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# PLANETARY EFFECTS IN THE MOTION OF NATURAL SATELLITES

*by Peter Musen*

*Goddard Space Flight Center*
*Greenbelt, Md.*

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## ABSTRACT

Addition of the direct and indirect planetary perturbations to the author's 1963 modification of Hansen's theory is proposed. The method described can determine the Saturnian effects upon the motion of the outer Jovian satellites. The expansion of the disturbing function and of its derivatives is reduced to a form convenient for programming.

# BASIC NOTATIONS

$\mathbf{r}'$ - the jovicentric position vector of the sun

$\bar{\mathbf{r}}'$ - the jovicentric position vector of the fictitious sun moving in Hansen's mean ellipse

$\nu'$ - Hansen's perturbations of the solar radius vector

$u'$ - the elevation of the sun relative to the plane of the Hansen mean ellipse

$\mathbf{P}', \mathbf{Q}', \mathbf{R}'$ - Gibbsian vectors of the solar mean ellipse

$a_0', e_0', n_0'$ - the semimajor axis, eccentricity, and mean daily motion associated with the solar mean ellipse

$g' = n_0' t + g_0'$ - the undisturbed mean anomaly of the sun

$n_0' \delta z'$ - Hansen's perturbations of the mean anomaly of the sun

$\ell' = g' + n_0' \delta z'$ - Hansen's disturbed mean anomaly of the sun

$\bar{f}'$ - Hansen's disturbed true anomaly of the sun

The similar notations

$$\mathbf{r}'', \quad \bar{\mathbf{r}}'', \quad \nu'', \quad u'', \quad \mathbf{P}'', \mathbf{Q}'', \mathbf{R}'',$$

$$a_0'', e_0'', n_0'', \quad g'' = n_0'' t + g_0'',$$

$$n_0'' \delta z'', \quad \ell'' = g'' + n_0'' \delta z'', \quad \bar{f}''$$

refer to the heliocentric motion of Saturn; and the notations

$$\mathbf{r}, \quad \bar{\mathbf{r}}, \quad \nu, \quad u, \quad \mathbf{P}, \mathbf{Q}, \mathbf{R}, \quad a_0, e_0, n_0,$$

$$g = n_0 t + g_0, \quad n_0 \delta z, \quad \ell = g + n_0 \delta z, \quad \bar{f}$$

refer to the jovicentric motion of the satellite

$\omega$ - the mean argument of the perigee of the satellite

$-\omega'$ - the mean longitude of the ascending node of the satellite. The plane of the solar mean ellipse is taken as the basic reference plane.

$I_0$ - the mean inclination of the orbital plane of the satellite

$\Lambda = [\lambda_{ij}]$ - the matrix which is a polynomial in Euler parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and which carries all the periodic oscillations of the osculating orbit plane of the satellite around its mean position

$\mathbf{D}$ - the jovicentric position vector of Saturn

$A', B', C'$ - the projections of $\mathbf{r}'$

$A'', B'', C''$ - the projections of $\mathbf{r}''$

$A, B, C$ - the projections of $\mathbf{D}$

on $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ respectively

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# PLANETARY EFFECTS IN THE MOTION OF NATURAL SATELLITES

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## INTRODUCTION

In this article we develop the theory of general planetary perturbations in the motions of the "natural" planetary satellites—one which can be applied, for example, to the outer Jovian satellites to determine the effects of Saturn. This topic represents a particular case of the four-body problem.

For the determination of the purely solar effects in the motion of the outer Jovian satellites, the author has suggested a modified form of Hansen's theory (Musen, 1963). In the motion of these satellites, the solar effects are strongly dominant over the effects of Saturn or any other additional small force. Thus the determination of the direct solar effects, under the assumption that the solar motion is Keplerian, constitutes the "main problem" of the theory of satellite motions.

A modified form of Hansen's theory as suggested by the author was programmed by Charnow (1966), and as a test the actual expansion of the solar perturbations and the ephemerides for 1967 and 1968 were computed for Jupiter X. The satellite was found by E. Roemer in close proximity to the predicted position. The representation, by this theory, of 30 years (1968) of Jupiter X observations suggests that it may be necessary to include Saturnian effects in the theory of motion.

We must expect the terms affected by small divisors in the integration process to be among the most significant terms in the expansion of the planetary perturbations. Such terms, in the past, were discovered through attempts to find critical linear combinations of the mean anomalies of all three bodies (i.e., combinations with very small mean motions). Obviously, such an approach requires luck, and reasonable doubts can be raised that *all* the significant planetary effects can be found in this way. We prefer to use a form of the theory and a computational scheme which permit the computer to select the significant terms automatically and to decide their importance on the basis of a purely numerical criterion.

Two types of planetary effects are to be included in the theory: (1) the direct planetary effects caused by the direct perturbative action of Saturn on the satellite, and (2) the indirect effects caused by the deviation of the solar motion from its Keplerian approximation. The indirect effects must be included in the solar part of the disturbing function. Thus they represent an immediate contribution to the main problem. In order to include the direct planetary effects in the theory presented here, necessary changes are made in the formulation of the main problem. The basic "small parameter" in front of the solar part of the disturbing function is larger than that for the planetary part. This means that the influence of the mutual perturbations of Jupiter and Saturn is greater in the indirect than in the direct planetary perturbations of the satellite. Under the influence of the planetary effects, the mean argument of the perigee and the mean longitude of the ascending node of the satellite will cease to be linear functions of time. They will also contain terms that are quadratic in time; and, theoretically, also terms of higher orders. Thus, under the influence of the planetary effects, the motion of these two angles will be slightly accelerated.

The form of the theory we are pursuing here is numerical. The values of the mean elements of all bodies can be substituted from the outset, and the final output will be trigonometric expansions with purely numerical coefficients. We employ ideas expressed by the author in his earlier work on the lunar perturbations of artificial satellites (Musen, 1961) and expand the disturbing functions into a series of polynomials in $A$, $B$, $C$, where $A$ is the projection of $\mathbf{D}$, the jovicentric position vector of Saturn, on the mean line of apsides of the Jovian satellite; $B$ is the projection of $\mathbf{D}$ on the direction normal to the mean line of apsides in the orbital plane; and $C$ is the projection of $\mathbf{D}$ on the line normal to the satellite's orbital plane. These polynomials are very simple and their trigonometric expansions can be obtained fairly easily with a computer.

The disturbing function for the direct planetary effects contains the odd negative powers of $|\mathbf{D}|$. Following Newcomb's idea in his work (Newcomb, 1907) on the planetary inequalities in the motion of the moon, we abandon the application of Laplace coefficients to obtain the trigonometric expansions of the powers of $|\mathbf{D}|$, and suggest instead the use of harmonic analysis. Of course, the Keplerian elliptic approximations to the jovicentric motion of the sun and to the heliocentric motion of Saturn will produce the most significant part in the expression for the direct planetary effects. However, we shall provide a device to carry the process to higher approximations, either in order to include them, if necessary, or to have a more precise idea of the magnitudes of the terms we omit. In this work, in accordance with Hansen, we use the satellite's osculating orbit plane as the basic reference plane for the expansion of the perturbations in the satellite's radius vector and mean anomaly. In the classical Hansen theory, the motion of the satellite's osculating orbital plane is referred to the osculating solar orbital plane. However, the elevations of the sun and of Saturn relative to their mean orbital planes are very small, so that for our purposes we can refer the motions of the osculating orbital planes of the sun to the mean orbital plane. Also in this work we can use the existing Hill's expansions (Hill, 1890, 1906) of the Hansen coordinates of Jupiter and Saturn.

## THE DIRECT PLANETARY EFFECTS IN THE ORBITAL PLANE OF THE SATELLITE

We take the mean orbital plane of the sun relative to Jupiter as the basic reference plane. The $x$-axis is directed toward the perijove of Hansen's mean ellipse of the sun, and the $z$-axis is normal to this plane. We use the following notations:

$\mathbf{r}'$ - the jovicentric position vector of the sun,

$\bar{\mathbf{r}}'$ - the jovicentric position vector of the fictitious auxiliary sun moving in Hansen's mean ellipse,

$\nu'$ - Hansen's perturbations of the solar radius vector,

$u'$ - the elevation of the sun relative to the plane of the Hansen mean ellipse,

$\mathbf{P}', \mathbf{Q}', \mathbf{R}'$ - Gibbsian vectors of the solar mean ellipse,

$a_0', e_0', n_0'$ - the semimajor axis, eccentricity, and mean daily motion associated with the solar mean ellipse,

$g' = n_0' t + g_0'$ - the undisturbed mean anomaly of the fictitious sun,

$n_0' \delta z'$ - Hansen's perturbations of the mean anomaly of the sun,

$\ell' = g' + n_0' \delta z'$ - Hansen's disturbed mean anomaly of the sun,

$\bar{f}'$ - Hansen's disturbed true anomaly of the sun.

The notations

$$\mathbf{r}'', \quad \bar{\mathbf{r}}'', \quad \nu'', \quad u'', \quad \mathbf{P}'', \mathbf{Q}'', \mathbf{R}'', \quad a_0'', e_0'', n_0''$$

$$g'' = n_0'' t + g_0'', \quad n_0'' \delta z, \quad \ell'' = g'' + n_0 \delta z'', \quad \bar{f}''$$

refer to the heliocentric motion of Saturn; and the notations

$$\mathbf{r}, \quad \bar{\mathbf{r}}, \quad \nu, \quad u, \quad \mathbf{P}, \mathbf{Q}, \mathbf{R}, \quad a_0, e_0, n_0$$

$$g = n_0 t + g_0, \quad n_0 \delta z, \quad \ell = g + n_0 \delta z, \quad \bar{f}$$

refer to the jovicentric motion of the satellite. The basic relations between the two types of position vectors (for example, $\mathbf{r}''$ and $\bar{\mathbf{r}}''$) are:

$$\mathbf{r}'' = (1 + \nu'')(\bar{\mathbf{r}}'' + \mathbf{R}'' u''), \tag{1}$$

$$\bar{\mathbf{r}}'' = \mathbf{P}'' \bar{r}'' \cos \bar{f}'' + \mathbf{Q}'' \bar{r}'' \sin \bar{f}'', \tag{2}$$

$$\mathbf{r}' = (1 + \nu')(\bar{\mathbf{r}}' + \mathbf{R}' u'), \tag{3}$$

$$\bar{\mathbf{r}} = \mathbf{P}' \bar{r}' \cos \bar{f}' + \mathbf{Q}' \bar{r}' \sin \bar{f}', \tag{4}$$

$$\mathbf{r} = (1 + \nu)\bar{\mathbf{r}}, \tag{5}$$

$$\bar{\mathbf{r}} = \mathbf{P} \bar{r} \cos \bar{f} + \mathbf{Q} \bar{r} \sin \bar{f}. \tag{6}$$

The expansions of **P**, **Q**, **R** into periodic series can be obtained from the relation

$$[\mathbf{P}, \mathbf{Q}, \mathbf{R}] = A_3(-\omega') \cdot \Lambda \cdot A_3(+\omega) \tag{7}$$

given by the author in his previous work (Musen, 1963), where $A_3(\alpha)$ is the matrix of rotation around the $z$-axis; the $\Lambda$-matrix is a polynomial in the Euler parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ which carry all the periodic oscillations of the osculating orbit plane around its mean position; and $\omega$, $-\omega'$ are the mean argument of the perijove and the mean longitude of the ascending node respectively.

We obtain from Equation 7:

$$P_x = +\left(\lambda_4^2 - \lambda_3^2\right)\cos(\omega - \omega') - 2\lambda_3\lambda_4\sin(\omega - \omega') + \left(\lambda_1^2 - \lambda_2^2\right)\cos(\omega + \omega') - 2\lambda_1\lambda_2\sin(\omega + \omega'), \tag{8}$$

$$P_y = +\left(\lambda_4^2 - \lambda_3^2\right)\sin(\omega - \omega') + 2\lambda_3\lambda_4\cos(\omega - \omega') - \left(\lambda_1^2 - \lambda_2^2\right)\sin(\omega + \omega') - 2\lambda_1\lambda_2\cos(\omega + \omega'), \tag{9}$$

$$P_z = +2\left(\lambda_2\lambda_4 + \lambda_1\lambda_3\right)\cos\omega + 2\left(\lambda_1\lambda_4 - \lambda_2\lambda_3\right)\sin\omega, \tag{10}$$

$$Q_x = -\left(\lambda_4^2 - \lambda_3^2\right)\sin(\omega - \omega') - 2\lambda_3\lambda_4\cos(\omega - \omega') - \left(\lambda_1^2 - \lambda_2^2\right)\sin(\omega + \omega') - 2\lambda_1\lambda_2\cos(\omega + \omega'), \tag{11}$$

$$Q_y = +\left(\lambda_4^2 - \lambda_3^2\right)\cos(\omega - \omega') - 2\lambda_3\lambda_4\sin(\omega - \omega') - \left(\lambda_1^2 - \lambda_2^2\right)\cos(\omega + \omega') + 2\lambda_1\lambda_2\sin(\omega + \omega'), \tag{12}$$

$$Q_z = +2\left(\lambda_1\lambda_4 - \lambda_2\lambda_3\right)\cos\omega - 2\left(\lambda_1\lambda_3 + \lambda_2\lambda_4\right)\sin\omega, \tag{13}$$

$$R_x = +2\left(\lambda_1\lambda_3 - \lambda_2\lambda_4\right)\cos\omega' - 2\left(\lambda_1\lambda_4 + \lambda_2\lambda_3\right)\sin\omega', \tag{14}$$

$$R_y = -2\left(\lambda_1\lambda_4 + \lambda_2\lambda_3\right)\cos\omega' - 2\left(\lambda_1\lambda_3 - \lambda_2\lambda_4\right)\sin\omega', \tag{15}$$

$$R_z = -\lambda_1^2 - \lambda_2^2 + \lambda_3^2 + \lambda_4^2. \tag{16}$$

We can set

$$\lambda_1 = \sin\frac{I_0}{2} + \delta\lambda_1, \tag{17}$$

$$\lambda_2 = \delta\lambda_2, \tag{18}$$

$$\lambda_3 = \delta\lambda_3, \tag{19}$$

$$\lambda_4 = \cos\frac{I_0}{2} + \delta\lambda_4, \tag{20}$$

where $I_0$ is the mean inclination of the satellite's orbital plane and $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$, $\delta\lambda_4$ are of the order of perturbations. Making use of Equations 17 through 20, we can write Equations 8 through 16 with sufficient accuracy in the form

$$\mathbf{P} = \mathbf{P}_0 + \delta\boldsymbol{\psi} \times \mathbf{P}_0 , \tag{21}$$

$$\mathbf{Q} = \mathbf{Q}_0 + \delta\boldsymbol{\psi} \times \mathbf{Q}_0 , \tag{22}$$

$$\mathbf{R} = \mathbf{R}_0 + \delta\boldsymbol{\psi} \times \mathbf{R}_0 , \tag{23}$$

where

$$\mathbf{P}_0 = \begin{Bmatrix} + \cos^2 \frac{I_0}{2} \cos(\omega - \omega') & + \sin^2 \frac{I_0}{2} \cos(\omega + \omega') \\ + \cos^2 \frac{I_0}{2} \sin(\omega - \omega') & - \sin^2 \frac{I_0}{2} \sin(\omega + \omega') \\ \multicolumn{2}{c}{\sin I \sin\omega} \end{Bmatrix} , \tag{24}$$

$$\mathbf{Q}_0 = \begin{Bmatrix} - \cos^2 \frac{I_0}{2} \sin(\omega - \omega') & - \sin^2 \frac{I_0}{2} \sin(\omega + \omega') \\ + \cos^2 \frac{I_0}{2} \cos(\omega - \omega') & - \sin^2 \frac{I_0}{2} \cos(\omega + \omega') \\ \multicolumn{2}{c}{\sin I_0 \cos\omega} \end{Bmatrix} , \tag{25}$$

and

$$\mathbf{R}_0 = \begin{Bmatrix} - \sin I_0 \sin\omega' \\ - \sin I_0 \cos\omega' \\ \cos I_0 \end{Bmatrix} \tag{26}$$

are the main long-period parts in **P**, **Q**, and **R**, and

$$\begin{aligned} \delta\boldsymbol{\psi} &= \left(\cos\frac{I_0}{2}\,\delta\lambda_1 - \sin\frac{I_0}{2}\,\delta\lambda_4\right)\left(\mathbf{P}_0 \cos\omega - \mathbf{Q}_0 \sin\omega\right) \\ &\quad - \left(\cos\frac{I_0}{2}\,\delta\lambda_2 - \sin\frac{I_0}{2}\,\delta\lambda_3\right)\left(\mathbf{P}_0 \sin\omega + \mathbf{Q}_0 \cos\omega\right) \\ &\quad + \left(\sin\frac{I_0}{2}\,\delta\lambda_2 + \cos\frac{I_0}{2}\,\delta\lambda_3\right)\mathbf{R}_0 \\ &= \mathbf{P}_0\,\delta\psi_1 + \mathbf{Q}_0\,\delta\psi_2 + \mathbf{R}_0\,\delta\psi_3 . \end{aligned} \tag{27}$$

Designating by $\mathbf{D}$ the jovicentric position vector of Saturn, we have

$$\mathbf{D} = \mathbf{r}'' + \mathbf{r}' \; ; \tag{28}$$

or, taking Equations 1 and 3 into account,

$$\mathbf{D} = (1 + \nu'')(\bar{\mathbf{r}}'' + \mathbf{R}'' u'') + (1 + \nu')(\bar{\mathbf{r}}' + \mathbf{R}' u') \; . \tag{29}$$

In order to achieve symmetry of formulas and computational procedures, we decompose $\mathbf{r}''$, $\mathbf{r}'$, and $\mathbf{D}$ along $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$:

$$\mathbf{r}'' = A''\mathbf{P} + B''\mathbf{Q} + C''\mathbf{R} \; , \tag{30}$$

$$\mathbf{r}' = A'\mathbf{P} + B'\mathbf{Q} + C'\mathbf{R} \; , \tag{31}$$

$$\mathbf{D} = A\mathbf{P} + B\mathbf{Q} + C\mathbf{R} \; , \tag{32}$$

$$A = A'' + A' \; , \qquad B = B'' + B' \; , \qquad C = C'' + C' \; . \tag{33}$$

The disturbing function $\Omega''$ associated with the direct Saturnian action, when expanded into a series in Legendre polynomials, can be written in the form

$$\begin{aligned} \Omega'' = {} & m'' \frac{a_0^2}{a_0''^3} \left[ \frac{3}{2} \left( \frac{a_0''}{D} \right)^5 \left( \frac{\mathbf{D}}{a_0''} \cdot \frac{\mathbf{r}}{a_0} \right)^2 - \frac{1}{2} \left( \frac{a_0''}{D} \right)^3 \left( \frac{r}{a_0} \right)^2 \right] \\ & + m'' \frac{a_0^3}{a_0''^4} \left[ \frac{5}{2} \left( \frac{a_0''}{D} \right)^7 \left( \frac{\mathbf{D}}{a_0''} \cdot \frac{\mathbf{r}}{a_0} \right)^3 - \frac{3}{2} \left( \frac{a_0''}{D} \right)^5 \left( \frac{\mathbf{D}}{a_0''} \cdot \frac{\mathbf{r}}{a_0} \right) \left( \frac{r}{a_0} \right)^3 \right] + \cdots \; , \end{aligned} \tag{34}$$

where $m''$ is the ratio of the mass of Saturn to the mass of Jupiter. Making use of Equations 32, and substituting 5 and 6 in the last equation, we obtain

$$\begin{aligned} a_0 \Omega'' = {} & \frac{m''}{1 + m''} \left( \frac{n_0''}{n_0} \right)^2 (1 + \nu)^2 \Bigg\{ \left[ \frac{3}{4} \left( \frac{a_0''}{D} \right)^5 \frac{A^2 + B^2}{a_0''^2} - \frac{1}{2} \left( \frac{a_0''}{D} \right)^3 \right] \left( \frac{\bar{r}}{a_0} \right)^2 \\ & + \frac{3}{4} \left( \frac{a_0''}{D} \right)^5 \frac{A^2 - B^2}{a_0''^2} \left( \frac{\bar{r}}{a_0} \right)^2 \cos 2\bar{f} + \frac{3}{2} \left( \frac{a_0''}{D} \right)^5 \frac{AB}{a_0''^2} \left( \frac{\bar{r}}{a_0} \right)^2 \sin 2\bar{f} \Bigg\} \\ & + \frac{m''}{1 + m''} \left( \frac{n_0''}{n_0} \right)^2 \frac{a_0}{a_0''} (1 + \nu)^3 \Bigg\{ \frac{A}{a_0''} \left[ \frac{15}{8} \left( \frac{a_0''}{D} \right)^7 \frac{A^2 + B^2}{a_0''^2} - \frac{3}{2} \left( \frac{a_0''}{D} \right)^5 \right] \left( \frac{\bar{r}}{a_0} \right)^3 \cos \bar{f} \end{aligned}$$

$$+\frac{B}{a_0''}\left[\frac{15}{8}\left(\frac{a_0''}{D}\right)^7\frac{A^2+B^2}{a_0''^2}-\frac{3}{2}\left(\frac{a_0''}{D}\right)^5\right]\left(\frac{\bar{r}}{a_0}\right)^3\sin\bar{f}$$

$$+\frac{5}{8}\left(\frac{a_0''}{D}\right)^7\frac{A\left(A^2-3B^2\right)}{a_0''^3}\left(\frac{\bar{r}}{a_0}\right)^3\cos 3\bar{f}$$

$$\left.+\frac{5}{8}\left(\frac{a_0''}{D}\right)^7\frac{B\left(3A^2-B^2\right)}{a_0''^3}\left(\frac{\bar{r}}{a_0}\right)^3\sin 3\bar{f}\right\}+\cdots, \tag{35}$$

where now $m''$ designates the ratio of the mass of Saturn to the mass of the sun.

A similar form of the expansion of the disturbing function was used by the author (Musen, 1961) in the theory of the lunar effects in the motions of artificial satellites. The application of harmonic analysis in $g'$ and $g''$ would be the easiest way to expand $D^{-3}$, $D^{-5}$, $D^{-7}$, $\cdots$ into trigonometric series. In order to make this harmonic analysis possible, we must expand $a_0\,\Omega''$ in powers of $\nu'$, $\nu''$, $u'$, $u''$ and $\delta\boldsymbol{\psi}$.

Setting

$$\mathbf{D}_0 = \bar{\mathbf{r}}'' + \bar{\mathbf{r}}', \tag{36}$$

$$\delta\mathbf{D} = \nu''\bar{\mathbf{r}}'' + \nu'\bar{\mathbf{r}}' + \mathbf{R}''u'' + \mathbf{R}'u', \tag{37}$$

we have

$$\mathbf{D} = \mathbf{D}_0 + \delta\mathbf{D} + \delta\boldsymbol{\psi}\times\mathbf{D}_0, \tag{38}$$

where $\mathbf{D}_0$ and also $\delta\boldsymbol{\psi}$ and $\delta\mathbf{D}$ are referred to the frame $(\mathbf{P}_0, \mathbf{Q}_0, \mathbf{R}_0)$. From now on there is no need for the subscripts on $\mathbf{P}_0$, $\mathbf{Q}_0$, $\mathbf{R}_0$, and $\mathbf{D}_0$. We omit them and use simply $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{D}$. The notations $A''$, $A'$, $A$ ... will be associated with this new meaning of $\mathbf{P}, \mathbf{Q}, \mathbf{R}$. We shall also change the meaning of

$$\frac{\bar{r}''}{a_0''}\begin{matrix}\cos\\ \sin\end{matrix}\bar{f}'', \qquad \frac{\bar{r}'}{a_0'}\begin{matrix}\cos\\ \sin\end{matrix}\bar{f}'$$

and define them as the coordinates associated with the undisturbed Keplerian motions, depending upon $g''$ and $g'$ respectively. Thus

$$\frac{\bar{r}''}{a_0''}\cos\bar{f}'' = \cos E'' - e_0'', \qquad \frac{\bar{r}'}{a_0'}\cos\bar{f}' = \cos E' - e_0',$$

$$\frac{\bar{r}''}{a_0''}\sin\bar{f}'' = \sqrt{1-e_0''^2}\,\sin E'' , \qquad \frac{\bar{r}'}{a_0'}\sin\bar{f}' = \sqrt{1-e_0'^2}\,\sin E' ,$$

$$E'' - e_0''\sin E'' = g'' , \qquad E' - e_0'\sin E' = g' .$$

Now, omitting the factor $1+\nu$ and assigning the new meaning to A, B, C and $\mathbf{D} = \bar{\mathbf{r}}'' + \bar{\mathbf{r}}'$, we set

$$a_0\,\Omega'' = \frac{m''}{1+m''}\left(\frac{n_0''}{n_0}\right)^2\left[\left(\frac{3}{4}\frac{a_0''^5}{D^5}\frac{A^2+B^2}{a_0''^2} - \frac{1}{2}\frac{a_0''^3}{D^3}\right)\left(\frac{\bar{r}}{a_0}\right)^2\right.$$

$$\left. + \frac{3}{4}\frac{a_0''^5}{D^5}\cdot\frac{A^2-B^2}{a_0''^2}\left(\frac{\bar{r}}{a_0}\right)^2\cos 2\bar{f} + \frac{3}{2}\frac{a_0''^5}{D^5}\cdot\frac{AB}{a_0''^2}\left(\frac{\bar{r}}{a_0}\right)^2\sin 2\bar{f}\right]$$

$$+ \frac{m''}{1+m''}\left(\frac{n_0''}{n_0}\right)^2\frac{a_0}{a_0''}\left[\frac{A}{a_0''}\left(\frac{15}{8}\frac{a_0''^7}{D^7}\cdot\frac{A^2+B^2}{a_0''^2} - \frac{3}{2}\frac{a_0''^5}{D^5}\right)\left(\frac{\bar{r}}{a_0}\right)^3\cos\bar{f}\right.$$

$$+ \frac{B}{a_0''}\cdot\left(\frac{15}{8}\frac{a_0''^7}{D^7}\cdot\frac{A^2+B^2}{a_0''^2} - \frac{3}{2}\frac{a_0''^5}{D^5}\right)\left(\frac{\bar{r}}{a_0}\right)^3\sin\bar{f}$$

$$+ \frac{5}{8}\frac{a_0''^7}{D^7}\cdot\frac{A\left(A^2-3B^2\right)}{a_0''^3}\left(\frac{\bar{r}}{a_0}\right)^3\cos 3\bar{f}$$

$$\left. + \frac{5}{8}\frac{a_0''^7}{D^7}\cdot\frac{B\left(3A^2-B^2\right)}{a_0'^3}\left(\frac{\bar{r}}{a_0}\right)^3\sin 3\bar{f}\right] + \cdots \quad (39)$$

Designating the old value of $a_0\,\Omega''$ by $a_0\,\Omega_o''$, we have

$$a_0\,\Omega_o'' = a_0\,\Omega'' + K\,a_0\,\Omega'' \quad (40)$$

where

$$K = \nu\bar{r}\frac{\partial}{\partial\bar{r}} + n_0'\,\delta z'\frac{\partial}{\partial g'} + n_0''\,\delta z''\frac{\partial}{\partial g''} + \left(\delta\frac{\mathbf{D}}{a_0''} + \delta\boldsymbol{\psi}\times\frac{\mathbf{D}}{a_0''}\right)\cdot\left(\nabla_{\mathbf{D}/a_0''} + \frac{\mathbf{D}}{a_0''}\cdot\frac{a_0''}{D}\frac{\partial}{\partial D}\right), \quad (41)$$

and

$$\nabla_{\mathbf{D}/a_0''} = \mathbf{P}\frac{\partial}{\partial A/a_0''} + \mathbf{Q}\frac{\partial}{\partial B/a_0''} + \mathbf{R}\frac{\partial}{\partial C/a_0''} . \quad (42)$$

Taking Equations 28 and 37 into account and neglecting the very small quantities $u'^2$, $u''^2$, $u'u''$, $u'\nu'$, $u''\nu''$, we have

$$K = \nu\bar{r}\frac{\partial}{\partial\bar{r}} + n_0'\,\delta z'\,\frac{\partial}{\partial g'} + n_0''\,\delta z''\,\frac{\partial}{\partial g''}$$

$$+ \nu'\,\frac{a_0'}{a_0''}\,\frac{\bar{\mathbf{r}}'}{a_0'}\cdot\left(\nabla_{D/a_0''} + \frac{\mathbf{D}}{a_0''}\cdot\frac{a_0''}{D}\,\frac{\partial}{\partial D/a_0''}\right)$$

$$+ \nu''\,\frac{\bar{\mathbf{r}}''}{a_0''}\cdot\left(\nabla_{D/a_0''} + \frac{\mathbf{D}}{a_0''}\cdot\frac{a_0''}{D}\,\frac{\partial}{\partial D/a_0''}\right)$$

$$+ \frac{a_0'}{a_0''}\,\frac{u'}{a_0'}\,\mathbf{R}'\cdot\left(\nabla_{D/a_0''} + \frac{\bar{\mathbf{r}}''}{a_0''}\cdot\frac{a_0''}{D}\,\frac{\partial}{\partial D/a_0''}\right)$$

$$+ \frac{u''}{a_0''}\,\mathbf{R}''\cdot\left(\nabla_{D/a_0''} + \frac{a_0'}{a_0''}\,\frac{\bar{\mathbf{r}}'}{a_0'}\,\frac{a_0''}{D}\,\frac{\partial}{\partial D/a_0}\right)$$

$$+ \delta\boldsymbol{\psi}\cdot\frac{\mathbf{D}}{a_0''}\times\nabla_{D/a_0''}\ . \qquad (43)$$

We introduce the following auxiliary notations:

$$a_1'' = \mathbf{P}''\cdot\mathbf{P} = \left[P_x''\cos(\omega-\omega') + P_y''\sin(\omega-\omega')\right]\cos^2\frac{I_0}{2}$$

$$+ \left[P_x''\cos(\omega+\omega') - P_y''\sin(\omega+\omega')\right]\sin^2\frac{I_0}{2} + P_z''\sin I_0\sin\omega\ , \qquad (44)$$

$$a_2'' = \mathbf{Q}''\cdot\mathbf{P} = \left[Q_x''\cos(\omega-\omega') + Q_y''\sin(\omega-\omega')\right]\cos^2\frac{I_0}{2}$$

$$+ \left[Q_x''\cos(\omega+\omega') - Q_y''\sin(\omega+\omega')\right]\sin^2\frac{I_0}{2} + Q_z''\sin I_0\sin\omega\ , \qquad (45)$$

$$a_3'' = \mathbf{R}''\cdot\mathbf{P} = \left[R_x''\cos(\omega-\omega') + R_y''\sin(\omega-\omega')\right]\cos^2\frac{I_0}{2}$$

$$+ \left[R_x''\cos(\omega+\omega') - R_y''\sin(\omega+\omega')\right]\sin^2\frac{I_0}{2} + R_z''\sin I_0\sin\omega\ , \qquad (46)$$

$$b_1'' = \mathbf{P}'' \cdot \mathbf{Q} = \left[P_y'' \cos(\omega - \omega') - P_x'' \sin(\omega - \omega')\right] \cos^2 \frac{I_0}{2}$$

$$- \left[P_y'' \cos(\omega + \omega') + P_x'' \sin(\omega + \omega')\right] \sin^2 \frac{I_0}{2} + P_z'' \sin I_0 \cos\omega \;, \qquad (47)$$

$$b_2'' = \mathbf{Q}'' \cdot \mathbf{Q} = \left[Q_y'' \cos(\omega - \omega') - Q_x'' \sin(\omega - \omega')\right] \cos^2 \frac{I_0}{2}$$

$$- \left[Q_y'' \cos(\omega + \omega') + Q_x'' \sin(\omega + \omega')\right] \sin^2 \frac{I_0}{2} + Q_z'' \sin I_0 \cos\omega \;, \qquad (48)$$

$$b_3'' = \mathbf{R}'' \cdot \mathbf{Q} = \left[R_y'' \cos(\omega - \omega') - R_x'' \sin(\omega - \omega')\right] \cos^2 \frac{I_0}{2}$$

$$- \left[R_y'' \cos(\omega + \omega') + R_x'' \sin(\omega + \omega')\right] \sin^2 \frac{I_0}{2} + R_z'' \sin I_0 \cos\omega \;, \qquad (49)$$

$$c_1'' = \mathbf{P}'' \cdot \mathbf{R} = -\left(P_y'' \cos\omega' + P_x'' \sin\omega'\right) \sin I_0 + P_z'' \cos I_0 \;, \qquad (50)$$

$$c_2'' = \mathbf{Q}'' \cdot \mathbf{R} = -\left(Q_y'' \cos\omega' + Q_x'' \sin\omega'\right) \sin I_0 + Q_z'' \cos I_0 \;, \qquad (51)$$

$$c_3'' = \mathbf{R}'' \cdot \mathbf{R} = -\left(R_y'' \cos\omega' + R_x'' \sin\omega'\right) \sin I_0 + R_z'' \cos I_0 \;, \qquad (52)$$

$$a_1' = \mathbf{P}' \cdot \mathbf{P} = P_x = +\cos^2 \frac{I_0}{2} \cos(\omega - \omega') + \sin^2 \frac{I_0}{2} \cos(\omega + \omega') \;, \qquad (53)$$

$$a_2' = \mathbf{Q}' \cdot \mathbf{P} = P_y = -\cos^2 \frac{I_0}{2} \sin(\omega - \omega') - \sin^2 \frac{I_0}{2} \sin(\omega + \omega') \;, \qquad (54)$$

$$a_3' = \mathbf{R}' \cdot \mathbf{P} = P_z = \sin I_0 \sin\omega \;, \qquad (55)$$

$$b_1' = \mathbf{P}' \cdot \mathbf{Q} = Q_x = -\cos^2 \frac{I_0}{2} \sin(\omega - \omega') - \sin^2 \frac{I_0}{2} \sin(\omega + \omega') \;, \qquad (56)$$

$$b_2' = \mathbf{Q}' \cdot \mathbf{Q} = Q_y = +\cos^2 \frac{I_0}{2} \cos(\omega - \omega') - \sin^2 \frac{I_0}{2} \cos(\omega + \omega') \;, \qquad (57)$$

$$b_3' = \mathbf{R}' \cdot \mathbf{Q} = \sin I_0 \cos\omega \;, \qquad (58)$$

$$c_1' = \mathbf{P}' \cdot \mathbf{R} = R_x = -\sin I_0 \sin\omega' \;, \qquad (59)$$

$$c_2' = \mathbf{Q}' \cdot \mathbf{R} = R_y = -\sin I_0 \cos\omega' \;, \qquad (60)$$

$$c_3' = \mathbf{R}' \cdot \mathbf{R} = \cos I_0 \ . \tag{61}$$

It is of interest to note that these auxiliary quantities are affected by the long-period perturbations only. We have, also,

$$\frac{A''}{a_0''} = a_1'' \frac{\bar{r}''}{a_0''} \cos \bar{f}'' + a_2'' \frac{\bar{r}''}{a_0''} \sin \bar{f}'' \ , \tag{62}$$

$$\frac{B''}{a_0''} = b_1'' \frac{\bar{r}''}{a_0''} \cos \bar{f}'' + b_2'' \frac{\bar{r}''}{a_0''} \sin \bar{f}'' \ , \tag{63}$$

$$\frac{C''}{a_0''} = c_1'' \frac{\bar{r}''}{a_0''} \cos \bar{f}'' + c_2'' \frac{\bar{r}''}{a_0''} \sin \bar{f}'' \ , \tag{64}$$

$$\frac{A'}{a_0''} = \frac{a_0'}{a_0''} \left( a_1' \frac{\bar{r}'}{a_0'} \cos \bar{f}' + a_2' \frac{\bar{r}'}{a_0'} \sin \bar{f}' \right) \ , \tag{65}$$

$$\frac{B'}{a_0''} = \frac{a_0'}{a_0''} \left( b_1' \frac{\bar{r}'}{a_0'} \cos \bar{f}' + b_2' \frac{\bar{r}'}{a_0'} \sin \bar{f}' \right) \ , \tag{66}$$

$$\frac{C'}{a_0''} = \frac{a_0'}{a_0''} \left( c_1' \frac{\bar{r}'}{a_0'} \cos \bar{f}' + c_2' \frac{\bar{r}'}{a_0'} \sin \bar{f}' \right) \ , \tag{67}$$

$$\frac{A}{a_0''} = \frac{A'' + A'}{a_0''} \ , \qquad \frac{B}{a_0''} = \frac{B'' + B'}{a_0''} \ , \qquad \frac{C}{a_0''} = \frac{C'' + C'}{a_0''} \ . \tag{33}$$

After we have obtained the expansions in Equations 62 through 67 we deduce the expansion

$$\frac{a_0'}{a_0''} \frac{\bar{\mathbf{r}}'}{a_0'} \cdot \frac{\mathbf{D}}{a_0''} = \frac{AA' + BB' + CC'}{a_0''^{2}} \ , \tag{68}$$

$$\frac{\bar{\mathbf{r}}''}{a_0''} \cdot \frac{\mathbf{D}}{a_0''} = \frac{AA'' + BB'' + CC''}{a_0''^{2}} \ , \tag{69}$$

$$\mathbf{R}' \cdot \frac{\bar{\mathbf{r}}''}{a_0''} = P_z'' \frac{\bar{r}''}{a_0''} \cos \bar{f}'' + Q_z'' \frac{\bar{r}''}{a_0''} \sin \bar{f}'' \ , \tag{70}$$

$$\mathbf{R}'' \cdot \frac{\bar{\mathbf{r}}'}{a_0'} = R_x'' \frac{\bar{r}'}{a_0'} \cos \bar{f}' + R_y'' \frac{\bar{r}'}{a_0'} \sin \bar{f}' \ . \tag{71}$$

All the expansions in Equations 53 through 71 are purely periodic, and depend upon the basic arguments $g'$, $g''$, $\omega$, and $\omega'$.

The scalar expression of the operator K to be used in the actual computations takes the form

$$
\begin{aligned}
K = {} & \nu \bar{r} \frac{\partial}{\partial \bar{r}} + n_0' \, \delta z' \frac{\partial}{\partial g'} + n_0'' \, \delta z'' \frac{\partial}{\partial g''} \\
& + \nu' \left( \frac{A'}{a_0''} \frac{\partial}{\partial A/a_0''} + \frac{B'}{a_0''} \frac{\partial}{\partial B/a_0''} + \frac{a_0'}{a_0''} \frac{\bar{\mathbf{r}}'}{a_0'} \cdot \frac{\mathbf{D}}{a_0''} \frac{a_0''}{D} \frac{\partial}{\partial D/a_0''} \right) \\
& + \nu'' \left( \frac{A''}{a_0''} \frac{\partial}{\partial A/a_0''} + \frac{B''}{a_0''} \frac{\partial}{\partial B/a_0''} + \frac{\bar{\mathbf{r}}''}{a_0''} \cdot \frac{\mathbf{D}}{a_0''} \frac{a_0''}{D} \frac{\partial}{\partial D/a_0''} \right) \\
& + \frac{a_0'}{a_0''} \frac{u'}{a_0'} \left( P_z \frac{\partial}{\partial A/a_0} + Q_z \frac{\partial}{\partial B/a_0} + \mathbf{R}' \cdot \frac{\bar{\mathbf{r}}''}{a_0''} \frac{a_0''}{D} \frac{\partial}{\partial D/a_0''} \right) \\
& + \frac{u''}{a_0''} \left( a_3'' \frac{\partial}{\partial A/a_0''} + b_3'' \frac{\partial}{\partial B/a_0''} + \frac{a_0'}{a_0''} \mathbf{R}'' \cdot \frac{\bar{\mathbf{r}}'}{a_0'} \frac{a_0''}{D} \frac{\partial}{\partial D/a_0''} \right) \\
& - \frac{C}{a_0''} \delta\psi_1 \frac{\partial}{\partial B/a_0''} + \frac{C}{a_0''} \delta\psi_2 \frac{\partial}{\partial A/a_0''} \\
& + \delta\psi_3 \left( \frac{A}{a_0''} \frac{\partial}{\partial B/a_0''} - \frac{B}{a_0''} \frac{\partial}{\partial A/a_0''} \right) .
\end{aligned}
\tag{72}
$$

The expansions of $\delta\psi_1$, $\delta\psi_2$, $\delta\psi_3$ to be substituted into Equation 72 can be obtained from Equation 27:

$$\delta\psi_1 = + \left( \cos \frac{I_0}{2} \delta\lambda_1 - \sin \frac{I_0}{2} \delta\lambda_2 \right) \cos\omega - \left( \cos \frac{I_0}{2} \delta\lambda_2 - \sin \frac{I_0}{2} \delta\lambda_3 \right) \sin\omega \, , \tag{27'}$$

$$\delta\psi_2 = - \left( \cos \frac{I_0}{2} \delta\lambda_1 - \sin \frac{I_0}{2} \delta\lambda_2 \right) \sin\omega - \left( \cos \frac{I_0}{2} \delta\lambda_2 - \sin \frac{I_0}{2} \delta\lambda_3 \right) \cos\omega \, , \tag{27''}$$

$$\delta\psi_3 = \sin \frac{I_0}{2} \delta\lambda_2 + \cos \frac{I_0}{2} \delta\lambda_3 \, . \tag{27'''}$$

Not all the perturbations in the expansion of K (Equation 72) are comparable in size. For each satellite we shall make a special decision about their relative importance, and in each particular case the expansion of the operator K and the result of its application to $a_0 \Omega''$ will always be reduced to only a few terms.

We obtained the differential equations governing the variations of Hansen's elements $h_0/h$,

$$\Upsilon = 2\frac{h}{h_0}\,\frac{e\cos\left(\chi-\pi_0-n_0 y t\right)-e_0}{1-e_0^2},$$

$$\Psi = 2\frac{h}{h_0}\,\frac{e\sin\left(\chi-\pi_0-n_0 y t\right)}{1-e_0^2},$$

by combining ideas from Hansen's planetary (1857-59) and lunar (1862) theories (Musen, 1963). We have, in the general case,

$$\frac{d\Upsilon}{dt} = +n_0 y\Psi + M_1\frac{\partial a_0\Omega}{\partial \ell} + N_1 r\frac{\partial a_0\Omega}{\partial r}, \tag{73}$$

$$\frac{d\Psi}{dt} = -n_0 y\left(\Upsilon + 2\frac{h}{h_0}\frac{e_0}{1-e_0^2}\right) + M_2\frac{\partial a_0\Omega}{\partial \ell} + N_2 r\frac{\partial a_0\Omega}{\partial r}, \tag{74}$$

$$\frac{d}{dt}\frac{h_0}{h} = M_3\frac{\partial a_0\Omega}{\partial \ell} + N_3 r\frac{\partial a_0\Omega}{\partial r}, \tag{75}$$

where $M_i$, $N_i$ $(i = 1, 2, 3)$ are defined by the equations:

$$M_1 = \frac{2n_0}{1-e_0^2}\left[\frac{1}{e_0}\left(1-e_0^2-\frac{r^2}{a_0^2}\right)-\frac{\nu}{1+\nu}\frac{1}{e_0}\left(1-e_0^2-\frac{\bar r}{a_0}\right)+\left(\frac{h^2}{h_0^2}-1\right)\frac{1}{e_0}\frac{r}{a_0}\left(1-\frac{r}{a_0}\right)\right], \tag{76}$$

$$N_1 = \frac{2n_0}{1-e_0^2}\frac{r}{a_0}\frac{\sin f}{\sqrt{1-e_0^2}}\left[1-\frac{\nu}{1+\nu}\frac{a_0}{r}-\left(\frac{h^2}{h_0^2}-1\right)\left(\frac{a_0}{r}-1\right)\right], \tag{77}$$

$$M_2 = \frac{2n_0}{1-e_0^2}\left[\frac{1}{\sqrt{1-e_0^2}}\int\left(2\frac{\bar r}{a_0}\cos f + 3e_0\right)d\ell-\frac{\nu}{1+\nu}\frac{r}{a_0}\sin f+\left(\frac{h^2}{h_0^2}-1\right)\frac{r^2}{a_0^2}\frac{\sin f}{1-e_0^2}\right], \tag{78}$$

$$N_2 = \frac{2n_0}{(1-e_0^2)^{3/2}}\left[-\frac{\bar r}{a_0}\cos\bar f-2e_0+\sqrt{1-e_0^2}\,\frac{\nu}{1+\nu}\frac{d}{d\ell}\frac{\bar r}{a_0}\sin\bar f\right.$$

$$\left.+\left(\frac{h^2}{h_0^2}-1\right)e_0\frac{\bar r}{a_0}\frac{\sin\bar f}{\sqrt{1-e_0^2}}\frac{d}{d\ell}\frac{\bar r}{a_0}\cos\bar f\right], \tag{79}$$

$$M_3 = +\frac{n_0}{1-e_0^2}\frac{\bar r^2}{a_0^2}, \tag{80}$$

$$N_3 = -\frac{n_0}{1-e_0^2}\frac{\bar r}{a_0}\frac{e_0\sin\bar f}{\sqrt{1-e_0^2}}. \tag{81}$$

Now, however, in order to separate the main part of the direct planetary effects from a less significant part, it will be convenient to change the meaning of some symbols again. So far the notations $\bar{r}$ and $\bar{f}$ have meant the coordinates which are functions of the disturbed mean anomaly $\ell$. Now we shall designate by the same symbols the coordinates which will be functions of the undisturbed mean anomaly $g$. Thus

$$\frac{\bar{r}}{a_0} \cos \bar{f} = \cos E - e_0 ,$$

$$\frac{\bar{r}}{a_0} \sin \bar{f} = \sqrt{1 - e_0^2} \sin E ;$$

and $E - e_0 \sin E = g$. This change in notations requires the expansion of the right sides of Equations 76 through 81 in powers of $n_0 \, \delta z$.

We now set

$$M_1 = \frac{2n_0}{1 - e_0^2} \frac{1}{e_0} \left(1 - e_0^2 - \frac{\bar{r}^2}{a_0^2}\right) , \tag{82}$$

$$\delta M_1 = \frac{2n_0}{1 - e_0^2} \left[- \frac{\nu}{e_0} \left(1 - e_0^2 - \frac{\bar{r}}{a_0}\right) - \frac{2}{e_0} \left(\frac{h_0}{h} - 1\right) \frac{\bar{r}}{a_0} \left(1 - \frac{\bar{r}}{a_0}\right)\right] , \tag{83}$$

$$N_1 = \frac{2n_0}{\left(1 - e_0^2\right)^{3/2}} \frac{\bar{r}}{a_0} \sin \bar{f} , \tag{84}$$

$$\delta N_1 = \frac{2n_0}{\left(1 - e_0^2\right)^{3/2}} \left[- \nu \frac{a_0}{\bar{r}} + 2 \left(\frac{a_0}{\bar{r}} - 1\right) \left(\frac{h_0}{h} - 1\right)\right] \frac{\bar{r}}{a_0} \sin \bar{f} , \tag{85}$$

$$M_2 = \frac{2n_0}{\left(1 - e_0^2\right)^{3/2}} \int \left(2 \frac{\bar{r}}{a_0} \cos \bar{f} + 3e_0\right) dg , \tag{86}$$

$$\delta M_2 = - \frac{2n_0}{1 - e_0^2} \left[\nu \frac{\bar{r}}{a_0} \sin \bar{f} - 2 \frac{\bar{r}}{a_0} \frac{\sin \bar{f}}{1 - e_0^2} \left(\frac{h_0}{h} - 1\right)\right] , \tag{87}$$

$$N_2 = - \frac{2n_0}{\left(1 - e_0^2\right)^{3/2}} \frac{\bar{r}}{a_0} \cos \bar{f} , \tag{88}$$

$$\delta N_2 = \frac{2n_0}{\left(1 - e_0^2\right)^{3/2}} \left[\nu \sqrt{1 - e_0^2} \frac{d}{dg} \frac{\bar{r}}{a_0} \sin \bar{f} + 2 \left(\frac{h_0}{h} - 1\right) \frac{e_0}{\sqrt{1 - e_0^2}} \frac{\bar{r}}{a_0} \sin \bar{f} \frac{d}{dg} \frac{\bar{r}}{a_0} \cos \bar{f}\right] ; \tag{89}$$

and, as before,

$$M_3 = \frac{n_0}{1 - e_0^2} \left(\frac{\overline{r}}{a_0}\right)^2 , \tag{90}$$

$$N_3 = - \frac{n_0}{\left(1 - e_0^2\right)^{3/2}} e_0 \frac{\overline{r}}{a_0} \sin \overline{f} , \tag{91}$$

with the new meaning of $\overline{r}$ and $\overline{f}$. Setting

$$T_i = M_i \frac{\partial a_0 \Omega''}{\partial g} + N_i \overline{r} \frac{\partial a_0 \Omega''}{\partial \overline{r}} , \qquad (i = 1, 2, 3) \tag{92}$$

and taking into consideration that the operators $\partial/\partial g$, $\overline{r}\,(\partial/\partial \overline{r})$ are commutative with the operator $K$, we obtain the following differential equations for the contributions to Hansen's elements under the influence of the direct planetary effects:

$$\frac{d\Delta \Upsilon}{dt} = \Gamma T_1 + \delta M_1 \frac{\partial a_0 \Omega''}{\partial g} + \delta N_1 \overline{r} \frac{\partial a_0 \Omega''}{\partial \overline{r}} , \tag{93}$$

$$\frac{d\Delta \Psi}{dt} = \Gamma T_2 + \delta M_2 \frac{\partial a_0 \Omega''}{\partial g} + \delta N_2 \overline{r} \frac{\partial a_0 \Omega''}{\partial \overline{r}} , \tag{94}$$

$$\frac{d}{dt} \Delta \frac{h_0}{h} = \Gamma T_3 , \tag{95}$$

where the operator $\Gamma$ is defined as

$$\Gamma = I + n_0 \delta z \frac{\partial}{\partial g} + K , \tag{96}$$

and $I$ is the identity operator. The values of $n_0 \delta z$, $\nu$, and $\delta\psi$ taken from the solution of the main problem can be used in the computation of Equations 93 through 96.

Our approach here differs from the approach to the main problem, in which we preferred the method of iteration to obtain the solution. In the main problem it is difficult to split the perturbations into different orders if the satellite theory is a purely numerical one. Generally speaking, the second iteration cycle only *starts* to produce the meaningful approximations to the amplitudes and the mean motions of the arguments of the periodic terms in the expansions of the coordinates. The uniformity of programming is an additional important factor which favors the use of the iteration process in the main problem. In the theory of the direct planetary effects the situation is different, because $(m''/1 + m'')(n_0''/n_0)^2$, the basic small parameter, is much smaller than the corresponding small parameter of the main problem.

The most significant part of the direct planetary effects is associated with the elliptic approximations to the motion of the sun and of Saturn. The terms which are *linear* with respect to perturbations in the right sides of Equations 93 through 96 are the only terms which might produce some small but noticeable additional effects if the computation is performed with an accuracy of $10^{-8}$ to $10^{-9}$. In practice the operator $\Gamma$, like $K$, will always be reduced to only a few terms. The purely secular part in Equation 94 must be transferred to the corresponding equation in the main problem in order to determine the mean motion $\int n_0 y\, dt$ of the pericenter relative to the ideal system of coordinates.

## THE DIRECT PLANETARY PERTURBATIONS OF THE POSITION OF THE ORBITAL PLANE OF THE SATELLITE

Previously (Musen, 1959) we have established the following equations governing the oscillations of the orbital plane around its mean position:

$$\frac{d\lambda_1}{dt} = +n_0 \sigma \lambda_2 + \frac{1}{2}(1+\nu)\frac{h}{h_0}\frac{a_0 n_0}{\sqrt{1-e_0^2}}\frac{\partial(a_0\Omega)}{\partial Z}\left[+\lambda_4 \frac{\bar{r}}{a_0}\cos(\bar{f}+\omega) - \lambda_3 \frac{\bar{r}}{a_0}\sin(\bar{f}+\omega)\right], \tag{97}$$

$$\frac{d\lambda_2}{dt} = -n_0 \sigma \lambda_1 + \frac{1}{2}(1+\nu)\frac{h}{h_0}\frac{a_0 n_0}{\sqrt{1-e_0^2}}\frac{\partial(a_0\Omega)}{\partial Z}\left[-\lambda_3 \frac{\bar{r}}{a_0}\cos(\bar{f}+\omega) - \lambda_4 \frac{\bar{r}}{a_0}\sin(\bar{f}+\omega)\right], \tag{98}$$

$$\frac{d\lambda_3}{dt} = +n_0 \sigma \lambda_4 + \frac{1}{2}(1+\nu)\frac{h}{h_0}\frac{a_0 n_0}{\sqrt{1-e_0^2}}\frac{\partial(a_0\Omega)}{\partial Z}\left[+\lambda_2 \frac{\bar{r}}{a_0}\cos(\bar{f}+\omega) + \lambda_1 \frac{\bar{r}}{a_0}\sin(\bar{f}+\omega)\right], \tag{99}$$

$$\frac{d\lambda_4}{dt} = -n_0 \sigma \lambda_3 + \frac{1}{2}(1+\nu)\frac{h}{h_0}\frac{a_0 n_0}{\sqrt{1-e_0^2}}\frac{\partial(a_0\Omega)}{\partial Z}\left[-\lambda_1 \frac{\bar{r}}{a_0}\cos(\bar{f}+\omega) + \lambda_2 \frac{\bar{r}}{a_0}\sin(\bar{f}+\omega)\right], \tag{100}$$

where $\partial\Omega/\partial Z$ designates the component of the disturbing function normal to the orbital plane. In our present case,

$$\frac{\partial\Omega''}{\partial Z} = m''\left(\frac{1}{|\mathbf{D}-\mathbf{r}|^3} - \frac{1}{D^3}\right)C, \tag{101}$$

or, in the expanded form,

$$\frac{\partial\Omega''}{\partial Z} = m' C\left\{\frac{3\mathbf{r}\cdot\mathbf{D}}{D^5} + \left[\frac{15}{2}\frac{(\mathbf{r}\cdot\mathbf{D})^2}{D^7} - \frac{3}{2}\frac{r^2}{D^5}\right] + \cdots\right\}. \tag{102}$$

At this point it is convenient to change the meaning of notations in the same manner as we did in the previous section. After this change we can set

$$\mathbf{r}\cdot\mathbf{D} = A(\bar{r}\cos\bar{f}) + B(\bar{r}\sin f)$$

and from Equation 102 we deduce with the new meaning of the notations,

$$\frac{\partial\left(a_0^2\,\Omega''\right)}{\partial Z} = +\frac{3m''}{1+m''}\left(\frac{n_0''}{n_0}\right)^2\left(\frac{a_0''}{D}\right)^5\left(\frac{AC}{a_0''^2}\,\frac{\bar r}{a_0}\cos\bar f + \frac{BC}{a_0''^2}\,\frac{\bar r}{a_0}\sin\bar f\right)$$

$$+\frac{m''}{1+m''}\left(\frac{n_0''}{n_0}\right)^2\left(\frac{a_0}{a_0''}\right)\left\{\frac{15}{4}\left(\frac{a_0''}{D}\right)^7\frac{\left(A^2-B^2\right)C}{a_0''^3}\left(\frac{\bar r}{a_0}\right)^2\cos 2\bar f\right.$$

$$\left.+\left[\frac{15}{4}\left(\frac{a_0''}{D}\right)^7\frac{\left(A^2+B^2\right)C}{a_0''^3}-\frac{3}{2}\left(\frac{a_0''}{D}\right)^5\frac{C}{a_0''}\right]\left(\frac{\bar r}{a_0}\right)^2\right\}+\cdots \qquad (103)$$

We set

$$\Lambda_c = \frac{1}{2}\,\frac{n_0}{\sqrt{1-e_0^2}}\,\frac{\partial\left(a_0^2\,\Omega''\right)}{\partial Z}\,\frac{\bar r}{a_0}\cos\left(\bar f+\omega\right), \qquad (104)$$

$$\Lambda_s = \frac{1}{2}\,\frac{n_0}{\sqrt{1-e_0^2}}\,\frac{\partial\left(a_0^2\,\Omega''\right)}{\partial Z}\,\frac{\bar r}{a_0}\sin\left(\bar f+\omega\right). \qquad (105)$$

Then Equations 97 through 100 for the contributions to $\lambda_i$ ($i = 1, 2, 3, 4$) as caused by the direct planetary effects, after some easy transformations, take the form

$$\frac{d\Delta\lambda_1}{dt} = +\left[\Gamma+\nu+\left(\frac{h_0}{h}-1\right)\right]\Lambda_c\cos\frac{I_0}{2} - \left(\Lambda_c\,\delta\lambda_4 - \Lambda_s\,\delta\lambda_3\right), \qquad (106)$$

$$\frac{d\Delta\lambda_2}{dt} = -\left[\Gamma+\nu+\left(\frac{h_0}{h}-1\right)\right]\Lambda_s\cos\frac{I_0}{2} - \left(\Lambda_c\,\delta\lambda_3 + \Lambda_s\,\delta\lambda_4\right), \qquad (107)$$

$$\frac{d\Delta\lambda_3}{dt} = +\left[\Gamma+\nu+\left(\frac{h_0}{h}-1\right)\right]\Lambda_s\sin\frac{I_0}{2} + \left(\Lambda_c\,\delta\lambda_2 + \Lambda_s\,\delta\lambda_1\right), \qquad (108)$$

$$\frac{d\Delta\lambda_4}{dt} = -\left[\Gamma+\nu+\left(\frac{h_0}{h}-1\right)\right]\Lambda_c\sin\frac{I_0}{2} - \left(\Lambda_c\,\delta\lambda_1 - \Lambda_s\,\delta\lambda_2\right). \qquad (109)$$

The purely secular terms should be transferred from Equations 107 and 108 to the corresponding equations of the main problem for the purpose of determination of the mean motion of the node and of the perigee. Normally the force component normal to the orbital plane is smaller than the components in the orbital plane. Thus the terms in the right sides of Equations 103 through 109 which contain the perturbations as factors will be very small and in most cases can be omitted.

## REQUIRED MODIFICATIONS IN THE MAIN PROBLEM

In this section we discuss the changes in the main problem which are necessary to account for the deviation of the solar motion from the elliptic Keplerian motion. The expansions of the disturbing function and of its derivatives as given in the author's previous work (Musen, 1963) require some modifications. The factor $1 + \nu'$ must now be attached to $\bar{r}'$, and $\bar{r}'$, $\bar{f}'$ must now be considered as functions of the disturbed mean anomaly $\ell'$. Furthermore, to make the arguments linear with respect to time from the outset, we must expand the disturbing function and its derivatives in powers not only of $n_0\,\delta z$ as before, but also in powers of $n_0'\,\delta z'$. To perform this expansion we must distinguish between the two uses of the symbols g and g′; that is, between their use as arguments in the expansion of the perturbations and their use as the "elliptic" g and g′ (constituting the main parts of $\ell$ and $\ell'$ respectively). The expansion in powers of $n_0\,\delta z$, $n_0'\,\delta z'$ requires formation of the derivatives with respect to the elliptic g and g′ only.

We resort to the standard Hansen device by introducing the temporary notations $\gamma$, $\gamma'$ for g and g′ respectively. After the expansion in powers of $n_0\,\delta z$ and $n_0'\,\delta z'$ has been completed, we remove this distinction and return to the notations g and g′. We generalize here the author's previous expansion (Musen, 1963) by introducing the disturbing function $\Omega^*$ of the same external form as in the previous work but with g, $\gamma$ and g′, $\gamma'$ separated. We have

$$a_0\,\Omega^* = a_0\,\Omega_1^* + a_0\,\Omega_2^* + a_0\,\Omega_3^* + \cdots , \tag{110}$$

where

$$a_0\,\Omega_1^* = \frac{m'}{1+m'}\left(\frac{n_0'}{n_0}\right)^2 q^*\left(\frac{3}{2}\,s^{*2} - \frac{1}{2}\,p^{*2}\right) , \tag{111}$$

$$a_0\,\Omega_2^* = \frac{m'}{1+m'}\left(\frac{n_0'}{n_0}\right)^2 \frac{a_0}{a_0'}\, q^*\left(\frac{5}{2}\,s^{*3} - \frac{3}{2}\,p^{*2}\,s^*\right) , \tag{112}$$

$$a_0\,\Omega_3^* = \frac{m'}{1+m'}\left(\frac{n_0'}{n_0}\right)^2 \left(\frac{a_0}{a_0'}\right)^2 q^*\left(\frac{35}{8}\,s^{*4} - \frac{15}{4}\,s^{*2}\,p^{*2} + \frac{3}{8}\,p^{*4}\right) , \tag{113}$$

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The symbols $s^*$, $p^*$, $q^*$ are defined by

$$s^* = +\left(\lambda_1^2 - \lambda_2^2\right)s_1^* - 2\lambda_1\lambda_2\,s_2^* + \left(\lambda_4^2 - \lambda_3^2\right)s_3^* - 2\lambda_3\lambda_4\,s_4^* , \tag{114}$$

$$p^* = \frac{1+\nu}{1+\nu'}\,\frac{\rho}{a_0}\cdot\frac{a_0'}{\rho'} , \tag{115}$$

$$q^* = \frac{1}{1+\nu'}\,\frac{a_0'}{\rho'} , \tag{116}$$

where

$$s_1^* = \frac{1+\nu}{1+\nu'} \frac{\rho}{a_0} \frac{a_0'}{\rho'} \cos(\phi + \phi' + \omega + \omega') , \qquad (117)$$

$$s_2^* = \frac{1+\nu}{1+\nu'} \frac{\rho}{a_0} \frac{a_0'}{\rho'} \sin(\phi + \phi' + \omega + \omega') , \qquad (118)$$

$$s_3^* = \frac{1+\nu}{1+\nu'} \frac{\rho}{a_0} \frac{a_0'}{\rho'} \cos(\phi - \phi' + \omega - \omega') , \qquad (119)$$

$$s_4^* = \frac{1+\nu}{1+\nu'} \frac{\rho}{a_0} \frac{a_0'}{\rho'} \sin(\phi - \phi' + \omega - \omega') , \qquad (120)$$

in which $\rho$, $\phi$, $\rho'$, $\phi'$ are defined by the standard formulas

$$\frac{\rho}{a_0} \cos\phi = \cos\epsilon - e_0 ,$$

$$\frac{\rho}{a_0} \sin\phi = \sqrt{1 - e_0^2} \sin\epsilon ,$$

$$\epsilon - e_0 \sin\epsilon = \gamma ,$$

$$\frac{\rho'}{a_0'} \cos\phi' = \cos\epsilon' - e_0' ,$$

$$\frac{\rho'}{a_0'} \sin\phi' = \sqrt{1 - e_0'^2} \sin\epsilon'$$

$$\epsilon' - e_0' \sin\epsilon' = \gamma' .$$

The expansions of

$$\frac{\rho}{a_0} \begin{matrix}\cos\\ \sin\end{matrix} \phi , \qquad \frac{\rho}{a_0}$$

and of

$$\frac{a_0'}{\rho'} \begin{matrix}\cos\\ \sin\end{matrix} \phi' , \qquad \frac{a_0'}{\rho'}$$

in terms of $\gamma$ and $\gamma'$ respectively, can be obtained either by using Bessel functions or by means of the numerical harmonic analysis. After these expansions have been obtained, we can easily deduce the expansions of Equations 111 through 120 in terms of the arguments $g$, $g'$, $g''$, $\omega$, $\omega'$.

We now set

$$a_0 \Omega = \overline{T' a_0 \Omega_1^*} , \qquad \rho \frac{\partial a_0 \Omega}{\partial \rho} = \sum_n (n+1) \overline{T' a_0 \Omega_1^*} , \tag{121}$$

where

$$T' = \sum_n \frac{(n_0' \delta z')^n}{n!} \frac{\partial^n}{\partial \gamma'^n}$$

is the Taylor operator, and the "bar-operator" designates the replacement of $\gamma'$ by $g'$. This modified value of the disturbing function must be used in association with the formulas and the differential equations developed for the main problem instead of its previous value as given in the earlier work (Musen, 1963). The form of the differential equations and of the integration procedure undergoes no change. However, the form of the solution will differ from the solution to the main problem. The series representing the coordinates will no longer be purely periodic. The inclusion of the perturbations of the sun and of Saturn will introduce very small mixed terms into the expansion of the coordinates. The mean motions of the node and of the perijove become the power series (in practice, polynomials) with respect to time. Consequently, Hansen's long-period arguments will become

$$\omega = \omega_0 + \int_0^t n_0 (y + \alpha - \eta) \, dt ,$$

$$\omega' = \omega_0' + \int_0^t n_0 (\alpha + \eta + y') \, dt .$$

## CONCLUSION

We have proposed here the theory and computational scheme which permit addition of the direct and the indirect planetary perturbations, with a high degree of accuracy, to the author's modification in 1963 of Hansen's theory. We suggest application of the method presented here to the determination of the Saturnian effects in the motion of the outer Jovian satellites. No precise statement can be made in advance concerning the selection of terms. Each Jovian satellite displays its own peculiarities of motion, so the final selection of the significant periodic terms must be left to the electronic computer. The expansion of the disturbing function and of its derivatives, also the differential equations of the problem, are reduced to a form convenient for programming.

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