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APPLICATIONS OF ANALYTIC FUNCTION THEORY TO ANALYSIS OF SINGLE-SIDEBAND ANGLE-MODULATED SYSTEMS

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16. Abstract

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SUMMARY

This paper applies the theory and notation of complex analytic time functions and stochastic processes to the investigation of the single-sideband angle-modulation process. Low-deviation modulation and linear product detection in the presence of noise are carefully examined for the case of sinusoidal modulation. Modulation by an arbitrary number of sinusoids or by modulated subcarriers is considered. Also treated is a method for increasing modulation efficiency. The paper concludes with an examination of the asymptotic (low-noise) performance of nonlinear detection of a single-sideband carrier which is heavily frequency modulated by a Gaussian process.

INTRODUCTION

The theory of complex analytic time functions and stochastic processes is applied herein to the mathematical examination of single-sideband (SSB) angle-modulated sinusoidal carriers. Emphasis is on application of the theory, rather than on derivation of it. Elements of the theory and the accompanying notation are fully developed in reference 1.

There are two purposes for this paper. First, and most important, is the desire to determine mathematically the communication efficiency of SSB angle-modulation. This determination is made separately for low-index sinusoidal modulation with product demodulation and for high-index random modulation with nonlinear (discriminator) demodulation. A secondary purpose is to investigate the application of the theory of analytic functions to the solution of SSB modulation and detection problems.

First the basic mathematical models for band-pass noise and signals and for their correlation and spectral functions are established on the basis of the complex analytic notation. Next, "modulation functions" are defined for the signals to be treated, and low-deviation modulation by subcarriers is discussed.

In the section on "Sinusoidal Single-Sideband Amplitude-Phase Modulation and Detection," four unique models for signals having one subcarrier are presented and their power structure is derived. An observation is made on the transmission efficiency achievable with the four signal models. Then a more efficient signal is synthesized from two of these models. One of the four models is examined and some observations are made for the case of an arbitrary number of modulating sinusoids. Then the consequences of modulated subcarriers are investigated. The section concludes with a modeling of linear product demodulation of one of the four signals in the presence of noise.

The section on "Gaussian Single-Sideband Frequency Modulation" begins with derivations of the average power of a Gaussian SSB-FM signal. Next the equivalent phase of the sum of the modulated signal plus Gaussian channel noise is investigated. A low-noise approximation to the equivalent phase process is used to obtain an asymptotic result for the phase noise autocorrelation function. The section concludes with a modeling of the nonlinear demodulation of the signal in the presence of noise under the assumptions of low noise and high modulation.

SYMBOLS AND NOTATION

The symbols and mathematical notation for the body of the paper are defined below. Symbols used only in the appendixes are not listed here. Since the appendixes are short and self-contained, each new symbol is defined within each appendix.

Subscripting, as applied to functions, is standard. Thus, the symbol $R_{xy}(\ )$ means the function $R(\ )$ which is related to functions $x(\ )$ and $y(\ )$. In some cases double subscripting is used. Thus, the symbol P_{s_i} indicates the ith value of P_s. General mathematical notation is standard except as detailed below.

Mathematical notation:

$E\{\ \}$	statistical expectation
$j = \sqrt{-1}$	
$L(\ )$	linear transformation
$Re(\ )$	real part
$\hat{(\ )}$	Hilbert transform
$\lvert\ \rvert$	absolute value
$\triangleq$	equality by definition

$\approx$	approximate equality
$( )^*$	complex conjugate
$\dot{( )}$	time derivative
$( )''$	second derivative

Real numbers:

A	amplitude constant, dimensionless
B	bandwidth constant, hertz
c	arbitrary constant, real or complex, dimensionless
N	noise power constant, watts
P, S	signal component power, watts
Q, η	noise spectral density amplitude, watts per radian per second
T	limit of integration, seconds
t, τ	time variable, seconds
β	amplitude constant, radians
θ	constant or dummy variable, radians
σ^2	variance of a stochastic process, radians^2
ϕ	constant or dummy variable, radians
ω	frequency variable or constant, radians per second

Deterministic functions and stochastic processes:

A( )	real envelope function of modulated signal (deterministic or stochastic, depending on context)
f( )	real low-pass message function (deterministic or stochastic, depending on context)
g( )	real low-pass message function (deterministic or stochastic, depending on context)
I( )	modified Bessel function
J( )	Bessel function
m( )	complex low-pass modulation function (deterministic or stochastic, depending on context)
N( )	noise spectral density
n( )	real band-pass stochastic noise process
R( )	correlation function or envelope function of signal plus noise, depending on context
r( )	sum function of signal plus noise
S( )	general spectral density (Fourier transform of R( ))
s( )	real band-pass signal function (deterministic or stochastic, depending on context)
x( )	real low-pass signal or noise function (deterministic or stochastic, depending on context)
y( )	real low-pass signal or noise function (deterministic or stochastic, depending on context)

$z(\ )$ complex low-pass signal or noise function (deterministic or stochastic, depending on context)

$\delta(\ )$ the delta function

$\theta(\ )$ subcarrier phase function or noise perturbation of the equivalent phase function of signal plus noise, depending on context

$\lambda(\ )$ first zone output function of an ideal limiter

$\nu(\ )$ complex band-pass stochastic noise process

$\Phi(\ )$ statistical characteristic function

$\phi(\ )$ real phase function of a modulated signal (deterministic or stochastic, depending on context)

$\psi(\ )$ complex band-pass signal function (deterministic or stochastic, depending on context)

ANALYTIC SIGNAL AND NOISE NOTATION

In the analysis of SSB angle-modulated systems, maximum use is made of analytic notation. That is, real signal and noise processes are expressed as the real parts of complex analytic processes to exploit the advantages inherent in such notation. Derivations of the properties of the analytic notation have been made elsewhere (refs. 1 to 6). Here, the properties of the notation are simply assumed.

Given a real time function $x(t)$, which may be either a deterministic function (signal) or sample function from a random process (noise or noise-modulated carrier), $x(t)$ may be represented as the real part of a complex analytic function $z(t)$, where

$$z(t) = x(t) + jy(t) \tag{1}$$

In the random case, it is assumed without loss of generality that $z(t)$, $x(t)$, and $y(t)$ are zero-mean and weakly stationary.

The modulated signal and noise processes dealt with in this paper are of the band-pass type. That is, the frequency spectra of both the signals and the accompanying noise are assumed to be band-limited. To describe such processes suitably it is convenient to write them in "polar form" as $z(t) \exp(j\omega_c t)$. The exponential function serves merely to translate the spectrum of $z(t)$ to the neighborhood of the carrier frequency ω_c. If $z(t)$ is assumed to have a low-pass spectrum, then $z(t) \exp(j\omega_c t)$ has a band-pass spectrum. Hence, in this paper it is assumed that $z(t)$ is low-pass. Such a restriction is not necessary in general and is used strictly for simplification.

Band-Pass Noise Processes

A real band-pass noise process $n(t)$ is denoted as

$$n(t) = \mathrm{Re}\left[\nu(t)\right] \tag{2}$$

where

$$\nu(t) = z(t) \exp(j\omega_c t) \tag{3}$$

Thus,

$$n(t) = x(t) \cos \omega_c t - y(t) \sin \omega_c t \tag{4}$$

Equation (4) is a well-known formulation for a band-pass noise process in terms of two low-pass processes, $x(t)$ and $y(t)$, and a reference frequency ω_c (ref. 7). The requirements on $x(t)$ and $y(t)$ to insure that $n(t)$ is zero-mean and weakly stationary are given in terms of the correlation functions as

$$\left.\begin{aligned} R_{xx}(\tau) &= R_{yy}(\tau) \\ R_{yx}(\tau) &= -R_{yx}(-\tau) \end{aligned}\right\} \tag{5}$$

where $x(t)$ and $y(t)$ are individually zero-mean.

If ω_c is in the set of frequencies for which the spectrum of $n(t)$ is nonzero, then equations (5) are satisfied if $x(t)$ and $y(t)$ are individually and jointly Gaussian (ref. 8). If ω_c is in the set of frequencies for which the spectrum of $n(t)$ is zero, then $x(t) + jy(t)$ is analytic, or conjugate analytic, and equations (5) are satisfied, as is the additional relation

$$R_{yx}(\tau) = \pm\hat{R}_{xx}(\tau) \tag{6}$$

where the caret denotes Hilbert transform (ref. 1). The plus and minus signs are chosen, respectively, according to whether the frequency domain of the spectrum of $n(t)$ is above or below ω_c.

In dealing with signals which have single sidebands with respect to a carrier frequency ω_c, it is convenient to reference the channel noise to the same frequency ω_c. In this case, the noise process is also single-sideband in nature and equation (6) applies. Although in this case it is not strictly necessary, the channel noise is taken as Gaussian.

For computational use in the remainder of the paper, the relations between the spectral densities of $x(t)$, $y(t)$, and $n(t)$ are now derived. An idealizing assumption is made for computational simplicity. It is assumed that the noise process $n(t)$ results from filtering a white Gaussian process with an ideal upper sideband filter. That is, the spectral density $S_{nn}(\omega)$ of $n(t)$ is taken as

$$S_{nn}(\omega) = \left\{ \begin{array}{ll} Q; & \left(0 \leqq |\omega| - \omega_c \leqq \omega_u\right) \\ 0; & (\text{all other} \quad \omega) \end{array} \right\} \tag{7}$$

where ω_c, the carrier reference frequency, is the "lower cutoff frequency" for $S_{nn}(\omega)$ and $\omega_c + \omega_u$ is the "upper cutoff frequency."

From equation (A11) of appendix A, the spectra of $x(t)$ and $y(t)$ are

$$S_{xx}(\omega) = S_{yy}(\omega) = \left\{ \begin{array}{ll} Q; & \left(|\omega| \leqq \omega_u\right) \\ 0; & (\text{all other} \quad \omega) \end{array} \right\} \tag{8}$$

Figure 1 is given to clarify the relations expressed in equations (7) and (8).

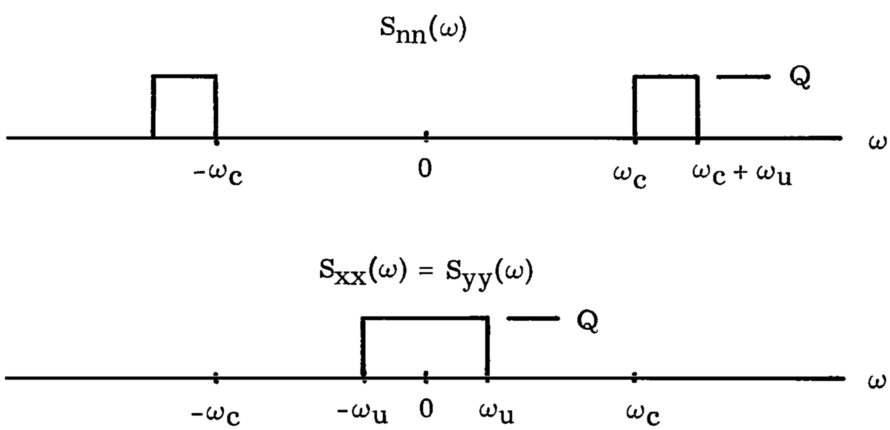

Figure 1.- Spectral relations.

Band-Pass Signal Processes

A real band-pass signal process $s(t)$ is handled in a manner similar to that for noise processes. That is, $s(t)$ is taken as the real part of some analytic $\psi(t)$, where $\psi(t)$ has the same form as the $\nu(t)$ in equation (3). That is, let

$$\left.\begin{aligned} s(t) &= \mathrm{Re}\left[\psi(t)\right] \\ \psi(t) &= m(t)\,\exp\left(j\omega_c t\right) \end{aligned}\right\} \tag{9}$$

where $m(t)$ is a "modulation function" which is, in general, complex.

The properties of $m(t)$ may be determined from examination of $s(t)$. The most general form for a "modulated carrier" is

$$s(t) = A(t)\cos\left[\omega_c t + \phi(t)\right] \tag{10}$$

where $A(t)$ is an amplitude function and $\phi(t)$ is a phase function. Various forms of modulation are obtained for different choices of $A(t)$ and $\phi(t)$. For $\phi(t)$ held constant, double-sideband amplitude modulation results. For $A(t)$ held constant, double-sideband phase or frequency modulation results. If $A(t)$ and $\phi(t)$ vary simultaneously and if they are related functionally, then various hybrid modulations are obtainable which have, in general, nonsymmetric sidebands.

To obtain equation (10) from equations (9) requires

$$m(t) = A(t)\exp\left[j\phi(t)\right] \tag{11}$$

Now $m(t)$ may be either deterministic or stochastic, depending on which type of modulated carrier is being considered. For the stochastic case, sufficient conditions for $s(t)$ to be weakly stationary are for $m(t)$ to be zero-mean and (from ref. 1):

$$E\left\{m(t+\tau)\,m(t)\right\} = 0 \tag{12}$$

It should be noted in passing that for the stochastic case, the phase function $\phi(t)$ of $m(t)$ may require a uniformly distributed random constant to satisfy equation (12). This matter is treated in detail in appendix B.

Given the modulation function $m(t)$, the autocorrelation and spectral density functions of the real signal $s(t)$, in the stochastic case, are obtained from equations (A16) and (A22), respectively, as

$$R_{ss}(\tau) = \frac{1}{2} \mathrm{Re}\left[R_{mm}(\tau) \exp(j\omega_c\tau)\right] \tag{13}$$

$$S_{ss}(\omega) = \frac{1}{4}\left[S_{mm}(\omega-\omega_c) + S^*_{mm}(\omega+\omega_c)\right] \tag{14}$$

Modulation Functions

Bedrosian (ref. 6) has given the modulation functions $m(t)$ for many types of modulated signals. Those of chief interest in this paper are the angle-modulated signals. By an angle-modulated signal, what is explicitly meant is a modulated carrier where a "message function" $f(t)$ is linearly related to the phase function $\phi(t)$. That is, for some $f(t)$ which is to be transmitted and recovered, $f(t)$ must be recoverable by a linear operation on $\phi(t)$. There are also certain hybrid signals where $f(t)$ is recovered linearly from $s(t)$.

For a deterministic message function $f(t)$, the most general modulation function $m(t)$ which results in an SSB angle-modulated signal is

$$m(t) = \exp\left\{cL\left[f(t)+j\hat{f}(t)\right]\right\} \tag{15}$$

where L denotes a linear transformation and c is a complex constant. Here $m(t)$ is the exponent function of an analytic function and, hence, is analytic.

For stochastic $s(t)$, it is sufficient for this analysis to consider $m(t)$ as simply

$$m(t) = \exp\left\{j\left[f(t) + jg(t)\right]\right\} \exp(j\theta) \tag{16}$$

where $f(t) + jg(t)$ is analytic, and θ is uniformly distributed in $[-\pi, \pi]$. From appendix B it is seen that $m(t)$, as given in equation (16), satisfies the stationarity requirement of equation (12) regardless of the statistics of $f(t) + jg(t)$.

For the stationary $m(t)$ of equation (16), the autocorrelation function is

$$\begin{aligned} R_{mm}(\tau) &\triangleq E\left\{m(t+\tau)\, m^*(t)\right\} \\ &= E\left\{\exp\left\{j\left[f(t+\tau) + jg(t+\tau) - f(t) + jg(t)\right]\right\}\right\} \\ &\triangleq \Phi(1,j,-1,j;\tau) \end{aligned} \tag{17}$$

where Φ denotes characteristic function (ref. 9).

SINUSOIDAL SINGLE-SIDEBAND AMPLITUDE-PHASE MODULATION AND DETECTION

Unique Modulated Carrier Forms

From equation (15) may be written four unique modulation functions for SSB (upper-sideband) sinusoidal modulation:

$$\left.\begin{aligned} m_1(t) &= \exp\left\{j\left[f(t) + j\hat{f}(t)\right]\right\} \\ m_2(t) &= \exp\left\{-j\left[f(t) + j\hat{f}(t)\right]\right\} \\ m_3(t) &= \exp\left\{1\left[f(t) + j\hat{f}(t)\right]\right\} \\ m_4(t) &= \exp\left\{-1\left[f(t) + j\hat{f}(t)\right]\right\} \end{aligned}\right\} \tag{18}$$

Now, denote

$$f(t) \triangleq \beta \cos \omega_m t \tag{19}$$

where β and ω_m are constants. Then the analytic signals $\psi_i(t)$, corresponding to the $m_i(t)$, for $i = 1, 2, 3,$ and 4 are

$$\left.\begin{aligned} \psi_1(t) &= \exp\left\{\beta \exp\left[j\left(\omega_m t + \frac{\pi}{2}\right)\right]\right\} \exp\left(j\omega_c t\right) \\ \psi_2(t) &= \exp\left\{-\beta \exp\left[j\left(\omega_m t + \frac{\pi}{2}\right)\right]\right\} \exp\left(j\omega_c t\right) \\ \psi_3(t) &= \exp\left\{\beta \exp\left[j\omega_m t\right]\right\} \exp\left(j\omega_c t\right) \\ \psi_4(t) &= \exp\left\{-\beta \exp\left[j\omega_m t\right]\right\} \exp\left(j\omega_c t\right) \end{aligned}\right\} \tag{20}$$

If the Maclaurin series expansion for the exponent function is used, equations (20) yield

$$\left.\begin{aligned}
\psi_1(t) &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \exp\left\{j\left[\left(\omega_c + n\omega_m\right)t + n\,\frac{\pi}{2}\right]\right\} \\
\psi_2(t) &= \sum_{n=0}^{\infty} (-1)^n \frac{\beta^n}{n!} \exp\left\{j\left[\left(\omega_c + n\omega_m\right)t + n\,\frac{\pi}{2}\right]\right\} \\
\psi_3(t) &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \exp\left\{j\left[\left(\omega_c + n\omega_m\right)t\right]\right\} \\
\psi_4(t) &= \sum_{n=0}^{\infty} (-1)^n \frac{\beta^n}{n!} \exp\left\{j\left[\left(\omega_c + n\omega_m\right)t\right]\right\}
\end{aligned}\right\} \tag{21}$$

Now the real signals $s_i(t)$ corresponding to the $\psi_i(t)$ are given in the form of equation (10) and in series form as

$$\left.\begin{aligned}
s_1(t) &= \exp\left(-\beta \sin \omega_m t\right)\cos\left(\omega_c t + \beta \cos \omega_m t\right) = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \cos\left[\left(\omega_c + n\omega_m\right)t + n\,\frac{\pi}{2}\right] \\
s_2(t) &= \exp\left(\beta \sin \omega_m t\right)\cos\left(\omega_c t - \beta \cos \omega_m t\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\beta^n}{n!} \cos\left[\left(\omega_c + n\omega_m\right)t + n\,\frac{\pi}{2}\right] \\
s_3(t) &= \exp\left(\beta \cos \omega_m t\right)\cos\left(\omega_c t + \beta \sin \omega_m t\right) = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \cos\left[\left(\omega_c + n\omega_m\right)t\right] \\
s_4(t) &= \exp\left(-\beta \cos \omega_m t\right)\cos\left(\omega_c t - \beta \sin \omega_m t\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\beta^n}{n!} \cos\left[\left(\omega_c + n\omega_m\right)t\right]
\end{aligned}\right\} \tag{22}$$

Although the autocorrelation functions of the physical signals are not of prime interest, the mean-squared values, or powers, are. In the deterministic case, the autocorrelation function of the ith analytic signal may be defined as a time average:

$$R_{\psi_i\psi_i}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \psi_i(t+\tau)\, \psi_i^*(t)\, dt \tag{23}$$

The autocorrelation function of the ith physical signal is

$$R_{s_i s_i}(\tau) = \frac{1}{2} \operatorname{Re}\left[R_{\psi_i \psi_i}(\tau)\right] \tag{24}$$

The power, or mean-squared value, of the ith physical signal is

$$P_{s_i} = R_{s_i s_i}(0) \tag{25}$$

Thus

$$P_{s_i} = \frac{1}{2} \operatorname{Re}\left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \psi_i(t)\ \psi_i^*(t)\ dt\right] \tag{26}$$

Due to the symmetry properties of $R_{\psi_i \psi_i}(\tau)$, equation (26) reduces to

$$P_{s_i} = \frac{1}{2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \psi_i(t)\ \psi_i^*(t)\ dt \tag{27}$$

Now, it is easily verified that

$$\psi_i(t)\ \psi_i^*(t) = \left\{\begin{array}{ll} \exp\left(-2\beta \sin \omega_m t\right); & (i = 1) \\ \exp\left(2\beta \sin \omega_m t\right); & (i = 2) \\ \exp\left(2\beta \cos \omega_m t\right); & (i = 3) \\ \exp\left(-2\beta \cos \omega_m t\right); & (i = 4) \end{array}\right\} \tag{28}$$

Thus $\psi_i(t)\ \psi_i^*(t)$ is periodic in 2π for θ_m, a dummy variable defined as

$$\theta_m = \omega_m t \tag{29}$$

It follows that

$$P_{s_i} = \frac{1}{2\pi} \int_0^{\pi} \exp\left(2\beta \cos \theta_m\right)\ d\theta_m = \frac{1}{2} I_0(2\beta); \quad (i = 1, 2, 3, 4) \tag{30}$$

where I_0 is the modified Bessel function (ref. 10).

More Efficient Modulation

From equations (22) it is seen that the power residing in the recoverable signal component, defined here as the first-order sideband, is

$$P_{1_i} = \frac{\beta^2}{2} \tag{31}$$

Thus, from equations (30) and (31) the ratio of recoverable sideband power to total power is

$$\frac{P_{1_i}}{P_{s_i}} = \frac{\beta^2}{I_0(2\beta)} \tag{32}$$

This function of modulation index β has an absolute maximum,

$$\max\left(\frac{P_{1_i}}{P_{s_i}}\right) = 0.475; \quad (\beta = 1.29) \tag{33}$$

If efficiency is defined as percentage of recoverable signal power, referenced to total power, then this modulation process has a maximum efficiency of 47.5 percent. It is desirable to ascertain if there exist methods of increasing this efficiency.

Figure 2 shows in three dimensions the phase relationship for the residual carrier and sideband components for the four signals of equations (22). It is easily determined from equations (22) and (30) that for the most efficient case, the sum of the powers in the residual carrier and the first- and second-order sidebands accounts for 99 percent of the total power. Thus, if two of the four signals can be combined so that the first-order components add and the carrier and second-order components subtract, then the overall efficiency can be increased. Figure 2 shows that this can be accomplished by subtracting $s_2(t)$ from $s_1(t)$ or $s_4(t)$ from $s_3(t)$. Furthermore, by adjusting the amplitudes of the two signals, the relative amplitudes of the carrier and first-order sideband can be controlled. Finally, by adjusting the modulation indices, the relative amplitudes of the carrier and second-order sideband can be controlled.

Although it is not the purpose of this paper to give a complete synthesis technique (ref. 11) for sinusoidally modulated SSB-AM-FM signals, it is interesting to see, for the signal constructed above, how solutions for β can be formulated. Define the combination signal as

$$s(t) = s_2(t) - s_1(t) \tag{34}$$

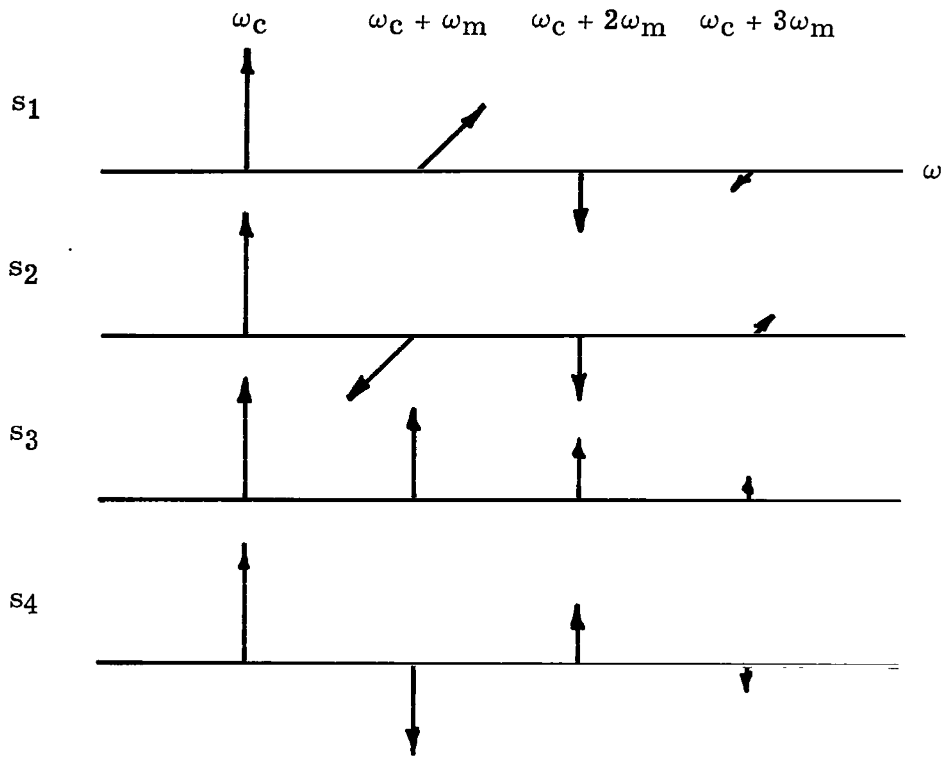

Figure 2.- Relative phasing.

where

$$\left.\begin{aligned} s_1(t) &= A_1 \exp\left(-\beta_1 \sin \omega_m t\right) \cos\left(\omega_c t + \beta_1 \cos \omega_m t\right) \\ s_2(t) &= A_2 \exp\left(\beta_2 \sin \omega_m t\right) \cos\left(\omega_c t - \beta_2 \cos \omega_m t\right) \end{aligned}\right\} \tag{35}$$

From equations (22) it can be determined that the power of the residual carrier is

$$P_c = \frac{1}{2}\left(A_2 - A_1\right)^2 \tag{36}$$

Likewise the powers of the first- and second-order sidebands are

$$\left.\begin{aligned} P_1 &= \frac{1}{2}\left(A_2\beta_2 + A_1\beta_1\right)^2 \\ P_2 &= \frac{1}{8}\left(A_2\beta_2^2 - A_1{\beta_1}^2\right)^2 \end{aligned}\right\} \tag{37}$$

Now define

$$\frac{A_2}{A_1} \triangleq C_\alpha \qquad\qquad \frac{\beta_2}{\beta_1} \triangleq C_\beta \tag{38}$$

Suppose constants k_1 and k_2 are defined as

$$\frac{P_1}{P_c} \triangleq {k_1}^2 \qquad\qquad \frac{P_2}{P_c} \triangleq {k_2}^2 \tag{39}$$

Then

$$\left.\begin{aligned} k_1 &= \frac{\beta_1\left(C_\alpha C_\beta + 1\right)}{C_\alpha - 1} \\ k_2 &= \frac{{\beta_1}^2\left(C_\alpha {C_\beta}^2 - 1\right)}{2\left(C_\alpha - 1\right)} \end{aligned}\right\} \quad \left(C_\alpha \neq 1\right) \tag{40}$$

Equations (40) may be solved implicitly for C_α and C_β in terms of design constants k_1 and k_2 with β_1 as parameter. Then β_1 may be varied to satisfy some other constraint, say, on the total power P_s.

An expression for the total power P_s in $s(t)$ may be obtained in the same way P_{s_i} was obtained for the single signal. Let the analytic signal $\psi_s(t)$ be

$$\psi_s(t) = \psi_2(t) - \psi_1(t) \tag{41}$$

where

$$\left.\begin{aligned} \psi_2(t) &= A_2 m_2(t)\exp(j\omega_c t) \\ \psi_1(t) &= A_1 m_1(t)\exp(j\omega_c t) \end{aligned}\right\} \tag{42}$$

and

$$\left.\begin{aligned} m_2(t) &= \exp\left\{-j\left[f_2(t) + j\hat{f}_2(t)\right]\right\} \\ m_1(t) &= \exp\left\{j\left[f_1(t) + j\hat{f}_1(t)\right]\right\} \end{aligned}\right\} \tag{43}$$

where

$$\left.\begin{aligned} f_i(t) &= \beta_i \cos \omega_m t \\ \hat{f}_i(t) &= \beta_i \sin \omega_m t \end{aligned}\right\} \tag{44}$$

Then

$$\psi_s(t) = \left[A_2 m_2(t) - A_1 m_1(t)\right] \exp(j\omega_c t) \tag{45}$$

and

$$\begin{aligned} \psi_s(t)\, \psi_s^*(t) &= \left[A_2 m_2(t) - A_1 m_1(t)\right]\left[A_2 m_2^*(t) - A_1 m_1^*(t)\right] \\ &= A_2^2 \exp\left[2\hat{f}_2(t)\right] + A_1^2 \exp\left[-2\hat{f}_1(t)\right] - 2A_1A_2 \exp\left[\hat{f}_2(t) - \hat{f}_1(t)\right] \cos\left[f_2(t) + f_1(t)\right] \\ &= A_2^2 \exp\left(2\beta_2 \sin \omega_m t\right) + A_1^2 \exp\left(-2\beta_1 \sin \omega_m t\right) \\ &\quad - 2A_1A_2 \exp\left[\left(\beta_2 - \beta_1\right)\sin \omega_m t\right]\cos\left[\left(\beta_2 + \beta_1\right)\cos \omega_m t\right] \end{aligned} \tag{46}$$

The total power is defined as a time average, as was done in equation (27):

$$P_s \triangleq \frac{1}{2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \psi_s(t)\, \psi_s^*(t)\, dt \tag{47}$$

Making the substitution as in equation (29), $\psi_s(\theta_m)\, \psi_s^*(\theta_m)$ is periodic in the θ_m interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Thus

$$P_s \triangleq \frac{1}{2}\, \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} \psi_s(\theta_m)\, \psi_s^*(\theta_m)\, d\theta_m \tag{48}$$

Now define the transformation

$$\phi = \theta_m - \frac{\pi}{2} \tag{49}$$

Then, advantage may be taken of the even symmetry of $\psi_s(\phi)\, \psi_s^*(\phi)$ in the $[-\pi, \pi]$ interval to define

$$P_s = \frac{1}{2\pi}\int_0^{\pi} \psi_s(\phi)\, \psi_s^*(\phi)\, d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi} \Big\{A_2^2 \exp\left(2\beta_2 \cos \phi\right) + A_1^2 \exp\left(-2\beta_1 \cos \phi\right)$$

$$- 2A_1A_2 \exp\left[\left(\beta_2 - \beta_1\right) \cos \phi\right] \cos\left[\left(\beta_2 + \beta_1\right) \sin \phi\right]\Big\} d\phi \qquad (50)$$

By applying several Bessel function identities, P_s is obtained as

$$P_s = \frac{1}{2}\Bigg\{A_2^2\, I_0\left(2\beta_2\right) + A_1^2 I_0\left(2\beta_1\right) - 2A_1A_2\Bigg[J_0\left(\beta_2+\beta_1\right) I_0\left(\beta_2-\beta_1\right)$$

$$+ 2\sum_{k=1}^{\infty} J_{2k}\left(\beta_2+\beta_1\right) I_{2k}\left(\beta_2-\beta_1\right)\Bigg]\Bigg\} \qquad (51)$$

Substituting the definitions of equations (38) yields

$$P_s = \frac{A_1^2}{2}\Bigg(C_\alpha^2\, I_0\left(2C_\beta\beta_1\right) + I_0\left(2\beta_1\right) - 2C_\alpha\Bigg\{J_0\left[\left(C_\beta+1\right)\beta_1\right] I_0\left[\left(C_\beta-1\right)\beta_1\right]$$

$$+ 2\sum_{k=1}^{\infty} J_{2k}\left[\left(C_\beta+1\right)\beta_1\right] I_{2k}\left[\left(C_\beta-1\right)\beta_1\right]\Bigg\}\Bigg) \qquad (52)$$

Thus the synthesis procedure for the combined SSB-PM signal is as follows: Solve equations (40) for C_α and C_β with parameter β_1, so that the k_1 and k_2 constraints are satisfied. Vary the solution with β_1 so that the constraint on total power P_s is satisfied.

Modulation by Multiple Sinusoids

So far, this section has dealt with modulation by a single sinusoid. The treatment is now generalized to an arbitrary number k of sinusoids. Let the modulation function $m(t)$ be

$$
\begin{aligned}
m(t) &= \exp\left\{j\left[\sum_{i=1}^{k} \beta_i \exp\left(j\omega_i t\right)\right]\right\} \\
&= \exp\left\{\sum_{i=1}^{k} \beta_i \exp\left[j\left(\omega_i t + \frac{\pi}{2}\right)\right]\right\} \\
&= \prod_{i=1}^{k} \exp\left\{\beta_i \exp\left[j\left(\omega_i t + \frac{\pi}{2}\right)\right]\right\} \\
&= \prod_{i=1}^{k}\left\{\sum_{n_i=0}^{\infty} \frac{\beta_i^{n_i}}{n_i!} \exp\left[jn_i\left(\omega_i t + \frac{\pi}{2}\right)\right]\right\}
\end{aligned}
\tag{53}
$$

where β_i and ω_i are the modulation index and frequency, respectively, of the ith subcarrier. Equation (53) shows that the complex modulation function for k sinusoids of frequency ω_i, for $i = 1, 2, \ldots k$, may be expressed as the product of k infinite series.

The Cauchy rule (ref. 12) for multiplication of two infinite series is

$$
\left[\sum_{m_1=0}^{\infty} {}^{2}a_{m_1}\right] \times \left[\sum_{m_2=0}^{\infty} {}^{1}a_{m_2}\right] = \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{n_2} {}^{1}a_{n_1}\, {}^{2}a_{n_2-n_1} \tag{54}
$$

where the superscript numbers simply serve to identify the terms of each series. This product rule may be extended to k-fold multiplication as

$$
\prod_{i=1}^{k}\left[\sum_{m_i=0}^{\infty} {}^{i}a_{m_i}\right] = \sum_{n_k=0}^{\infty} \sum_{n_{k-1}=0}^{n_k} \sum_{n_{k-2}=0}^{n_k-n_{k-1}} \cdots \sum_{n_1=0}^{n_k-n_{k-1}-\cdots-n_2} {}^{k}a_{n_k-n_{k-1}-\cdots-n_1} \prod_{i=1}^{k-1} {}^{i}a_{n_{k-i}} \tag{55}
$$

Substituting equation (53) into equation (55) gives

$$m(t) = \sum_{n_k=0}^{\infty} \sum_{n_{k-1}=0}^{n_k} \cdots \sum_{n_1=0}^{n_k-n_{k-1}-\cdots-n_2} \left(\frac{(\beta_k)^{n_k-n_{k-1}-\cdots-n_1}}{(n_k - n_{k-1} - \cdots - n_1)!} \left[\prod_{i=1}^{k-1} \frac{(\beta_i)^{n_{k-i}}}{(n_{k-i})!} \right] \right.$$

$$\left. \times \exp\left\{ j\left[\left(\omega_k t + \frac{\pi}{2}\right)(n_k - n_{k-1} - \cdots - n_1) + \sum_{i=1}^{k-1} \left(\omega_i t + \frac{\pi}{2}\right) n_{k-i} \right] \right\} \right) \tag{56}$$

Multiplying $m(t)$ in equation (56) by $\exp(j\omega_c t)$ and taking the real part of the product gives the resulting physical signal $s(t)$:

$$s(t) = \left[\prod_{i=1}^{k} \exp(-\beta_i \sin \omega_i t) \right] \cos\left(\omega_c t + \sum_{i=1}^{k} \beta_i \cos \omega_i t \right)$$

$$= \sum_{n_k=0}^{\infty} \sum_{n_{k-1}=0}^{n_k} \cdots \sum_{n_1=0}^{n_k-n_{k-1}-\cdots-n_2} \left\{ \frac{(\beta_k)^{n_k-n_{k-1}-\cdots-n_1}}{(n_k - n_{k-1} - \cdots - n_i)!} \left[\prod_{i=1}^{k-1} \frac{(\beta_i)^{n_{k-i}}}{(n_{k-i})!} \right] \right.$$

$$\left. \times \cos\left[\omega_c t + (n_k - n_{k-1} - \cdots - n_1)\left(\omega_k t + \frac{\pi}{2}\right) + \sum_{i=1}^{k-1} n_{k-i} \left(\omega_i t + \frac{\pi}{2}\right) \right] \right\} \tag{57}$$

From equation (57), any arbitrary modulation term, say the jth, may be isolated. The jth term is explicitly the one for which the frequency is $\omega_c + \omega_j$ and for which

$$\left. \begin{aligned} n_k &= 1 \\ n_{k-i} &= \begin{cases} 1; & (i = j) \\ 0; & (i \neq j) \end{cases} \end{aligned} \right\} \tag{58}$$

Denoting the jth modulation term as $s_j(t)$, equations (57) and (58) yield

$$s_j(t) = \beta_j \cos\left[(\omega_c + \omega_j)t + \frac{\pi}{2} \right] \tag{59}$$

Equation (59) shows the major difference between the power behavior of this hybrid modulation and classical angle modulation. In the usual double-sideband angle-modulated signal, the amplitude of the jth modulation term is a product of Bessel functions whose arguments include all the "modulation indices" β_i (ref. 13). Thus, the amplitude of the jth term is reduced by the presence of the other modulating sinusoids so that the total signal power is constant and is just equal to the unmodulated carrier power. In this hybrid SSB modulation scheme, the amplitude of the jth term is a function of β_j only. Hence, the total signal power increases with the addition of modulating sinusoids. This behavior is attributable to the amplitude modulation present in the hybrid process.

Subcarrier Modulation

As in classic double-sideband modulation schemes, it is possible to modulate the modulating sinusoids to obtain subcarriers which then are processed as in the previous section to obtain an SSB spectrum. In general the subcarriers may be amplitude modulated, angle modulated, or hybrid modulated to obtain subcarriers which, themselves, have SSB spectra. However, the following development is limited to classically angle-modulated subcarriers.

Now, modify the message function $f(t)$ of equation (19) to be

$$f(t) = \beta \cos\left[\omega_m t + \theta(t)\right] \tag{60}$$

where $\theta(t)$ represents the phase function of the subcarrier. It is assumed that $\theta(t)$ is linearly related to some "submessage function" of $f(t)$. The Hilbert transform of $f(t)$ is given as

$$\hat{f}(t) = \beta \sin\left[\omega_m t + \theta(t)\right] \tag{61}$$

provided the spectrum of $\exp\left[j\theta(t)\right]$ is zero for $\omega < -\omega_m$ (ref. 14). Then it is easily seen that the four signals of equations (22) are modified to

$$\begin{aligned} s_1(t) &= \exp\left\{-\beta \sin\left[\omega_m t + \theta(t)\right]\right\} \cos\left\{\omega_c t + \beta \cos\left[\omega_m t + \theta(t)\right]\right\} \\ &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \cos\left[\left(\omega_c + n\omega_m\right)t + n\theta(t) + n\frac{\pi}{2}\right] \end{aligned} \tag{62a}$$

$$s_2(t) = \exp\left\{\beta \sin\left[\omega_m t + \theta(t)\right]\right\} \cos\left\{\omega_c t - \beta \cos\left[\omega_m t + \theta(t)\right]\right\}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\beta^n}{n!} \cos\left[\left(\omega_c + n\omega_m\right)t + n\theta(t) + n\frac{\pi}{2}\right] \tag{62b}$$

$$s_3(t) = \exp\left\{\beta \cos\left[\omega_m t + \theta(t)\right]\right\} \cos\left\{\omega_c t + \beta \sin\left[\omega_m t + \theta(t)\right]\right\}$$

$$= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \cos\left[\left(\omega_c + n\omega_m\right)t + n\theta(t)\right] \tag{62c}$$

$$s_4(t) = \exp\left\{-\beta \cos\left[\omega_m t + \theta(t)\right]\right\} \cos\left\{\omega_c t - \beta \sin\left[\omega_m t + \theta(t)\right]\right\}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\beta^n}{n!} \cos\left[\left(\omega_c + n\omega_m\right)t + n\theta(t)\right] \tag{62d}$$

The power spectral density for any of the four signals, whether modulated or unmodulated, may be represented as

$$S(\omega) = \sum_{n=0}^{\infty} S_n(\omega) \tag{63}$$

where for the case of unmodulated subcarriers,

$$S_n(\omega) = \left(\frac{\beta^n}{n!}\right)^2 \left[\frac{\pi}{2}\, \delta\left(\omega - \omega_c - n\omega_m\right) + \frac{\pi}{2}\, \delta\left(\omega + \omega_c + n\omega_m\right)\right] \tag{64}$$

and for the modulated case,

$$S_n(\omega) = \left(\frac{\beta^n}{n!}\right)^2 \left[\frac{1}{4}\, S_{m_n}\left(\omega - \omega_c - n\omega_m\right) + \frac{1}{4}\, S^*_{m_n}\left(\omega + \omega_c + n\omega_m\right)\right] \tag{65}$$

where $S_{m_n}(\omega)$ is the spectral density of

$$m_n(t) = \exp\left[jn\theta(t)\right] \tag{66}$$

It is easily verified, using Parseval's relation, that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)\, d\omega = \frac{1}{2}\left(\frac{\beta^n}{n!}\right)^2 \tag{67}$$

for both equation (64) and equation (65). Thus, as is intuitively expected, the presence of angle modulation on the subcarrier does not change the total power in each spectral component, but just spreads the spectrum.

Linear Product Demodulation

The product demodulation of a sinusoidally phase-modulated SSB signal in the presence of noise is considered next. Figure 3 shows the model of the demodulator.

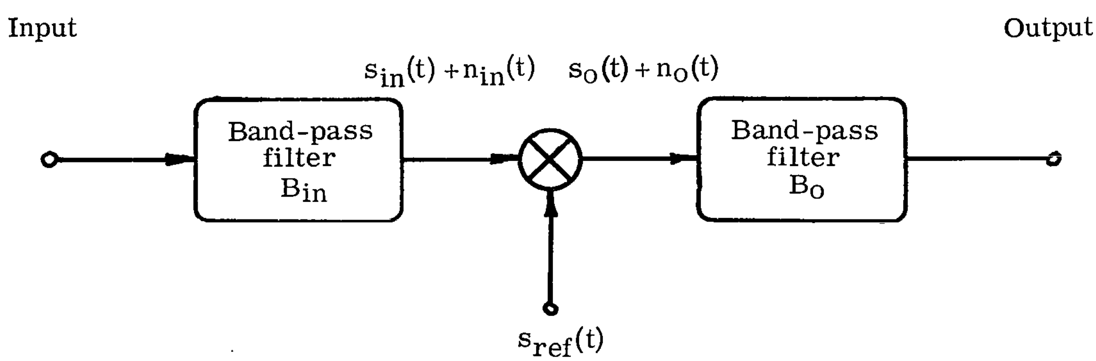

Figure 3.- Linear product demodulator.

The input signal and noise terms are taken at the output of an ideal band-pass filter which drives an ideal product device. The input signal $s_{in}(t)$ is assumed, ideally, to be undistorted and is taken as

$$s_{in}(t) = A \exp\left(-\beta \sin \omega_m t\right)\cos\left(\omega_c t + \beta \cos \omega_m t\right) \tag{68}$$

The noise is taken as ideally band-limited white Gaussian noise:

$$n_{in}(t) = x(t) \cos \omega_c t - y(t) \sin \omega_c t \tag{69}$$

The reference signal $s_{ref}(t)$, which drives the other input port of the product device, is taken as

$$s_{ref}(t) = -2 \sin \omega_c t \tag{70}$$

Note that the reference signal is taken to be phase coherent with the input signal. This is not a loss of generality, since signals of the type dealt with here have a residual carrier component of frequency ω_c which may be detected and used to synchronize the reference signal. Now the output signal $s_o(t)$, the result of demodulating the first-order sideband, taken at the input to the band-pass output filter, is, neglecting terms of frequency $2\omega_c$,

$$s_o(t) = A\beta \cos \omega_m t \tag{71}$$

Likewise, the output noise term $n_o(t)$, taken at the input to the band-pass output filter, is

$$n_o(t) = y(t) \tag{72}$$

Now, the input signal $s_{in}(t)$ has only an upper sideband. Hence it is sensible to restrict the filter bandwidth B_{in} to the upper sideband. As a matter of fact, the filter need not pass all the signal components. It need pass only the component at frequency $\omega_c + \omega_m$. In case a subcarrier system is considered, the filter should be wide enough to pass the modulated subcarrier without appreciable distortion. However, the filter need not pass the residual carrier (assuming it is detected in a separate channel) or higher order modulation components.

Let $n_{in}(t)$ be described by its spectral density $N_{in}(\omega)$. Since $n_{in}(t)$ is a real process, $N_{in}(\omega)$ is necessarily two-sided. That is $N_{in}(\omega)$ exists for positive and negative ω. Under the ideal filtering assumption, $N_{in}(\omega)$ has constant value, say $\frac{1}{2}\eta$, in the frequency neighborhood occupied by the signal component at frequency $\omega_c + \omega_m$. For all other frequencies, $N_{in}(\omega)$ is zero. Figure 4 gives $N_{in}(\omega)$ graphically.

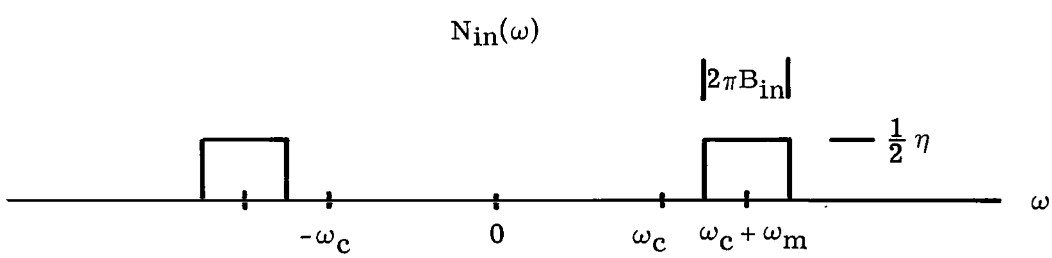

Figure 4.- Noise spectral density $N_{in}(\omega)$.

The input signal-to-noise ratio (SNR), the ratio of total modulated carrier power to total noise power, at the input to the product device is

$$\frac{S_{in}}{N_{in}} = \frac{\frac{A^2}{2} I_0(2\beta)}{\frac{1}{2} \eta \, 2B_{in}} = \frac{A^2}{2} \frac{I_0(2\beta)}{\eta B_{in}} \tag{73}$$

The output noise spectral density $N_0(\omega)$ is just the spectral density of $y(t)$, or $N_y(\omega)$. From appendix A it may be determined that $N_y(\omega)$ is equal to a constant, $\frac{1}{2}\eta$, in the frequency domain of the signal component at frequency ω_m and is zero elsewhere. Figure 5 shows $N_y(\omega)$.

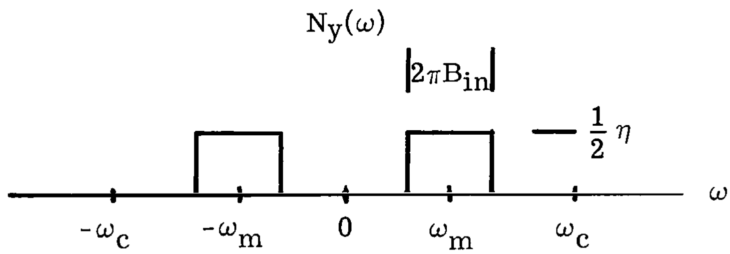

Figure 5.- Noise spectral density $N_y(\omega)$.

The output SNR at the output of the product device is computed for the noise existing in the output bandwidth B_o. It is assumed that $B_o \leqq B_{in}$. Then

$$\frac{S_o}{N_o} = \frac{\frac{(A\beta)^2}{2}}{\frac{1}{2} \eta \, 2B_o} = \frac{A^2}{2} \frac{\beta^2}{\eta B_o} = \frac{B_{in}}{B_o} \left[\frac{\beta^2}{I_0(2\beta)} \right] \frac{A^2}{2} \frac{I_0(2\beta)}{\eta B_{in}} = \frac{B_{in}}{B_o} \left[\frac{\beta^2}{I_0(2\beta)} \right] \frac{S_{in}}{N_{in}} \tag{74}$$

The function of β in brackets may be termed the "modulation gain" of this particular modulation system. This function is the ratio of recoverable sideband power to total modulated carrier power given in equation (32). Thus the modulation gain factor has an absolute maximum of 0.475 for $\beta = 1.29$. Hence, it is seen that this type of modulation system is inherently a "low-index" system.

In this section, analytic function theory has been applied to investigate the synthesis and detection in noise of single-sideband signals, phase-modulated by sinusoids. Mathematical expressions for recoverable signal power and for total modulated carrier power were derived for the case of modulation by a single sinusoid. It was shown that the power expressions are insensitive to angle modulation of the modulating sinusoid (subcarrier). It was also shown that for modulation by multiple sinusoids, the total modulated carrier power increases with increasing numbers of sinusoids, and that the amplitude of each is independent of the others. It was determined that for linear product demodulation, this type of modulation system has highest efficiency for low modulation index (1.29). A synthesis procedure for signals of greater modulation efficiency was outlined.

GAUSSIAN SINGLE-SIDEBAND FREQUENCY MODULATION

In this section, SSB frequency modulation of a carrier by a Gaussian process is considered. Nonlinear (discriminator) detection of such a signal in the presence of noise is modeled mathematically for the high SNR case. Gaussian modulation is chosen since other message functions are often approximated by using a Gaussian process of suitable spectral shape. Frequency modulation is chosen as being representative of the "wide-band" angle-modulation process.

Properties of the Modulated Carrier

Since the signal treated here is frequency modulated, the message function is $\dot{\phi}_s(t)$ where $\phi_s(t)$ is the phase function of the modulated carrier, as given in equation (10). Clearly, if $\dot{\phi}_s(t)$ is specified to be a Gaussian process, then $\phi_s(t)$ is also Gaussian, since the time derivative is linear. Thus, let $\phi_s(t)$ be defined as

$$\phi_s(t) = f(t) \tag{75}$$

where $f(t)$ is a Gaussian process, assumed weakly stationary and with zero mean. Let $f(t) + jg(t)$ be an analytic Gaussian process. Then the analytic modulation function $m(t)$ is given by equation (16) and repeated here for convenience:

$$m(t) = \exp\left\{j\left[f(t) + jg(t)\right]\right\} \exp(j\theta) \tag{76}$$

where θ is uniformly distributed on $[-\pi, \pi]$. Then the analytic signal is

$$\psi(t) = Am(t) \exp\left(j\omega_c t\right) \tag{77}$$

and the autocorrelation function is

$$R_{\psi\psi}(\tau) = A^2\left[\Phi(1,j,-1,j;\tau)\right] \exp\left(j\omega_c\tau\right) \tag{78}$$

where A is the amplitude constant and Φ the characteristic function (ref. 9) given as

$$\Phi(1,j,-1,j;\tau) = E\left\{\exp\left\{j\left[f(t+\tau) + jg(t+\tau) - f(t) + jg(t)\right]\right\}\right\} \tag{79}$$

From equation (B22), $R_{\psi\psi}(\tau)$ is developed as

$$R_{\psi\psi}(\tau) = A^2 \exp\left\{R_{gg}(0) - R_{ff}(0) + R_{gg}(\tau) + R_{ff}(\tau) + j\left[\omega_c\tau + R_{gf}(\tau) - R_{fg}(\tau)\right]\right\}$$

$$= A^2 \exp\left[2R_{ff}(\tau)\right] \exp\left\{j\left[\omega_c\tau + 2R_{gf}(\tau)\right]\right\} \tag{80}$$

It is interesting to note that the autocorrelation function for a double-sideband Gaussian angle-modulated carrier is obtainable from equation (80) as a special case. If $g(t) = 0$, the real part of the analytic signal of equation (77) becomes

$$s(t) = A \cos\left[\omega_c t + f(t)\right] \tag{81}$$

for which

$$R_{ss}(\tau) = \frac{1}{2} \operatorname{Re}\left[R_{\psi\psi}(\tau)\right]$$

$$= \frac{A^2}{2} \exp\left[-R_{ff}(0)\right] \exp\left[R_{ff}(\tau)\right] \cos \omega_c\tau \tag{82}$$

Now, for the SSB signal, taking the real parts of the various analytic expressions gives

$$s(t) = A \exp\left[-g(t)\right] \cos\left[\omega_c t + f(t) + \theta\right] \tag{83}$$

and

$$R_{ss}(\tau) = \frac{A^2}{2} \exp\left[2R_{ff}(\tau)\right] \cos\left[\omega_c\tau + 2R_{gf}(\tau)\right] \tag{84}$$

The "average power" of $s(t)$, given by its variance σ_s^2, is denoted P_s and is

$$P_s = \sigma_s^2 = R_{ss}(0) = \frac{A^2}{2} \exp\left[2R_{ff}(0)\right] \tag{85}$$

since $R_{gf}(\tau)$ is an odd function. But $R_{ff}(0)$ is simply the variance σ_f^2 of the phase function $f(t)$. Hence,

$$P_s = \frac{A^2}{2} \exp\left(2\sigma_f^2\right) \tag{86}$$

Thus, equation (86) shows that the average signal power increases as the exponent of twice the message variance. This result is analogous to equation (30), which gives the average power for sinusoidal modulation.

The Phase Process of Signal Plus Noise

The frequency demodulator which is to be treated is sensitive not only to the modulated signal but also to the additive noise (Gaussian) in the signal channel. The demodulator detects the time derivative of what may be called the "equivalent phase" of the signal plus channel noise. Appendix C treats the equivalent-phase concept in detail.

Let the noise $n(t)$ and signal $s(t)$ be given in the standard forms of equations (4) and (10), respectively. It is possible, in general, to represent the sum $r(t)$ of the frequency-modulated signal $s(t)$ plus noise $n(t)$ as

$$r(t) = s(t) + n(t) = R(t) \cos\left[\omega_c t + \phi(t) + \theta(t) + \theta\right] \tag{87}$$

where $R(t)$ is nonnegative, $\phi(t)$ is the signal phase function, $\theta(t)$ is the perturbation of the equivalent phase due to additive noise, and θ is a uniformly distributed constant.

In the case at hand,

$$\left.\begin{aligned} A(t) &= A \exp\left[-g(t)\right] \\ \phi(t) &= f(t) \end{aligned}\right\} \tag{88}$$

where $f(t)$ and $g(t)$ are jointly Gaussian and $f(t) + jg(t)$ is analytic. By substituting in equation (C11), the low-noise approximation to $\theta(t)$ is obtained as

$$\theta(t) \approx \frac{1}{A} \exp\left[g(t)\right] \left\{y(t) \cos\left[f(t)\right] - x(t) \sin\left[f(t)\right]\right\} \tag{89}$$

Noting equation (B23), $R_{\theta\theta}(\tau)$ is obtained from equation (C18) as

$$R_{\theta\theta}(\tau) = \frac{1}{2} \operatorname{Re}\left\{\frac{1}{A^2}\left[R_{zz}(\tau)\ \Phi(1,j,-1,j;\tau)\right]^*\right\} \tag{90}$$

where the characteristic function is evaluated in equation (B22). Then

$$\begin{aligned} R_{\theta\theta}(\tau) = \frac{1}{A^2} \exp&\left[R_{gg}(0) - R_{ff}(0) + R_{gg}(\tau) + R_{ff}(\tau)\right] \\ &\times \left\{R_{xx}(\tau) \cos\left[2R_{gf}(\tau)\right] - R_{yx}(\tau) \sin\left[2R_{gf}(\tau)\right]\right\} \end{aligned} \tag{91}$$

Equation (91) is not the most compact form but is given so that a specialization may be made to classical double-sideband FM. Notice that the autocorrelation function for the phase-noise process of a classical FM signal in noise is obtained by letting $g(t) = 0$ as in equation (81). Then, from equation (91),

$$R_{\theta\theta}(\tau) = \frac{1}{A^2} \exp\left[-R_{ff}(0)\right] R_{xx}(\tau) \exp\left[R_{ff}(\tau)\right] \tag{92}$$

Now, it is common practice in classical FM to assume that, in the low-noise case, the phase noise is simply

$$\theta(t) = \frac{y(t)}{A} \tag{93}$$

which gives

$$R_{\theta\theta}(\tau) = \frac{1}{A^2} R_{yy}(\tau) \tag{94}$$

This result is not rigorously correct. However, for high-index modulation, $\exp R_{ff}(\tau)$ is much broader than $R_{xx}(\tau)$. Hence an approximate result, good for high-index modulation, is

$$\begin{aligned} R_{\theta\theta}(\tau) &\approx \frac{1}{A^2} \exp\left[-R_{ff}(0)\right] R_{xx}(\tau) \exp\left[R_{ff}(0)\right] \\ &= \frac{1}{A^2} R_{xx}(\tau) \\ &= \frac{1}{A^2} R_{yy}(\tau) \end{aligned} \tag{95}$$

For SSB-FM, equation (91) may be written more compactly as

$$R_{\theta\theta}(\tau) = \frac{1}{A^2} \exp\left[2R_{ff}(\tau)\right] \left\{R_{xx}(\tau) \cos\left[2\hat{R}_{ff}(\tau)\right] - \hat{R}_{xx}(\tau) \sin\left[2\hat{R}_{ff}(\tau)\right]\right\} \tag{96}$$

As in classical FM, if the signal channel bandwidth is sufficiently wide compared with the bandwidth of the modulation function $f(t)$, then equation (96) reduces to

$$R_{\theta\theta}(\tau) \approx \frac{1}{A^2} \exp\left[2R_{ff}(0)\right] R_{xx}(\tau) \tag{97}$$

Limiter-Discriminator Demodulation

Now the response of an idealized limiter-discriminator to an SSB-FM signal plus band-limited white Gaussian noise will be determined. The model of the demodulator is given in figure 6, where all filters are taken to be ideal.

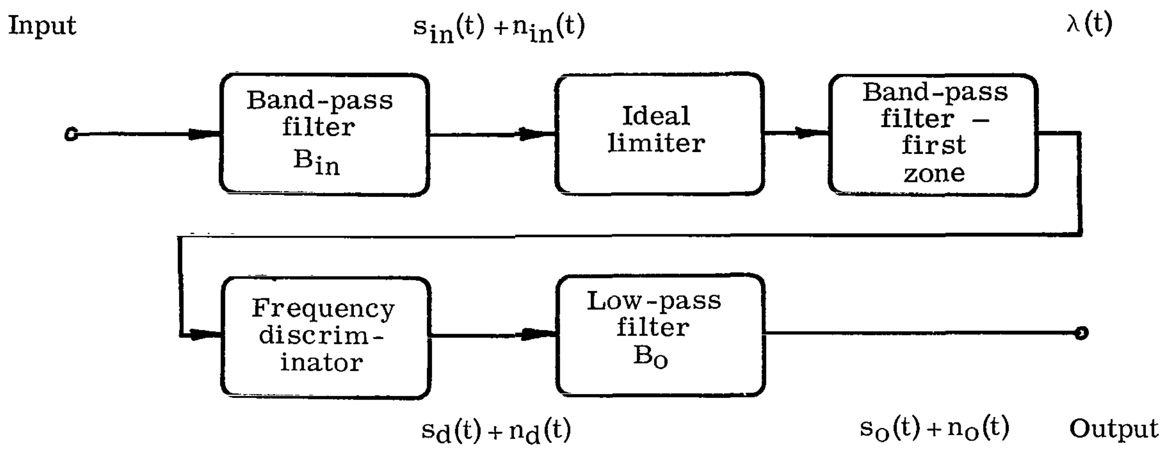

Figure 6.- Demodulator model.

Since the signal $s_{in}(t)$ at the input to the band-pass filter has only an upper sideband, it is sensible to use a filter which passes only the upper sideband. Then the noise process $n_{in}(t)$ at the input to the ideal limiter also has only an upper sideband, referenced to ω_c, the signal carrier frequency. The band-pass filter at the output of the limiter is zonal. That is, the filter passes only the first zone output of the limiter, which is a sinusoidal carrier of frequency ω_c, having only phase modulation (ref. 15). Then the signal $\lambda(t)$ is represented as

$$\lambda(t) = \cos\left[\omega_c t + f(t) + \theta(t) + \theta\right] \tag{98}$$

where $f(t)$ is Gaussian and $\theta(t)$ is the phase-noise perturbation.

Since the objective is to determine the best performance of the limiter-discriminator in demodulating SSB-FM in the presence of noise, the low-noise approximation for $\theta(t)$ is made. Also, it is assumed that the input bandwidth B_{in} is sufficiently large with respect to the spectral width of $f(t)$ so that $R_{\theta\theta}(\tau)$ is well approximated by equation (97).

The signal and noise outputs of the discriminator, $s_d(t)$ and $n_d(t)$, respectively, are given by

$$\left.\begin{aligned} s_d(t) &= \frac{d}{dt}\left[f(t)\right] = \dot{f}(t) \\ n_d(t) &= \frac{d}{dt}\left[\theta(t)\right] = \dot{\theta}(t) \end{aligned}\right\} \tag{99}$$

The output filter is of the low-pass type, under the assumption that the message process $\dot{f}(t)$ is low-pass. The bandwidth of the filter B_o is chosen so as to pass $\dot{f}(t)$, or most of the power in the spectrum of $\dot{f}(t)$, while not passing any more noise than necessary. Thus the output signal power S_o is essentially

$$S_o = R_{s_d s_d}(0) = -\frac{d^2}{d\tau^2}\left[R_{ff}(\tau)\right]\Big|_{\tau=0} \triangleq -R_{ff}''(0) \tag{100}$$

To determine the output noise power N_o requires first the determination of the noise spectral density, and hence the autocorrelation function of $n_d(t)$:

$$R_{n_d n_d}(\tau) = -\frac{d^2}{d\tau^2}\left[R_{\theta\theta}(\tau)\right] = -R_{\theta\theta}''(\tau) \approx -\frac{1}{A^2}\exp\left[2R_{ff}(0)\right] R_{xx}''(\tau) \tag{101}$$

Thus, the spectral density $N_{nd}(\omega)$ of $n_d(t)$ is

$$N_{nd}(\omega) = \frac{1}{A^2}\exp\left[2R_{ff}(0)\right]\omega^2 N_x(\omega) \tag{102}$$

where $N_x(\omega)$ is the spectral density of $x(t)$. If the input spectral density $N_{in}(\omega)$ is equal to a constant, say $\frac{1}{2}\eta$ over the bandwidth B_{in}, then from appendix A it may be seen that $N_x(\omega)$ is also constant over the bandwidth B_o and is also equal to $\frac{1}{2}\eta$. Then

$$N_{n_d}(\omega) = \frac{\frac{1}{2}\eta}{A^2}\exp\left[2R_{ff}(0)\right]\omega^2 \tag{103}$$

The output noise N_o in the bandwidth B_o is just

$$N_o = \frac{1}{2\pi}\int_{-2\pi B_o}^{2\pi B_o} N_{nd}(\omega)d\omega = \frac{1}{2\pi}\frac{\frac{1}{2}\eta}{A^2}\exp\left[2R_{ff}(0)\right]\frac{2}{3}\left(2\pi B_o\right)^3 \tag{104}$$

The output SNR is then

$$\frac{S_o}{N_o} = \frac{-R''_{ff}(0)}{\frac{1}{2\pi}\left(-\frac{\frac{1}{2}\eta}{A^2}\right)\exp\left[2R_{ff}(0)\right]\frac{2}{3}\left(2\pi B_o\right)^3} \tag{105}$$

To evaluate equation (105) and relate S_o/N_o to the input SNR, a spectral density for $f(t)$ is chosen. Let the spectral density for $f(t)$ be ideal low-pass as shown in

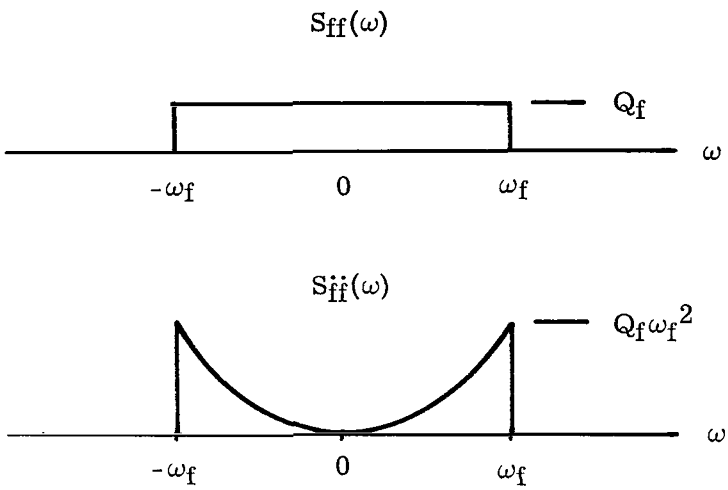

Figure 7.- Spectral densities of $f(t)$ and $\dot{f}(t)$.

figure 7. Then $\dot{f}(t)$ is also low-pass. The autocorrelation function is then

$$R_{ff}(\tau) = \frac{Q_f}{\pi}\,\frac{\sin\,\omega_f t}{\tau} \tag{106}$$

where Q_f is the amplitude of the spectral density $S_{ff}(\omega)$, and

$$-R''_{ff}(0) = \frac{\omega_f{}^3 Q_f}{3\pi} = \frac{\omega_f{}^2}{3}\,R_{ff}(0) \tag{107}$$

Obviously, the optimum choice for output bandwidth in this idealized case is

$$\omega_f = 2\pi B_o \tag{108}$$

Then, for an ideal flat-output filter,

$$\frac{S_o}{N_o} = \frac{\frac{\omega_f^2}{3} R_{ff}(0)}{\frac{1}{2\pi}\frac{\frac{1}{2}\eta}{A^2}\exp\left[2R_{ff}(0)\right]\frac{2}{3}\omega_f^3}$$

$$= 2\,\frac{\omega_{in}}{\omega_f}\,\frac{R_{ff}(0)}{\exp\left[4R_{ff}(0)\right]}\,\frac{\frac{A^2}{2}\exp\left[2R_{ff}(0)\right]}{2\,\frac{1}{2\pi}\,\frac{1}{2}\,\eta\omega_{in}} \tag{109}$$

But by defining

$$\omega_{in} \triangleq 2\pi B_{in} \tag{110}$$

it is seen that

$$\frac{\frac{A^2}{2}\exp\left[2R_{ff}(0)\right]}{2\,\frac{1}{2\pi}\,\frac{1}{2}\,\eta\omega_{in}} = \frac{S_{in}}{N_{in}} \tag{111}$$

where S_{in}/N_{in} is the input SNR in bandwidth B_{in}. Hence

$$\frac{S_o}{N_o} = 2\,\frac{B_{in}}{B_o}\,R_{ff}(0)\,\exp\left[-4R_{ff}(0)\right]\frac{S_{in}}{N_{in}} \tag{112}$$

Since $R_{ff}(0)$ is the mean-squared value, or variance, of the phase-function $f(t)$, equation (112) shows that for high-index modulation, SSB-FM delivers modulation loss rather than modulation gain.

CONCLUSIONS

This analysis produced two types of results. First, it showed that mathematical investigation of complicated single-sideband angle-modulation and demodulation problems is made tractable through use of the theory of analytic functions. Second, it gave technical results concerning the communications efficiency of two SSB angle-modulated signals. For the case of high-index SSB frequency modulation by a Gaussian random process, the results were negative. Investigation showed that even in the low-noise case, limiter-discriminator detection of such a signal produces modulation loss, rather than gain. On the other hand, it was determined that low-index SSB phase-modulation, using subcarriers and linear product demodulation, is theoretically feasible. In fact, a synthesis procedure for such a signal with high modulation efficiency was outlined.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., May 14, 1969.

APPENDIX A

SPECTRAL RELATIONS FOR BAND-PASS NOISE AND SIGNALS

Noise Processes

In computational applications, the spectral density of a band-pass process $n(t)$ is usually given. When $n(t)$ is expressed as

$$n(t) = x(t) \cos \omega_c t - y(t) \sin \omega_c t \tag{A1}$$

the spectral densities of $x(t)$ and $y(t)$ must be determined. This determination may be made in general for a noise process of arbitrary sideband structure as follows.

Let $n(t)$ be the real part of an analytic process $\nu(t)$, where

$$\left.\begin{aligned} \nu(t) &= z(t) \exp\left(j\omega_c t\right) \\ z(t) &= x(t) + jy(t) \end{aligned}\right\} \tag{A2}$$

It is assumed without loss of generality that $n(t)$, $x(t)$, $y(t)$, $z(t)$, and $\nu(t)$ are zero-mean weakly stationary processes. The autocorrelation function of $z(t)$ is

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) + j\left[R_{yx}(\tau) - R_{xy}(\tau)\right] \tag{A3}$$

Now, it is necessary for the weak stationarity of $n(t)$ that

$$\left.\begin{aligned} R_{xx}(\tau) &= R_{yy}(\tau) \\ R_{yx}(\tau) &= -R_{yx}(-\tau) \end{aligned}\right\} \tag{A4}$$

Thus,

$$\left.\begin{aligned} R_{zz}(\tau) &= 2\left[R_{xx}(\tau) + jR_{yx}(\tau)\right] \\ R_{\nu\nu}(\tau) &= R_{zz}(\tau) \exp\left(j\omega_c \tau\right) \\ &= 2\left[R_{nn}(\tau) + j\hat{R}_{nn}(\tau)\right] \end{aligned}\right\} \tag{A5}$$

where the caret denotes Hilbert transform. The last equality of equations (A5) follows from the analyticity of $\nu(t)$.

The spectral densities are defined as the Fourier transforms of the respective autocorrelation functions and are given as

$$\left.\begin{aligned} S_{zz}(\omega) &= 2\left[S_{xx}(\omega) + jS_{yx}(\omega)\right] \\ &= S_{\nu\nu}\left(\omega+\omega_c\right) \\ S_{\nu\nu}(\omega) &= 2\left[1 + \mathrm{sgn}(\omega)\right] S_{nn}(\omega) \end{aligned}\right\} \tag{A6}$$

The second equality of equations (A6) follows from multiplying $R_{\nu\nu}(\tau)$ by $\exp\left[-j\omega_c\tau\right]$ and applying the convolution theorem.

Now by a result of Fourier transform theory (ref. 16),

$$S_{zz}^*(-\omega) = 2\left[S_{xx}(\omega) - jS_{yx}(\omega)\right] \tag{A7}$$

By addition and subtraction of equations (A5) and (A7), it follows that

$$\left.\begin{aligned} S_{xx}(\omega) &= \frac{S_{zz}(\omega) + S_{zz}^*(-\omega)}{4} \\ S_{yx}(\omega) &= j\left[\frac{S_{zz}^*(-\omega) - S_{zz}(\omega)}{4}\right] \end{aligned}\right\} \tag{A8}$$

From equations (A6)

$$\left.\begin{aligned} S_{zz}(\omega) &= 2\left[1 + \mathrm{sgn}\left(\omega+\omega_c\right)\right] S_{nn}\left(\omega+\omega_c\right) \\ S_{zz}^*(-\omega) &= 2\left[1 + \mathrm{sgn}\left(-\omega+\omega_c\right)\right] S_{nn}\left(-\omega+\omega_c\right) \end{aligned}\right\} \tag{A9}$$

Thus

$$\begin{aligned} S_{xx}(\omega) &= S_{yy}(\omega) \\ &= \frac{2\left[1 + \mathrm{sgn}\left(\omega+\omega_c\right)\right] S_{nn}\left(\omega+\omega_c\right) + 2\left[1 + \mathrm{sgn}\left(-\omega+\omega_c\right)\right] S_{nn}\left(-\omega+\omega_c\right)}{4} \\ &= \frac{\left[1 + \mathrm{sgn}\left(\omega+\omega_c\right)\right] S_{nn}\left(\omega+\omega_c\right) + \left[1 - \mathrm{sgn}\left(\omega-\omega_c\right)\right] S_{nn}\left(\omega-\omega_c\right)}{2} \end{aligned} \tag{A10}$$

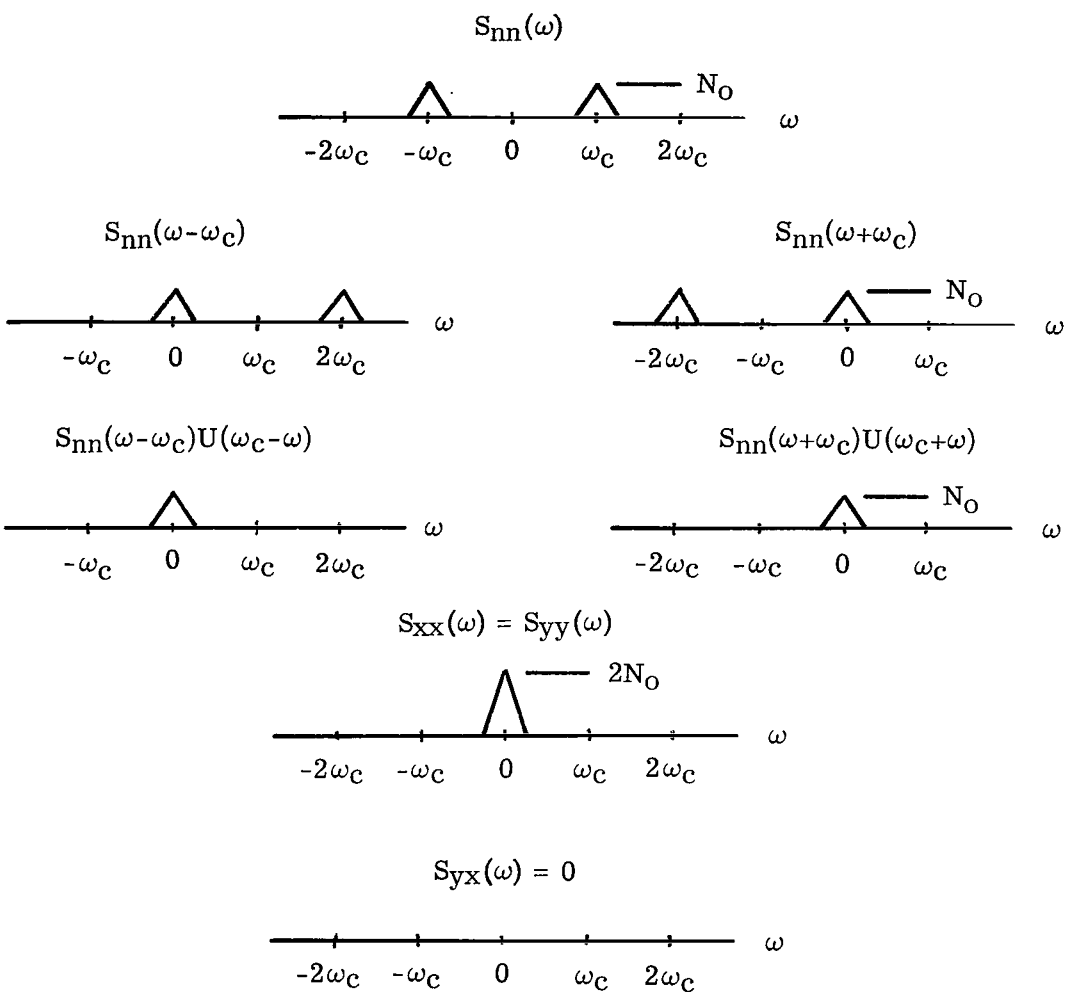

Figure A1.- Spectral relations for noise.

by the "evenness" of $S_{nn}(\omega)$ and the "oddness" of $\operatorname{sgn}(\omega)$. Equation (A10) can be written in terms of the unit step function $U(\omega)$ as

$$S_{xx}(\omega) = S_{yy}(\omega) = S_{nn}\left(\omega+\omega_c\right)U\left(\omega_c+\omega\right) + S_{nn}\left(\omega-\omega_c\right)U\left(\omega_c-\omega\right) \tag{A11}$$

An interesting observation about the correlation of $x(t)$ with $y(t)$ may be made by consideration of the cross spectral density $S_{yx}(\omega)$. From equations (A8) and (A9),

$$S_{yx}(\omega) = j\left[S_{nn}\left(\omega-\omega_c\right)U\left(\omega_c-\omega\right) - S_{nn}\left(\omega+\omega_c\right)U\left(\omega_c+\omega\right)\right] \tag{A12}$$

Since $S_{nn}(\omega)$ is real, $S_{yx}(\omega)$ is purely imaginary, a consequence of the "oddness" of $R_{yx}(\tau)$. Now, although it may not be obvious from equation (A12), if $S_{nn}(\omega)$ is locally symmetric about ω_c, then $S_{yx}(\omega)$ is identically zero. Thus $R_{yx}(\tau)$ must be identically zero. Hence $x(t)$ and $y(t)$ are orthogonal. Since $x(t)$ and $y(t)$ are assumed to be zero-mean they are also uncorrelated. To illustrate equations (A11) and (A12) for a process which is locally symmetric about ω_c, figure A1 is given.

Signal Processes

For signal processes, the computational problem is generally the inverse of the problem encountered with noise. That is, the spectra associated with the modulation function are generally known and the spectrum of the modulated carrier is to be derived.

Let a real band-pass process $s(t)$ be given as the real part of an analytic process $\psi(t)$ by

$$\left.\begin{aligned} s(t) &= A(t)\cos\left[\omega_c t + \phi(t)\right] \\ \psi(t) &= m(t)\exp\left(j\omega_c t\right) \end{aligned}\right\} \tag{A13}$$

The "modulation process" $m(t)$ may be represented, as for noise processes, by

$$\left.\begin{aligned} m(t) &= x(t) + jy(t) \\ x(t) &= A(t)\cos\phi(t) \\ y(t) &= A(t)\sin\phi(t) \end{aligned}\right\} \tag{A14}$$

As with noise processes, it is required that $s(t)$, $x(t)$, $y(t)$, $m(t)$, and $\psi(t)$ be zero-mean, weakly stationary processes.

Because of the weak stationarity of $s(t)$, equations (A4) hold for the present case also. Then

$$R_{mm}(\tau) = 2\left[R_{xx}(\tau) + jR_{yx}(\tau)\right] \tag{A15}$$

Also,

$$R_{\psi\psi}(\tau) = 2\left[R_{ss}(\tau) + j\hat{R}_{ss}(\tau)\right] = R_{mm}(\tau)\exp\left(j\omega_c\tau\right) \tag{A16}$$

Thus

$$\left.\begin{aligned} S_{\psi\psi}(\omega) &= 2(1 + \operatorname{sgn}\omega)\,S_{ss}(\omega) = S_{mm}\left(\omega-\omega_c\right) \\ S^{*}_{\psi\psi}(-\omega) &= 2(1 - \operatorname{sgn}\omega)\,S_{ss}(\omega) = S^{*}_{mm}\left(\omega+\omega_c\right) \end{aligned}\right\} \tag{A17}$$

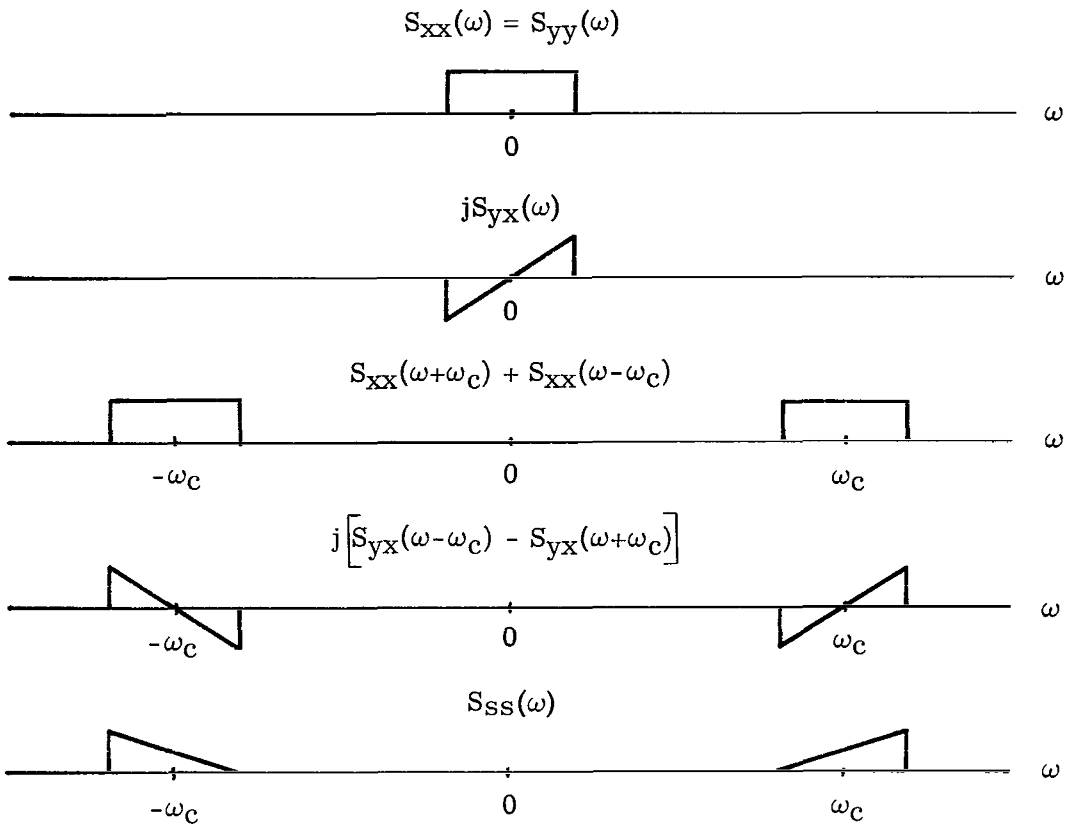

Figure A2.- Spectral relations for signal.

It follows that

$$\left.\begin{aligned} 2(1 + \operatorname{sgn}\omega)\, S_{ss}(\omega) &= 2\left[S_{xx}\left(\omega-\omega_c\right) + jS_{yx}\left(\omega-\omega_c\right)\right] \\ 2(1 - \operatorname{sgn}\omega)\, S_{ss}(\omega) &= 2\left[S_{xx}\left(\omega+\omega_c\right) - jS_{yx}\left(\omega+\omega_c\right)\right] \end{aligned}\right\} \tag{A18}$$

Then

$$\left.\begin{aligned} 2S_{ss}(\omega) &= S_{xx}\left(\omega-\omega_c\right) + jS_{yx}\left(\omega-\omega_c\right); \quad (\omega > 0) \\ 2S_{ss}(\omega) &= S_{xx}\left(\omega+\omega_c\right) - jS_{yx}\left(\omega+\omega_c\right); \quad (\omega < 0) \end{aligned}\right\} \tag{A19}$$

Because of the analyticity of $\psi(t)$,

$$S_{mm}(\omega) = 0; \quad \left(|\omega| \geqq \omega_c\right) \tag{A20}$$

Thus

$$S_{ss}(\omega) = \frac{1}{2}\left\{\left[S_{xx}\left(\omega-\omega_c\right) + S_{xx}\left(\omega+\omega_c\right)\right] + j\left[S_{yx}\left(\omega-\omega_c\right) - S_{yx}\left(\omega+\omega_c\right)\right]\right\} \tag{A21}$$

It should be noted that because $R_{yx}(\tau)$ is real and odd, $S_{yx}(\omega)$ is odd and imaginary. Hence $S_{ss}(\omega)$ is real and even, as required of spectral densities of real processes. To illustrate equation (A21), figure A2 is given.

In computational work it may be easier to compute $S_{ss}(\omega)$ directly from equations (A17) as

$$S_{ss}(\omega) = \frac{1}{4}\left[S_{mm}\left(\omega-\omega_c\right) + S_{mm}^{*}\left(\omega+\omega_c\right)\right] \tag{A22}$$

APPENDIX B

STATIONARITY CONDITIONS FOR STOCHASTIC SINGLE-SIDEBAND ANGLE MODULATION

The General Case

The analytic form of an SSB angle-modulated signal s(t) may be written as

$$\psi(t) = m(t)\exp\left(j\omega_c t\right) \tag{B1}$$

For s(t), the real part of $\psi(t)$, to be weakly stationary requires that

$$E\left\{m(t)\right\} = E\left\{m(t+\tau)\, m(t)\right\} = 0 \tag{B2}$$

Now consider m(t) of the form

$$m(t) = \exp\left\{j\left[f(t) + jg(t)\right]\right\} \tag{B3}$$

where $f(t) + jg(t)$ is analytic. Because the exponential function is entire it follows that m(t) is analytic, and hence $\psi(t)$ represents a signal having only an upper sideband. The conditions of equation (B2) are then written

$$E\left\{m(t)\right\} \equiv E\left\{\exp\left\{j\left[f(t) + jg(t)\right]\right\}\right\} \triangleq \Phi(1,j) \triangleq 0 \tag{B4}$$

$$E\left\{m(t+\tau)\, m(t)\right\} \equiv E\left\{\exp\left\{j\left[f(t+\tau) + jg(t+\tau) + f(t) + jg(t)\right]\right\}\right\} \triangleq \Phi(1,j,1,j;\tau) \triangleq 0 \tag{B5}$$

where Φ denotes characteristic function.

Thus, the stationarity requirement of equation (B2) is written as

$$\Phi(1,j) = \Phi(1,j,1,j;\tau) = 0 \tag{B6}$$

where the ordering of the stochastic variables is understood to be that implicit in equations (B4) and (B5).

The Gaussian Case

Now let $f(t) + jg(t)$ be a complex Gaussian process. Then f(t) and g(t) are individually and jointly Gaussian. Assuming that f(t) and g(t) are zero-mean, they are also individually and jointly stationary in the wide sense. The characteristic functions may be expressed (ref. 9) as

$$\Phi\left(\omega_1,\omega_2,\ldots,\omega_n\right) \triangleq E\left\{\exp\left[j\left(\omega_1 z_1 + \omega_2 z_2 + \ldots + \omega_n z_n\right)\right]\right\}$$

$$\equiv \exp\left(-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\omega_j\omega_k\mu_{jk}\right) \tag{B7}$$

where

$$\mu_{jk} \triangleq E\left\{z_j z_k\right\} \tag{B8}$$

The terms $\omega_j\omega_k\mu_{jk}$ are easily tabulated in matrix form as

$$\left[\omega_j\omega_k\mu_{jk}\right] = \begin{bmatrix} R_{ff}(0) & jR_{gf}(0) & R_{ff}(\tau) & jR_{fg}(\tau) \\ jR_{gf}(0) & -R_{gg}(0) & jR_{gf}(\tau) & -R_{gg}(\tau) \\ R_{ff}(\tau) & jR_{gf}(\tau) & R_{ff}(0) & jR_{gf}(0) \\ jR_{fg}(\tau) & -R_{gg}(\tau) & jR_{gf}(0) & -R_{gg}(0) \end{bmatrix} \tag{B9}$$

for the characteristic function $\Phi(1,j,1,j;\tau)$. Also,

$$\left[\omega_j\omega_k\mu_{jk}\right] = \begin{bmatrix} R_{ff}(0) & jR_{fg}(0) \\ jR_{fg}(0) & -R_{gg}(0) \end{bmatrix} \tag{B10}$$

for the characteristic function $\Phi(1,j)$.

Now, from equations (B7) and (B9),

$$\Phi(1,j,1,j;\tau) = \exp\left\{R_{gg}(0) - R_{ff}(0) + R_{gg}(\tau) - R_{ff}(\tau) - j\left[2R_{gf}(0) + R_{fg}(\tau) + R_{gf}(\tau)\right]\right\} \tag{B11}$$

From equations (B7) and (B10)

$$\Phi(1,j) = \exp\left\{\frac{1}{2}\left[R_{gg}(0) - R_{ff}(0)\right] - jR_{gf}(0)\right\} \tag{B12}$$

It is obvious from equations (B11) and (B12) that the stationarity conditions of equation (B6) are not satisfied for $m(t)$ of the form of equation (B3), where $f(t) + jg(t)$ is Gaussian.

Consider a modified modulation function $m'(t)$ such that

$$m'(t) = m(t) \exp(j\theta) \tag{B13}$$

where θ is a random constant uniformly distributed in $[-\pi, \pi]$ and statistically independent of $m(t)$. Then

$$E\left\{m'(t)\right\} = E\left\{m(t) \exp(j\theta)\right\} = E\left\{m(t)\right\} E\left\{\exp(j\theta)\right\} \tag{B14}$$

Now

$$E\left\{m(t)\right\} = \Phi(1,j) \neq 0 \tag{B15}$$

which implies

$$E\left\{m'(t)\right\} = 0 \quad \text{if and only if} \quad E\left\{\exp(j\theta)\right\} = 0 \tag{B16}$$

Likewise

$$E\left\{m'(t+\tau)m'(t)\right\} = \Phi(1,j,1,j;\tau)\ E\left\{\exp(j2\theta)\right\} \tag{B17}$$

which implies

$$E\left\{m'(t+\tau)\ m'(t)\right\} = 0 \quad \text{if and only if} \quad E\left\{\exp(j2\theta)\right\} = 0 \tag{B18}$$

It is readily verified that for θ uniformly distributed in $[-\pi, \pi]$,

$$E\left\{\exp(j\theta)\right\} = E\left\{\exp(j2\theta)\right\} = 0 \tag{B19}$$

Hence,

$$m'(t) = \exp\left\{j\left[f(t) + jg(t)\right]\right\} \exp(j\theta) \tag{B20}$$

satisfies equation (B2), the stationarity requirement on $s(t)$, where $f(t) + jg(t)$ is analytic and Gaussian.

Expansion of the characteristic function $\Phi(1,j,-1,j;\tau)$ is required for the text. The matrix tabulation of the $\omega_j\omega_k\mu_{jk}$ terms for this characteristic function is

$$\left[\omega_j\omega_k\mu_{jk}\right] = \begin{bmatrix} R_{ff}(0) & jR_{fg}(0) & -R_{ff}(\tau) & jR_{fg}(\tau) \\ jR_{gf}(0) & -R_{gg}(0) & -jR_{gf}(\tau) & -R_{gg}(\tau) \\ -R_{ff}(\tau) & -jR_{gf}(\tau) & R_{ff}(0) & -jR_{fg}(0) \\ jR_{fg}(\tau) & -R_{gg}(\tau) & -jR_{fg}(0) & -R_{gg}(0) \end{bmatrix} \tag{B21}$$

It follows that

$$\begin{aligned} \Phi(1,j,-1,j;\tau) &= \exp\left\{R_{gg}(0) - R_{ff}(0) + R_{gg}(\tau) + R_{ff}(\tau) + j\left[R_{gf}(\tau) - R_{fg}(\tau)\right]\right\} \\ &= \exp\left\{2\left[R_{ff}(\tau) + jR_{gf}(\tau)\right]\right\} \end{aligned} \tag{B22}$$

Also, it is easily shown that

$$\begin{aligned} \left[\Phi(1,j,-1,j;\tau)\right]^* &= \Phi(-1,-j,1,-j;-\tau) \\ &= \Phi(1,-j,-1,-j;\tau) \end{aligned} \tag{B23}$$

APPENDIX C

THE EQUIVALENT PHASE PROCESS OF THE SUM OF AN ANGLE-MODULATED CARRIER AND GAUSSIAN NOISE

Let an angle-modulated sinusoid be represented in standard form as

$$s(t) = A(t) \cos\left[\omega_c t + \phi(t)\right] \tag{C1}$$

where $\phi(t)$ represents the angle modulation and $A(t)$ is a nonnegative "envelope function." Let an additive noise process be given in standard form as

$$n(t) = x(t) \cos \omega_c t - y(t) \sin \omega_c t \tag{C2}$$

The analytic forms $\psi(t)$ and $\nu(t)$, of $s(t)$ and $n(t)$, respectively, are

$$\left.\begin{aligned} \psi(t) &= A(t) \exp\left[j\phi(t)\right] \exp\left(j\omega_c t\right) \\ \nu(t) &= z(t) \exp\left(j\omega_c t\right); \quad \left[z(t) = x(t) + jy(t)\right] \end{aligned}\right\} \tag{C3}$$

Now, it is desirable to represent the sum of $s(t)$ and $n(t)$ as a single modulated sinusoid $r(t)$ of the form

$$r(t) = s(t) + n(t) = R(t) \cos\left[\omega_c t + \phi(t) + \theta(t)\right] \tag{C4}$$

in the manner of Rice (ref. 17). In this representation, $R(t)$ is a nonnegative envelope function, $\phi(t)$ is the angle modulation of the original signal, and $\theta(t)$ is a perturbation due to the additive noise $n(t)$. The analytic form $\rho(t)$ of $r(t)$ is

$$\rho(t) = R(t) \exp\left[j\phi(t)\right] \exp\left[j\theta(t)\right] \exp\left(j\omega_c t\right) \tag{C5}$$

Now, the sum of $\psi(t)$ and $\nu(t)$ is

$$\psi(t) + \nu(t) = \left\{A(t) \exp\left[j\phi(t)\right] + z(t)\right\} \exp\left(j\omega_c t\right) \tag{C6}$$

Hence, equating equations (C5) and (C6) and dividing by $\exp\left(j\omega_c t\right)$ gives

$$R(t) \exp\left[j\phi(t)\right] \exp\left[j\theta(t)\right] = A(t) \exp\left[j\phi(t)\right] + z(t) \tag{C7}$$

and

$$R(t)\ \exp\left[j\theta(t)\right] = A(t) + z(t)\ \exp\left[-j\phi(t)\right] \tag{C8}$$

Equating real and imaginary parts of equation (C8) gives

$$\left.\begin{aligned} R(t)\cos\theta(t) &= A(t) + x(t)\cos\phi(t) + y(t)\sin\phi(t) \\ R(t)\sin\theta(t) &= y(t)\cos\phi(t) - x(t)\sin\phi(t) \end{aligned}\right\} \tag{C9}$$

It follows that $R(t)$ is nonnegative and that

$$\theta(t) = \tan^{-1}\left[\frac{y(t)\cos\phi(t) - x(t)\sin\phi(t)}{A(t) + x(t)\cos\phi(t) + y(t)\sin\phi(t)}\right] \tag{C10}$$

It is seen that, in general, $\theta(t)$, the perturbation of the sum signal $r(t)$, is a non-simple function of the noise and modulation components. However, an approximation to $\theta(t)$ may be obtained for the case where the mean-squared values of $x(t)$ and $y(t)$ are much less than the mean-squared value of $A(t)$. The usual "small angle" approximation is

$$\theta(t) \approx y(t)\frac{\cos\phi(t)}{A(t)} - x(t)\frac{\sin\phi(t)}{A(t)};\quad \left[A(t) \neq 0\right] \tag{C11}$$

It is assumed that in nontrivial cases the probability of $A(t) = 0$ is zero, and hence $\theta(t)$ is given for small $x(t)$ and $y(t)$ by equation (C11) almost all the time.

Having now an approximate model for the "phase-noise" component of $r(t)$, it is next desirable to develop the autocorrelation function of $\theta(t)$ with a view toward obtaining the spectral density:

$$\theta(t+\tau)\ \theta(t) \approx \left[\frac{y(t+\tau)\cos\phi(t+\tau)}{A(t+\tau)} - \frac{x(t+\tau)\sin\phi(t+\tau)}{A(t+\tau)}\right]\left[\frac{y(t)\cos\phi(t)}{A(t)} - \frac{x(t)\sin\phi(t)}{A(t)}\right]$$

$$= \left[\frac{y(t+\tau)\ y(t)\cos\phi(t+\tau)\cos\phi(t)}{A(t+\tau)\ A(t)} + \frac{x(t+\tau)\ x(t)\sin\phi(t+\tau)\sin\phi(t)}{A(t+\tau)\ A(t)}\right.$$

$$\left. - \frac{x(t+\tau)\ y(t)\sin\phi(t+\tau)\cos\phi(t)}{A(t+\tau)\ A(t)} - \frac{y(t+\tau)\ x(t)\cos\phi(t+\tau)\sin\phi(t)}{A(t+\tau)\ A(t)}\right] \tag{C12}$$

Now, assuming that $x(t)$ and $y(t)$ are both statistically independent of $A(t)$ and $\phi(t)$, $R_{\theta\theta}(\tau)$ is

$$R_{\theta\theta}(\tau) = E\left\{\theta(t+\tau)\ \theta(t)\right\}$$

$$= R_{yy}(\tau)\ E\left\{\frac{\cos\ \phi(t+\tau)\ \cos\ \phi(t)}{A(t+\tau)\ A(t)}\right\} + R_{xx}(\tau)\ E\left\{\frac{\sin\ \phi(t+\tau)\ \sin\ \phi(t)}{A(t+\tau)\ A(t)}\right\}$$

$$- R_{xy}(\tau)\ E\left\{\frac{\sin\ \phi(t+\tau)\ \cos\ \phi(t)}{A(t+\tau)\ A(t)}\right\} - R_{yx}(\tau)\ E\left\{\frac{\cos\ \phi(t+\tau)\ \sin\ \phi(t)}{A(t+\tau)\ A(t)}\right\} \tag{C13}$$

Because $n(t)$ is Gaussian,

$$R_{\theta\theta}(\tau) = R_{xx}(\tau)\ E\left\{\frac{\cos\ \phi(t+\tau)\ \cos\ \phi(t) + \sin\ \phi(t+\tau)\ \sin\ \phi(t)}{A(t+\tau)\ A(t)}\right\}$$

$$+ R_{yx}(\tau)\ E\left\{\frac{\sin\ \phi(t+\tau)\ \cos\ \phi(t) - \cos\ \phi(t+\tau)\ \sin\ \phi(t)}{A(t+\tau)\ A(t)}\right\}$$

$$= R_{xx}(\tau)\ E\left\{\frac{\cos\left[\phi(t) - \phi(t+\tau)\right]}{A(t+\tau)\ A(t)}\right\} - R_{yx}(\tau)\ E\left\{\frac{\sin\left[\phi(t) - \phi(t+\tau)\right]}{A(t+\tau)\ A(t)}\right\} \tag{C14}$$

Now, since $A(t)$ is nonnegative, it may be represented as

$$A(t) \triangleq A\ \exp\left[-\alpha(t)\right] \tag{C15}$$

Then, denoting

$$\Phi(-1,-j,1,-j;\tau) \triangleq E\left\{\exp\left\{j\left[-\phi(t+\tau) - j\alpha(t+\tau) + \phi(t) - j\alpha(t)\right]\right\}\right\} \tag{C16}$$

it follows that since

$$R_{zz}(\tau) = 2\left[R_{xx}(\tau) + jR_{yx}(\tau)\right] \tag{C17}$$

then

$$R_{\theta\theta}(\tau) = \frac{1}{2}\ \mathrm{Re}\left\{\frac{1}{A^2}\ R_{zz}(\tau)\ \Phi(-1,-j,1,-j;\tau)\right\} \tag{C18}$$

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