since the homogeneous part of each equation is always the scalar wave equation, the characteristics are not revised in successive approximations. Higher approximations may then be thought of as giving successive terms in the Taylor series expansion of the exact solution about the linearized (freestream) characteristics (Van Dyke (1952)).

Such a scheme has been extremely useful for predicting disturbances near bodies immersed in supersonic streams, especially for predicting surface pressures. However, since the characteristics are not revised in higher-order solutions, the predictions become increasingly inaccurate as the distance from the body is increased. Such behavior is a result of the "nonuniform validity" of the linearized solution. I.e., the solution obtained by such a scheme is not equally accurate at all distances from the body. Physically, this is due to the fact that the actual (exact) characteristics and the linearized (freestream) characteristics are getting farther apart as they are followed away from the body. This deficiency is serious if we are concerned with accurately describing the flow field at large distances from the body, and is interesting from a theoretical viewpoint as it illustrates the nature of (at least one type of) nonlinearity which must be accounted for to achieve uniformly valid solutions.

The essential nonlinearities in the solution at large distances from the body may be accounted for in two ways. We could solve the purely linear problem for the solution on the linearized characteristics, and simultaneously correct the placement of these characteristics in physical space due to the perturbations of the solution. Or, we may try to rescale the independent variables of the problem in such a way as to bring the essential (or cumulative) nonlinearities into the lower-order problem. Hopefully, this rescaling will also provide some simplification in the other terms of the equation, so that the new equation will be mathematically tractable. In this second approach, the first-order equation will be nonlinear (as it must be to achieve our goal), but higher-order equations will again be linear.

This second method will be used here. The specific purpose of the present treatise is to discuss the nature of the second approximation, or the second term in the asymptotic series representation of the perturbation solution, valid uniformly to large distances from the body.

The purpose of such a second-order theory is two-fold. First, it may provide increased numerical accuracy for solutions simple enough to be calculated analytically. But, secondly, and more importantly, a study of the second-order solution is one way to verify the uniform validity of the first-order solution, thus showing that the entire procedure is indeed a rational approximation to the exact solution in some limit. Further, such a study gives greater insight into the nature of the first approximation, and provides limits on the accuracy of that theory.

1.B. Relevant previous researches

The background of the linear theory of supersonic aerodynamics is well established in the literature and textbooks, and will not be discussed here. (See, e.g., Ward (1955) or Heaslet and Lomax (1954)).

Local second-order effects were considered by Van Dyke (1952). He solved the potential equation by iteration (i.e., using the first-order solution to evaluate quadratic terms) to obtain a second approximation. He found complete particular integrals for the nonhomogeneous terms in the cases of planar and axisymmetric flows, thus reducing the second-order problem to an equivalent first-order problem. His solutions gave marked improvement for the calculation of surface pressures, but were not uniformly valid at large distances from the body. He recognized this fact, and realized that it was due to the inability of the theory to revise the characteristics in successive approximations.

Friedrichs (1948) first obtained solutions accurate to second order and uniformly valid to large distances for the planar flow over airfoils, using the fact that the flow is (to second order) a simple wave. Correction of his expression for the shock slope in terms of the perturbation quantities on either side and inclusion of effects due to the broad third-order waves behind the trailing shocks (which have a cumulative effect of second order) were discussed by Lighthill (1954).

Whitham (1952) rendered the first-order solution for the flow over slender, axisymmetric bodies uniformly valid to large distances by applying the coordinate straining technique of Lighthill. This essentially uses the linear solution to correct the placement of the characteristics in the physical plane.

Hayes (1954) used the method of rescaling the independent variables to order terms according to their cumulative (rather than local) effects to obtain the proper form of the first-order wave structure for more general flows. For flows about finite planar systems, the method relies upon the method of "coincident signals" of Hayes (1947) to supply the appropriate boundary condition in terms of the large-distance asymptotic behavior of the local solution.

Work currently in progress by Landahl et al, (1968), and somewhat parallel to the present study, has looked briefly at second-order corrections to the wave structure theory of Hayes. They have independently noted the need for an intermediate homogeneous wave structure solution for flows about finite systems (which we term of one-and-one-half order), which is determined by the local second-order solution. This result is used to support the need for experimental surveys in the absence of a complete second-order local theory.

The nonuniform validity of the linear theory at large distances is equivalent to the singular behavior of the linear solution near the Mach cone in problems of the conical type. This is because the relevant parameter is the ratio of the distance a characteristic has travelled from the body to the axial distance along the body from which the characteristic started, and this ratio becomes very large for waves near the leading Mach cone. This nonuniformity may be rectified using either of the two methods previously mentioned for the wave structure problem. Lighthill (1949) applied the method of strained coordinates to the problem to determine the shock wave position to first order (i.e., its lowest-order deviation from the freestream Mach cone). Broderick (1949) used a separate expansion scheme valid near the Mach cone (essentially the method of matched asymptotic expansions) to prove that his expression for the second-order surface pressure on a circular cone, derived without consideration of the shock, was correct to that order, and also verified

Lighthill's expression for the position of the shock to first order. Bulakh (1961), in the Soviet Union, showed Broderick's result for a general conical flow, and again verified Lighthill's result for the shock wave position to first order.

The current work also bears a formal resemblence to the second-order transonic theory of Hayes (1966). The effects which exert a cumulative influence upon the supersonic wave system are locally important in transonic flows. Thus, the similarity coefficients of the two theories (in the sense of Van Dyke (1958) as giving the form of the solution) are essentially the same.

1.C. General nature and scope of present theory

The present treatise presents a perturbation theory for steady, inviscid supersonic flows which is uniformly valid to large distances from the body of interest. The theory is basically an extension to include second-order effects of the first-order wave structure theory of Hayes (1954b). Uniform validity of the solution to large distances is obtained by rescaling the independent variables of the problem such that nonlinear terms are ordered according to their cumulative (rather than local) effects in an outer, or wave structure, region. The solution so obtained is uniformly valid to the body surface for planar flows, but the inner boundary condition for flows about finite systems must be obtained by matching with an appropriate local solution.

The complete theory of planar flows is presented, along with a sample solution. The results obtained confirm those of Friedrichs (1948), obtained by a characteristics method, as corrected and amplified by Lighthill (1954), except for the position of the rear shock at great distances from the airfoil, where the Lighthill result is in error (see Appendix B).

For the case of flows about finite systems, the matching with the local second-order solution reveals the necessity of a one-and-one-half-order wave structure solution, satisfying homogeneous equations. This corresponds to a first-order solution over a slightly modified body. The third-order local solution is required to obtain the second-order wave structure solution. The complete general solution of the second-order wave structure problem has been found, but difficulty in obtaining the third-order local solution to provide the inner boundary condition precludes calculation of any examples (except the rather specialized case of a right circular cone at zero angle of attack).

In the following chapter, the basic equations are developed, and a brief discussion of the scaling of the wave structure region is included. Chapter Three specializes the analysis to planar flows. The general form of the solution is given, as well as some specific features noted from the sample solution. Flows about finite systems are treated in Chapter Four, including the matching problem with the local solution. The most important results are summarized in Chapter Five, along with some general remarks on future researches. Details not essential to the development of the text are included in the appendices.

Chapter Two

The Development of Basic Equations

2.A. Potential Equation

As the heading of this section suggests, we shall work almost exclusively with a velocity potential. This allows us to describe the vector velocity field by a single scalar equation, but not without the loss of some generality. That such a potential exists depends heavily upon our assumption of small perturbations, and particularly that there be no strong shocks in the flow. For, while the flow is initially irrotational, and considered to be inviscid, we cannot completely disregard the entropy increments caused by shock waves, and it is vorticity arising from the variation in strength of these discontinuities that would eventually force us to abandon the use of a velocity potential if we were interested in a theory more accurate than the second order.

It can be shown that, for our purposes, a velocity potential exists everywhere in the flow field to the required order except behind the trailing shocks, where entropy and velocity (pressure) perturbations are of the same order. (See Appendix A). This is essentially because the entropy increment at a weak shock wave is of third order in the strength of the wave, and we are concerned with only second-order effects. We are interested in the third-order quantities behind the trailing shocks only because they have a cumulative influence upon the position of these shocks which is of second order (See section 3.C.).

We consider a steady, homocompositional, homoenergetic, inviscid fluid flow. The equations of continuity, momentum conservation, and isentropy for such a flow may be written

$$\nabla\cdot(\rho\vec{q}) = 0, \tag{2.01}$$

$$\vec{q}\cdot\nabla\vec{q} + \frac{1}{\rho}\nabla p = 0, \tag{2.02}$$

$$\vec{q}\cdot\nabla S = 0, \tag{2.03}$$

where ρ is the fluid density, $\vec{q}$ is the vector velocity, p is the pressure, and S is the specific entropy.

We form a modified continuity equation by converting the density gradient to a pressure gradient, using the condition of isentropy:

$$\rho\nabla\cdot\vec{q} + \frac{1}{a^2}\vec{q}\cdot\nabla p = 0, \tag{2.04}$$

where a is the acoustic velocity.

Then, eliminating the pressure gradient between this and the momentum equation (2.02) yeilds

$$\vec{q}\cdot(\vec{q}\cdot\nabla\vec{q}) - a^2\nabla\cdot\vec{q} = 0. \tag{2.05}$$

Or, introducing the velocity potential, Φ , such that

$$\vec{q} = \nabla\Phi , \tag{2.06}$$

we have

$$a^2\nabla^2\Phi = \nabla\Phi\cdot\nabla\left(\tfrac{1}{2}\nabla\Phi^2\right). \tag{2.07}$$

This is the fundamental equation for all steady, irrotational flows. It is an exact equation, subject only to the requirements of steadiness, homocompositionality, homoenergy, and homoentropy. It is, however, nonlinear and extremely difficult to treat with any generality. (Note that the nonlinearity arises not only from the right-hand-side of the above equation, but also from the fact that a^2 is also a function of the potential, Φ).

Therefore, additional assumptions must be made to cast the problem as one that is mathematically tractable. A standard technique (and the one we shall use) is to assume the flow may be described as everywhere a small perturbation upon a uniform streaming motion. This allows us not only to expand the right-hand-side of the equation, neglecting higher-order products of the perturbations, but also to expand the acoustic velocity in powers of the perturbations. Thus

$$a^2 = a_\infty^2 + 2(\Gamma-1)\frac{\delta p}{\rho_\infty} + \frac{1-\Gamma+\sigma\Gamma}{a_\infty^2}\left(\frac{\delta p}{\rho_\infty}\right)^2 + O(\delta p^3, \delta S) , \tag{2.08}$$

where Γ and σ are thermodynamic parameters of the fluid (always understood to be evaluated in the undisturbed stream) and defined by

$$\Gamma = \frac{1}{a}\left(\frac{\partial \rho a}{\partial \rho}\right)_S , \tag{2.09}$$

$$\sigma = \frac{\rho}{\Gamma}\left(\frac{\partial \Gamma}{\partial \rho}\right)_S . \tag{2.10}$$

The symbol δ designates the difference of the quantity it precedes from its value in the undisturbed stream, and the subscript $(\ )_\infty$ refers to evaluation in the undisturbed stream. For a calorically perfect gas, $\Gamma = \frac{\gamma+1}{2}$ and $\sigma = 0$.

In order to use this expansion in the potential equation, we must have a relation between the pressure perturbations and the velocity (potential) perturbations. Such a Bernoulli relation may be derived as follows. For a homocompositional, inviscid flow, the Crocco-Vaszonyi theorem states

$$\vec{q} \times (\nabla \times \vec{q}) = \nabla H - T \nabla S , \tag{2.11}$$

where H is the total enthalpy and T is the temperature of the fluid. This may be combined with the momentum equation to give, for a homoenergetic (i.e., H = constant) flow,

$$\frac{\nabla p}{\rho} = -\frac{\nabla q^2}{2} - T \nabla S , \tag{2.12}$$

which may be expanded and integrated to give

$$\int \frac{dp}{\rho} = \frac{U^2 - q^2}{2} - T_\infty \delta S + O(\delta S^2, \delta p\,\delta S) , \tag{2.13}$$

where U is the undisturbed, or free-stream, velocity. We may also write a formal, thermodynamic expansion of this integral,

$$\int \frac{dp}{\rho} = \frac{\delta p}{\rho_\infty} - \frac{1}{2a_\infty^2}\left(\frac{\delta p}{\rho_\infty}\right)^2 + \frac{\Gamma}{3a_\infty^4}\left(\frac{\delta p}{\rho_\infty}\right)^3 + O(\delta p^4, \delta p\,\delta S) . \tag{2.14}$$

And comparison of the last two results, by an iterative process, yields

$$\begin{aligned} \frac{\delta p}{\rho_\infty} = &-\frac{U^2\tau}{\beta}\phi_{\bar\xi} - \frac{U^2\tau^2}{2}(\nabla\phi^2 - \phi_{\bar\xi}^2) \\ &+ \frac{U^2M^2\tau^3}{\beta}\phi_{\bar\xi}\left(\frac{\nabla\phi^2 - \phi_{\xi}^2}{2} + \frac{\Gamma M^2}{3\beta^2}\phi_{\xi}^2\right) - T_\infty\delta S + \cdots . \end{aligned} \tag{2.15}$$

This is the required Bernoulli equation which relates pressure perturbations to changes in velocity potential. Here we have introduced the perturbation potential, ϕ .

$$\vec{q} = \nabla\Phi = U\left\{\hat{i} + \frac{\tau c}{\beta}\nabla\phi\right\} , \tag{2.16}$$

where τ is a measure of the maximum body slope, c is the body length (chord), and $\beta = \sqrt{M^2 - 1}$ with $M = U/a_\infty$. We have also introduced the Prandtl-Glauert coordinates

$$
\begin{aligned}
\bar{\xi} &= x/c , \\
\bar{\eta} &= \beta y/c , \\
\bar{\zeta} &= \beta z/c ,
\end{aligned}
\tag{2.17}
$$

in place of a cartesian system (x,y,z), with its x-axis aligned to the freestream, and have used the notation

$$\nabla = \hat{j}\frac{\partial}{\partial \bar{\eta}} + \hat{k}\frac{\partial}{\partial \bar{\zeta}} .$$

The Bernoulli equation may be substituted back into equation (2.08) to give

$$a^2 = a_\infty^2 - \frac{2(\Gamma-1)U^2\tau}{\beta}\phi_{\bar{\xi}} - \frac{U^2\tau^2}{\beta^2}\left\{(\Gamma-1)(\phi_{\bar{\xi}}^2 + \beta^2\nabla\phi^2) - \sigma\Gamma M^2\phi_{\bar{\xi}}^2\right\} + \cdots . \tag{2.18}$$

Finally, this may be substituted back into the fundamental potential equation (2.07), the right-hand-side of that equation expanded in terms of the perturbation potential, and we arrive at

$$
\begin{aligned}
\nabla^2\phi - \phi_{\bar{\xi}\bar{\xi}} = \frac{M^2\tau}{\beta}\Big\{ & \frac{2\Gamma M^2}{\beta^2}\phi_{\bar{\xi}}\phi_{\bar{\xi}\bar{\xi}} + 2(\nabla\phi\cdot\nabla\phi_{\bar{\xi}} - \phi_{\bar{\xi}}\phi_{\bar{\xi}\bar{\xi}}) \\
& + 2(\Gamma-1)\phi_{\bar{\xi}}(\nabla^2\phi - \phi_{\bar{\xi}\bar{\xi}})\Big\} + \frac{M^2\tau^2}{\beta^2}\Big\{\Gamma(\phi_{\bar{\xi}\bar{\xi}} + \beta^2\nabla^2\phi)(\nabla\phi^2 - \phi_{\bar{\xi}}^2) \\
& + \Gamma M^2(1-\sigma)\phi_{\bar{\xi}}^2(\nabla^2\phi - \phi_{\bar{\xi}\bar{\xi}}) + \frac{\Gamma M^4}{\beta^2}(1-\sigma)\phi_{\bar{\xi}}^2\phi_{\bar{\xi}\bar{\xi}} \\
& + 2\phi_{\bar{\xi}}\nabla\phi\cdot\nabla\phi_{\bar{\xi}} + \phi_{\bar{\xi}}^2\nabla^2\phi - \phi_{\bar{\xi}\bar{\xi}}\nabla\phi^2 + \beta^2\left(\tfrac{1}{2}\nabla\phi\cdot\nabla(\nabla\phi)^2 - \nabla\phi^2\nabla^2\phi\right)\Big\} + \cdots .
\end{aligned}
\tag{2.19}
$$

This equation for the perturbation potential, ϕ, is the starting point for all perturbation theories, and in the above form is accurate to third order in the local perturbation quantities. It is an extension of the equation given by Van Dyke (1952) to include all third-order quantities (not just those important near a slender body), and is derived for a general fluid (not a calorically perfect gas).

2.B. Shock relations

A peculiar feature of nonlinear partial differential equations is the presence of regions in the physical plane where the solution is multiply defined, due to the overlap of characteristics coming from different points on the boundary or initial data. In our case, this difficulty is overcome by the insertion of shocks, or discontinuities in the solution (or its derivatives), across which the solution jumps from one branch to another in the characteristic plane. The intermediate portion of the solution is then omitted to render the solution unique in the physical plane.

These shocks are inserted according to the classical Hugoniot conditions for conservation of mass, momentum, and energy fluxes across the discontinuity surface, and the requirement that the tangential velocity be continuous across the surface.

We describe the shock location as

$$\bar{\xi} = \bar{\xi}_{sh}(\bar{\eta}, \bar{\zeta}) \; , \qquad (2.20)$$

and use square brackets, [ ], to denote the jump in any quantity across the discontinuity surface. Thus, the requirements of continuous tangential velocity and normal mass flux may be written

$$[\phi] = 0 \; , \qquad (2.21)$$

and

$$[\rho q_n] = 0 \; , \qquad (2.22)$$

respectively, where q_n is the velocity component along the normal to the discontinuity surface. The conservation of energy is automatically assured through our use of the Bernoulli equation (2.15) to relate pressure, velocity, and entropy perturbations. Finally, in place of the usual momentum relation, we shall use a quantitative expression for the entropy jump

$$T_\infty [S] = \frac{-\Gamma}{6 a_\infty} [q_n]^3 + O\left([q_n]^4\right) , \qquad (2.23)$$

which is valid for weak shock waves. (See, e.g., Hayes (1954a.).)

We may expand the density as

$$\rho = \rho_\infty \left\{ 1 + \frac{1}{a_\infty^2} \frac{\delta p}{\rho_\infty} - \frac{\Gamma - 1}{a_\infty^4} \left(\frac{\delta p}{\rho_\infty} \right)^2 + \frac{2\mu\Gamma + 3}{3 a_\infty^6} \left(\frac{\delta p}{\rho_\infty} \right)^3 - \frac{\nu}{a_\infty^2} T_\infty \delta S + \cdots \right\} , \qquad (2.24)$$

where

$$\mu = \frac{1}{2}(4\Gamma - \sigma - 7) \; , \qquad (2.25)$$

and

$$\nu = -\frac{a^2}{\rho T}\left(\frac{\partial \rho}{\partial S}\right)_p , \tag{2.26}$$

and these last two parameters are always assumed to be evaluated in the freestream. For a calorically perfect gas, $\mu = \gamma - 5/2$ and $\nu = \gamma - 1$.

Substituting from the Bernoulli result (2.15) gives

$$\begin{aligned}\rho = \rho_\infty \Big\{ 1 &- \frac{M^2\tau}{\beta}\phi_\xi - \frac{M^2\tau^2}{\beta^2}\Big\{\frac{\beta^2}{2}(\nabla\phi^2 - \phi_\xi^2) + (\Gamma-1)M^2\phi_\xi^2\Big\} \\ &+ \frac{M^4\tau^3}{\beta^3}\Big\{\frac{3-2\Gamma}{2}\beta^2\phi_\xi(\nabla\phi^2-\phi_\xi^2) + \frac{M^2(\Gamma-2\mu\Gamma-3)}{3}\phi_\xi^3\Big\} \\ &- \frac{(\nu+1)T_\infty}{a_\infty^2}\delta S + \cdots \Big\} ,\end{aligned} \tag{2.27}$$

and since

$$q_n = \frac{U\{1 + \frac{\tau}{\beta}\phi_\xi - \tau\beta\nabla\phi\cdot\nabla\xi_{sh}\}}{\sqrt{1 + \beta^2\nabla\xi_{sh}^2}} , \tag{2.28}$$

we may write (2.22) as

$$\begin{aligned}\frac{\rho_\infty U}{\sqrt{1+\beta^2\nabla\xi_{sh}^2}}\Big[&-\tau\beta\{\phi_\xi + \nabla\phi\cdot\nabla\xi_{sh}\} - M^2\tau^2\Big\{\frac{1}{2}(\nabla\phi^2 - \phi_\xi^2) \\ &- \phi_\xi(\phi_\xi + \nabla\phi\cdot\nabla\xi_{sh}) + \frac{\Gamma M^2}{\beta^2}\phi_\xi^2\Big\} \\ &+ \frac{M^2\tau^3}{\beta}\Big\{(1-\Gamma)M^2\phi_\xi(\nabla\phi^2 - \phi_\xi^2) - \frac{2\Gamma M^4(\mu+1)}{3\beta^2}\phi_\xi^3 \\ &+ (\phi_\xi + \nabla\phi\cdot\nabla\xi_{sh})\Big(\frac{\beta^2}{2}(\nabla\phi^2-\phi_\xi^2) + (\Gamma-1)M^2\phi_\xi^2\Big) - \frac{(\nu+1)T_\infty}{a_\infty^2}\delta S\Big] = 0 .\end{aligned} \tag{2.29}$$

The term involving the entropy jump may be eliminated using (2.23) to give

$$\begin{aligned}\Big[\phi_\xi &+ \nabla\phi\cdot\nabla\xi_{sh} + \frac{M^2\tau}{\beta}\Big\{\frac{1}{2}(\nabla\phi^2 - \phi_\xi^2) - \phi_\xi(\phi_\xi + \nabla\phi\cdot\nabla\xi_{sh}) + \frac{\Gamma M^2}{\beta^2}\phi_\xi^2\Big\} \\ &+ \frac{M^2\tau^2}{\beta^2}\Big\{(\Gamma-1)M^2\phi_\xi(\nabla\phi^2 - \phi_\xi^2) - \Big((\Gamma-1)M^2\phi_\xi^2 + \frac{\beta^2}{2}(\nabla\phi^2 - \phi_\xi^2)\Big)\cdot \\ &\cdot(\phi_\xi + \nabla\phi\cdot\nabla\xi_{sh}) + \frac{2\Gamma M^4}{3\beta^2}(\mu+1)\phi_\xi^3\Big\}\Big] \\ &- \frac{(\nu+1)\Gamma M^3\tau^2}{6\beta^4(1+\beta^2\nabla\xi_{sh}^2)^{3/2}}\Big[\phi_\xi - \beta^2\nabla\phi\cdot\nabla\xi_{sh}\Big]^3 = 0 .\end{aligned} \tag{2.30}$$

This last equation and that expressing continuity of the potential across the discontinuity, (2.21), will be used to determine the locations of any shocks in the flow field, and the jumps in perturbation quantities across them.

2.C. Boundary and Initial Conditions

The appropriate boundary condition for inviscid flows is that the velocity normal to the surface at every point on a physical body vanish. In addition, we require the initial condition that all perturbations vanish upstream of the body. (In linearized theory, this initial condition is applied on the leading Mach cone; we shall apply it immediately upstream of the leading shock surfaces, which are slightly ahead of the leading Mach cone.) These conditions form a well posed problem for the wave-type equations with which we shall be dealing.

For planar flows, the wave structure solution is uniformly valid to the body surface, and the boundary condition may be applied directly. The mean surface approximation is used, and the conditions are applied along the line $\bar{\eta} = 0$.

For flows about finite bodies, the boundary condition is used to determine a local solution, which is valid in the vicinity of the body. The large-distance asymptotic behavior of this solution is then matched with the small-distance asymptotic behavior of the wave structure solution to determine the potential in this outer region. Matching of the potential and not merely its radial derivative (which might be though to be a sufficient condition) is required to completely determine the positions of any shocks in the flow.

2.D. General nature of scaling concepts

The appropriate scaling of variables in the wave structure region, i.e., that region in which cumulative nonlinearities must be taken into account, may be seen in several ways. For simplicity of exposition, we will illustrate the arguments only for the case of planar flows. Similar arguments may be followed for more general flows, but the essential ideas are more easily lost in algebraic complexity. A discussion of the scaling problem for supersonic airfoil theory is also included in Van Dyke's book (1965), to which the following paragraphs owe.

The most physical argument is one based upon the hyperbolic nature of the equation describing the flow, and the failure of successive approximations to correct the characteristics to more closely approximate those of the full nonlinear equation. Since the straightforward perturbation procedure attempts to correct the solution by expanding it in a Taylor series about the linearized characteristics, the procedure will become inaccurate when the distance between the actual characteristic and the linearized characteristic starting from the same point on the body becomes of order unity, for then the first correction is of the same order as the linear approximation. Since for the plane flow over a slender obstacle of thickness of $O(\tau)$ the perturbation in signal velocity (or characteristic velocity) is of $O(\tau)$ and constant along the characteristics in the linear solution, the slope of the exact characteristic is everywhere of $O(\tau)$ different from that of the linearized one passing through the same point. Thus, if we follow the linearized characteristic a distance of $O(1/\tau)$ away from the body, we will arrive at a point in the flow that is of $O(1)$ distant from the point we would have found by following the exact characteristic. This suggests that if we rescale the lateral variable -- i.e., the one which measures distance along the characteristic -- such that it is of $O(1)$ when the physical variable is of $O(1/\tau)$, the equation should rearrange itself to provide an accurate description of this long range, cumulative nonlinearity.

A second argument (which, of course, arrives at the same result) is more formal in nature, and is seen by observing the nature of the nonuniformity in the straightforward perturbation solution. For the planar flow over a bump described by

$$y_b = \tau c f(x/c) \quad , \tag{2.31}$$

the straightforward expansion gives for the perturbation potential

$$\phi = \phi_0 + \frac{M^2\tau}{\beta^3}\phi_1 + \cdots \quad , \tag{2.32}$$

the solutions (Van Dyke (1952)):

$$\phi_0 = -f(x-\beta y) \quad , \tag{2.33}$$

$$\phi_1 = -\frac{1}{M^2} f(x-\beta y) f' + \left(\frac{\Gamma M^2}{2\beta^2} - 1\right)\int_0^{x-\beta y} \{f'(t)\}^2 dt + \frac{\Gamma M^2}{2\beta^2} f'^2 \cdot \beta y/c \quad . \tag{2.34}$$

Now, the solution (2.32) is a uniformly valid asymptotic expansion as $\tau \to 0$ if $\frac{M^2\tau}{\beta}\phi_1$ grows no faster than ϕ_0 in any region. However, due to the presence of the third term in the solution (2.34) for ϕ_1 , the contribution from the second-order term will be of the same order as that from the first-order term when $y \sim 1/\tau$. This, too, suggests that in the region of $O(1/\tau)$ distant from the body, a new scaling of variables is required.

Chapter Three

The Theory of Planar Flows

3.A. Equations and Boundary Conditions

For planar flows, we assume the flow properties to be independent of $\bar{\zeta}$, and introduce the contracted, semi-characteristic coordinates

$$\xi = \bar{\xi} - \bar{\eta}, \qquad \eta = K\bar{\eta}, \tag{3.01}$$

where

$$K = \frac{\Gamma M_T^4}{\beta^3} \tag{3.02}$$

is the fundamental lateral scale factor which forms the basis for Hayes' (1954) first-order similitude. The new variables, ξ and η, correspond essentially to a phase and an age variable, respectively, in a geometric analysis of the analogous one-dimensional unsteady problem. (Or, they may be thought of as the short- and long-time variables in a multiple time scale analysis.)

The equation for the perturbation potential (2.19) then becomes

$$\phi_{\xi\eta} + \phi_\xi \phi_{\xi\xi} = \frac{M_T^2}{\beta}\left\{\frac{\Gamma M_T^2}{\beta^2}\phi_{\eta\eta} + 2(\phi_\eta\phi_{\xi\xi} + \phi_\xi\phi_{\xi\eta}) + 4(\Gamma-1)\phi_\xi\phi_{\xi\eta} + (\sigma-1)\phi_\xi^2\phi_{\xi\xi}\right\} + \cdots. \tag{3.03}$$

We introduce the expansion for the potential

$$\phi = \phi_0 + \frac{M_T^2}{\beta}\phi_1 + \cdots, \tag{3.04}$$

whence we arrive at the hierarchy of equations

$$\phi_{0_{\xi\eta}} + \phi_{0_\xi}\phi_{0_{\xi\xi}} = 0, \tag{3.05}$$

$$\phi_{1_{\xi\eta}} + (\phi_{0_\xi}\phi_{1_\xi})_\xi = \frac{\Gamma M^2}{\beta^2}\phi_{0_{\eta\eta}} + 2(\phi_{0_\eta}\phi_{0_{\xi\xi}} + \phi_{0_\xi}\phi_{0_{\xi\eta}}) + 4(\Gamma-1)\phi_{0_\xi}\phi_{0_{\xi\eta}} + (\sigma-1)\phi_{0_\xi}^2\phi_{0_{\xi\xi}}. \tag{3.06}$$

We note that the first-order equation (for ϕ_0) is nonlinear. In addition to the first term (which is essentially the wave equation in semi-characteristic coordinates for waves of primarily one family), there is now a term to allow correction of the characteristics. This latter term is, of course, absent in the linear theory. The second-order equation is again linear (as would be the case for all higher approximations) with nonhomogeneous terms which are known functions of the first-order solution. The extreme simplicity of the first-order equation is a major factor in our ability to obtain useful results.

We expand the shock shape in a manner similar to that for the potential,

$$\Xi_{sh} = \Xi_0(\eta) + \frac{M_T^2}{\beta}\Xi_1(\eta) + \cdots , \tag{3.07}$$

and the shock relations (2.30), (2.21) become

$$[\phi_{0\eta} + \phi_{0\xi}^2 - \Xi_{0\eta}\phi_{0\xi}] = 0 , \tag{3.08}$$

$$[\phi_{0\eta} + \Xi_{0\eta}\phi_{0\xi}] = 0 ; \tag{3.09}$$

and

$$\begin{aligned}[\phi_{1\eta} &+ 2\phi_{0\xi}\phi_{1\xi} - \xi_{0\eta}\phi_{0\xi} + \Xi_1(\phi_{0\xi\eta} + 2\phi_{0\xi}\phi_{0\xi\xi} - \Xi_{0\eta}\phi_{0\xi\xi}) \\ &+ \xi_{0\eta}\phi_{0\xi}^2 - 2\phi_{0\xi}\phi_{0\eta} + \frac{\Gamma M^2}{\beta^2}\xi_{0\eta}\phi_{0\eta} + \frac{2(\mu+1)}{3}\phi_{0\xi}^3] - \frac{\nu+1}{6}[\phi_{0\xi}]^3 = 0,\end{aligned} \tag{3.10}$$

$$[\phi_{1\eta} + \xi_{0\eta}\phi_{1\xi} + \xi_{1\eta}\phi_{0\xi} + \Xi_1(\phi_{0\xi\eta} + \Xi_{0\eta}\phi_{0\xi\xi})] = 0 ; \tag{3.11}$$

where (2.21) has been differentiated along the shock surface to produce (3.09) and (3.11), and all quantities are evaluated at the surface Ξ_0 , in analogy with the mean surface approximation used for the boundary conditions. The differentiation of (2.21) has the effect of introducing an arbitrary constant into the expression for the shock location. This constant is eliminated in the first-order result by starting the shock at the first point of characteristic overlap. In the second-order result, (3.11) is used only to simplify (3.10), and the nondifferentiated form of (2.21) (eqn. (3.18)) is used to locate the shock.

The first-order shock relations (3.08) and (3.09) may be combined to give

$$\Xi_{o\eta} = \frac{1}{2}\frac{[\phi_{o\xi}^2]}{[\phi_{o\xi}]} = \overline{\phi_{o\xi}} \;, \tag{3.12}$$

and

$$[\phi_{o\eta} + \tfrac{1}{2}\phi_{o\xi}^2] = 0 \,. \tag{3.13}$$

The first result states the well known fact that (to first order) the shock velocity is the mean of the characteristic velocities on either side of the discontinuity, and the latter may be interpreted as stating that the Riemann invariant of right-running (returning) waves is unchanged in crossing the shock. (See below). This latter result, combined with the first integral of the first-order equation (3.05),

$$\phi_{o\eta} + \tfrac{1}{2}\phi_{o\xi}^2 = fn(\eta) \quad , \tag{3.14}$$

and the fact that all perturbations must vanish everywhere upstream, gives the relation

$$\phi_{o\eta} + \tfrac{1}{2}\phi_{o\xi}^2 = 0 \;, \tag{3.15}$$

which will prove most useful in simplifying the second-order problem.

The second-order equation (3.06), with the use of (3.15) and (3.05) may be simplified to

$$\phi_{1\xi\eta} + (\phi_{o\xi}\phi_{1\xi})_\xi = \left(-\tfrac{7}{2} + \frac{\Gamma M^2}{2\beta^2} - \mu\right)\phi_{o\xi}^2\,\phi_{o\xi\xi} \;. \tag{3.16}$$

The second-order shock relations (3.10) and (3.11) may be combined to give

$$\begin{aligned}[\phi_{1\eta} + \phi_{o\xi}\phi_{1\xi}] = \left(-\tfrac{7}{6} + \frac{\Gamma M^2}{6\beta^2} - \tfrac{\mu}{3}\right)[\phi_{o\xi}^3] \\ + \left(\tfrac{1}{6} - \frac{\Gamma M^2}{24\beta^2} + \tfrac{7}{12}\right)[\phi_{o\xi}]^3 \;,\end{aligned} \tag{3.17}$$

where the result

$$[\phi_1 + \Xi_1\phi_{o\xi}] = 0 \quad , \tag{3.18}$$

obtainable directly from (2.21), and the identity

$$[E_{o\eta}\phi_{o\xi}^2] = \frac{2}{3}[\phi_{o\xi}^3] - \frac{1}{6}[\phi_{o\xi}]^3 \tag{3.19}$$

have been used, as well as the simplifications of (3.05) and (3.15). The identity (3.19) may be derived from (3.09) by multiplying within the brackets by $\phi_{o\xi}$, and using the fact that

$$[a \cdot b] = [a]\bar{b} + \bar{a}[b] \quad , \tag{3.20}$$

where the bar denotes taking of the mean of the quantity on either side of the discontinuity. Equation (3.17) essentially gives the change in the Riemann invariant for the crossing waves, and (3.18) will be used to determine E_1 .

The wave structure equations presented here correspond to an orderly expansion of the characteristic relations for steady, inviscid flow. These equations state that

$$\frac{\sqrt{M^2-1}}{\rho q^2}\,dp \pm d\theta = 0 \tag{3.21}$$

along

$$\frac{dy}{dx} = \tan\left(\theta \pm \sin^{-1} 1/M\right) \quad , \tag{3.22}$$

respectively. In the above form, the relations are valid for a rotational flow. When expressed in terms of our wave structure variables, the characteristic equations become

$$d\left\{-2\tau\phi_\xi + \frac{M_T^2\tau^2}{\beta}\left\{\left(\frac{2}{M^2} - \frac{\Gamma M^2}{2\beta^2}\right)\phi_\xi^2 + \frac{\Gamma M^2}{\beta^2}\phi_\eta\right\} + \cdots\right\} = 0 \tag{3.23}$$

along

$$\frac{dE}{d\eta} = \phi_\xi - \frac{M_T^2}{\beta}\left\{\left(-3 + \frac{\Gamma M^2}{2\beta^2} - \mu\right)\phi_\xi^2 + \phi_\eta\right\} + \cdots \quad , \tag{3.24}$$

and

$$d\left\{-K\tau\left(\phi_\eta + \tfrac{1}{2}\phi_\xi^2\right) - K\frac{M_T^2\tau^2}{\beta}\left\{\left(\frac{2\mu+7}{6} - \frac{\Gamma M^2}{6\beta^2} - \frac{1}{2M^2}\right)\phi_\xi^3 - \frac{1}{M^2}\phi_\xi\phi_\eta\right\} - \frac{\beta T_\infty}{U^2}\delta S + \cdots\right\} = 0 \tag{3.25}$$

along

$$\frac{d\Xi}{d\eta} = \frac{1}{K}\left\{-2 + \frac{M_T^2}{\beta}\left(2 - \frac{\Gamma M^2}{\beta^2}\right)\phi_\xi + \cdots\right\} . \tag{3.26}$$

Thus, the shock relations (3.13) and (3.17) may be interpreted as giving the change in the Riemann invariant of the crossing system of characteristics. This change is zero to first order, but is non-zero in the second approximation, due to the inability of the shock to match perfectly with the conditions in a simple wave. This locally third-order effect, when followed to large distances from the body, contributes a cumulative effect of the second order. From (3.17) and (3.25), the change in the invariant, Q, for the crossing wave is

$$[Q] = \left(\frac{1}{6M^2} - \frac{\Gamma M^2}{24\beta^2} + \frac{1}{12}\right)[\phi_{0_\xi}]^3 . \tag{3.27}$$

This quantity will play an important role in the determination of the location of the rear shock at very large distances from the airfoil. (See Appendix B).

Since the flow under consideration is two-dimensional, the wave system is nearly planar not only in the wave structure region, but near the body surface as well. This fact is manifested mathematically in the uniform validity of the wave structure solution in the region of the body surface, with the result that the boundary conditions may be applied directly to the wave structure solution, without the necessity of introducing an inner (or local) solution.

We consider the shape of the body surface to be described by

$$y_b = \tau c f(\bar{\xi}) , \tag{3.28}$$

where $f(\bar{\xi})$ is the body shape function, and the subscript $(\ )_b$ refers to the body surface. Then the appropriate boundary condition is that the streamline slope,

$$\tan\theta = -\tau\phi_\xi + \frac{M_T^2}{\beta}\left(\frac{1}{M^2}\phi_\xi^2 + \frac{\Gamma M^2}{\beta^2}\phi_\eta\right) + \cdots , \tag{3.29}$$

be tangent to the surface at every point on the body.

Since we are considering bodies which lie everywhere near the mean plane $\eta = 0$, this condition may be satisfied at the surface by an appropriate Taylor expansion about values on that plane. (This is the so-called "mean-surface approximation.") Thus, substitution of (3.04) gives

$$\phi_{0_\xi} = -f'(\Xi) \ , \tag{3.30}$$

$$\phi_{1_\xi} = f\phi_{0_{\xi\xi}} - \frac{1}{M^2}\left(f\phi_{0_{\xi\xi}} + f'\phi_{0_\xi}\right) - \frac{\Gamma M^2}{2\beta^2}\phi_{0_\xi}^2 \ , \tag{3.31}$$

as the required boundary conditions, to be applied at $\eta = 0$.

To summarize, we here collect the equations, shock relations, and boundary conditions in their final form for the first-order problem,

$$\phi_{0_{\xi\eta}} + \phi_{0_\xi}\phi_{0_{\xi\xi}} = 0 \ , \tag{3.05}$$

$$\left[\phi_{0_\eta} + \tfrac{1}{2}\phi_{0_\xi}^2\right] = 0 \ , \tag{3.13}$$

$$\Xi_{0_\eta} = \overline{\phi_{0_\xi}} \ , \tag{3.12}$$

$$\phi_{0_\xi} = -f'(\xi) \ , \quad \text{on} \quad \eta = 0 \ ; \tag{3.30}$$

and for the second-order problem,

$$\phi_{1_{\xi\eta}} + \left(\phi_{0_\xi}\phi_{1_\xi}\right)_\xi = \left(-\tfrac{7}{2} + \frac{\Gamma M^2}{2\beta^2} - \mu\right)\phi_{0_\xi}^2\phi_{0_{\xi\xi}} \ , \tag{3.16}$$

$$\left[\phi_{1_\eta} + \phi_{0_\xi}\phi_{1_\xi}\right] = \left(-\frac{7}{6} + \frac{\Gamma M^2}{6\beta^2} - \frac{\mu}{3}\right)\left[\phi_{0_\xi}^3\right] + \left(\frac{1}{6} - \frac{\Gamma M^2}{24\beta^2} + \frac{\nu}{12}\right)\left[\phi_{0_\xi}\right]^3 , \tag{3.17}$$

$$\Xi_1 = -[\phi_1]/[\phi_{0_\xi}] \ , \tag{3.18}$$

$$\phi_{1_\xi} = f\phi_{0_{\xi\xi}} - \frac{1}{M^2}\left(f\phi_{0_{\xi\xi}} + f'\phi_{0_\xi}\right) - \frac{\Gamma M^2}{2\beta^2}\phi_{0_\xi}^2 \ , \text{ on } \eta = 0 \ . \tag{3.31}$$

3.B. General solution

The first-order equation (3.05) has characteristics described by

$$\left.\frac{d\xi}{d\eta}\right)_{ch} = \phi_{o_\xi} \quad , \tag{3.32}$$

along which ϕ_{o_ξ} is constant. Thus, the characteristics are straight lines in the ξ-η plane. The value of ϕ_{o_ξ} on a particular characteristic is determined from the value of f' at the value of ξ corresponding to $\eta = 0$. We denote this value of ξ as ξ_b. Thus the solution may be written

$$\phi_{o_\xi} = -f'(\xi_b) \quad , \tag{3.33}$$

where

$$\xi_b - \xi - \eta f'(\xi_b) = 0 \tag{3.34}$$

determines ξ_b implicitly. This is equivalent to Whitham's result, which states that the linear solution is correct even at large distances from the body if the linearized characteristics are replaced by the exact ones (or at least by ones accurate to first order). The above result reduces to the linear result near the body (i.e., for small η), where ξ_b approaches ξ and the linearized and first-order characteristics are indistinguishable.

If we are interested in the actual value of the potential, we have

$$\phi_o = -f(\xi_b) + \frac{1}{2}\eta f'^2(\xi_b) \quad , \tag{3.35}$$

where the relation (3.13) has been used to eliminate the arbitrary function of η arising from the integration. The locations of any shocks which may appear in the flow are then determined by integrating (3.12) in any region where the characteristics overlap in the physical plane, starting at the point where any such overlap first occurs.

The second-order equation has the same characteristics as the first-order equation, and along these the homogeneous part of the second-order potential is constant. Integrating the second-order equation (3.16) once with respect to ξ yields

$$\phi_{1_\eta} + \phi_{o_\xi}\phi_{1_\xi} = \left(-\tfrac{7}{6} + \frac{\Gamma M^2}{3\beta^2} - \mu/3\right)\phi_{o_\xi}^3 + s(\eta) \quad , \tag{3.36}$$

where $s(\eta)$ is an, as yet, arbitrary function, introduced by the integration. Since the velocity perturbations must vanish everywhere ahead of the leading shock, $s(\eta)$ must be identically zero in that region. Indeed, since it is a function only of η, it would have to be zero everywhere were it not for the fact that it can be incremented at shocks. This is seen from the shock relation (3.17), which states

$$[\phi_{1\eta}+\phi_{0\xi}\phi_{1\xi}] = \left(-\frac{7}{6}+\frac{\Gamma M^2}{6\beta^2}-\frac{\mu}{3}\right)[\phi_{0\xi}^3] + \left(\frac{1}{6}-\frac{\Gamma M^2}{24\beta^2}+\frac{\nu}{12}\right)[\phi_{0\xi}]^3. \quad (3.17)$$

Thus, while the function $s(\eta)$ cannot depend upon ξ in any inviscid region, it must be allowed to increase by

$$\left(\frac{1}{6}-\frac{\Gamma M^2}{24\beta^2}+\frac{\nu}{12}\right)[\phi_{0\xi}]^3.$$

at each shock wave in the flow field. The quantity

$$\phi_{1\eta}+\phi_{0\xi}\phi_{1\xi}$$

is related to the Riemann invariant of the wave family crossing the shock, and the amount by which it changes is a measure of the departure of the wave system under consideration from a simple one.

Thus, the equations (3.36) and (3.17), by comparison require

$$\phi_{1\eta}+\phi_{0\xi}\phi_{1\xi} = \left(-\frac{7}{6}+\frac{\Gamma M^2}{6\beta^2}-\frac{\mu}{3}\right)\phi_{0\xi}^3 + \left(\frac{1}{6}-\frac{\Gamma M^2}{24\beta^2}+\frac{\nu}{12}\right)\sum_{\xi_0}[\phi_{0\xi}]^3, \quad (3.37)$$

where the summation is over all shocks upstream of the point of consideration at any value of η.

Now, the first-order solution, $\phi_{0\xi}$, is a function only of ξ_0, whence

$$\phi_{1\eta}(\xi_0,\eta) = \left(\frac{7}{6}-\frac{\Gamma M^2}{6\beta^2}+\frac{\mu}{3}\right)f'^3(\xi_0) + \left(-\frac{1}{6}+\frac{\Gamma M^2}{24\beta^2}-\frac{\nu}{12}\right)\sum_{\xi_0}[f'(\xi_0)]^3, \quad (3.38)$$

and

$$\phi_1(\xi_0,\eta) = \left(\frac{7}{6}-\frac{\Gamma M^2}{6\beta^2}+\frac{\mu}{3}\right)\eta f'^3(\xi_0) + \left(-\frac{1}{6}+\frac{\Gamma M^2}{24\beta^2}-\frac{\nu}{12}\right)\int\sum_{\xi_0}[f']^3 d\eta + h(\xi_0), \quad (3.39)$$

where $h(\xi_0)$ is a function of integration to be determined by the boundary condition.

Differentiating the solution with respect to ξ, and evaluating it at $\eta = 0$, we have

$$\phi_{1_\xi}(\xi_b, 0) = h'(\xi_b) \quad , \tag{3.40}$$

whence the boundary condition (3.31) requires

$$h'(\xi_b) = -ff'' + \frac{1}{M^2}(ff'' + f'^2) - \frac{\Gamma M^2}{2\beta^2} f'^2 , \tag{3.41}$$

and we have, finally,

$$\begin{aligned}\phi_1(\xi_b, \eta) = \left(\frac{7}{6} - \frac{\Gamma M^2}{6\beta^2} + \frac{\mu}{3}\right)\eta f'^3(\xi_b) + \left(-\frac{1}{6} + \frac{\Gamma M^2}{24\beta^2} - \frac{\nu}{12}\right)\int \sum_{\xi_0} [f']^3 d\eta \\ + \int\left(-ff'' + \frac{1}{M^2}(ff'' + f'^2) - \frac{\Gamma M^2}{2\beta^2} f'^2\right) d\xi_b .\end{aligned} \tag{3.42}$$

This gives the complete second-order potential in terms of the body shape function, $f(\xi_b)$, in the first-order characteristic coordinate system (ξ_b, η) which is known from the first-order solution.

The second-order correction to the shapes of shocks in the flow is then obtained by evaluating ϕ_{0_ξ} and ϕ_1 along the first-order curve, and using

$$\xi_1 = -[\phi_1] / [\phi_{0_\xi}] \quad . \tag{3.18}$$

The uniform validity of the solution may now be seen from a comparison of the general solutions for the first- and second-order problems. We thus have for large distances that

$$\phi_{0_\xi} \sim -f'(\xi_b) \quad , \tag{3.43}$$

while

$$\phi_{1_\xi} \sim \left(-\frac{7}{2} + \frac{\Gamma M^2}{\beta^2} - \mu\right) f'^2(\xi_b) \quad . \tag{3.44}$$

Thus the ratio of second-order perturbation quantities to those of first order along any characteristic is

$$\frac{\frac{M_T^2}{\beta}\phi_{1_\xi}}{\phi_{0_\xi}} \sim \frac{M_T^2}{\beta}\left(\frac{7}{2} - \frac{\Gamma M^2}{\beta^2} + \mu\right) f'(\xi_b) \quad , \tag{3.45}$$

which approaches zero uniformly as $\tau \to 0$.

3.C. Sample solution

The general form of the wave structure solution for planar flows is presented in the previous section, and the complete solution may be constructed once the body shape function, $f(\xi_b)$, is known. This has been done in Appendix B for the particularly simple example of a symmetric, circular-arc airfoil at zero angle of attack, and some particular features of the solution will now be presented.

Since the flow will be symmetric about the chord-line of the section, only the flow in the upper half-plane is considered. Thus, we consider the shape function to be given by

$$\begin{aligned} f(\xi_b) &= 0 \qquad , \qquad \xi_b < -1/2 \, , \\ &= \frac{1-4\xi_b^2}{2} \qquad , \qquad -1/2 < \xi_b < 1/2 \, , \\ &= 0 \qquad , \qquad 1/2 < \xi_b \, . \end{aligned} \tag{3.46}$$

The first-order solution, far from the airfoil (i.e., in the limit of large η) is then

$$\phi_{0_\xi} \sim \xi/\eta \tag{3.47}$$

in the region between the shocks described by

$$\xi_0 \sim \mp\sqrt{\eta} \, , \tag{3.48}$$

and is zero elsewhere (i.e., upstream and downstream of the leading and trailing shocks, respectively).

The second-order solution between the shocks behaves, for large η , as

$$\phi_{1_\xi} \sim \left(-\frac{7}{2} + \frac{\Gamma M^2}{2\beta^2} - \mu\right) \xi^2/\eta^2 + \frac{1}{2}\left(1 - 1/M^2\right)\frac{1}{\eta} \, , \tag{3.49}$$

while the shocks are displaced by an amount

$$\xi_{1_L} = \xi_{1_t} \sim -\frac{1}{6}\left(\frac{\Gamma M^2}{\beta^2} + 2\nu\right)\sqrt{\eta} \, . \tag{3.50}$$

The dominant second-order effect at large distances is a displacement of the shock wave which strengthens the leading wave and weakens the trailing wave. These results agree with those of Friedrichs (1948) when our analysis is specialized to the case of a calorically perfect gas, and his expression for the shock slope is corrected as noted by Lighthill (1954).

Behind the trailing shock, the analysis gives as the potential

$$\phi_1 = \left(\frac{4}{3} - \frac{\Gamma M^2}{3\beta^2} + \frac{2\nu}{3}\right)\frac{1}{\sqrt{1+4\eta}} - \left(\frac{\Gamma M^2}{3\beta^2} + \frac{2\nu}{3}\right), \qquad (3.51)$$

which is a function of η alone, whence the pressure perturbations are of third order in this region. But the form of (3.51) suggests that there is a right-running wave that is locally of third order, but has a cumulative effect of the second order, and hence must be included in the analysis. This effect was first noted by Lighthill (1954), and must be accounted for if certain global integrals relating to lift and drag are to be correct to second order (See Appendix C). This wave eventually interacts with the trailing shock below the airfoil (or is "reflected" and interacts with the trailing shock above the airfoil in our symmetric case), and alters its position to second order. This interaction takes place only very far from the body, and we introduce for this "Lighthill region" the new scaling

$$\tilde{\xi} = K\xi, \qquad \tilde{\eta} = K^2\eta. \qquad (3.52)$$

Since entropy and pressure perturbations are of the same order behind the shocks near the airfoil, the strength of this wave must be determined using the roational method of characteristics. In the region of interaction with the trailing shock (3.52), however, the entropy perturbations are again negligible, and the potential behind the trailing shock is

$$\phi_1 = \left(\frac{4}{3M^2} - \frac{\Gamma M^2}{3\beta^2} + \frac{2\nu}{3}\right)\frac{1}{\sqrt{1+2\tilde{\xi}}} - \frac{4}{3}\left(\frac{1}{M^2} + \nu\right), \qquad (3.53)$$

and the second-order shock displacement is

$$\tilde{\xi}_1 = \sqrt{\tilde{\eta}}\left\{\left(\frac{4}{3M^2} - \frac{\Gamma M^2}{3\beta^2} + \frac{2\nu}{3}\right)\frac{1}{\sqrt{1+2\tilde{\eta}^{1/2}}} - \frac{4}{3M^2} + \frac{\Gamma M^2}{6\beta^2} - \nu\right\}, \qquad (3.54)$$

which at very, very great distances from the airfoil (i.e., for large $\tilde{\eta}$), approaches

$$\tilde{\xi}_1 \sim \sqrt{\tilde{\eta}}\left(-\frac{4}{3M^2}+\frac{\Gamma M^2}{6\beta^2}-\nu\right) . \tag{3.55}$$

This result differs slightly from that of Lighthill (1954) for the position of the trailing shock, which is in error. (See Appendix B).

3.D. Structure of Shock Origin

Since the characteristics are not revised in our second-order approximation, we cannot hope for the theory to give detailed information about the wave structure near the origin of a shock within the flow field. The theory does approximate such local behavior in a global sense, however, and gives a second-order displacement of the point of formation which corresponds to the axial displacement of the continuation of the shock shape valid away from the point of formation, to the same lateral position as the first-order shock was first formed. This is illustrated in Appendix D, where the geometry of the true characteristics is investigated in the neighborhood of such a point.

A shock is formed in the flow field when characteristics coming from different points on the boundary data first cross. The shock then forms a barrier which the characteristics are not allowed to cross, thus rendering the solution unique in regions where the overlapping characteristics would otherwise have caused the solution to be multivalued. The case examined in Appendix D is that of the shock formed in the planar flow over a step formed by first a compressive, then an expansive, arc of a parabola. The geometry and first-order characteristics are shown in Figure 3.01. The first-order theory predicts a shock to be formed with finite strength at $(0, \frac{1}{2})$, upstream of which the flow is undisturbed, and downstream of which the solution from points on the body for which $\xi_b > 1$ is valid. The second-order theory then predicts the shock to originate at the slightly displaced point,

$$\xi = \frac{M_T^2}{\beta}\xi_1(\tfrac{1}{2}) = \frac{M_T^2}{\beta}\left(2+\frac{1}{M^2}-\frac{2\Gamma M^2}{3\beta^2}+\frac{2\mu}{3}\right), \qquad \eta = \tfrac{1}{2} . \tag{3.56}$$

To interpret this in terms of local structure, we must look at the geometry of the actual characteristics in the region near the point of formation. When second-order effects are included in the characteristic relations, the characteristics coming from the forward part of the body $0 < \xi_b < 1$ form an envelope described parametrically by

$$\xi = -\frac{M_T^2}{\beta}\left(6+\frac{3}{M^2}-\frac{2\Gamma M^2}{\beta^2}+2\mu\right)\xi_b^2 , \tag{3.57}$$
$$\eta = \tfrac{1}{2} + \frac{M_T^2}{\beta}\left(6+\frac{3}{M^2}-\frac{2\Gamma M^2}{\beta^2}+2\mu\right)\xi_b .$$

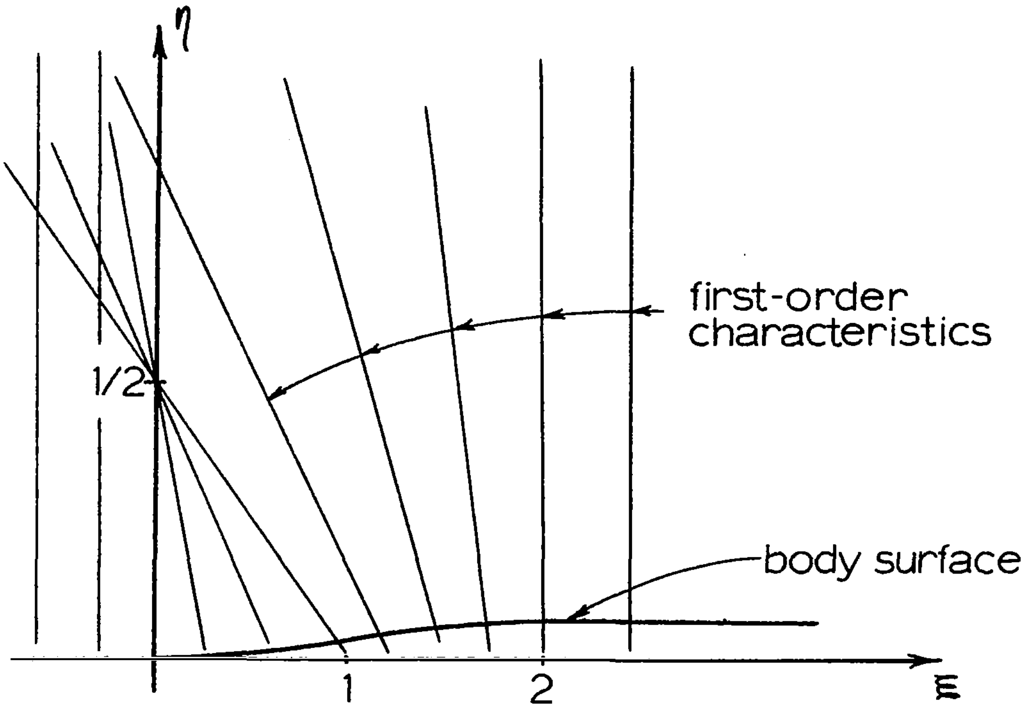

Figure 3.01: Geometry of Shock Origin Problem.

When the coefficient,

$$\varkappa = 6 + \frac{3}{M^2} - \frac{2\Gamma M^2}{\beta^2} + 2\mu \ , \qquad (3.58)$$

is positive, the envelope appears as in Figure 3.02, and a shock is formed to intercept the characteristics before the envelope is formed. The shock begins with zero strength and the flow is undisturbed ahead of the shock. Beyond the point corresponding to $\xi_b = 1$, the shock is determined by the appropriate relations between the free-stream characteristics and those coming from the expansive part of the body ($\xi_b > 1$), as in the regular second-order theory. If this slope is continued back to $\eta = 1/2$, the shock in virtual origin is exactly

$$\xi_{1_v} = \frac{1}{3}\varkappa = \xi_1 \ , \qquad (3.59)$$

whence, the second-order theory is correct in this global sense.

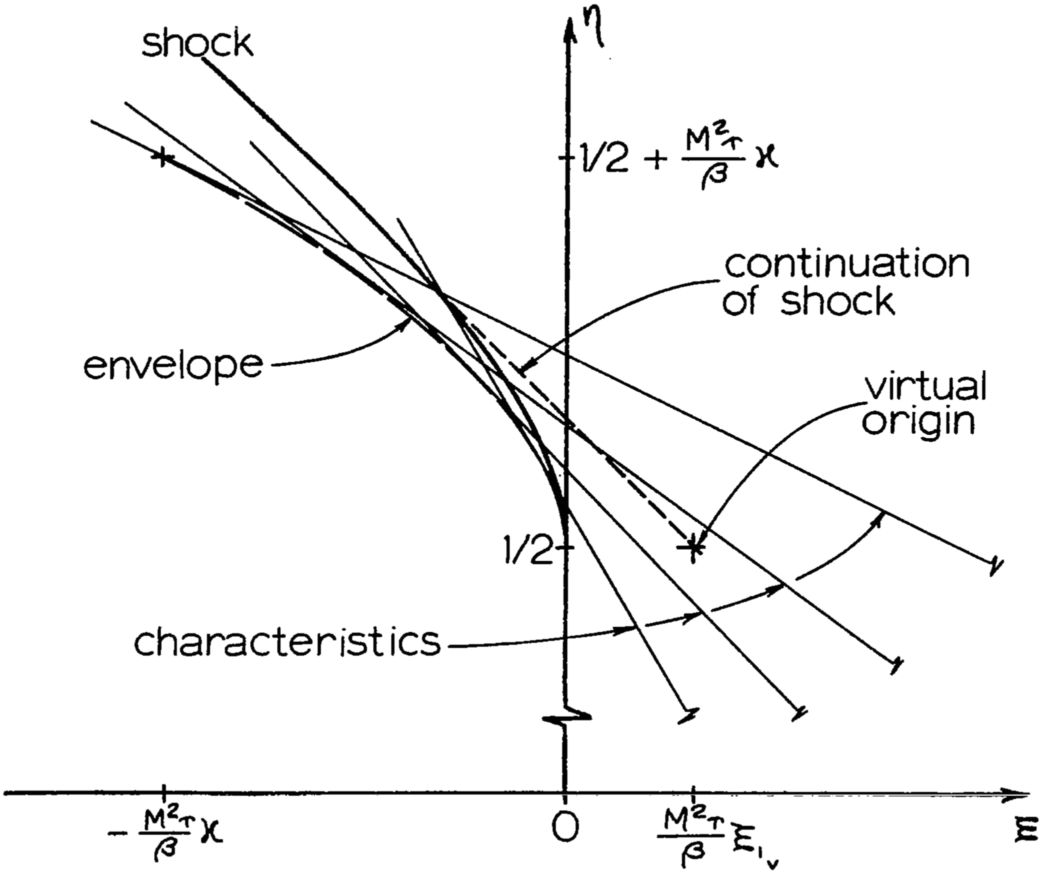

Figure 3.02: Structure of Shock Origin ($\chi > 0$).

When the coefficient, χ , is negative, the envelope appears as in Figure 3.03. The shock begins with zero strength and its early development occurs with disturbed flow on both sides. Since the shock is now formed for $\eta < \frac{1}{2}$, the virtual origin at $\eta = \frac{1}{2}$ is merely the value of Ξ_{sh} at that point. It is

$$\Xi_{1_v} = \frac{1}{3}\chi = \Xi_1 \quad , \tag{3.60}$$

again showing the second-order theory to be globally correct.

When the coefficint, χ , is zero, the shock again, as in the first-order theory, starts at the point ($0, \frac{1}{2}$), and the predicted second-order displacement is zero.

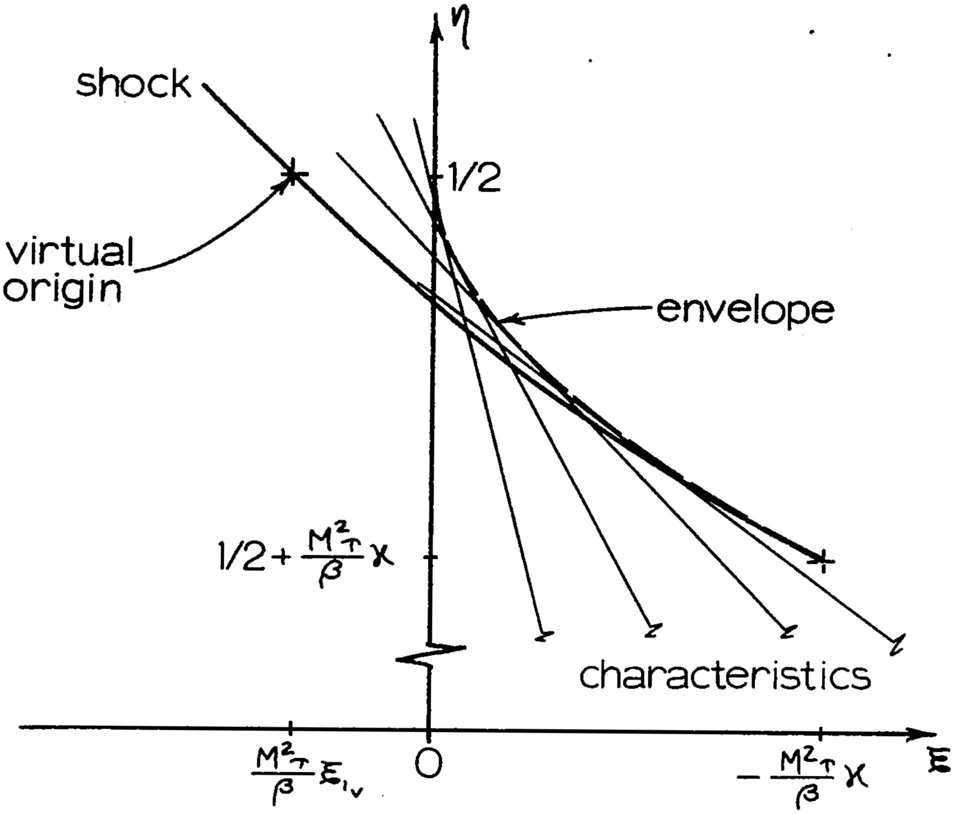

Figure 3.03: Structure of Shock Origin ($\chi < 0$).

Chapter Four

The Theory for Finite Systems

4.A. Introduction

The study of planar flows, while interesting pedagogically, is of little practical value, since any body we are likely to conceive for supersonic flight will most certainly not look two-dimensional from the large distances with which we are primarily concerned. Thus, it is important that we should consider the problem of wave structure for flows about bodies which are of finite extent (and, perhaps, look somewhat like real airplanes).

The easiest problems of this type to analyze are those of axially symmetric flows, since these may still be described with only two independent variables. The consideration of axially symmetric flows is important for another reason, however, and this is because they hold the key to description of the wave structure for flows about more general bodies. This is suggested by the result of Hayes (1947) that, for finite planar systems, the perturbations in any azimuthal plane look, to first order, like those from some equivalent body of revolution, if the observer is sufficiently distant from the system. (This is the so-called supersonic "area rule", first used for calculating wave drags). A similar result may be obtained for bodies whose every lateral dimension is small (slender bodies), with the additional simplification that the "equivalent" body of revolution is the same in all azimuthal planes. Thus, since a general body may be thought to be made up of suitably arranged slender and planar bodies, the result holds for quite general finite bodies.

Now these results are based upon the large-distance asymptotic behavior of the local (straightforward perturbation) solution -- and this is exactly what is used to provide the inner boundary condition on the wave structure solution. Thus, the first-order wave structure depends upon the azimuthal angle only as a parameter, which enters by matching with the local solution in the appropriate plane (Hayes (1954)). We shall refer to this property as quasi-axisymmetry.

This property suggests that the appropriate coordinate system in which to study the wave systems for finite bodies is a cylindrical one, aligned with the freestream. Thus, the equations for the wave structure are expressed in a cylindrical coordinate system, with the lateral (or radial) coordinate properly contracted to allow consideration of cumulative nonlinearities. The inner boundary condition is then provided by asymptotic matching with the local solution.

We are hampered in this sense by a lack of local second-order solutions of any generality. Only for flows over bodies of revolution does a reasonably complete second-order theory exist. Thus, we shall consider in detail the matching for these flows, and then briefly discuss the nature of problems encountered for flows about non-axisymmetric and planar bodies.

Finally, it will be seen that the local second-order solution determines only a one-and-one-half-order solution in the wave structure region. The third-order local solution is required to determine the second-order wave structure solution. Third-order local solutions can be obtained only for the simplest conical geometries at the present time.

4.B. Equations and Shock Relations

The quasi-axisymmetry of the first-order wave structure suggests the coordinate system

$$\begin{aligned}\xi &= \bar{\xi} - \pi , \qquad \pi = \sqrt{\bar{\eta}^2 + \bar{\zeta}^2} ,\\ \eta &= 2K\sqrt{\pi} ,\\ \theta &= \tan^{-1}(\bar{\zeta}/\bar{\eta}) ,\end{aligned} \tag{4.01}$$

for description of the wave systems emanating from finite systems. It is essentially a cylindrical coordinate system, aligned with the freestream, with the radial coordinate contracted to allow consideration of cumulative nonlinear effects. We introduce the reduced potential

$$\psi = \sqrt{\pi}\,\phi , \tag{4.02}$$

which automatically accounts for the geometric attenuation of cylindrical waves, and is of order unity throughout the wave structure region.

Thus, the potential equation becomes

$$\begin{aligned}\psi_{\xi\eta} + \psi_\xi \psi_{\xi\xi} = K\frac{M_T^2}{\beta}\frac{1}{\eta}\Big\{ &\frac{\Gamma M^2}{\beta^2}\psi_{\eta\eta} + 2(\psi_\xi\psi_{\xi\eta} + \psi_\eta\psi_{\xi\xi}) \\ &+ 4(\Gamma-1)\psi_\xi\psi_{\xi\eta} + (\sigma-1)\psi_\xi^2\psi_{\xi\xi} \\ &- \frac{2}{\eta}(\psi\psi_{\xi\xi} + \psi_\xi^2) + \frac{\Gamma M^2}{\beta^2}\frac{1}{\eta^2}(\psi + 4\psi_{\theta\theta})\Big\} + \cdots .\end{aligned} \tag{4.03}$$

We introduce the expansion for the potential

$$\psi = \psi_0 + \frac{M_T^2}{\beta}\psi_{1/2} + K\frac{M_T^2}{\beta}\psi_1 + \cdots , \tag{4.04}$$

(where the $\psi_{1/2}$ term is included to satisfy the matching requirement with the local second-order solution) and arrive at the set of equations

$$\Psi_{0_{\xi\eta}} + \Psi_{0_\xi}\Psi_{0_{\xi\xi}} = 0, \tag{4.05}$$

$$\Psi_{\frac{1}{2}_{\xi\eta}} + (\Psi_{0_\xi}\Psi_{\frac{1}{2}_\xi})_\xi = 0, \tag{4.06}$$

$$\begin{aligned}\Psi_{1_{\xi\eta}} + (\Psi_{0_\xi}\Psi_{1_\xi})_\xi = \frac{1}{\eta}\Big\{&\frac{\Gamma M^2}{\beta^2}\Psi_{0_{\eta\eta}} + 2(\Psi_{0_\xi}\Psi_{0_{\xi\eta}} + \Psi_{0_\eta}\Psi_{0_{\xi\xi}}) \\ &+ 4(\Gamma-1)\Psi_{0_\xi}\Psi_{0_{\xi\eta}} + (\sigma-1)\Psi_{0_\xi}^2\Psi_{0_{\xi\xi}} \\ &- \frac{2}{\eta}(\Psi_0\Psi_{0_{\xi\xi}} + \Psi_{0_\xi}^2) + \frac{\Gamma M^2}{\beta^2}\frac{1}{\eta^2}(\Psi_0 + 4\Psi_{0_{\theta\theta}})\Big\} \\ &- \frac{\beta^2}{\Gamma M^2}\Psi_{\frac{1}{2}_\xi}\Psi_{\frac{1}{2}_{\xi\xi}}.\end{aligned} \tag{4.07}$$

Similarly, the shock relations (2.30) and (2.21), when transformed as above, with the introduction of

$$\Xi_{sh}(\eta,\theta) = \Xi_0(\eta,\theta) + \frac{M_T^2}{\beta}\Xi_{\frac{1}{2}}(\eta,\theta) + K\frac{M_T^2}{\beta}\Xi_1(\eta,\theta) + \cdots, \tag{4.08}$$

become

$$[\Psi_{0_\eta} + \Psi_{0_\xi}^2 - \Xi_{0_\eta}\Psi_{0_\xi}] = 0, \tag{4.09}$$

$$[\Psi_{0_\eta} + \Xi_{0_\eta}\Psi_{0_\xi}] = 0; \tag{4.10}$$

and

$$\begin{aligned}[-\frac{1}{\eta}\Psi_{\frac{1}{2}} + \Psi_{\frac{1}{2}_\eta} + 2\Psi_{0_\xi}\Psi_{\frac{1}{2}_\xi} - \Xi_{0_\eta}\Psi_{\frac{1}{2}_\xi} - \Xi_{\frac{1}{2}_\eta}\Psi_{0_\xi} \\ - \Xi_{\frac{1}{2}}(\Xi_{0_\eta}\Psi_{0_{\xi\xi}} - \Psi_{0_{\xi\eta}} - 2\Psi_{0_\xi}\Psi_{0_{\xi\xi}} + \frac{1}{\eta}\Psi_{0_\xi})] = 0,\end{aligned} \tag{4.11}$$

$$\begin{aligned}[-\frac{1}{\eta}\Psi_{\frac{1}{2}} - \frac{1}{\eta}\Xi_{\frac{1}{2}}\Psi_{0_\xi} + \Psi_{\frac{1}{2}_\eta} + \Xi_{0_\eta}\Psi_{\frac{1}{2}_\xi} \\ + \Xi_{\frac{1}{2}_\eta}\Psi_{0_\xi} + \Xi_{\frac{1}{2}}(\Psi_{0_{\xi\eta}} + \Xi_{0_\eta}\Psi_{0_{\xi\xi}})] = 0;\end{aligned} \tag{4.12}$$

and

$$
\begin{aligned}
&\Big[-\frac{1}{\eta}\psi_1 + \psi_{1\eta} + 2\psi_{0\xi}\psi_{1\xi} - \Xi_{0\eta}\psi_{1\xi} - \xi_{1\eta}\psi_{0\xi} - \xi_1\big(\xi_{0\eta}\psi_{0\xi\xi} \\
&\quad - \psi_{0\xi\eta} + \frac{1}{\eta}\psi_{0\xi} - 2\psi_{0\xi}\psi_{0\xi\xi}\big) - \frac{\beta^2}{\Gamma M^2}\xi_{1/2}\big(\xi_{0\eta}\psi_{1/2\,\xi\xi} + \xi_{1/2}\psi_{0\xi\xi} \\
&\quad - \psi_{1/2\,\xi\eta} + \frac{1}{\eta}\psi_{1/2\,\xi} - 2\psi_{1/2\,\xi}\psi_{0\xi\xi} - 2\psi_{0\xi}\psi_{1/2\,\xi\xi}\big) - \frac{\beta^2}{\Gamma M^2}\Xi_{1/2}^2\Big(\frac{1}{2\eta}\psi_{0\xi\xi} \\
&\quad - \frac{1}{2}\psi_{0\xi\xi\eta} - \psi_{0\xi}\psi_{0\xi\xi\xi} - \psi_{0\xi\xi}^2 + \frac{1}{2}\xi_{0\eta}\psi_{0\xi\xi\xi}\Big) - \frac{2}{\eta}\Big(2\psi_{0\xi}\psi_{0\eta} \\
&\quad - \xi_{0\eta}\psi_{0\xi}^2 - \frac{\Gamma M^2}{\beta^2}\xi_{0\eta}\psi_{0\eta} - \frac{2(\mu+1)}{3}\psi_{0\xi}^3 - \frac{4\Gamma M^2}{\beta^2}\frac{1}{\eta^2}\xi_{0\theta}\psi_{0\theta} - \frac{2}{\eta}\psi_0\psi_{0\xi} \\
&\quad + \frac{\Gamma M^2}{\beta^2}\frac{1}{\eta}\xi_{0\eta}\psi_0\Big) + \frac{\Gamma M^2}{\beta^2}\big(\psi_{1/2\,\xi}^2 - \xi_{1/2\,\eta}\psi_{1/2\,\xi}\big)\Big] - \frac{(\nu+1)}{3}\frac{1}{\eta}\left[\psi_{0\xi}\right]^3 = 0,
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
&\Big[-\frac{1}{\eta}\psi_1 - \frac{1}{\eta}\xi_1\psi_{0\xi} + \psi_{1\eta} + \xi_{0\eta}\psi_{1\xi} + \Xi_{1\eta}\psi_{0\xi} \\
&\quad + \Xi_1\big(\psi_{0\xi\eta} + \xi_{0\eta}\psi_{0\xi\xi}\big) + \frac{\beta^2}{\Gamma M^2}\xi_{1/2}\Big(\psi_{1/2\,\xi\eta} - \frac{1}{\eta}\psi_{1/2\,\xi} \\
&\quad + \xi_{1/2\,\eta}\psi_{0\xi\xi} + \xi_{0\eta}\psi_{1/2\,\xi\xi}\Big) + \frac{\beta^2}{\Gamma M^2}\Xi_{1/2}^2\Big(\frac{1}{2}\psi_{0\xi\xi\eta} \\
&\quad + \frac{1}{2}\xi_{0\eta}\psi_{0\xi\xi\xi} - \frac{1}{2\eta}\psi_{0\xi\xi}\Big) + \frac{\beta^2}{\Gamma M^2}\xi_{1/2\,\eta}\psi_{1/2\,\xi}\Big] = 0;
\end{aligned}
\tag{4.14}
$$

where (2.21) has been differentiated along the shock surface, and all quantities have been evaluated on the first-order shock surface, $\Xi = \xi_0$, by a mean-surface approximation.

The first-order relations, (4.09) and (4.10) may be combined to give

$$
\left[\psi_{0\eta} + \frac{1}{2}\psi_{0\xi}^2\right] = 0, \tag{4.15}
$$

and

$$
\Xi_{0\eta} = \frac{1}{2}\,\frac{\left[\psi_{0\xi}^2\right]}{\left[\psi_{0\xi}\right]} = \overline{\psi_{0\xi}} \quad , \tag{4.16}
$$

where $\overline{\psi_{0\xi}}$ is the mean of $\psi_{0\xi}$ on either side of the surface. This latter relationship is recognized as the bisector law, while the former may be combined with the first integral of the equation of motion (4.05) to give

$$
\psi_{0\eta} + \frac{1}{2}\psi_{0\xi}^2 = 0 \tag{4.17}
$$

everywhere in the flow field, in analogy with the planar result. This relation will again be extremely useful in simplifying the second-order problem.

The one-and-one-half-order shock relations (4.11) and (4.12) may be combined to give

$$[\psi_{\frac{1}{2}\eta} + \psi_{0_\xi}\psi_{\frac{1}{2}\xi}] = 0\,, \tag{4.18}$$

where the result

$$[\psi_{\frac{1}{2}} + \Xi_{\frac{1}{2}}\psi_{0_\xi}] = 0\,, \tag{4.19}$$

obtainable directly from (2.21) and the fact that $\psi_{\frac{1}{2}}$ satisfies (4.06) have been used. The former result, (4.18), combined with the first integral of (4.06) give the one-and-one-half-order analogue of (4.17):

$$\psi_{\frac{1}{2}\eta} + \psi_{0_\xi}\psi_{\frac{1}{2}\xi} = 0\,, \tag{4.20}$$

everywhere in the flow field.

Combining the two second-order shock relations yeilds

$$\begin{aligned}[\psi_{1_\eta} + \psi_{0_\xi}\psi_{1_\xi}] = {} & \frac{1}{\eta}\left(-\frac{7}{3} + \frac{\Gamma M^2}{3\beta^2} - \frac{2\mu}{3}\right)[\psi_{0_\xi}^{\,3}] \\ & + \frac{1}{\eta}\left(\frac{1}{3} - \frac{\Gamma M^2}{12\beta^2} + \frac{\nu}{6}\right)[\psi_{0_\xi}]^3 - \frac{2}{\eta^2}[\psi_0\psi_{0_\xi}] \\ & - \frac{\beta^2}{2\Gamma M^2}[\psi_{\frac{1}{2}\xi}^{\,2}] - \frac{4\Gamma M^2}{\beta^2}\frac{1}{\eta^3}[\Xi_{0_\theta}\psi_{0_\theta}]\,,\end{aligned} \tag{4.21}$$

where the result

$$\left[\psi_1 + \Xi_1\psi_{0_\xi} + \frac{\beta^2}{\Gamma M^2}\left(\Xi_{\frac{1}{2}}\psi_{\frac{1}{2}\xi} + \frac{1}{2}\Xi_{\frac{1}{2}}^2\psi_{0_{\xi\xi}}\right)\right] = 0 \tag{4.22}$$

has been used, as well as (4.05), (4.06), (4.17) and the result

$$\Xi_{0_\eta}[\psi_{0_\xi}^{\,2}] = -\frac{1}{6}[\psi_{0_\xi}]^3 + \frac{2}{3}[\psi_{0_\xi}^{\,3}]\,. \tag{4.23}$$

The result (4.17) may be used to simplify the second-order equation to the form

$$\psi_{1_{\xi\eta}} + (\psi_{0_\xi}\psi_{1_\xi})_\xi = \left(-7+\frac{\Gamma M^2}{\beta^2}-2\mu\right)\frac{1}{\eta}\psi_{0_\xi}^2\psi_{0_{\xi\xi}} - \frac{2}{\eta^2}\left(\psi_{0_\xi}^2+\psi_0\psi_{0_{\xi\xi}}\right) + \frac{\Gamma M^2}{\beta^2}\frac{1}{\eta^3}\left(\psi_0+4\psi_{0_{\theta\theta}}\right) - \frac{\beta^2}{\Gamma M^2}\psi_{\frac{1}{2}_\xi}\psi_{\frac{1}{2}_{\xi\xi}} . \tag{4.24}$$

For convenience, we here summarize the equations and shock relations in final form for the first-order problem:

$$\psi_{0_{\xi\eta}} + \psi_{0_\xi}\psi_{0_{\xi\xi}} = 0, \tag{4.05}$$

$$\left[\psi_{0_\eta} + \frac{1}{2}\psi_{0_\xi}^2\right] = 0, \tag{4.15}$$

$$\Xi_{0_\eta} = \overline{\psi_{0_\xi}} \quad ; \tag{4.16}$$

for the one-and-one-half-order problem:

$$\psi_{\frac{1}{2}_{\xi\eta}} + \left(\psi_{0_\xi}\psi_{\frac{1}{2}_\xi}\right)_\xi = 0, \tag{4.06}$$

$$\left[\psi_{\frac{1}{2}_\eta} + \psi_{0_\xi}\psi_{\frac{1}{2}_\xi}\right] = 0, \tag{4.18}$$

$$\Xi_{\frac{1}{2}} = -\left[\psi_{\frac{1}{2}}\right]/\left[\psi_{0_\xi}\right] \quad ; \tag{4.25}$$

and for the second-order problem:

$$\psi_{1_{\xi\eta}} + (\psi_{0_\xi}\psi_{1_\xi})_\xi = \left(-7+\frac{\Gamma M^2}{\beta^2}-2\mu\right)\frac{1}{\eta}\psi_{0_\xi}^2\psi_{0_{\xi\xi}} - \frac{2}{\eta^2}\left(\psi_{0_\xi}^2+\psi_0\psi_{0_{\xi\xi}}\right) + \frac{\Gamma M^2}{\beta^2}\frac{1}{\eta^3}\left(\psi_0+4\psi_{0_{\theta\theta}}\right) - \frac{\beta^2}{\Gamma M^2}\psi_{\frac{1}{2}_\xi}\psi_{\frac{1}{2}_{\xi\xi}} , \tag{4.24}$$

$$\left[\psi_{1_\eta}+\psi_{0_\xi}\psi_{1_\xi}\right] = \frac{1}{\eta}\left(-\frac{7}{3}+\frac{\Gamma M^2}{3\beta^2}-\frac{2\mu}{3}\right)\left[\psi_{0_\xi}^3\right] + \frac{1}{\eta}\left(\frac{1}{3}-\frac{\Gamma M^2}{12\beta^2}+\frac{\nu}{6}\right)\left[\psi_{0_\xi}\right]^3 - \frac{2}{\eta^2}\left[\psi_0\psi_{0_\xi}\right] - \frac{\beta^2}{2\Gamma M^2}\left[\psi_{\frac{1}{2}_\xi}^2\right] - \frac{4\Gamma M^2}{\beta^2}\frac{1}{\eta^3}\left[\Xi_{0_\theta}\psi_{0_\theta}\right], \tag{4.21}$$

$$\Xi_1 = -\frac{1}{[\psi_{0_\xi}]}\left[\psi_1 + \frac{\beta^2}{\gamma M^2}\left(\Xi_{1/2}\psi_{1/2_\xi} + \frac{1}{2}\Xi_{1/2}^2\psi_{0_{\xi\xi}}\right)\right] . \tag{4.26}$$

It is of interest to note that as the wave propagates away from the body, geometric effects and wave interaction effects remain of comparable magnitude in the second-order equation. This is seen from the fact that, at large distances, the wave region will be approximately parabolic in the ξ-η plane; i.e., η will scale as the square of ξ . (This is because the characteristics are straight lines in the Ξ-η plane, and at large distances the body appears as a point. Thus the shocks, which define the wave extent, are approximately the bisectors of a centered fan and parallel lines (the freestream characteristics); they are therefore nearly parabolic.) Thus the term

$$\sim \frac{1}{\eta}\psi_{0_\xi}^2\psi_{0_{\xi\xi}}$$

which arises from the wave interaction, must remain of the same order of magnitude as the term arising from the changing geometry,

$$\sim \frac{1}{\eta^3}\left(\psi_0 + 4\psi_{0_{\theta\theta}}\right) .$$

Thus, at even very large distances from the body, second-order wave structure effects are never dominated by the geometry, and both effects must be considered.

4.C. General Solution

The first-order solution is almost completely analogous to that in the planar case. The equation (4.05) has characteristics in any azimuthal plane, θ , given by

$$\left.\frac{d\Xi}{d\eta}\right)_{ch} = \psi_{0_\xi} \quad , \tag{4.27}$$

along which ψ_{0_ξ} is constant, whence they are straight lines in the azimuthal Ξ-η plane. (Due to the fact that $\eta \sim \sqrt{\bar{r}}$, however, they will no longer be straight in the physical plane, but, rather, parabolic.) Thus we will again have

$$\psi_{0_\xi}^{\cdot} = -F'(\Xi_b;\theta) \quad , \tag{4.28}$$

where

$$\xi_b - \xi - \eta F'(\xi_b;\theta) = 0 , \tag{4.29}$$

and $F(\xi_b;\theta)$ is a function to be determined by matching with the first-order local solution in the appropriate azimuthal plane. The potential is then given by

$$\psi_0(\xi_b,\eta;\theta) = -F(\xi_b;\theta) + \frac{1}{2}\eta F'^2(\xi_b;\theta) . \tag{4.30}$$

Shocks are inserted exactly as in the planar case, by integrating (4.16) in regions of overlap of characteristics in the ξ-η plane.

The azimuthal angle, θ , thus enters the first-order solution only as a parameter, and the wave structure in any azimuthal plane is the same as that for some equivalent body of revolution. This equivalent body is defined by matching with the local solution in the azimuthal plane of interest.

The one-and-one-half-order equation has the same characteristics (4.27), and the potential, $\psi_{1/2}$, is constant along these. The equation may be written

$$\psi_{1/2\,\eta}(\xi_b,\eta) = 0 , \tag{4.31}$$

where the arbitrary function of integration has been set to zero by (4.18) and the application of the initial condition. Thus,

$$\psi_{1/2} = - G(\xi_b;\theta) , \tag{4.32}$$

where $G(\xi_b;\theta)$ is to be determined by matching with the local second-order solution. The correction to the first-order shock shape is given by (4.25).

Again, as in the first-order solution, the azimuthal angle enters only as a parameter, and determines the plane in which the matching is to be done to determine $G(\xi_b;\theta)$.

The characteristics of the second-order equation are again given by (4.27), and the homogeneous part of the solution is constant along them. Integration of the equation (4.24) with respect to ξ gives.

$$\begin{aligned}\psi_{1\eta} + \psi_{0\xi}\psi_{1\xi} &= \left(-\frac{7}{3} + \frac{\Gamma M^2}{3\beta^2} - \frac{2\mu}{3}\right)\frac{1}{\eta}\psi_{0\xi}^{\,3} - \frac{2}{\eta^2}\psi_0\psi_{0\xi} \\ &\quad + \frac{\Gamma M^2}{\beta^2}\frac{1}{\eta^3}\int\left(\psi_0 + 4\psi_{0\theta\theta}\right)d\xi - \frac{\beta^2}{2\Gamma M^2}\psi_{1/2\,\xi}^{\,2} + S(\eta) ,\end{aligned} \tag{4.33}$$

where the shock relation (4.21) determines

$$s(\eta) = \sum_{\Xi_0} \frac{1}{\eta}\left(\frac{1}{3} - \frac{\Gamma M^2}{12\beta^2} + \frac{\nu}{6}\right)[\psi_{0_\xi}]^3, \tag{4.34}$$

and the summation is again over all upstream shocks at any value of η . Thus, the first integral may be written

$$\begin{aligned}\psi_{1_\eta}(\Xi_b,\eta;\theta) = {} & \left(\frac{10}{3} - \frac{\Gamma M^2}{2\beta^2} + \frac{2\mu}{3}\right)\frac{1}{\eta}F'^3(\xi_b;\theta) + \left(\frac{\Gamma M^2}{\beta^2} - 2\right)\frac{FF'}{\eta^2} \\ & - \frac{\Gamma M^2}{\beta^2}\left\{\frac{1}{\eta^3}\int (F + 4F_{\theta\theta})\,d\Xi_b + \frac{1}{2\eta^2}\int \left(F'^2 + 4(F'^2)_{\theta\theta}\right) d\Xi_b\right\} \\ & - \frac{\beta^2}{2\Gamma M^2}\frac{G'^2}{(1-\eta F'')^2} + \left(-\frac{1}{3} + \frac{\Gamma M^2}{12\beta^2} - \frac{\nu}{6}\right)\frac{1}{\eta}\sum_{\xi_0}[F']^3,\end{aligned} \tag{4.35}$$

which may be integrated to give

$$\begin{aligned}\psi_1(\Xi_b,\eta,\theta) = {} & \left(\frac{10}{3} - \frac{\Gamma M^2}{2\beta^2} + \frac{2\mu}{3}\right)F'^3(\xi_b;\theta)\ln\eta + \left(2 - \frac{\Gamma M^2}{\beta^2}\right)\frac{FF'}{\eta} \\ & + \frac{\Gamma M^2}{2\beta^2}\left\{\frac{1}{\eta^2}\int (F + 4F_{\theta\theta})\,d\Xi_b + \frac{1}{\eta}\int \left(F'^2 + 4(F'^2)_{\theta\theta}\right) d\Xi_b\right\} \\ & - \frac{\beta^2}{2\Gamma M^2}\frac{G'^2}{(1-\eta F'')F''} + \left(-\frac{1}{3} + \frac{\Gamma M^2}{12\beta^2} - \frac{\nu}{6}\right)\int \frac{1}{\eta}\sum_{\xi_0}[F']^3 d\eta - H(\Xi_b;\theta).\end{aligned} \tag{4.36}$$

The function $H(\Xi_b;\theta)$ is to be determined by matching with the third-order local solution.

The second-order correction to the shock shape is given by

$$\Xi_1 = -\frac{1}{[\psi_{0_\xi}]}\left[\psi_1 + \frac{\beta^2}{\Gamma M^2}\left(\Xi_{\frac{1}{2}}\psi_{\frac{1}{2}_\xi} + \frac{1}{2}\Xi_{\frac{1}{2}}^2\psi_{0_{\xi\xi}}\right)\right]. \tag{4.26}$$

The azimuthal angle enters the second-order solution parametrically in the determination of $H(\xi_b;\theta)$, and also through the dependence of F and G upon θ in the particular integral terms of the solution.

The uniform validity of the finite-body solution at large distances from the body may be seen from a comparison of the orders of magnitude of the various terms as we follow any characteristic to infinity. Thus, for large distances, the first-order solution gives

$$\psi_{0_\xi} \sim -F'(\Xi_b;\theta), \tag{4.37}$$

while the one-and-one-half-order solution gives

$$\psi_{\frac{1}{2}\xi} \sim \frac{1}{\eta} \frac{G'(\xi_b;\Theta)}{F''(\xi_b;\Theta)} , \qquad (4.38)$$

and the second-order solution gives

$$\psi_{1\xi} \sim \frac{\ln \eta}{\eta} F'^2(\xi_b;\Theta) . \qquad (4.39)$$

Thus, it is seen that, in the limit as $\tau \to 0$, the one-and-one-half-order and second-order contributions to the solution are always small compared to that of the first-order solution. Unlike the planar case, the higher-order solutions actually become small compared to the first-order solution at large distances (by a ratio of approximately the inverse square root of the distance from the body). Note also that the second-order solution grows logarithmically compared to the one-and-one-half-order solution. Thus, in a rigorous sense, the true second-order solution must be included whenever the one-and-one-half-order solution is, although in practical cases, logarithmic terms may usually be considered to be of order unity.

The wave structure solution is not, however, uniformly valid for small η. This may be seen from the equation for the second-order potential (4.24) which has nonhomogeneous terms in reciprocal powers of η which will become large for small values of η. Or, more directly, the nonuniformity may be seen by comparing the second-order solution (4.36) with the first-order solution (4.30). This comparison shows that the second-order contribution to the potential will be of the same order as that of the first-order solution when

$$\eta = O(\tau) ,$$

or, equivalently,

$$\bar{\pi} = O(1) .$$

Thus, in the region $\bar{\pi} = O(1)$, a separate expansion must be carried out, and the solution of this problem matched with that of the wave structure problem. The solution in this region is, of course, the classical, "straightforward perturbation" solution.

The reverse of the above procedure, i.e., an examination of the nonuniformity of the local (straightforward perturbation solution), is exactly what led to our choice of the contraction of radial coordinate to be used in the wave structure region scaling (as was done in Section 2.D. for the planar flow case).

4.D. Matching with Local Solution

In order to see how the functions $F(\xi_b;\theta)$, $G(\xi_b;\theta)$, and $H(\xi_b;\theta)$ are determined, it is necessary to look at the local solution in some detail. The simplest problem which illustrates the nature of the matching involved is that of the flow over an axisymmetric disturbance on an infinite cylinder, aligned to the freestream. Thus, if the body surface is described by

$$r_b = r_0 + \tau c f(x/c) , \tag{4.40}$$

the perturbation potential, expressed as

$$\phi = \phi_0 + \frac{M_T^2}{\beta}\phi_1 + \cdots , \tag{4.41}$$

is determined to be

$$\phi_0 = -\int_0^{\bar{\xi}-\bar{r}} \frac{\mathcal{F}(\lambda)\, d\lambda}{\sqrt{(\bar{\xi}-\lambda)^2 - \bar{r}^2}} , \tag{4.42}$$

and

$$\phi_1 = \phi_{0_{\bar{\xi}}}\left(\phi_0 + \frac{\Gamma M^2}{\beta^2}\bar{r}\,\phi_{0_{\bar{r}}}\right) - \int_0^{\bar{\xi}-\bar{r}} \frac{\mathcal{G}(\lambda)\, d\lambda}{\sqrt{(\bar{\xi}-\lambda)^2 - \bar{r}^2}} . \tag{4.43}$$

The source functions, $\mathcal{F}(\lambda)$ and $\mathcal{G}(\lambda)$ are determined by the first- and second-order boundary conditions

$$\int_0^{\bar{\xi}-\bar{r}_0} \frac{(\bar{\xi}-\lambda)\,\mathcal{F}'(\lambda)\, d\lambda}{\sqrt{(\bar{\xi}-\lambda)^2 - \bar{r}_0^2}} = \bar{r}_0 f'(\bar{\xi}) , \tag{4.44}$$

and

$$\begin{aligned} \int_0^{\bar{\xi}-\bar{r}_0} \frac{(\bar{\xi}-\lambda)\,\mathcal{G}'(\lambda)\, d\lambda}{\sqrt{(\bar{\xi}-\lambda)^2 - \bar{r}_0^2}} = \Big\{ & -\frac{\beta^2}{M^2}\bar{r}\left(\phi_{0_{\bar{r}}}\phi_{0_{\bar{\xi}}} + f\phi_{0_{\bar{r}\bar{r}}}\right) \\ & -\bar{r}\,\phi_{0_{\bar{r}\bar{\xi}}}\left(\phi_0 + \frac{\Gamma M^2}{\beta^2}\bar{r}\,\phi_{0_{\bar{r}\bar{r}}}\right) \\ & -\frac{\Gamma M^2}{\beta^2}\bar{r}\,\phi_{0_{\bar{\xi}}}\left(\phi_{0_{\bar{r}\bar{r}}} + \bar{r}\,\phi_{0_{\bar{r}\bar{r}\bar{r}}}\right)\Big\}\Big|_{\bar{r}=\bar{r}_0} . \end{aligned} \tag{4.45}$$

(See Van Dyke (1952)). The local solution is completely determined once these Volterra integral equations are solved for $\mathcal{F}(\lambda)$ and $\mathcal{M}(\lambda)$. This solution may then be used to determine the wave structure solution by the matching principle of Lagerstrom which (following Van Dyke (1964)) states that

> the m-term inner expansion of (the n-term outer expansion) = the n-term outer expansion of (the m-term inner expansion).

We will here carry the matching only to terms of second order in the local solution (one-and-one-half-order in the wave structure solution), since the third-order local solution is not known. Thus, the one-and-one-half-order outer expansion of the second-order inner solution is

$$\left(\phi_{(1)_i}\right)^{(\frac{1}{2})_o} \sim -\frac{2K}{\eta}\int_0^{\xi}\frac{\mathcal{F}(\lambda)\,d\lambda}{\sqrt{2(\xi-\lambda)}} - K\left\{\int_0^{\xi}\frac{\mathcal{F}'(\lambda)\,d\lambda}{\sqrt{2(\xi-\lambda)}}\right\}^2 - \frac{2K}{\eta}\frac{M_T^2}{\beta}\int_0^{\xi}\frac{\mathcal{M}(\lambda)\,d\lambda}{\sqrt{2(\xi-\lambda)}} + O(\tau^3), \tag{4.46}$$

where a subscript in parentheses indicates the complete solution for the potential to the order indicated, and the subscript on the parentheses indicates the region in which the solution is valid. Thus, e.g., the subscript $(1)_i$ signifies the solution in the inner (or local) region, up to and including terms of second order in that region.

The one-and-one-half-order wave structure solution, expanded to terms of second order in the local region is then

$$\left(\phi_{(\frac{1}{2})_o}\right)^{(1)_i} \sim \frac{1}{\sqrt{\bar{\kappa}}}\left\{-F(\xi) - K\sqrt{\bar{\kappa}}\,F'^2(\xi) - \frac{M_T^2}{\beta}G(\xi) + \cdots\right\}. \tag{4.47}$$

Expressing (4.47) in terms of the wave structure variables for comparison with (4.46) gives

$$\left(\phi_{(\frac{1}{2})_o}\right)^{(1)_i} \sim -\frac{2K}{\eta}F - KF'^2 - \frac{2K}{\eta}\frac{M_T^2}{\beta}G + \cdots, \tag{4.48}$$

whence the matching requires

$$F(\xi_b) = \int_0^{\xi_b}\frac{\mathcal{F}(\lambda)\,d\lambda}{\sqrt{2(\xi_b-\lambda)}}, \tag{4.49}$$

$$G(\xi_b) = \int_0^{\xi_b} \frac{\mathcal{A}(\lambda)\, d\lambda}{\sqrt{2(\xi_b - \lambda)}} \quad . \tag{4.50}$$

Note, in particular, that the local second-order solution determines only the one-and-one-half-order solution in the wave structure region. To determine the second-order wave structure, we would require the local solution to third order, the homogeneous part of which would contribute a term like

$$-\frac{2K}{\eta}\frac{M_T^4\tau^2}{\beta^2}\int_0^{\xi} \frac{\mathcal{H}(\lambda)\, d\lambda}{\sqrt{2(\xi-\lambda)}} \tag{4.51}$$

in (4.46), which would match with a term like

$$-\frac{2K}{\eta}\frac{M_T^4\tau^2}{\beta^2} H(\xi_b) \tag{4.52}$$

in (4.48) to determine the homogeneous term in the second-order solution (4.36).

4.E. General Finite Bodies

The treatment of flows about more general finite bodies is severely limited by the lack of a sufficiently general second-order local theory. This is primarily a result of the inability to find a complete particular integral for the nonhomogeneous terms of the equation when non-axisymmetric terms are included. However, a few general remarks may be made about these flows.

a. Quasi-cylindrical flows

For the flow over a nearly axisymmetric perturbation on an infinite cylinder aligned with the flow, it is useful to think of the body surface as described by a Fourier series. To illustrate, we will consider only the first term (e.g., the dipole) of such a series, though all terms are of equal importance in determining the wave structure, since higher-order moments do not decay as in the slender body case (see following section). Thus the body is assumed to be described by

$$\bar{r}_b = \bar{r}_0 + \tau\beta\left\{ f_1(\bar{\xi}) + f_2(\bar{\xi})\cos\theta \right\} . \tag{4.53}$$

The first-order solution will then be of the form

$$\phi_0 = -\int_0^{\bar{\xi}-\bar{r}} \frac{\mathcal{F}_1(\lambda)\,d\lambda}{\sqrt{(\bar{\xi}-\lambda)^2-\bar{r}^2}} - \cos\Theta \int_0^{\bar{\xi}-\bar{r}} \left\{ \frac{\bar{\xi}-\lambda+\sqrt{(\bar{\xi}-\lambda)^2-\bar{r}^2}}{\bar{r}} + \frac{\bar{r}}{\bar{\xi}-\lambda+\sqrt{(\bar{\xi}-\lambda)^2-\bar{r}^2}} \right\} \frac{\mathcal{F}_2(\lambda)\,d\lambda}{\sqrt{(\bar{\xi}-\lambda)^2-\bar{r}^2}}, \tag{4.54}$$

where $\mathcal{F}_1(\lambda)$ and $\mathcal{F}_2(\lambda)$ are related to the shape functions $f_1(\xi_b)$ and $f_2(\xi_b)$ by Volterra integral equations through the boundary condition, which may be applied at $\bar{r}_0$. The second-order local solution will have a similar form for the homogeneous terms, plus particular integrals of the nonhomogeneous terms which we are unable to calculate:

$$\phi_1 = -\int_0^{\bar{\xi}-\bar{r}} \frac{\mathcal{G}_1(\lambda)\,d\lambda}{\sqrt{(\bar{\xi}-\lambda)^2-\bar{r}^2}} - \cos\Theta \int_0^{\bar{\xi}-\bar{r}} \left\{ \frac{\bar{\xi}-\lambda+\sqrt{(\bar{\xi}-\lambda)^2-\bar{r}^2}}{\bar{r}} + \frac{\bar{r}}{\bar{\xi}-\lambda+\sqrt{(\bar{\xi}-\lambda)^2-\bar{r}^2}} \right\} \frac{\mathcal{G}_2(\lambda)\,d\lambda}{\sqrt{(\bar{\xi}-\lambda)^2-\bar{r}^2}} + (\text{particular integral terms}). \tag{4.55}$$

Matching this with the wave structure solution results in

$$F(\xi_b;\Theta) = \int_0^{\xi_b} \frac{\mathcal{F}_1(\lambda) + 2\cos\Theta\,\mathcal{F}_2(\lambda)}{\sqrt{2(\xi_b-\lambda)}}\,d\lambda, \tag{4.56}$$

and

$$G(\xi_b;\Theta) = \int_0^{\xi_b} \frac{\mathcal{G}_1(\lambda) + 2\cos\Theta\,\mathcal{G}_2(\lambda)}{\sqrt{2(\xi_b-\lambda)}}\,d\lambda\ . \tag{4.57}$$

The second-order source functions, $\mathcal{G}_1(\lambda)$ and $\mathcal{G}_2(\lambda)$ cannot be determined, however, since the complete particular integral (to be used in applying the boundary conditions) is not known. The second-order outer behavior of the first-order local solution matches appropriately with the inner behavior of the wave structure solution, but the check cannot be made for the nonhomogeneous terms of the second-order local solution.

As in the axially symmetric case, the local third-order solution is required to determine the wave structure solution to second order.

b. Slender body flows

For flows over bodies whose every cross-stream dimension is small, the following redefinitions are useful. We define the slender-body perturbation potential

$$\psi = \frac{1}{\tau\beta}\phi\ , \tag{4.58}$$

and the new parameter

$$K_{sl} = \frac{\Gamma M^4 \tau^2}{\beta^2}\ , \tag{4.59}$$

whence, with the expansion

$$\psi = \psi_0 + M^2_\tau{}^2\psi_1 + \cdots\ , \tag{4.60}$$

the equations of Section 4.B. again describe the wave structure when ϕ is everywhere replaced by ψ and K is replaced by K_{sl}.

In this case, an additional scaling region must be considered within the local region, in the immediate vicinity of the body. In this slender-body region, radial velocity perturbations are large compared with axial ones. The solution in this region must then be matched with the local solution, which in turn provides the inner boundary condition for the wave structure solution. The first-order slender-body solution is a harmonic function in the cross plane, plus an additive function of the axial variable. Thus, non-axisymmetric terms decay rapidly, and the first-order local solution is axisymmetric. The slender-body dipole distribution contributes to the second-order local solution. Since the first-order local solution is axisymmetric, none of the difficulties of the preceding (quasi-cylindrical) section appear, and the wave structure solution, in principle, may be determined (to one-and-one-half order). The slender-body quadripole distribution, as well as particular integral terms arising from the non-axisymmetry of the local second-order solution will enter into the third-order local solution, which is needed to determine the second-order wave structure solution.

c. Finite planar systems

For flows over finite planar bodies, the local theory exists only to first order (Hayes (1947)). The so-called "area rule" is a large-distance local theory -- which is exactly what we need for matching with the wave structure solution. It is based upon the fact that the disturbance at a point far from a finite planar body is primarily due to sources near the intersection of the forward Mach cone from the point of interest and the planar body, and that this region may be shrunk to a point, as viewed from the distant point of interest.

For the thickness, or symmetrical, solution, we need consider only volume sources, and the first-order disturbance potential at a point $(\xi, \bar{\eta}, \bar{\zeta})$, is of the form

$$\phi_0 = -\iint_{\Omega} \frac{\sigma_t(x_1, x_2)\, dx_1\, dx_2}{\sqrt{(\xi - x_1)^2 - (\bar{\eta} - x_2)^2 - \bar{\zeta}^2}} , \tag{4.61}$$

where the integration is carried out over the region upstream of the intersection of the forward Mach cone from the point of interest and the plane of the body (assumed to be $\bar{\zeta} = 0$), and σ_t is proportional to the local downwash due to the thickness distribution. The one-term wave structure expansion of this integral may be easily shown to be

$$\left(\phi_{(0)_i}\right)^{(0)_0} \sim -\frac{2K}{\eta} \int_0^{\xi} \frac{\tilde{\sigma}_t(\lambda')\, d\lambda'}{\sqrt{2(\xi - \lambda')}} , \tag{4.62}$$

where

$$\lambda' = x_1 - x_2 \cos\theta \tag{4.63}$$

and

$$\tilde{\sigma}_t(\lambda') = \int \sigma_t(\lambda', x_2)\, dx_2 . \tag{4.64}$$

Thus, we note that at large distances, the perturbation potential from a planar distribution of sources is identical to that of an equivalent axisymmetric distribution, given by $\tilde{\sigma}_t(\lambda')$. Note that $\tilde{\sigma}_t(\lambda')$ will depend upon the azimuthal angle, θ, through the definition of λ'.

Similarly, for the lifting problem, the potential due to a distribution of elementary horseshoe vortices may be represented as

$$\phi_0 = \iint_{\Omega} \frac{\bar{\zeta}(\xi - x_1)\, \sigma_l(x_1, x_2)\, dx_1\, dx_2}{\left\{(\bar{\eta} - x_2)^2 + \bar{\zeta}^2\right\} \sqrt{(\xi - x_1)^2 - (\bar{\eta} - x_2)^2 - \bar{\zeta}^2}} , \tag{4.65}$$

where σ_l is proportional to the local lift coefficient. In the wave structure region this takes the form

$$\left(\phi_{(0)_i}\right)^{(0)_0} \sim \frac{2K}{\eta} \int_0^{\xi} \frac{\sin\theta\, \tilde{\sigma}_l(\lambda')\, d\lambda'}{\sqrt{2(\xi - \lambda')}} . \tag{4.66}$$

In this case, the azimuthal angle enters not only in describing the cuts across which the integration will be carried out, but explicitly in the integrand.

Again, the second-order local problem cannot be solved due to the lack of a complete particular integral, but it may be noted that the homogeneous potential will also be of the form (4.61) or (4.65) and will, in the matching with the wave structure solution determine the one-and-one-half-order solution $G(\xi_b;\theta)$.

4.F. Lighthill region

The broad third-order wave found in the planar case behind the trailing shocks also exists in the finite body case, with a cumulative effect of second order. The wave in this case is even broader in the physical space, and the Lighthill region defined by

$$\begin{aligned} \tilde{\xi} &= 4K^2\xi \\ \tilde{\eta} &= 16K^4\eta \ , \end{aligned} \tag{4.67}$$

corresponds to

$$\pi = O(\tau^{-10}) \tag{4.68}$$

in the physical plane. The strength of the wave cannot be determined as it was in the planar case, however, due to the fact that the geometry now enters the characteristic equations, precluding direct integration. The strength might be determined for specific examples, but the great distance at which the interaction with the rear shock wave occurs justifies, in some sense, its neglect.

Chapter Five

Summary and Concluding Remarks

A second-order perturbation theory for the inviscid, supersonic flow of a general fluid past a body has been presented, with particular emphasis on the behavior of the wave systems at large distances from the body.

The theory justifies confidence in the first-order theory (which is a standard tool in all sonic boom work), by showing that it is the first term in an asymptotic approximation scheme, valid in the limit of vanishing body thickness, and gives additional insight into the nature of that approximation. The theory also provides increased numerical accuracy for flows about bodies of finite thickness in the examples calculated, provided the disturbances are not too large.

The theory of planar flows generally confirms the results of Friedrichs (1948), as corrected and extended by Lighthill (1954). An error is pointed out in the latter work, wherein the position of the rear shock at very great distances is incorrectly given. It is also demonstrated that for shocks originating within the flow field, the predicted second-order shift in point of origin corresponds to the shift in virtual origin necessary to account for the local structure near that point (which structure the present theory does not predict).

The theory of flows about finite systems is essentially new. Of particular importance is the fact that the local solution (i.e., that valid near the body, where the cylindrical nature of the waves is essential) must be known to third order to determine the second-order wave structure solution. An intermediate, homogeneous solution (termed of one-and-one-half order) is determined from matching with the local second-order solution. Numerical results from the calculation of the shock position in a conical flow indicate that appreciable increase in numerical accuracy is obtained only when the full second-order solution is calculated. The one-and-one-half-order solution offers little improvement over the first-order theory.

The wave structure theory, as presented, is essentially complete for the large distance behavior of these systems. Extension of the theory to third order would probably not be worthwhile. Such a theory would not necessarily give increased numerical accuracy even if its algebraic hurdles could be overcome, and the essential nature of the higher approximations is illustrated in the second-order theory.

The most useful avenue for further study is consideration of the second approximation for non-axisymmetric bodies in the local region. Only the outer asymptotic behavior of this solution is required to determine the corresponding wave structure. Determination of the local third-order solution is probably not feasible.

Appendix A

The Existence of a Velocity Potential

An irrotational vector field, i.e., one having zero curl, may be described as everywhere the gradient of a single scalar function, called the potential. The irrotationality of the field is a necessary and sufficient condition for the existence of such a potential. We will now investigate the conditions under which the velocity field in our particular problem is irrotational.

Since we assume the flow to be inviscid and initially uniform (and hence irrotational), vorticity may be generated within the flow field only at curved shocks, where entropy gradients are incurred. The vorticity is related to these entropy gradients for a homocompositional, inviscid flow by the Crocco-Vaszonyi theorem

$$\vec{q} \times \vec{\zeta} = -T \nabla S + \nabla H \,, \tag{A.01}$$

where

$$\vec{\zeta} = \nabla \times \vec{q} \tag{A.02}$$

is the vorticity. Thus, for our homoenergetic flow, the magnitude of the vorticity may be estimated by

$$\zeta \sim \frac{T_\infty \nabla S}{U} \,, \tag{A.03}$$

which, using our quantitative expression for the entropy jump at weak shock waves, (2.23), becomes

$$\zeta \sim \frac{\Gamma}{6 U a_\infty} \left| \nabla [q_n]^3 \right| . \tag{A.04}$$

Using equation (2.28) for the normal component of the velocity, this becomes

$$\zeta \sim \frac{K \tau^2 U}{6 \beta^2} \nabla \left\{ [\phi_{0_\xi}]^3 \right\} . \tag{A.05}$$

This gives for the magnitude of the vorticity in the planar case,

$$\zeta \sim \frac{K^2 \tau^2 \beta U}{c}, \tag{A.06}$$

and for the finite system case,

$$\zeta \sim \frac{K^6 \tau^2 \beta U}{c}. \tag{A.07}$$

We can now calculate the magnitude of the vorticity in a more general (rotational) flow. This is a flow for which a potential does not exist, hence the magnitude of the vorticity is given by terms of the sort

$$(u_y - v_x).$$

These may be estimated by calculating the cross derivative terms in the potential analysis. The vorticity is then found to be of order

$$\frac{K^3 \tau U}{c} \tag{A.08}$$

in the planar case, and of order

$$\frac{K^5 M^4 \tau^3 U}{c\beta^2} \tag{A.09}$$

in the finite body case only in the third-order solutions.

Thus, the velocity potential may be used with no loss in generality for describing these flows to second order.

From the above, it is seen that vorticity must be taken into account in the third-order problem, and a velocity potential cannot be used to describe the third-order wave behind the body. However, in the region of interaction of this wave with the rear shock on the other side of the body, the shock strength has decayed sufficiently that the vorticity is of order

$$\frac{K^4 \tau^2 \beta U}{c} \tag{A.10}$$

for the planar case, whereas it is of order

$$\frac{K^5 \tau U}{c} \tag{A.11}$$

only in the third approximation in the corresponding rotational flow. Thus, the velocity potential may be used to study this interaction, which appears as a second-order effect in this region.

For flows about finite systems, the Lighthill region is even more distant, with a correspondingly greater decrease in the vorticity. In this region it is of order

$$\frac{\underline{K}^{22}\tau^2\beta U}{c}, \tag{A.12}$$

whence it must be accounted for only in the ninth approximation. The vorticity in the second approximation to the corresponding rotational flow is of order

$$\frac{\underline{K}^{14}M^2\tau^2 U}{\beta c}. \tag{A.13}$$

Appendix B

An Example: The Circular-arc Airfoil

We consider the solution for the wave structure emanating from a symmetric, biconvex airfoil, of circular arc section. The body shape function is then

$$
\begin{aligned}
f(\xi_b) &= 0\,, & \xi_b &< -\tfrac{1}{2}\,, \\
&= \frac{1-4\xi_b^2}{2}\,, & -\tfrac{1}{2} < \xi_b &< \tfrac{1}{2}\,, \\
&= 0\,, & \tfrac{1}{2} &< \xi_b\,,
\end{aligned}
\tag{B.01}
$$

and only the flow above the airfoil will be considered, as the flow is symmetric about the axis $\eta = 0$.

From equation (3.35) we have that $\phi_0 = 0$ in the regions defined by the vertical characteristics for $|\xi_b| > \tfrac{1}{2}$, and

$$\phi_0 = 2(1+4\eta)\,\xi_b^2 - \tfrac{1}{2} \tag{B.02}$$

in the region swept out by characteristics from the body where, from (3.34)

$$\xi_b = \frac{\xi}{1+4\eta}\,, \tag{B.03}$$

whence

$$\phi_0 = \frac{2\xi^2}{1+4\eta} - \tfrac{1}{2}\,. \tag{B.04}$$

These characteristics form a fan-like pattern and overlap the vertical characteristics from ahead and behind the body. Shocks must be inserted to render the solutions unique, and, by (3.12) must obey

$$\frac{d\xi_0}{d\eta} = \frac{2\xi}{1+4\eta}\,, \tag{B.05}$$

whence

$$\xi_0 = \mp \frac{1}{2}\sqrt{1+4\eta} \quad , \tag{B.06}$$

where the upper and lower signs refer to the leading and trailing shocks respectively. It is seen that the first-order wave pattern is symmetric about the vertical axis ($\xi = 0$) for this symmetric (fore and aft) profile. The strengths of the shocks are given as a function of η by

$$[\phi_{0_\xi}] = \frac{-2}{\sqrt{1+4\eta}} \quad , \tag{B.07}$$

and are seen to vary as the inverse square root of the distance, for large distances, a well-known result for planar flows about general sections.

The second-order solution is also zero along the freestream characteristics upstream of the leading shock, and is by (3.42) and (B.03)

$$\begin{aligned} \phi_1 = {} & 64\left(-\frac{7}{6} + \frac{\Gamma M^2}{6\beta^2} - \frac{\mu}{3}\right)\frac{\eta\xi^3}{(1+4\eta)^3} \\ & + \left(\frac{2}{3} - \frac{\Gamma M^2}{6\beta^2} + \frac{\nu}{3}\right)\frac{1}{\sqrt{1+4\eta}} \\ & + \left(2 - \frac{2}{M^2}\right)\frac{\xi}{1+4\eta} + \left(-\frac{8}{3} + \frac{8}{M^2} - \frac{8\Gamma M^2}{3\beta^2}\right)\frac{\xi^3}{(1+4\eta)^3} \\ & - \frac{\Gamma M^2}{6\beta^2} - \frac{\nu}{3} \quad , \end{aligned} \tag{B.08}$$

in the region between the shocks, and

$$\phi_1 = \left(\frac{4}{3} - \frac{\Gamma M^2}{3\beta^2} + \frac{2\nu}{3}\right)\frac{1}{\sqrt{1+4\eta}} - \left(\frac{\Gamma M^2}{3\beta^2} + \frac{2\nu}{3}\right) \tag{B.09}$$

behind the trailing shock. The second-order shock displacement is, by (3.18)

$$\begin{aligned} \xi_1 = {} & -\left(\frac{\Gamma M^2}{12\beta^2} + \frac{\nu}{6}\right)\sqrt{1+4\eta} + 1 + \frac{1}{2M^2} - \frac{\Gamma M^2}{4\beta^2} \\ & + \frac{\mu}{3} + \frac{\nu}{6} + \left(-1 - \frac{1}{2M^2} + \frac{\Gamma M^2}{3\beta^2} - \frac{\mu}{3}\right)\frac{1}{1+4\eta} \quad , \end{aligned} \tag{B.10}$$

for either the leading or trailing shock. The second-order increment to the strength of the shock is thus

$$\frac{M^2\tau}{\beta}\left[\phi_{1_\xi}(\Xi_0,\eta) + \Xi_1\phi_{0_{\xi\xi}}(\Xi_0,\eta)\right] = \frac{M^2\tau}{\beta}\left\{-\left(\frac{\Gamma M^2}{12\beta^2}+\frac{\nu}{6}\right)\frac{4}{\sqrt{1+4\eta}} + \left(6-\frac{\Gamma M^2}{\beta^2}+\frac{4\mu}{3}+\frac{2\nu}{3}\right)\frac{1}{1+4\eta} + \left(-7/8+\frac{8\Gamma M^2}{\beta^2} - 16\mu\right)\frac{\eta}{(1+4\eta)^2} + \left(-6+\frac{4}{M^2}-\frac{2\Gamma M^2}{3\beta^2}-\frac{4\mu}{3}\right)\frac{1}{(1+4\eta)^2}\right\}, \tag{B.11}$$

which, for large distances behaves as

$$-\frac{M^2\tau}{\beta}\left(\frac{\Gamma M^2}{6\beta^2}+\nu/3\right)\frac{1}{\sqrt{\eta}} , \tag{B.12}$$

and is seen to increase the strength of the leading shock, and decrease that of the trailing shock (since the coefficient in parentheses is positive for typical gases).

Behind the trailing shock, the potential is a function of η alone, whence the pressure in this region is zero to second order. That ϕ_1 is not zero, but a function of η, hints that there is a wave of the right-running (or returning) family that is locally of third order in strength, but has a cumulative effect of second order. The behavior of this wave cannot be calculated using the velocity potential, however, since the velocity and entropy perturbations are of the same order in this region.

This wave is of interest because it must be included to give results consistent to second order for certain global integrals relating to lift and drag of the airfoil, and also because it eventually intersects the rear shock on the other side of the airfoil, altering its position to second order. (In the symmetric case under consideration, the wave may be imagined to be reflected at the line of symmetry (the ξ-axis) and thus eventually interact with the upper rear shock.) See figure B.01.

This interaction occurs only at very great distances from the airfoil, and at such distances that the shock has decayed sufficiently for a velocity potential to again exist to the order required to study the interaction. Thus, a potential analysis may be used to study the interaction, but vorticity must be taken into account in determining the initial strength of the wave.

The characteristic equations for steady, planar, inviscid (but rotational) supersonic flow are that

$$\frac{\sqrt{M^2-1}}{\rho q^2}\,dp \pm d\theta = 0 , \tag{B.13}$$

along

$$\frac{dy}{dx} = \tan\left(\theta \pm \sin^{-1} 1/M\right) . \tag{B.14}$$

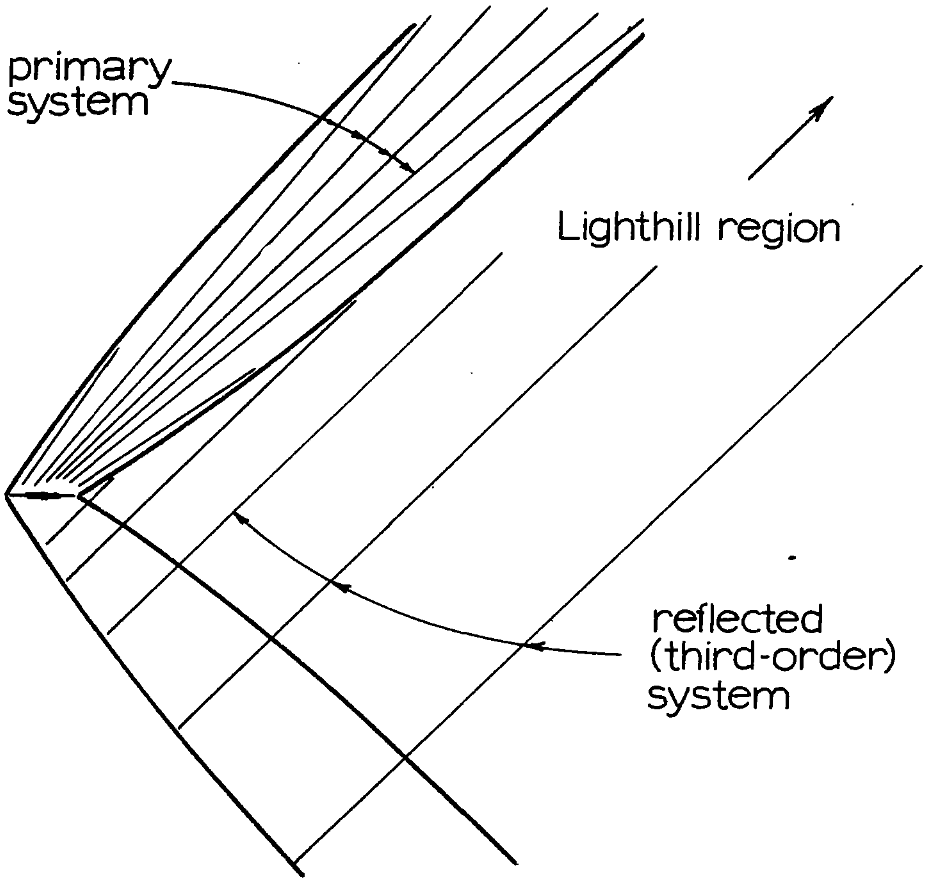

Figure B.01: Wave Systems associated with Body in Supersonic Flight.

Substituting expansions in terms of our wave structure variables, we find in the region behind the trailing shock, that the quantity

$$\int \frac{\sqrt{M^2-1}}{\rho q^2} dp - \theta = -\frac{K M_T^{2\,2}}{\beta} \phi_{1\eta} - \frac{\beta T_\infty}{U^2} \delta S \qquad (B.15)$$

is constant along the lines

$$\frac{d\xi}{d\eta} = -2/K \,. \qquad (B.16)$$

The invariant (B.15) may be calculated at the rear shock and is, for our example,

$$\frac{\mathbb{K} M_T^2 \tau^2}{\beta}\left\{\frac{8}{3M^2}-\frac{2\Gamma M^2}{3\beta^2}+\frac{4\nu}{3}\right\}\frac{1}{(1+4\eta)^{3/2}} , \tag{B.17}$$

which, upon reflection from the ξ- axis, gives a wave of strength

$$\frac{M_T^2\tau^2}{\beta}\phi_{\xi} = \frac{\mathbb{K} M_T^2 \tau^2}{\beta}\left\{-\frac{4}{3M^2}+\frac{\Gamma M^2}{3\beta^2}-\frac{2\nu}{3}\right\}\frac{1}{(1+2\mathbb{K}\xi)^{3/2}} , \tag{B.18}$$

which is propagated along lines of constant ξ (linearized characteristics).

Now, the region in which this wave interacts with the rear shock is so distant from the airfoil that the shock has decayed to the point where a potential again exists; i.e., the entropy perturbations are again small compared to the velocity perturbations in the wave (B.18). (See Appendix A) In this region (which we shall call the Lighthill region, after the discoverer of its importance (1954)), we define the new independent variables

$$\begin{aligned}\tilde{\xi} &= \mathbb{K}\xi , \\ \tilde{\eta} &= \mathbb{K}^2\eta ,\end{aligned} \tag{B.19}$$

whence

$$\tilde{\xi}_1 = \sqrt{\tilde{\eta}}\,[\phi_1] , \tag{B.20}$$

where the jump in potential is to be calculated from (B.09) and (B.18). From the latter we have

$$\phi_{1\tilde{\xi}} = \left(-\frac{4}{3M^2}+\frac{\Gamma M^2}{3\beta^2}-\frac{2\nu}{3}\right)\frac{1}{(1+2\tilde{\xi})^{3/2}} , \tag{B.21}$$

or

$$\phi_1 = \left(\frac{4}{3M^2}-\frac{\Gamma M^2}{3\beta^2}+\frac{2\nu}{3}\right)\frac{1}{\sqrt{1+2\tilde{\xi}}} - \left(\frac{4}{3M^2}+\frac{4\nu}{3}\right) , \tag{B.22}$$

where the constant has been determined by matching the shock shape in this region with that in the wave structure region, using the asymptotic principle. This procedure results in

$$\tilde{\Xi}_1 = \sqrt{\tilde{\eta}}\left\{\left(-\frac{4}{3M^2}+\frac{\Gamma M^2}{6\beta^2}-\nu\right)+\left(\frac{4}{3M^2}-\frac{\Gamma M^2}{3\beta^2}+\frac{2\nu}{3}\right)\frac{1}{\sqrt{1+2\tilde{\eta}^{1/2}}}\right\} \tag{B.23}$$

for the second-order correction to the location of the trailing shock. For very, very large distances (i.e., large $\tilde{\eta}$), this becomes

$$\tilde{\Xi}_1 \sim \sqrt{\tilde{\eta}}\left(-\frac{4}{3M^2}+\frac{\Gamma M^2}{6\beta^2}-\nu\right) \tag{B.24}$$

This last result differs from the expression given by Lighthill (1954) for the analogous quantity in his analysis (equation (4.49)). The quantity given there seems to correspond to the shift in the rear shock beyond that calculated earlier by Friedrichs, and thus must be added to the original coefficient K (defined in his text between equations (4.09) and (4.10)). Also, since only the rear shock is affected (a fact noted earlier in the same article by Lighthill) this quantity must be multiplied by the ratio of total mass flux excess in the simple wave to that in the rear portion (coming from downstream of the peak value of the airfoil). Thus, if for the upper shock Lighthill's equation (4.49) is replaced by

$$K + \frac{\int_0^c y_l'^{\,2}\,dx}{\int_{x_p}^c y_u'^{\,2}\,dx}\left\{\frac{1+(\gamma-1)M_\infty^2}{M_\infty^2-1}\right\},$$

our results are brought into agreement.

Appendix C

Mass and Momentum Fluxes in the Wave Systems

That the theory presented here gives a consistent representation of the flow at large distances from the body may be verified by considering the asymptotic form of the solution as it relates to the total forces on the body and to conservation of certain global integrals (such as mass flux). Since the broad third-order wave behind the body (in the Lighthill region) has a cumulative effect of second order, this must be included in the analysis. We shall again restrict ourselves to planar flows, where the complete second-order solution may be obtained, and the strength of the third-order wave behind the trailing shocks may be determined.

We shall consider integrating mass and momentum fluxes around contours sufficiently far removed from the body that the asymptotic forms of the solutions may be used, and show how these integrals correspond to conservation of mass and momentum, thus giving the lift and drag forces on the body. Consider the control volume shown in figure C.01, around which we will integrate mass and momentum fluxes.

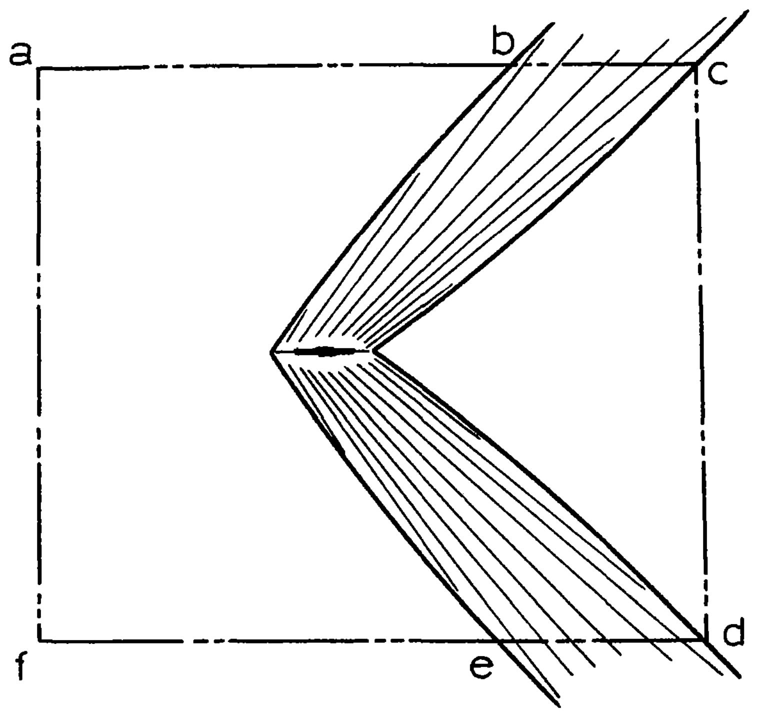

Figure C.01: Control Volume

There will be no contribution to the flux integrals from a-b or e-f since the flow is parallel to the control surfaces; there will be a contribution from the difference of fluxes across c-d and f-a due to the increased entropy behind the wave systems; and there will be a contribution from the integration across the primary wave systems b-c and d-e. The effect of the third-order wave behind the body will be accounted for as it alters the position of the trailing shocks, and hence the contribution from the primary wave systems.

For the symmetric circular-arc airfoil, we need consider only the upper half-plane. The excess mass flux in the primary wave system is

$$(\Delta m)_{b-c} = \int_{\xi_l}^{\xi_t} \rho v \, dx = -\rho_\infty U \tau c \left\{ \phi(\xi_t) - \phi(\xi_l) \right\} . \tag{C.01}$$

Thus, since

$$\phi(\xi_l) = 0 , \tag{C.02}$$

and

$$\begin{aligned} \phi(\xi_t) &= \phi_0(\xi_0) + \frac{M_T^2}{\beta}\left\{ \phi_1(\xi_0) + \xi_1 \phi_{0_\xi}(\xi_0) \right\} \\ &= -\frac{M_T^2}{\beta}\left(\frac{4}{3M^2} + \frac{4\nu}{3} \right) , \end{aligned} \tag{C.03}$$

we have

$$(\Delta m)_{b-c} = \frac{\rho_\infty U c M_T^2 \tau^2}{\beta}\left(\frac{4}{3M^2} + \frac{4\nu}{3} \right) . \tag{C.04}$$

The excess mass flux in the entropy wake region, c-d, over that in the freestream, f-a, is given by

$$(\Delta m)_{c-d} = \int_0^\infty \left\{ \frac{\partial(\rho u)}{\partial S} \delta S \right\} dy = -\frac{\rho_\infty U}{a_\infty^2}\left(\nu + \frac{1}{M^2} \right) T_\infty \int_0^\infty \delta S \, dy , \tag{C.05}$$

where the surface has been taken far enough downstream that pressure perturbations are negligible. Since entropy is conserved along streamlines, the integral may be taken at the shocks, and using our expression for the entropy jump at a shock, (C.05) may be written

$$(\Delta m)_{c-d} = \frac{\rho_\infty U c M_T^2 \tau^2}{\beta}\left\{ -\frac{4}{3M^2} - \frac{4\nu}{3} \right\} , \tag{C.06}$$

and is seen to exactly balance the excess mass flux in the primary wave system, thus confirming that mass is conserved in the solution. Note that in (C.03) the value of the potential in the Lighthill region, (3.53), has been used.

The drag of the airfoil must be equal to the resultant of the axial pressure force and the deficit of axial momentum flux through the control surface. In the upper primary wave system, this is given by

$$(\Delta p_x)_{b-c} = \int_{\xi_l}^{\xi_t} \rho v (u - U)\, dx = \rho_\infty U^2 c \frac{\tau^2}{\beta} \int_{\xi_l}^{\xi_t} \phi_\xi^{\,2}\, d\xi \,, \tag{C.07}$$

which, for our circular-arc airfoil is

$$(\Delta p_x)_{b-c} = \frac{4}{3} \frac{\rho_\infty U^2 c \tau^2}{\beta} \left\{ \frac{\xi_t^{\,3} - \xi_l^{\,3}}{(1+4\eta)^2} \right\} \,, \tag{C.08}$$

which approaches zero as $(1+4\eta)^{-1/2}$ for large η, and thus contributes nothing to the drag. The deficit of axial momentum flux in the entropy wake is

$$(\Delta p_x)_{c-d} = \int_0^\infty \rho u (u - U)\, dy = \rho_\infty T_\infty \int_0^\infty \delta S\, dy \,, \tag{C.09}$$

where, again, the plane of integration is assumed sufficiently far downstream that pressure perturbations are negligible. Thus,

$$(\Delta p_x)_{c-d} = \frac{4}{3} \frac{\rho_\infty U^2 c \tau^2}{\beta} \tag{C.10}$$

which is exactly the drag on the upper surface of the airfoil which is obtained by integrating the surface pressure times the axial component of the normal to the surface.

The airfoil treated in the preceeding paragraphs has no net lift, since it is symmetric. To investigate lift forces, we consider an airfoil with the same circular-arc upper surface, but with a flat lower surface. Thus there will be no wave system from the lower surface, and the airfoil will have a net (negative) second-order lift. The control surface for this airfoil is shown in figure C.02; note that we have included integration across the third-order tail wave, which now is propagated to infinity below the section.

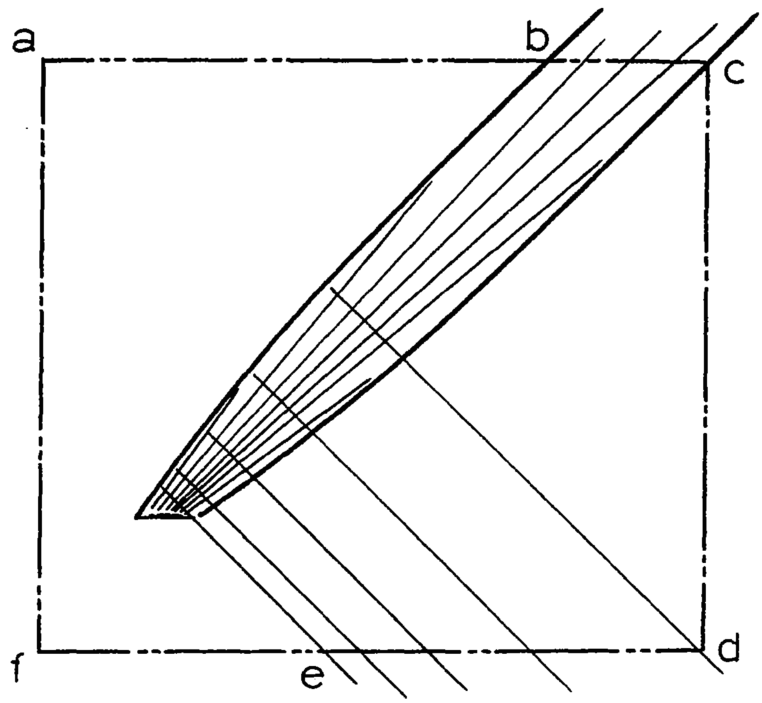

Figure C.02: Control Volume for Lifting Airfoil

Here, the lift force must be balanced exactly by the resultant of vertical pressure forces and momentum fluxes, integrated around the control volume. We need consider only the integration across the primary wave system, b-c, and the reflected (Lighthill) wave system, d-e, since elsewhere the contributions are negligible. The contribution due to integration across the primary wave system is

$$(\Delta P_y)_{b-c} = -\int_{\xi_l}^{\xi_t} \left\{ p - p_\infty + \rho v^2 \right\} dx = \frac{\rho_\infty U_c^2 \tau}{\beta} \left\{ \phi(\xi_t) - \phi(\xi_l) \right\}, \qquad \text{(C.11)}$$

which for our case is

$$(\Delta P_y)_{b-c} = -\frac{\rho_\infty U_c^2 M^2 \tau^2}{\beta^2} \left\{ \frac{\Gamma M^2}{3\beta^2} + \frac{2\nu}{3} \right\} . \qquad \text{(C.12)}$$

In this case, there is no "Lighthill region" effect upon the position of the trailing shock, since there is no reflected wave coming from below the airfoil. Integration across the third-order wave coming from the upper, primary system

$$\begin{aligned}(\Delta P_y)_{d-e} &= \int_0^{\infty}\left\{p - p_\infty + \rho v^2\right\} dx \\ &= -\frac{\rho_\infty U^2 c M^2 \tau^2}{\beta^2}\left\{\phi_1(\tilde{\xi}\to\infty) - \phi_1(0)\right\},\end{aligned} \tag{C.13}$$

which for our case is

$$(\Delta P_y)_{d-e} = \frac{\rho_\infty U^2 c M^2 \tau^2}{\beta^2}\left\{\frac{4}{3M^2} - \frac{\Gamma M^2}{3\beta^2} + \frac{2\nu}{3}\right\}. \tag{C.14}$$

The sum of these two integrated pressure forces is thus

$$\Delta P_y = -\frac{2\rho_\infty U^2 c \tau^2}{\beta^4}\left\{\Gamma M^4 - 2\beta^2\right\}, \tag{C.15}$$

which is the familiar Busemann result for the second-order lift of the section considered.

Appendix D

The Structure of Shock Origin

We consider the second-order structure near the point of formation of a shock within the flow field, and its effect upon the point at which the shock is first formed. Since the characteristics are not revised after the first approximation, we cannot hope for the second-order theory to give detailed local information at the point of formation; however, we would hope that the solution at least approximates any such local behavior in some global sense. Such is the case, and we will, in fact, show that the shock displacement predicted by the second-order theory at the point of formation corresponds exactly to the shift in virtual origin the shock must have to account for this local behavior.

To demonstrate this fact, we consider the planar flow over a parabolic compression-expansion bump. The first-order theory predicts formation of a shock of finite initial strength for this case within the flow field. (See Figure D.01)

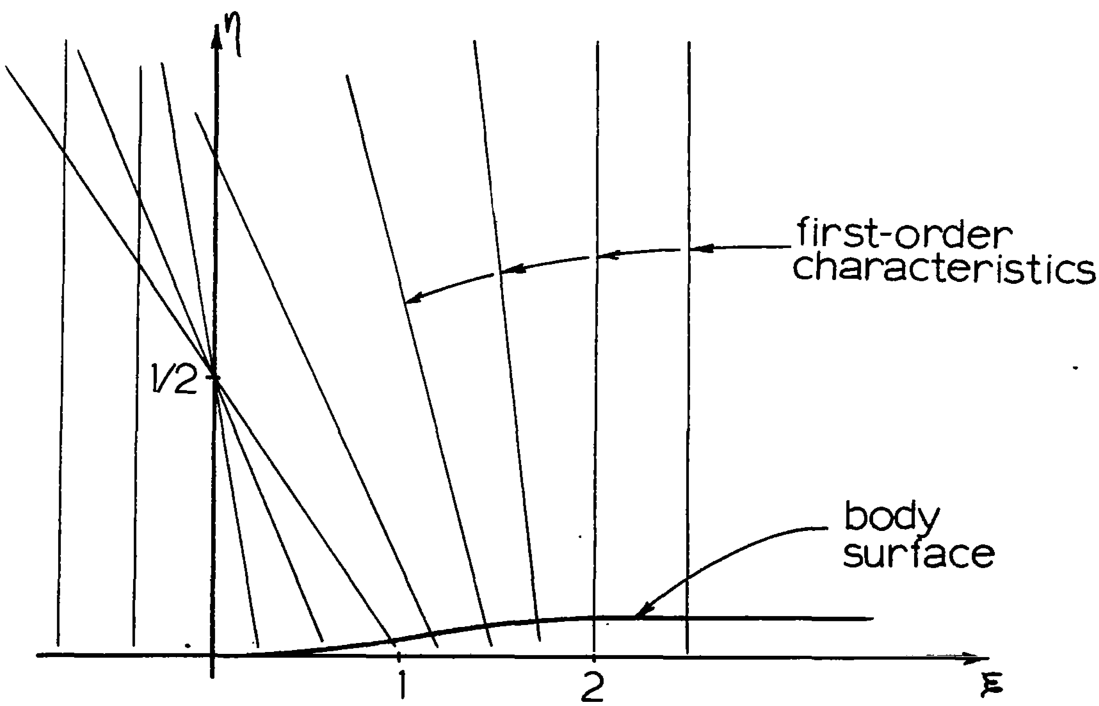

Figure D.01: Geometry of Shock Origin Problem.

First, the wave structure solution will be found, and the second-order displacement of the shock at its point of formation will be calculated. The geometry of the actual characteristics (accurate to second order) in the vicinity of the point of formation will then be considered to show how this displacement corresponds to the true local behavior.

1. Second-order wave structure solution

We consider the planar flow past the curved wall described by

$$
\begin{aligned}
f(\Xi_b) &= 0 \,, & \Xi_b &< 0 \\
&= \Xi_b^2 \,, & 0 < \Xi_b &< 1 \,, \\
&= 2-(2-\Xi_b)^2 \,, & 1 < \Xi_b &< 2 \,, \\
&= 2 \,, & 2 &< \Xi_b \,.
\end{aligned}
\tag{D.01}
$$

Then, we have by (3.35) that for $0 < \Xi_b < 1$,

$$
\phi_0 = \frac{-\Xi^2}{1-2\eta} \,, \tag{D.02}
$$

and for $1 < \Xi_b < 2$,

$$
\phi_0 = -2 + \frac{(2-\Xi)^2}{1+2\eta} \,. \tag{D.03}
$$

A shock starts at $(0, 1/2)$ with finite strength and obeys

$$
\Xi_0 = 2 - \sqrt{2(1+2\eta)} \,. \tag{D.04}
$$

The second-order solution, by equation (3.42) is seen to be, for $0 < \Xi_b < 1$ (and $\eta < 1/2$)

$$
\phi_1 = \left(\frac{28}{3} - \frac{4\Gamma M^2}{3\beta^2} + \frac{8\mu}{3}\right)\eta\,\Xi_b^3 + \left(-\frac{2}{3} + \frac{2}{M^2} - \frac{2\Gamma M^2}{3\beta^2}\right)\Xi_b^3 \tag{D.05}
$$

and for $1 < \Xi_b < 2$,

$$
\begin{aligned}
\phi_1 = {} & \left(\frac{28}{3} - \frac{4\Gamma M^2}{3\beta^2} + \frac{8\mu}{3}\right)\eta(2-\Xi_b)^2 + \left(\frac{16}{3} - \frac{4\Gamma M^2}{3\beta^2} + \frac{8\nu}{3}\right)\left(\frac{1}{\sqrt{2(1+2\eta)}} - \frac{1}{2}\right) \\
& + \left(\frac{2}{3} - \frac{2}{M^2} + \frac{2\Gamma M^2}{3\beta^2}\right)(2-\Xi_b)^3 + 4\left(\frac{1}{M^2} - 1\right)(2-\Xi_b) \\
& + \frac{8}{3} - \frac{4\Gamma M^2}{3\beta^2} \,.
\end{aligned}
\tag{D.06}
$$

Evaluating this along the position of the first-order shock yields

$$\phi_1(\xi_0,\eta) = \left(-\frac{8}{3}+\frac{8}{M^2}-\frac{4\Gamma M^2}{3\beta^2}+\frac{8\nu}{3}\right)\frac{1}{\sqrt{2(1+2\eta)}} + \left\{\left(\frac{224}{3}-\frac{32\Gamma M^2}{3\beta^2} + \frac{64\mu}{3}\right)\eta + \frac{16}{3} - \frac{16}{M^2} + \frac{16\Gamma M^2}{3\beta^2}\right\}\frac{1}{\{2(1+2\eta)\}^{3/2}} - \frac{2\Gamma M^2}{3\beta^2} - \frac{4\nu}{3} \quad , \tag{D.07}$$

and since

$$\phi_{0_\xi}(\xi_0,\eta) = \frac{-4}{\sqrt{2(1+2\eta)}} \quad , \tag{D.08}$$

the second-order shock displacement is given by (3.18) as

$$\xi_1 = -\left(\frac{\Gamma M^2}{6\beta^2}+\frac{\nu}{3}\right)\sqrt{2(1+2\eta)} + \left(-\frac{2}{3}+\frac{2}{M^2}-\frac{\Gamma M^2}{3\beta^2}+\frac{2\nu}{3}\right) + \left\{\left(\frac{28}{3}-\frac{4\Gamma M^2}{3\beta^2}+\frac{8\mu}{3}\right)\eta + \frac{2}{3} - \frac{2}{M^2} + \frac{2\Gamma M^2}{3\beta^2}\right\}\frac{1}{1+2\eta} \, . \tag{D.09}$$

Finally, the displacement due to second-order effects at the point of formation is

$$\xi_1(1/2) = 2 + \frac{1}{M^2} - \frac{2\Gamma M^2}{3\beta^2} + \frac{2\mu}{3} \, . \tag{D.10}$$

2. Interpretation in terms of local structure

The effect of second-order corrections to the characteristics is to give a local structure to the point at which all the characteristics from the compressive portion of the body converged according to the first-order theory. The geometry of these characteristics (accurate to second-order) and the shock wave in the neighborhood of the point of shock formation is now considered. The left-running characteristics are given in terms of our wave structure perturbation potential as

$$\left.\frac{d\xi}{d\eta}\right)_{ch} = \phi_\xi + \frac{M_T^2}{\beta}\left(\frac{\gamma}{2} - \frac{\Gamma M^2}{2\beta^2} + \mu\right)\phi_\xi^2 + \cdots \, . \tag{D.11}$$

Since the flow may be described as a simple wave to second order, the flow deflection angle

$$\theta = -\tau\phi_\xi + \frac{M^2\tau^2}{\beta}\left(\frac{1}{M^2}\phi_\xi^2 - \frac{\Gamma M^2}{2\beta^2}\phi_\xi^2\right) + \cdots , \qquad \text{(D.12)}$$

and, hence, also ϕ_ξ , are constant along these characteristics. Application of the boundary condition at the body surface gives

$$\phi_\xi = -f'(\xi_b) + \frac{M^2\tau}{\beta}\left\{-ff'' + \frac{1}{M^2}(f'^2 + ff'') - \frac{\Gamma M^2}{2\beta^2}f'^2\right\} , \qquad \text{(D.13)}$$

whence the equation of the characteristics becomes

$$\left.\frac{d\xi}{d\eta}\right)_{ch} = -f' + \frac{M^2\tau}{\beta}\left\{-ff'' + \frac{1}{M^2}(f'^2 + ff'') + \left(\frac{3}{2} - \frac{\Gamma M^2}{\beta^2} + \mu\right)f'^2\right\} + \cdots . \qquad \text{(D.14)}$$

Thus, for the parabolic bend described by

$$f(\xi_b) = \xi_b^2 , \qquad \text{(D.15)}$$

we have

$$\left.\frac{d\xi}{d\eta}\right)_{ch} = -2\xi_b + \frac{M^2\tau}{\beta}\left(12 + \frac{6}{M^2} - \frac{4\Gamma M^2}{\beta^2} + 4\mu\right)\xi_b^2 + \cdots , \qquad \text{(D.16)}$$

which may be integrated to give the equation of the family of characteristics

$$\xi - \xi_b + 2\xi_b\eta - \frac{M^2\tau}{\beta}\left(12 + \frac{6}{M^2} - \frac{4\Gamma M^2}{\beta^2} + 4\mu\right)\xi_b^2\eta = 0 , \qquad \text{(D.17)}$$

accurate to second order. Detailed information about the characteristics from only the compressive portion of the body is essential in describing the early stages of formation of the shock, as will later be seen.

The shock will first be formed in the vicinity of the envelope of these characteristics. This envelope is determined by eliminating ξ_b between the preceeding equation and the equation obtained by differentiating it with respect to the parameter, Ξ_b :

$$-1 + 2\eta - \frac{M^2\tau}{\beta}\left(24 + \frac{12}{M^2} - \frac{8\Gamma M^2}{\beta^2} + 8\mu\right)\Xi_b\eta = 0. \tag{D.18}$$

Thus, the envelope is parametrically described by

$$\begin{aligned} \Xi &= -\frac{M^2\tau}{\beta}\left(6 + \frac{3}{M^2} - \frac{2\Gamma M^2}{\beta^2} + 2\mu\right)\Xi_b^2 \,, \\ \eta &= \tfrac{1}{2} + \frac{M^2\tau}{\beta}\left(6 + \frac{3}{M^2} - \frac{2\Gamma M^2}{\beta^2} + 2\mu\right)\Xi_b \,; \end{aligned} \tag{D.19}$$

from which it is seen to be the upper half of the parabola with vertex at $(0, \tfrac{1}{2})$ and opening to the left if the factor

$$\varkappa = 6 + \frac{3}{M^2} - \frac{2\Gamma M^2}{\beta^2} + 2\mu \tag{D.20}$$

is positive, or the reflection about the origin (vertex) of that curve if $\varkappa$ is negative. In the vicinity of the origin of the shock, then, the problem naturally divides itself into two cases depending upon the algebraic sign of $\varkappa$. (For air, treated as a calorically perfect gas, $\varkappa = 0$ when M $\doteq$ 1.33, is negative for lower supersonic Mach numbers, and is positive for greater Mach numbers.) For positive $\varkappa$, the shock starts with zero strength at $(0, \tfrac{1}{2})$ -- the same point at which the finite-strength first-order theory shock originated -- and builds up to full strength in a region of scale $0(\tau)$ in the Ξ-η plane. For negative $\varkappa$, the shock again starts with zero strength at the point on the envelope corresponding to the characteristic from the steepest portion of the body, and develops for a short distance (again of $0(\tau)$ in the Ξ-η plane) with disturbed flow on both sides. See figures D.02 and D.03.

If $\varkappa = 0$, the envelope collapses to the point $(0, \tfrac{1}{2})$ as in the first-order theory, and the shock starts from that point with finite strength. Since the total length scale in this local development is short, in considering the path of the shock, we will use simply the characteristic-bisector rule with negligible error.

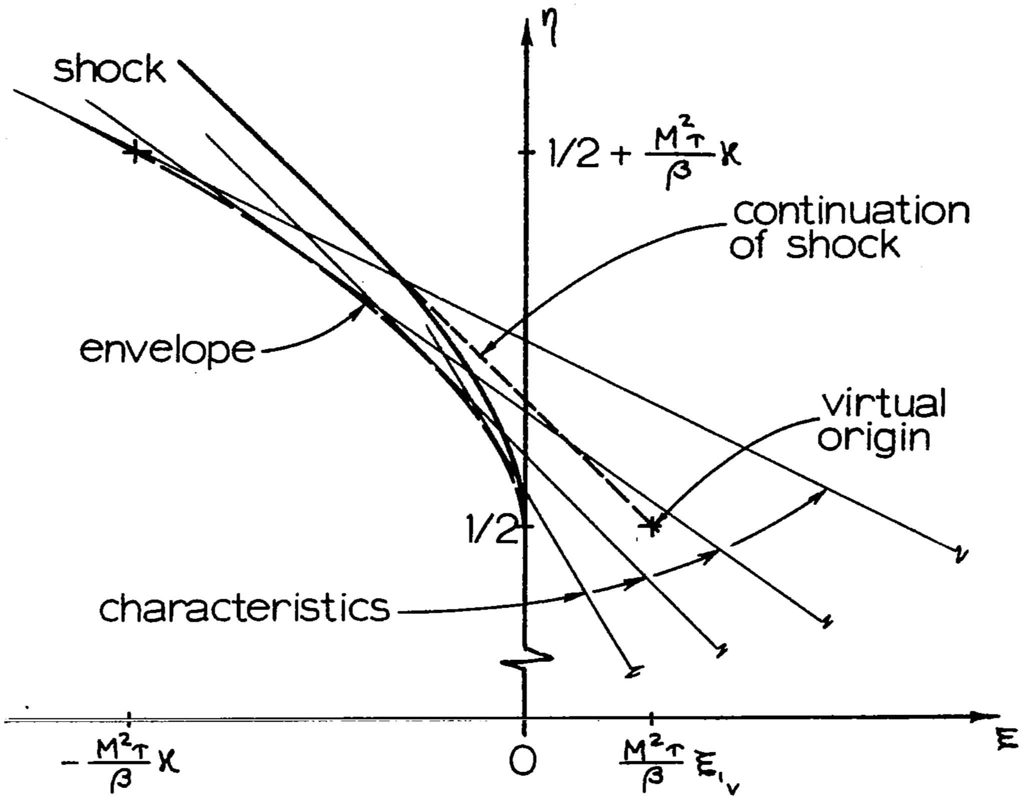

Figure D.02: Structure of Shock Origin ($\chi > 0$).

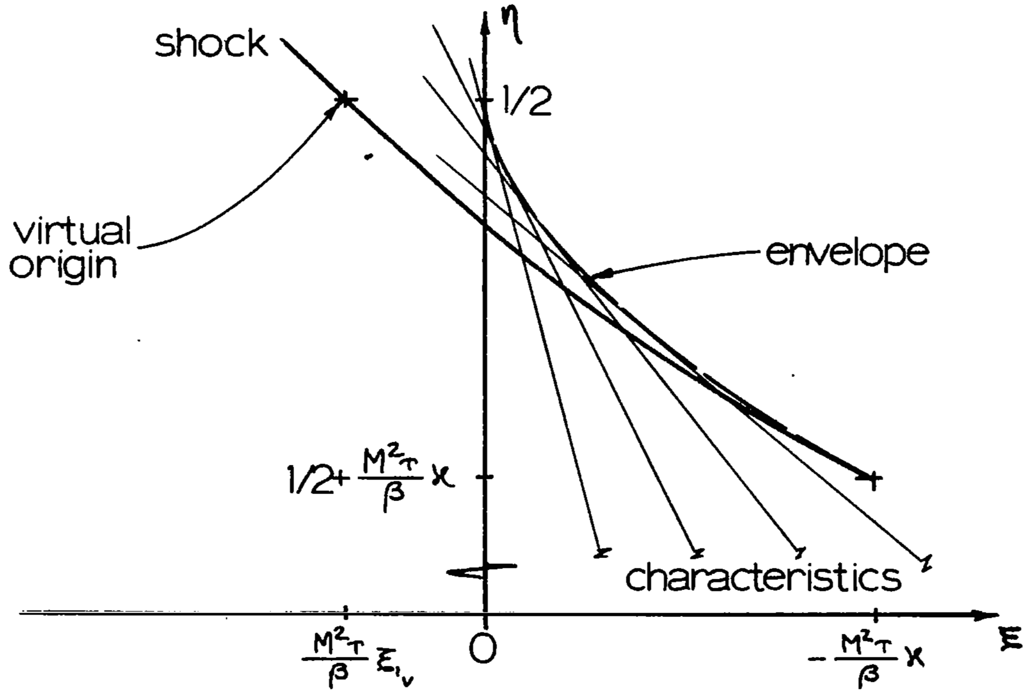

Figure D.03: Structure of Shock Origin ($\chi < 0$)

a. Case I, $\chi > 0$:

For the case when χ is positive, the shock starts at $(0, \frac{1}{2})$ with zero strength, and its path is determined as the bisector of the freestream characteristics and those coming from the body for $0 < \xi_b < 1$. The shock will intercept these characteristics coming from the body before they can form the envelope calculated in the preceding section. (See Figure D.02). We set up a local coordinate system with origin at the point $(0, \frac{1}{2})$ and

$$x = \frac{1}{\frac{M^2 T}{\beta}\chi}\,\xi\,,$$
$$y = \frac{1}{\frac{M^2 T}{\beta}\chi}\left(\eta - \tfrac{1}{2}\right). \qquad \text{(D.21)}$$

Then, the characteristics coming from the body are described by

$$x = -2\,\xi_b\left(y - \tfrac{1}{2}\xi_b\right), \qquad 0 < \xi_b < 1, \qquad \text{(D.22)}$$

and the slope of the shock at every point is determined by the bisector rule

$$\left.\frac{dx}{dy}\right)_{sh} = \frac{1}{2}\left(-2\xi_b + 0\right) = -\xi_b. \qquad \text{(D.23)}$$

Eliminating ξ_b from this last equation using (D.22) gives

$$\frac{dx}{dy} = -y - \sqrt{y^2 + x}\,. \qquad \text{(D.24)}$$

This equation cannot easily be solved in general due to its nonlinear nature, but the unique solution passing through the origin is easily verified to be

$$x = -\frac{3}{4}y^2. \qquad \text{(D.25)}$$

The shock follows this curve until the point where it intersects the characteristic coming from $\xi_b = 1$, (i.e., from the inflection point on the body). This point is found to be

$$x = -1/3 ,$$
$$y = 2/3 . \tag{D.26}$$

From this point on, the shock is determined to satisfy the appropriate relations between the freestream characteristics and those coming from the expansive portion of the body, as in the wave structure solution. If we analytically continue this shock shape back to the value $\eta = 1/2$ (i.e., y = 0), which is the origin according to first-order theory, we find

$$x_1 = 1/3$$

or equivalently,

$$\xi_{1_V} = 2 + \frac{1}{M^2} - \frac{2\Gamma M^2}{3\beta^2} + \frac{2\mu}{3} , \tag{D.27}$$

which is precisely the displacement predicted by the second-order wave structure theory. Thus, the displacement of the origin of the shock predicted by the second-order wave structure theory corresponds to the displacement of the virtual origin the shock must have to account for the local formation process.

b. Case II, $\varkappa < 0$:

For the case when $\varkappa$ is negative, the shock starts at the point on the envelope of characteristics corresponding to $\xi_b = 1$ with zero strength. The path of the shock is then determined as the bisector of the characteristics coming from the compressive ($0 < \xi_b < 1$) and expansive ($1 < \xi_b < 2$) portions of the body, respectively. (See Figure D.03). The shock will again intercept the characteristics coming from the forward portion of the body before the envelope is formed. Using the same local coordinate system as in Case I, the shock slope obeys

$$\left.\frac{dx}{dy}\right)_{sh} = y - \sqrt{y^2 - x} - 1 , \tag{D.28}$$

when the body has been assumed straight for $\xi_b > 1$ with no loss in accuracy. Again, a general solution to this equation is not found, but the particular solution satisfying the appropriate initial conditions is

$$x = \frac{3}{4} y^2 - \frac{1}{2} y - 1/4 . \tag{D.29}$$

The point after which the shock again becomes determined by the freestream characteristics and those coming from the expansive portion of the body is

$$x = 0, \qquad y = -1/3. \tag{D.30}$$

For this case, since the shock is formed for $\eta < 1/2$, we merely note the value of Ξ_{sh} when $\eta = 1/2$. It is

$$\Xi_{1/2} = 2 + \frac{1}{M^2} - \frac{2\Gamma M^2}{3\beta^2} + \frac{2\mu}{3}, \tag{D.31}$$

again corresponding to the displacement of the virtual origin of the shock predicted by the second-order wave structure theory.

Appendix E

Some Numerical Comparisons

The primary importance of higher-order asymptotic theories is the justification they provide for the lower-order theories, by showing that they are the first step(s) in a rational approximation to the solution in some limit. Since the series thus produced are not necessarily convergent, the inclusion of higher-order terms can worsen numerical accuracy for calculations at finite values of the appropriate small parameter -- a fact for which asymptotic series are somewhat notorious. It is, however, still a matter of interest to see how the inclusion of such higher-order terms affects numerical results, and, for that reason two such examples are included here. The two simplest examples in each, planar and axisymmetric flows, are considered because (1) exact inviscid calculations were readily available for comparison, and (2) only the simplest geometries are conducive to solution in the axisymmetric case. The shock angles predicted for these flows are calculated and compared with the exact inviscid solutions in the following sections.

E.1. Semi-infinite Wedge (Planar flow)

The body shape function for a semi-infinite wedge of semi-angle τ is

$$\begin{aligned} f(\xi_b) &= 0 , \qquad \xi_b < 0 , \\ &= 1 , \qquad 0 < \xi_b . \end{aligned} \tag{E.01}$$

Thus, in the region of disturbed flow, the first-order potential is

$$\phi_{o_\xi} = -1 , \tag{E.02}$$

and the shock separating this region from the undisturbed stream obeys

$$\xi_o = -\gamma/2 . \tag{E.03}$$

The second-order potential, evaluated at the first-order shock location, $\xi = \xi_o$, is then

$$\phi_1(\xi_0, \eta) = \left(1 + \frac{1}{2M^2} - \frac{3\Gamma M^2}{8\beta^2} + \mu/3 - \nu/12\right)\eta \quad , \tag{E.04}$$

whence

$$\xi_1 = \left(1 + \frac{1}{2M^2} - \frac{3\Gamma M^2}{8\beta^2} + \mu/3 - \nu/12\right)\eta \quad . \tag{E.05}$$

The shock location, to second order is thus

$$\xi_{sh} = -\frac{\eta}{2}\left\{1 - \frac{M_T^2}{\beta}\left(2 + \frac{1}{M^2} - \frac{3\Gamma M^2}{4\beta^2} + \frac{2\mu}{3} - \nu/6\right)\right\} , \tag{E.06}$$

whence

$$\cot\theta_{sh} = \left.\frac{dx}{dy}\right)_{sh} = \beta\left\{1 - \frac{\bar{K}}{2}\left(1 - \frac{M_T^2}{\beta}\left(2 + \frac{1}{M^2} - \frac{3\Gamma M^2}{4\beta^2} + \frac{2\mu}{3} - \frac{\nu}{6}\right)\right)\right\} . \tag{E.07}$$

Numerical values for the shock angle for flows in air (treated as a calorically perfect gas with $\Gamma = 1.2$, $\mu = -1.1$, and $\nu = 0.4$) are compared with results of exact inviscid calculations from the Ames Research Staff (1953) in Figures E.01 and E.02. The second-order theory is seen to give considerable improvement in numerical accuracy even for quite high Mach numbers (e.g., $M_\infty = 5$).

E.2. Semi-infinite Cone (Axisymmetric flow)

The calculation for the cone is much more complicated than the wedge for two reasons. First, an additional scaling region near the body must be considered, and this "slender body" solution used to determine the local solution by matching. Secondly, both of these inner solutions must be determined to third order to determine the true second-order wave structure solution.

All of the details of this extremely tedious calculation will not be given here. The general nature of the slender body theory from the viewpoint of matched asymptotic expansions may be found in Ashley and Landahl (1965). No peculiar difficulties are encountered in extending the solution to third order (for the case of a cone!) beyond algebraic tedium. Only the radial derivative of the third-order solution was obtained, since this determines the local solution. (The complete solution is necessary if we are interested in pressures near the body.)

For the flow over a slender cone of semi-angle τ , the first-order shock strength is of $O(\tau^4)$, whence a velocity potential exists to the required third order (which describes perturbations of $O(\tau^6)$ in the local region. This local solution is of the form

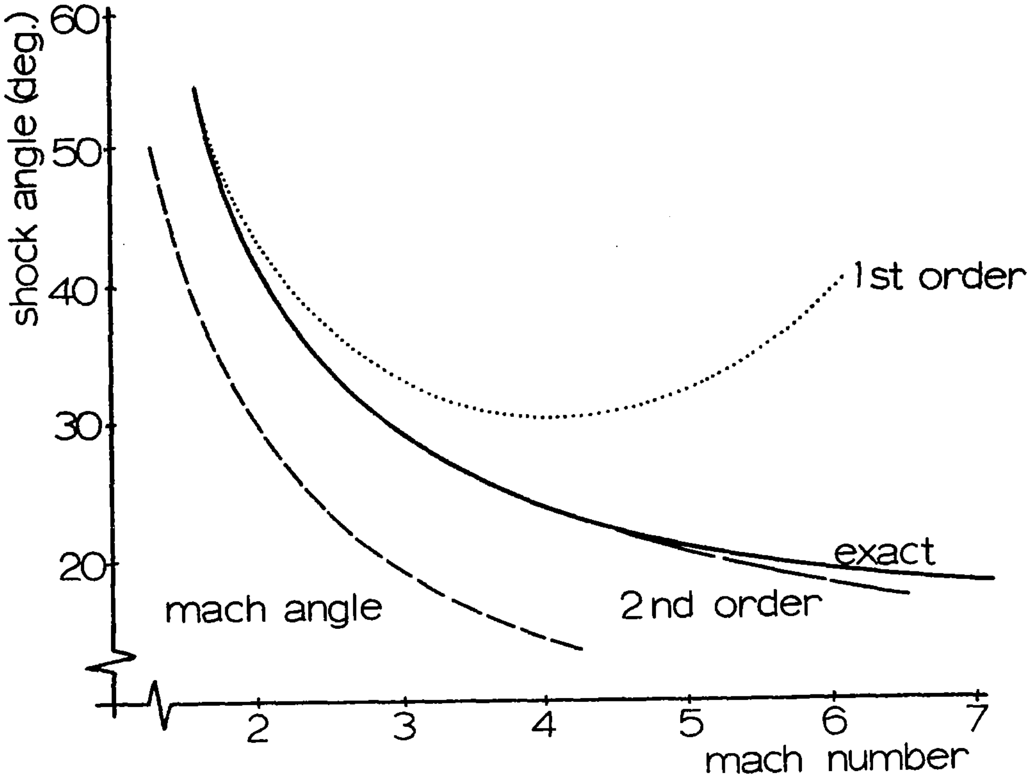

Figure E.01: Shock Angle on Twelve-degree Wedge.

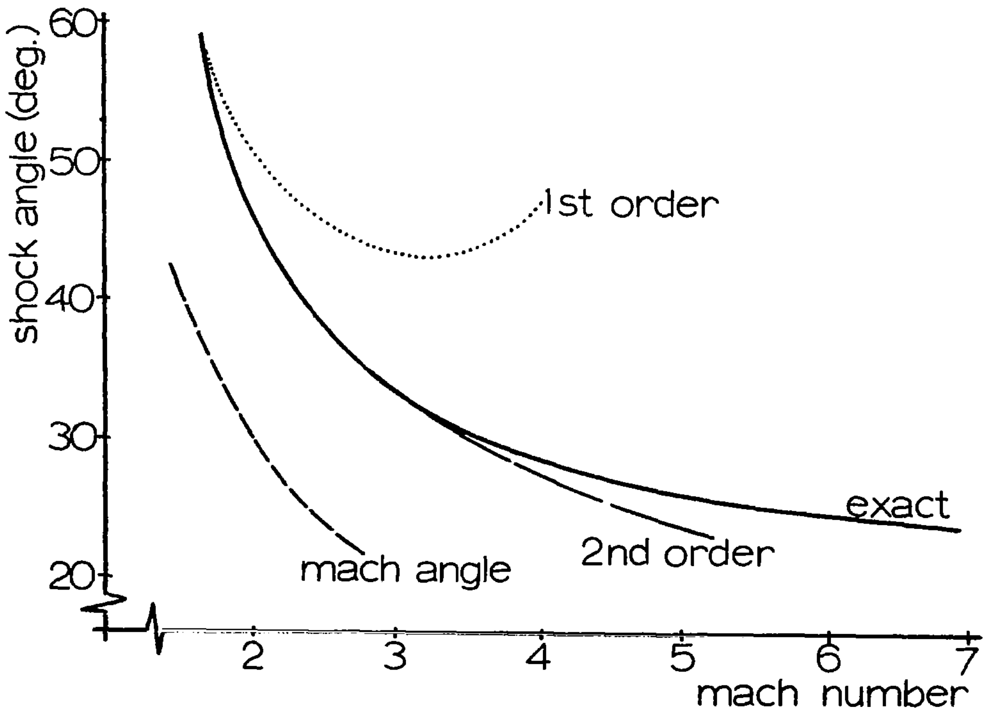

Figure E.02: Shock Angle on Sixteen-degree Wedge.

$$\frac{1}{\Xi}\varphi_{\pi} = \mathcal{A}\frac{\sqrt{1-t^2}}{t} + M_T^2\tau^2\left\{\mathcal{B}\frac{\sqrt{1-t^2}}{t} + \cdots\right\}$$
$$+ M_T^2\tau^4\left\{\mathcal{C}\frac{\sqrt{1-t^2}}{t} + \frac{\Gamma M^4}{\beta^2}\left(-10 + \frac{3\Gamma M^2}{2\beta^2} - 2\mu\right)\frac{\sqrt{1-t^2}}{t}\ln\left(\frac{t^2}{1-t^2}\right)\right\}, \tag{E.08}$$

where the conical variable

$$t = \pi/\Xi \tag{E.09}$$

has been introduced, and particular integral terms which do not enter directly into the determination of the wave structure solution have been omitted for clarity. Of these omitted terms, some will automatically match with particular integral terms in the wave structure solution, and the others will be negligible. The constants in this local solution are determined from the slender body matching to be

$$\begin{aligned}
\mathcal{A} &= 1 \,, \\
\mathcal{B} &= (2 - 1/M^2)\ln\frac{2}{\tau\beta} - 1 - 1/2M^2 - \frac{\Gamma M^2}{\beta^2} \,, \\
\mathcal{C} &= -\frac{2\Gamma M^4}{\beta^2}\left\{\mathcal{B} + 1/2 + \frac{\sigma-3}{2}\ln 2 + \frac{\Gamma M^2}{\beta^2}\left(\frac{3}{4} + \ln 2\right)\right\} \\
&\quad + 2M^2(1/4 - \mathcal{B}) + 4\frac{\Gamma^2 M^4}{\beta^2}\ln 2 + \ln\frac{2}{\tau\beta}\left\{\left(-6 + 6M^2 + \frac{1}{M^2} - 2\Gamma M^2\right)\ln\frac{2}{\tau\beta} - 1 - 4M^2 + 1/M^2 - \Gamma M^2\right\} \\
&\quad - \frac{\Gamma M^2}{\beta^2} + 1/8M^2 - \Gamma M^2/4 - 1/2 \,.
\end{aligned} \tag{E.10}$$

The large distance (i.e., $t \to 1$) behavior of the local solution is given by

$$\varphi \sim -\frac{2\bar{K}_{S1}}{\eta}\frac{(2\Xi)^{3/2}}{3} - \frac{2\bar{K}_{S1}}{\eta}M_T^2\tau^2\,\mathcal{B}\frac{(2\Xi)^{3/2}}{3}$$
$$-\frac{2\bar{K}_{S1}^2}{\eta}M_T^2\tau^2\left\{\frac{\beta^2}{\Gamma M^4}\mathcal{C} + \left(-10 + \frac{3\Gamma M^2}{2\beta^2} - 2\mu\right)\ln\frac{\eta}{2\bar{K}_{S1}\sqrt{2\xi}}\right\}\frac{(2\xi)^{3/2}}{3}, \tag{E.11}$$

which must match with the small-distance behavior of the wave structure solution,

$$\left(\frac{1}{\sqrt{\pi}}\psi\right) \sim -\frac{2\bar{K}_{S1}}{\eta}F - \frac{2\bar{K}_{S1}}{\eta}M_T^2\tau^2 G - \frac{2\bar{K}_{S1}^2}{\eta}M_T^2\tau^2\left\{H + \left(\frac{-10}{3} + \frac{\Gamma M^2}{2\beta^2} - \frac{2\mu}{3}\right)F'^3\ln\eta\right\} + \cdots, \tag{E.12}$$

where, again terms which do not enter directly into the determination of the unknown functions in the wave structure solution have been omitted. The wave structure solution is thus determined by the conditions

$$
\begin{aligned}
F(\Xi_b) &= \frac{(2\Xi_b)^{3/2}}{3}, \\
G(\Xi_b) &= \mathcal{B}\,\frac{(2\Xi_b)^{3/2}}{3}, \qquad \text{(E.13)} \\
H(\Xi_b) &= \left\{\frac{\beta^2}{\Gamma M^4}\mathcal{C} + \left(10 - \frac{3\Gamma M^2}{2\beta^2} + 2\mu\right)\ln 2K_{S1}\sqrt{2\Xi_b}\right\}\frac{(2\Xi_b)^{3/2}}{3},
\end{aligned}
$$

where

$$\sqrt{2\Xi_b} = \eta\left\{1 + \sqrt{1 + 2\Xi/\eta^2}\right\}. \qquad \text{(E.14)}$$

The first-order shock position is then found to be

$$\Xi_0 = -\frac{3}{8}\eta^2. \qquad \text{(E.15)}$$

The one-and-one-half-order correction is

$$\Xi_{1/2} = -\frac{3}{4}\mathcal{B}\eta^2. \qquad \text{(E.16)}$$

And the second-order correction is

$$
\begin{aligned}
\Xi_1 = \Big\{ &-\frac{3}{4}\frac{\beta^2}{\Gamma M^4}\mathcal{C} + \left(-\frac{15}{2} + \frac{9\Gamma M^2}{8\beta^2} - \frac{3\mu}{2}\right)\ln 3K_{S1} \\
&+ 2 - \frac{151}{320}\frac{\Gamma M^2}{\beta^2} - 7/8 - \frac{3}{2}\frac{\beta^2}{\Gamma M^2}\mathcal{B}^2\Big\}\eta^2. \qquad \text{(E.17)}
\end{aligned}
$$

Thus, the shock angle is given by

$$
\begin{aligned}
\cot\theta_{Sh} = \beta\Big\{1 - \frac{3}{2}K_{S1}^2\Big\{1 + 2M_T^{2}{}^{2}\mathcal{B} + K_{S1}M_T^{2}{}^{2}\Big\{2\frac{\beta^2}{\Gamma M^4}\mathcal{C} \\
+ \left(10 - 3\frac{\Gamma M^2}{\beta^2} + 4\mu\right)\ln 3K_{S1} - 16/3 + \frac{151}{120}\frac{\Gamma M^2}{\beta^2} \qquad \text{(E.18)} \\
+ 7/3 + 4\frac{\beta^2}{\Gamma M^2}\mathcal{B}^2\Big\}\Big\}\Big\}.
\end{aligned}
$$

Numerical values for the shock angle for flows in air are again compared with those from exact inviscid calculations (Ames Research Staff (1953)), and are presented in Figures E.03 and E.04.

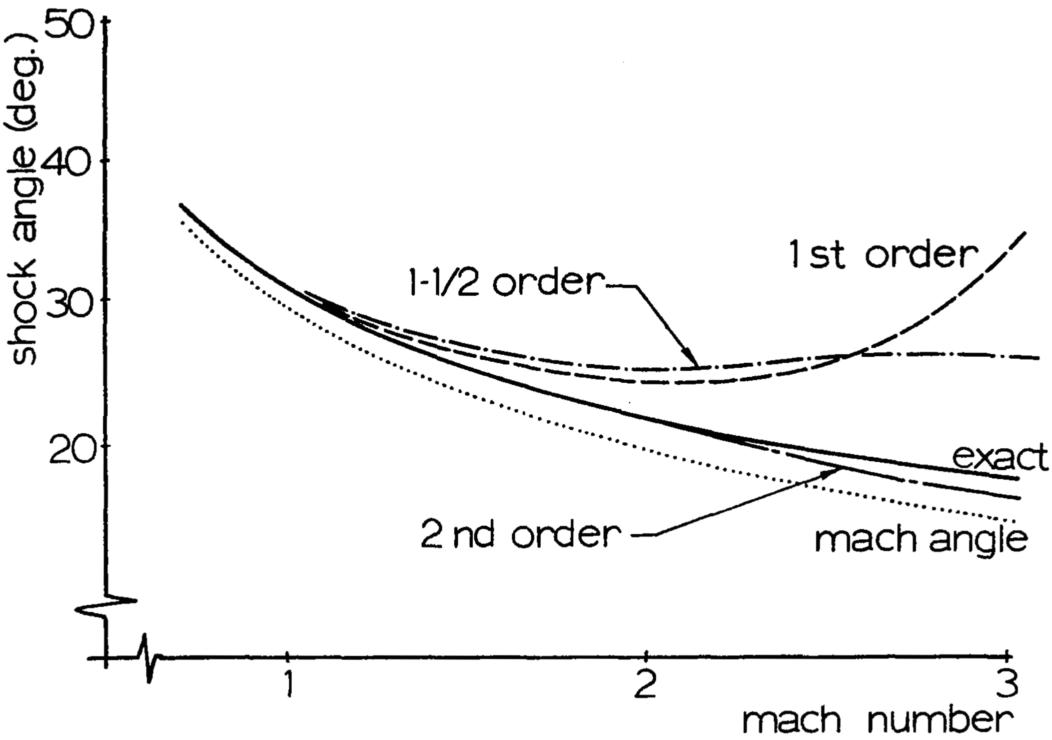

Figure E.03: Shock Angle on Ten-degree Cone.

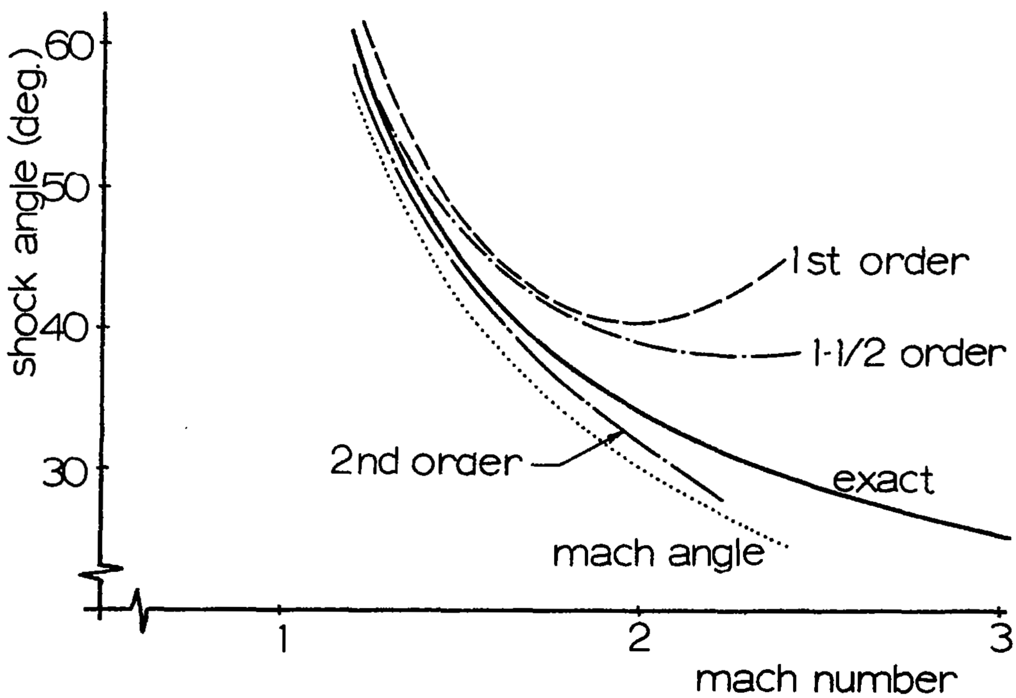

Figure E.04: Shock Angle on Fifteen-degree Cone.

Note that, generally, the one-and-one-half-order solution does not give results that are appreciably better than the first-order theory, and we must go to the full second-order theory to achieve noticeable improvement. That the second-order theory, which is based upon the local third-order theory gives increased accuracy for these finite values of body thickness, is probably due in part to the fact that the theory is of second order near the shock. Third-order theories are often very bad for numerical calculations at finite values of the asymptotic parameter.

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