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SOME GENERALIZED POWER SERIES INVERSIONS

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ABSTRACT

A function $G(\xi, \eta, \tau) = 0$ is expanded as a tri-variate power series and a general inversion algorithm derived. Various important special cases are examined, including a two dimensional generalization of the Schlömilch-Cesàro recursion formula for the high order derivatives of a function of a function.

SOME GENERALIZED POWER SERIES INVERSIONS

INTRODUCTION

The use of recursive power series on digital computers as a numerical technique is well known in the literature. Moreover, with the development of compilers for the formal manipulation of symbolic expressions the method of recursive power series has also become an important tool in analytical studies. However, most efforts in these fields have been directed toward efficient algorithms for various highly specialized problems. It is the purpose of this paper to present a very powerful, general algorithm which itself may be specialized to solve a variety of specific problems.

A GENERAL TWO DIMENSIONAL INVERSION

Consider a function of three variables

$$G(\xi, \eta, \tau) = 0$$

$$G(\xi_0, \eta_0, \tau_0) = 0 \tag{1}$$

which may be represented by a power series in a region about ξ_0, η_0, τ_0. Then let

$$x = \xi - \xi_0 \qquad y = \eta - \eta_0 \qquad t = \tau - \tau_0$$

so that

$$x_0 = y_0 = t_0 = 0$$

and

$$G\left(x+\xi_0,\ y+\eta_0,\ t+\tau_0\right) = F(x, y, t) = 0\ . \tag{2}$$

Then

$$0 \equiv F(x, y, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{i,j,k}\, x^i y^j t^k$$

$$= \sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k=j}^{i} a_{j,k-j,i-k}\, x^j y^{k-j} t^{i-k} \tag{3}$$

where

$$i!\ j!\ k!\ a_{ijk} = \frac{\partial^{\,i+j+k}}{\partial x^i\,\partial y^j\,\partial t^k} F(x, y, t)\Big|_0\ , \qquad a_{000} = 0\ .$$

Put

$$W_i \equiv x^i = \sum_{k=i}^{\infty} W_{ik}\, t^k$$

$$V_i = y^i = \sum_{k=i}^{\infty} V_{ik}\, t^k\ . \tag{4}$$

Writing

$$W_i = WW_{i-1}\ , \qquad V_i = VV_{i-1}$$

and forming the Cauchy product yields the recurrence formulas

$$W_{ij} = \sum_{k=1}^{j-i+1} W_{1k} W_{i-1,j-k}$$

$$V_{ij} = \sum_{k=1}^{j-i+1} V_{1k} V_{i-1,j-k} \quad . \tag{5}$$

Note that

$$W_{0j} = V_{0j} = \delta_{0j} \; ; \qquad W_{jk} = 0 \, , \qquad j > k$$

where δ_{ij} is the well-known Kronecker delta. Substituting Equations (4) into (3) we obtain

$$0 \equiv F(x, y, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=i+j}^{\infty} D_{ijk} t^k \tag{6}$$

where

$$D_{ijk} = \sum_{m=0}^{k-i-j} a_{ijm} G_{i,j,k-m}$$

$$G_{ijk} = \sum_{n=j}^{k-i} W_{i,k-n} V_{jn} \quad . \tag{7}$$

Observe that

$$G_{0jk} = V_{jk} \qquad G_{i0k} = W_{ik} \qquad G_{00k} = \delta_{0k} \quad .$$

Equating to zero the coefficients of powers of t in Equation (6) yields

$$0 = \sum_{k=0}^{i} \sum_{j=0}^{i-k} D_{kji} \; ; \qquad i = 0, 1, 2, \cdots \tag{8}$$

Using Equations (7) we have for $i = 0, 1, 2, \cdots$

$$0 = \sum_{k=0}^{i} \sum_{j=0}^{i-k} \sum_{m=0}^{i-k-j} a_{kjm} G_{k,j,i-m}$$

$$= \sum_{k=0}^{i} \sum_{j=0}^{i-k} \sum_{m=0}^{i-k-j} \sum_{n=j}^{i-m-k} a_{kjm} W_{k,i-m-n} V_{jn} \; . \tag{9}$$

Rearranging (9) gives, for $i = 0, 1, 2, \cdots$

$$-\left(a_{010} V_{1i} + a_{100} W_{1i}\right) = a_{00i} + \sum_{m=1}^{i-1} \left[a_{01m} V_{1,i-m} + a_{10m} W_{1,i-m}\right]$$

$$+ \sum_{j=2}^{i} \sum_{m=0}^{i-j} \left[a_{0jm} V_{j,i-m} + a_{j0m} W_{j,i-m}\right]$$

$$+ \sum_{m=0}^{i-2} \sum_{n=1}^{i-m-1} a_{11m} W_{1,i-m-n} V_{1,n}$$

$$+ \sum_{j=2}^{i-1} \sum_{m=0}^{i-j-1} \sum_{n=j}^{i-m-1} a_{1jm} W_{1,i-m-n} V_{j,n}$$

$$+\sum_{j=2}^{i-1}\sum_{m=0}^{i-j-1}\sum_{n=1}^{i-m-j} a_{j1m} W_{j,\,i-m-n} V_{1,n}$$

$$+\sum_{k=2}^{i-2}\sum_{j=2}^{i-k}\sum_{m=0}^{i-k-j}\sum_{n=j}^{i-m-k} a_{kjm} W_{k,\,i-m-n} V_{j,n} \qquad (10)$$

where it is understood that a sum is identically zero if the upper summation index is less than the lower index. Hence if x or y is known as a power series in t, Equation (10) yields y or x respectively as a power series in t.

If we are given two functions in the form

$$0 \equiv F(x, y, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{ijk}\, x^i y^j t^k$$

$$0 \equiv H(x, y, t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{ijk}\, x^i y^j t^k \qquad (11)$$

where

$$a_{010} b_{100} - a_{100} b_{010} \neq 0$$

$$a_{000} = b_{000} = 0$$

there will be two equations of the form (10) which may be solved simultaneously for V_{1i} and W_{1i}, thereby giving

$$x = \sum_{j=1}^{\infty} W_{1j}\, t^j$$

$$y = \sum_{j=1}^{\infty} V_{1j} t^j . \qquad (12)$$

The algorithm for computing the W_{1j}, V_{1j} when given a_{ijk}, b_{ijk} is as follows: For $i = 1, 2, \cdots, j - 2$ compute $W_{j-i,j}$ and $V_{j-i,j}$ from Equations (5). Then solve the two equations of the form (10) for W_{1j} and V_{1j}.

To indicate the flexible nature of Equation (2), let

$$F(x, y, t) = g(x, y, t) - f(t) \equiv 0$$

where

$$g(x, y, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} g_{ijk} x^i y^j t^k$$

$$f(t) = \sum_{i=0}^{\infty} f_i t^i$$

and where $F(x, y, t)$ is given as an expansion by Equation (3) and x and y are given by Equations (4). Then suppose we are given g_{ijk}, W_{1j} and V_{1j} for $i, j, k = 0, 1, 2, \cdots$ and we desire f_j. We have

$$a_{ij0} = g_{ij0}$$

$$a_{00k} = g_{00k} - f_k$$

$$a_{ijk} = g_{ijk} , \qquad i, j, k \neq 0$$

and Equation (10) becomes, for $i = 0, 1, 2, \cdots$

$$f_i = g_{00i} + g_{010} V_{1i} + g_{100} W_{1i} + \sum_{m=1}^{i-1} \left\{ g_{01m} V_{1,i-m} + g_{10m} W_{1,i-m} \right\}$$

$$+ \sum_{j=2}^{i} \sum_{m=0}^{i-j} \left\{ g_{0jm} V_{j,i-m} + g_{j0m} W_{j,i-m} \right\} + \sum_{m=0}^{i-2} \sum_{n=1}^{i-m-1} g_{11m} W_{1,i-m-n} V_{1n}$$

$$+ \sum_{j=2}^{i-1} \sum_{m=0}^{i-j-1} \sum_{n=j}^{i-m-1} g_{1jm} W_{1,i-m-n} V_{jn} + \sum_{j=2}^{i-1} \sum_{m=0}^{i-j-1} \sum_{n=1}^{i-m-j} g_{j1m} W_{j,i-m-n} V_{1n}$$

$$+ \sum_{k=2}^{i-2} \sum_{j=2}^{i-k} \sum_{m=0}^{i-k-j} \sum_{n=j}^{i-m-k} g_{kjm} W_{k,i-m-n} V_{jn} .$$

TWO-DIMENSIONAL SCHLÖMILCH-CESÀRO FORMULA

An important special case of our general result is the two dimensional analogue of the Schlömilch-Cesàro recursion [1] which gives the total derivative of a function of two variables which in turn are functions of a single variable. Szebehely [4] has generalized the Faà di Bruno formula [2] to this problem but his result is somewhat cumbersome for numerical application. To obtain our special case let Equation (2) take the separable form

$$F(x, y, t) = h(x, y) - f(t) \equiv 0 \tag{13}$$

where x and y are functions of t and where

$$x_0 = y_0 = t_0 = 0 .$$

Put

$$h(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} A_{j, i-j} x^j y^{i-j}$$

$$f(t) = \sum_{i=0}^{\infty} f_i t^i \quad . \tag{14}$$

Then by Equation (3)

$$a_{ijk} = \begin{cases} 0 & \text{for} \quad i, j, \text{ or } k \neq 0 \\ A_{ij} & \text{for} \quad k = 0 \\ -f_k & \text{for} \quad i = j = 0 \\ A_{00} - f_0 & \text{for} \quad i = j = k = 0 \, . \end{cases} \tag{15}$$

Equation (9) then reduces to, for $i = 1, 2, \cdots$

$$f_i = \sum_{k=1}^{i} \left[A_{k0} W_{k,i} + A_{0k} V_{k,i} \right]$$

$$+ \sum_{k=1}^{i-1} \sum_{j=1}^{i-k} \sum_{m=k+j}^{i} A_{kj} W_{k,m-j} V_{j,i+j-m} \tag{16}$$

where our previous summation convention has been used and

$$a_{000} = A_{00} - f_0 = 0 \, .$$

Then given A_{ij}, W_{1j} and V_{1j}, we can calculate f_j from Equations (16) and (5).

LOWER DIMENSIONAL SPECIALIZATIONS

If we let y be identically zero in Equation (2) and sin plify Equation (10) we obtain a special case of our more general result which itself has many important and interesting special cases. Equation (2) becomes

$$g(x, t) = 0$$

$$x_0 = t_0 = 0 \tag{17}$$

and $y = 0$ implies

$$V_{00} = 1, \qquad V_{ij} = 0, \qquad i, j \neq 0$$

Equation (10) reduces to, for $i = 1, 2, \cdots$

$$a_{10} W_{1i} = -a_{0i} - \sum_{j=2}^{i} a_{j0} W_{ji} - \sum_{j=1}^{i-1} \sum_{k=1}^{i-j} a_{jk} W_{j,i-k} \tag{18}$$

where a_{ij} replaces a_{i0j} and $a_{00} = 0$.

Equations (17) and (18) give a general result which contains the following special cases of interest; the expression of the solution x of the equation $f(x) = h(t)$ as a power series in t as derived by Thacher [5], the recursive inversion of power series, and a recursive Schlömilch-Cesàro algorithm to compute the derivatives of a single-variate function of a function.

To obtain Thacher's result we assume Equation (17) is separable in the form

$$g(x, t) = f(x) - h(t) = 0 \tag{19}$$

where x is a function of t. We then have

$$a_{j0} = \left(\frac{1}{j!}\right) \frac{d^j f}{dx^j}\bigg|_0 , \qquad a_{0j} = -\left(\frac{1}{j!}\right) \frac{d^j h}{dt^j}\bigg|_0$$

$$a_{ij} = \left(\frac{1}{i!\,j!}\right) \frac{\partial^{\,i+j} g}{\partial x^i \partial t^j}\bigg|_0 = 0 , \qquad i, j \neq 0 .$$

Equation (18) becomes

$$a_{10} W_{1i} = - a_{0i} - \sum_{j=2}^{i} a_{j0} W_{ji} . \tag{20}$$

Denoting

$$f(x) = \sum_{i=0}^{\infty} f_i x^i$$

$$h(t) = \sum_{i=0}^{\infty} h_i t^i \tag{21}$$

we have

$$f_j = a_{j0} , \qquad h_j = - a_{0j}$$

$$f_1 W_{11} = h_1$$

$$f_1 W_{1i} = h_i - \sum_{j=2}^{i} f_j W_{ji} , \qquad i = 2, 3, \cdots . \tag{22}$$

Thus, when f and h are given as power series and the root $x = 0$ is known at $t = 0$, the solution of Equation (19) is given as

$$x = \sum_{i=1}^{\infty} W_{1i} t^i .$$

For the further specialization

$$h(t) \equiv t$$

or

$$h_i = \delta_{i1}$$

we have the well-known power series inversion problem whereby given the series

$$t = \sum_{i=1}^{\infty} f_i x^i$$

we desire

$$x = \sum_{i=1}^{\infty} W_{1i} t^i .$$

The need to compute high order derivatives of a function of a function

$$f(x) = f\left[x(t)\right] = h(t) \tag{23}$$

occurs frequently. Let

$$f(x) = \sum_{i=1}^{\infty} f_i x^i$$

$$x(t) = \sum_{i=1}^{\infty} W_{1i} t^i$$

$$h(t) = \sum_{i=1}^{\infty} h_i t^i \quad . \tag{24}$$

Then given f_i and W_{1i}, we desire h_i. An algorithm presented by both Schlömilch and Cesàro [3] gives

$$h_i = \sum_{j=1}^{i} f_i P_{ij}$$

where P_{ij} is a polynomial in W_{1k}, $k = 1, \cdots, i$, established by the formula

$$P_{i+1,k} = \frac{d}{dt} P_{i,k} + W_{11} P_{i,k-1} \, , \qquad k = 1, 2, \cdots, i+1$$

where

$$P_{i+1,0} = 0 \, , \qquad P_{i+1,k} = 0 \, , \qquad k > i+1 \, .$$

However, Equation (22) may be written

$$h_i = \sum_{j=1}^{i} f_i W_{ji} \tag{25}$$

where the W_{ji} are determined recursively in terms of the W_{1i} from Equation (5) and no further differentiation is necessary.

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