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INJECTION OF AN INVISCID SEPARATED JET AT AN OBLIQUE ANGLE TO A MOVING STREAM

by Marvin E. Goldstein and Willis Braun

Lewis Research Center
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SUMMARY

An analytical solution has been obtained to the problem of a two-dimensional incompressible jet injected into a moving stream from an orifice set at an oblique angle to the stream. It is assumed that the jet separates from the downstream edge of the orifice. The solution is valid when the normalized difference between the total pressure in the jet and the total pressure in the main stream is not too large. Typical flow patterns are shown to illustrate the effects of varying both the jet offset ratio and the total pressure within the jet. The analysis shows that for orifices tilted into the main stream small increases in the total pressure in the jet result in large increases in the jet penetration and thickness.

INTRODUCTION

The flow field which results from the oblique injection of a jet into a moving stream is of considerable interest in a number of fluid mechanical devices. Among these are ground-effect machines, jet flaps, wing fans on VTOL aircraft and fuel injection systems in combustion chambers.

The fluid mechanics of jet penetration into moving streams is by no means fully understood. Some insight into this phenomenon can be gained by considering the injection of two-dimensional inviscid incompressible jets into moving streams since such flows are simple enough to be amendable to mathematical analysis. Of course, viscous effects can be quite significant in real fluid flows. However, in order to take viscous effects into account the usual procedure is to first perform an inviscid analysis and then, provided viscous effects do not modify the flow significantly in the wake region, to modify the flow by superimposing viscous boundary layers. In any event, it is hoped that the inviscid analysis will reveal some of the important features of the flow field and thereby lead to an increased understanding of the phenomena involved.

Relatively few analyses of inviscid flows of this type have been performed. This is due at least partially to the fact that analysis of such flows involves the solution of a nonlinear problem in which the shapes of the boundaries are unknown. However, in the very special case where the total pressure in the jet is equal to the total pressure in the main stream, the classical theory of inviscid jets can be used to obtain solutions. Flows in which the total pressure in the jet equals the total pressure in the main stream are discussed by Ehrich in reference 1. Ehrich considers a large number of possible flow configurations for jets issuing from both slots and orifices. It has been found, however, that there is a serious error in his results for jets issuing from orifices. (Since these results emerge as a special case of the analysis performed herein the corrected solutions will be given.) The error was caused by taking the wrong sign for the square root in the expression for the contraction ratio.

There is another limiting case, called the "strong jet approximation," in which the analysis can be considerably simplified. This is the case where the total pressure in the jet is very much larger than that in the main stream. An analysis of this type of jet was first carried out by G. I. Taylor (ref. 2). Taylor obtained an analytical solution by introducing an additional approximation. This latter approximation was removed by Ackerberg and Pal (ref. 3), who developed a variational principle for the problem and thereby obtained a numerical solution. Although making the strong jet approximation considerably simplifies the analysis, the resulting problem is still nonlinear and involves an unknown boundary so that an exact analytical solution does not seem feasible. In any event, as was pointed out by Taylor (ref. 2), the viscous spreading of real jets which have a total pressure much larger than that in the main stream is so large that the inviscid solutions show no relation to actual experiments.

The general problem of an inviscid jet issuing into a flowing stream from a two-dimensional vaned slot was reformulated by Ting, Libby, and Ruger (ref. 4) in terms of two simultaneous nonlinear singular integral equations. Because of the extreme difficulty involved in solving such equations, the authors considered two limiting cases of the equations. The first corresponded to the strong jet approximation and the second to the case where the total pressure in the main stream differs by only a small amount from the total pressure in the jet. Because of the nature of the boundary conditions associated with the vaned slot, the authors could not linearize the problem even in the case of small total pressure difference. Hence, their formulation of this problem is still in terms of a very difficult nonlinear singular integral equation. The authors give a numerical procedure for obtaining a solution to this equation; but, as pointed out by Ting and Ruger (ref. 5), no attempts at carrying out the solution have been successful. It was also shown in reference 5 that no ordinary perturbation procedure could be used to obtain the solution to the problem of a jet issuing from a vaned slot. However, it will be shown subsequently that a systematic perturbation procedure can be used to linearize the problem of a jet issuing from an orifice into a moving stream in the case where the normalized total pressure difference is small.

In this report an explicit analytic (closed form) solution is obtained for the flow field resulting from a jet issuing from an orifice into a moving stream. The orifice is set at an oblique angle to the flow and it is assumed that the jet separates from the downstream edge of the orifice to form a stagnent wake. The flow configuration is shown in figure 1. The flow will be assumed to be two-dimensional, inviscid, and incompressible. In addition, it will be required that in a certain sense (to be specified more precisely below) the normalized difference between the total pressure in the jet and the total pressure in the main stream be small. The upstream boundary of the jet is the stream line emanating from the upstream edge of the orifice. The velocity, in general, will not be continuous across this stream line.

The problem is solved by expanding the solutions in a small parameter related to the difference in total pressure between the jet and the main stream. The zeroth-order solution corresponds to equal total pressures and can be written down immediately by a simple application of classical techniques. The solution to the zeroth-order problem in fact corresponds to one of the solutions obtained in reference 1. However, it was found that by using a somewhat different procedure than that used in reference 1 a simpler and more convenient form of this solution could be obtained.

Since the boundary shapes for the first-order (different total pressure) problem are unknown, a technique similar to that employed in thin airfoil theory is used to transform the first-order boundary conditions to the zeroth-order boundary. This transformed problem still involves a combination of boundary and jump conditions which cannot be handled by ordinary techniques. Therefore, a new procedure was developed to transform this problem into a standard problem for a sectionally analytic function. (A sectionally analytic function is one which is holomorphic in each of two adjoining regions and has a specified jump in value across the boundary of these regions.) The procedure consists of introducing a new dependent variable in such a way that the new variable has to satisfy only jump and symmetry conditions instead of the combination of jump conditions and boundary conditions that the original variable satisfies. In order to introduce this new variable, several mappings between certain complex planes are introduced. The solution is then obtained by using the theory of sectionally analytic functions.

It should be emphasized that the techniques developed herein are quite general and can be applied to a wide variety of jet injection problems. Since it is impossible to tell from the inviscid analysis whether separation will occur at the downstream edge of the orifice, the case with no separation will be considered in a future report. It is shown in appendix A that, by a simple rescaling, the results obtained herein can be applied to the case where the density of the fluid in the jet is different from that in the main stream.

It is perhaps worth pointing out why an ordinary perturbation procedure works for jet injection problems from orifices but will not work for jet injection problems from slots and vaned slots. The reason is a consequence of the fact that in the mathematical solution the flow fields associated with slots and vaned slots have stagnation points at the up-stream edge of the slot, whereas the flow fields associated with orifices do not. The nature of this stagnation point changes discontinuously as the total pressure in the jet is changed from the free-stream value. This discontinuous change results in a singularity at the stagnation point in the first-order solution when an ordinary expansion procedure is used. This difficulty can be overcome by using the method of matched asymptotic expansions. The inner expansion can, in fact, be obtained by rescaling the results given in reference 5, and the outer expansion can be obtained by the procedure given in this report.

ANALYSIS

Formulation and Boundary Conditions

It will be assumed that the flow is inviscid, incompressible, and irrotational. The jet configuration is illustrated in figure 1. The analysis is limited to the case in which the difference between the total pressure in the jet P_j and the total pressure in the main stream P_∞ is not too large; or more specifically, to the case in which

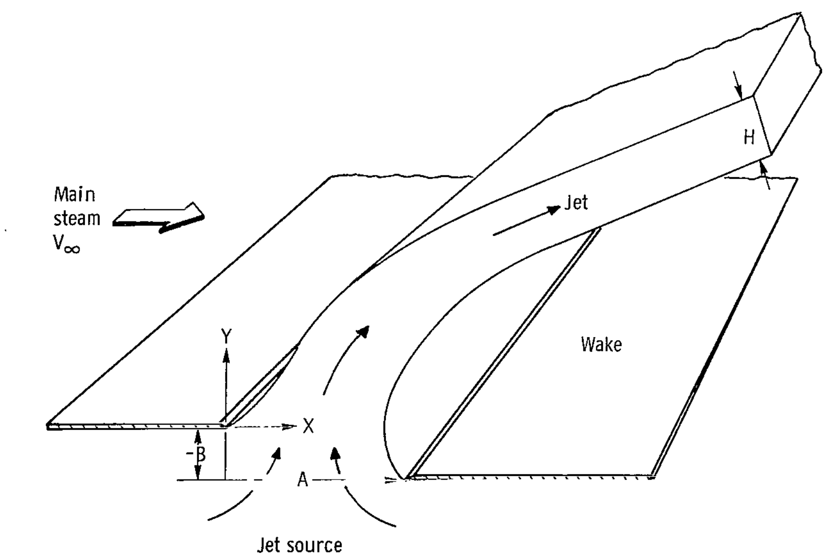


Figure 1. - Schematic configuration of jet penetrating into flowing stream.

$$|\epsilon| << 1$$

re

$$\epsilon \equiv \frac{P_j - P_\infty}{\frac{1}{2} \rho V_\infty^2} \tag{1}$$

is the density of the fluid, and V_∞ is the velocity of the main stream at infinity.

Let l be a convenient reference length which will be specified in the course of the alysis. The X and Y components of the velocity, U and V respectively, will be made dimensionless by V_∞, and the stream function Ψ and the velocity potential Φ will be made dimensionless by $V_\infty l$. Thus, the dimensionless quantities u, v, ψ, and φ are defined by $u \equiv U/V_\infty$, $v \equiv V/V_\infty$, $\psi \equiv \Psi/V_\infty l$, and $\varphi \equiv \Phi/V_\infty l$. The dimensionless complex conjugate velocity ζ and the dimensionless complex potential W are defined, as usual, by

$$\zeta = u - iv$$

and

$$W = \varphi + i\psi$$

With all lengths made dimensionless by l (i.e., $x = X/l$, $y = Y/l$, $a = A/l$, $b = B/l$, and $h = H/l$), the flow configuration is shown in the physical plane (with the complex variable z defined by $z = x + iy$) in figure 2.

The stream of fluid issuing from the orifice formed by the two parallel walls $\widehat{KD}$ and $\widehat{EK}$ meets the main stream at the point D and forms a common stream line, which is denoted by S in figure 2. At the point E the jet separates from the wall $\widehat{EK}$ and a

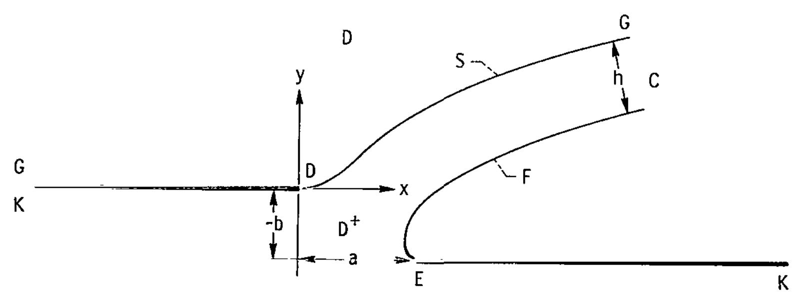


Figure 2. - Flow configuration in physical plane (z-plane).

stream line (denoted by F in fig. 2) marks the boundary between the jet and the downstream wake region, which is assumed to be composed of stagnent fluid. The shapes of the curves S and F are a priori unknown and must be determined from the analysis. Points on the curves S and F will be denoted, respectively, by $z^S = x^S + iy^S$ and $z^F + x^F + iy^F$.

In order to satisfy the requirement that there be no discontinuities in static pressure anywhere within the flow field, it is necessary (as will be shown subsequently) to allow the velocity to be discontinuous across S. For this reason the stream line S will be called the slip line. The region within the jet and orifice is denoted by D^+, and the remaining region of the flow (i.e., the main stream) by D^-. Since the velocity (and as a consequence, the velocity potential) is discontinuous across S, it is convenient to use the superscripts + and - to distinguish these regions. Thus,

$$\zeta(z) = \begin{cases} \zeta^+(z); & z \in D^+ \\ \zeta^-(z); & z \in D^- \end{cases}$$

and

$$W(z) = \begin{cases} W^+(z); & z \in D^+ \\ W^-(z); & z \in D^- \end{cases}$$

Then ζ^+ and W^+ are holomorphic in the interior of D^+, and ζ^- and W^- are holomorphic in the interior of D^-.

The following argument will show that the velocity must be discontinuous across S. Bernoulli's equation can be written for the flow inside the jet as

$$\frac{p(x,y)}{\frac{1}{2}\rho V_\infty^2} + |\zeta^+(z)|^2 = \frac{P_j}{\frac{1}{2}\rho V_\infty^2} \qquad z \in D^+ \tag{2}$$

and for the flow external to the jet, Bernoulli's equation is

$$\frac{p(x,y)}{\frac{1}{2}\rho V_\infty^2} + |\zeta^-(z)|^2 = \frac{P_\infty}{\frac{1}{2}\rho V_\infty^2} \qquad z \in D^- \tag{3}$$

Now, in view of the fact that S is a common stream line to the external and internal flow, the points z in both these equations can be chosen to be the same point z^S on the curve S. Since the static pressure must not be discontinuous across S, it follows from equations (2) and (3) that

$$|\zeta^+(z^S)|^2 - |\zeta^-(z^S)|^2 = \frac{P_j - P_\infty}{\frac{1}{2}\rho V_\infty^2} = \epsilon \tag{4}$$

at every point z^S of S.

On the other hand, since there is no flow in the wake, it is clear that the static pressure along the free-stream line F is constant and equal to the pressure in the wake. Because the velocity in the jet must become uniform across the jet far downstream (i.e., at the point C in fig. 2), it follows that there cannot be any pressure difference across the jet far downstream. It can be concluded from this that the pressure in the wake must equal p_∞, the static pressure of the main stream at infinity. Hence, equation (2) shows that at any point z^F of F

$$|\zeta^+(z^F)|^2 = \frac{P_j - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 + \frac{P_j - \left(p_\infty + \frac{1}{2}\rho V_\infty^2\right)}{\frac{1}{2}\rho V_\infty^2} = 1 + \frac{P_j - P_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 + \epsilon \tag{5}$$

Since S is a common stream line to the internal and external flows it is clear that $\mathcal{Im}\, W^+(z^S)$ and $\mathcal{Im}\, W^-(z^S)$ are both constants. Moreover, the arbitrariness in the definition of W can be partly removed by choosing these constants to be zero (ref. 6). Hence,

$$\mathcal{Im}\, W^+(z) = \mathcal{Im}\, W^-(z) = 0 \qquad \text{for } z \in S \tag{6}$$

The remaining arbitrariness in W can be removed by choosing

$$W^+(0) = W^-(0) = 0 \tag{7}$$

The fact that F is a stream line implies that there exists a real constant ψ^F such that

$$\mathcal{Im}\, W^+(z) = \psi^F \qquad \text{for } z \in F \tag{8}$$

The conditions imposed on the velocity at infinity are (in view of the manner of nondimensionalization; also see fig. 2)

$$\zeta^{+}(z) \to 0 \qquad \text{for} \quad z \to K \tag{9}$$

and

$$\zeta^{-}(z) \to 1 \qquad \text{for} \quad z \to G \tag{10}$$

The remaining boundary condition is that the normal component of the velocity vanish on the solid boundaries. These conditions are sufficient to completely determine the solution. They are summarized below for convenient reference.

$$\left.\begin{array}{ll}
\left.\begin{array}{l}
|\zeta^{+}(z)|^{2} - |\zeta^{-}(z)|^{2} = \epsilon \\
\mathcal{I}m\, W^{+}(z) = \mathcal{I}m\, W^{-}(z) = 0
\end{array}\right\} z \in S & \\
\left.\begin{array}{l}
|\zeta^{+}(z)|^{2} = 1 + \epsilon \\
\mathcal{I}m\, W^{+}(z) = \psi^{F}
\end{array}\right\} z \in F & \\
\mathcal{I}m\, \zeta^{+}(z) = 0; \qquad z \in \widehat{KD} & \\
\mathcal{I}m\, \zeta^{+}(z) = 0; \qquad z \in \widehat{EK} & \\
\mathcal{I}m\, \zeta^{-}(z) = 0; \qquad z \in \widehat{GD} &
\end{array}\right\} \tag{11}$$

Asymptotic Expansions

For small values of ϵ, the functions $\zeta^{\pm}$ and $W^{\pm}$ can be expanded in an asymptotic power series in ϵ. In view of the fact that the shape of the slip line and of the free-stream line depend on ϵ, these expansions imply that the coordinates of S and F, z^{S} and z^{F}, respectively, and the asymptotic jet width h must also be expanded in powers of ϵ. Hence,

$$\left.\begin{aligned}
\zeta^{\pm} &= \zeta_0 + \epsilon\zeta_1^{\pm} + \ldots \\
W^{\pm} &= W_0 + \epsilon W_1^{\pm} + \ldots \\
z^{S} &= z_0^{S} + \epsilon z_1^{S} + \ldots \\
z^{F} &= z_0^{F} + \epsilon z_1^{F} + \ldots \\
h &= h_0 + \epsilon h_1 + \ldots \\
\psi^{F} &= \psi_0^{F} + \epsilon\psi_1^{F} + \ldots
\end{aligned}\right\} \tag{12}$$

It should be pointed out that the expansions for z^S and z^F do not imply that the complex variable z is being expanded. These expansions are only for the shape of the boundary of the jet. In fact, only one of the expansions (12) is independent, and the remaining expansions are determined in terms of it. Thus, for example, since

$$\zeta^{\pm} = \frac{dW^{\pm}}{dz}$$

it follows that

$$\zeta_0 + \epsilon\zeta_1^{\pm} + \ldots = \frac{dW_0}{dz} + \epsilon \frac{dW_1^{\pm}}{dz} + \ldots$$

Hence, equating like powers of ϵ shows

$$\left.\begin{aligned}
\zeta_0 &= \frac{dW_0}{dz} \\
\zeta_1^{\pm} &= \frac{dW_1^{\pm}}{dz} \\
&\ldots\ldots \\
&\ldots\ldots \\
&\ldots\ldots
\end{aligned}\right\} \tag{13}$$

Therefore, once the coefficients of the various powers of ϵ are known in the expansion for $\zeta^{\pm}$, the coefficients of the various powers of ϵ in the expansion for $W^{\pm}$ are determined by equations (13).

The reason for omitting the superscript + or - in the zeroth-order terms of the first two expansions is that (as will be shown subsequently) the zeroth-order solutions are not discontinuous across the curve S and so there is a single function ζ_0 which is homomorphic in the entire flow field (of course, the same is true for W_0).

The reference length l will now be chosen in such a way that

$$h_0 = 1$$

Thus, l is the zeroth-order asymptotic thickness of the jet. This is denoted symbolically by putting

$$l = H_0 \tag{14}$$

The last expansion (eq. (12)) is then

$$h = 1 + \epsilon h_1 + \ldots \tag{15}$$

Zeroth-Order Solution

When the expansions (12) are substituted into the boundary conditions (7) and (9) to (11)) and only the zeroth-order terms are retained, the following boundary conditions for the zeroth-order solution are obtained: First, the first boundary condition (11) shows, as has already been anticipated, that the zeroth-order solution must be continuous across the slip line and, hence, that it is characterized by functions which are holomorphic everywhere within the flow field. The remaining conditions show that

$$\left.\begin{aligned} W_0(0) &= 0 \\ \mathscr{Im}\, W_0\left(z_0^S\right) &= 0 \end{aligned}\right\} \tag{16}$$

$$
\left.\begin{array}{l}
\left.\begin{array}{r}
\left|\zeta_0\left(z_0^F\right)\right| = 1 \\
\mathcal{I}m\, W_0\left(z_0^F\right) = \psi_0^F
\end{array}\right\} \\
\mathcal{I}m\, \zeta_0(z) = 0 \begin{cases} z \in \widehat{KD} \\ z \in \widehat{EK} \\ z \in \widehat{GD} \end{cases} \\
\zeta_0(z) \to 0 \qquad \text{for } z \to K \\
\zeta_0(z) \to 1 \qquad \text{for } z \to G
\end{array}\right\} \tag{17}
$$

The conditions (16) merely serve to show that, because of the manner in which the arbitrary constants have been adjusted in the complex potential, the stream line emanating from the point D is to be taken as the zero stream line.

Now the change in the stream function across the jet must be equal to the volume flow rate through the jet. Hence, if Q_0 denotes the zeroth-order volume flow through the jet, in view of the second condition (16) and the second condition (17), it follows that

$$
Q_0 = 0 - \psi_0^F
$$

The first boundary condition (eq. 17)) shows that far downstream in the jet (i.e., at the point C) the zeroth-order velocity goes to 1. In view of the normalization (15) the asymptotic thickness of this portion of the jet must also be 1. It follows from these remarks that $Q_0 = 1$. Hence,

$$
\psi_0^F = -1 \tag{18}
$$

Now, the boundary value problem posed by the boundary conditions (17) is a simple free-stream-line problem which can be readily solved by the Helmholtz-Kirchhoff technique. In fact, the solution to this problem has already been carried out by Ehrich (ref. 1). His solution, however, is somewhat inconvenient for our purposes. The procedure used herein for obtaining the solution is (ref. 6) to draw the region of flow in the hodograph plane and in the complex potential plane, and then to find the appropriate mapping of these two planes into some convenient intermediate plane (say the T-plane). The

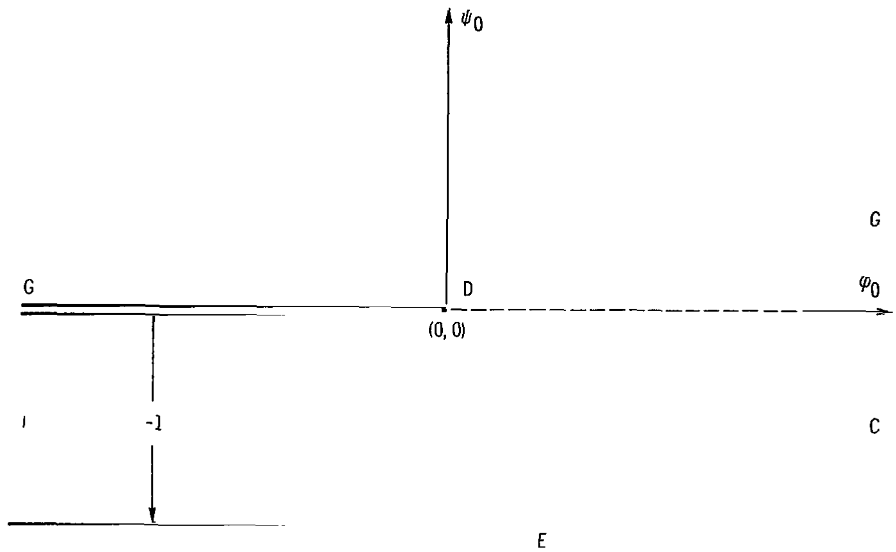


Figure 3. - Zeroth-order complex potential plane (W_0-plane).

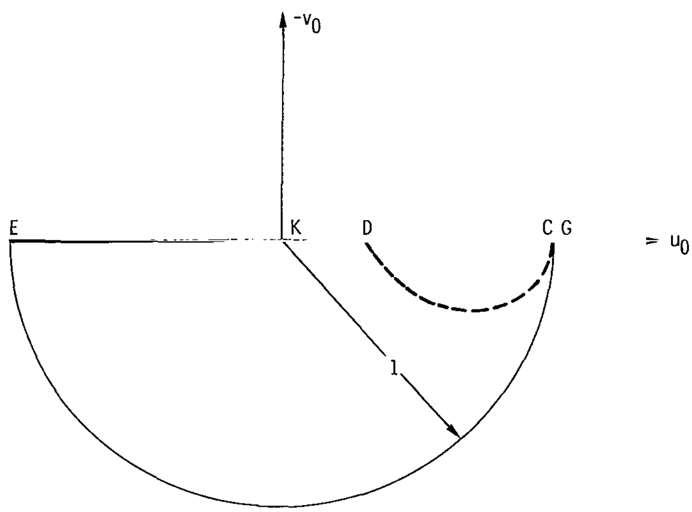


Figure 4. - Zeroth-order hodograph (ζ_0-plane).

shapes of these regions can readily be deduced from the boundary conditions (17), and they are shown in figures 3 and 4 (we have put $W_0 = \varphi_0 + i\psi_0$ in fig. 3). The corresponding points in the various planes are designated by the same letters. The zeroth-order "slip line" is shown dashed in these figures since it does not correspond to a line of discontinuity and can therefore be ignored as far as obtaining the zeroth-order solution is concerned. The intermediate T-plane is chosen in such a way that the region of flow maps into the upper half-plane in the manner indicated in figure 5. We shall denote the real and imaginary parts of the variable T by ξ and η, respectively. The region

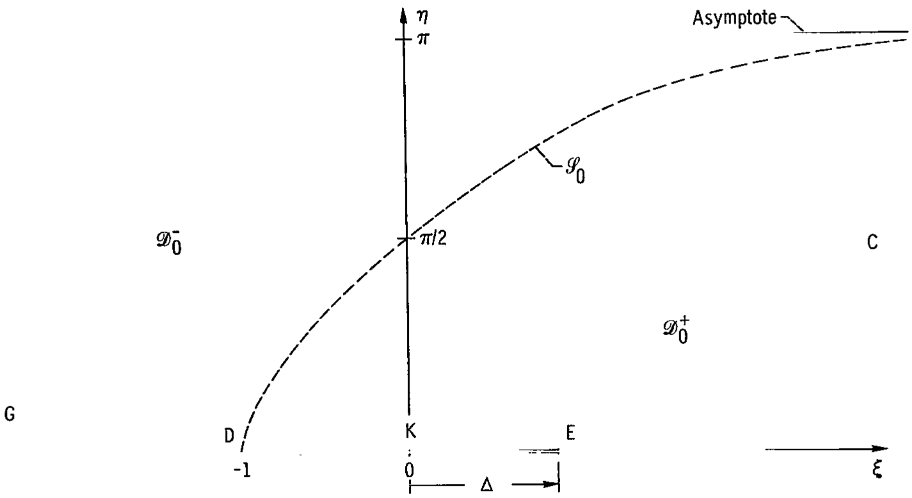


Figure 5. - Intermediate plane (T-plane).

of the T-plane into which the zeroth-order flow field interior to the jet and orifice maps is denoted by $\mathscr{D}_0^+$, and the region of the T-plane into which the zeroth-order main stream maps is denoted by $\mathscr{D}_0^-$. The dividing line between these two regions (which is being called for convenience the zeroth-order slip line even though no slip occurs in the zeroth-order solution) is denoted by $\mathscr{S}_0$.

A simple application of the Schwartz-Christoffel transformation (ref. 7) shows that the mappings which properly transform the W_0-plane and the ζ_0-plane into the upper half T-plane in the manner indicated in the figures are respectively defined by

$$\frac{dW_0}{dT} = \frac{1}{\pi} \frac{T + 1}{T} \qquad \eta \geq 0 \tag{19}$$

and

$$\frac{d \ln \zeta_0}{dT} = \frac{i\sqrt{\Delta}}{T\sqrt{T - \Delta}} \qquad \eta \geq 0 \tag{20}$$

Or, performing the indicated integrations,

$$W_0 = \frac{1}{\pi}(T + 1 + \ln T) - i \qquad \eta \geq 0 \tag{21}$$

$$\zeta_0 = \frac{T - 2\Delta - 2i\sqrt{\Delta}\sqrt{T - \Delta}}{T} \qquad \eta \geq 0 \tag{22}$$

$$\frac{1}{\zeta_0} = \frac{T - 2\Delta + 2i\sqrt{\Delta}\sqrt{T - \Delta}}{T} \qquad \eta \geq 0 \tag{23}$$

In order to find the equation for $\mathscr{S}_0$, the zeroth-order slip line in the T-plane, notice figures 3 and 5 show that $\mathcal{Im}\, W_0(T) = 0$, whenever $T \in \mathscr{S}_0$. Now equation (21) shows, since $T = \xi + i\eta$,

$$W_0(T) = \frac{1}{\pi}\left[\xi + 1 + i\eta - i\pi + \ln\left(\xi^2 + \eta^2\right)^{1/2} + i\tan^{-1}\frac{\eta}{\xi}\right] \qquad 0 \leq \tan^{-1}\frac{\eta}{\xi} \leq \pi$$

Hence, setting the imaginary part of this expression equal to zero we find that

$$\frac{1}{\pi}\left(\eta - \pi + \tan^{-1}\frac{\eta}{\xi}\right) = 0 \qquad \text{for} \quad T \in \mathscr{S}_0$$

This equation has two solutions. The first

$$\eta = 0 \qquad \text{and} \qquad \xi < -1$$

corresponds to the negative real axis in the W_0-plane and is of no interest to us. The other solution is

$$\tan(\pi - \eta) = \frac{\eta}{\xi} \qquad 0 < \eta < \pi$$

This solution corresponds to the positive real axis in the W_0-plane and therefore determines the equation of $\mathscr{S}_0$. Thus,

$$\xi = \frac{\eta}{\tan(\pi - \eta)} = -\frac{\eta}{\tan\eta} \qquad 0 < \eta < \pi \qquad \text{for} \quad T \in \mathscr{S}_0$$

Hence, the parametric equation for $\mathscr{S}_0$ is

$$T = -\frac{\eta}{\tan\eta} + i\eta = -\frac{\eta}{\sin\eta}\,e^{-i\eta} \qquad 0 < \eta < \pi \tag{24}$$

It will be necessary in what follows to have an expression for $\sqrt{T - \Delta}$ for $T \in \mathscr{S}_0$ in terms of the parametric variable η. In order to obtain this expression, notice that

$$\sqrt{T - \Delta} = e^{(1/2)\ln(T-\Delta)}$$

Hence, using equation (24) in this expression shows that for $T \in \mathscr{S}_0$

$$\sqrt{T - \Delta} = \sqrt{\frac{\left(\mu^2 + \eta^2\right)^{1/2} - \mu}{2}} + i\sqrt{\frac{\left(\mu^2 + \eta^2\right)^{1/2} + \mu}{2}} \qquad \text{for } T \in \mathscr{S}_0 \tag{25}$$

where

$$\mu(\eta) \equiv \eta \cot \eta + \Delta \tag{26}$$

It follows from the first equation (13) that the points in the physical plane (fig. 2) are related to the points in the T-plane by

$$z(T) = \int \frac{1}{\zeta_0(T)} \frac{dW_0}{dT} dT + \text{Constant} \tag{27}$$

Substituting equations (19) and (23) into this formula shows that

$$z(T) = \frac{1}{\pi}\left[I_1(T) + I_2(T)\right] \tag{28}$$

where

$$I_1(T) \equiv \int_{-1}^{T} \frac{T - 2\Delta + 2i\sqrt{\Delta}\sqrt{T - \Delta}}{T} dT = \int_{-1}^{T} \frac{1}{\zeta_0(T)} dT \tag{29}$$

$$I_2(T) = \int_{-1}^{T} \frac{T - 2\Delta + 2i\sqrt{\Delta}\sqrt{T - \Delta}}{T^2} dT \tag{30}$$

and we have used the fact, indicated in figure 2, that the origin of the coordinate system in the physical plane is to be at the point D. On carrying out the indicated integration in equations (29) and (30) we find that

$$I_1(T) = (T + 1) + 4i\sqrt{\Delta}\,\sqrt{T - \Delta} - 4\,\Delta\,\ln\left(i\sqrt{T - \Delta} + \sqrt{\Delta}\right)$$

$$+ 4\sqrt{\Delta}\,\sqrt{1 + \Delta} + 2\,\Delta\,\ln\left(\sqrt{1 + \Delta} - \sqrt{\Delta}\right)^2 + 4\,\Delta\,i\pi \qquad (31)$$

and that

$$I_2(T) = \frac{2\,\Delta}{T} - \frac{2i\sqrt{\Delta}\,\sqrt{T - \Delta}}{T} + 2\,\ln\left(i\sqrt{T - \Delta} + \sqrt{\Delta}\right)$$

$$+ 2\,\Delta + 2\sqrt{\Delta}\,\sqrt{1 + \Delta} - \ln\left(\sqrt{1 + \Delta} - \sqrt{\Delta}\right)^2 - 2\pi i$$

Hence, substituting these relations into equation (28) yields

$$z(T) = \frac{1}{\pi}\left[T + \frac{2\,\Delta}{T} + 2i\sqrt{\Delta}\left(2 - \frac{1}{T}\right)\sqrt{T - \Delta} + 2(1 - 2\,\Delta)\ln\left(i\sqrt{T - \Delta} + \sqrt{\Delta}\right)\right]$$

$$+ \frac{1}{\pi}\left[(2\,\Delta + 1) + (2\,\Delta - 1)\ln\left(\sqrt{1 + \Delta} - \sqrt{\Delta}\right)^2 + 6\sqrt{\Delta}\,\sqrt{1 + \Delta} + 2i\pi(2\,\Delta - 1)\right] \qquad (32)$$

By definition (see figs. 2 and 5),

$$z(\Delta) = a + ib \qquad (33)$$

Hence, equation (32) shows, after equating real and imaginary parts,

$$\left.\begin{aligned} a &= \frac{1}{\pi}\left[3(\Delta + 1) + (2\,\Delta - 1)\ln\left(\frac{\sqrt{1 + \Delta} - \sqrt{\Delta}}{\sqrt{\Delta}}\right)^2 + 6\sqrt{\Delta}\,\sqrt{1 + \Delta}\right] \\ b &= 2(2\,\Delta - 1) \end{aligned}\right\} \qquad (34)$$

There is an error in Ehrich's expression for a which corresponds to having the wrong coefficient in the $\sqrt{1 + \Delta}$ term in equation (34). By using these latter two equations, equation (32) can be written as

$$z(T) = a + ib + \frac{1}{\pi}\left[(T - \Delta)\left(1 - \frac{2}{T}\right) + 2i\sqrt{\Delta}\left(2 - \frac{1}{T}\right)\sqrt{T - \Delta} - b \ln\left(\frac{i\sqrt{T - \Delta} + \sqrt{\Delta}}{\sqrt{\Delta}}\right)\right] \tag{35}$$

We notice for future reference that equation (31) shows

$$I_1(\Delta) = \Delta + 1 + 2 \Delta \ln\left(\frac{\sqrt{1 + \Delta} - \sqrt{\Delta}}{\sqrt{\Delta}}\right)^2 + 4\sqrt{\Delta}\,\sqrt{1 + \Delta} + 4\pi i\,\Delta \tag{36}$$

Formulation of First-Order Problem in Physical Plane

The mapping $T \to z$ defined by equation (35) maps the upper half T-plane approximately into the region of flow in the physical plane. The domain $\mathscr{D}_0^+$ is mapped into the cross-hatched region of the physical plane shown in figure 6. The curve $\mathscr{S}_0$ and the line segment $\widehat{EC}$ are mapped into the dashed boundaries S_0 and F_0, respectively, of this region. This region, of course, differs from the true interior of the jet, whose boundaries S and F are indicated by the solid lines (curved) in the figure. However, when $|\epsilon|$ is sufficiently small, S and F do not differ very much from the zeroth-order boundaries of the jet S_0 and F_0, respectively.

Now the first group of the boundary conditions (11) are specified on the curves S and F in the physical plane, whose shapes are not known at this stage of the solution. However, the curves S_0 and F_0, whose shapes are known, differ from S and F, respectively, by quantities which are of order ϵ. Since insofar as the first-order solution is concerned, the boundary conditions only have to be satisfied up to and including

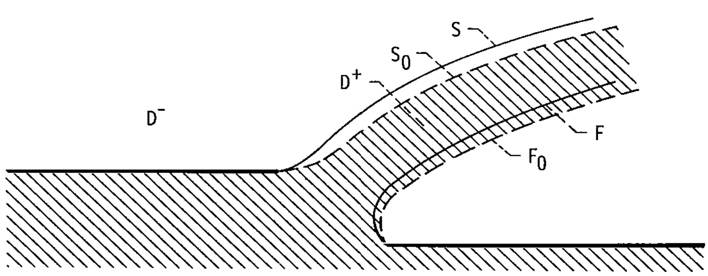


Figure 6. - Comparison of zeroth-order and true boundaries in physical plane.

terms of order ϵ, we shall attempt to transfer the boundary conditions correct to terms of order ϵ from S and F to S_0 and F_0, respectively. To this end recall that the solutions have been divided into two parts (indicated by the superscripts + and -), one of which is holomorphic in D^+ and the other holomorphic in D^-. We shall assume where necessary (as is done in thin-airfoil theory) that each of these portions of the solution can be analytically continued across S and F to S_0 and F_0, respectively. Thus, the values of $\zeta^{\pm}$ and $W^{\pm}$ at a point z^S of S can be expressed in terms of their values at the neighboring point z_0^S of S_0 by performing a Taylor series expansion of these quantities about z_0^S. Of course, similar remarks apply to F. Hence,

$$\zeta^{\pm}\left(z^S\right) = \zeta^{\pm}\left(z_0^S\right) + \left(\frac{d\zeta^{\pm}}{dz}\right)_{z=z_0^S}\left(z^S - z_0^S\right) + \ldots$$

$$W^{\pm}\left(z^S\right) = W^{\pm}\left(z_0^S\right) + \zeta^{\pm}\left(z_0^S\right)\left(z^S - z_0^S\right) + \ldots$$

$$\zeta^{+}\left(z^F\right) = \zeta^{+}\left(z_0^F\right) + \left(\frac{d\zeta^{+}}{dz}\right)_{z=z_0^F}\left(z^F - z_0^F\right) + \ldots$$

$$W^{+}\left(z^F\right) = W^{+}\left(z_0^F\right) + \zeta^{+}\left(z_0^F\right)\left(z^F - z_0^F\right) + \ldots$$

Substituting the asymptotic expansions (12) into these Taylor series and retaining the terms of $O(\epsilon)$ yields

$$\zeta^{\pm}\left(z^S\right) = \zeta_0\left(z_0^S\right) + \epsilon\left[\zeta_1^{\pm}\left(z_0^S\right) + \left(\frac{d\zeta_0}{dz}\right)_{z=z_0^S} z_1^S\right] + O\left(\epsilon^2\right) \tag{37}$$

$$W^{\pm}\left(z^S\right) = W_0\left(z_0^S\right) + \epsilon\left[W_1^{\pm}\left(z_0^S\right) + \zeta_0\left(z_0^S\right)z_1^S\right] + O\left(\epsilon^2\right) \tag{38}$$

$$\zeta^{+}\left(z^F\right) = \zeta_0\left(z_0^F\right) + \epsilon\left[\zeta_1^{+}\left(z_0^F\right) + \left(\frac{d\zeta_0}{dz}\right)_{z=z_0^F} z_1^F\right] + O\left(\epsilon^2\right) \tag{39}$$

$$W^{+}\left(z^{F}\right) = W_0\left(z_0^F\right) + \epsilon\left[W_1^{+}\left(z_0^F\right) + \zeta_0\left(z_0^F\right) z_1^F\right] + O\left(\epsilon^2\right) \tag{40}$$

Hence, we have succeeded in obtaining expressions for the values of the dependent variables $W^{\pm}$ and $\zeta^{\pm}$ at the points of the unknown boundaries S and F in terms of their values on known boundaries S_0 and F_0 with an error of order ϵ^2. When these expressions are substituted into the boundary conditions (11) and terms of $O(\epsilon^2)$ are neglected, expressions will be obtained which give the correct boundary conditions for the first-order solutions on the known boundaries S_0 and F_0. It is clear from the form of these boundary conditions that it will first be necessary to obtain expressions for $|\zeta^{\pm}(z^S)|^2$ and $|\zeta^{\pm}(z^F)|^2$ correct to terms of $O(\epsilon^2)$ before this can be carried out. But it follows immediately from equation (37) that

$$\begin{aligned}
\left|\zeta^{\pm}\left(z^S\right)\right|^2 &= \left|\zeta_0\left(z_0^S\right)\right|^2 + \epsilon\zeta_0\left(z_0^S\right)\overline{\left[\zeta_1^{\pm}\left(z_0^S\right) + \left(\frac{d\zeta_0}{dz}\right)_{z=z_0^S} z_1^S\right]} \\
&\qquad + \epsilon\,\overline{\zeta_0\left(z_0^S\right)}\left[\zeta_1^{\pm}\left(z_0^S\right) + \left(\frac{d\zeta_0}{dz}\right)_{z=z_0^S} z_1^S\right] + O\left(\epsilon^2\right) \\
&= \left|\zeta_0\left(z_0^S\right)\right|^2 + \epsilon\left|\zeta_0\left(z_0^S\right)\right|^2\overline{\left[\frac{\zeta_1^{\pm}\left(z_0^S\right)}{\zeta_0\left(z_0^S\right)} + \frac{1}{\zeta_0\left(z_0^S\right)}\left(\frac{d\zeta_0}{dz}\right)_{z=z_0^S} z_1^S\right]} \\
&\qquad + \epsilon\left|\zeta_0\left(z_0^S\right)\right|^2\left[\frac{\zeta_1^{\pm}\left(z_0^S\right)}{\zeta_0\left(z_0^S\right)} + \frac{1}{\zeta_0\left(z_0^S\right)}\left(\frac{d\zeta_0}{dz}\right)_{z=z_0^S} z_1^S\right] + O\left(\epsilon^2\right)
\end{aligned}$$

Hence,

$$\left|\zeta^{\pm}\left(z^{S}\right)\right|^{2} = \left|\zeta_0\left(z_0^S\right)\right|^{2} + 2\epsilon\left|\zeta_0\left(z_0^S\right)\right|^{2} \mathcal{Re}\left[\frac{\zeta_1^{\pm}\left(z_0^S\right)}{\zeta_0\left(z_0^S\right)} + \frac{1}{\zeta_0\left(z_0^S\right)}\left(\frac{d\zeta_0}{dz}\right)_{z=z_0^S} z_1^S\right] + O\left(\epsilon^2\right) \tag{41}$$

In a completely analogous way, it follows that

$$\left|\zeta^{+}\left(z^{F}\right)\right|^{2} = \left|\zeta_0\left(z_0^F\right)\right|^{2} + 2\epsilon\left|\zeta_0\left(z_0^F\right)\right|^{2} \mathcal{Re}\left[\frac{\zeta_1^{+}\left(z_0^F\right)}{\zeta_0\left(z_0^F\right)} + \frac{1}{\zeta_0\left(z_0^F\right)}\left(\frac{d\zeta_0}{dz}\right)_{z=z_0^F} z_1^F\right] + O\left(\epsilon^2\right) \tag{42}$$

Substituting the expansions (38) and (40) to (42) and the last of expansions (12) into the first group of boundary conditions (11), equating the coefficients of ϵ to the first power, and using the zeroth-order boundary conditions (17) to eliminate $\left|\zeta_0\left(z_0^F\right)\right|$ yield

$$\mathcal{Re}\left[\frac{\zeta_1^{+}\left(z_0^S\right)}{\zeta_0\left(z_0^S\right)} - \frac{\zeta_1^{-}\left(z_0^S\right)}{\zeta_0\left(z_0^S\right)}\right] = \frac{1}{2}\,\frac{1}{\left|\zeta_0\left(z_0^S\right)\right|^{2}} \tag{43}$$

$$\mathcal{Re}\left[\frac{\zeta_1^{+}\left(z_0^F\right)}{\zeta_0\left(z_0^F\right)} + \frac{1}{\zeta_0\left(z_0^F\right)}\left(\frac{d\zeta_0}{dz}\right)_{z=z_0^F} z_1^F\right] = \frac{1}{2} \tag{44}$$

$$\mathcal{Im}\left[W_1^{+}\left(z_0^S\right) + \zeta_0\left(z_0^S\right) z_1^S\right] = 0 \tag{45}$$

$$\mathcal{Im}\left[W_1^{-}\left(z_0^S\right) + \zeta_0\left(z_0^S\right) z_1^S\right] = 0 \tag{46}$$

$$\mathcal{Im}\left[W_1^{+}\left(z_0^F\right) + \zeta_0\left(z_0^F\right) z_1^F\right] = \psi_1^F \tag{47}$$

Thus, equations (43) to (47) are the boundary conditions for the first-order solutions on the boundaries S and F "transferred" to the zeroth-order boundaries S_0 and F_0. Hence, the first-order boundary value problem has been transformed from one in which the shapes of the boundaries are unknown to one in which they are known. Notice, however, that these boundary conditions involve the variables $W_1^{\pm}$, $\zeta_1^{\pm}$, z_1^S, and z_1^F. But $\zeta_1^{\pm}$ is completely determined in terms of $W_1^{\pm}$ by the second equation (13). In view of this the conditions (43), (45), and (46) may be thought of as two boundary conditions connecting the variable ζ_1^+ with the variable ζ_1^- across S_0 (or equivalently the variable W_1^+ with the variable W_1^-) plus an equation which determines z_1^S once $\zeta_1^{\pm}$ are known. Thus, subtracting equation (46) from (45) yields

$$\mathcal{Im}\, W_1^+\left(z_0^S\right) = \mathcal{Im}\, W_1^-\left(z_0^S\right) \tag{48}$$

Then, in view of the second equation (13), equations (43) and (48) are the boundary conditions on S_0 which connect the solution ζ_1^+ in D^+ with the solution ζ_1^- in D^-, and equation (45) serves to determine z_1^S once ζ_1^+ is known. (Actually, z_1^S will be determined in a slightly different fashion.) Similarly, the variable z_1^F can be eliminated between the conditions (44) and (47) to yield a single boundary condition for ζ_1^+ (or W_1^+) on F_0. To this end notice it follows from equation (13) that

$$\frac{1}{\zeta_0}\frac{d\zeta_0}{dz} = \frac{1}{\zeta_0}\frac{d\zeta_0}{dW_0}\frac{dW_0}{dz} = \frac{1}{\zeta_0}\frac{d\zeta_0}{dW_0}\zeta_0 = \frac{d\ln\zeta_0}{dW_0}\zeta_0$$

Hence, in particular

$$\frac{1}{\zeta_0\left(z_0^F\right)}\left(\frac{d\zeta_0}{dz}\right)_{z=z_0^F} = \zeta_0\left(z_0^F\right)\left(\frac{d\ln\zeta_0}{dW_0}\right)_{z=z_0^F}$$

But

$$i\left(\frac{d\ln\zeta_0}{dW_0}\right)_{z=z_0^F} = i\cdot i\,\frac{d\theta_0}{d\varphi_0} = -\frac{d\theta_0}{d\varphi_0}$$

where

$$\theta_0 \equiv \arg \zeta_0$$

This shows that $i\left(d \ln \zeta_0 / dW_0\right)_{z=z_0^F}$ is real. (In fact, it shows that $i\left(d \ln \zeta_0/dW_0\right)_{z=z_0^F}$ is the curvature of the stream line S_0.) Hence,

$$\mathcal{R}e\left[\frac{1}{\zeta_0\left(z_0^F\right)}\left(\frac{d\zeta_0}{dz}\right)_{z=z_0^F} z_1^F\right] = \mathcal{R}e\left[\left(\frac{d \ln \zeta_0}{dW_0}\right)_{z=z_0^F} \zeta_0\left(z_0^F\right)z_1^F\right]$$

$$= \mathcal{R}e\left\{i\left(\frac{d \ln \zeta_0}{dW_0}\right)_{z=z_0^F}\left[\frac{\zeta_0\left(z_0^F\right)z_1^F}{i}\right]\right\}$$

$$= i\left(\frac{d \ln \zeta_0}{dW_0}\right)_{z=z_0^F} \mathcal{R}e\left[\frac{\zeta_0\left(z_0^F\right)z_1^F}{i}\right]$$

$$= i\left(\frac{d \ln \zeta_0}{dW_0}\right)_{z=z_0^F} \mathcal{I}m\left[\zeta_0\left(z_0^F\right)z_1^F\right]$$

Using equation (47) to eliminate $\mathcal{I}m\left[\zeta_0\left(z_0^F\right)z_1^F\right]$ from this relation yields

$$\mathcal{R}e\left[\frac{1}{\zeta_0\left(z_0^F\right)}\left(\frac{d\zeta_0}{dz}\right)_{z=z_0^F} z_1^F\right] = -i\left(\frac{d \ln \zeta_0}{dW_0}\right)_{z=z_0^F}\left[\mathcal{I}m\, W_1^+\left(z_0^F\right) - \psi_1^F\right]$$

Substituting this into equation (44) yields

$$\mathcal{R}e\left[\frac{\zeta_1^+\left(z_0^F\right)}{\zeta_0\left(z_0^F\right)}\right] - i\left(\frac{d\ln\zeta_0}{dW_0}\right)_{z=z_0^F}\left[\mathcal{I}m\, W_1^+\left(z_0^F\right) - \psi_1^F\right] = \frac{1}{2}$$

Or, dividing both sides by $i\left(d\ln\zeta_0/dW_0\right)_{z=z_0^F}$,

$$\mathcal{R}e\left\{\frac{\zeta_0\left(z_0^F\right)}{i}\left(\frac{dW_0}{d\zeta_0}\right)_{z=z_0^F}\left[\frac{\zeta_1^+\left(z_0^F\right)}{\zeta_0\left(z_0^F\right)}\right]\right\} - \mathcal{I}m\, W_1^+\left(z_0^F\right) + \psi_1^F$$

$$= \frac{1}{2i}\zeta_0\left(z_0^F\right)\left(\frac{dW_0}{d\zeta_0}\right)_{z=z_0^F} = \mathcal{R}e\left[\frac{1}{2i}\zeta_0\left(z_0^F\right)\left(\frac{dW_0}{d\zeta_0}\right)_{z=z_0^F}\right]$$

Hence,

$$\mathcal{I}m\left\{\zeta_0\left(z_0^F\right)\left[\frac{\zeta_1^+\left(z_0^F\right)}{\zeta_0\left(z_0^F\right)}\right]\left(\frac{dW_0}{d\zeta_0}\right)_{z=z_0^F} - W_1^+\left(z_0^F\right)\right\} + \psi_1^F = \mathcal{I}m\,\frac{1}{2}\zeta_0\left(z_0^F\right)\left(\frac{dW_0}{d\zeta_0}\right)_{z=z_0^F} \qquad (49)$$

is the boundary condition for ζ_1^+ (or W_1^+) on F_0. Thus, the boundary conditions on the curves S_0 and F_0 are given by equations (43), (48), and (49). The boundary conditions for the remaining (solid) boundaries are easily deduced by substituting the first asymptotic expansion (12) into the second group of boundary conditions (11) and equating the coefficients of ϵ to the first power. For convenience, we now collect in one place the complete set of first-order boundary conditions.

$$
\left.
\begin{array}{ll}
\left.
\begin{array}{l}
\mathcal{Re}\left[\dfrac{\zeta_1^+(z)}{\zeta_0(z)} - \dfrac{\zeta_1^-(z)}{\zeta_0(z)}\right] = \dfrac{1}{2|\zeta_0(z)|^2} \\[2ex]
\mathcal{Im}\left[W_1^+(z) - W_1^-(z)\right] = 0
\end{array}
\right\} z \in S_0 & \\[4ex]
\mathcal{Im}\left[\zeta_0(z)\dfrac{dW_0}{d\zeta_0}(z)\dfrac{\zeta_1^+(z)}{\zeta_0(z)} - W_1^+(z)\right] = \mathcal{Im}\left\{\dfrac{1}{2}\zeta_0(z)\left[\dfrac{dW_0}{d\zeta_0}(z)\right]\right\} - \psi_1^F & z \in F_0 \\[4ex]
\left.
\begin{array}{ll}
\mathcal{Im}\,\zeta_1^+(z) = 0 & z \in \widehat{KD} \\[2ex]
\mathcal{Im}\,\zeta_1^+(z) = 0 & z \in \widehat{EK} \\[2ex]
\mathcal{Im}\,\zeta_1^-(z) = 0 & z \in \widehat{GD}
\end{array}
\right\} & \\[4ex]
\zeta_1^+(z) \to 0 \qquad z \to K & \\[2ex]
\zeta_1^-(z) \to 0 \qquad z \to G &
\end{array}
\right\} \tag{50}
$$

Transformation of First-Order Problem to W_0-Plane

Now these boundary conditions completely specify a boundary value problem (or more precisely, two boundary value problems connected along the curve S_0) in the zeroth-order region of flow in the physical plane. However, under the change of variable $z \to W_0$ defined by equations (21) and (35), this boundary value problem can be transformed into one in the zeroth-order region of flow in the W_0-plane (which is indicated in fig. 3). The boundary conditions in the W_0-plane are

$$
\left.
\begin{array}{l}
\mathcal{Re}\left[\dfrac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - \dfrac{\zeta_1^-(W_0)}{\zeta_0(W_0)}\right] = \dfrac{1}{2|\zeta_0(W_0)|^2} \\[2ex]
\mathcal{Im}\left[W_1^+(W_0) - W_1^-(W_0)\right] = 0
\end{array}
\right\} W_0 \in \widehat{DC} \tag{51}
$$

$$\mathcal{I}m\left[\zeta_0(W_0)\,\frac{dW_0}{d\zeta_0}\,\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - W_1^+(W_0)\right] = \mathcal{I}m\left[\frac{1}{2}\,\zeta_0(W_0)\,\frac{dW_0}{d\zeta_0}\right] - \psi_1^F \qquad W_0 \in \widehat{EC} \tag{52}$$

$$\left.\begin{aligned}
&\mathcal{I}m\,\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} = 0 \qquad W_0 \in \widehat{KD} \\
&\mathcal{I}m\,\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} = 0 \qquad W_0 \in \widehat{EK} \\
&\mathcal{I}m\,\frac{\zeta_1^-(W_0)}{\zeta_0(W_0)} = 0 \qquad W_0 \in \widehat{GD}
\end{aligned}\right\} \tag{53}$$

where the boundary conditions (17) have been combined with the boundary conditions (50) to obtain the boundary conditions (53). Naturally, the variables ζ_1^+ and W_1^+ are defined on the unit strip below the real axis, and the variables ζ_1^- and W_1^- are defined on the upper half-plane.

The two equations (51) can be combined to give a single jump condition for ζ_1 across $\widehat{DC}$. In order to see this, notice it follows from figure 3 and the definition of a derivative of a holomorphic function that, for $W_0 \in \widehat{DC}$.

$$\begin{aligned}
0 &= \frac{\partial}{\partial\varphi_0}\,\mathcal{I}m\left[W_1^+(W_0) - W_1^-(W_0)\right] \\
&= \frac{\partial}{\partial\varphi_0}\,\mathcal{I}m\left[W_1^+(\varphi_0) - W_1^-(\varphi_0)\right] = \mathcal{I}m\,\frac{\partial}{\partial\varphi_0}\left[W_1^+(\varphi_0) - W_1^-(\varphi_0)\right] \\
&= \mathcal{I}m\,\frac{d}{dW_0}\left[W_1^+(W_0) - W_1^-(W_0)\right]
\end{aligned}$$

Hence, using equations (13) shows that

$$0 = \mathcal{I}m \frac{1}{\zeta_0(W_0)}\left[\frac{dW_1^+}{dz}(W_0) - \frac{dW_1^-}{dz}(W_0)\right] = \mathcal{I}m\left[\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - \frac{\zeta_1^-(W_0)}{\zeta_0(W_0)}\right]$$

Multiplying this result by i and adding it to the first equation (51) gives

$$\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - \frac{\zeta_1^-(W_0)}{\zeta_0(W_0)} = \frac{1}{2|\zeta_0(W_0)|^2} \qquad \text{for } W_0 \in \widehat{DC} \tag{54}$$

It is also necessary to transform the boundary condition (52). To this end notice that for $W_0 \in \widehat{EC}$

$$\frac{\partial}{\partial\varphi_0}\mathcal{I}m\left[\zeta_0(W_0)\frac{dW_0}{d\zeta_0}\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - W_1^+(W_0)\right] = \frac{\partial}{\partial\varphi_0}\left[\mathcal{I}m\frac{1}{2}\zeta_0(W_0)\frac{dW_0}{d\zeta_0}\right]$$

Hence,

$$\mathcal{I}m\frac{\partial}{\partial\varphi_0}\left[\zeta_0(W_0)\frac{dW_0}{d\zeta_0}\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - W_1^+(W_0)\right] = \mathcal{I}m\left[\frac{1}{2}\frac{\partial}{\partial\varphi_0}\zeta_0(W_0)\frac{dW_0}{d\zeta_0}\right]$$

Therefore, using the definition of a derivative of a complex variable

$$\mathcal{I}m\frac{d}{dW_0}\left[\zeta_0(W_0)\frac{dW_0}{d\zeta_0}\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - W_1^+(W_0)\right] = \mathcal{I}m\frac{1}{2}\frac{d}{dW_0}\left[\zeta_0(W_0)\frac{dW_0}{d\zeta_0}\right]$$

Differentiating by parts yields

$$\mathcal{I}m\left\{\zeta_0(W_0)\frac{d}{dW_0}\left[\frac{dW_0}{d\zeta_0}\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)}\right] + \frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - \frac{dW_1^+}{dW_0}\right\} = \mathcal{I}m\left[\frac{1}{2} + \zeta_0(W_0)\frac{d}{dW_0}\left(\frac{1}{2}\frac{dW_0}{d\zeta_0}\right)\right]$$

Upon using both equations (13) we find

$$\mathcal{I}m\left\{\zeta_0(W_0)\frac{d}{dW_0}\left[\frac{dW_0}{d\zeta_0}\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)}\right]+\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)}-\frac{1}{\zeta_0(W_0)}\frac{dW_1^+}{dz}\right\}$$

$$=\mathcal{I}m\ \zeta_0(W_0)\frac{d}{dW_0}\left(\frac{1}{2}\frac{dW_0}{d\zeta_0}\right)=\mathcal{I}m\left\{\zeta_0(W_0)\frac{d}{dW_0}\left[\frac{dW_0}{d\zeta_0}\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)}\right]\right\}$$

Therefore,

$$\mathcal{I}m\ \zeta_0(W_0)\left\{\frac{d}{dW_0}\frac{dW_0}{d\zeta_0}\left[\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)}-\frac{1}{2}\right]\right\}=0 \qquad \text{for } W_0\in\widehat{EC}$$

In view of this equation, let us define Λ^+ on the unit strip below the real axis in the W_0-plane by

$$\Lambda^+(W_0)\equiv\zeta_0(W_0)\frac{d}{dW_0}\left[\frac{dW_0}{d\zeta_0}\left(\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)}-\frac{1}{2}\right)\right] \qquad -1\leq\psi_0\leq 0 \tag{55}$$

Clearly, Λ^+ is holomorphic on its domain of definition, and

$$\mathcal{I}m\,\Lambda^+(W_0)=0 \qquad \text{for } W_0\in\widehat{EC} \tag{56}$$

Now it follows from boundary conditions (17) that $\zeta_0(W_0)$ is real for $W_0\in\widehat{KD}$. Hence, it is readily seen from figure 3 and the definition of a derivative of a holomorphic function that for $W_0\in\widehat{KD}$

$$\frac{dW_0}{d\zeta_0}=\frac{1}{(d\zeta_0/dW_0)}=\frac{1}{\partial u_0/\partial\varphi_0}$$

where we have put

$$u_0=\mathcal{R}e\,\zeta_0$$

This shows that $(dW_0/d\zeta_0)$ is also real on $\widehat{KD}$. Hence, it follows from the first boundary condition (53) that for $W_0\in\widehat{KD}$

$$0 = \frac{dW_0}{d\zeta_0} \mathcal{Im} \frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} = \frac{dW_0}{d\zeta_0} \mathcal{Im} \left[\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - \frac{1}{2}\right] = \mathcal{Im} \left\{\frac{dW_0}{d\zeta_0}\left[\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - \frac{1}{2}\right]\right\}$$

Differentiating this result along $\widehat{HD}$

$$0 = \frac{\partial}{\partial\varphi_0}\left\{\mathcal{Im} \frac{dW_0}{d\zeta_0}\left[\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - \frac{1}{2}\right]\right\}$$

$$= \mathcal{Im} \frac{\partial}{\partial\varphi_0}\left\{\frac{dW_0}{d\zeta_0}\left[\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - \frac{1}{2}\right]\right\}$$

$$= \mathcal{Im} \frac{d}{dW_0}\left\{\frac{dW_0}{d\zeta_0}\left[\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - \frac{1}{2}\right]\right\}$$

Therefore, in view of the fact that $\zeta_0(W_0)$ is real for $W_0 \in \widehat{KD}$, we conclude from definition (55) that

$$\mathcal{Im}\, \Lambda^+(W_0) = 0 \qquad \text{for} \quad W_0 \in \widehat{KD} \tag{57}$$

A completely analogous argument shows that

$$\mathcal{Im}\, \Lambda^+(W_0) = 0 \qquad \text{for} \quad W_0 \in \widehat{KE} \tag{58}$$

Now define Λ^- on the upper half W_0-plane by

$$\Lambda^-(W_0) \equiv \zeta_0(W_0) \frac{d}{dW_0}\left\{\frac{dW_0}{d\zeta_0}\left[\frac{\zeta_1^-(W_0)}{\zeta_0(W_0)} + \frac{1}{2\zeta_0^2(W_0)} - \frac{1}{2}\right]\right\} - 2 \qquad 0 \leq \psi_0 \tag{59}$$

Then Λ^- is holomorphic in the upper half-plane. An argument analogous to that used to deduce condition (57) suffices to show that

$$\mathcal{Im}\, \Lambda^-(W_0) = 0 \qquad \text{for} \quad W_0 \in \widehat{GD} \tag{60}$$

It follows from the jump condition (54) and figure 3 that, for $W_0 \in \widehat{DC}$,

$$\zeta_0(W_0)\left(\left\{\frac{d}{dW_0}\left[\frac{dW_0}{d\zeta_0}\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)}\right]\right\} - \left\{\frac{d}{dW_0}\left[\frac{dW_0}{d\zeta_0}\frac{\zeta_1^-(W_0)}{\zeta_0(W_0)}\right]\right\}\right)$$

$$= \zeta_0(W_0)\frac{\partial}{\partial\varphi_0}\left\{\frac{dW_0}{d\zeta_0}\left[\frac{\zeta_1^+(W_0)}{\zeta_0(W_0)} - \frac{\zeta_1^-(W_0)}{\zeta_0(W_0)}\right]\right\} = \zeta_0(W_0)\frac{1}{2}\frac{\partial}{\partial\varphi_0}\left[\frac{dW_0}{d\zeta_0}\frac{1}{|\zeta_0(W_0)|^2}\right]$$

Hence, in view of definitions (55) and (59) this shows that the jump condition of $\Lambda^\pm$ is

$$\Lambda^+(W_0) - \Lambda^-(W_0) = \frac{1}{2}\zeta_0(W_0)\frac{\partial}{\partial\varphi_0}\left\{\frac{dW_0}{d\zeta_0}\left[\frac{1}{|\zeta_0(W_0)|^2} - \frac{1}{\zeta_0^2(W_0)}\right]\right\} + 2 \qquad \text{for } W_0 \in \widehat{DC}$$

The right side of this equation can be put into a more convenient form as follows: Notice that, for $W_0 \in \widehat{DC}$,

$$\frac{\partial}{\partial\varphi_0}\left\{\frac{dW_0}{d\zeta_0}\left[\frac{1}{|\zeta_0(W_0)|^2} - \frac{1}{\zeta_0^2(W_0)}\right]\right\} = \frac{\partial}{\partial\varphi_0}\left\{\frac{1}{\zeta_0(W_0)}\frac{dW_0}{d\zeta_0}\left[\frac{1}{\overline{\zeta_0(W_0)}} - \frac{1}{\zeta_0(W_0)}\right]\right\}$$

$$= -2i\frac{\partial}{\partial\varphi_0}\left[\frac{1}{\zeta_0(W_0)}\frac{dW_0}{d\zeta_0}\,\mathcal{I}m\,\frac{1}{\zeta_0(W_0)}\right]$$

Hence,

$$\Lambda^+(W_0) - \Lambda^-(W_0) = -i\zeta_0(W_0)\frac{\partial}{\partial\varphi_0}\left[\frac{1}{\zeta_0(W_0)}\frac{dW_0}{d\zeta_0}\,\mathcal{I}m\,\frac{1}{\zeta_0(W_0)}\right] + 2 \qquad \text{for } W_0 \in \widehat{DC} \tag{61}$$

and upon collecting equations (56) to (58) and equation (60) we have

$$\mathcal{I}m\, \Lambda^+(W_0) = 0 \qquad \text{for} \begin{cases} W_0 \in \widehat{EC} \\ W_0 \in \widehat{KD} \\ W_0 \in \widehat{KE} \end{cases} \tag{62}$$

$$\mathcal{I}m\, \Lambda^-(W_0) = 0 \qquad \text{for } W_0 \in \widehat{GD} \tag{63}$$

Hence, in view of equations (61) to (63), we have transformed the boundary value problem in the W_0-plane for the holomorphic functions $\zeta_1^\pm$ (or $W_1^\pm$) into a boundary value problem for the holomorphic functions $\Lambda^\pm$. Clearly, once the functions $\Lambda^\pm$ are known, the original functions $\zeta_1^\pm$ can be obtained by simple quadrature from definitions (55) and (59). However, it is necessary to transform the boundary value problem once more before a solution can be obtained.

Transformation of Boundary Value Problem for $\Lambda^\pm$ into T-Plane and Solution of Problem

Under the change of variable $W_0 \to T$ defined by equation (21), the boundary value problem for $\Lambda^\pm$ in the W_0-plane posed by equations (61) to (63) is transformed into the following boundary value problem for the holomorphic functions $\Lambda^\pm$ in the T-plane (shown in fig. 5).

$$\left.\begin{aligned} &\Lambda^+(T) - \Lambda^-(T) = -i\zeta_0(T) \frac{\partial}{\partial\varphi_0}\left[\frac{dW_0}{d\zeta_0} \frac{1}{\zeta_0(T)} \mathcal{I}m \frac{1}{\zeta_0(T)}\right] + 2 \qquad T \in \mathcal{S}_0 \\ &\mathcal{I}m\, \Lambda^+(\xi + i0) = 0 \qquad \xi \geq -1 \\ &\mathcal{I}m\, \Lambda^-(\xi + i0) = 0 \qquad \xi \leq -1 \end{aligned}\right\} \tag{64}$$

Clearly, the domains of definition of Λ^+ and Λ^- are $\mathcal{D}_0^+$ and $\mathcal{D}_0^-$, respectively.

It is more convenient to work with the sectionally analytic function Ω defined on the upper half T-plane in terms of $\Lambda^\pm$ as follows

$$
\Omega(T) \equiv \begin{cases} \Omega^{+}(T) \equiv \dfrac{dW_0}{dT}\Lambda^{+}(T) & T \in \mathscr{D}_0^{+} \\ \\ \Omega^{-}(T) \equiv \dfrac{dW_0}{dT}\Lambda^{-}(T) & T \in \mathscr{D}_0^{-} \end{cases} \tag{65}
$$

Since it is clear from equation (19) that dW_0/dT is real when T is real, it follows from the conditions (64) that Ω must satisfy the following conditions:

$$
\left.\begin{aligned} &\Omega^{+}(T) - \Omega^{-}(T) = \Gamma(T) \qquad \text{for } T \in \mathscr{S}_0 \\ &\mathscr{Im}\,\Omega(\xi + i0) = 0 \qquad \text{for } -\infty < \xi < +\infty \end{aligned}\right\} \tag{66}
$$

where we have put

$$
\Gamma(T) \equiv -\frac{dW_0}{dT}\left\{ i\zeta_0(T)\frac{\partial}{\partial\varphi_0}\left[\frac{1}{\zeta_0(T)}\frac{dW_0}{d\zeta_0}\,\mathscr{Im}\,\frac{1}{\zeta_0(T)}\right] - 2\right\} \qquad \text{for } T \in \mathscr{S}_0 \tag{67}
$$

Now suppose Θ is a sectionally analytic function which satisfies the conditions (66). Then, if ω is any function which is holomorphic in the interior of the upper half-plane and which is real on the real axis, the function $\Theta + \omega$ also satisfied the boundary conditions (66). First, suppose that ω has no singularities on the real axis. Then, the Schwartz reflection principle shows that ω has an analytic continuation to the entire T-plane and therefore has a Taylor series expansion about the origin which has real coefficients. Thus, ω can be represented in the form

$$
\sum_{n=0}^{m} c_n T^n \qquad c_n \text{ real}
$$

If $m = +\infty$, ω has an essential singularity at $T = +\infty$; if m is finite, ω has a pole of order m at $T = +\infty$. It will be shown subsequently that the behavior of ζ at $T = \infty$ dictates that m be finite. If ω has singularities on the real axis, the requirement that ω be real there shows that they cannot be branch points. Hence, these singularities must be poles or perhaps essential singularities. However, an investigation of the solution shows that, if ζ is to be bounded, the only singularity which can be allowed is a simple pole at the origin. In view of equation (19) the most general solution to the boundary value problem (66) with certain restrictions of boundedness imposed can be written as

$$\Omega(T) = \Theta(T) + \sum_{n=0}^{m} c_n T^n + \alpha \frac{dW_0}{dT} \qquad \mathcal{R}e\, T > 0 \tag{68}$$

where c_n and α are real constants and Θ is a sectionally analytic function which vanishes at infinity, is bounded on the real axis, and satisfies the boundary value problem (66).

The function Θ can be constructed as follows: An investigation of the behavior of $\Gamma(T)$ at $T = \infty$ and $T = -1$ shows that it vanishes at these points like some power of T. Hence, the Plemelj formulas (ref. 8) show that the Cauchy integral

$$\frac{1}{2\pi i} \int_{\mathscr{S}_0} \frac{\Gamma(\tau)}{\tau - T}\, d\tau \tag{69}$$

(where the integration is to be performed in a counterclockwise direction along $\mathscr{S}_0$) is a sectionally analytic function which is bounded on the real axis, vanishes at infinity, and which satisfies the jump condition (66). However, this function is not necessarily real for real values of T. But this can be compensated (as shown in ref. 8) by adding the function

$$-\frac{1}{2\pi i} \int_{\mathscr{S}_0} \frac{\overline{\Gamma(\tau)\, d\tau}}{\overline{\tau} - T}$$

to equation (69). (Notice that, if f is holomorphic in the upper half-plane, the function $\overline{f}$ defined by $\overline{f}(T) = \overline{f(\overline{T})}$ is also holomorphic there, and $f(\xi) + \overline{f}(\xi) = f(\xi) + \overline{f(\xi)}$ is real). Thus, the function Θ with the required properties is defined by

$$\Theta(T) \equiv \frac{1}{2\pi i} \int_{\mathscr{S}_0} \frac{\Gamma(\tau)d\tau}{\tau - T} - \frac{1}{2\pi i} \int_{\mathscr{S}_0} \frac{\overline{\Gamma(\tau)\, d\tau}}{\overline{\tau} - T} \qquad \mathcal{R}e\, T \geq 0 \tag{70}$$

In order to complete the solution it is necessary to determine the real constants $c_n (n = 0, 1, 2, \ldots)$ and α. Before this can be done, however, it is necessary to use the preceding results to find $\zeta_1^{\pm}$ as a function of T. In view of the mapping $T \rightarrow z$ defined by equation (35), which transforms the T-plane into the physical plane, this will complete the solution.

Apropos of these remarks notice that definitions (55) and (65) combined with equation (68) show that for $T \in \mathscr{D}_0^+$

$$\zeta_0(T) \frac{dW_0}{dT} \frac{d}{dW_0}\left[\frac{dW_0}{d\zeta_0}\left(\frac{\zeta_1^+(T)}{\zeta_0(T)} - \frac{1}{2}\right)\right] = \Theta(T) + \sum_{n=0}^{m} c_n T^n + \alpha \frac{dW_0}{dT} \tag{71}$$

Since equations (19) and (20) show that $dW_0/d\zeta_0 = 0$ at $T = -1$, integrating both sides of this equation yields

$$\frac{dW_0}{d\zeta_0}\left[\frac{\zeta_1^+(T)}{\zeta_0(T)} - \frac{1}{2}\right] = \int_{-1}^{T} \frac{1}{\zeta_0(T)} \Theta(T)dT$$

$$+ \sum_{n=0}^{m} c_n \int_{-1}^{T} \frac{1}{\zeta_0(T)} T^n dT + \alpha \int_{-1}^{T} \frac{1}{\zeta_0(T)} \frac{dW_0}{dT} dT \tag{72}$$

Substituting in equations (19) and (20) and rearranging yield

$$\frac{\zeta_1^+(T)}{\zeta_0(T)} = \frac{1}{2} + i\pi\sqrt{\Delta} \frac{\zeta_0(T)}{(T + 1)\sqrt{T - \Delta}} \int_{-1}^{T} \frac{1}{\zeta_0(T)} \Theta(T)dT$$

$$+ i\pi\sqrt{\Delta} \frac{\zeta_0(T)}{(T + 1)\sqrt{T - \Delta}} \sum_{n=0}^{m} c_n \int_{-1}^{T} \frac{1}{\zeta_0(T)} T^n dT$$

$$+ i\sqrt{\Delta}\, \alpha \frac{\zeta_0(T)}{(T + 1)\sqrt{T - \Delta}} \int_{-1}^{T} \frac{1}{\zeta_0(T)} \frac{T + 1}{T} dT \tag{73}$$

Since $\Gamma(T) = O(T^{\beta})$ with some $\beta < 0$ for $T \rightarrow \infty$, it follows from the properties of the Cauchy-type integral (ref. 8) that

$$\Theta(T) = o(1) \qquad \text{for} \qquad T \rightarrow \infty$$

It is easy to see from equation (22) that $\zeta_0(T) \rightarrow 1$ as $T \rightarrow \infty$. Hence, as $T \rightarrow \infty$,

$$\int_{-1}^{T} \frac{1}{\zeta_0(T)} \Theta(T)\, dT = o(T)$$

$$\int_{-1}^{T} \frac{1}{\zeta_0(T)} T^n\, dT \sim T^{n+1}$$

and

$$\int_{-1}^{T} \frac{1}{\zeta_0(T)} \frac{T+1}{T}\, dT \sim T$$

Using these results in equation (73) shows that

$$\frac{\zeta_1^+(T)}{\zeta_0(T)} \sim \frac{1}{2} + i\pi\sqrt{\Delta}\, o\left(\frac{1}{\sqrt{T}}\right) + i\pi\sqrt{\Delta} \sum_{n=0}^{m} c_n \frac{T^n}{\sqrt{T}} + i\sqrt{\Delta}\, \alpha \frac{1}{\sqrt{T}} \qquad \text{for} \qquad T \rightarrow \infty$$

Thus, in order that ζ_1^+ be bounded for $T \rightarrow \infty$, it is necessary to set

$$c_n = 0 \qquad \text{for} \qquad n > 0 \tag{74}$$

Equation (73) can now be written as

$$\frac{\zeta_1^+(T)}{\zeta_0(T)} = \frac{1}{2} + i\pi\sqrt{\Delta} \frac{\zeta_0(T)}{(T+1)\sqrt{T-\Delta}} \left[\int_{\Delta}^{T} \frac{1}{\zeta_0(T)} \Theta(T)\, dT + c_0 \int_{\Delta}^{T} \frac{1}{\zeta_0(T)}\, dT + \frac{\alpha}{\pi} \int_{\Delta}^{T} \frac{1}{\zeta_0(T)} \frac{T+1}{T}\, dT\right] + \frac{i\pi\sqrt{\Delta}\, \zeta_0(T)}{(T+1)\sqrt{T-\Delta}}$$
$$\times \left[\int_{-1}^{\Delta} \frac{1}{\zeta_0(T)} \Theta(T)\, dT + c_0 \int_{-1}^{\Delta} \frac{1}{\zeta_0(T)}\, dT + \frac{\alpha}{\pi} \int_{-1}^{\Delta} \frac{1}{\zeta_0(T)} \frac{T+1}{T}\, dT\right] \tag{75}$$

Since $\zeta_0(\Delta) = -1$ and since $\Theta(\Delta)$ is bounded, it follows from equation (75) that

$$\frac{\zeta_1^+(T)}{\zeta_0(T)} \sim \frac{1}{2} + O(\sqrt{T - \Delta}) - \frac{i\pi\sqrt{\Delta}}{2(\Delta + 1)} \frac{1}{\sqrt{T - \Delta}} \left[\int_{-1}^{\Delta} \frac{1}{\zeta_0(T)} \Theta(T)\, dT \right.$$

$$\left. + c_0 \int_{-1}^{\Delta} \frac{1}{\zeta_0(T)}\, dT + \frac{\alpha}{\pi} \int_{-1}^{\Delta} \frac{1}{\zeta_0(T)} \frac{T + 1}{T}\, dT \right] \qquad \text{for} \qquad T \to \Delta$$

Hence, in order that ζ_1^+ be bounded at $T = \Delta$, it is necessary that c_0 and α satisfy the two real equations

$$\int_{-1}^{\Delta} \frac{1}{\zeta_0(T)} \Theta(T)\, dT + c_0 \int_{-1}^{\Delta} \frac{1}{\zeta_0(T)}\, dT + \frac{\alpha}{\pi} \int_{-1}^{\Delta} \frac{1}{\zeta_0(T)} \frac{T + 1}{T}\, dT = 0 \tag{76}$$

Using equation (27) and the fact that the origin of the coordinate system in the physical plane is at D (fig. 2), it follows that

$$z(T) = \frac{1}{\pi} \int_{-1}^{T} \frac{1}{\zeta_0(T)} \frac{T + 1}{T}\, dT \tag{77}$$

Hence, it follows from this and definition (29) that equation (76) can be written as

$$\int_{-1}^{\Delta} \frac{1}{\zeta_0(T)} \Theta(T)\, dT + c_0 I_1(\Delta) + \alpha z(\Delta) = 0 \tag{78}$$

The constant α appearing in this equation is directly related to a geometric parameter of the flow. In order to see this, notice first that the flow in the jet far downstream becomes uniform. Hence, in view of equation (5) (see fig. 2),

$$\lim_{z \to c} \zeta^+(z) = \left| \zeta^+(z^F) \right| = \sqrt{1 + \epsilon}$$

Therefore, continuity requirements dictate that the dimensionless net volume flow through the jet Q be equal to

$$h\sqrt{1+\epsilon}$$

But Q must also be equal to the change in the stream function across the jet. Hence, the boundary conditions (11) show that

$$0 - \psi_F = Q = h\sqrt{1+\epsilon}$$

Expanding both sides of this expression in powers of ϵ by using the expansion (15), the last expansion (12), and equation (18) shows that

$$1 - \epsilon\psi_1^F + \ldots = (1 + \epsilon h_1 + \ldots)\left(1 + \frac{1}{2}\epsilon + \ldots\right)$$

Upon equating like powers of ϵ, we find that

$$\psi_1^F = -\left(h_1 + \frac{1}{2}\right) \tag{79}$$

Another expression for ψ_1^F can be obtained by first obtaining an expression for the first-order complex potential within the jet W_1^+ from equation (71). To this end, notice it follows from equations (13) that

$$\frac{\zeta_1^+}{\zeta_0} = \frac{1}{\zeta_0}\frac{dW_1^+}{dz} = \frac{dW_1^+}{dW_0}\frac{1}{\zeta_0}\frac{dW_0}{dz} = \frac{dW_1^+}{dW_0}$$

Hence,

$$\zeta_0 \frac{dW_0}{dT}\frac{d}{dW_0}\left[\frac{dW_0}{d\zeta_0}\left(\frac{\zeta_1^+}{\zeta_0} - \frac{1}{2}\right)\right] = \zeta_0 \frac{d}{dT}\left[\frac{dW_0}{d\zeta_0}\frac{d}{dW_0}\left(W_1^+ - \frac{1}{2}W_0\right)\right]$$

$$= \frac{d}{dT}\left[\zeta_0 \frac{d}{d\zeta_0}\left(W_1^+ - \frac{1}{2}W_0\right)\right] - \frac{d\zeta_0}{dT}\frac{d}{d\zeta_0}\left(W_1^+ - \frac{1}{2}W_0\right)$$

$$= \frac{d}{dT}\left[\zeta_0^2 \frac{d}{d\zeta_0}\left(\frac{W_1^+ - \frac{1}{2}W_0}{\zeta_0}\right)\right]$$

$$= \frac{d}{dT}\left[\frac{dW_0}{d \ln \zeta_0}\left(\frac{\zeta_1^+}{\zeta_0} - \frac{1}{2}\right) - W_1^+ + \frac{1}{2}W_0\right]$$

Substituting this result together with equation (74) into equation (71) yields

$$\frac{d}{dT}\left\{\frac{dW_0}{d\ln\zeta_0}\left[\frac{\zeta_1^+(T)}{\zeta_0(T)}-\frac{1}{2}\right]-W_1^+(T)+\frac{1}{2}W_0(T)\right\}=\Theta(T)+c_0+\alpha\frac{dW_0}{dT}$$

Integrating this expression between -1 and T and recalling that $dW_0/d\ln\zeta_0 = 0$ at $T = -1$ and that the normalization condition (7) implies $W_1^+(T) = W_0(T) = 0$ at $T = -1$, we find

$$\frac{dW_0}{d\ln\zeta_0}\left[\frac{\zeta_1^+(T)}{\zeta_0(T)}-\frac{1}{2}\right]-W_1^+(T)+\frac{1}{2}W_0(T)=\int_{-1}^{T}\Theta(T)\,dT+c_0(T+1)+\alpha W_0(T) \tag{80}$$

Using equation (72) together with equation (74) to eliminate $\zeta_1^+(T)/\zeta_0(T)$ from this equation yields

$$\begin{aligned}\frac{1}{\zeta_0(T)}\left[W_1^+(T)-\frac{1}{2}W_0(T)\right]&=\left[\int_{-1}^{T}\frac{1}{\zeta_0(T)}\Theta(T)\,dt-\frac{1}{\zeta_0(T)}\int_{-1}^{T}\Theta(T)\,dT\right]\\&\quad+c_0\left[\int_{-1}^{T}\frac{1}{\zeta_0(T)}dT-\frac{1}{\zeta_0(T)}(T+1)\right]+\alpha\left[z(T)-\frac{1}{\zeta_0(T)}W_0(T)\right]\end{aligned} \tag{81}$$

It follows from this equation, after using equations (76) and (77), that

$$W_1^+(\Delta)=\left(\frac{1}{2}-\alpha\right)W_0(\Delta)-\int_{-1}^{\Delta}\Theta(T)\,dT-c_0(\Delta+1) \tag{82}$$

Since $\Theta(T)$ is real for real values of T (see fig. 5, p. 13) and since $\mathcal{Im}W_0(\Delta) = -1$, equation (82) shows that

$$\mathcal{Im}W_1^+(\Delta)=\alpha-\frac{1}{2}$$

An examination of figures 2, 5, and 6 (pp. 5, 13, and 17) together with equation (47) easily shows that

$$\mathcal{Im}W_1^+(\Delta)=\psi_1^F$$

Hence,

$$\psi_1^F = \alpha - \frac{1}{2}$$

Finally, combining this with equation (79) shows that

$$\alpha = -h_1 \tag{83}$$

We now return to the problem of solving equation (78) for α and c_0, or, in view of equation (83), for h_1 and c_0. Substituting equations (23), (33), (34), (36), and (83) into equation (78) shows that

$$\int_{-1}^{\Delta} \frac{T - 2\Delta + 2i\sqrt{\Delta}\sqrt{T - \Delta}}{T}\,\Theta(T)\,dT + c_0\left[\Delta + 1 + 2\Delta\ \ln\left(\frac{\sqrt{1 + \Delta} - \sqrt{\Delta}}{\sqrt{\Delta}}\right)^2 + 4\sqrt{\Delta}\sqrt{1 + \Delta}\right]$$

$$- \frac{h_1}{\pi}\left[3(\Delta + 1) + (2\Delta - 1)\ln\left(\frac{\sqrt{1 + \Delta} - \sqrt{\Delta}}{\sqrt{\Delta}}\right)^2 + 6\sqrt{\Delta}\sqrt{1 + \Delta}\right]$$

$$+ 4i\pi\Delta c_0 - 2ih_1(2\Delta - 1) = 0 \tag{84}$$

Now

$$\int_{-1}^{\Delta} \frac{T - 2\Delta + 2i\sqrt{\Delta}\sqrt{T - \Delta}}{T}\,\Theta(T)\,dT = \int_{-1}^{\Delta} \Theta(\xi)\,d\xi - 2\sqrt{\Delta}\int_{-1}^{\Delta} \frac{(\sqrt{\Delta} - i\sqrt{T - \Delta})}{T}\,\Theta(T)\,dT$$

In view of the singularity in the denominator, the second integral must first be carried out over the path shown in figure 7, and then the limit $\delta \rightarrow 0$ can be taken. After performing these operations, we find that

$$2\sqrt{\Delta}\int_{-1}^{\Delta} \frac{(\sqrt{\Delta} - i\sqrt{T - \Delta})}{T}\,\Theta(T)\,dT = 2\sqrt{\Delta}\ \text{P.V.}\int_{-1}^{\Delta} \frac{(\sqrt{\Delta} + \sqrt{\Delta - \xi})}{\xi}\,\Theta(\xi)\,d\xi - i\pi 4\Delta\Theta(0)$$

where the notation P.V. indicates that the Cauchy principal value of the singular integral

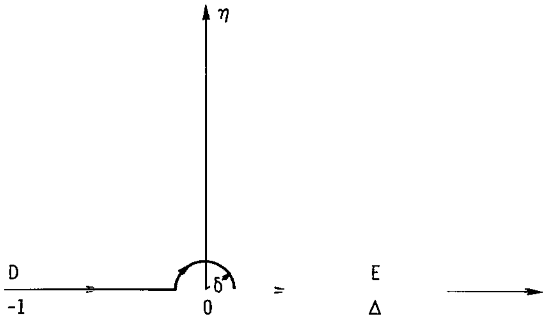


Figure 7. - Path of integration for $\int_{-1}^{\Delta} \frac{(\sqrt{\Delta} - i\sqrt{T - \Delta})}{T}\Theta(T)dT$ in T-plane.

is to be taken. Substituting these relations into equation (84), recalling that $\Theta(T)$ is real for real values of T, and taking the real and imaginary parts of the resulting expression yield

$$\int_{-1}^{\Delta} \Theta(\xi)\, d\xi - 2\sqrt{\Delta}\ \text{P.V.} \int_{-1}^{\Delta} \frac{\left(\sqrt{\Delta} + \sqrt{\Delta + \xi}\right)}{\xi} \Theta(\xi)\, d\xi + c_0\left(\Delta + 1 + 4\sqrt{\Delta}\sqrt{1 + \Delta}\right)$$

$$- \frac{3h_1}{\pi}\left(\Delta + 1 + 2\sqrt{\Delta}\sqrt{1 + \Delta}\right) + \left[c_0 2\Delta - \frac{h_1}{\pi}(2\Delta - 1)\right] \ln\left(\frac{\sqrt{1 + \Delta} - \sqrt{\Delta}}{\sqrt{\Delta}}\right)^2 = 0$$

and

$$2\pi\Delta c_0 - h_1(2\Delta - 1) + 2\pi\Delta\Theta(0) = 0$$

Upon eliminating c_0 between these two equations, we find

$$h_1 = \frac{-2\pi\Delta}{(\Delta+1)\left(4\Delta+1+4\sqrt{\Delta}\sqrt{\Delta+1}\right)}\left\{2\sqrt{\Delta}\ \text{P.V.} \int_{-1}^{\Delta} \frac{\left(\sqrt{\Delta} + \sqrt{\Delta - \xi}\right)}{\xi}\Theta(\xi)\, d\xi\right.$$

$$\left. - \int_{-1}^{\Delta} \Theta(\xi)\, d\xi + \left[\Delta + 1 + 2\Delta \ln\left(\frac{\sqrt{1 + \Delta} - \sqrt{\Delta}}{\sqrt{\Delta}}\right)^2 + 4\sqrt{\Delta}\sqrt{1 + \Delta}\right]\Theta(0)\right\} \qquad (85)$$

and

$$c_0 = \frac{(2\Delta - 1)}{2\pi\Delta} h_1 - \Theta(O) \tag{86}$$

Since equation (35) sets up a one-to-one correspondence between points of the physical plane and points of the T-plane, it is clear that equations (71) and (81) (with the use of definitions (67) and (70)) allow us to compute the first-order perturbation to the velocity and stream function at each point of the physical plane within the jet. Similar results are obtained by exactly the same procedure for the flow quantities in the main stream, but we shall not list the equations here. However, an inspection of these equations does show that the choice of the constants c_n and α made above is sufficient to guarantee that ζ^- is bounded.

In view of the fact that once the shape of the jet is known it is quite easy to sketch in the streamline patterns, the most important quantities to be obtained from the analysis are the shapes of the curves S and F in the physical plane (see fig. 2, p. 5). However, since the viscous spreading of the jet is controlled by the pressure (or equivalently the velocity) distribution along the slip line S, that quantity is also of some importance. Hence, explicit formulas will now be obtained for these quantities by using the formulas derived above.

Derivation of Boundary Values

In view of the one-to-one nature of the mapping involved, it is clear that, if $\mathcal{Im}W^+(z) = \psi^F$, z must be a point on the stream line which passes through the point E in figure 2. In addition, since the velocity potential is increasing in the direction E→ C along F, it is clear that, if $\mathcal{Re}W^+(z) > \mathcal{Re}W^+(a + ib)$ (where $a + ib$ are of course the coordinates of the point E), z must be a point of the free-stream line F. Hence, to within an error of order ϵ^2, the point z will be on the free-stream line F (i.e., it will be on the first-order position of F) if

$$\mathcal{Im}W^+(z) = \psi_0^F + \epsilon\psi_1^F = -1 + \epsilon\psi_1^F$$

and if

$$\mathcal{Re}W^+(z) > \mathcal{Re}W_0(a + ib) + \epsilon\mathcal{Re}W_1^+(a + ib)$$

where equation (18) has been used. It is also clear that $\mathcal{Re}W_0(z_0^F) > \mathcal{Re}W_0(a + ib)$ for any point $z_0^F \in F_0$. In view of these considerations, it follows from equations (79), (82), and (83) and the fact that $\mathscr{Im}\, W_0\left(z_0^F\right) = -1$ that the point $z^F = z_0^F + \epsilon z_1^F$ will be on the first-order position of F if z^F satisfies the equation

$$W^+(z^F) = W_0\left(z_0^F\right)(1 - \epsilon\psi_1^F) - \epsilon \int_{-1}^{\Delta} \Theta(T)\, dT - \epsilon c_0(\Delta + 1) \tag{87}$$

It is clear from this equation that, when $z_0^F = a + ib$, $W^+(z^F) = W^+(a + ib)$ and that $\mathscr{Re}\, W^+(z^F) \to \infty$ as $z_0^F \to \infty$. Hence, the point z^F traverses the stream line F (to within terms of order ϵ^2) as z_0^F traverses the zeroth-order free-stream line F_0.

Analogous considerations show that the point $z^S = z_0^S + \epsilon z_1^S$, which is determined by the equation

$$W(z^S) = W_0(z_0^S)(1 - \epsilon\psi_1^F) \tag{88}$$

traverses the first-order slip line S as the point z_0^S traverses the zeroth-order slip line S_0.

By substituting equations (87) and (88) into the expansions (40) and (38), respectively, the first-order distances z_1^F and z_1^S from the zeroth-order free-stream line to the free-stream line and from the zeroth-order slip line to the slip line, respectively, are found to be

$$z_1^F = -\frac{1}{\zeta_0(T)}\left[W_1^+(T) + \psi_1^F W_0(T) + \int_{-1}^{\Delta} \Theta(T)\, dT + c_0(\Delta + 1)\right] \qquad T \in \widehat{EC} \tag{89}$$

and

$$z_1^S = -\frac{1}{\zeta_0(T)}\left[W_1^+(T) + \psi_1^F W_0(T)\right] \qquad T \in \mathscr{S}_0 \tag{90}$$

where the fact has been used that the curves S_0 and F_0 in the physical plane are the conformal images under the mapping $T \to z$ defined by equation (35) of the curve $\mathscr{S}_0$ and the line EC, respectively, in the T-plane. The expansions (12) show that

$$z^F = z(T) + \epsilon z_1^F \qquad T \in \widehat{EC} \tag{91}$$

$$z^S = z(T) + \epsilon z_1^S \qquad T \in \mathscr{S}_0 \tag{92}$$

In addition, equation (41) shows that the magnitude of the velocity at each point z^S of the slip line is given to within terms of order ϵ^2 by

$$|\zeta^+(z^S)|^2 = |\zeta_0(T)|^2 + 2\epsilon|\zeta_0(T)|^2 \mathcal{Re}\left[\frac{\zeta_1^+(T)}{\zeta_0(T)} + \frac{1}{\zeta_0(T)}\frac{d\zeta_0}{dz} z_1^S\right] \qquad T \in \mathscr{S}_0 \tag{93}$$

Upon substituting equations (89) and (90) into equations (91) and (92), respectively, and substituting equation (81) into the resulting expressions, we find, after using equation (79), that

$$z^S = z(T)(1 + \epsilon h_1) - \epsilon\left\{\int_{-1}^{T} \frac{1}{\zeta_0(T)}\left[\Theta^+(T) + c_0\right]dT - \frac{1}{\zeta_0(T)}\int_{-1}^{T}\left[\Theta^+(T) + c_0\right]dT\right\} \qquad T \in \mathscr{S}_0 \tag{94}$$

$$z^F = z(\Delta) + \left[z(T) - z(\Delta)\right](1 + \epsilon h_1)$$
$$- \epsilon\left\{\int_{\Delta}^{T} \frac{1}{\zeta_0(T)}\left[\Theta(T) + c_0\right]dT - \frac{1}{\zeta_0(T)}\int_{\Delta}^{T}\left[\Theta(T) + c_0\right]dT\right\} \qquad T \in \widehat{EC} \tag{95}$$

where equation (77) was also used in obtaining equation (95), $\Theta^+(T)$ denotes the limiting value of $\Theta(T)$ as T approaches $\mathscr{S}_0$ from within $\mathscr{D}_0^+$, and the line integrals in equation (94) are to be taken along $\mathscr{S}_0$

It follows from the first equation (13) and equation (90) that

$$\frac{\zeta_1^+(T)}{\zeta_0(T)} + \frac{1}{\zeta_0(T)}\frac{d\zeta_0}{dz} z_1^S = \frac{\zeta_1^+(T)}{\zeta_0(T)} + \frac{d\zeta_0}{dW_0} z_1^S$$

$$= \frac{\zeta_1^+(T)}{\zeta_0(T)} - \frac{1}{\zeta_0(T)}\frac{d\zeta_0}{dW_0}\left[W_1^+(T) + \psi_1^F W_0(T)\right]$$

$$= \frac{d \ln \zeta_0}{dW_0}\left[\frac{dW_0}{d \ln \zeta_0}\left(\frac{\zeta_1^+(T)}{\zeta_0(T)} - \frac{1}{2}\right) - W_1^+(T) + \frac{1}{2}W_0(T)\right] + \frac{1}{2} - \alpha\frac{d \ln \zeta_0}{dW_0} W_0(T)$$

Upon substituting equation (80) into this expression, we find that

$$\frac{\zeta_1^+(T)}{\zeta_0(T)} + \frac{1}{\zeta_0(T)} \frac{d\zeta_0}{dz} z_1^S = \frac{d \ln \zeta_0}{dW_0} \left[\int_{-1}^{T} \Theta(T)\, dT + c_0(T + 1) \right] + \frac{1}{2}$$

Substituting this into equation (93) shows that

$$|\zeta^+(z^S)|^2 = |\zeta_0(T)|^2 \left\{ 1 + \epsilon + 2\epsilon \mathcal{R}e \frac{d \ln \zeta_0}{dW_0} \int_{-1}^{T} \left[\Theta^+(T) + c_0\right] dT \right\} \qquad T \in \mathscr{S}_0 \tag{96}$$

where the line integral is along $\mathscr{S}_0$.

The distance $\hat{S}$ measured along the curve S is given by

$$\hat{S} = H_0 \int_{-1}^{T} \left| \frac{dz^S}{dT} \right| |dT| \qquad T \in \mathscr{S}_0 \tag{97}$$

where the integral is taken along the curve $\mathscr{S}_0$. Now, upon differentiating equation (94), we find after using the first equation (13) that

$$\frac{dz^S}{dT} = \frac{dW_0}{dT} \frac{1}{\zeta_0(T)} \left\{ 1 + \epsilon h_1 - \epsilon \frac{d \ln \zeta_0}{dW_0} \int_{-1}^{T} \left[\Theta^+(T) + c_0\right] dT \right\} \qquad T \in \mathscr{S}_0$$

Hence,

$$\left| \frac{dz^S}{dT} \right| = \frac{1}{|\zeta_0(T)|} \left| \frac{dW_0}{dT} \right| \left\{ 1 + 2\epsilon h_1 - 2\epsilon \mathcal{R}e \frac{d \ln \zeta_0}{dW_0} \int_{-1}^{T} \left[\Theta^+(T) + c_0\right] dT \right\}^{1/2} \qquad T \in \mathscr{S}_0$$

Substituting equation (96) into equation (79) gives, to within terms of order ϵ^2,

$$\left| \frac{dz^S}{dT} \right| = \left(1 - \epsilon \psi_1^F\right) \left| \frac{dW_0}{dT} \right| \frac{1}{|\zeta^+(z^S)|} \qquad T \in \mathscr{S}_0$$

Substituting this into equation (97) shows that

$$\frac{\hat{S}}{H_0} = \left(1 - \epsilon\psi_1^F\right)\int_{-1}^{T} \frac{1}{\left|\zeta^+(z^S)\right|}\left|\frac{dW_0}{dT}\right|\left|dT\right| \qquad T \in \mathscr{S}_0 \tag{98}$$

All the necessary results have now been obtained. However, it is convenient to rewrite some of these in more explicit form. This is done in appendix B. For convenience, the most important equations of this section are now summarized.

Summary of Equations

Coordinates of downstream edge of nozzle:

$$\left.\begin{aligned} a &= \frac{1}{\pi}\left[3(\Delta + 1) + (2\Delta - 1)\ln\left(\frac{\sqrt{1+\Delta} - \sqrt{\Delta}}{\bar{\Delta}}\right)^2 + 6\sqrt{\Delta}\sqrt{1+\Delta}\right] \\ b &= 2(2\Delta - 1) \end{aligned}\right\} \quad 0 < \Delta \tag{34}$$

Zeroth-order velocity:

$$\frac{1}{\zeta_0(T)} = \frac{T - 2\Delta + 2i\sqrt{\Delta}\sqrt{T - \Delta}}{T} \qquad \mathscr{Im}\, T \geqq 0 \tag{23}$$

Transformation from T-plane to physical plane:

$$z(T) + a + ib + \frac{1}{\pi}\left\{(T - \Delta)\left(1 - \frac{2}{T}\right) + 2i\sqrt{\Delta}\left(2 - \frac{1}{T}\right)\sqrt{T - \Delta} - b\ln\left(\frac{i\sqrt{T - \Delta} + \sqrt{\Delta}}{\sqrt{\Delta}}\right)\right\} \tag{35}$$

Change in stream function across jet:

$$\psi^F = -1 - \epsilon\left(h_1 + \frac{1}{2}\right) = 1 - + \epsilon\psi_1^F \tag{79}$$

Asymptotic jet width:

$$h = 1 + \epsilon h_1 \tag{15}$$

where

$$h_1 = \frac{-2\pi\ \Delta}{(\Delta+1)\left(4\ \Delta+1+4\sqrt{\Delta}\sqrt{\Delta+1}\right)}\left\{2\sqrt{\Delta}\ \text{P. V.} \int_{-1}^{\Delta} \frac{\left(\sqrt{\Delta}+\sqrt{\Delta-\xi}\right)}{\xi}\Theta(\xi)d\xi\right.$$

$$\left.-\int_{-1}^{\Delta}\Theta(\xi)d\xi + \left[\Delta + 1 + 2\ \Delta \ln\left(\frac{\sqrt{1+\Delta}-\sqrt{\Delta}}{\sqrt{\Delta}}\right)^2 + 4\sqrt{\Delta}\sqrt{1+\Delta}\right]\Theta(O)\right\} \tag{85}$$

Intermediate variables:

$$c_0 = \frac{(2\ \Delta - 1)}{2\pi\ \Delta} h_1 - \Theta(O) \tag{86}$$

$$\Gamma(\eta) = -\frac{1}{\pi}\frac{T+1}{T}\left(\frac{1}{\zeta_0(T)}\left\{\frac{T\sqrt{T-\Delta}}{\sqrt{\Delta}\,(T+1)}\left(1+\frac{1}{2}\frac{T+1}{T-\Delta}\right)\mathcal{Im}\frac{1}{\zeta_0(T)} - 2i\,\mathcal{Im}\frac{1}{\zeta_0(T)}\right.\right.$$

$$\left.\left.-(T+1)\sqrt{T-\Delta}\ \mathcal{Re}\left[\frac{1}{\zeta_0(T)}\ \frac{1}{(T+1)\sqrt{T-\Delta}}\right]\right\} - 2\right) \qquad T = -\frac{\eta}{\sin\eta}e^{-i\eta} \tag{99}$$

$$\Theta^+(\eta) = \frac{1}{2}\Gamma(\eta) + \frac{\text{P. V.}}{2\pi i}\int_0^{\pi}\frac{\Gamma(\gamma)M(\gamma)d\gamma}{\gamma\cot\gamma - \eta\cot\eta - i(\gamma-\eta)}$$

$$-\frac{1}{2\pi i}\int_0^{\pi}\frac{\overline{\Gamma(\gamma)M(\gamma)}d\gamma}{\gamma\cot\gamma - \eta\cot\eta + i(\gamma+\eta)} \qquad 0 \le \eta \le \pi \tag{B4}$$

where

$$M(\eta) \equiv \frac{1}{\sin^2\eta}(\eta - \cos\eta\sin\eta) + i \tag{100}$$

$$\Theta(\xi) = \frac{1}{\pi}\,\mathcal{I}m\int_0^{\pi}\frac{\Gamma(\gamma)M(\gamma)}{\gamma\cot\gamma + \xi - i\gamma}\,d\gamma \qquad -\infty < \xi < +\infty \tag{B5}$$

$$\left.\begin{aligned} z_0^S(\eta) &\equiv z\left(-\frac{\eta}{\sin\eta}\,e^{-i\eta}\right) \\ \zeta_0^S(\eta) &\equiv \zeta_0\left(-\frac{\eta}{\sin\eta}\,e^{-i\eta}\right) \end{aligned}\right\} \tag{B6}$$

$$V_S^+(\eta) = \left|\zeta^+(z^S)\right| V_\infty \tag{B8}$$

$$J(\eta) \equiv (1 - \eta\cot\eta + i\eta)\left[\sqrt{\frac{\left(\mu^2 + \eta^2\right)^{1/2} - \mu}{2}} + i\sqrt{\frac{\left(\mu^2 + \eta^2\right)^{1/2} + \mu}{2}}\right] \tag{B9}$$

$$\mu = \eta\cot\eta + \Delta \tag{26}$$

Position of slip line:

$$z^S = z_0^S(\eta)(1 + \epsilon h_1) - \epsilon\int_0^{\eta}\left[\frac{1}{\zeta_0(\gamma)} - \frac{1}{\zeta_0(\eta)}\right]\left[\Theta^+(\gamma) + c_0\right]M(\gamma)\,d\gamma \qquad 0 \leq \eta \leq \pi \tag{B7}$$

Velocity along slip line:

$$\frac{\left[V_s^+(\eta)\right]^2}{V_\infty^2} = C_{ps} = \left|\zeta_0^S(\eta)\right|^2\Bigg(1 + \epsilon$$

$$+ 2\epsilon\pi\sqrt{\Delta}\;\mathcal{R}e\left\{\frac{i}{J(\eta)}\int_0^{\eta}\left[\Theta^+(\gamma) + c_0\right]M(\gamma)\,d\gamma\right\}\Bigg) \qquad 0 \leq \eta < \pi \tag{B10}$$

Distance along slip line:

$$\frac{\hat{S}}{H_0} = \frac{1 - \epsilon\psi_1^F}{\pi} \int_0^{\eta} \frac{V_\infty}{V_s^+(\gamma)} \frac{(1 - \gamma \cot \gamma)^2 + \gamma^2}{\gamma} d\gamma \qquad 0 \leq \eta < \pi \tag{B13}$$

Position of free-stream line:

$$z^F = a + ib + \left[z(\xi) - (a + ib)\right](1 + \epsilon h_1)$$

$$- \epsilon \int_{\Delta}^{\xi} \left[\frac{1}{\zeta_0(\xi_1)} - \frac{1}{\zeta_0(\xi)}\right]\left[\Theta(\xi_1) + c_0\right]d\xi_1 \qquad \Delta \leq \xi < \infty \tag{B14}$$

Equations (19) and (20) have been combined with equation (B1) to obtain equation (99).

RESULTS AND DISCUSSION

The numerical calculations were performed by using complex arithmetic. Hence, there is no need to separate the real and imaginary parts of the various formulas given in the preceding section. Equations (34) are used to calculate the orifice offset ratio $B/A = b/a$ for various values of the parameter Δ. However, there is not a one-to-one correspondence between the values of B/A and the values of Δ. In view of this fact, it is more convenient to present the results in terms of the orifice orientation angle defined as $\tan^{-1} B/A$ since there is a one-to-one correspondence between this latter quantity and the parameter Δ. A plot of the orifice angle against the parameter Δ is presented in figure 8. The orifice angle completely fixes the geometry of the problem. Hence, once the geometry of the orifice is set, the parameter Δ can be determined from figure 8. This parameter is the one which appears naturally in the formulas which are used to calculate the various physical quantities of interest. The only other parameter appearing in the problem is ϵ, which gives a measure of the difference between the total pressure in the jet and the total pressure in the main stream. This parameter is defined by equation (1) as

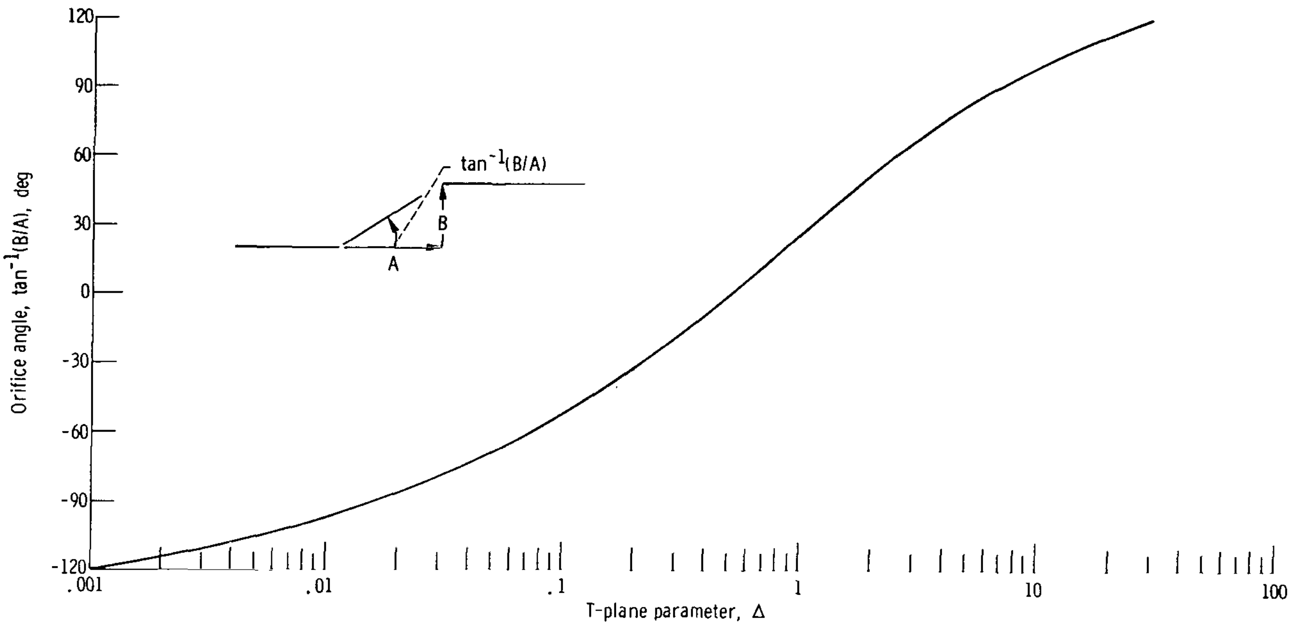


Figure 8. - Orifice angle in z-plane as function of T-plane parameter Δ.

$$\epsilon = \frac{P_j - P_\infty}{\frac{1}{2}\rho V_\infty^2}$$

Equations (23) and (113) are used to calculate $\Gamma(\eta)$ for various values of Δ, and these values of $\Gamma(\eta)$ are used together with equation (114) to calculate $\Theta^+(\eta)$ and $\Theta(\xi)$ for various values of Δ from equations (102) and (103), respectively. All the physical quantities presented in the plots are determined by these functions. Substituting the values of $\Theta(\xi)$ into equation (85) allows the quantity h_1 to be found as a function of the parameter Δ. Combining this with equation (15) gives $h = H/H_0$ as a function of Δ and ϵ. Now for two-dimensional jets, the jet contraction ratio is defined as the asymptotic jet thickness divided by the length of the orifice. Hence, the jet contraction ratio is

$$\frac{H}{\sqrt{A^2 + B^2}} = \frac{h}{\sqrt{a^2 + b^2}}$$

Substituting equations (15) and (34) into this formula gives the jet contraction ratio as a function of Δ and ϵ or, in view of figure 8, as a function of $\tan^{-1} B/A$ and ϵ. These results are presented in figure 9. It can be seen from figure 9 that, for positive values

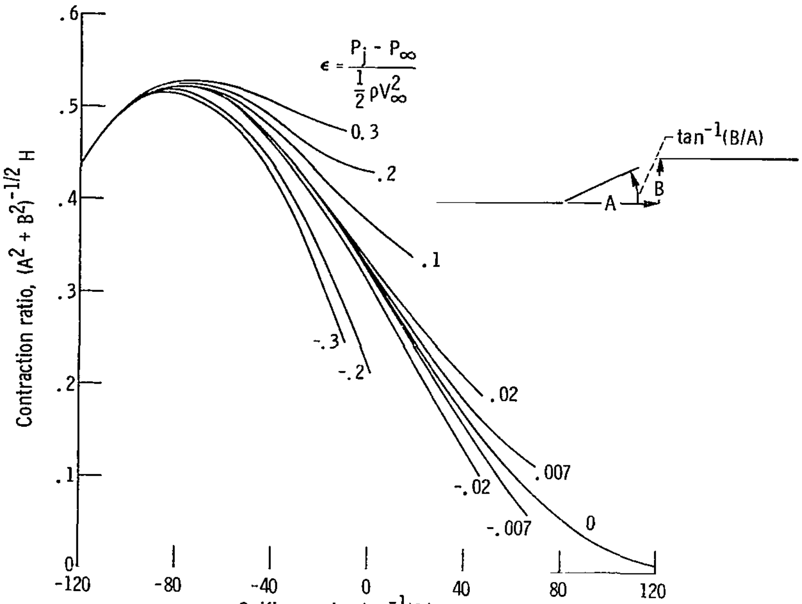


Figure 9. - Jet contraction ratio.

of the orifice angle, small changes in ϵ result in large changes in the contraction ratio, the effect becoming more marked as the orifice angle is increased. The opposite conclusion holds for negative values of the orifice angle. Figure 9 also shows that, for a given orifice angle, increasing ϵ always results in an increase in the jet contraction ratio. This increase is negligible, however, for orifice angles less than -100°. Figure 9 shows that the jet contraction ratio is a maximum for an orifice angle of approximately -80° and falls off markedly when the orifice angle is changed.

The parametric equations (with parameter η) for the slip line are obtained by substituting equations (86), (35), (114), and (23) into equation (105) and using definitions (104) and the expression for $\Theta^+(\eta)$ discussed above. The parametric equations (with parameter ξ) of the free-stream line are obtained by substituting equations (23), (35), and (86) together with the values of $\Theta(\xi)$ discussed above into equation (112). The resulting expressions (for the slip line and for the free-stream line) determine the boundary of the jet. The shapes of the jet boundaries for various values of the parameters ϵ and B/A are shown in figures 10 to 19. A number of these figures consist of two parts. When this occurs, part (a) of the figure is for the case when the total pressure in the jet exceeds that in the main stream ($\epsilon \geq 0$) and part (b) is for the case when the total pressure in the main stream exceeds that in the jet ($\epsilon \leq 0$). Figure 10 corresponds to a jet injected normal to the main stream ($B = 0$). Figures 11 to 15 are for negative orifice angles (i.e., jet injected downstream), and figures 16 to 19 are for positive orifice angles (i.e., jet injected upstream). Figures 10 and 16 show that, when the orifice

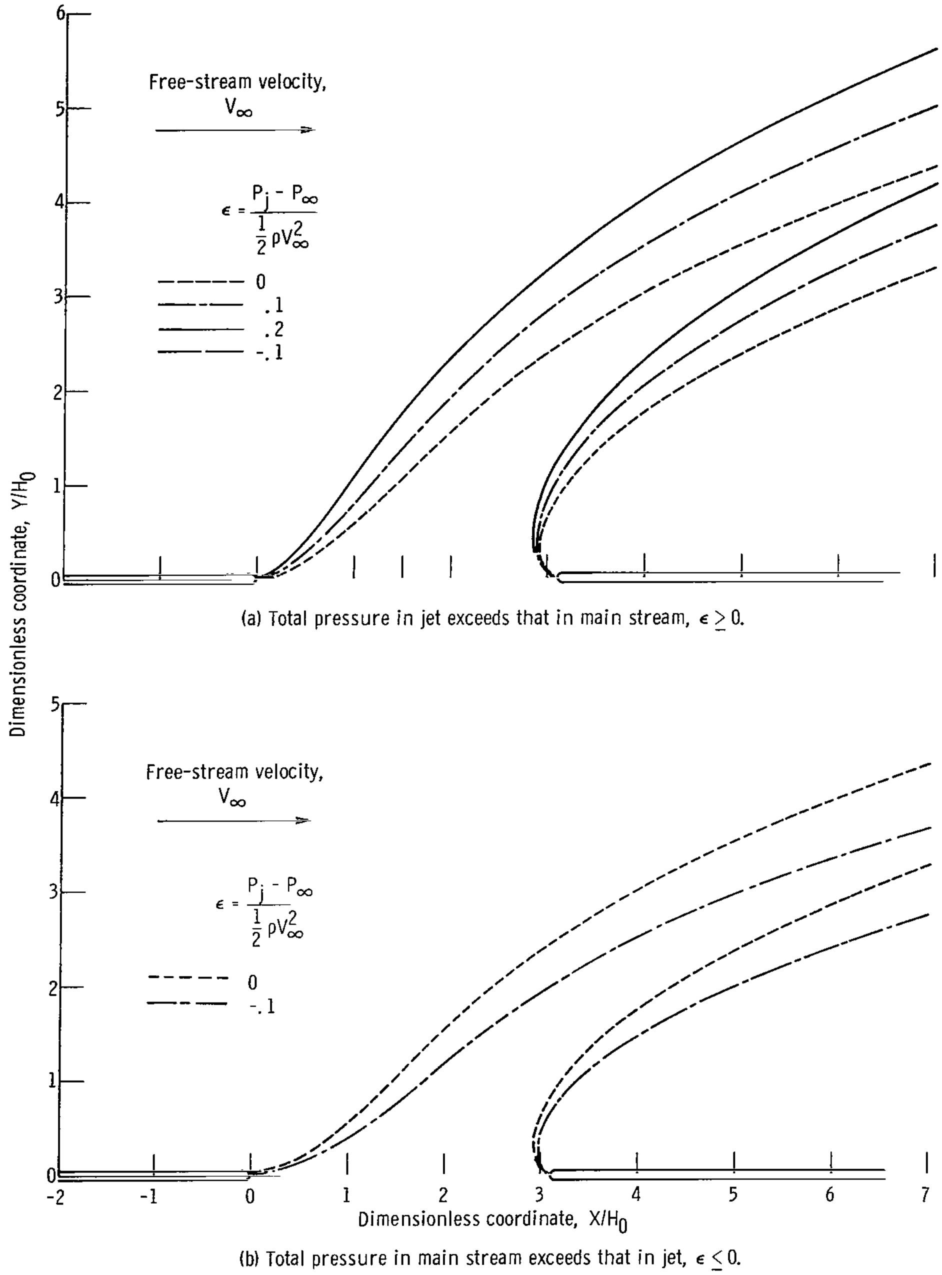


Figure 10. - Jet contour for orifice offset ratio, B/A = 0. Jet injected normal to main stream.

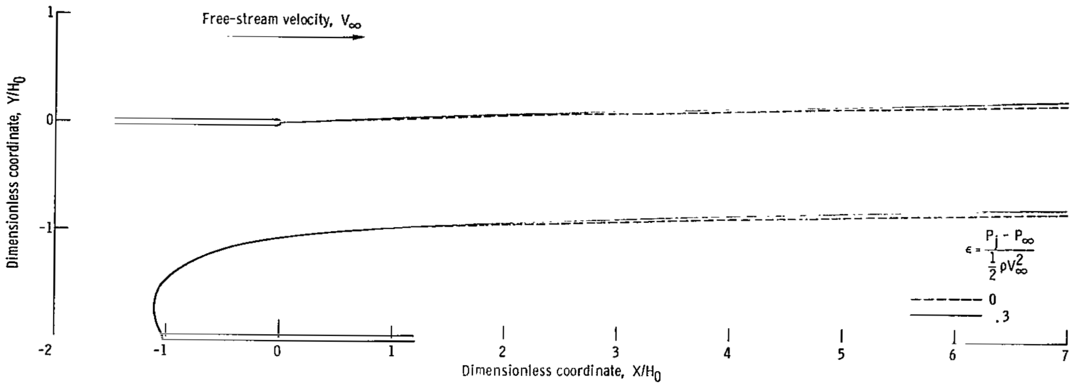


Figure 11. - Jet contour for orifice offset ratio, B/A = 2. Jet injected downstream. Total pressure in jet exceeds that in main stream, $\epsilon \geq 0$.

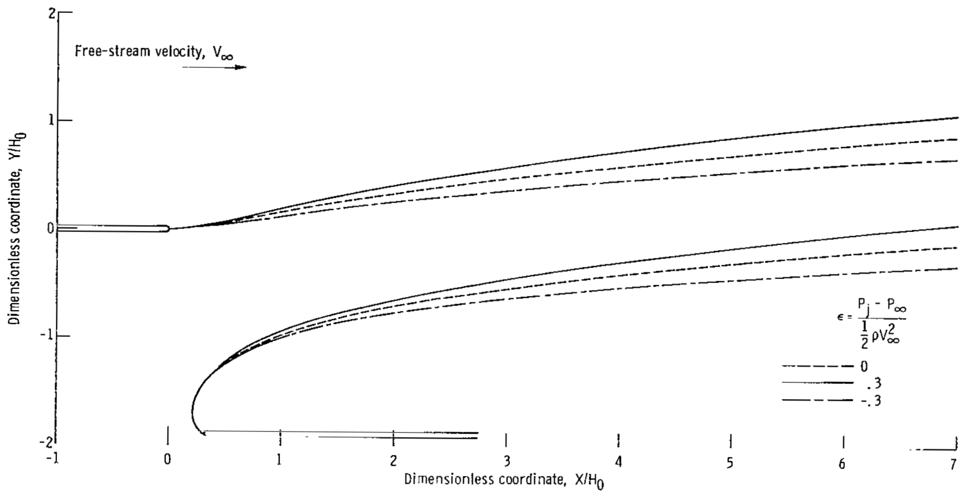


Figure 12. - Jet contour for orifice offset ratio, B/A = -6. Jet injected downstream.

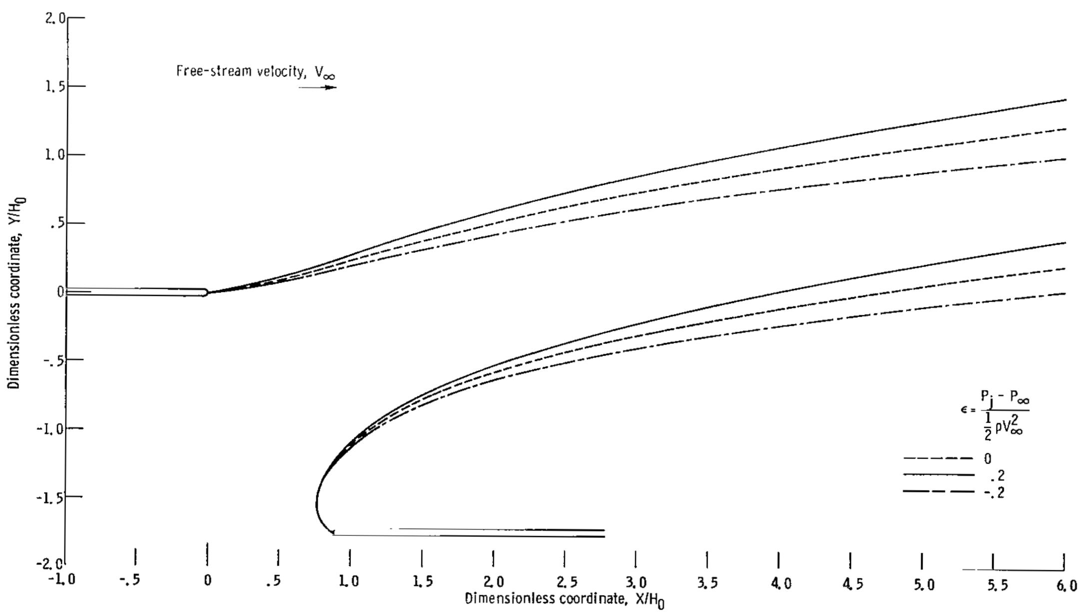


Figure 13. - Jet contour for orifice offset ratio, B/A = -2. Jet injected downstream.

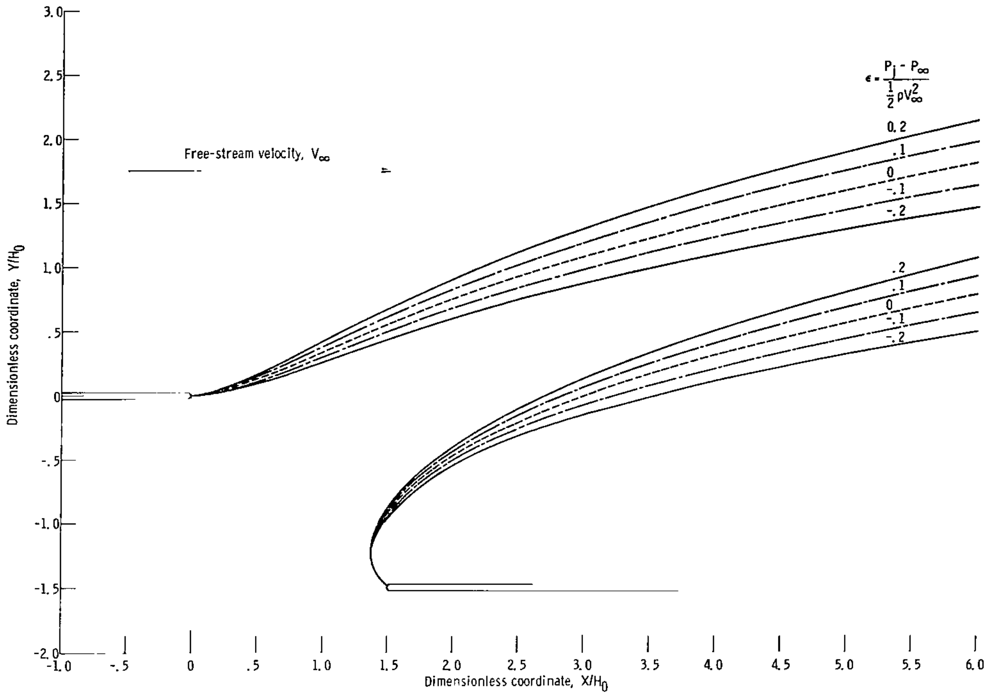
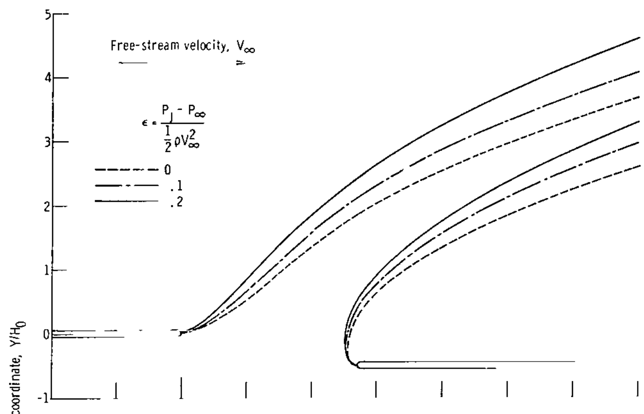
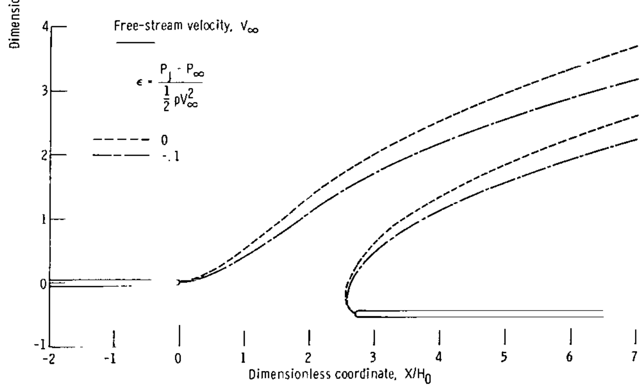


Figure 14. - Jet contour for orifice offset ratio, B/A = -1. Jet injected downstream.



(a) Total pressure in jet exceeds that in main stream, $\epsilon \geq 0$.



(b) Total pressure in main stream exceeds that in jet, $\epsilon \leq 0$.

Figure 15. - Jet contour for orifice offset ratio, B/A = -0.1862. Jet injected downstream.

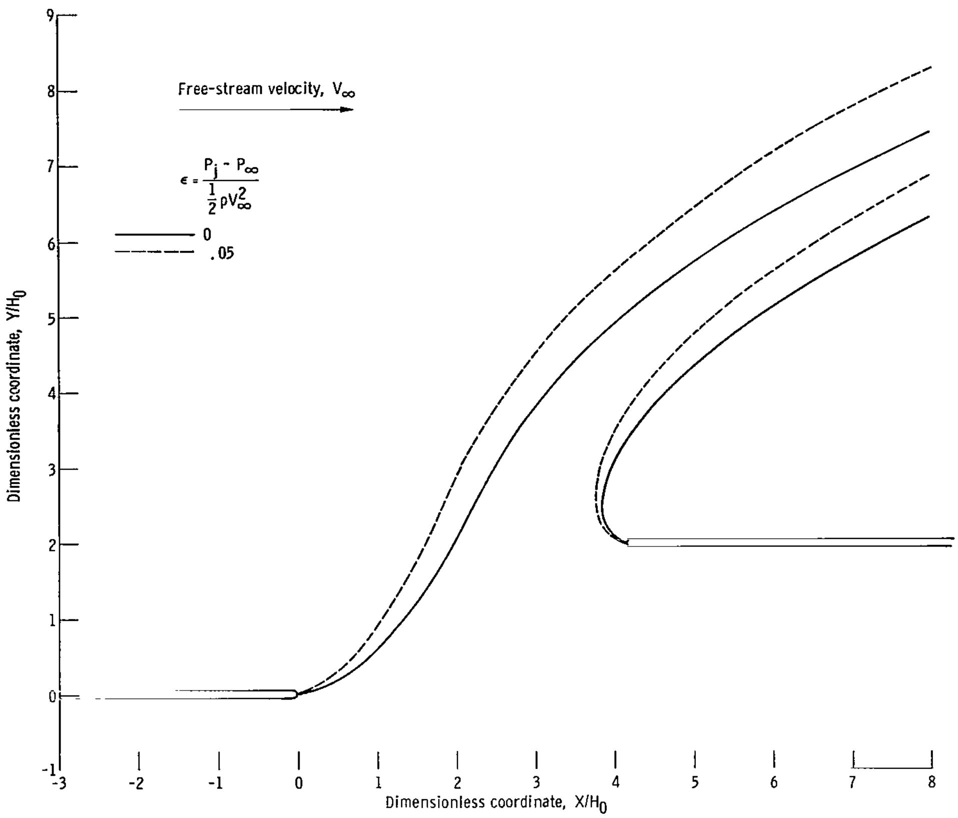


Figure 16. - Jet contour for orifice offset ratio, B/A = 0.5. Jet injected upstream.

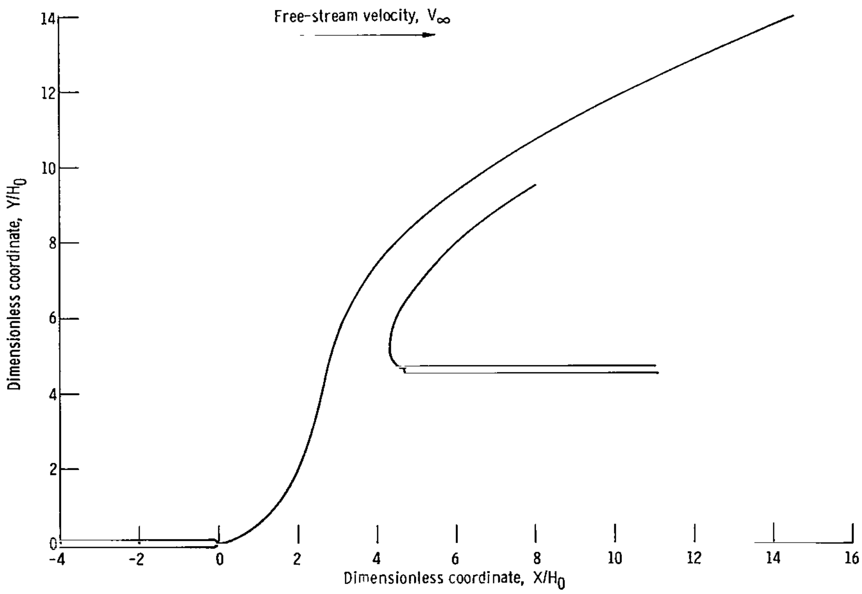


Figure 17. - Jet contour for orifice offset ratio, B/A = 1. Jet injected upstream. Total pressure change within jet, $\epsilon = 0$.

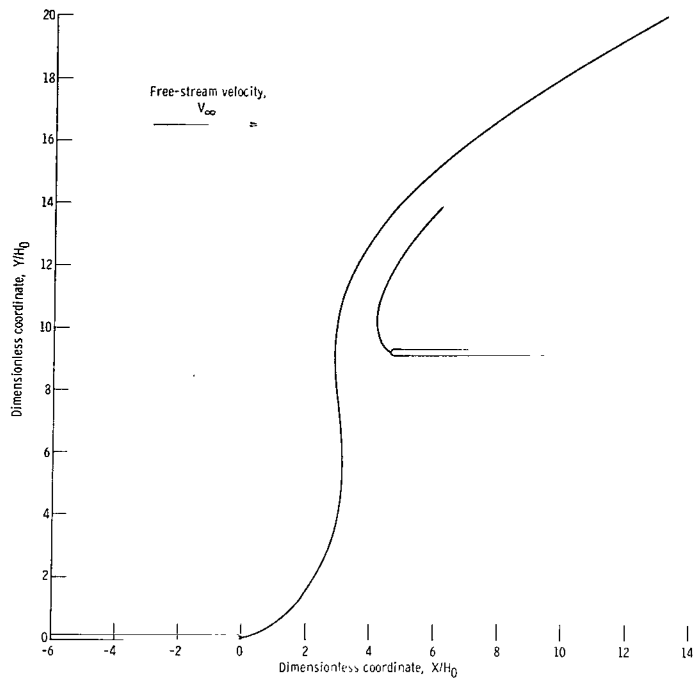


Figure 18. - Jet contour for orifice offset ratio, B/A = 2. Jet injected upstream. Total pressure change within jet, ϵ = 0.

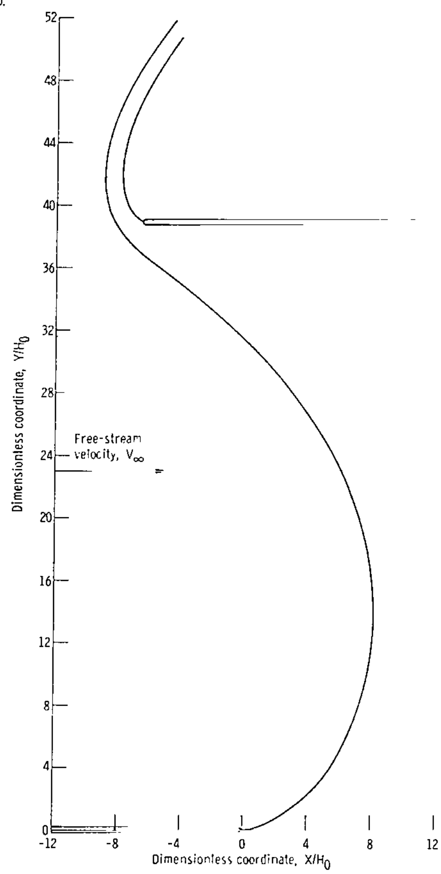


Figure 19. - Jet contour for orifice offset ratio, B/A = -6. Jet injected upstream; total pressure change within jet, ϵ = 0.

angle is greater than or equal to zero, a small change in the total pressure within the jet results in a large change in both the jet penetration and jet thickness. This effect becomes more pronounced as the orifice angle is increased. In fact, it becomes so pronounced that it is felt that the analysis may be invalid for orifice angles larger than 45°. For this reason, only the curves for $\epsilon = 0$ are given for these orifice angles. On the other hand, figures 11 to 15 show that, for sufficiently large negative values of the orifice angle, changes of the total pressure within the jet have almost no effect on the jet penetration or on the jet thickness. Although the scale of the figures is too small to show this, the numerical results do show that the jet always leaves the wall in a tangential direction. Figure 9 shows that turning the jet into the main stream tends to markedly decrease the contraction ratio, which is indicative of a decrease in the flow in the jet. The indications are, however, that a slight increase in the total pressure in the jet can easily compensate for this decreased flow. The jet penetration increases with increasing orifice angle.

The pressure coefficient on the slip line is obtained as a function of the distance along the slip line in parametric form from equations (108) and (111) after using definitions (104), (106), and (107) and substituting in equations (114), (23), (79), and (86). The results of these calculations are shown in figures 20 to 27. Each figure is drawn for a different orifice angle. These curves contain all the information necessary for calculating the viscous boundary layer along the slip line. The figures show that the velocity within the jet at the upstream edge of the orifice decreases with both increasing orifice angle and decreasing ϵ. It can also be seen from these curves that the pressure coefficient nearly reaches its asymptotic value in a distance of 10 jet diameters.

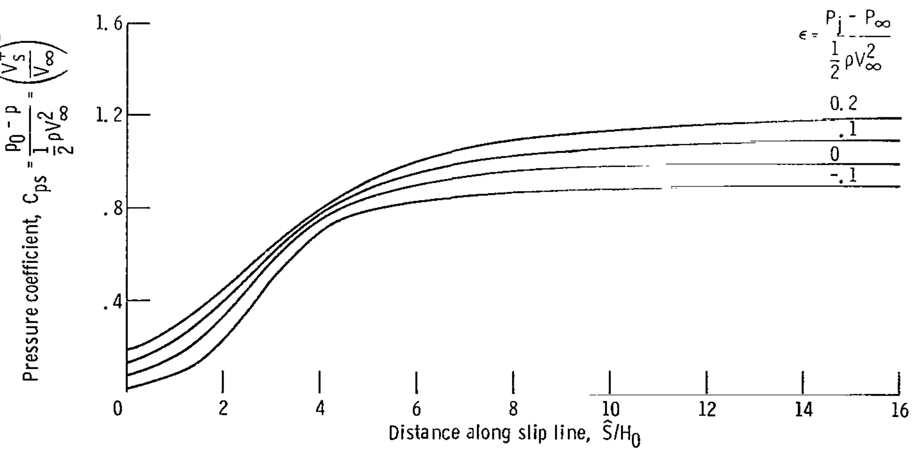


Figure 20. - Pressure coefficient on slip line of jet for orifice offset ratio, B/A = 0.

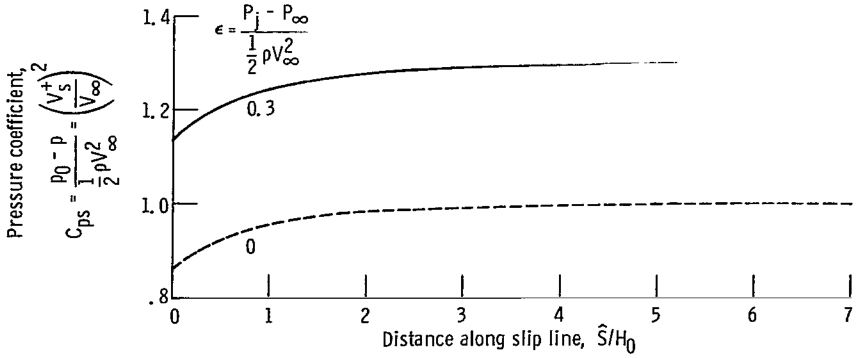


Figure 21. - Pressure coefficient on slip line of jet for orifice offset ratio of B/A = 2 (third quadrant).

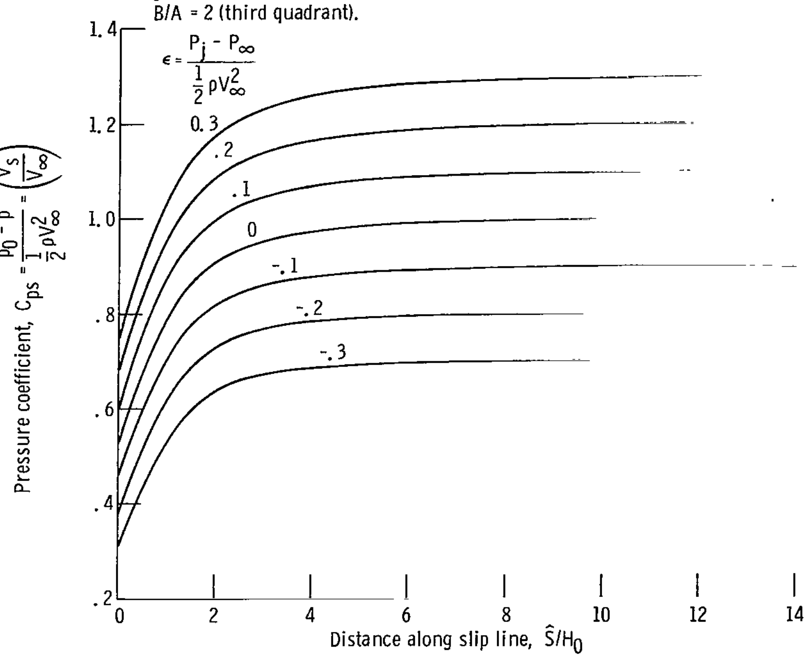


Figure 22. - Pressure coefficient on slip line of jet for orifice offset ratio, B/A = -6 (fourth quadrant).

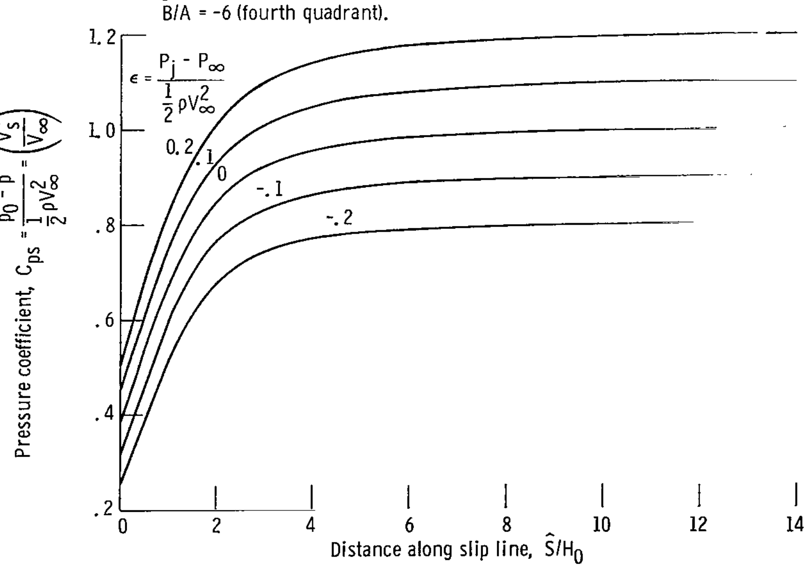


Figure 23. - Pressure coefficient on slip line of jet for orifice offset ratio, B/A = -2 (fourth quadrant).

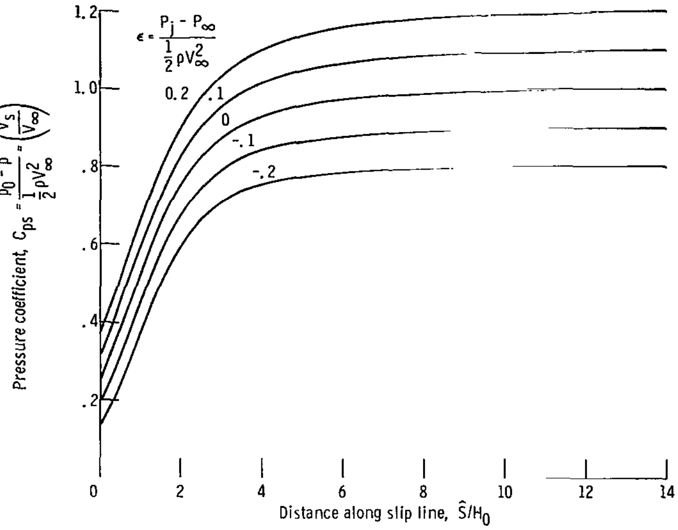


Figure 24. - Pressure coefficient on slip line of jet for orifice offset ratio, B/A = -1 (fourth quadrant).

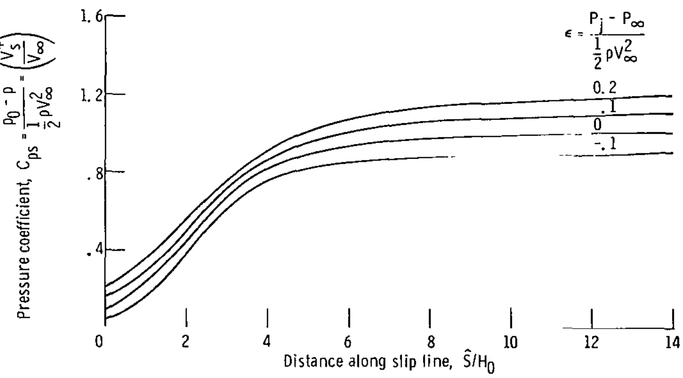


Figure 25. - Pressure coefficient on slip line of jet for orifice offset ratio, B/A = -0.1862 (fourth quadrant).

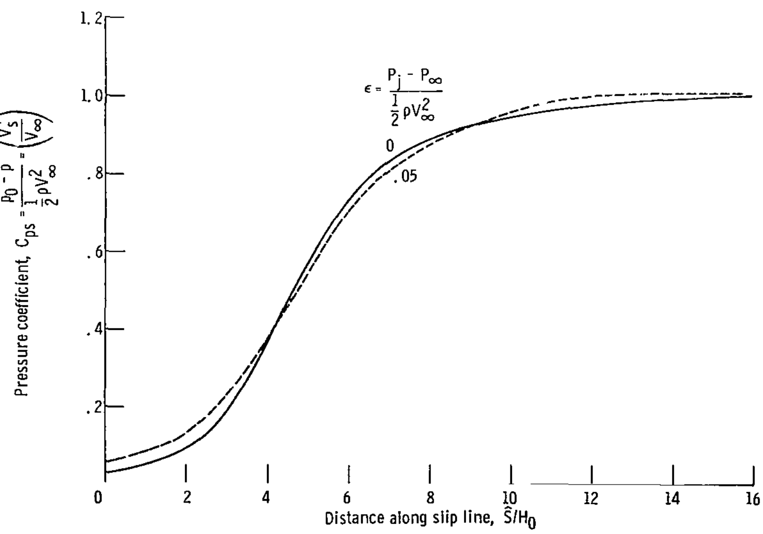


Figure 26. - Pressure coefficient on slip line of jet for orifice offset ratio, B/A = 0.5 (first quadrant).

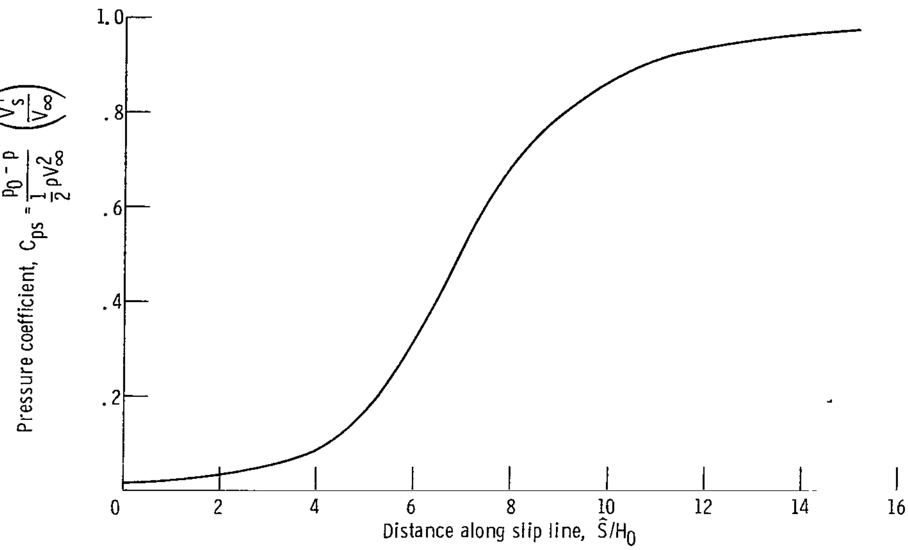


Figure 27. - Pressure coefficient on slip line of jet for orifice offset ratio, B/A = 1 (first quadrant). Total pressure change within jet, $\epsilon = 0$.

CONCLUDING REMARKS

A procedure has been developed to obtain a solution to the problem of a two-dimensional jet injected from an orifice at an oblique angle to a moving stream for the case where the total pressure in the jet does not differ too much from the total pressure in the main stream. The analysis shows that, for orifices tilted into the main stream, small increases in the total pressure in the jet result in large increases in the jet penetration and in the jet thickness.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, June 24, 1969,
129-01-07-07-22.

APPENDIX A

FLUIDS OF UNEQUAL DENSITY

A situation often encountered is the flow of a jet of high-density fluid into a low-density stream. When the jet and main stream are not composed of the same fluid, the foregoing analysis may be applied after a simple rescaling. This fact will now be proved.

Previously, the problem was formulated in terms of the dimensionless velocity

$$\zeta = u - iv = \frac{U - iV}{V_\infty}$$

and its complex potential

$$W = \varphi + i\psi = \frac{\Phi + i\Psi}{V_\infty H_0}$$

In order to consider fluids of unequal density let ζ be redefined as

$$\zeta(z) = \begin{cases} \zeta^+(z) = \sqrt{\dfrac{\rho_j}{\rho_\infty}}\,\dfrac{(U - iV)}{V_\infty} & z \in D^+ \\[2ex] \zeta^-(z) = \dfrac{U - iV}{V_\infty} & z \in D^- \end{cases} \tag{A1}$$

where ρ_j is the density of the fluid in the jet and ρ_∞ is the density in the stream. To preserve the relation

$$\zeta = \frac{dW}{dz}$$

it is also necessary to redefine the potential W as

$$W = \begin{cases} W^{+}(z) = \sqrt{\dfrac{\rho_j}{\rho_\infty}}\,\dfrac{(\Phi + i\Psi)}{V_\infty H_0} & z \in D^{+} \\[2ex] W^{-}(z) = \dfrac{\Phi + i\Psi}{V_\infty H_0} & z \in D^{-} \end{cases} \tag{A2}$$

The dimensionless coordinates are, as before,

$$x = \frac{X}{H_0}$$

$$y = \frac{Y}{H_0}$$

$$a = \frac{A}{H_0}$$

$$b = \frac{B}{H_0}$$

$$h = \frac{H}{H_0}$$

The perturbation parameter ϵ is defined more generally as

$$\epsilon \equiv \frac{P_j - P_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{\rho_j\left(V_s^{+}\right)^2 - \rho_\infty\left(V_s^{-}\right)^2}{\rho_\infty V_\infty^2} = \left|\zeta^{+}(z^S)\right|^2 - \left|\zeta^{-}(z^S)\right|^2 \tag{4}$$

which shows that the jump condition on the slip line has the same form as for fluids of equal density.

Likewise, the value of the dimensionless velocity on the free-stream line is given again by

$$\left|\zeta^+(z^F)\right|^2 = \frac{\rho_j\left(V_F^+\right)^2}{\rho_\infty V_\infty^2} = \frac{P_j - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 1 + \frac{P_j - P_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 1 + \epsilon \tag{5}$$

Very far from the orifice, the dimensionless velocity takes the same limits as in the equal density case, namely,

$$\zeta^+(z) \to 0 \qquad z \to K \tag{9}$$

$$\zeta^-(z) \to 1 \qquad z \to G \tag{10}$$

Similarly, the homogeneous conditions on the velocity components at the solid boundaries remain unaltered:

$$\mathcal{Im}\ \zeta^+(z) = 0 \qquad z \in \widehat{KD}$$

$$\mathcal{Im}\ \zeta^+(z) = 0 \qquad z \in \widehat{EK}$$

$$\mathcal{Im}\ \zeta^-(z) = 0 \qquad z \in \widehat{GD}$$

Lastly, the boundary values of the stream function are again given by

$$\mathcal{Im}\, W^+(z) = \mathcal{Im}\, W^-(z) = 0 \qquad z \in S$$

$$\mathcal{Im}\, W^+(z) = \psi^F \qquad z \in F$$

The value of ψ^F is to be determined in the course of the analysis exactly as before.

Thus, it is seen that the boundary conditions on the velocities and the stream function are identical to those obtained when the densities are equal (eq. (11)). Since the differential equation for the flow (Laplace's equation) remains unaltered as well, the analysis proceeds exactly as before, with the same solutions for the zeroth- and first-order problems.

The interpretation of the results must be modified in one instance through inclusion of the density ratio. The pressure ratio (eq. (10)) is now

$$C_{ps} \equiv \frac{p_o - p(x_S, y_S)}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{\rho_j\left[V_S^+(\eta)\right]^2}{\rho_\infty V_\infty^2} = \left|\zeta^+(z^S)\right|^2 \tag{A3}$$

Equation (A3) shows that the pressure coefficient given in figures 20 to 27 must be reinterpreted as the ratio of the dynamic pressures when the fluid densities are unequal.

Another quantity of interest is the mass flow in the jet

$$\dot{M}_j = \rho_\infty V_\infty H_0 \sqrt{\frac{\rho_j}{\rho_\infty}} \left|\psi^F\right| = \sqrt{\rho_j \rho_\infty}\, V_\infty H_0 \left[1 + \epsilon\left(h_1 + \frac{1}{2}\right)\right] = \sqrt{\rho_j \rho_\infty (A^2 + B^2)}(1 + \epsilon)\ \text{C.R.}$$

where C.R. is the contraction ratio computed to first order (fig. 9).

APPENDIX B

EXPLICIT FORMULAS FOR CALCULATING BOUNDARY VALUES

Carrying out the differentiation by parts in equation (67) yields

$$\Gamma(T) = -\frac{dW_0}{dT}\left\{i\zeta_0\left[\frac{\partial}{\partial\varphi_0}\left(\frac{1}{\zeta_0}\frac{dW_0}{d\zeta_0}\right)\mathscr{I}m\,\frac{1}{\zeta_0} + \frac{1}{\zeta_0}\frac{dW_0}{d\zeta_0}\,\mathscr{I}m\left(\frac{\partial}{\partial\varphi_0}\frac{1}{\zeta_0}\right)\right] - 2\right\} \qquad T \in \mathscr{S}_0$$

It follows from the definition of a derivative of a holomorphic function that this can be written as

$$\Gamma(T) = -\frac{dW_0}{dT}\left\{i\zeta_0\left[\frac{d}{dW_0}\left(\frac{1}{\zeta_0}\frac{dW_0}{d\zeta_0}\right)\mathscr{I}m\,\frac{1}{\zeta_0} + \frac{1}{\zeta_0}\frac{dW_0}{d\zeta_0}\,\mathscr{I}m\left(\frac{d}{dW_0}\frac{1}{\zeta_0}\right)\right] - 2\right\} \qquad T \in \mathscr{S}_0 \tag{B1}$$

Applying the Plemelj formulas (ref. 8) to equation (70), we see that

$$\Theta(T) \rightarrow \Theta^+(T) = \frac{1}{2}\,\Gamma(T) + \frac{\mathrm{P.V.}}{2\pi i}\int_{\mathscr{S}_0}\frac{\Gamma(\tau)d\tau}{\tau - T} - \frac{1}{2\pi i}\int_{\mathscr{S}_0}\frac{\overline{\Gamma(\tau)}\,\overline{d\tau}}{\overline{\tau} - T} \qquad T \in \mathscr{S}_0$$

where the integration is to be performed in a counterclockwise direction along $\mathscr{S}_0$. In view of equation (24), however, this can be written as

$$\Theta^+(\eta) = \frac{1}{2}\,\Gamma(\eta) + \frac{\mathrm{P.V.}}{2\pi i}\int_{\pi}^{0}\frac{\Gamma(\gamma)}{\frac{\gamma}{\sin\gamma}e^{-i\gamma} - \frac{\eta}{\sin\eta}e^{-i\eta}}\,d\left(\frac{\gamma}{\sin\gamma}e^{-i\gamma}\right)$$

$$-\frac{1}{2\pi i}\int_{\pi}^{0}\frac{\overline{\Gamma(\gamma)}}{\frac{\gamma}{\sin\gamma}e^{i\gamma} - \frac{\eta}{\sin\eta}e^{-i\eta}}\,d\left(\frac{\gamma}{\sin\gamma}e^{i\gamma}\right) \qquad 0 \leq \eta < \pi \tag{B2}$$

where, for brevity, we have written $\Theta^+(\eta)$ in place of $\Theta^+\left[(-\eta/\sin\eta)\, e^{-i\eta}\right]$, $\Gamma(\eta)$ in place of $\Gamma\left[-(\eta/\sin\eta)\, e^{-i\eta}\right]$, etc. Since

$$d\left(\frac{\gamma}{\sin\gamma}\, e^{-i\gamma}\right) = -\left(\frac{\gamma - \cos\gamma\,\sin\gamma}{\sin^2\gamma} + i\right) d\gamma \tag{B3}$$

equation (B2) can be written as

$$\Theta^+(\eta) = \frac{1}{2}\,\Gamma(\eta) + \frac{\text{P.V.}}{2\pi i}\int_0^{\pi} \frac{\Gamma(\gamma)\left[\frac{1}{\sin^2\gamma}(\gamma - \cos\gamma\,\sin\gamma) + i\right] d\gamma}{\gamma\cot\gamma - \eta\cot\eta - i(\gamma - \eta)}$$

$$- \frac{1}{2\pi i}\int_0^{\pi} \frac{\overline{\Gamma(\gamma)}\left[\frac{1}{\sin^2\gamma}(\gamma - \cos\gamma\,\sin\gamma) - i\right] d\gamma}{\gamma\cot\gamma - \eta\cot\eta + i(\gamma + \eta)} \qquad 0 \le \eta < \pi \tag{B4}$$

For $T = \xi + i0$, equation (70) becomes

$$\Theta(\xi) = \frac{1}{2\pi i}\int_{\pi}^{0} \frac{\Gamma(\gamma)\, d\left(\frac{\gamma}{\sin\gamma}\, e^{-i\gamma}\right)}{\frac{\gamma}{\sin\gamma}\, e^{-i\gamma} + \xi} - \frac{1}{2\pi i}\int_{\pi}^{0} \frac{\overline{\Gamma(\gamma)}\, d\left(\frac{\gamma}{\sin\gamma}\, e^{i\gamma}\right)}{\frac{\gamma}{\sin\gamma}\, e^{i\gamma} + \xi}$$

$$= \frac{1}{\pi}\,\mathcal{I}m \int_{\pi}^{0} \frac{\Gamma(\gamma)}{\frac{\gamma}{\sin\gamma}\, e^{-i\gamma} + \xi}\, d\left(\frac{\gamma}{\sin\gamma}\, e^{-i\gamma}\right)$$

and in view of equation (B3) this can be written as

$$\Theta(\xi) = \frac{1}{\pi}\,\mathscr{Im}\int_0^{\pi} \frac{\Gamma(\gamma)\left[\dfrac{\gamma - \cos\gamma\,\sin\gamma}{\sin^2\gamma} + i\right]d\gamma}{\gamma\,\cot\gamma + \xi - i\gamma} \qquad -\infty < \xi < +\infty \tag{B5}$$

Upon defining $z_0^S(\eta)$ and $\zeta_0^S(\eta)$ by

$$\left.\begin{aligned} z_0^S(\eta) &\equiv z\left(\frac{-\eta}{\sin\eta}\,e^{-i\eta}\right) \\ \zeta_0^S(\eta) &\equiv \zeta_0\left(\frac{-\eta}{\sin\eta}\,e^{-i\eta}\right) \end{aligned}\right\} \qquad 0 \le \eta < \pi \tag{B6}$$

and using equation (B3), equation (94) becomes

$$z^S = z_0^S(\eta)(1 + \epsilon h_1)$$

$$- \epsilon \int_0^{\eta} \left[\frac{1}{\zeta_0(\gamma)} - \frac{1}{\zeta_0(\eta)}\right]\left[\Theta^+(\gamma) + c_0\right]\left(\gamma - \frac{\cos\gamma\,\sin\gamma}{\sin^2\gamma} + i\right)d\gamma \qquad 0 \le \eta < \pi \tag{B7}$$

Upon defining $V_S^+(\eta)$ by

$$V_S^+(\eta) = V_\infty\left|\zeta^+(z^S)\right| \qquad 0 \le \eta < \pi \tag{B8}$$

and $J(\eta)$ by

$$J(\eta) = (1 - \eta\,\cot\eta + i\eta)\left[\sqrt{\frac{\left(\mu^2 + \eta^2\right)^{1/2} - \mu}{2}} + i\sqrt{\frac{\left(\mu^2 + \eta^2\right)^{1/2} + \mu}{2}}\right] \tag{B9}$$

where μ is defined by equation (26), it follows from equations (19), (20), (24), (25), and (B6) that equation (96) can be written as

$$\frac{\left[V_s^+(\eta)\right]^2}{V_\infty^2} = \left|\zeta_0^S(\eta)\right|^2 \left(1 + \epsilon\right.$$

$$\left. + 2\epsilon\pi\sqrt{\Delta}\ \mathcal{R}e\left\{\frac{i}{J(\eta)}\int_0^\eta \left[\Theta^+(\gamma) + c_0\right]\left(\frac{\gamma - \cos\gamma\sin\gamma}{\sin^2\gamma} + i\right)d\gamma\right\}\right) \qquad 0 \leq \eta < \pi \qquad \text{(B10)}$$

Let p_o be the pressure far inside the orifice (the point K in fig. 2). Since the velocity is zero there, it is clear that $p_o = P_j$. Hence, it follows from equation (2) that the pressure at the point x_s, y_s on the slip line $p(x_s, y_s)$ is given by

$$\frac{p_o - p(x_s, y_s)}{\frac{1}{2}\rho V_\infty^2} = \left|\zeta^+(z_s)\right|^2 = \frac{\left[V_s^+(\eta)\right]^2}{V_\infty^2} \qquad \text{(B11)}$$

Hence, let the pressure coefficient on the slip line C_{ps} be defined by

$$C_{ps} \equiv \frac{p_o - p(x_s, y_s)}{\frac{1}{2}\rho V_\infty^2} \qquad \text{(B12)}$$

It is clear from equation (19) that

$$\left|\frac{dW_0}{dT}\right| = \frac{1}{\pi}\left|\frac{T + 1}{T}\right| = \frac{1}{\pi}\ \frac{(|T| + 2\ \mathcal{R}e\ T + 1)^{1/2}}{|T|}$$

Hence, it follows from equation (24) that

$$\left|\frac{dW_0}{dT}\right| = \frac{\sin\eta}{\pi\eta}\left(\frac{\eta^2}{\sin^2\eta} - 2\eta\cot\eta + 1\right)^{1/2} \qquad \text{for } T \in \mathscr{S}_0$$

Equations (24) and (101) show that for $T \in \mathscr{S}_0$

$$|dT| = \left|\cot\eta - \frac{\eta}{\sin^2\eta} - i\right| d\eta = \frac{1}{\sin\eta}\left(1 - 2\eta\cot\eta + \frac{\eta^2}{\sin^2\eta}\right)^{1/2} d\eta$$

Hence, for $T \in \mathscr{S}_0$

$$\left|\frac{dW_0}{dT}\right| |dT| = \frac{1}{\pi\eta}\left(\frac{\eta^2}{\sin^2\eta} - 2\eta\cot\eta + 1\right) d\eta = \frac{1}{\pi\eta}\left[(\eta\cot\eta - 1)^2 + \eta^2\right] d\eta$$

Using this result in equation (98) together with definition (B8),

$$\frac{\hat{S}}{H_0} = \frac{\left(1 - \epsilon\psi_1^F\right)V_\infty}{\pi} \int_0^\eta \frac{1}{V_s^+(\gamma)} \frac{\left[(1 - \gamma\cot\gamma)^2 + \gamma^2\right]}{\gamma} d\gamma \tag{B13}$$

Finally, it follows from equations (95) and (33) and figure 5 that

$$z^F = a + ib + \left[z(\xi) - (a + ib)\right](1 + \epsilon h_1)$$

$$- \epsilon \int_\Delta^\xi \left[\frac{1}{\zeta_0(\xi_1)} - \frac{1}{\zeta_0(\xi)}\right]\left[\Theta(\xi_1) + c_0\right] d\xi_1 \qquad \Delta \leq \xi < \infty \tag{B14}$$

APPENDIX C

SYMBOLS

A	horizontal distance between edges of orifice
a	A/l
B	vertical distance between edges of orifice
b	B/l
C_{ps}	pressure coefficient along slip line
c_n	$n = 0, 1, 2, \ldots$ constants
$D^{\pm}$	flow regions in physical plane
$\mathscr{D}_0^{\pm}$	regions in T-plane
F	free streamline in physical plane
H	asymptotic jet width
h	H/l
I_1	function defined by eq. (29)
J	function defined by eq. (107)
l	characteristic length (set equal to H_0)
M	function defined by eq. (114)
O	order symbol
o	order symbol
P_j	total pressure in jet
P_∞	total pressure in main stream
P.V.	Cauchy principal value
p	static pressure
p_o	static pressure at jet source
p_∞	static pressure far upstream from jet
Q	dimensionless volume flow through jet
S	slip line in physical plane
$\hat{S}$	distance along slip line
$\mathscr{S}_0$	slip line in T-plane
T	intermediate variable, $\xi + i\eta$
U	X-component of velocity
u	U/V_∞
V	Y-component of velocity
V_s^+	velocity along slip line inside of jet
V_∞	free-stream velocity
v	V/V_∞
W	dimensionless complex potential, $\varphi + i\psi$
X	coordinate in physical plane
x	X/l
Y	coordinate in physical plane
y	Y/l
z	dimensionless complex physical coordinate, $x + iy$
z^S	dimensionless coordinate of points on slip line
z^F	dimensionless coordinate of points on free-stream line
α	constant
Γ	function defined by eq. (67)
γ	dummy variable to replace η

Δ	location of edge of orifice in T-plane
ϵ	$(P_j - P_\infty)/\frac{1}{2}\rho V_\infty^2$
ζ	dimensionless complex conjugate velocity, u - iv
η	coordinate in T-plane
Θ	function defined by eq. (70)
θ_0	arg. ζ_0
$\Lambda^\pm$	defined by eqs. (55) and (59)
μ	function defined by eq. (26)
ξ	coordinate in T-plane
ρ	density
τ	dummy variable in T-plane
Φ	velocity potential
φ	$\Phi/l V_\infty$
Ψ	stream function
ψ	$\Psi/l V_\infty$
ψ^F	value of dimensionless stream function on free-stream line
Ω	function defined by eq. (65)

Subscripts:

0	zeroth-order quantity
1	first-order quantity

Superscripts:

F	value of quantity on free-stream line
S	value of quantity on slip line
+	value of quantity inside jet and orifice
-	value of quantity in main stream
—	complex conjugate (over bar)

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