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Gentlemen:

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In accordance with NASA Research Grant NGR 41-001-016 to Clemson University the following is the final report. Numbers enclosed in brackets refer to the List of References at the end of the report.

The objective of the research sponsored by this NASA Grant was to study the existence, stability and computation of periodic and almost periodic solutions of vector differential equations $\dot{x} = Ax + F(x,t)$ and the analogous functional differential equations. The mathematics used for such systems treats this system as a perturbation of the linear system $\dot{x} = Ax$. Those cases where the construction of almost periodic solutions leads to a trigonometric series with coefficients in which arbitrarily small divisors occur were of particular interest. For convenience we refer to these cases as the "small divisor" problem.

The existence, stability and computation of an almost periodic solution is well known [4] if the perturbation $F(t,x)$ is small enough and either the eigenvalues of A have nonzero real parts or the eigenvalues of a certain Jacobian matrix have nonzero real parts. Other cases were generally intractable except by using the relatively crude methods which are known for the "small divisor" problem. The methods [1] for the small divisor problems are limited to very small perturbations and are necessarily troublesome to execute.

We have discovered during this investigation a very significant generalization of the problem. It is possible in many cases to solve a nonlinear initial value problem

$$\dot{x} = f(t,x), \quad x(\tau) = \gamma$$

for all γ in some region of R^n. It is then possible to study the existence, stability and computation of almost periodic or periodic solutions of perturbed nonlinear systems $\dot{x} = f(t,x) + g(t,x)$. Many of the theorems concerning the perturbed linear system may be proved for the perturbed nonlinear system. One obvious benefit of such theorems is that previously the internal nonlinearities of such systems were treated as noise; and consequently small external perturbations would nullify the existence of the desired periodic or almost periodic solutions. Another benefit is that this technique provides a method to handle many problems previously tractable only by small divisor methods. This approach to perturbed nonlinear systems was discovered independently by L. E. May [5]; however, his analysis requires that the unperturbed system is essentially a linear system. Two papers illustrating the new approach to perturbed systems have been written and are described below, (4), (5). Work is progressing on other aspects of this technique, and new papers will be submitted soon.

The work on "small divisor" problems was somewhat disappointing. One paper was printed as a NASA contractor's report and is described below. In addition we have developed a formal recursive procedure for constructing almost periodic solutions in some situations; however, we have been unable to prove the convergence of the series generated by this procedure. The work which was done to establish the size of allowable perturbations was very complicated and it was found that in general the allowable perturbations were extremely small.

The existence, stability and computation of periodic and almost periodic solutions to functional differential equations is analogous to that for the ordinary differential equation. One paper was submitted concerning the problem; however, much work is left undone. In particular, the generalization to perturbed nonlinear systems mentioned above for ordinary differential equations has not yet been done; however, no particular difficulties are forseen for this case.

The following papers have been submitted for publication as a report of research performed under this grant.

(1) "A family of solutions of certain nonautonomous differential equations by series of exponential functions" by T. G. Proctor and H. H. Suber. <u>NASA Contractor Report NASA CR-1294</u>. This paper presents two theorems concerning the computation of periodic or almost periodic solutions of a differential equation. The first result makes use of a technique developed by Golomb [3] to construct a periodic solution to a nonlinear periodic differential equation, $\dot{x} = f(t,x)$, f odd in t, using a recursive technique to find the coefficients in a

trigonometric series. The technique is also applicable to quasiperiodic equations; however, we were able to guarantee convergence of the trigonometric series only in the periodic case. The coefficients can be calculated using a computer; however, we found the process time-consuming. The second result gives the existence and construction of an almost periodic solution to a differential equation of the form $\dot{x} = f(t,x)$ where f is odd and almost periodic in t and is small. This problem has the distinction of being the easiest small divisor problem in differential equations which has appeared in the literature. The paper is written so that the reader can determine the size of the allowable perturbation function (in this case $f(t,x)$); however, even in this relatively simple problem the resulting calculations are quite time-consuming. The two theorems described above are then applied to the problem of reducing a 2-vector linear differential equation $\dot{y} = P(t)y$ to the easily handled form $\dot{z} = Az$.

(2) "Uniqueness and successive approximations for functional differential equations" by T. G. Proctor. This paper will appear in the Journal of Mathematical Analysis and Applications. The theory of functional differential equations is not as complete as that of ordinary differential equations. This paper was written to provide a basis for studying the existence and computation of solutions of functional differential equations possessing certain properties, for example, periodicity. The first result establishes a differential inequality analogous to the Kamke differential inequalities for ordinary differential equations. The use of such a theorem is illustrated repeatedly in (5) below. The theorem is then applied to the existence and construction of solutions to the initial value problem

$$y(t) = \phi(t), \quad \alpha \leq t \leq t_0,$$
$$\dot{y}(t) = F(t,y(\cdot)), \quad t_0 < t \leq t_0 + a$$

where $\phi(t)$ is a given function and F is a delay functional. In a later paper we plan to use the analysis above to study the existence and construction of periodic solutions to a periodic functional differential equation.

(3) "Characteristic multipliers for some periodic differential equations" by T. G. Proctor. This paper will appear in the Proceedings of the American Mathematical Society. This paper gives two new theorems which give the construction of periodic solutions to nonlinear differential equations. The smallness condition on the perturbation function is replaced by finding an upper and lower trial solution. These theorems are quite useful in constructing examples of systems which

possess periodic solutions in systems where the perturbation is not small. The same construction is valid for almost periodic differential equations; however, the hypothesis is hard to verify in this case. The results are applied to the problem of finding characteristic multipliers for the 2-vector linear differential equation $\dot{x} = P(t)x$, P with period T. A knowledge of the characteristic multipliers and vectors enables one to determine the solutions for all t by calculating the solution values for $0 \leq t \leq T$.

(4) "Periodic solutions for perturbed nonlinear differential equations" by T. G. Proctor. This paper has been submitted to the Proceedings of the American Mathematical Society. Here we generalize the basic results in periodically perturbed linear systems to periodic perturbed nonlinear systems. The results should give new information concerning the motion of nonlinear springs, etc., in the presence of noise. The advantages of such a theory have been pointed out above. The chief difficulty comes in solving the unperturbed nonlinear initial value problem; however, this is known in many applications. The results can also be applied to the determination of characteristic multipliers of periodic 2-vector systems $\dot{x} = P(t)x$ as in (4) above. Here one reduces the problem to a Ricatti equation with periodic coefficients. If these coefficients are nearly constant the problem is easily solved since solutions to Ricatti equations with constant coefficients are well known.

(5) "Integral manifolds for perturbed nonlinear systems" by H. H. Suber. This is Dr. Suber's dissertation and has been approved by Clemson University. Currently Dr. Suber is condensing the results and will submit them for publication soon.

The main theorem in this paper generalizes basic results concerning the existence of integral manifolds for perturbed conditionally stable linear systems of ordinary differential equations [2]. Sufficient conditions are given for the existence of an integral manifold for perturbed nonlinear system, in a neighborhood of a critical point, periodic orbit or periodic surface, in the case where the unperturbed state exhibits conditionally stable behavior. Although the proof of the theorem is constructive in nature it is not expected that the technique will be of great value for computational purposes. However, estimates for allowable size of perturbation are readily available.

One copy of each of the papers above is enclosed as part of this report.

Dr. Suber's final two years were sponsored under this grant during which time he contributed papers (1) and (5) above. Talks covering the papers (1), (2) and (3) were presented to the Mathematical Association of America (Spring 1968) and the American Mathematical Society (Fall 1969). Three one-hour presentations concerning the papers above were given at Langley Research Center during the period of the grant.

The work on this project may be summarized by saying that we have established several techniques for proving the existence and for computing periodic solutions for perturbed linear or nonlinear differential and functional differential equations. One small divisor problem has been treated; however, no real progress has been made on this class of problems.

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List of References

[1] Arnold, V. I. Small divisors and the problem of stability of motion in classical and celestial mechanics, Uspehi Mat. Nauk XVIII 6 (114), 1963: 92-192.

[2] Coddington, E. A. and Levinson, N. Theory of Ordinary Differential Equations. McGraw Hill Book Company, Inc., 1955.

[3] Golomb, M. Solution of certain nonautonomous differential systems by series of exponential functions. Illinois J. Math. 3, 1959:45-65.

(List of References, continued)

[4] Hale, J. K. Oscillations in Nonlinear Systems, New York, McGraw Hill Book Company, Inc., 1963.

[5] May, L. E. Perturbation problems in fully nonlinear systems. Dissertation, North Carolina State University at Raleigh, 1969.

UNIQUENESS AND SUCCESSIVE APPROXIMATIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS*

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1. INTRODUCTION

Differential equations which express $y'(t)$ as a function of past and present values of $y(t)$ have been called delay functional differential equations. Many of the ideas from the theory of ordinary differential equations have been generalized for this type of equation including the basic ideas of existence and uniqueness of solutions of initial value problems.

Let α and t_o be numbers where $-\infty \leq \alpha < \infty$ and $t_o (\geq \alpha)$ is a finite number. In case $\alpha = -\infty$ read $[\alpha, t]$ as $(\alpha, t]$. Suppose that $\phi(t)$ is a prescribed continuous n-vector function on $[\alpha, t_o]$ and we wish to find a continuous function $y(t)$ on some interval $[\alpha, t_o + a]$, $a > 0$, such that

$$\begin{aligned} y(t) &= \phi(t), & \alpha \leq t \leq t_o, \\ y'(t) &= F(t, y(\cdot)), & t_o < t \leq t_o + a, \end{aligned} \tag{1}$$

where $F(t, \psi(\cdot))$ is a function (functional) defined for t in $[t_o, t_o + a]$ and ψ in $C(t)$ and taking values in R^n and where $C(t)$ is all continuous functions from $[\alpha, t]$ into some set D in R^n. It is known [1] - [3] that for appropriate sets D if F is

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continuous and satisfies a Lipschitz condition in ψ, then the initial value problem has a unique solution and this solution may be constructed by successive approximations.

In this paper we give theorems analogous to the Nagumo and Wintner theorems for ordinary differential equations [4] which prove the solution of the initial value problem (1) for other conditions on F is unique and can be constructed by successive approximations.

2. A DIFFERENTIAL INEQUALITY

First we give a lemma on differential inequalities which is analogous to the Kamke differential inequality theorems for ordinary differential equations [4], [5]. Let D be a region in R^n and let F be a function taking values in R^n for $t_o \leq t \leq t_o + a$ and ψ in $C_D(t)$, where $C_D(t)$ is all continuous functions from $[\alpha, t]$ into D, with the properties that $\lim_{n\to\infty} F(t_n, \psi_n(\cdot)) = F(t, \psi(\cdot))$ whenever $t, t_1, \cdots$ are in $[t_o, t_o + a]$, $\psi, \psi_1 \cdots$ belong to $C_D(t_o + a)$ and $t_n \to t$, $\psi_n \to \psi$ (in the sup norm topology). (If $\alpha = -\infty$, $C_D(t)$ is all continuous functions from $(-\infty, t]$ into some compact subset of D.) Further, let (t, ψ_1), (t, ψ_2) in the domain of F and $\psi_1(s) \leq \psi_2(s)$ for $\alpha \leq s \leq t$ imply $F(t, \psi_1(\cdot)) \leq F(t, \psi_2(\cdot))$. Here a vector inequality < means that every component of the left vector is less than the corresponding component of the right vector, etc. Also note that we use the same notation for a function ψ given on an interval $[\alpha, t]$ and its restriction to a subinterval $[\alpha, t'] \subset [\alpha, t]$.

LEMMA 1. Let z be a continuous function from $[\alpha, t_o + a]$ into D and be such that

$$D^+z(t) < F(t, z(\cdot)), \qquad t_o \leq t < t_o + a; \tag{2}$$

and let $y(t)$ be a continuous function with the properties

(a) $y(t) \geq z(t), \qquad \alpha \leq t \leq t_o,$

(b) $y'(t) = F(t, y(\cdot)), \qquad t_o \leq t \leq t_o + a.$

Then $y(t) \geq z(t)$ for $t_o < t \leq t_o + a$. (Here $D^+z(t) = \lim\sup[z(t + h) - z(t)]/h$ as $h \to 0^+$.)

Proof. Suppose $y(t) \geq z(t)$ on some largest interval $[t_o, \delta]$ in $[t_o, t_o + a]$. (Here δ could be t_o.) If for some $k \in \{1, 2, \cdots, n\}$ $z_k(\delta) = y_k(\delta)$ then

$$D^+z_k(\delta) < F_k(\delta, z(\cdot)) \leq F_k(\delta, y(\cdot)) = y_k'(\delta),$$

which implies $z(t) \leq y(t)$ for values of t in a right neighborhood of δ, thus $\delta = t_o + a$.

The monotonic behavior of F is crucial since $y(t) = 2^{3/4}t^{1/2}$ is a solution of $y'(t) = 1/y(t/2)$ on $1 \leq t < 4^{1/7}$ and $z(t) = t^4$ satisfies (1) on $1 \leq t < 4^{1/7}$; however, $z(t) > y(t)$ for $2^{3/14} < t < 4^{1/7}$. Thus, the Kamke differential inequality type theorem requires additional conditions. The continuity condition on F is sufficient to guarantee the existence of at least one solution to (1), see [1], [2]. The differential inequality theorem above then will infer the existence of a right maximal solution to (1) as Coppel [5] does for ordinary differential equations. Then the strict inequality in (2) may be replaced with $\leq$ provided $y(t)$ is the maximal solution of (1), see Coppel [5].

3. A UNIQUENESS THEOREM.

Let ϕ be a given continuous n vector valued function on $[\alpha, t_o]$ and let $C_b(t)$ be the collection of continuous functions $f(s)$ on $[\alpha, t]$ which agree with ϕ on $[\alpha, t_o]$ and which lie in $\bar{S}(\phi(t_o), b)$ for $t_o < s \leq t \leq t_o + a$, ($\bar{S}(y_o, b) = \{y: |y-y_o| \leq b\}$). Let F be a function taking values in R^n for t in $[t_o, t_o + a]$ and ψ in $C_b(t)$ with the property that $\lim_{n\to\infty} F(t_n, \psi_n(\cdot)) = F(t, \psi(\cdot))$ whenever $t, t_1, \cdots,$ are in $[t_o, t_o + a]$, $\psi, \psi_1, \cdots,$ belong to $C_b\ (t_o + a)$ and $t_n \to t$, $\psi_n \to \psi$ (in the sup norm topology).

In addition to the continuity requirement above, we want to impose a restriction on F which is analogous to the Nagumo condition. For this purpose we define a set of linear functionals. Let W be the set of all linear functionals w, of the type

$$w(t, x(\cdot)) = \int_{\alpha}^{t_o+a} x(s)d_s\eta(t, s),$$

defined for $t_o < t \leq t_o + a$, x a continuous function on $[\alpha, t_o + a]$ where $\eta(t, s)$ is as described below and where the functional differential equation

$$\begin{aligned} x(t) &= 0, && \alpha \leq t \leq t_o, \\ x(t)' &= w(t, x(\cdot)), && t_o < t \leq t_o + a, \end{aligned} \tag{3}$$

has a solution $f_w(t)$ on some interval $[\alpha, t_o + \delta_w]$, where $\delta_w > 0$, such that f_w does not vanish on $(t_o, t_o + \delta_w]$ and $\lim_{t\to t_o+} f_w(t)/t - t_o \neq 0$. Here $\eta(t, s)$ is

$$\left.\begin{array}{rl} \text{(i)} & \text{defined, real-valued on } (t_o,\ t_o + a] \times [\alpha,\ t_o + a], \\ \text{(ii)} & \text{constant for } t \le s, \\ \text{(iii)} & \text{nondecreasing in } s, \\ \text{(iv)} & \text{continuous in } t \text{ uniformly with respect to } s. \end{array}\right\} \quad (4)$$

Some examples of such linear functionals w in which $f_w(t) = t^p$, $0 < p \le 1$ are

(a) $w(t,\ x(\cdot)) = \frac{2}{t^2}\int_{t-\varepsilon}^{t} x(s)ds$ where $\alpha = -\varepsilon$, $t_o = 0$, $\delta_w = \varepsilon$.

(b) $w(t,\ x(\cdot)) = \frac{1}{t}[Ax(t) + Bx(t/2)]$ where $\alpha = t_o = 0$, A, B, such that $p = A + B/2^p$, and

(c) $w(t,\ x(\cdot)) = \frac{1}{t^2}\ x(t^2)$ where $\alpha = t_o = 0$.

THEOREM 1. <u>Let</u> F <u>be as specified above and satisfy</u>

$$|F(t,\ \psi_1(\cdot)) - F(t,\ \psi_2(\cdot))| \le w(t, |(\psi_1 - \psi_2)|(\cdot)), \qquad (5)$$

<u>for</u> $(t,\ \psi_1)$, $(t,\ \psi_2)$ <u>in the domain of</u> F, <u>for</u> $t > t_o$ <u>and for some</u> w <u>in</u> W. <u>Then the initial value problem</u> (1) <u>has at most one solution on any interval</u> $[\alpha,\ t_o + \varepsilon]$ <u>for small</u> $\varepsilon > 0$.

Proof. Assume there are two different solutions $y_1(t)$ and $y_2(t)$ on some interval $[\alpha,\ t_o + \varepsilon]$. By shrinking ε we may suppose that for $y(t) = y_1(t) - y_2(t)$, $|y(t_o + \varepsilon)| \neq 0$ and $\varepsilon \le \delta_w$. Consider the solution $x(t) = 0$, $\alpha \le t \le t_o$ $x(t) = \frac{1}{2}|y(t_o + \varepsilon)|f_w(t)/|f_w(t_o + \varepsilon)|$, $t_o < t \le t_o + \varepsilon$ of the initial value problem (3). By (5) we have

$$D^+|y(t)| \le w(t, |y(\cdot)|), \qquad t_o < t \le t_o + \varepsilon.$$

For any $0 < \sigma < \varepsilon$ the initial value problem

$$z(t) = \begin{cases} 0, & \alpha \leq t \leq t_o, \\ x(t), & t_o \leq t \leq t_o + \sigma, \end{cases}$$

$$z'(t) = w(t, z(t)), \quad t_o + \sigma < t \leq t_o + \varepsilon,$$

has a unique solution, namely $x(t)$, since $w(t, x(\cdot))$ satisfies a Lipschitz condition in x for t in $[t_o + \sigma, t_o + \varepsilon]$. The remarks following Lemma 1 then imply that we cannot have $x(t) > |y(t)|$ on a right neighborhood of t_o. Hence, there is a decreasing sequence of points $t_k \to t_o$ such that

$$|y(t_k)| \geq x(t_k), \qquad k = 1, 2, \cdots.$$

Thus $x(t_k)/(t_k - t_o) \to 0$ but this is impossible since $\lim f_w(t)/t - t_o \neq 0$ as $t \to t_o^+$. Hence $y(t_o + \varepsilon) = 0$.

Remark 1. The proof of the theorem makes use of the linearity of the functional w and the form of the solutions of (3). It is not readily apparent how one would generalize Kamke's uniqueness theorem [4] since if $w(t, x(\cdot))$ is a nonlinear functional, a solution of (1) passing through $\frac{1}{2}|y(t_o + \varepsilon)|$ may not exist.

Remark 2. It is possible to relax the requirements on the integrator function $\eta(t, s)$ in the case of nth order scalar functional differential equations as shown by Theorem 3 below.

4. SUCCESSIVE APPROXIMATIONS.

Next we wish to show that the sequence of successive approximations given by

$$\left.\begin{aligned} y_n(t) &= \phi(t), \qquad \alpha \le t \le t_o, \qquad n = 1, 2, \cdots, \\ y_1(t) &= \phi(t_o), \\ y_n(t) &= \phi(t_o) + \int_{t_o}^{t} F(s, y_{n-1}(\cdot))ds, \end{aligned}\right\} \begin{array}{c} t_o < t \le t_o + \beta, \\ n = 1, 2, \cdots, \end{array} \tag{6}$$

converges for t in $[\alpha, t_o + \beta]$ to a solution of (1) where $\beta = \min\{a, \frac{b}{M}, \delta_w\}$ and where M is such that $|F(t, \psi(\cdot))| \le M$ when $t_o \le t \le t_o + \min\{a, \delta_w\}$ and ψ is in $C_b(t_o + \min\{a, \delta_w\})$. It is easy to see that (6) defines a sequence.

THEOREM 2. *Let F be as specified above and satisfy*

$$|F(t, \psi_1(\cdot)) - F(t, \psi_2(\cdot))| \le w(t, |\psi_1 - \psi_2|(\cdot)),$$

for (t, ψ_1), (t, ψ_2) *in the domain of* F *and* $t > t_o$, *for some* w *in* W. *Then the sequence* (5) *converges to the solution of* (1) *on* $[t_o, t_o + \beta]$.

Proof. The sequence (6) is uniformly bounded and uniformly equicontinuous on $[t_o, t_o + \beta]$ so by Ascoli's theorem there is a subsequence $\{y_{n_k}(t)\}_{k=1}^{\infty}$ converging to some function y(t) uniformly on $[t_o, t_o + \beta]$. Hence the sequence $\{y_{n_k+1}(t)\}_{k=1}^{\infty}$ converges uniformly to some function $y^*(t)$ on $[t_o, t_o + \beta]$. If we can show $\lambda(t) \equiv 0$, where

$$\lambda(t) = \limsup_{n \to \infty} |\omega_n(t)|, \quad \omega_n(t) = y_n(t) - y_{n-1}(t),$$

then $y = y^* =$ the unique solution of (1) and thus the whole sequence $\{y_n(t)\}_{n=1}^{\infty}$ converges to $y(t)$ uniformly on $[t_o, t_o + \beta]$ (see Hartman [4], pp. 4 and 41). The remainder of the proof amounts to first showing $D^+\lambda(t) \leq w(t, \lambda(t))$. Then if we suppose $\lambda(t) \not\equiv 0$ we may repeat the strategy in Theorem 1 to get a contradiction. The details in the case of ordinary differential equations are given in [6] and [7].

5. A UNIQUENESS THEOREM FOR SCALAR PROBLEMS.

Let ϕ be a real-valued function with n-1 continuous derivatives defined on $[\alpha, t_o]$; and for t in $[t_o, t_o + a]$ let $C_b(t)$ be the set of all real-valued functions f with n-1 continuous derivatives defined on $[\alpha, t]$ which agree with ϕ on $[\alpha, t_o]$ and satisfy

$$|f^{(k)}(t) - \phi^{(k)}(t_o)| \leq b, \qquad k = 0, 1, \cdots, n-1,$$

t in $(t_o, t_o + a]$, ($f^{(k)}$ is the <u>kth</u> derivative of f). Let $U(t, x(\cdot))$ be real-valued for t in $[t_o, t_o + a]$ and x in $C_b(t)$ and be such that $\lim_{n\to\infty} U(t_n, \psi_n(\cdot)) = U(t, \psi(\cdot))$ whenever $t, t_1, t_2, \cdots$ is in $[t_o, t_o+a]$, $\psi, \psi_1, \cdots$ belong to $C_b(t_o + a)$ and $t_n \to t$, $\psi_n \to \psi$ (in the sup norm topology).

An <u>nth</u> order initial value problem for delay functional differential equations is to find a function y in $C_b(t_o + a)$ such that

$$y^{(n)}(t) = U(t, y(\cdot)), \quad t_o < t \leq t_o + a. \tag{7}$$

Of course this type of problem is a special case of the initial value problem (1); however in this form it is possible to give new hypothesis on U so that (7) has at most one solution on any interval $[t_o, t_o + \varepsilon]$ for small $\varepsilon > 0$. The following theorem generalizes a result due to Wintner [8] which is given in expanded form in Hartman [4], page 34.

THEOREM 3. *Let* $\eta_k(t, s)$, $k = 0, 1, 2, \cdots, n-1$ *satisfy conditions (i) - (iii) of* (4) *and let* $U(t, x(\cdot))$ *satisfy*

$$|U(t, x(\cdot)) - U(t, y(\cdot))| \leq \sum_{k=o}^{n-1} \lambda_k(t) \int_{t_o}^{t_o+a} |x^{(k)}(s)-y^{(k)}(s)|d_s\eta_k(t,s)$$

for t *in* $(t_o, t_o + a]$, *where the* $\eta_k(t)$ *are non-negative functions such that*

$$\sum_{k=o}^{n-1} \lambda_k(t) \int_{t_o}^{t_o+a} \frac{(s-t_o)^{n-k}}{(n-k)!} \, d_s\eta_k(t, s) \leq 1, \quad t_o < t \leq t_o + a.$$

Then there is at most one solution to (7) *on any interval* $[\alpha_o, t_o + \varepsilon]$ *for small* $\varepsilon > 0$.

The proof is an easy extension of the proof given in Hartman [4], page 558.

REFERENCES

1. R. D. Driver. Existence and stability of solutions of a delay-differential system. Archives for Rational Mechanics and Analysis 10, 410-426 (1962).

2. R. D. Driver. Existence theory for a delay-differential system. Contributions to Differential Equations 1, 317-336 (1963).

3. R. D. Driver. Existence and continuous dependence of solutions of a neutral functional-differential equation. Archives for Rational Mechanics and Analysis 19, 149-166 (1965).

4. Phillip Hartmen. Ordinary Differential Equations. New York, John Wiley and Sons, 1964.

5. W. A. Coppel. Stability and Asymptotic Behavior of Differential Equations. Boston, D. C. Heath and Company, 1965.

6. E. A. Coddington, and N. Levinson. Theory of Ordinary Differential Equations. New York, McGraw-Hill Book Company, 1955.

7. F. Brauer, and S. Sternberg. Local uniqueness, existence in the large, and the convergence of Successive approximations. American Journal of Mathematics 80, 421-430 (1958).

8. A. Wintner. On the local uniqueness of the initial value problem of the differential equation $d^nx/dt^n = f(t,x)$. Boll. Un. Mat. Ital. 11, 496-498 (1956).

CHARACTERISTIC MULTIPLIERS FOR SOME PERIODIC DIFFERENTIAL EQUATIONS

T. G. Proctor*

1. Introduction. Let $P(t)$ be a 2×2 matrix with elements which are continuous real valued functions of period T and consider the differential equation

$$\dot{x} = P(t)x, \tag{1}$$

where x is a vector with two components. It is well known [7] that there are numbers λ_1 and λ_2, called characteristic multipliers, and corresponding solutions $x_1(t)$, $x_2(t)$ of (1), called normal solutions, which satisfy for $i = 1, 2$,

$$x_i(t + T) = \lambda_i x_i(t), \ -\infty < t < \infty, \quad \lambda_1\lambda_2 = \exp \int_0^T \text{trace } P(t)dt.$$

If $\lambda_1 \neq \lambda_2$ and in some cases when $\lambda_1 = \lambda_2$ any such normal solutions $x_1(t)$, $x_2(t)$ are independent. In this case a knowledge of the characteristic multipliers and the values of $x_1(t)$, $x_2(t)$ for $0 \leq t \leq T$ gives information for every solution of (1) for all t. It is clear that corresponding statements can be made for the second order equation

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0,$$

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with continuous periodic coefficient functions $p(t)$, $q(t)$ since this results from the case

$$x = \begin{vmatrix} y \\ \frac{dy}{dt} \end{vmatrix}, \qquad P(t) = \begin{vmatrix} 0 & 1 \\ -q(t) & -p(t) \end{vmatrix}.$$

Calculation of the characteristic multipliers is not routine since in general one does not know even one non-trivial solution of (1). However it is possible to obtain convergent series representations for the solutions and thus calculate approximate values for the multipliers [2], [8].

An alternative procedure for obtaining the characteristic multipliers and the corresponding normal solutions for (1) is possible whenever an associated Riccati differential equation has a periodic solution. If we make the change of coordinates $x^1 = z^1 + y(t)z^2$, $x^2 = z^2$ in (1) where $y(t)$ is a solution of the Riccati equation

$$\begin{gathered} \frac{dy}{dt} = a(t) + b(t)y + c(t)y^2, \\ a(t) = p_{12}(t), \quad b(t) = p_{11}(t) - p_{22}(t), \quad c(t) = -p_{21}(t), \end{gathered} \tag{2}$$

the differential equation in z can be integrated. This gives

THEOREM 1. (a) If $x(t) = \text{column } (x^1(t), x^2(t))$ is a solution of (1) then $y(t) = x^1(t)/x^2(t)$ is a solution of (2) on any interval on which $x^2(t)$ does not vanish. (b) If $y(t)$ is a solution of (2) on an interval I containing the number k then

$$
(3)\quad
\begin{aligned}
x^1(t) &= y(t)\,\exp \int_k^t [p_{21}(s)y(s) + p_{22}(s)]ds \\
x^2(t) &= \exp \int_k^t [p_{21}(s)y(s) + p_{22}(s)]ds
\end{aligned}
$$

is a solution of (1) on I. (c) If $y(t)$ is a solution of (2) with period nT and f is the mean value of $p_{21}(t)y(t) + p_{22}(t)$ over the period nT, where n is a positive integer, then e^{nTf} is a characteristic multiplier for (1) for the period nT and (3) is a normal solution of (1) corresponding to this multiplier.

In sections 2 and 3 of this paper we will prove two theorems with rather restrictive hypotheses which give the existence of a periodic solution of (2). The theorem in section 2 can be viewed as a special case of the theorem in section 3. We also mention other known techniques for constructing periodic solutions to the Riccati differential equation.

Similar analysis for the differential equation (1) where $P(t)$ is $n \times n$ and x is an n vector leads to the study of a matrix Riccati differential equation and the analysis is more difficult. The technique of using a Riccati differential equation has been used by Gelmand [3] and Andrianov [1] for the case of quasiperiodic coefficients $p_{ij}(t)$.

2. $b(t)$ has nonzero mean value. Let H be the set of all continuous real valued functions with period T, let $b \in H$, let $B(t) = \int_0^t b(s)ds$ and in this section we will assume $B(T) \neq 0$. Also suppose $u, \ell \in H$ and satisfy $\ell(t) \leq u(t)$ for

all t and let K be the subset of H consisting of functions f where $\ell(t) \leq f(t) \leq u(t)$. Further assume $q(t, z)$ is a continuous real valued function defined for $\ell(t) \leq z \leq u(t)$ such that for any fixed z, $q \in H$ and define $\mathcal{J}: K \to H$ by

$$\mathcal{J}h(t) = \frac{1}{e^{-B(T)}-1} \int_t^{t+T} q(s, h(s)) \exp \int_s^t b(v)dv\, ds.$$

THEOREM 2. If (1) $\mathcal{J}u(t) \leq u(t)$, $\ell(t) \leq \mathcal{J}\ell(t)$ for all t and (2) $e^{-B(T)} - 1 > 0$ and $z \leq w$ implies $q(t, z) \leq q(t, w)$ or (3) $e^{-B(T)} - 1 < 0$ and $z \leq w$ implies $q(t, z) \geq q(t, w)$, then the sequences $\{\mathcal{J}^m u(t)\}_{m=1}^{\infty}$ and $\{\mathcal{J}^m \ell(t)\}_{m=1}^{\infty}$ converge uniformly to periodic solutions of

$$(4) \qquad \frac{dz}{dt} = b(t)z + q(t, z).$$

Proof. The hypothesis (1) and (2) or (1) and (3) imply that if $h, k \in K$ and $h(t) \leq k(t)$ for all t then $\mathcal{J}h(t) \leq \mathcal{J}k(t)$ for all t. Thus the sequences $\{\mathcal{J}^m \ell(t)\}_{m=0}^{\infty}$ and $\{\mathcal{J}^m u(t)\}_{m=0}^{\infty}$ are nondecreasing and nonincreasing respectively, uniformly bounded and equicontinuous.

Remarks. If $z(t)$ is a periodic solution of (4) where $q(t, z) = c(t)(w(t) + z)^2$,

$$w(t) = \frac{1}{e^{-B(T)}-1} \int_t^{t+T} a(s) \exp \int_s^t b(v)dv\, ds$$

we have that $y(t) = w(t) + z(t)$ is a periodic solution of (2). If $g(t)$ is a periodic solution of (4) where $q(t, y) = a(t) + c(t)y^2$ then $y(t)$ is a periodic solution of (2).

Example. If we regard the right member of $\dot{y} = (y - 1) \times (y - 2)$ as having period 2π the corresponding transformation $\mathcal{J}$ is given by

$$\mathcal{J}h(t) = \frac{e^{-3t}}{e^{6\pi}-1}\int_t^{t+2\pi} e^{3s}[h^2(s) + 2]ds.$$

We note $u(t) = 5/4$, $\ell(t) = 0$ satisfy the conditions of the theorem and $\mathcal{J}^m\ell(t)$, $\mathcal{J}^m u(t)$ converge to 1 as $m \to \infty$.

Several theorems on periodic solutions of (2) are given in Hale [6, pp. 28-31] which result from the contraction mapping theorem and successive approximations. Using Theorem 5.1 [6] a periodic solution exists if (1) $|c(t)|$ is small enough $(q(t, z) = c(t)(w(t) + z)^2$, $y = w(t) + z(t))$ or (2) if $|a(t)|$ is small enough $(q(t, y) = a(t) + c(t)y^2)$. We state without proof an obvious modification (for the scalar case) of Theorem 5.2 [6] because of its relation to (1). Let $h \in H$ and have mean value zero. Let r and σ be positive numbers and let $q(t, y, \mu)$ be a continuous real valued function for all t, $-r \leq y \leq r$ and $|\mu| \leq \sigma$. Further let q be periodic for fixed (y, μ) and satisfy for $|y|$, $|z| \leq \rho$, $|\mu| \leq \sigma$

$$|q(t, y, \mu) - q(t, z, \mu)| \leq \eta(\mu, \rho)|y - z|, \quad q(t, 0, 0) = 0,$$

where $\eta(\mu, \rho)$ is continuous and nondecreasing in μ and in ρ for $|\mu| < \sigma$, $0 \leq \rho \leq r$.

THEOREM 3. There is a δ, $0 < \delta < \sigma$, so that for $|\mu| < \delta$

$$\frac{dy}{dt} = \mu[b(t) + q(t, y, \mu) + h(t)]$$

has a periodic solution which is the limit of successive approximations.

COROLLARY. Let $p(t, \mu)$, $c(t, \mu)$ be continuous real valued functions defined on $R \times [0, \sigma]$ which satisfy $p(t, 0) = c(t, 0) = 0$, $p, c \in H$ for fixed μ, let $q, r, h \in H$, h with zero mean value, r with nonzero mean value and let

$$P(t) = P(t, \mu) = \mu \begin{pmatrix} q(t) & p(t, \mu) + h(t) \\ -c(t, \mu) & q(t) - r(t) \end{pmatrix} .$$

Then for μ sufficiently small there is a periodic solution $y(t)$ of (2) and e^{Tf} is a characteristic multiplier for (1) where f is the mean value of $p_{11}(t)y(t) + p_{22}(t)$.

By replacing t by t/μ we see that an alternative statement for this corollary is that there is a number λ and a solution of

$$\dot{x} = \begin{pmatrix} q(t/\mu) & p(t/\mu, \mu) + h(t/\mu) \\ -c(t/\mu, \mu) & q(t/\mu) - r(t/\mu) \end{pmatrix} x$$

satisfying $x(t + \mu T) = \lambda x(t)$ when μ is small enough.

Wasow [9] and Golomb [4] have developed a recursive scheme for constructing periodic solutions for the quasi-linear differential equation $y = b(t)y + q(t, y, \mu)$ when $b(t)$ has nonzero mean value and μ is small. The resulting theorems provide another technique for obtaining periodic solutions of (2) and thus obtaining characteristic multipliers for (1). Golomb [5] also shows that in certain cases this recursive scheme leads directly to a calculation of the characteristic multipliers and corresponding normal solutions.

3. General Case. Let H' be any closed subspace of H in the uniform topology and for any $f \in H$ define $Mf = \frac{1}{T}\int_0^T f(t)dt$. Let ℓ, u be in H' with $\ell(t) \leq u(t)$ for all t and let K be all functions in H' lying between ℓ and u. Further suppose $q(t, y, \mu)$ is defined and continuous for $\ell(t) \leq y \leq u(t)$ $|\mu| \leq \sigma$ for some $\sigma > 0$ and has period T for fixed (y, μ). For any number a satisfying $\ell(0) \leq a \leq u(0)$ and $|\mu| \leq \sigma$ define $\mathcal{J}_{a,\mu}$: $K \to H$ by

$$\mathcal{J}_{a,\mu}f(t) = a + \int_0^t [q(s, f(s), \mu) - M\{q(t, f(t), \mu)\}]ds.$$

THEOREM 4. For fixed (a, μ) if $\mathcal{J}_a(K) \subset H'$, $\mathcal{J}_{a,\mu}u(t) \leq u(t)$, $\ell(t) \leq \mathcal{J}_{a,\mu}\ell(t)$ for all t and if $y, z \in K$, $y(t) \leq z(t)$ implies $\mathcal{J}_{a,\mu}y(t) \leq \mathcal{J}_{a,\mu}z(t)$ for all t then the sequences $\{\mathcal{J}^m_{a,\mu}\ell(t)\}_{m=0}^{\infty}$, $\{\mathcal{J}^m_{a,\mu}u(t)\}_{m=0}^{\infty}$ converge uniformly to functions $\ell^*(t, a, \mu)$, $u^*(t, a, \mu)$ satisfying

$$\frac{dy}{dt}(t, a, \mu) = q(t, y(t, a, \mu), \mu) - M\{q(t, y(t,a,\mu), \mu)\}.$$

The proof is identical to that of Theorem 2.

COROLLARY. If $\ell^*(t, a, \mu)$ exists and $M\{q(t,\ell^*(t,a,\mu), \mu)\} = 0$ then ℓ^* is a periodic solution of

$$\frac{dy}{dt} = q(t, y, \mu).$$

An analogous statement holds for $u^*(t, a, \mu)$.

As an example of this theorem and corollary consider $q(t, y, \mu) = y \sin t$, $a = 1$, and H' the set of all even functions in H, $\ell(t) \equiv 1$ and $u(t) = e^{1-\cos t}$.

Hale [6, pp. 38-44] gives several theorems for periodic solutions of $\frac{dy}{dt} = b(t)y + q(t, y, \mu)$ in the case where $b(t)$ has zero mean value and q is small for small μ. These theorems again use successive approximation and the contraction mapping theorem. In the application of these theorems to the differential equations the symmetry properties of $q(t, y, \mu)$ play an important role in the analysis to determine if the bifurcation equations $M_q(t, y(t)) = 0$ have a solution. One of the simplest cases arises when $b(t) = 0$ and $q(t, y, \mu)$ is odd in t. In this particular case the recursive method developed by Wasow and Golomb also gives a convergent series expansion for the solution for small μ.

4. Final Remarks. Note that if T is the least period of $P(t)$ in (1) then the coefficient functions a, b, c in (2) have least period 0 or T/n for some positive integer n. By applying the construction procedures used in Theorems 2, 3, or 4, we construct solutions of (2) with period T/n or 0. Then by replacing nT by T in Theorem 1, we obtain a characteristic multiplier for the period T.

For a given matrix $Q(t)$ we notice that all matrices $P(t) = Q(t) + K(t)$ have the same associated Ricatti differential equation (2) if $K(t)$ has the form

$$K(t) = \begin{pmatrix} p(t) & 0 \\ 0 & p(t) \end{pmatrix} .$$

If for some such $K(t)$ equation (1) can be solved, Theorem 1 may be used to solve equation (2). This in turn furnishes a method to solve $\dot{x} = Q(t)x$. This remark applies even if none of the functions in $Q(t)$ and $K(t)$ are periodic. However in the case of periodic $P(t)$ suppose $\dot{x} = P(t)x$ has a normal solution, column $(x^1(t), x^2(t))$ where $x^2(t)$ does not vanish. If λ is the corresponding characteristic multiplier, $y(t) = x^1(t)/x^2(t)$ and $g = Mp(t)$, then λe^{Tg} is a characteristic multiplier for $\dot{x} = Q(t)x$ and a corresponding normal solution is given by (3) where p_{22} is replaced by $p_{22} - p$.

Since the proof of Theorem 1a and b does not require the periodicity of $P(t)$, theorems which imply the existence of bounded or almost periodic solutions of (2) or the asymptotic form of certain solutions of (2) also give information concerning the solutions of (1).

REFERENCES

1. L. Ya. Adrianova, The reducibility of systems of n linear differential equations with quasiperiodic coefficients, Vestnik, Leningrad Univ. 17 (1962), 14-26

2. N. P. Erugin, Linear Systems of Ordinary Differential Equations, Academic Press Inc., New York, 1966.

3. A. E. Gelmand, On the reducibility of a class of systems of differential equations with quasiperiodic coefficients, Dokl. Akad. Nauk SSSR 116 (1957), 535-537.

4. M. Golomb, Solution of certain nonautonomous differential systems by series of exponential functions, Illinois J. Math. 2 (1959), 45-65.

5. M. Golomb, Expansion and boundedness theorems for solutions of linear differential systems with periodic or almost periodic coefficients, Archive for Rational Mechanics and Analysis, 2 (1958), 283-308.

6. J. K. Hale, Oscillations in Nonlinear Systems, McGraw-Hill Book Company, New York, 1963.

7. P. Hartman, Ordinary Differential Equations, John Wiley and Sons, Inc., New York, 1964.

8. I. G. Malkin, Theory of stability of motion, AEC Translation 3352, U. S. Department of Commerce, Washington, 1952.

9. W. Wasow, Solution of certain nonlinear differential equations by series of exponential functions, Illinois J. Math. 3 (1959), 45-65.

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T. G. Proctor, Periodic solutions for perturbed nonlinear differential equations.

Abstract: The existence of periodic solutions of a periodically perturbed system of nonlinear differential equations is established. The construction of such solutions is proved in a more restricted situation. These results generalize well known results for perturbed linear differential equations. Examples are given.

PERIODIC SOLUTIONS FOR PERTURBED NONLINEAR DIFFERENTIAL EQUATIONS

T. G. Proctor*

1. Introduction. In this paper we investigate the existence and construction of periodic solutions of a periodically perturbed system of nonlinear differential equations. The perturbed system is studied using an integral equation introduced by Alekseev [1], [2] which is a generalization of the variation of constants integral equation. The techniques used are analogous to those used in establishing the existence of periodic solutions in perturbed linear systems [3]. Almost periodic perturbations of nonlinear systems have been studied by May [4] using a similar technique; however, our systems will not necessarily meet his requirements.

2. Existence of periodic solutions. Let P, δ, σ be positive numbers with $\delta < \sigma$, let S_δ and S_σ be the closed spheres of radii δ and σ respectively in R^n and let $f(t,x)$ be a C^1 function from $R \times S_\sigma$ into R^n with period P in t. We make the following assumptions concerning the function f.

(i) γ in S_δ implies the solution $x(t,\tau,\gamma)$ of the unperturbed initial value problem

$$\dot{x} = f(t,x), \qquad x(\tau) = \gamma$$

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exists for $0 \leq t-\tau \leq P$.

(ii) There is a set $K \subset S_\delta$ so the function F given by

$$F(\gamma) = \gamma - x(P,0,\gamma)$$

is a homeomorphism of K onto F(K).

(iii) $f(t,0) = 0, \qquad 0 \leq t \leq P$.

We denote the continuous functions with period P from R into S_δ by $\mathcal{S}_\delta$. Let the perturbation function $g(t,x,\varepsilon)$ be a continuous function from $R \times S_\delta \times [0,\varepsilon_0]$ into R^n and satisfy

(iv) g has period P in t, $\qquad g(t,x,0) \equiv 0$.

(v) For k in $\mathcal{S}_\delta$ and

$$\gamma(k,\varepsilon) = \int_0^P \frac{\partial x}{\partial \gamma}(P,s,k(s))g(s,k(s),\varepsilon)ds,$$

we assume $\gamma(k,\varepsilon)$ is in F(K) for $0 \leq \varepsilon \leq \varepsilon_0$.

Let $\mathcal{S}$ be the Banach space of all continuous functions with period P from R into R^n with the supremum norm and let T be an operator from $\mathcal{S}_\delta$ into $\mathcal{S}$ defined by

$$Tk(t) = x(t,0,F^{-1}(\gamma(k,\varepsilon)) + \int_0^t \frac{\partial x}{\partial \gamma}(t,s,k(s))g(s,k(s),\varepsilon)ds$$

for $0 \leq t \leq P, \quad 0 \leq \varepsilon \leq \varepsilon_0$.

THEOREM 1. For ε small enough T has a fixed point $y(t)$. Further $y(t)$ is a periodic solution of the perturbed differential equation

$$\dot{y} = f(t,y) + g(t,y,\varepsilon).$$

Proof. The hypothesis on f implies $x(t,\tau,\gamma)$ and $\frac{\partial x}{\partial \gamma}(t,\tau,\gamma)$ are continuous for γ in S_δ and $0 \le t-\tau \le P$. Conditions (iii) and (iv) imply

$$\lim_{\varepsilon \to 0} x(t,0,F^{-1}(\gamma(k,\varepsilon))) = 0$$

uniformly for $0 \le t \le P$ and k in S_δ so choose ε_1 so that $TS_\delta \subseteq S_\delta$ for $0 \le \varepsilon \le \varepsilon_1$. It is easily checked that T is continuous, S_δ is closed and convex and TS_δ has a compact closure in S_δ; therefore by the Schauder fixed point theorem there is a function $y(t) = Ty(t)$. This representation shows $y(t)$ has a derivative which is given by

$$\dot{y}(t) = f(t,x(t)) + g(t,y(t),\varepsilon) + \int_0^t f_x(t,x(t,s,y(s)))\frac{\partial x}{\partial \gamma}(t,s,y(s))g(s,y(s),\varepsilon)ds$$

where $x(t) = x(t,0,F^{-1}(\gamma(y,\varepsilon)))$. Also we have

$$f(t,y(t)-f(t,x(t)) =$$

$$= \int_0^t \frac{d}{ds} f(t,x(t,s,y(s)))ds$$

$$= \int_0^t f_x(t,x(t,s,y(s))\frac{\partial x}{\partial \gamma}(t,s,y(s))[\dot{y}(s)-f(s,y(s))]ds.$$

Therefore for

$$W(t) = \dot{y}(t) - g(t,y(t))-f(t,y(t)),$$

we have

$$W(t) = -\int_0^t f_x(t,x(t,s,y(s)))\frac{\partial x}{\partial \gamma}(t,s,y(s))W(s)ds$$

The only solution of this equation on $[0,P]$ is $W(t) \equiv 0$.

Example 1. The initial value problem $\dot{x} = x^2$, $x(0) = \gamma$ has solution

$$x(t,\tau,\gamma) = \frac{\gamma}{1-\gamma(t-\tau)}, \quad -\frac{1}{2P} \leq \gamma \leq \frac{1}{2P} \qquad 0 \leq t-\tau \leq P.$$

For $K = [-\frac{1}{2P},0]$ the function

$$F(\gamma) = \gamma - \frac{\gamma}{1-\gamma P} \qquad \gamma \text{ in } K$$

is continuous and has continuous inverse

$$F^{-1}(\alpha) = \frac{\alpha-\sqrt{\alpha^2-4\alpha/P}}{2}$$

for $-1/6P \le \alpha \le 0$. The requirement (v) on g can be written as k in $S_{\frac{1}{2P}}$ implies

$$-1/6P \le \int_0^P \frac{1}{[1-k(s)(P-s)]^2}\, g(s,k(s),\varepsilon)ds < 0.$$

Theorem 1 establishes the existence of a periodic solution to a perturbed nonlinear differential equation; however, no construction is given for such a solution. In many cases it seems unrealistic to suppose that T is a contraction operator since F^{-1} may not be Lipshitz as in the example. The following section does provide a method to construct periodic solutions in a special situation.

3. Construction of periodic solutions. Let Ω be a region in R^n, and let $f(t,x)$ be a C' function from $R\times\Omega$ into R^n with period $P > 0$ in t and let $\ell(t)$, $u(t)$ be continuous functions from R into Ω with period T with $\ell(t) \le u(t)$, $0 \le t \le P$ where a vector inequality $\ell \le u$ means the components ℓ_i, u_i of the vectors ℓ, u satisfy $\ell_i \le u_i$, $i = 1,2,\dots,n$. Let

$$S = \{x \text{ in } R^n:\ \ \ell(t)\le x\le u(t) \text{ for some } t\}$$

then for γ in S we assume

(i') the solution $x(t,\tau,\gamma)$ of the initial value problem

$$\dot{x} = f(t,x), \qquad x(\tau) = \gamma$$

exists for $0 \leq t-\tau \leq P$, τ any real number; and

(ii') the function F given by

$$F(\gamma) = \gamma - x(P,0,\gamma)$$

is a homeomorphism of S onto $F(S)$.

Let $g(t,x)$ be a continuous function from $R\times S$ into R^n with period P in t and let $\mathcal{S}^*$ be the set of continuous functions $k(t)$ from R into R^n with period P satisfying $\ell(t) \leq k(t) \leq u(t)$ for all t. For k in $\mathcal{S}^*$ and

$$\gamma(k) = \int_0^P \frac{\partial x}{\partial \gamma}(P,s,k(s))g(s,k(s))ds$$

we assume $\gamma(k)$ is in $F(S)$. And we define an operator T on $\mathcal{S}^*$ into $\mathcal{S}$ by

$$Tk(t) = x(t,0,F^{-1}(\gamma(k))) + \int_0^t \frac{\partial x}{\partial \gamma}(t,s,k(s))g(s,k(s))ds$$

for $0 \leq t \leq P$.

THEOREM 2. If $\ell(t) \leq T\ell(t)$, $Tu(t) \leq u(t)$ for $0 \leq t \leq P$ and if k, h in S^* with $k(t) \leq h(t)$, $0 \leq t \leq P$ implies $Tk(t) \leq Th(t)$, $0 \leq t \leq P$, the sequences $\{T^m\ell\}_{m=0}^{\infty}$, $T^m u \;_{m=0}$ converge uniformly to fixed points of T. If y is a fixed point of T then $y(t)$ satisfies

$$\dot{y} = f(t,y) + g(t,y).$$

Proof. The sequences $\{T^m\ell(t)\}_{m=0}^{\infty}$ and $\{T^m u(t)\}_{m=0}^{\infty}$ are nondecreasing and nonincreasing respectively, uniformly bounded and equicontinuous. Hence they converge uniformly to limit functions which are fixed points of T. The proof that such a fixed point is a solution of the perturbed differential equation is identical to that given for Theorem 1.

Example 2. The initial value problem $\dot{x} = x(1-x)$, $x(0) = \gamma$ has solution

$$x(t,\tau,\gamma) = \frac{\gamma e^{t-\tau}}{1+\gamma(e^{t-\tau}-1)}$$

for $\gamma > 0$, $0 \leq t-\tau \leq P$. For $\gamma^* = (e^{P/2}-1)/(e^P-1)$

the function

$$F(\gamma) = \gamma - \frac{e^P}{1+\gamma(e^P-1)}, \qquad \gamma^* \leq \gamma \leq 1,$$

has continuous inverse

$$F^{-1}(\alpha) = \frac{(e^P-1)(1+\alpha)+\sqrt{(e^P-1)^2(1+\alpha)^2+4\alpha(e^P-1)}}{2(e^P-1)},$$

$F(\gamma^*) \leq \alpha \leq 0$. Let $h(t)$ be a nonnegative function of period P such that

$$\int_0^P h(S)ds \leq \gamma^*(e^{P/2}-1),$$

let

$$g(t,x) = \frac{-h(t)}{1+x^2},$$

and let $\ell(t) = 0$, $u(t) = 1$, $0 \leq t \leq P$. It is easy to verify the conditions of the theorem if P is small enough.

4. Final remarks. The hypothesis ii and ii' of Theorems 1 and 3 respectively is analogous to the noncriticality requirement [3] made for unperturbed linear systems. The existence of periodic solutions to perturbed nonlinear systems corresponding to the critical case can be treated using the methods above. However, if one imposes hypotheses similar to those used in Theorem 1 the question of a solution to the resulting bifurcation equations is not easily established. This follows since we do not know the dependence of the fixed point $y(t)$ as a function of ε. Hypotheses similar to those used in Theorem 2 are extremely hard to verify since this requires that the difference between a function $g(t,x)$ and its mean value be monotone over a class of periodic functions $x(t)$.

REFERENCES

[1] V. M. Alekseev. An estimate for the perturbation of the solutions of ordinary differential equations I (Russian). Vestnik Moskov Univ. Ser. I. Mat. Meh. No. 2 (1961), 28-36.

[2] V. M. Alekseev. An estimate for the perturbation of the solutions of ordinary differential equations II (Russian). Vestnik Moskov Univ. Ser. I. Mat. Meh. No. 3 (1961), 3-10.

[3] J. K. Hale. Oscillations in Nonlinear Systems. McGraw-Hill Book Company, Inc., New York (1963).

[4] L. E. May. Perturbation problems in fully nonlinear systems. Dissertation, North Carolina State University, Raleigh, N. C., 1969.

Footnotes

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A FAMILY OF SOLUTIONS OF CERTAIN NONAUTONOMOUS DIFFERENTIAL EQUATIONS BY SERIES OF EXPONENTIAL FUNCTIONS

by T. G. Proctor and H. H. Suber

1. INTRODUCTION

We consider in this paper the construction of solutions for certain nonautonomous differential equations. The first result makes use of a technique developed by Golomb [5] and Wasow [10] for constructing solutions of some nonlinear differential equations by means of series of exponential functions. The technique as employed here gives explicit formulae for a family of periodic solutions of a Ricatti equation with odd periodic coefficient and finite Fourier series expansion.

Following this is a theorem concerning the existence of a family of almost periodic solutions of the vector differential equation

$$\dot{y} = g(t, y).$$

Here y is an m-vector; g(t, y) is quasi-periodic and odd in t and satisfies certain other conditions. (A quasi-periodic function is a function almost periodic in t with a finite base of frequencies $\omega_1, \omega_2, \cdot \cdot \cdot, \omega_n$.) The theorem is a generalization of a result concerning periodic solutions when g(t, y) is periodic in t [2], in particular, the Ricatti case mentioned above. The proof of the theorem utilizes a method devised by Kolmogorov [7] to overcome the problem of arbitrarily small divisors and gives a method of constructing approximations to the almost periodic solutions. Since we assume that g has a finite base of frequencies we can present the system of equations in an autonomous form by considering a higher dimensional version of the differential equation. The theorem is as follows:

Let x be an n vector, let y be an m vector and consider the differential equations

$$\dot{x} = \omega \quad , \qquad \dot{y} = f(x, y) \ , \tag{1.1}$$

where $f(x, y) = -f(-x, y)$, and where the components of $f(x, y)$ are analytic for $|y| \leq R_1$ and $|\mathrm{Im} x| \leq R_2$, and where f has period 2π in each of the components of x. Suppose that the vector $\Omega = (\omega_1, \omega_2, \cdot \cdot \cdot, \omega_n)$ satisfies an inequality

$$|k \cdot \Omega| \geq \frac{K}{|k|^{\nu}} \tag{1.2}$$

for some positive constants K and ν and all vectors $k = (k_1, k_2, \cdot \cdot \cdot, k_n)$ with integer components where $k \cdot \Omega \equiv \sum_{i=1}^{n} k_i \omega_i$ and $|k| \equiv \sum_{i=1}^{n} |k_i| \neq 0$. Then if $f(x, y)$ is sufficiently small for $|y| \leq R_1$ and $|\mathrm{Im} x| \leq R_2$, there is a neighborhood of $y = 0$ such that all solutions of (1.1) starting in this neighborhood are almost periodic with base frequencies Ω.

We note the requirement on Ω (inequality (1.2)) is not stringent. If $\nu > n$, such a constant K exists for almost all Ω (in the sense of Lebesgue measure) [2]. The proof of this theorem is given by constructing an infinite sequence of coordinate transformations so that in the limiting set of coordinates the differential equations can be integrated in a neighborhood of $y = 0$.

The last section gives an application of the results mentioned above.

2. THE PERIODIC CASE

Let I_k denote the set of all k-tuples of non-negative integers. The elements of I_k can be counted according to the following technique. Let $(n_1, n_2, \cdots, n_k) \in I_k$, let $m = 2^{n_1}3^{n_2} \cdots p_i^{n_i} \cdots p_k^{n_k}$ where p_i is the ith prime number $i = 1, 2, \cdots, k$, then denote $(n_1, n_2, \cdots, n_k)$ by N_m. Note that $N_1 = (0, 0, \cdots, 0)$, $N_2 = (1, 0, \cdots, 0)$, etc. The natural ordering of the non-negative integers then orders all the elements of I_k, $N_1, N_2, \cdots$. For $N_n, N_m \in I_k$ we make the following remarks:

i) m is prime iff N_m is of the form $(0,0, \cdots,1, \cdots 0)$.

ii) Define addition in I_k component-wise. Then $N_n + N_m = N_\ell$ iff $nm = \ell$.

iii) We say that $N_n \leq N_m$ iff $n \leq m$ and $N_n < N_m$ iff $n < m$. From ii) it is clear that $N_n + N_m = N_\ell$ implies that $N_n \leq N_\ell$ and $N_m \leq N_\ell$.

For ω real let $\Omega = (\omega, -\omega, 2\omega, -2\omega, \cdot \cdot \cdot, k\omega, -k\omega)$. Now for $N_m = (n_1, n_2, \cdot \cdot \cdot, n_{2k-1}, n_{2k}) \in I_{2k}$ let m' be the integer so that $N_{m'} = (n_2, n_1, \cdot \cdot \cdot, n_{2k}, n_{2k-1})$. We observe that $N_m \cdot \Omega = -N_{m'} \cdot \Omega$. Let O_k denote the class of odd periodic functions of the real variable t with period $2\pi\omega$ which have finite Fourier series containing only terms of the type $f_j^* e^{ij\omega t}$ for $j = \pm 1, \pm 2, \cdot \cdot \cdot, \pm k$. Using the above notation we may represent functions in O_k in the form

$$f(t) = \sum_{\substack{N_n \varepsilon I_{2k} \\ n \text{ prime}}} f_n e^{iN_n \cdot \Omega t} \tag{2.1}$$

where $f_n = -f_n$ since f is odd.

Theorem 2.1. For $\eta > 0$ sufficiently small the differential equation

$$y' = \eta(a(t) + b(t)y + c(t)y^2), \tag{2.2}$$

where a, b and $c \in O_k$, has an even periodic solution $y(t)$ of the form

$$y(t) = \sum_{N_n \varepsilon I_{2k}} y_n e^{iN_n \cdot \Omega t} \quad . \tag{2.3}$$

Proof. Assume that (2.2) has a solution of the form indicated. Then formally we have

$$\sum_{N_n \varepsilon I_{2k}} iN_n \cdot \Omega y_n e^{iN_n \cdot \Omega t} =$$

$$\eta \sum_{N_n \varepsilon I_{2k}} \left[a_n + \sum_{\substack{mp=n \\ p \text{ prime}}} y_m b_p + \sum_{\substack{\ell mp=n \\ p \text{ prime}}} y_\ell y_m c_p \right] e^{iN_n \cdot \Omega t}, \qquad (2.4)$$

where a_j, b_j and c_j, j prime are the coefficients of the series representations (2.1) of a, b, and c respectively.

Now in case $N_n \cdot \Omega \neq 0$ we write

$$y_n = \begin{cases} \dfrac{\eta}{iN_n \cdot \Omega} a_n, & n \text{ prime} \\ \\ \dfrac{\eta}{iN_n \cdot \Omega} \left[\sum\limits_{\substack{mp=n \\ p \text{ prime}}} y_m b_p + \sum\limits_{\substack{\ell mp=n \\ p \text{ prime}}} y_\ell y_m c_p \right]. & n \text{ not prime} \end{cases} \qquad (2.5)$$

and for $N_n \cdot \Omega = 0$,

$$y_n = 0 . \qquad (2.5a)$$

Suppose that the terms containing $e^{iN_n \cdot \Omega t}$ on the right side of (2.4) vanish whenever $N_n \cdot \Omega = 0$. Then by remark iii) above we see that (2.5) defines y_n recursively so that (2.3) will be a formal solution. To show that this is indeed the case we present the following.

<u>Lemma 2.2</u>. Let $\{y_n\}$ be the sequence of numbers defined by 2.5. Then $y_n = y_{n^*}$.

Proof. The proof is by induction. If $N_n \cdot \Omega = 0$ then clearly $y_n = y_{n'}$ and so in particular for $n = 1$. Now suppose that $y_m = y_{m'}$ for all $m < n$. Then we may write

$$y_n = \frac{1}{iN_n \cdot \Omega}\left[\sum_{mp=n} y_m b_p + \sum_{\ell mp=n} y_\ell y_m c_p\right]$$

$$= \frac{1}{-iN_{n'} \cdot \Omega}\left[\sum_{mp=n} y_{m'}(-b_{p'}) + \sum_{\ell mp=n} y_{\ell'} y_{m'}(-c_{p'})\right]$$

where p is prime. But $mp = n$ iff $m'p' = n'$. So we see that $y_n = y_{n'}$ which proves the lemma.

Lemma 2.3. If $N_n \in I_{2k}$ is such that $N_n \cdot \Omega = 0$ then

$$\sum_{mp=n} y_m b_p + \sum_{\ell mp=n} y_\ell y_m c_p + \sum_{mp=n'} y_m b_p + \sum_{\ell mp=n'} y_\ell y_m c_p = 0,$$

where p is prime.

Proof. By Lemma 2.2 we have $y_n = y_{n'}$ for all n and since b and $c \in O_k$, we may write

$$\sum_{mp=n} y_m b_p = \sum_{mp=n} y_{m'}(-b_{p'}) = -\sum_{mp=n'} y_m b_p$$

and

$$\sum_{\ell mp=n} y_\ell y_m c_p = \sum_{\ell mp=n} y_{\ell'} y_{m'}(-c_{p'}) = -\sum_{\ell mp=n'} y_\ell y_m c_p,$$

where p is prime.

Now in order to show that (2.2) is a solution of equation (2.1) we will prove that formal series (2.3) with y_n defined by (2.5) converges uniformly and absolutely for all t and η sufficiently small.

Let C represent the complex plane, for $z = (z_1, z_2, \cdots, z_{2k}) \varepsilon C^{2k}$ and $N_n \varepsilon I_{2k}$ define $z^{N_n} = z_1^{n_1} z_2^{n_2} \cdots z_{2k}^{n_{2k}}$ and $|z| = \max_j |z_j|$. A function f mapping C^{2k} into C is analytic in the polydisk $\{z \varepsilon C^{2k} : |z_i| \leq r\}$ of radius r about the origin iff f has the representation,

$$f(z) = \sum_{N_n \varepsilon I_{2k}} a_n z^{N_n}$$

where the sum is uniformly and absolutely convergent in the polydisk. In case f is analytic, Cauchy's inequality gives for $|z| \leq \delta$

$$|a_n| \leq M/\delta^{|N_n|}$$

where $|N_n| = n_1 + n_2 + \cdots + n_{2k}$ and $M = \sup_{|z|=\delta} |f(z)|$.

Now for $z \varepsilon C^{2k}$ let

$$a^*(z) = \sum_{j=1}^{2k} |a_j| z_j \ ,$$

$$b^*(z) = \sum_{j=1}^{2k} |b_j| z_j \ , \tag{2.6}$$

$$c^*(z) = \sum_{j=1}^{2k} |c_j| z_j \ ;$$

and let $u(z) = f(a^*(z), b^*(z), c^*(z))$ where

$$f(a,b,c) = \begin{cases} \dfrac{1-\eta b}{2\eta c} - \dfrac{1-\eta b}{2\eta c}\left[1 - \dfrac{4\eta^2 ac}{(1-\eta b)^2}\right]^{\frac{1}{2}}, & c \neq 0, \\[2ex] \dfrac{\eta a}{1-\eta b}\ , & c = 0. \end{cases}$$

Note that u is the solution of the equation

$$u(z) = \eta[c^*u^2(z) + b^*u(z) + a^*] \tag{2.7}$$

which vanishes when $a^* = b^* = c^* = 0$. We see that $u(z)$ is an analytic function of z in any region which does not include zeros of the function

$$g(z) = (1-\eta b^*(z))^2 - 4\eta^2 a^*(z)c^*(z).$$

Now for $\delta > 0$ choose $\eta_o > 0$ so that $|g(z)| > 0$ whenever $|z| \leq 1 + \delta$; e.g. for $L = \max_j\{|a_j|,|b_j|,|c_j|\}$, let $\eta_o < \frac{1}{8Lk(1+\delta)}$, then $\eta_o|a^*|$, $\eta_o|b^*|$, $\eta_o|c^*| < \frac{1}{4}$ and we see that in this case $|g(z)| > 0$. Now for all η, $0 < \eta < \eta_o$ we have u analytic in the polydisk $|z| \leq 1 + \delta$. Hence, in this polydisk u has the representation

$$u(z) = \sum_{N_n \varepsilon I_{2k}} u_n z^{N_n}, \tag{2.8}$$

where

$$|u_n| \leq M/(1+\delta)^{|N_n|}$$

with $M = \sup_{|z_j|=1+\delta} |u(z)|$.

On the other hand, substituting from (2.8) into (2.7) and using (2.6) we obtain

$$u_n = \begin{cases} \eta |a_n| & n \text{ prime} \\ \eta \sum\limits_{\substack{mp=n \\ p \text{ prime}}} u_m |b_p| + \eta \sum\limits_{\substack{\ell mp=n \\ p \text{ prime}}} u_\ell u_m c_p, & n \text{ not prime.} \end{cases} \tag{2.9}$$

Comparing this with the recursion formula (2.4), with $y_1 = 0$, we see immediately that

$$|y_n| \leq \frac{1}{\omega}|u_n|, \quad n = 1, 2, \cdots.$$

Since $|e^{iN_n \cdot \Omega t}| = 1$ for all t we have

$$\left| \sum_{N_n \varepsilon I_{2k}} y_n e^{iN_n \cdot \Omega t} \right| \leq \frac{1}{\omega} \sum_{N_n \varepsilon I_{2k}} |u_n| ,$$

$$\leq \frac{M}{\omega} \sum_{N_n \varepsilon I_{2k}} \frac{1}{(1+\delta)^{|N_n|}} ,$$

$$\leq \frac{M}{\omega} \left(\frac{1+\delta}{\delta}\right)^{2k} ,$$

which not only proves absolute and uniform convergence, but also gives a bound for the solution $y(t)$.

Remarks:

i) In the proof of Lemma 2.2 we showed that $y_n = y_{n'}$ for all $N_n \varepsilon I_{2k}$ such that $N_n \cdot \Omega \neq 0$. From this we conclude that the solution found above is even in t. Note also that the solution has zero mean value.

ii) The particular order relation used here for I_k is not essential to the proof. See Golomb [5] and Wasow [10] for different schemes.

iii) It is possible to use the result in this section directly to obtain solutions with mean value other than zero. Let $f(t, y)$ represent the right side of equation (2.1) and suppose that $y(t)$ is the solution of

$$y' = f(t, y)$$

given above. For any fixed constant c, let $z = y + c$ in (2.1). The theorem gives a technique for obtaining a solution of the new equation

$$z' = f^*(t, z),$$

where $f^*(t, z) = f(t, z - c)$ with zero mean value. This in turn gives a solution to the original equation with mean value $-c$.

iv) Let y be an n-vector, let $p_\ell(t)$ be an n-vector with components $p_\ell^{(j)}(t) \in O_k$, $j = 1,2,\cdots,n$, $\ell = 1,2,\cdots$. Then the differential equation

$$y' = \eta \sum_{\substack{N_\ell \in I_{2k} \\ \ell < \infty}} p_\ell(t) y^{N_\ell} \qquad (2.10)$$

where the right side converges for $|y| \leq r$ may be solved using the techniques of this section. The only essential difference occurs when one attempts to find an analytic solution of the corresponding equation (2.8). Here one may use the implicit function theorem to show existence of such a solution for η sufficiently small.

v) The existence of periodic solutions of equations of the form (2.10), for η small is shown by Hale [6, p. 45].

3. THE QUASIPERIODIC CASE

For any positive integer n let J_n denote the set of all n-tuples of integers, for $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \varepsilon J_n$ let $|\alpha| = \sum_{i=1}^{n} |\alpha_i|$ and let C^m be all m vectors $(y_1,\cdots,y_m)$ where each component is a complex number. For simplicity we will treat only the case where y is a m = 1 vector.

We shall be concerned in this section with functions defined and analytic on (x, y) subsets of $C^n \times C^1$ into C^1 which are periodic of period 2π in each component of x. These subsets will be of the form

$$D(r, \rho) = \{(x,y) \varepsilon C^{1+n}: |Imx| \leq \rho, |y| \leq r\}$$

where the norm $|\ |$ of the vector x denotes the maximum of the absolute value of its components. We denote the class of such functions by $P(r, \rho)$ and note that any $g \varepsilon P(r, \rho)$ has a Fourier-Taylor series representation

$$g(x, y) = \sum_{|\alpha|,|\beta|=0}^{\infty} g_{\alpha\beta} e^{i\alpha\cdot x} y^{\beta},$$

where the $g_{\alpha\beta}$ are complex numbers and where the sum is taken over all $\alpha \varepsilon J_n$ and $\beta \varepsilon I_1$.

Several lemmas are listed below without proof. The proofs are elementary and are similar to those given in [2].

Lemma 3.1. Let $h \varepsilon P(R_1, R_2)$ and let $|h(x,y)| \leq M$, $M > 0$ in $D(R_1, R_2)$. The Fourier-Taylor coefficients, given by

$$h_{\alpha\beta} = \frac{1}{(2\pi)^n} \iint \cdots \int h_\beta(x) e^{-i\alpha \cdot x} dx_1 dx_2 \cdots dx_n, \quad \alpha \varepsilon J_n, \quad \beta \varepsilon I_1$$

where

$$h_\beta(x) = \frac{1}{\beta!} \frac{\partial^\beta}{\partial y^\beta} h(x, y)\bigg|_{y=0} ,$$

and where the jth integral is taken from $x_j = 0$ to $x_j = 2\pi$, satisfy the inequality

$$|h_{\alpha\beta}| \leq \frac{Me^{-|\alpha|R_2}}{R_1{}^{|\beta|}} .$$

If $h(-x, y) = -h(x, y)$ in the above we have $h_{-\alpha\beta} = -h_{\alpha\beta}$ and conversely. If $h(-x, y) = h(x, y)$ we have $h_{-\alpha\beta} = h_{\alpha\beta}$ and conversely.

Lemma 3.2. If the elements of the sequence $\{h_{\alpha\beta}\}$ $\alpha \varepsilon J_n$, $\beta \varepsilon I_1$ satisfy

$$|h_{\alpha\beta}| \leq \frac{Me^{-|\alpha|R_2}}{R_1{}^{|\beta|}} ,$$

then

$$h(x,y) = \sum_{|\alpha|,|\beta|=0}^{\infty} h_{\alpha\beta} e^{i\alpha\cdot x} y^{\beta}$$

is analytic for $|y| \leq R_1 e^{-\delta}$, $|\mathrm{Im}\, x| \leq R_2 - \delta$ for any positive $\delta < 1$ such that $R_2 - \delta > 0$; and in this domain we have

$$|h(x, y)| \leq \frac{2^{2n+1} M}{\delta^{n+1}} \quad .$$

<u>Lemma 3.3</u>. For all positive numbers m, ν, δ we have

$$m^{\nu} \leq \left(\frac{\nu}{e}\right)^{\nu} \frac{e^{m\delta}}{\delta^{\nu}} \quad .$$

<u>Lemma 3.4</u>. (Cauchy's Inequality). If the complex valued function h(z) is analytic and bounded by M for $|z| \leq R$, $M,R > 0$, then for $|z| \leq Re^{-\delta}$, $0 < \delta < 1$, we have

$$\left|\frac{dh}{dz}\right| \leq \frac{2M}{R\delta}$$

The proof of the main result of this section depends almost entirely on the following considerations.

Let $f \in P(R_1, R_2)$ and satisfy $f(-x, y) = -f(x, y)$, let f satisfy

$$|f(x, y)| \leq M \equiv \delta^{2(n+\nu)+7} \tag{3.1}$$

in $D(R_1, R_2)$ where δ is specified below, let ω satisfy (1.2) for some positive constants K and ν and all $\alpha \in J_n$ with $|\alpha| \neq 0$ and consider the differential equations

$$\dot{x} = \omega, \qquad \dot{y} = f(x, y). \tag{3.2}$$

Lemma 3.5. For each x in $|\mathrm{Im}\,x| \leq R_2 - 2\delta$ there exists an invertible transformation U defined on a subset of C into C, given by $U(\eta) = y$ where

$$y = \eta + u(x, \eta), \quad |u(x, \eta)| \leq \frac{M}{\delta^{n+\nu+2}} \tag{3.3}$$

for $|\eta| \leq R_1 e^{-4\delta}$, $|\mathrm{Im}\,x| \leq R_2 - 2\delta$. Putting $\eta = U^{-1}(y)$ in (3.3) we obtain the differential equations (3.2) in the new coordinates

$$\dot{x} = \omega, \qquad \dot{\eta} = f^*(x, \eta); \tag{3.4}$$

and in $D(R_1 e^{-4\delta}, R_2 - 2\delta)$

$$|f^*(x, \eta)| \leq M^{3/2}. \tag{3.5}$$

Further we have $f^*(x, \eta) = -f^*(-x, \eta)$ and $f^* \varepsilon\, P(R_1 e^{-4\delta}, R_2 - 2\delta)$. Here δ is taken as a positive number satisfying

$$\delta \leq \min\left\{\left(\frac{R_1}{16}\right)^{1/n+\nu+4}, R_2, \frac{R_1 K}{2^{2n+3}}\left(\frac{e}{\nu}\right)^{\nu}, \frac{K}{2^{2n+1}}\left(\frac{e}{\nu}\right)^{\nu}, \left(\frac{1}{2}\right)^{1/n+\nu+\delta}\right\}.$$

Proof. a) Definition of $u(x, \eta)$: Choose $u(x, \eta)$ as the solution with mean value zero of

$$\frac{\partial u}{\partial x}\,\omega = f(x, \eta) \tag{3.6}$$

where $\frac{\partial u}{\partial x}$ is the vector with elements $\frac{\partial u}{\partial x_j}$ in the ith row and jth column. This gives

$$u(x, \eta) = \sum_{|\alpha|,|\beta|=0}^{\infty} {}_{\alpha\beta} e^{i\alpha\cdot x} \eta^{\beta}$$

where

$$u_{\alpha\beta} = \frac{f_{\alpha\beta}}{i\alpha\cdot\omega} \qquad \alpha \neq 0, \qquad u_{o\beta} = 0.$$

Since $|f(x, \eta)| \leq M$ for $|y| \leq R_1$, $|\mathrm{Im} x| \leq R_2$, Lemma 3.1, inequality (1.2) and the above imply

$$|u_{\alpha\beta}| \leq \frac{Me^{-|\alpha|R_2}}{R_1^{|\beta|}} \quad \frac{|\alpha|^{\nu}}{K} .$$

By Lemma 3.3

$$|u_{\alpha\beta}| \leq \frac{M}{K\delta^{\nu}} \left(\frac{\nu}{e}\right)^{\nu} \frac{e^{-|\alpha|(R_2-\delta)}}{R_1^{|\beta|}}$$

Hence, using Lemma 3.2 we have $u(x, \eta)$ defined and analytic for $|\eta| \leq R_1 e^{-\delta}$, $|\mathrm{Im} x| \leq R_2 - 2\delta$ and bounded in this domain by

$$|u(x, \eta)| \leq \frac{2^{2n+1}}{\delta^{n+1+\nu}} \frac{M}{K} \left(\frac{\nu}{e}\right)^{\nu} . \tag{3.7}$$

Thus, inequality (3.3) is valid and we note that $u(x, \eta) = u(-x, \eta)$ and $u \in P(R_1 e^{-\delta}, R_2 - 2\delta)$.

b) The transformation U : By equation (3.3) the set $D = \{\eta \in C : |\eta| \leq R_1 e^{-2\delta}\}$ is mapped into a set containing $A = \{y \in C : |y| \leq R_1 e^{-3\delta}\}$; i.e. $U(D) \supseteq A$. Since

$$\frac{\partial u}{\partial \eta} \geq \frac{1}{2} > 0$$

for $|\eta| \leq R_1 e^{-2\delta}$, $|\mathrm{Im} x| \leq R_2 - 2\delta$, we see that U^{-1} is defined on A. Thus for $|\eta| \leq R_1 e^{-4\delta}$, $|\mathrm{Im} x| \leq R_2 - 2\delta$, $u(x, \eta)$ is defined and (3.7) holds.

c) The function $f^*(x, \eta)$: Substituting from (3.3) into (3.2) gives

$$(1 + \frac{\partial u}{\partial \eta})\dot{\eta} = f(x, y) - f(x, \eta),$$

so that

$$f^*(x,\eta) = (1 + \frac{\partial u}{\partial \eta}(x,\eta))^{-1}(f(x,\eta + u(x,\eta)) - f(x,\eta)); \quad (3.8)$$

and we note here that $-f^*(-x,\eta) = f^*(x,\eta)$.

Now we have

$$|(1 + \frac{\partial u}{\partial \eta})|^{-1} \leq 2 \; ; \quad (3.9)$$

hence

$$|f^*(x, \eta)| \leq 2 \, |f(x,\eta + u(x, \eta)) - f(x, \eta)|.$$

But

$$| f(x,\eta + u(x,\eta)) - f(x,\eta)| \leq \sup \{|\frac{\partial f}{\partial \eta}|\}_{\delta n+1+\nu} \frac{2^{2n+1}M}{K} (\frac{\nu}{e})^{\nu},$$

where the supremum is taken over $|y| \leq R_1 e^{-\delta}$, $|\mathrm{Im} x| \leq R_2$. By Cauchy's inequality

$$\left|\frac{\partial f}{\partial y}\right| \leq \frac{2M}{R_1 \delta},$$

thus

$$f^*(x, \eta) \leq \frac{2^{2n+3}}{KR_1 \delta^{n+\nu+2}} \left(\frac{\nu}{e}\right)^{\nu} M^2 \leq M^{3/2},$$

and the proof of the lemma is complete.

Theorem 3.1. Let f be as in Lemma 3.5. Then if M (and thus δ) is sufficiently small for each x in $|\mathrm{Im} x| \leq R_2/2$ there exists an invertible transformation, V, defined and analytic on $\{\eta \in C \; : |\eta| \leq R_1 e^{-R_2}\}$ into C^m, given by $V(\eta) = y$ where

$$y = \eta + v(x, \eta). \tag{3.10}$$

Denoting the inverse transformation $\eta = V^{-1}(y)$ we obtain the differential equations (3.3) in new coordinates

$$\dot{x} = \omega, \quad \dot{\eta} = 0.$$

Furthermore we have $v(-x, \eta) = v(x, \eta)$.

Proof. Choose $\delta_1 > 0$ so that for $\delta_j = \delta_{j-1}^{3/2}$ we have

$$\sum_{j=1}^{\infty} \delta_j \leq \frac{R_2}{4} .$$

$$\delta_1 \leq \left\{ \left(\frac{R_1}{16}\right)^{1/n+\nu+4} , \frac{R_1 K}{2^{2n+3}}\left(\frac{e}{\nu}\right)^{\nu}, \frac{K}{2^{2n+1}}\left(\frac{e}{\nu}\right)^{\nu}, \left(\frac{1}{2}\right)^{1/n+\nu+5} \right\} .$$

Apply Lemma 3.5 iteratively j times. Let $u_i(x, y)$ denote the function in the transformation of coordinates at the ith step and let $f_i(x, \eta)$ denote the corresponding right side of the differential equation. We obtain the composite map $F_j(x, \eta) = y$ where

$$F_j(x,\eta) = \eta + u_j(x,\eta) + u_{j-1}(x,u_j(x,\eta))+\cdots+ u_1(x,\eta + u_j(x,\eta)+\cdots+$$

$$u_2(x,\eta + u_j(x, \eta) + \cdots + u_3(x, \eta)),$$

defined for $|\eta| \leq R_1 \exp[-4 \sum_{i=1}^{j} \delta_i]$, $|\mathrm{Im} x| \leq R_2 - 2 \sum_{i=1}^{j} \delta_i$,

where in the associated differential equations

$$\dot{x} = \omega \quad , \qquad \dot{\eta} = f_j(x, \eta) \ ,$$

the functions $f_j(x, \eta)$ satisfy

$$|f_j(x, \eta)| \leq M_j = \delta_j^{\,2(n+\nu)+7}$$

We observe that the composite transformations are defined for all j in $|\eta| \leq R_1 e^{-R_2}$, $|\mathrm{Im} x| \leq R_2/2$, and that

$$\sum_{i=1}^{\infty} |u_i| < \infty;$$

thus the limiting composite transformation

$$F(x, \eta) = \lim_{j\to\infty} F_j(x, \eta) = \eta + v(x, \eta), \tag{3.11}$$

will exist in the above domain. In the coordinates defined by (3.11) the differential equation (3.3) becomes

$$\dot{x} = \omega \quad , \qquad \dot{\eta} = 0 \ .$$

4. APPLICATION

Adrianov [1] and Gelmand [4] outlined a procedure for finding a transformation $x = Z(t)y$ so a given differential equation

$$\frac{dx}{dt} = Q(t)x,$$

where $Z(t)$ $Q(t)$ are almost periodic $n \times n$ matrices and where P satisfies certain conditions and x in an n-vector, becomes

$$\dot{y} = Ay, \qquad A \text{ constant}$$

in the new coordinates. We shall follow this procedure and use Theorem 3.6 to effect the same transformation in circumstances where the earlier work fails to apply.

Let H be the class of all functions

$$f(t) = g(\omega_1 t, \omega_2 t, \cdot \cdot \cdot, \omega_n t)$$

where $g(u_1, u_2, \cdot \cdot \cdot, u_n)$ is real analytic and has period 2π in each u_i, $i = 1, 2, \cdot \cdot \cdot, n$. Consider the differential equations

$$\frac{dx_1}{dt} = [q(t) + \eta q_{11}(t)]x_1 + \eta q_{12}(t)x_2,$$

$$\frac{dx_2}{dt} = \eta q_{21}(t)x_1 + [q(t) + \eta q_{22}(t)]x_2, \tag{4.1}$$

where $\eta > 0$ and q, $q_{ij} \in H$ and $q_{ij}(t) = -q_{ij}(-t)$, $i,j = 1, 2,$ and where $\omega = (\omega_1, \omega_2, \cdot \cdot \cdot, \omega_n)$ satisfies inequality (1.2). We make the change of coordinates

$$x_1 = y_1 + \tau y_2, \qquad x_2 = y_2 \tag{4.2}$$

where τ is any almost periodic solution of the differential equation

$$\dot{\tau} = \eta[q_{12} + (q_{11} - q_{22})\tau - q_{21}\tau^2]. \tag{4.3}$$

In the new coordinates (4.1) becomes

$$\frac{dy_1}{dt} = [q(t) + \eta(q_{11} - \tau q_{21})]y_1$$

$$\frac{dy_2}{dt} = \eta q_{21}y_1 + \eta(q_{21}\tau + q_{22} + q)y_2 \tag{4.4}$$

Theorem 3.6 guarantees that for η small enough equation (4.3) has almost periodic solutions, which belong to H. Equation (4.4) may now be integrated to obtain

$$y_1 = a^*(t)e^{a_o t}c_1$$

$$y_2 = b^*(t)e^{a_o t}c_1 + b^{**}(t)e^{b_o t}c_2$$

where a_o and b_o are the mean values of $q + \eta q_{11} - \eta\tau q_{21}$ and $q + \eta q_{22} + \eta q_{21}\tau$ respectively and a^*, b^* and $b^{**} \varepsilon H$.

Reversing the change of coordinates (4.2) we obtain a fundamental matrix solution of (4.1) of the form

$$\phi(t) = P(t)e^{At}, \qquad A = \begin{bmatrix} a_o & 0 \\ 0 & b_o \end{bmatrix} ,$$

where the elements of P(t) are almost periodic and belong to H.

The change of coordinates

$$x = P(t)z$$

in (4.1) yields

$$\dot{Z} = AZ .$$

The case treated by Gelmand [4] required that the linear term in the resulting differential equation for τ have mean value which dominates the other elements in order that there exists an almost periodic solution of this equation. Thus Theorem 3.6 permits us to consider a new situation.

If in 4.1 we require that q, $q_{ij} \varepsilon O_k$ (defined in section 2) $i, j = 1, 2$, then we may use Theorem 2.1 to obtain an explicit representation for periodic solutions of equation (4.3) for η sufficiently small. Then we may integrate equations (4.4) and obtain explicit solutions of (4.1). Note that if

$$\tau(t) = \sum_{n=1}^{\infty} \tau_n e^{iN_n \cdot \Omega t}$$

is the solution of (4.3) given by Theorem 2.1 and $\tau_1 = 0$ we have periodic solution of (4.1). The existence of these solutions was shown by Epstein [3].

REFERENCES

1. Adrianova, L. Ya. The Reducibility of Systems of N Linear Differential Equations with Quasi-Periodic Coefficients. Vestnik, Leningrad. Univ. 17, No. 7, 14-26 (1962).

2. Arnold, V. I. Small Divisors and The Problem of Stability of Motion in Classical and in Celestial Mechanics. Uspehi Mat. Nauk XVIII, 6(114)(L963): 92-192.

3. Epstein, I. Periodic Solutions of Differential Equations. Proc. Amer. Math. Soc. 13 (1962), 690-694.

4. Gelmand, A. E. On the Reducibility of a Class of Systems of Differential Equations with Quasi-Periodic coefficients. Dokl. Akad. Nauk SSSR 116 (4), (1957): 535-537.

5. Golomb, M. Solution of Certain Nonautonomous Differential Systems by Series of Exponential Functions. Illinois J. Math. 3(1959): 45-65.

6. Hale, J. K. Oscillations in Non-Linear Systems. New York. McGraw-Hill Book Company, Inc. 1963.

7. Kolmogorov, A. N. On the Conservation of Conditionally Periodic Motion in the Presence of Small Changes in the Hamiltonian. Dokl. Akad. Nauk SSSR 98(1954): 527-530.

8. Siegel, C. L. Iteration of Analytic Functions. Annals of Math. 42(4) 1942: 607-612.

9. Siegel, C. L. Analytische Hamiltonshe Differential-Gleichungen. Math. Ann. 128(1954): 144-170.

10. Wasow, W. Solution of Certain Non-Linear Differential Equations By Series of Exponential Functions. Illinois J. Math. 2(1958): 254-260.

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INTEGRAL MANIFOLDS FOR

PERTURBED NONLINEAR SYSTEMS

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INTRODUCTION

This dissertation establishes the existence of integral manifolds for a system of perturbed nonlinear differential equations in a neighborhood of a critical point.

The notion of an integral manifold for a system of differential equations, as used in this discussion, is given here. Let U be an open set in R^p and consider the system of differential equations

$$(1) \qquad \dot{z} = Z(t,z),$$

where z, Z are real p-vectors and Z is continuous in $R \times U$. Let $z(t,\tau,\zeta)$ denote solutions of (1) satisfying the initial condition $z(\tau,\tau,\zeta) = \zeta$. For a fixed number τ suppose there exists a set S^+ which can be represented in the form

$$S^+ = \{(t,z) \text{ in } R \times R^p \mid z_i = v_i(t,z_{k+1},\ldots,z_p),\ i=1,\ldots,k,\ t \geq \tau\},$$

where $v = (v_1,\ldots,v_k)$, $k < p$, is defined and continuous on $[\tau,\infty) \times V$, V a subset of R^{p-k}. The set S^+ will be called a positive (integral) manifold for system (1) if any solution, $z(t,\tau,\zeta)$ of (1), with (τ,ζ) in S^+ exists for all $t \geq \tau$ and is such that $(t,z(t,\tau,\zeta))$ is in S^+ for all $t \geq \tau$.

Similarly a set

$$S^- = \{(t,z) \text{ in } R\times R^p \mid z_i = u_i(t,z_{k+1},\ldots,z_p),\ i=1,\ldots,\ell,\ t\leq\tau\},$$

where $u = (u_1,\ldots,u_\ell)$, $\ell < p$, is defined and continuous on $(-\infty,\tau]\times W$, W a subset of $R^{p-\ell}$, will be called a negative (integral) manifold for system (1) if any solution, $z(t,\tau,\zeta)$ of (1), with (τ,ζ) in S^- exists for all $t \leq \tau$ and is such that $(t,z(t,\tau,\zeta))$ is in S^- for all $t \leq \tau$.

Suppose that system (1) possesses a positive manifold S^+ with v bounded on its domain. Consider the perturbed system

$$(2) \qquad \dot{z} = Z(t,z) + Z^*(t,z),$$

where Z^* is a real p-vector. Sufficient conditions on the function Z^* are given to insure the existence of a positive manifold

$$S^* = \{(t,z) \text{ in } R^{1+p} \mid z_i = v_i^*(t,z_{k+1},\ldots,z_p),\ i=1,\ldots,k, t\geq\tau\},$$

with $v^* = (v_1^*,\ldots,v_k^*)$ bounded on its domain. A similar result is obtained for negative manifolds.

A restatement of the problem in a special case serves as a model. The system

$$\dot{x} = -x^3, \quad \dot{y} = y^3,$$

has local solutions

$$x(t,\tau,\alpha) = \frac{\alpha}{(1+2\alpha^2(t-\tau))^{1/2}}, \; y(t,\tau,\beta) = \frac{\beta}{(1-2\beta^2(t-\tau))^{1/2}}.$$

The phase portrait manifold of (3) is given in Figure 1a. Notice that

$$S^+ = \{(t,x,y) | 0 \leq x, \; y=0, \; t \geq 0\},$$

is a positive manifold for (3) and

$$S^- = \{(t,x,y) | x=0, \; 0 \leq y, \; t \leq 0\},$$

is a negative manifold for (3). The problem here is to find sufficient conditions on the perturbation functions $X(x,y)$, $Y(x,y)$ so that the system

$$\dot{x} = -x^3 + X(x,y), \quad \dot{y} = y^3 + Y(x,y),$$

has positive and negative manifolds

$$S_*^+ = \{(t,x,y)\,|\,0\leq x<\varepsilon,\ y=v^+(x),\ t\geq 0\}\ ,$$
$$S_*^- = \{(t,x,y)\,|\,x=v^-(y),\,0\leq y<\eta,\ t\leq 0\}\ ,$$

with v^+ and v^- bounded on their respective domains. The phase portrait of this system may then be represented schematically by Figure 1b.

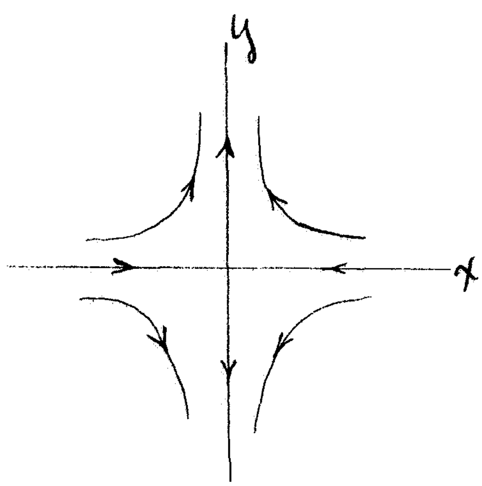

Figure 1a

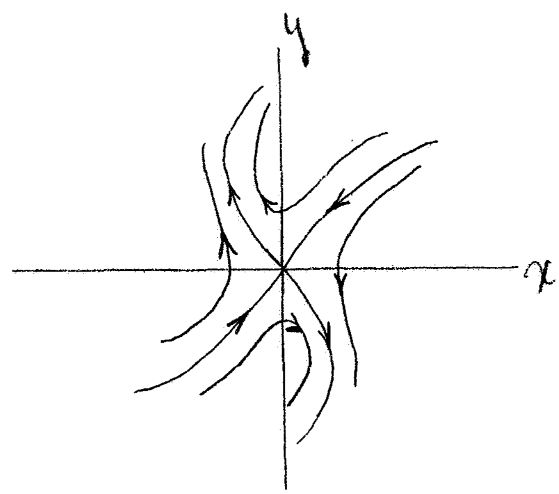

Figure 1b

In previous work on this problem the typical approach has been to assume that system (1) has a manifold S. In many cases solutions on S are periodic in t. Then under the proper conditions, see [9], there exists a reversible transformation, T, defined in a neighborhood of S which carries (1) into a system of the form

$$
\begin{aligned}
\dot{\theta} &= d + \Theta(t,\theta,x,y), \\
\dot{x} &= Ax + F(t,\theta,x,y), \\
\dot{y} &= By + G(t,\theta,x,y),
\end{aligned}
\tag{4}
$$

where d is a constant ℓ-vector, θ an ℓ-vector, x an m-vector, y an n-vector, $\ell+m+n = p$. Here A is an $m\times m$ matrix with eigenvalues which have negative real parts, B is an $n\times n$ matrix, with eigenvalues which have positive real parts, and the functions Θ, F and G satisfy certain smoothness and order conditions and are multiply periodic in θ with period $\omega = (\omega_1,\ldots,\omega_\ell)$, $\omega_i > 0$, $i = 1,2,\ldots,\ell$, i.e. $\Theta(t,\theta+\omega,x,y) = \Theta(t,\theta,x,y)$ where $\theta + \omega = (\theta_1+\omega_1,\ldots,\theta_\ell+\omega_\ell)$.

Equations (4) represent system (1) in a neighborhood of a critical point, periodic orbit or periodic surface depending on whether the θ-equation is absent, $\ell = 1$, or $\ell > 1$. In case $\ell = 0$ system (4) may be considered a linearization of the original system. The transformation T sends the perturbation function Z^* into the triple (Θ^*,F^*,G^*) and the problem then becomes one of finding a manifold with some specified properties for the system

$$
\begin{aligned}
\theta &= d + \tilde{\Theta}(t,\theta,x,y), \\
\dot{x} &= Ax + \tilde{F}(t,\theta,x,y), \\
\dot{y} &= By + \tilde{G}(t,\theta,x,y),
\end{aligned}
\tag{5}
$$

where $\tilde{\Theta} = \Theta + \Theta^*$, etc. One usually begins with a system in the form of (5). Functions describing the manifolds for (5) are then obtained as solutions of a certain improper integral equation, see for example ([13], p.137), formulated by the use of the classical variation of constants technique. The linear term in (5) gives rise to factors of the form exp(tA) which guarantee the convergence of these integrals. (An exception to this is work by Kelley [15]). This approach was introduced by Bogolinbov and Mitropolsky [3], [4] and subsequently developed by Hale [12] and Kelley[15]. Of interest here is the case in which the linear terms in (5) are replaced by nonlinear terms. Again one is led to consider an improper integral equation, obtained in this case from a generalization of the variation of constants formula due to V. M. Alekseev [1]. Special hypotheses are introduced to insure convergence of the improper integrals.

Some work on the nonlinear problem has been done by L. E. May [17]. The hypotheses used by May seem to admit only systems which behave in an essentially linear manner in the unperturbed state. The theorem presented here, which includes the model problem above, requires no linear or approximate linear behavior in the unperturbed state.

Other people who have worked on the problem of integral manifolds are Diliberto [10], [11], Levinson [16] and Sacker [18].

PRELIMINARY CONSIDERATIONS

The symbol $|\,|$ will be used to represent any fixed norm on R^{ℓ}, R^m or R^n as well as the corresponding operator norm. In several of the proofs in this dissertation it is expeditious to use some specific norm, say $|\,|_1$, for the finite dimensional space involved. In each such case norm equivalence in finite dimensional spaces is envoked to obtain the desired result, i.e., there exist numbers N_1, $N_2 > 0$ such that $N_1|\,|_1 \leq |\,| \leq N_2|\,|_1$.

For a square matrix A define $\mu[A]$ by

$$\mu[A] = \lim_{h \to +0} \frac{|I+hA|-1}{h},$$

where I is the identity matrix. For a proof that this limit always exists and for further discussion of the function μ see [8].

For $\varepsilon > 0$ and m a positive integer, let $B(\varepsilon,m)$ represent the open ball of radius ε in R^m. Let $U \subset R^k$, $V \subset R^{\ell}$ and suppose that $f: U \times V \to R^p$. Further suppose that all <u>mth</u> order derivatives of f with respect to its first k variables exist and are continuous on $U \times V$ and that all <u>nth</u> order derivatives of f with respect to its last ℓ variables exist and are continuous on $U \times V$. Then f is said to be $C^m(U) \cap C^n(V): R^p$.

Let W be any open set in R^n and J be an interval in R. Suppose that f, g are $C(J) \cap C'(W):R^n$. Then solutions of the initial value problem

$$(6) \qquad \dot{x} = f(t,x), \quad x(\tau) = \zeta,$$

$$(7) \qquad \dot{y} = f(t,y) + g(t,y), \quad y(\tau) = (\zeta),$$

are locally unique when (τ,ζ) is in $J \times W$. Further if $x(t) = x(t,\tau,\zeta)$ is the solution of (6) then it is well known [7] that the matrix $\Phi(t,\tau,\zeta) = \frac{\partial x}{\partial \zeta}(t,\tau,\zeta)$ is a fundamental solution of the variational equation

$$\dot{z} = f_x(t,x(t,\tau,\zeta))z,$$

where f_x is the matrix with i,j<u>th</u> element $\frac{\partial f_i}{\partial x_j}$. It is also known that $\frac{\partial x}{\partial \tau}(t,\tau,\zeta) = -\Phi(t,\tau,\zeta)f(t,\zeta)$, [17]. Now suppose that $y(t) = y(t,\tau,\zeta)$ is the solution of (7). Then one may write

$$\begin{aligned}\frac{d}{ds}x(t,s,y(s)) &= \Phi(t,s,y(s))[-f(s,y(s))+\dot{y}(s)] \\ &= \Phi(t,s,y(s))g(s,y(s)),\end{aligned}$$

and therefore

$$x(t) = y(t) + \int_t^\tau \Phi(t,s,y(s))g(s,y(s))ds$$

This relation, which is a generalization of the variation of constants formula, is due to V. M. Alekseev [1], [2]. For further discussion of this relation see [5], [6], [7] and [19].

THE UNPERTURBED SYSTEM

In this section sufficient conditions are given for existence of positive and negative manifolds for the unperturbed system. Let $\ell \geq 0$, $m > 0$, $n > 0$ be integers. Consider the system

$$\begin{aligned} \dot{\theta} &= d, \\ \dot{x} &= f(t,x), \qquad (8) \\ \dot{y} &= g(t,y), \end{aligned}$$

where d is a constant ℓ-vector and f and g satisfy

(Ii) f is $C(R) \cap C^1(B(\varepsilon,m)):R^m$ for some $\varepsilon > 0$;

(Iii) $f(t,0) = 0$, t in R;

(Iiii) there exist α, $a > 0$ such that
$\mu[f_x(t,x)] \leq -a|x|^\alpha$, t in R, x in $B(\varepsilon,m)$;

(IIi) g is $C(R) \cap C^1(B(\eta,n)):R^n$ for some $\eta > 0$;

(IIii) $g(t,0) = 0$, t in R; and,

(IIiii) there exist β, $b > 0$ such that
$b|y|^\beta \leq -\mu[-g_y(t,y)]$, t in R, y in $B(\eta,n)$.

Theorem 1. Let (Ii) through (IIiii) hold. Then for τ a fixed number

$$S^+ = \{(t,\theta,x,y)\,|\,\theta \text{ arbitrary}, |x|<\varepsilon, y=0, t\geq\tau\},$$

is a positive manifold for system (8) and

$$S^- = \{(t,\theta,x,y)|\theta \text{ arbitrary, } x=0,\ |y|<\eta,\ t\leq\tau\},$$

is a negative manifold for system (8). Further if $(\theta(t),x(t),y(t)) = (\theta(t,\tau,\nu),x(t,\tau,\zeta),0)$ is any solution of system (8) with $(\tau,\nu,\zeta,0)$ in S^+ then

$$(9) \qquad |x(t,\tau,\zeta)| \leq \frac{|\zeta|}{(1+A|\zeta|^{\alpha}(t-\tau))^{1/\alpha}},$$

for some $A > 0$ and all $t \geq \tau$; and if $(\theta(t),x(t),y(t)) = (\theta(t,\tau,\nu),0,y(t,\tau,\sigma))$ is any solution of (8) with $(\tau,\nu,0,\sigma)$ in S^- then

$$(10) \qquad |y(t,\tau,\sigma)| \leq \frac{|\sigma|}{(1-B|\sigma|^{\beta}(t-\tau))^{1/\beta}},$$

for some $B > 0$ and all $t \leq \tau$.

Proof. It is sufficient to show that solutions of $\dot{x} = f(t,x)$ exist for $t \geq \tau$ and are bounded as in (9), and that solutions of $\dot{y} = (g(t,y))$ exist for $t \leq \tau$ and are bounded as in (10).

From (Ii) it follows that for each τ in R and $|\zeta| < \varepsilon$ there exists a unique solution $x(t) = x(t,\tau,\zeta)$ with $x(\tau,\tau,\zeta) = \zeta$ of $\dot{x} = f(t,x)$ in a neighborhood V of τ. So for each t in V and $|\zeta| < \varepsilon$ one may write

$$D_R|x(t)| = \lim_{h\to+0} \frac{|x(t)+h\dot{x}(t)|-|x(t)|}{h}$$

$$= \lim_{h\to+0} \frac{|x(t)+hf(t,x(t))|-|x(t)|}{h},$$

where $D_R|x(t)|$ denotes the right derivative of $|x|$ at t. Since $f(t,0) = 0$, t in R, the mean value theorem gives

$$f(t,x(t)) = \int_0^1 f_x(t,sx(t))ds\cdot x(t).$$

Therefore

$$D_R|x(t)| \leq \lim_{h\to+0} \frac{1}{h}\left(\left|I+h\int_0^1 f_x(t,sx(t))ds\right|-1\right)|x(t)|$$

$$\leq \lim_{h\to+0} \int_0^1 \frac{|I+hf_x(t,sx(t))|-1}{h}\, ds\,|x(t)|.$$

Let $\{h_i\}_{i=1}^{\infty}$ be a sequence of positive numbers with limit zero and let

$$J_i(s) = \frac{|I+h_i f_x(t,sx(t))|-1}{h_i}, \quad i = 1,2,\ldots, \quad s \text{ in } [0,1].$$

Since

$$|J_i(s)| \leq \left|\left|\frac{1}{h_i}I+f_x(t,sx(t))\right|-\frac{1}{h_i}|I|\right|$$

$$\leq \left|\frac{1}{h_i}(I+f_x(t,sx(t))-\frac{1}{h_i}I\right| = |f_x(t,sx(t))|,$$

it follows that $\{J_i(s)\}_{i=1}^{\infty}$ is a uniformly bounded sequence sequence of uniformly continuous functions on $[0,1]$. Further $\lim_{i\to\infty} J_i(s) = \mu[f_x(t,sx(t))]$. Thus by the Lebesgue Dominated Convergence Theorem

$$\lim_{n\to+0} \int_0^1 \frac{|I+hf_x(t,sx(t))|-1}{h}\, ds = \int_0^1 \mu[f_x(t,sx(t))]ds,$$

and

$$D_R|x(t)| \leq \int_0^1 \mu[f_x(t,sx(t))]ds\,|x(t)| \leq -a\int_0^1 |sx(t)|^{\alpha}ds\,|x(t)|$$

$$= -\frac{a}{\alpha+1}|x(t)|^{\alpha+1}.$$

The initial value problem $\dot{u} = -\frac{au^{\alpha+1}}{\alpha+1}$, $\quad u(\tau) = |\zeta|$, has solution

$$u = \frac{|\zeta|}{(1+A|\zeta|^{\alpha}(t-\tau))^{1/\alpha}},$$

where $A = \frac{a\alpha}{\alpha+1}$. Using a standard comparison theorem ([14], Theorem 4.1, p. 26) one finds that

$$|x(t)| \leq \frac{|\zeta|}{(1+A|\zeta|^{\alpha}(t-\tau))^{1/\alpha}}, \quad t \geq \tau.$$

Hypothesis (IIi) implies that for each τ in R and $|\sigma| < \eta$ there exists a unique solution $y(t) = y(t,\tau,\sigma)$ of $\dot{y} = g(t,y)$ in a neighborhood V of τ. So for t in V and $|\sigma| < \eta$ it follows that

$$D_L|y(t)| = -\lim_{h\to+0} \frac{|y(t)-h\dot{y}(t)|-|y(t)|}{h}$$

$$\geq -\lim_{h\to+0} \frac{1}{h}\left| \left|\left(I-h\int_0^1 g_y(t,sy(t))ds\right)y(t)\right| - |y(t)|\right|$$

$$\geq -\lim_{h\to+0} \int_0^1 \frac{|I-hg_y(t,sy(t))|-1}{h}\, ds\,|y(t)|,$$

where $D_L y(t)$ denotes the left derivative of $|y|$ at t. As above one may apply the Lebesgue Dominated Convergence Theorem to show that

$$D_L|y(t)| \geq -\int_0^1 \mu[-g_y(t,sy(t))]ds\,|y(t)| \geq \int_0^1 b\,|sy(t)|^{\beta} ds\,|y(t)|.$$

Thus

$$D_L|y(t)| \geq \frac{b}{\beta+1}|y(t)|^{\beta+1},$$

and inequality (10) follows as above with $B = \frac{b\beta}{\beta+1}$.

THE PERTURBED SYSTEM

Again let $\ell \geq 0$, $m > 0$, $n > 0$ be integers. Consider the real system of ordinary differential equations

$$
\begin{aligned}
\dot{\theta} &= d + \Theta(t,\theta,x,y), \\
\dot{x} &= f(t,x) + F(t,\theta,x,y), \qquad (11) \\
\dot{y} &= g(t,y) + G(t,\theta,x,y),
\end{aligned}
$$

where d is a constant ℓ-vector and the following assumptions hold:

(IIIi) f is $C(R) \cap C^1(B(\varepsilon_0,m)):R^m$ for some ε_0 in $(0,1)$;

(IIIii) $f(t,0) = 0$, t in R;

(IIIiii) there exist α, $a > 0$ such that
$\mu[f_x(t,x)] \leq -a|x|^\alpha$, t in R, x in $B(\varepsilon_0,m)$;

(IVi) g is $C(R) \cap C^2(B(\eta,n)):R^n$ for some η in $(0,1)$;

(IVii) $g(t,0) = 0$, $g_y(t,0) = 0$, t in R;

(IViii) there exist β, $b > 0$ such that
$b|y|^\beta \leq -\mu[-g_y(t,y)]$, t in R, y in $B(\eta,n)$;

(IViv) there exist γ, $c > 0$ such that
$\max_{i,j,k} \left| \frac{\partial^2 g_i}{\partial y_j \partial y_k}(t,y) \right| \leq c|y|^\gamma$, t in R, y in $B(\eta,n)$;

(Vi) Θ, F and G are defined and $C(R) \cap C^1(R^\ell \times B(\varepsilon_0,m)\times B(\eta,n)):R^k$ for $k = \ell$, m and n, respectively;

(Vii) Θ, F and G are multiply periodic in θ with period $\omega = (\omega_1,\ldots,\omega_\ell)$, $\omega_i > 0$, $i = 1,2,\ldots,\ell$;

(Viii) $\Theta(t,\theta,0,0)$, $F(t,\theta,0,0)$, $G(t,\theta,0,0)$ vanish for (t,θ) in $R^{1+\ell}$; and,

(Viv) there exist positive numbers p, q, r and L such that

$$|\Theta_\theta|,\ |\Theta_x|,\ |\Theta_y| \leq L(|x|+|y|)^p,$$
$$|F_\theta|,\ |F_x|,\ |F_y| \leq L(|x|+|y|)^q,$$
$$|G_\theta|,\ |G_x|,\ |G_y| \leq L(|x|+|y|)^r,$$
for $|x| < \varepsilon_0$, $|y| < \eta$.

Theorem 2. Let (IIIi) through (Viv) hold with $\gamma + 1 > \alpha$, $\min\{p,q,r\} > \alpha$. Then for all ε sufficiently small and τ any real number there exists a function v in $C([\tau,\infty)\times R^\ell \times B(\varepsilon,m)):R^n$, $v(t,\theta,x)$ multiply periodic in θ with period ω, $v(t,\theta,0) = 0$ for (t,θ) in $R^{1+\ell}$, such that

$$S_\varepsilon^+ = \{(t,\theta,x,y)\,|\,\theta \text{ arbitrary},\ |x|<\varepsilon,\ y=v(t,\theta,x),\ t\geq\tau\}$$

is a positive manifold for system (11). In case $\ell = 0$ $\min\{p,q,r\}$ is replaced by $\min\{q,r\}$.

This theorem is the main result of this thesis. The proof of Theorem 2 will be given after several results are established. First an outline of the proof will be presented.

Let $0 < \delta < \min\{1,\eta/\varepsilon\}$ and for τ in R, $0 < \varepsilon < \varepsilon_0$, define

$$\Omega(\varepsilon) = \{v \text{ in } C([\tau,\infty)) \cap C^1(R^\ell \times B(\varepsilon,m)):R^n | \\ v(t,\theta,x) \text{ satisfies (VIi)-(VIiii)}\}.$$

(VIi) v has multiple period ω in θ;

(VIii) $v(t,\theta,0) = 0$, θ in R^ℓ, $t \leq \tau$; and,

(VIiii) $$\max \sup_{\tau \leq t \leq \infty} \sup_{\theta \text{ in } R^\ell} \sup_{|x| \leq \varepsilon} \{|v(t,\theta,x)|, |v_\theta(t,\theta,x)|, \\ |v_x(t,\theta,x)|\} \leq \delta.$$

For v in $\Omega(\varepsilon)$ define $\Theta^v(t,\theta,x) = \Theta(t,\theta,x,v(t,\theta,x))$ and define F^v and G^v similarly. Let $(\psi^v(t,\tau,\theta,x),\xi^v(t,\tau,\theta,x))$ denote solutions of

$$\begin{aligned} \dot{\theta} &= d + \Theta^v(t,\theta,x), \\ \dot{x} &= f(t,x) + F^v(t,\theta,x), \end{aligned} \tag{12v}$$

with initial condition (θ,x) at $t = \tau$. When the initial conditions are clearly understood the solutions will be abbreviated $(\psi^v(t),\xi^v(t))$. It will be shown that an operator T, given by

(13)

$$(Tv)(t,\theta,x) = \int_{\infty}^{0} \Phi(t,s+t,v(s+t,\psi^{V}(s+t,t,\theta,x),\xi^{V}(s+t,t,\theta,x)))$$

$$\times G^{V}(s+t,\psi^{V}(s+t,t,\theta,x),\xi^{V}(s+t,t,\theta,x))ds,$$

where $\Phi(t,\tau,\sigma) = \frac{\partial}{\partial\sigma} y(t,\tau,\sigma)$ and $y(t,\tau,\sigma)$ is the solution of $\dot{y} = g(t,y)$ with $y(\tau,\tau,\sigma) = \sigma$, is defined on $\Omega(\varepsilon)$ into $\Omega(\varepsilon)$.

Note that

(14) $$(\psi^{V}(t,\tau,\theta+\omega,x),\xi^{V}(t,\tau,\theta+\omega,x)) = (\psi^{V}(t,\tau,\theta,x),\xi^{V}(t,\tau,\theta,x)) + (\omega,0),$$

implies that $(Tv)(t,\theta+\omega,x) = (Tv)(t,\theta,x)$ for v in $\Omega(\varepsilon)$.

The set $\Omega(\varepsilon)$ may be thought of as a subset of the Banach space of bounded continuous functions, $v(t,\theta,x)$, on $V(\varepsilon) = [\tau,\infty)\times R^{\ell}\times B(\varepsilon,m)$ with multiple period ω in θ and norm $||v|| = \sup_{V(\varepsilon)}|v(t,\theta,x)|$. It will be shown that for ε sufficiently small T is a contraction map on $\Omega(\varepsilon)$ and that there exists a function u in $\overline{\Omega(\varepsilon)}$ such that the operator T may be extended to u and Tu = u. The function $\omega(t) = u(t,\psi^{V}(t),\xi^{V}(t))$ can be shown to satisfy the differential equation

$$\dot{y} = g(t,y) + G(t,\psi^v(t),\xi^v(t),y),$$

and it follows that u is the function needed to describe the positive manifold S_ε^+ of Theorem 2.

The first of the lemmas which will be needed is

Lemma 3. Let the hypotheses of Theorem 2 hold. For $\varepsilon > 0$ sufficiently small, all θ, $|x| < \varepsilon$, v in $\Omega(\varepsilon)$, the solution $(\psi^v(t),\xi^v(t))$ of (12v) with initial condition (θ,x) at $t = \tau$ exists for $t \geq \tau$ and satisfies

$$|\xi^v(t)| \leq \frac{|x|}{(1+A|x|^\alpha(t-\tau))^{1/\alpha}},$$
$$|\xi_\theta^v(t)|,\ |\xi_x^v(t)| \leq M,$$
$$|\psi_\theta^v(t)|,\ |\psi_x^v(t)| \leq M,$$

for some positive constants A and M independent of the choice of v in $\Omega(\varepsilon)$.

Proof. Let

$$\varepsilon_1 \leq \min\left\{\varepsilon_0, \left|\frac{a}{(\alpha+1)2^{q+1}L}\right|^{\frac{1}{q-\alpha}}\right\}.$$

Then for $|x| < \varepsilon_1$ and all θ solutions $(\psi^v(t),\xi^v(t))$ of (12v) exist and satisfy $|\xi^v(t)| < \varepsilon_1$ in a neighborhood U of τ. So for t in U it follows that

$$D_R|\xi^v| = \lim_{h\to+0} \frac{|\xi^v+h\dot{\xi}^v|-|\xi^v|}{h}$$

$$= \lim_{h\to+0} \frac{|\xi^v+h(f(t,\xi^v)+F^v(t,\psi^v,\xi^v))|-|\xi^v|}{h}$$

$$\leq \lim_{h\to+0} \frac{|\xi^v+hf(t,\xi^v)|-|\xi^v|}{h} + |F^v(t,\psi^v,\xi^v)|.$$

As in the proof of Theorem 1 one finds that

$$\lim_{h\to+0} \frac{|\xi^v+hf(t,\xi^v)|-|\xi^v|}{h} \leq \frac{-a}{\alpha+1} |\xi^v|^{\alpha+1}.$$

From (VIii) and (VIiii) it follows that

$$|v(t,\theta,x)| \leq \delta|x| \leq |x|,$$

and from (Viii) and (Viv)

$$|F^v(t,\psi^v,\xi^v)| \leq 2^{q+1}L|\xi^v|^{q+1}.$$

Therefore

$$D_R|\xi^v| \leq \frac{-a}{\alpha+1}|\xi^v|^{\alpha+1} + 2^{q+1}L|\xi^v|^{q+1}$$

$$\leq \frac{-1}{\alpha+1}(a-2^{q+1}(\alpha+1)L\varepsilon_1^{q-\alpha})|\xi^v|^{\alpha+1}$$

$$\leq \frac{-A}{a}|\xi^v|^{\alpha+1},$$

where $A = \frac{\alpha}{\alpha+1}(a-(\alpha+1)2^{q+1}L\varepsilon_1^{q-\alpha}) > 0$. Thus, as in Theorem 1,

$$|\xi^V(t,\tau,\theta,x)| \leq \frac{|x|}{(1+A|x|^{\alpha}(t-\tau))^{1/\alpha}}, \quad t \geq \tau.$$

The existence and continuity of ξ_θ^V, ξ_x^V, ψ_θ^V and ψ_x^V in $V(\varepsilon_1)$ are well known, see ([17], p. 25). Let $(\lambda(t),\chi(t)) = (\psi_\theta^V(t),\xi_\theta^V(t))$ or $(\psi_x^V(t),\xi_x^V(t))$, then from (12) it can be seen that

$$\dot{\lambda} = (\Theta_\theta+\Theta_y v_\theta)\lambda + (\Theta_x+\Theta_y v_x)\chi,$$
$$\dot{\chi} = (f_x+F_x+F_y v_x)\chi + (F_\theta+F_y v_\theta)\lambda,$$

where $\Theta_\theta = \Theta_\theta(t,\psi^V(t,\tau,\theta,x),\xi^V(t,\tau,\theta,x),v(t,\psi^V(t,\tau,\theta,x),$
$\xi^V(t,\tau,\theta,x)))$,

etc. Therefore

$$\begin{aligned}
|\dot{\lambda}| \leq{}& L[(|\xi^V|+|v(t,\psi^V,\xi^V)|)^p+(|\xi^V|+|v(t,\psi^V,\xi^V)|)^p \\
&|v_\theta|]|\lambda| \\
&+ L[(|\xi^V|+|v(t,\psi^V,\xi^V)|)^p+(|\xi^V|+|v(t,\psi^V,\xi^V)|)^p \\
&|v_x|]|\chi| \\
\leq{}& 2^{p+1}L|\xi^V|^p|\lambda| + 2^{p+1}L|\xi^V|^p|\chi|,
\end{aligned}$$

and $D_R|\chi| = \lim_{h\to+0} \frac{|\chi+h\dot{\chi}|-|\chi|}{h}$,

$$\leq \lim_{h\to+0} \frac{|I+hf_x|-1}{h}|\chi| + |F_x+F_yv_x||\chi| + |F_\theta+F_yv_\theta||\lambda|$$

$$\leq \mu[f_x(t,\xi^v)]|\chi| + 2^{q+1}L|\xi^v|^q|\chi| + 2^{q+1}L|\xi^v|^q|\lambda|$$

$$\leq -A_0|\xi^v|^\alpha|\chi| + 2^{q+1}L|\xi^v|^q|\lambda|,$$

where $A_0 = a - 2^{q+1}L\varepsilon_1^{q-\alpha} > 0$. Since $D_R|\lambda| \leq |\dot{\lambda}|$ and since $p > \alpha$ implies $\int_\tau^\infty |\xi^v(s)|^p ds < \infty$ it follows that there exist constants K_1, $K_2 > 0$ such that

$$|\lambda(t)| \leq K_1 + K_2 \int_\tau^t |\chi(s)||\xi^v(s)|^p ds, \tag{15}$$

so that

$$D_R|\chi(t)| \leq -A_0|\xi^v(t)|^\alpha|\chi(t)| + 2^{q+1}L|\xi^v|^q \times (K_1+K_2\int_\tau^t |\chi(s)||\xi^v(s)|^p ds).$$

Now consider the scalar equation

$$\dot{\nu} = -A_0|\xi^v|^\alpha\nu + K_3|\xi^v|^q,$$

where

$$0 < K_3 = 2^{q+1}L(K_1+K_2 \int_\tau^\infty 2|\xi^v(s)|^p ds) < \infty.$$

It is clear that for $t \geq \tau$

$$\nu(t) = \nu(\tau)\exp[-A_0\int_\tau^t|\xi^V|^\alpha ds] + K_3\int_\tau^t|\xi^V|^q\exp[-A_0\int_s^t|\xi^V|^\alpha du]ds,$$

$$\leq \nu(\tau) + K_3\int_\tau^t|\xi^V(s)|^q ds.$$

Now for

$$\varepsilon < \min\left\{\varepsilon_1, \left|\frac{A(q-\alpha)}{K_3\alpha}\right|^{\frac{1}{q-\alpha}}\right\},$$

and $|x| < \varepsilon$,

$$K_3\int_\tau^\infty|\xi^V(s,\tau,\theta,x)|^q ds \leq K_3\int_\tau^\infty\frac{|x|^q \quad ds}{(1+A|x|^\alpha(s-\tau))^{q/\alpha}} = \frac{K_3\alpha|x|^{q-\alpha}}{A(q-\alpha)}.$$

Then if $0 \leq \nu(\tau) \leq 1$ it follows that $0 \leq \nu(\tau) \leq 2$ for all $t \geq \tau$. Suppose $\nu(\tau)$ is chosen so that $\nu(\tau) = |\chi(\tau)|$ then $|\chi(t)| \leq \nu(t)$ for all $t \geq \tau$. For suppose this is not the case. Then there exist $t \geq \tau$ such that $|\chi(t)| > \nu(t)$. Let

$$t_1 = \sup_{\hat{t}}\{\hat{t}\,|\,|\chi(t)|-\nu(t)\leq 0,\ \tau\leq t\leq\hat{t}\},$$

then $|\chi(t_1)| = \nu(t_1)$ and

$$D_R|\chi(t_1)| \leq -A_0|\xi^V(t_1)|^\alpha\nu(t_1)$$

$$+ 2^{q+1}L|\xi^V(t_1)|^q(K_1+K_2\int^t|\xi^V(s)|^p\nu(s)ds)$$

$$-A_0|\xi^V(t_1)|^\alpha\nu(t_1) + K_3|\xi^V(t_1)|^q = D_R\nu(t_1).$$

Thus

$$D_R(|\chi(t_1)|-\nu(t_1)) < 0,$$

and this contradicts the definition of t_1. Therefore from (15) it follows that

$$|\lambda(t)| \le K_1 + K_2\int_\tau^t 2|\xi^\nu(s)|^p ds$$
$$\le K_1 + 2K_2\int_\tau^\infty |\xi^\nu(s)|^p ds = M < \infty.$$

<u>Lemma 4</u>. Let (IVi) through (IViv) hold and let $y(t,\tau,\sigma)$ be the solution of $\dot{y} = g(t,y)$ with $y(\tau,\tau,\sigma) = \sigma$. Denote $\frac{\partial}{\partial\sigma}y(t,\tau,\sigma)$ by $\Phi(t,\tau,\sigma)$. Then for $|\sigma| < \eta$ and $t \le \tau$

$$|\Phi(t,\tau,\sigma)| \le 1,$$

and

$$\frac{\partial}{\partial\sigma_j}\Phi(t,\tau,\sigma)| \le \Gamma|t-\tau||\sigma|^\gamma,\ \Gamma > 0,\ j = 1,2,\ldots\ .$$

<u>Proof</u>. For solutions $z(t)$ of the linear system

$$\dot{z} = C(t)z,$$

it is known that ([18], Theorem 3, p. 58) for $t \le \tau$

$$|z(t)| \le |z(\tau)| \exp\int_\tau^t -\mu[-C(s)]ds.$$

Since $\Phi(t,\tau,\sigma)$ is a fundamental matrix for

$$\dot{z} = g_y(t,y(t,\tau,\sigma))z,$$

it follows that

$$|\Phi(t,\tau,\sigma)| \le \exp\int_{\tau}^{t} -\mu[-g_y(s,y(s,\tau,\sigma))]ds$$
$$\le \exp\int_{t}^{\tau} -b|y(s,\tau,\sigma)|^{\beta}ds \le 1.$$

Since g is $C(R) \cap C^2(B(\eta,n)):R^n$ the partial derivatives of $\Phi(t,\tau,\sigma)$ with respect to each component σ_j of σ exist and satisfy

$$\dot{Z} = g_y(t,y(t,\tau,\sigma))Z + A_j(t),$$

where

$$A_j(t) = M_j(t)\Phi(t,\tau,\sigma),$$

and $M_j(t)$ is the n×n matrix with k, ℓth element given by

$$\sum_{i=1}^{n} \frac{\partial^2 g_k}{\partial y_i \partial y_\ell}(t,y(t,\tau,s))\frac{\partial y_i}{\partial \sigma_j}(t,\tau,\sigma),$$

See ([14], Theorem 3.1, p. 95). Thus, by the usual procedure for solving a nonhomogeneous system of linear differential equations ([14], pp. 48-49) it can be seen that

$$\frac{\partial \Phi}{\partial \sigma_j}(t,\tau,\sigma) = \int_{\tau}^{t} \Phi(t,s,\sigma)A_j(s)ds.$$

Therefore for $t \leq \tau$

$$\left|\frac{\partial}{\partial \sigma_j}\Phi(t,\tau,\sigma)\right| \leq \int_t^\tau |\Phi(t,s,\sigma)|\,|A_j(s)|ds \leq \int_t^\tau A_j(s)\ ds.$$

Making use of norm equivalence in R^n there exists $N > 0$ such that

$$\begin{aligned}|A_j(s)| &\leq N \max_{i,k,\ell}\left|\frac{\partial^2 g_k}{\partial y_i \partial y_\ell}(s,y(s,\tau,\sigma))\right| |\Phi(s,\tau,\sigma)|^2 \\ &\leq Nc|y(s,\tau,\sigma)|^\gamma.\end{aligned}$$

Therefore by inequality (10)

$$\begin{aligned}\left|\frac{\partial}{\partial \sigma_j}\Phi(t,\tau,\sigma)\right| &\leq Nc\int_t^\tau |y(s,\tau,\sigma)|^\gamma ds \leq Nc|\sigma|^\gamma \int_t^\tau \frac{1}{(1-B|\sigma|^\gamma(s-\tau))^{\gamma/\beta}}ds \\ &\leq Nc|\sigma|^\gamma(\tau-t).\end{aligned}$$

Choose $\Gamma = Nc$ and the result follows.

Theorem 5. Let the hypotheses of Theorem 2 hold. Then for $\varepsilon > 0$ sufficiently small the operator T given by equation (13) is defined on $\Omega(\varepsilon)$ and maps $\Omega(\varepsilon)$ into itself.

Proof. Let $\varepsilon_2 > 0$ be small enough so that Lemmas 3 and 4 hold. Then for (t,θ,x) in $V(\varepsilon_2)$

$$
\begin{aligned}
|Tv(t,\theta,x)| &\le \int_0^\infty L(|\xi^v(t+s,t,\theta,x)| \\
&\quad + |v(t+s,\psi^v(t+s,t,\theta,x),\xi^v(t+s,t,\theta,x))|)^{r+1}ds \\
&\le 2^{r+1}L\int_0^\infty |\xi^v(t+s,t,\theta,x)|^{r+1}ds \\
&\le 2^{r+1}L\int_0^\infty \frac{|x|^{r+1}}{(1+A|x|^\alpha s)^{(r+1)/\alpha}}\,ds \\
&= \frac{2^{r+1}L\alpha|x|^{r+1-\alpha}}{A(r+1-\alpha)} < \infty.
\end{aligned}
$$

Let

$$
\varepsilon_3 = \min\left\{ \varepsilon_2, \left|\frac{A\delta(r+1-\alpha)}{2^{r+1}L\alpha}\right|^{\frac{1}{r+1-\alpha}} \right\},
$$

then for $\varepsilon < \varepsilon_3$ T is defined on $\Omega(\varepsilon)$, $Tv(t,\theta,0) = 0$ and $|Tv(t,\theta,x)| < \delta$. Also since

$$
\frac{|x|^{r+1}}{(1+A|x|^\alpha s)^{(r+1)/\alpha}} \le \frac{1}{(1+As)^{(r+1)/\alpha}} ,
$$

and $\int_0^\infty \frac{1}{(1+As)^{(r+1)/\alpha}}\,ds < \infty$ it follows that Tv converges uniformly for (t,θ,x) in $V(\varepsilon)$ which implies that Tv is continuous on $V(\varepsilon)$. It has been noted previously that Tv has period ω in θ for v in $\Omega(\varepsilon)$. It remains to be shown that $\frac{\partial}{\partial\theta}(Tv)$, $\frac{\partial}{\partial x}(Tv)$ exist and are continuous and bounded by δ for (t,θ,x) in $V(\varepsilon)$.

In the following the summation convention will be used. Let λ represent either the θ or the x vector. Then the i,j<u>th</u> element of the matrix $\frac{\partial}{\partial x}(\Phi G^v)$ is given by

$$\frac{\partial}{\partial \lambda_j}(\Phi_{ik}G_k^v) = \frac{\partial \Phi_{ik}}{\partial \sigma_\ell}\left[\frac{\partial v_\ell}{\partial \theta_m}\frac{\partial \psi_m^v}{\partial \lambda_j} + \frac{\partial v_\ell}{\partial x_m}\frac{\partial \xi_m^v}{\partial \lambda_j}\right]G_k^v$$
$$+ \Phi_{ik}\left[\frac{\partial G_k}{\partial \theta_\ell}\frac{\partial \psi_\ell^v}{\partial \lambda_j} + \frac{\partial G_k}{\partial x_\ell}\frac{\partial \xi_\ell^v}{\partial y_j}\right.$$
$$\left. + \frac{\partial G_k}{\partial y_\ell}\left\{\frac{\partial v_\ell}{\partial \theta_m}\frac{\partial \psi_m}{\partial \lambda_j} + \frac{\partial v_\ell}{\partial x_m}\frac{\partial \xi_m^v}{\partial \lambda_j}\right\}\right],$$

where $\phi G^v = \phi(t,t+s,v(t+s,\psi^v(t+s,t,\theta,x),\xi^v(t+s,t,\theta,x)))$
$\times G^v(t+s,\psi^v(t+s,t,\theta,x),\xi^v(t+s,t,\theta,x))$.

From Lemmas 3 and 4, (Viii), (Viv) and norm equivalence for R^n there exists $N^* > 0$ such that

$$\begin{aligned}\frac{\partial}{\partial \lambda}(\phi G^v) &\le NN^*\Gamma(s)|v|^\gamma(2M)L(|\xi^v|+|v|)^{r+1} \\ &\quad + N^*[L(|\xi^v|+|v|)^r M+L(|\xi^v|+|v|)^r 3M] \\ &\le M_0\frac{s|x|^{\gamma+r+1}}{(1+A|x|^\alpha s)^{(\gamma+r+1)/\alpha}} \\ &\quad + M_0\frac{|x|^r}{(1+A|x|^\alpha s)^{r/\alpha}},\end{aligned}$$

where $M_0 = 2^r N^*LM\max\{4N\Gamma,3\}$. Since $r > \max\{2\alpha-(\gamma+1),\alpha\}$ if follows that

$$\int_0^\infty \frac{s|x|^{\gamma+r+1}}{(1+A|x|^\alpha s)^{(\gamma+r+1)/\alpha}}ds = \frac{\alpha^2|x|^{\gamma+r+1-2\alpha}}{A^2(\alpha-(\gamma+r+1))(2\alpha-(\gamma+r+1))}$$

$$\int_0^\infty \frac{|x|^r}{(1+A|x|^\alpha s)^{r/\alpha}}ds = \frac{|x|^{r-\alpha}}{A(r+1-\alpha)} ,$$

and

$$\int_0^\infty \frac{s|x|^{\gamma+r+1}}{(1+A|x|^\alpha s)^{(\gamma+r+1)/\alpha}}ds \le \int_0^\infty \frac{s}{(1+As)^{(\gamma+r+1)/\alpha}}ds < \infty,$$

$$\int_0^\infty \frac{|x|}{(1+A|x|^\alpha s)^{r/\alpha}}ds \le \int_0^\infty \frac{1}{(1+As)^{r/\alpha}}ds < \infty.$$

Therefore $\int_\infty^0 \frac{\partial}{\partial\lambda}(\Phi G^V)ds$ converges uniformly for (t,θ,x) in $V(\varepsilon_2)$ and is dominated by a continuous real valued function $D(\varepsilon)$ defined on $[0,\varepsilon_2)$ with $\lim_{\varepsilon\to 0} D(\varepsilon) = 0$. If $\varepsilon_3 > 0$ is such that $D(\varepsilon_3) < \delta$ then for (t,θ,x) in $V(\varepsilon_3)$

$$\frac{\partial}{\partial\lambda}(Tv) = \int_\infty^0 \frac{\partial}{\partial\lambda}(\Phi G^V)ds,$$

and

$$\left|\frac{\partial}{\partial\lambda}(Tv)\right| < \delta.$$

The following lemmas will be used to show that T is a contraction map on $\Omega(\varepsilon)$.

Lemma 6. Let the hypotheses of Theorem 2 hold and let $\varepsilon_4 > 0$ be such that Theorem 5 holds. Further let v, w be in $\Omega(\varepsilon_4)$ and suppose that $(\psi^v(t,\tau,\theta,x),\xi^v(t,\tau,\theta,x))$, $(\psi^v(t,\tau,\theta,x),\xi^w(t,\tau,\theta,x))$ are the corresponding solutions of (12v), (12w). Then there exist $K > 0$, independent of v, w in $\Omega(\varepsilon_4)$ such that

$$|\psi^v-\psi^w| + |\xi^v-\xi^w| \leq K||v-w||,$$

where $||v-w|| = \sup\limits_{V(\varepsilon_4)} |v(t,\tau,x)-w(t,\theta,x)|$.

Proof. For (t,θ,x,y), $(t,\hat{\theta},\hat{x},\hat{y})$ in $V(\varepsilon_4)\times B(\eta,n)$ and

$$|\bar{x}| = \max\{|x|,|\hat{x}|\}, \qquad |\bar{y}| = \max\{|y|,|\hat{y}|\},$$

it is clear that

$$|\Theta(t,\hat{\theta},\hat{x},\hat{y})-\Theta(t,\theta,x,y)| \leq L(|\bar{x}|+|\bar{y}|)^p(|\theta-\hat{\theta}|+|x-\hat{x}|+|y-\hat{y}|).$$

Let $h(t) = \max\{|\xi^v(t)|,|\xi^w(t)|\}$, then $\max\{|v(t,\psi^v(t),\xi^v(t))|,|w(t,\psi^w(t),\xi^w(t))|\} \leq h(t)$. But

$$\begin{aligned}|v(t,\psi^v,\psi^v)-w(t,\psi^w,\psi^w)|& \\ &\leq |v(t,\psi^v,\xi^v)-v(t,\psi^w,\xi^w)|+|v(t,\psi^w,\xi^w)-w(t,\psi^w,\xi^w)| \\ &\leq (|\psi^v-\psi^w|+|\xi^v-\xi^w|+||v-w||),\end{aligned}$$

implying that

$$|\Theta^v(t,\psi^v,\xi^v)-\Theta^w(t,\psi^w,\xi^w)|$$
$$\le 2^p Lh^p(t)(2(|\psi^v-\psi^w|+|\xi^v-\xi^w|)+||v-w||)$$
$$\le 2^{p+1}Lh^p(|\psi^v-\psi^w|+|\xi^v-\xi^w|+||v-w||).$$

In the same way,

$$|F^v(t,\psi^v,\xi^v)-F^w(t,\psi^w,\xi^w)|$$
$$\le 2^{q+1}Lh^q(t)(|\psi^v-\psi^w|+|\xi^v-\xi^w|+||v-w||).$$

Let $H(t) = Lh(t)\max\{2^{p+1},2^{q+1}\}$. Note that Lemma 3 implies $\int_\tau^\infty H^p(s)ds < \infty$ and $\int_\tau^\infty H^q(s)ds < \infty$.

Now from (12)

$$(16)\quad D_R|\psi^v-\psi^w| \le \left|\frac{d}{dt}(\psi^v-\psi^w)\right|$$
$$\le |\Theta^v(t,\psi^v,\xi^v)-\Theta^w(t,\psi^w,\xi^w)|$$
$$\le H^p(t)(|\psi^v-\psi^w|+|\xi^v-\xi^w|+||v-w||),$$

and

$$
\begin{aligned}
D_R|\xi^v-\xi^w| &= \lim_{h\to+0} \frac{|(\xi^v-\xi^w)+h(\dot{\xi}^v-\dot{\xi}^w)|-|\xi^v-\xi^w|}{h} \\
&\le \lim_{h\to+0} \frac{|(\xi^v-\xi^w)+h(f(t,\xi^v)-f(t,\xi^w))|-|\xi^v-\xi^w|}{h} \\
&\qquad + |F^v(t,\psi^v,\xi^v)-F^w(t,\psi^w,\xi^w)|.
\end{aligned}
$$

The mean value theorem gives, for some ν in $[0,1]$

$$
f(t,\xi^v) - f(t,\xi^w) = f_x(t,\nu\xi^v+(1-\nu)\xi^w)(\xi^v-\xi^w),
$$

and therefore

$$
\begin{aligned}
D_R|\xi^v-\xi^w| &\le \mu[f_x(t,\nu\xi^v+(1-\nu)\xi^w)]|\xi^v-\xi^w| \\
&\qquad + H^q(t)(|\psi^v-\psi^w|+|\xi^v-\xi^w|+||v-w||).
\end{aligned}
$$

But

$$
\mu[f_x(t,\nu\xi^v+(1-\nu)\xi^w)] \le -a|\nu\xi^v+(1-\nu)\xi^w|^\alpha \le 0,
$$

so that

$$
D_R|\xi^v-\xi^w| \le H^q(t)(|\psi^v-\psi^w|+|\xi^v-\xi^w|+||v-w||),\ t \ge \tau.
$$

The above inequality, together with (16), gives

$$
D_R(|\psi^v-\psi^w|+|\xi^v-\xi^w|) \le K(t)(|\psi^v-\psi^w|+|\xi^v-\xi^w|+||v-w||),
$$

where $K(t) = H^q(t) + H^p(t)$. Since $|\psi^v(\tau)-\psi^w(\tau)| = 0 = |\xi^v(\tau)-\xi^w(\tau)|$ it follows that

$$|\psi^v-\psi^w| + |\xi^v-\xi^w| \le ||v-w||\int_\tau^t K(s)\exp\{\int_s^t K(u)du\}ds$$

$$\le K||v-w||,$$

where $K = \exp\int_\tau^\infty K(s)ds < \infty$.

Lemma 7. Let the hypotheses of Theorem 2 hold and let $\Phi(t,\tau,\sigma)$ be as in Lemma 4. Then for all (t,θ,x,y), $(t,\hat\theta,\hat x,\hat y)$ in $(-\infty,\tau]\times R^\ell\times B(\varepsilon_4,m)\times B(\eta,m)$ there exists $P > 0$ such that

$$\begin{aligned}|\Phi(t,\tau,y)G(t,\theta,x,y)-\Phi(t,\tau,\hat y)G(t,\hat\theta,\hat x,\hat y)|\\ \le P\{(|\bar x|+|\bar y|)^r+(\tau-t)(|\bar x|+|\bar y|)^{r+1}|\bar y|^\gamma\}\\ \times\{|\theta-\hat\theta|+|x-\hat x|+|y-\hat y|\},\end{aligned}$$

where $|\bar x| = \max\{|x|,|\hat x|\}$, $|\bar y| = \max\{|y|,|\hat y|\}$.

Proof. The mean value theorem gives

$$\begin{aligned}|\Phi(t,\tau,y)G(t,\theta,x,y)-\Phi(t,\tau,\hat y)G(t,\hat\theta,\hat x,\hat y)\\ \le \sup\left\{\left|\frac{\partial}{\partial\theta}(\Phi G)\right|+\left|\frac{\partial}{\partial x}(\Phi G)\right|+\left|\frac{\partial}{\partial y}(\Phi G)\right|\right\}\{|\theta-\hat\theta|+|x-\hat x|+|y-\hat y|\},\end{aligned}$$

where the supremum is taken over $R^\ell\times B(\varepsilon_4,m)\times B(\eta,m)$.

For $\lambda = \theta$ or x it follows from Lemma 4 and (Viv) that

$$\left|\frac{\partial}{\partial\lambda}(\Phi(t,\tau,y)G(t,\theta,x,y))\right| \leq L(|x|+|y|)^r.$$

Again norm equivalence for R^n implies there exists $N_1 > 0$ such that

$$\begin{aligned}&\left|\frac{\partial}{\partial y}(\Phi(t,\tau,y)G(t,\theta,x,y)\right| \\ &\qquad \leq N_1|G|\max_j \left|\frac{\partial\Phi}{\partial y_j}(t,\tau,y)\right| + |\Phi(t,\tau,y)|\left|\frac{\partial}{\partial y}G(t,\theta,x,y)\right|.\end{aligned}$$

By applying Lemma 4 and (Viv) it can be seen that

$$\begin{aligned}\left|\frac{\partial}{\partial y}(\Phi(t,\tau,y)G(t,\theta,x,y)\right| &\leq N_1L(|x|+|y|)^{r+1}\Gamma|y|^{\gamma}(t-\tau) \\ &\quad + L(|x|+|y|)^r.\end{aligned}$$

Put $P = \max\{N_1L\Gamma, 3L\}$ and the result follows.

Theorem 8. Let the hypotheses of Theorem 2 hold. Then for all $\varepsilon > 0$ sufficiently small the operator T, defined by equation (13), is a contraction map on $\Omega(\varepsilon)$.

Proof. Let $\varepsilon_5 > 0$ be such that Theorem 5 holds and let v, w be in $\Omega(\varepsilon_5)$. Then for $h(t) = \max\{|\xi^v(t)|, |\xi^w(t)|\}$ it follows from Lemmas 3, 6 and 7 and equation (13) that

$$
\begin{aligned}
|Tv-Tw| \le \int_0^\infty & |\Phi(t,t+s,v(t+s,\psi^v(t+s,t,\theta,x),\xi^v(t+s,t,\theta,x))) \\
& \times G^v(t+s,\psi^v(t+s,t,\theta,x),\xi^v(t+s,t,\theta,x)) \\
& - \Phi(t,t+s,w(t+s,\psi^w(t+s,t,\theta,x),\xi^w(t+s,t,\theta,x))) \\
& \times G^w(t+s,\psi^w(t+s,t,\theta,x),\xi^w(t+s,t,\theta,x)))|ds \\
\le P\int_0^\infty & (2^r h^r(s)+2^{r+1}sh^{r+\gamma+1}(s)) \\
& \times(|\psi^v-\psi^w|+|\xi^v-\xi^w|+||v-w||)ds \\
\le 2^{r+1}P(K+1) & \int_0^\infty (h^r(s)+sh^{r+\gamma+1}(s))ds\,||v-w|| \\
\le 2^{r+1}P(K+1) & \left|\int_0^\infty \frac{|x|^r}{(1+A|x|^\alpha s)^{r/\alpha}}\,ds \right. \\
& \left. + \int_0^\infty \frac{s|x|^r}{(1+A|x|^\alpha s)^{(r+\gamma+1)/\alpha}}\,ds\right| ||v-w|| \\
\le 2^{r+1}P(K+1) & D(|x|)||v-w||,
\end{aligned}
$$

where $D(\varepsilon)$ is described in Theorem 5. Let $\varepsilon_6 > 0$ be such that $2^{r+1}P(K+1)D(\varepsilon_6) < 1$. Then for $\varepsilon = \min\{\varepsilon_5,\varepsilon_6\}$ the result follows.

Proof of Theorem 2. Since T maps $\Omega(\varepsilon)$ into itself and is a contraction map it follows that the sequence $\{v_n\}_{n=1}^\infty$, v_1 in $\Omega(\varepsilon)$, $v_2 = Tv_1,\ldots,v_{n+1} = Tv_n,\ldots$ is Cauchy and therefore converges to a continuous function u in $\overline{\Omega(\varepsilon)}$. Since $\{v_n\}_{n=1}^\infty$ is uniformly Lipschitz with Lipschitz constant δ it follows that the limit function

u is likewise Lipschitz. By extending the definitions of Θ^v, G^v and F^v to allow v to be continuous it can be seen that the initial value problem (12u) has the unique solution $(\psi^u(t,\tau,\theta,x),\xi^u(t,\tau,\theta,x))$. Further, since $\Theta^{v_n}(t,\theta,x) \to \Theta^u(t,\tau,x)$ and $F^{v_n}(t,\theta,x) \to F^u(t,\theta,x)$ uniformly on $[\tau,\infty)\times R^\ell\times B(\varepsilon,m)\times B(\eta,n)$ it follows ([14], Theorem 2.4, p. 4) that $(\psi^{v_n},\xi^{v_n}) \to (\psi^u,\xi^u)$ uniformly on $I\times I\times R^\ell\times B(\varepsilon,m)$ where I is any compact interval in $[\tau,\infty)$. Observe that the bounds computed for ξ^v in Lemma 3 hold for ξ^u.

Consider the sequence of functions $\{J_n\}_{n=1}^\infty$, where

$$\begin{aligned} J_n(s) = {} & \Phi(t,t+s,v_n(t+s,\psi^{v_n}(t+s,t,\theta,x),\xi^{v_n}(t+s,t,\theta,x))) \\ & \times G^{v_n}(t+s,\psi^{v_n}(t+s,t,\theta,x),\xi^{v_n}(t+s,t,\theta,x)), \\ & \qquad n = 1,2,\ldots \end{aligned}$$

From the above remarks and from the continuity of and G it follows that

$$\begin{aligned} \lim_{n\to\infty} J_n(s) = {} & \Phi(t,t+s,u(t+s,\psi^u(t+s,t,\theta,x),\xi^u(t+s,t,\theta,x))) \\ & \times G^u(t+s,\psi^u(t+s,t,\theta,x),\xi^u(t+s,t,\theta,x)), \end{aligned}$$

for $\tau \le t \le s < \infty$.

It follows from Lemmas 3, 6 and (Viv) that

$$|J_n(s)| \le |G^{v_n}(t+s,\psi^{v_n}(t+s,t,\theta,x),\xi^{v_n}(t+s,t,\theta,x))|,$$

$$\le 2^{r+1}L|\xi^{v_n}(t+s,t,\theta,x)| \le \frac{2^{r+1}L|x|^{r+1}}{(1+A|x|^{\alpha}s)^{r+1/\alpha}},$$

$$n = 1,2,\ldots,$$

therefore the Lebesgue Dominated Convergence Theorem implies

$$(18) \quad u(t,\theta,x) = \lim_{n\to\infty} v_n(t,\theta,x) = \lim_{n\to\infty} \int_{\infty}^{0} J_{n-1}(s)ds$$
$$= \int_{\infty}^{0} \lim_{n\to\infty} J_{n-1}(s)ds$$
$$= \int_{\infty}^{0} \Phi(t,t+s,u(t+s,\psi^u(t+s,t,\theta,x),\xi^u(t+s,t,\theta,x)))$$
$$\times G^u(t+s,\psi^u(t+s,t,\theta,x),\xi^u(t+s,t,\theta,x))ds$$

By using this representation of u it will now be shown that the function $\omega(t) = u(t,\psi^u(t),\xi^u(t))$ satisfies the differential equation

$$\dot{y} = g(t,y) + G(t,\psi^u(t),\xi^u(t),y).$$

From the definition of $\omega(t)$ it follows that

$$\omega(t) = \int_{\infty}^{0} \Phi(t, t+s, u(t+s, \psi^u(t+s, t, \psi^u(t), \xi^u(t)), \xi^u(t+s, t, \psi^u(t), \xi^u(t))) \times G^u(t+s, \psi^u(t+s, t, \psi^u(t), \xi^u(t)), \xi^u(t+s, t, \psi^u(t), \xi^u(t))ds,$$

but from the uniqueness of solutions of (12u),

$$(\xi^u(t+s, t, \psi^u(t, \tau, \theta, x), \xi^u(t, \tau, \theta, x)), \xi^u(t+s, t, \psi^u(t, \tau, \theta, x), \xi^u(t, \tau, \theta, x)) = (\psi^u(t+s, \tau, \theta, x), \xi^u(t+s, \tau, \theta, x)).$$

Thus

$$\begin{aligned}\omega(t) &= \int_{\infty}^{0} \Phi(t, t+s, u(t+s, \psi^u(t+s, \tau, \theta, x), \xi^u(t+s, \tau, \theta, x))) \times G^u(t+s, \psi^u(t+s, \tau, \theta, x), \xi^u(t+s, \tau, \theta, x))ds \\ &= \int_{\infty}^{t} \Phi(t, s, u(s, \psi^u(s), \xi^u(s))) \times G(s, \psi^u(s), \xi^u(s), u(s, \psi^u(s)\xi^u(s)))ds \\ &= \int_{\infty}^{t} \Phi(t, s, \omega(s))G(s, \psi^u(s), \xi^u(s), \omega(s))ds.\end{aligned}$$

Let $\hat{t}$ be fixed such that $t \leq \hat{t} < \infty$ and let

$$Q^{\hat{t}}(t) = \int_{t}^{\hat{t}} \Phi(t, s, \omega(s))G(s, \psi^u(s), \xi^u(s), \omega(s))ds,$$

and

$$Q_{\hat{t}}(t) = \int_{\hat{t}}^{\infty} \Phi(t,s,\omega(s))G(s,\psi^u(s),\xi^u(s),\omega(s))ds.$$

Since $\frac{\partial\Phi}{\partial t}(t,s,\sigma) = g_y(t,y(t,s,\sigma))\Phi(t,s,\sigma)$ is continuous for $t \le s \le \tau$ and $|\sigma| < \eta$ and since $\Phi(t,t,\sigma) = I$

$$\frac{d}{dt}Q^{\hat{t}}(t) = -G(t,\psi^u(t),\xi^u(t),\omega(t)) + \int_t^{\hat{t}} g_y(t,y(t,s,\omega(s)))\Phi(t,s,\omega(s)) \times G(s,\psi^u(s),\xi^u(s),\omega(s))ds.$$

Also since

$$(19) \quad \left|\int_{\hat{t}}^{\infty} g_y(t,y(t,s,\omega(s)))\Phi(t,s,\omega(s)) \times G(s,\psi^u(s),\xi^u(s),\omega(s))ds\right|$$

$$\le \int_{\hat{t}}^{\infty} L\frac{c}{\gamma+1}|y(t,s,\omega(s))|^{\gamma+1}(|\xi^u(s)|+|\omega(s)|)^{r+1}ds$$

$$\le \frac{2^{r+1}Lc}{\gamma+1} \int_{\hat{t}}^{\infty} |\xi^u(s)|^{r+\gamma+2}ds$$

$$\le \frac{2^{r+1}Lc}{\gamma+1} \int_{\hat{t}}^{\infty} \frac{1}{(1+A(s-\tau))^{(r+\gamma+2)/\alpha}}\, ds$$

implies that

$$\frac{d}{dt}Q_{\hat{t}}(t) = \int_{\hat{t}}^{\infty} g_y(t,y(t,s,\omega(s)))\Phi(t,s,\omega(s))$$
$$\times G(s,\psi^u(s),\xi^u(s),\omega(s))ds,$$

it follows that

$$(20) \quad \dot{\omega}(t) = \frac{d}{dt}Q^{\hat{t}}(t) + \frac{d}{dt}Q_{\hat{t}}(t)$$
$$= -G(t,\psi^u(t),\xi^u(t),\omega(t))$$
$$+ \int_{t}^{\infty} g_y(t,y(t,s,\omega(s)))\Phi(t,s,\omega(s))$$
$$\times G(s,\psi^u(s),\xi^u(s),\omega(s))ds.$$

Now observe that

$$\int_{\infty}^{t} \frac{d}{ds} g(t,y(t,s,\omega(s)))ds = g(t,\omega(t))$$
$$- \lim_{\substack{s\to\infty \\ s\geq t}} g(t,y(t,s,\omega(s))).$$

But

$$\overline{\lim_{\substack{s\to\infty \\ s\geq t}}} |y(t,s,\omega(s))| \leq \overline{\lim_{\substack{s\to\infty \\ s\geq t}}} \frac{|\omega(s)|}{(1-B(t-s)|\omega(s)|^{\beta})^{1/\beta}} = 0,$$

and since $g(t,0) = 0$, t in R, and g is continuous, it follows that $\lim_{\substack{s\to\infty \\ s\geq t}} g(t,y(t,s,\omega(s))) = 0$. Also

$$\frac{d}{ds}g(t,y(t,s,\omega(s))) = g_y(t,y(t,y(t,s,\omega(s)))$$
$$\left[\frac{\partial y}{\partial s}(t,s,\omega(s))+\frac{\partial y}{\partial \sigma}(t,s,\omega(s))\dot{\omega}(s)\right]$$
$$= g_y(t,y(t,s,\omega(s))\Phi(t,s,\omega(s))$$
$$\times[-g(s,\omega(s))+\dot{\omega}(s)].$$

Therefore

$$g(t,\omega(t)) = \int_{\infty}^{t} g_y(t,y(t,s,\omega(s)))\Phi(t,s,\omega(s)$$
$$\times[\dot{\omega}(s)-g(s,\omega(s))]ds.$$

This equation, together with (20), gives

$$\dot{\omega}(t) - g(t,\omega(t)) - G(t,\psi^u(t),\xi^u(t),\omega(t))$$
$$= -\int_{t}^{\infty} g_y(t,y(t,s,\omega(s)))\Phi(t,s,\omega(s))$$
$$\times[\dot{\omega}(s)-g(s,\omega(s))-G(s,\psi^u(s),\xi^u(s),\omega(s))]ds$$

Let $\phi(t) = \dot{\omega}(t) - g(t,\omega(t)) - G(t,\psi^u(t),\xi^u(t),\sigma(t))$, then ϕ satisfies

$$(21) \qquad \phi(t) = \int_{t}^{\infty} g_y(t,y(t,s,\omega(s))\Phi(t,s,\omega(s))\phi(s)ds.$$

It will be shown that $\phi(t) = 0$, t in $[\tau,\infty)$, is the only solution of (21) and the proof of Theorem 2 will be complete. First observe that $\phi(t)$ is uniformly bounded on $[\tau,\infty)$. For it has already been shown that $\dot{\omega}(t) + G(t,\psi^u(t),\xi^u(t),\omega(t))$ is uniformly bounded, see (19) and (20), and (IVii), (IViv) and the definition of ω imply that $g(t,\omega(t))$ is uniformly bounded.

From Lemmas 3, 4, (IVii), (IViv) and the definition of ω

$$|\phi(t)| \le \int_t^\infty \frac{c}{\gamma+1}|\omega(s)|^{\gamma+1}|\phi(s)|ds \le \frac{c}{\gamma+1}\int_t^\infty |\xi^u(s)|^{\gamma+1}|\phi(s)|ds$$

$$\le \frac{c}{\gamma+1}\int_t^\infty \frac{1}{(1+A(s-\tau))^{(\gamma+1)/\alpha}}|\phi(s)|ds$$

$$\le \frac{c\alpha}{(\gamma+1)(\gamma+1-\alpha)} \quad \frac{1}{(1+A(t-\tau))^{(\gamma+1-\alpha)/\alpha}} \sup_{s\ge t}|\phi(s)|.$$

Choose $\hat{t} \ge \tau$ so large that

$\frac{c\alpha}{(\gamma+1)(\gamma+1-\alpha)} \quad \frac{1}{(1+A(\hat{t}-\tau))^{(\gamma+1-\alpha)/\alpha}} < 1.$ Then for all $t \ge \hat{t}$

$$|\phi(t)| < \sup_{s\ge\hat{t}}|\phi(s)|,$$

but this implies that $\phi(t) = 0$, $t \ge \hat{t}$. Thus

$$|\phi(t)| \leq \frac{c}{\gamma+1}\int_t^{\hat{t}} |\phi(s)|ds,$$

and therefore

$$\frac{d}{dt}\left[\exp\left\{\frac{c}{\gamma+1}\ t\right\}\int_{\hat{t}}^{t} |\phi(s)|ds\right] \leq 0.$$

But this gives

$$\int_{\hat{t}}^{t} |\phi(s)|ds \geq 0,\ t \leq \hat{t},$$

and therefore $\phi(s) = 0$, $t \leq s \leq \hat{t}$.

Corollary 9. Solutions of $\dot{y} = g(t,y) + G(t,\psi^u(t),\xi^u(t),y)$ on S_ε^+ are $O(t^{-1/\alpha})$ as $t \to \infty$.

Corollary 10. The positive manifold S_ε^+ is unique.

Proof. Equation (18) shows that T may be extended to $\overline{\Omega(\varepsilon)}$ and since Lemma 6 holds in $\overline{\Omega(\varepsilon)}$ the result follows from inequality (17).

The corresponding theorem for negative manifolds will be stated here. Consider the following hypotheses.

(VIIi) f is $C(R) \cap C^2(B(\varepsilon,m)):R^m$ for some ε in $(0,1)$;

(VIIii) $g(t,0) = 0$, $f_x(t,0) = 0$, t in R;

(VIIiii) there exist α, $a > 0$ such that

$\mu[f_x(t,s)] \leq -a|x|^{\alpha}$, t in R, x in $B(\varepsilon,m)$;

(VIIiv) there exist κ, $K > 0$ such that

$$\max_{i,j,k}\left|\frac{\partial f_i}{\partial x_j \partial x_k}(t,x)\right| \leq K|x|^{\kappa},\ t \text{ in } R,$$

x in $B(\varepsilon,m)$;

(VIIIi) g is $C(R)\cap C^1(B(\eta,n)):R^n$ for some η in $(0,1)$;

(VIIIii) $g(t,0) = 0$, t in R; and,

(VIIIiii) there exist β, $b > 0$ such that

$b|y|^{\beta} \leq -\mu[-g_y(t,y)]$, t in R, y in $B(\eta,n)$.

<u>Theorem 11</u>. Let (VIIi) through (VIIIiii) and (Vi) through (Viv) hold with $\kappa + 1 > \beta$ and $\min\{p,q,r\} > \beta$. Then for $\eta > 0$ sufficiently small and τ any real number there exists a unique function w in $C((-\infty,\tau]\times R^{\ell}\times B(\eta,m)):R^m$, $w(t,\theta,y)$ multiply periodic in θ with period ω, $w(t,\theta,0) = 0$ for (t,θ) in $(-\infty,\tau]\times R^{\ell}$, such that

$$S_{\eta}^{-} = \{(t,\theta,x,y)\,|\,\theta \text{ arbitrary},\ x=w(t,\theta,y),\ |y|<\eta,\ t\leq\tau\},$$

is a negative manifold for system (11). If the θ-equation is absent then $\min\{p,q,r\} = \min\{q,r\}$.

The proof of Theorem 11 follows closely that of Theorem 2 with the exception that one considers $t < \tau$ instead of $t \geq \tau$. For w in the proper class of functions one proceeds to prove Theorem 11 by finding a solution for the system

$$\text{(21w)} \qquad \begin{aligned} \dot\theta &= d + \Theta(t,\theta,w(t,\theta,y),y), \\ \dot y &= g(t,y) + G(t,\theta,w(t,\theta,y),y), \end{aligned}$$

$$\begin{aligned} w(t,\theta,x) = \int_0^{-\infty} &\Psi(t,t+s,w(t+s,\phi^W(t+s,t,\theta,y)\nu^W(t+s,t,\theta,y))) \\ &\times F(t+s,\phi^W(t+s,t,\theta,y),\nu^W(t+s,t,\theta,y), \\ &\quad w(t+s,\phi^W(t+s,t,\theta,y),\nu^W(t+s,t,\theta,y)))ds, \end{aligned}$$

where $(\phi^W(t,\tau,\theta,y),\nu^W(t,\tau,\theta,y))$ is the unique solution of (21w) with initial condition (θ,y) at $t = \tau$ and $\Psi(t,\tau,\zeta) = \frac{\partial x}{\partial \zeta}(t,\tau,\zeta)$, $x(t,\tau,\zeta)$ is the unique solution of $\dot x = f(t,x)$ satisfying the initial condition ζ at $t = \tau$.

<u>Corollary 12</u>. Solutions of $\dot x = f(t,x) + F(t,\phi^W(t),x,\nu^W(t))$ are $O(|t|^{-1/\beta})$ as $t \to -\infty$.

<u>Remark</u>. The hypothesis of Theorem 2 which restricts $\gamma + 1 > \alpha$ can be relaxed in some cases to include $\gamma + 1 = \alpha$.

Suppose that $f(t,x)$ satisfies (IIIi) through (IIIiii) with $\alpha = \gamma + 1$. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of positive numbers with $\alpha_k < \alpha$ and $\lim_{k\to\infty} \alpha_k = \alpha$. Suppose there exists a sequence $\{f^{(k)}(t,x)\}_{k=1}^{\infty}$ of $C(R)\cap C^1(B(\varepsilon,m)):R^m$ functions with $f^{(k)}(t,0) = 0$, t in R, and $\mu[f_x^{(k)}(t,x)] \leq -a|x|^{\alpha_k}$ for each $k = 1,2,\ldots$. Finally suppose that $\lim_{k\to\infty} f^{(k)}(t,x) = f(t,x)$ uniformly on $I\times B(\varepsilon,m)$, I any compact subset of R.

By applying Theorem 2 it can be seen that the system

$$
\begin{aligned}
\dot{\theta} &= d + \Theta(t,\theta,x,y), \\
\dot{x} &= f^{(k)}(t,x) + F(t,\theta,x,y), \qquad (22k)\\
\dot{y} &= g(t,y) + G(t,\theta,x,y),
\end{aligned}
$$

has a positive manifold

$$S^+_{\varepsilon,k} = \{(t,\theta,x,y)\,|\,\theta \text{ arbitrary, } |x|<\varepsilon,\ y=u_k(t,\theta,x), t\geq\tau\},$$

provided $\min\{p,q,r\} > \alpha$ and ε sufficiently small.

Let $(\psi_k(t),\xi_k(t))$ represent the unique solution of

$$\begin{aligned}\dot{\theta} &= d + \Theta(t,\theta,x,u_k(t,\theta,x)),\\ \dot{x} &= f^{(k)}(t,x) + F(t,\theta,x,u_k(t,\theta,x)),\end{aligned} \tag{23k}$$

with initial value (θ,x) at $t = \tau$. Then $\omega_k(t) = u_k(t,\psi_k(t),\xi_k(t))$ is the unique solution of

$$\dot{y} = g(t,y) + G(t,\psi_k(t),\xi_k(t),y),$$

with initial value $u_k(\tau,\theta,x)$ at $t = \tau$.

Since the functions

$$\begin{aligned}\Theta_k(t) &= d + \Theta(t,\psi_k(t),\xi_k(t),\omega_k(t)),\\ F_k(t) &= f^{(k)}(t,\xi_k(t)) + F(t,\psi_k(t),\xi_k(t),\omega_k(t)),\\ G_k(t) &= g(t,\omega_k(t)) + G(t,\psi_k(t),\xi_k(t),\omega_k(t)),\end{aligned}$$

are uniformly bounded on $[\tau,\infty)$ it follows that the sequence $\{(\psi_k,\xi_k,\omega_k)\}_{k=1}^{\infty}$ is uniformly Lipschitz and therefore equicontinuous. Now for each $k = 1,2,\ldots,$ $(\psi_k(\tau),\xi_k(\tau),\omega_k(\tau)) = (\theta,x,u_k(\tau,\theta,x))$. But $\{u_k(\tau,\theta,x)\}_{k=1}^{\infty}$ is a bounded sequence and therefore has a limit point, say u_0. Let $\{u_{k(i)}(\tau,\theta,x)\}_{i=1}^{\infty}$ be a convergent subsequence of $\{u_k(\tau,\theta,x)\}_{k=1}^{\infty}$ with $\lim\limits_{i\to\infty} u_{k(i)}(\tau,\theta,x) = u_0$. Since $\{(\psi_k,\xi_k,\omega_k)\}_{k=1}^{\infty}$ is equicontinuous and since solutions of (22k) are unique if follows that $\{(\psi_i,\xi_i,\omega_i)\}_{i=1}^{\infty}$, where $i = k(i)$, converges uniformly on $[\tau,\infty)$ to a function (ψ_0,ξ_0,ω_0). Therefore, from the continuity of Θ_k, F_k and G_k it follows that $\{(\Theta_k,F_k,G_k)\} \to (\Theta_0,F_0,G_0)$ uniformly on $[\tau,\infty)$. Thus (ψ_0,ξ_0,ω_0) is the unique solution of (22o) with initial value (θ,x,u_0) at $t = \tau$.

Observe that the asymptotic behavior of the solutions of the x and y equations in the limit is the same as in the case $\alpha < \gamma + 1$. For it is clear that

$$|\xi_k(t,\tau,\theta,x)| \leq \frac{|x|}{(1+A_k|x|^{\alpha_k}(t-\tau))^{1/\alpha_k}}$$

$$|\omega_k(t)| \leq \delta|\xi_k(t)|$$

for some $A_k > 0$, $k = 1,2,\ldots$. Since $A_k \to A$, $\alpha_k \to \alpha$ as $k \to \infty$ it follows that

$$|\xi_0(t,\tau,\theta,x)| \leq \frac{|x|}{(1+A|x|^{\alpha}(t-\tau))^{1/\alpha}}$$

$$|\omega_0(t)| \leq \delta|\xi_0(t)|.$$

In case system (22k) is autonomous it is clear that $\{u_k(t,\theta,x)\} = \{u_k(\theta,x)\}$ converges uniformly on $R^\ell \times B(\varepsilon,m)$. Thus for autonomous systems one may obtain a representation of the form $u_0 = Tu_0$ for the integral manifold even in the case $\gamma + 1 = \alpha$.

In view of the above discussion it can be seen that Theorem 11 can be extended to include the case $\beta = \kappa + 1$.

Now it can be seen that the model problem discussed in the introduction has the predicted solution. In that case $\alpha = \beta = 2$ and $\gamma = \kappa = 1$. So if X and Y are $o((|x|+|y|)^3)$ one may apply an extended version of Theorem 2 by letting $f^{(k)}(t,x) = -x^{3-\frac{1}{k}}$ to obtain a positive manifold for the perturbed system. Similarly one applies an extended version of Theorem 11 by letting $g^{(k)}(t,x) = y^{3-\frac{1}{k}}$ in order to obtain a negative manifold for the perturbed system.

CONCLUDING REMARKS

The results of this dissertation may be considered as a parallel development of the works of several authors, [3], [12], [15] and in particular they closely resemble Theorem 4.1, p. 330 in Coddington and Levinson [7]. These authors obtain results similar to Theorem 2 for a system whose unperturbed state is linear, e.g., system (4) in the absence of the θ-equation. Two aspects of the presentation in Coddington and Levinson are not given here. First, is is shown that solutions which do not start on the manifold eventually leave a neighborhood of the origin. Also it is shown that if the perturbation function is differentiable, as is the case here, then the manifold is likewise differentiable, i.e., the function which describes the manifold is differentiable. Although the manifold in the nonlinear case here has been shown to be Lipschitz, the problem of showing differentiability seems more difficult than in the linear case.

Another aspect of the results in this dissertation which are not pointed out explicitly in the theorems is the similarity of behavior of solutions of the unperturbed and perturbed systems on S^+ and S_ε^+, respectively. Let $(t,\theta(t),x(t),0)$ be in S^+ and $(t,\psi(t),\xi(t),\omega(t))$ be in S_ε^+. Then $|x(t)-\xi(t)|$ and $|\omega(t)|$ approach zero as t becomes large. In the absence of the θ-equation this may be interpreted as asymptotic equivalence of systems (8) and (11). This resembles results obtained by Marlin and Struble [19] who give sufficient conditions for the existence of solutions for a perturbed nonlinear system which are asymptotic to solutions of the unperturbed state. Their hypotheses do not cover the case treated here, nor conversely.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Alekseev, V. M., "An estimate for the perturbations of the solutions of ordinary differential equations" (Russian), Vestnik Moskov. Univ. Ser. I Mat. Meh., No. 2 (1961), 28-36.

2. Alekseev, V. M., "An estimate for the perturbations of the solutions of ordinary differential equations II" (Russian), Vestnik Moskov. Univ. Ser. I Mat. Meh., No. 3 (1961), 3-10.

3. Bogolinbov, N. and Mitropolski, Y. A., "The method of integral manifolds in nonlinear mechanics", Symposium on nonlinear vibrations, Kiev, U.S.S.R., September, 1961.

4. Bogolinbov, N. and Mitropolski, Y. A., "Asymptotic methods in the theory of nonlinear oscillations", Moscow, 1955; revised, 1958 (Russian), Goz. Iz. Fiz. Mat. Lit.; translated by Gordon and Breach, 1962.

5. Brauer, F., "Perturbations of nonlinear systems of differential equations, I", J. Math. Anal. Appl. 17 (1966), 198-206.

6. Brauer, F. "Perturbations of nonlinear systems of differential equations, II", J. Math. Anal. Ap Appl. 17 (1967), 418-434.

7. Coddington, E. and Levinson, N., "Theory of ordinary differential equations", McGraw-Hill, New York, 1955.

8. Coppel, W. A., "Stability and asymptotic behavior of differential equations", Heath, Boston, 1965.

9. Diliberto, S. P. and Hufford, G., Perturbation theorems of nonlinear differential equations, "Contributions to the theory of nonlinear oscillations", Vol. 3, pp. 257-261. Annals of Mathematics Studies, No. 36, Princeton Universi University Press, Princeton, N. J., 1956.

10. Diliberto, S. P., "Perturbation theorems for periodic surfaces, I. Definitions and main theorems", Rend. Cir. Mat. Palermo, Ser. 2, 9 (1960) 265-299.

11. Diliberto, S. P. "Perturbation theorems for periodic surfaces, II", Rend. Cir. Mat. Palermo, Ser. 2, 10 (1961), 111-161.

12. Hale, J., "Integral manifolds of perturbed differential systems", Ann. Mat. 73 (1961), 496-531.

13. Hale, J., "Oscillations in nonlinear systems", McGraw-Hill, New York, 1963.

14. Hartman, P. "Ordinary differential equations", John Wiley & Sons, Inc., New York, 1964.

15. Kelley, A., "The stable, center-stable, center, center-unstable, unstable manifolds", J. Diff. Eqs. 3 (1967), 546-570.

16. Levinson, N. "Small periodic perturbations of an autonomous system with a stable orbit", Ann. Math. 52 (1950), 727-738.

17. May, L. E., "Perturbation problems in fully non-linear systems", Dissertation, North Carolina State University at Raleigh, 1969.

18. Sacker, R. J., "A new approach to the perturbation theory of invariant surfaces", Commun. Pure App. Math. 18 (1965), 717-732.

19. Struble, R. A. and Marlin, J. A., "Asymptotic equivalence of nonlinear systems", to appear. J. Diff. Eqs.

ABSTRACT

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The purpose of this dissertation is to give a proof for the existence of an integral manifold for a system of perturbed nonlinear differential equations in a neighborhood of a critical point, periodic orbit, or periodic surface. Analogous studies of integral manifolds where the unperturbed system is linear have been done before. Functions describing the manifolds, in the linear case, are usually obtained as solutions of a certain improper integral equation formulated by the use of the classical variation of constants technique. To prove that a solution to this integral equation exists, use is made of certain exponential bounds induced by the unperturbed linear system. Of interest here is the case in which the unperturbed state of the system has no linear part. Again one is led to consider solutions of an improper integral equation, obtained in this case from a generalization of the variation of constants formula due to V. M. Alekseev. In general, one can not expect exponential bounds analogous to those present in the linear case; however, by assuming certain smoothness and order type properties of the functions involved it is possible to demonstrate

the existence of a unique solution of the integral equation. Additional work is done to show that the solution of the integral equation gives rise to a solution of the differntial equation.

It is also shown that on the manifold of the perturbed system solutions are asymptotic to solutions of the unperturbed system on the corresponding manifold.