N69-38141
TMX-53802

NASA TECHNICAL
MEMORANDUM

REPORT No. 53802

RELATIVISTIC PHYSICS OF CLOCKS AND THE SYNCHRONOUS ORBIT CLOCK EXPERIMENT

By Nat Edmonson, Jr. and Fred D. Wills
Space Sciences Laboratory

June 1, 1969

NASA

George C. Marshall Space Flight Center
Marshall Space Flight Center, Alabama

MSFC - Form 3190 (September 1968)

TECHNICAL REPORT STANDARD TITLE PAGE

1. REPORT NO. TM X-53802	2. GOVERNMENT ACCESSION NO.	3. RECIPIENT'S CATALOG NO.
4. TITLE AND SUBTITLE Relativistic Physics of Clocks and The Synchronous Orbit Clock Experiment		5. REPORT DATE June 1, 1969 6. PERFORMING ORGANIZATION CODE
7. AUTHOR(S) Nat Edmonson, Jr. and Fred D. Wills		8. PERFORMING ORGANIZATION REPORT #
9. PERFORMING ORGANIZATION NAME AND ADDRESS National Aeronautics and Space Administration George C. Marshall Space Flight Center Marshall Space Flight Center, Alabama 35812		10. WORK UNIT NO. 11. CONTRACT OR GRANT NO. 13. TYPE OF REPORT & PERIOD COVERED
12. SPONSORING AGENCY NAME AND ADDRESS		Technical Memorandum (TMX) 14. SPONSORING AGENCY CODE

15. SUPPLEMENTARY NOTES

Prepared by Space Sciences Laboratory of the Science and Engineering Directorate

16. ABSTRACT

The basic principles of special relativity and inertial reference frames are discussed. The Lorentz transformation is given and expressions for the proper time in an inertial frame of reference are developed. A concept of the space-time metric is developed from the properties of non-inertial systems. Expressions for differential distance and time intervals are developed from the space-time metric. Some of the basic properties of curvilinear coordinates, tensors, and covariant differentiation are expounded. Elementary properties of the Riemann curvature tensor and the development of the empty space gravitational field equations are presented. Comments are made about the gravitational field equations and some consequential space-time metrics. The combined effects of uniform rotation and the gravitational field on the space-time metric, and hence proper time, are discussed and developed. Finally, expressions are developed which measure the gravitational and relative motion effects on frequency comparisons between two "identical" atomic oscillators — one earth fixed and the other in synchronous orbit above the earth-fixed oscillator. The first-order Doppler effect is theoretically eliminated from the comparison. Perturbations will be treated in a later paper.

17. KEY WORDS	18. DISTRIBUTION STATEMENT For Public Release as announced in STAR Fred D. Wills

19. SECURITY CLASSIF. (of this report)	20. SECURITY CLASSIF. (of this page)	21. NO. OF PAGES	22. PRICE
UNCLASSIFIED	UNCLASSIFIED	79	$3.00

MSFC - Form 3292 (May 1969)

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

TECHNICAL MEMORANDUM X-53802

RELATIVISTIC PHYSICS OF CLOCKS AND THE SYNCHRONOUS ORBIT CLOCK EXPERIMENT

SUMMARY

The basic principles of special relativity and inertial reference frames are discussed. The Lorentz transformation is given and expressions for the proper time in an inertial frame of reference are developed. A concept of the space-time metric is developed from the properties of non-inertial systems. Expressions for differential distance and time intervals are developed from the space-time metric. Some of the basic properties of curvilinear coordinates, tensors, and covariant differentiation are expounded. Elementary properties of the Riemann curvature tensor and the development of the empty space gravitational field equations are presented. Comments are made about the gravitational field equations and some consequential space-time metrics. The combined effects of uniform rotation and the gravitational field on the space-time metric, and hence proper time, are discussed and developed. Finally, expressions are developed which measure the gravitational and relative motion effects on frequency comparisons between two "identical" atomic oscillators — one earth fixed and the other in synchronous orbit above the earth-fixed oscillator. The first-order Doppler effect is theoretically eliminated from the comparison. Perturbations will be treated in a later paper.

I. INTRODUCTION

This report is the first of a series of studies to be made in support of the space experimental program for testing the theory of relativity. To make this and subsequent reports of maximum value to participants in this program, a résumé of the theory of relativity is included. For those who wish to go into the theory in greater detail, this résumé will serve as an outline for the study of works by L. D. Landau and E. M. Lifshitz, Robert B. Leighton, P. G. Bergmann, G. C. McVittie, V. Fock, C. Møller, and R. C. Tolman; these works are listed in the Bibliography, along with other documents that are related to this report.

II. PRINCIPLES OF SPECIAL RELATIVITY

To order natural phonomena intelligently, one must, of necessity, have a system or frame of reference. A system of reference is understood to be composed of a set of coordinates that serve to indicate the spatial position of a naturally occurring phenomenon with identical clocks fixed in this system for the purpose of indicating time.

Suppose there exists a point mass that moves freely with constant velocity, i.e., without influence from external forces. A reference system that describes the force-free motion of such a point mass will be defined as an inertial reference system.

Now suppose another reference system that moves uniformly relative to an inertial system of reference. Such a system is likewise inertial since every free motion in this system will be linear and uniform. By definition, many inertial frames that move uniformly to one another can be obtained arbitrarily.

Therefore, one must postulate the first principle of special relativity: the laws of nature are identical in all inertial systems of reference. This means precisely that the equations describing any law of nature have one and the same form when written in terms of coordinates and time in different inertial systems.

Many of the interactions of natural phenomena are classically assumed to propagate instantaneously from one phenomenon to another. This is a very good first approximation in quite a few cases of such naturally occurring phenomena. Careful experimentation will show that instantaneous interactions are nonexistent in nature. Therefore, one cannot base a correct description of the laws of nature on an assumption of instantaneous propagation of interactions. Any change in a locally occurring phenomenon will, in fact, influence another phenomenon a finite distance away only after the lapse of a certain interval of time. If the distance between the two bodies is divided by the lapsed time interval the velocity of propagation of the interaction is obtained.

We should, in the strictest sense, speak of the maximum velocity of propagation of interaction. It is only after the maximum velocity of propagation of interaction that a change in a natural phenomenon begins to manifest itself in another naturally occurring phenomenon. It is clear that motions involved in naturally occurring phenomena cannot exceed the maximum velocity of propagation of interactions. If such motions should occur, one could realize an interaction with a velocity exceeding the maximum possible velocity of propagation.

The second principle of special relativity will now be postulated: the maximum velocity of propagation of interactions is a constant in free space and is the same in all inertial systems of reference. The constant is the velocity of light in a vacuum and is usually designated by the small letter c.

Incidentally the laws of classical mechanics must still be valid in the limit $c \rightarrow \infty$ in any subsequent theory that develops based upon the two above postulates. This, of course, implies that an instantaneous velocity exists for the propagation of an interaction, and that time is absolute in classical mechanics. This would mean that the properties of time are assumed to be independent of the system of reference, i.e., that any two phenomena occurring simultaneously for any particular observer occur simultaneously for all others or that the interval of time between two given events must be identical for all systems of reference.

It is appropriate at this time to define what is meant by the synchronization of clocks. Suppose that an arbitrary number of identical clocks are positioned at a fixed set of coordinates within an inertial frame of reference. Also assume that the positioning (with a standard reference of length) does not affect the identity of the clocks. Let one of the clocks and an observer be co-located at the origin. A light pulse may be sent from the origin to each of the other positioned clocks and returned to the origin for a comparison of the reading on each of the other clocks. By allowing for the propagation time of the light pulse, the clocks fixed in the inertial system can be synchronized by defining all of their readings to be some common value of time, t_0. All the clocks fixed in the inertial system are then said to be synchronized.

As a result of the two principles of relativity, time cannot be absolute because it elapses differently in different systems of reference. There is meaning in the statement that a definite time interval has elapsed between any two given events if and only if a frame of reference to which this statement applies is indicated. Figure 1 shows that simultaneous events in one reference frame will not necessarily be simultaneous in other frames.

Consider two inertial frames of reference, K and K', as illustrated in Figure 1, where K' moves with constant speed v relative to K along the x(x') axis. In the K' system let BA=CA, and let light signals initiate from point A in both directions toward B and C. The speed of light for all inertial systems is c; hence, the propagation velocity of the light signal in K' is c. The signals will reach B and C simultaneously in the K' system, but the arrivals of the signals at B and C are not simultaneous for an observer in the K system. According to our postulates of relativity, the velocity of propagation of the signal emanating from point A in the K' system has the same value c in the K system. But, relative to the K system, the point B moves toward the signal source while point C moves away

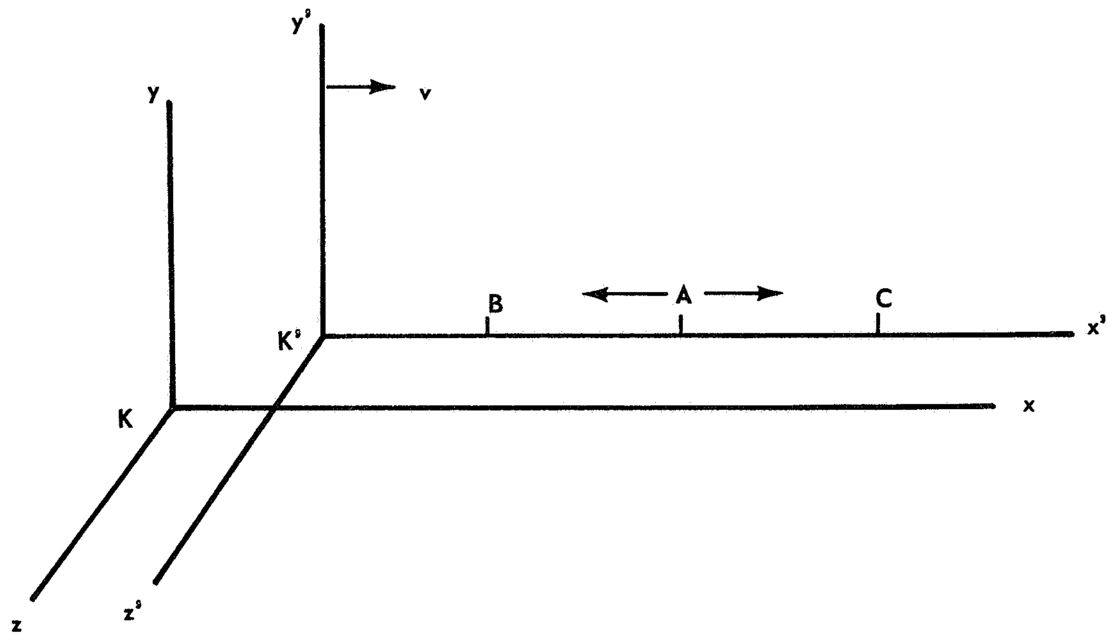

FIGURE 1. RELATIVELY MOVING INERTIAL SYSTEMS

from the signal. Hence, in the K system, the signal will reach point B earlier than point C. Thus the two events are simultaneous to K' but not simultaneous to K.

The following definitions and concepts will be needed:

1. Event: A physical phenomenon described and characterized by the place where it occurred and the time when it occurred relative to a coordinate frame of reference.

2. World Points: Events that are specified in four-dimensional space by three space coordinates and time.

3. World Line: Each physical process in four-dimensional space corresponds with a certain line. This line is called the world line. Mathematically it can be expressed as a scalar function $[f(x, y, z, t)]$ of the space coordinates and time set equal to zero.

In Figure 1, the axes x and x' coincide, while the y and z axes are parallel to y' and z', respectively. We designate the time in the system K and K' by t and t'.

Let the first event consist of sending out a signal, propagating with the velocity of light, from a point having four-dimensional coordinates x_1, y_1, z_1, t_1 in the K system. We observe the propagation of this signal in the K system. Let the second event consist of the arrival of the signal at point x_2, y_2, z_2, t_2. The signal propagates with velocity c; the distance covered by it is, therefore, $c(t_2 - t_1)$. On the other hand, this same distance equals $[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$. Thus, we can write the following relation between the two events in the K system:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2 = 0. \qquad \text{(II-1)}$$

The same two events, i.e., the propagation and reception of the signal can be observed from the K' system: let the coordinates of the first event in the K' system be x'_1, y'_1, z'_1, t'_1, and of the second event be x'_2, y'_2, z'_2, t'_2. Since the velocity of light is the same in the K and K' systems, we have similarly to equation (II-1),

$$(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 - c^2(t'_2 - t'_1)^2 = 0. \qquad \text{(II-2)}$$

If x_1, y_1, z_1, t_1, and x_2, y_2, z_2, t_2 are the coordinates of any two events in the K system, then the quantity

$$s_{12} = [c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2]^{1/2} \qquad \text{(II-3)}$$

is called the interval between these two events. Likewise, if x'_1, y'_1, z'_1, t'_1, and x'_2, y'_2, z'_2, t'_2 are the coordinates between any two events in the K' system, then the quantity

$$s'_{12} = [c^2(t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 - (y'_2 - y'_1)^2 - (z'_2 - z'_1)^2]^{1/2} \qquad \text{(II-4)}$$

is called the interval between these two events. Thus, it follows from the principle of invariance of the velocity of light that if the interval between two events is zero in one coordinate system, then it is equal to zero in all other inertial coordinate systems.

If two events are infinitely close to each other, then the intervals ds, ds' between them are defined by

$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2), \qquad \text{(II-5)}$$

$$ds'^2 = c^2dt'^2 - (dx'^2 + dy'^2 + dz'^2). \qquad \text{(II-6)}$$

As shown in equations (II-1) and (II-2), if $ds = 0$ in one inertial system, then $ds' = 0$ in any other system. On the other hand, ds and ds' are infinitesimals of the same order. Their difference can at most be linear and constant. Hence, it follows that $ds^2 = a(ds')^2$ where "a" is a function that can depend only on the absolute value of the relative velocity of the two inertial systems. It cannot depend on the coordinates or the time, since different points in space and different moments in time would not be equivalent, which would be in contradiction to the homogeneity of space and time. Also it cannot depend on the direction of the relative velocity, for this would contradict the isotropy of space.

Consider three reference systems K, K_1, K_2, and let v_1 and v_2 be the velocities of systems K_1 and K_2 relative to K. Then,

$$ds^2 = a(v_1)\ (ds_1)^2 = a(v_2)\ (ds_2)^2, \tag{II-7}$$

and, similarly, one can write

$$(ds_1)^2 = a(v_{12})\ (ds_2)^2, \tag{II-8}$$

where v_{12} is the absolute value of the velocity of K_2 relative to K_1. Comparing equations (II-7) and (II-8), one finds that it is necessary to have

$$\frac{a(v_2)}{a(v_1)} = a(v_{12})\,. \tag{II-9}$$

But v_{12} depends not only on the absolute values of the vectors $\bar{v}_1$ and $\bar{v}_2$, but also on the angle between them. However, the angle does not appear in the formula on the left side of equation (II-9). Therefore, the formula can be correct only if the function $a(v)$ reduces to a constant that is equal to unity according to the same formula. Hence,

$$ds^2 = (ds')^2, \tag{II-10}$$

and from the equality of the infinitesimal intervals there follows the equality of finite intervals: $s = s'$.

Thus one arrives at a very important result: the interval between two events is the same in all inertial systems of reference, i.e., it is invariant under transformation from one inertial system to another. This invariance is the mathematical expression of the constancy of the velocity of light.

III. THE LORENTZ TRANSFORMATION AND PROPER TIME

If one suspects that time is absolute or is some independent, universal quantity not connected with space, he might deduce from Figure 1, that the coordinate transformations are

$$x' = x - vt; \; y' = y; \; z' = z; \; t' = t. \tag{III-1}$$

This transformation is generally called the Galilean transformation. The non-invariance of Maxwell's field equations of electrodynamics under such a transformation led Lorentz (1904) to suspect that time must, in fact, be treated on an equal basis with the three length dimensions of space, instead of as an independent universal scalar parameter. The null results of the Michelson-Morley experiment added quite a bit of support to this position. Lorentz and Einstein (1905) realized independently that the Galilean transformation must be replaced with a new relationship between the two inertial frames of reference that satisfies the requirements of equations (II-1) and (II-2), and which are in agreement with the Galilean transformation if the two systems are moving very slowly with respect to each other.

An inspection of equations (II-1) and (II-2) and the Galilean transformation reveals that y and z create no problems, but that there are some terms involving x and t that must, somehow, be made to disappear. This must be done without disturbing the combination $(x - vt)$ in the transformation, since the speed of one system with respect to the other must mean the rate at which a point, fixed in one system, appears to move past the other system. Because of the homogeniety and isotropy of space, the transformation must be linear, i.e., straight lines in one system must transform into straight lines of the other system. Hence, we can at most modify the first equation of the Galilean transformation by a constant factor. Thus, an attempt shall be made to retain the x transformation in the relatively simple form

$$x' = \gamma\,(x - vt) \tag{III-2}$$

where γ is some constant factor very nearly equal to unity under the conditions of everyday experience. Further thought reveals that the equation $t' = t$ cannot be correct, since no rearrangement of space coordinates alone can give wave pulses that are simultaneously concentric spheres in both systems. The simplest modification of the time transformation is one which contains only x and t linearly:

$$t' = At + Bx \tag{III-3}$$

where A should be nearly unity and B nearly zero under ordinary familiar conditions.

The two systems K and K' of Figure 1 are in uniform relative motion. Now suppose that at some instant the two origins coincide. At that instant let two observers, each at rest in K and K' respectively, define $t_1 = t'_1 = 0$, and let a short light pulse be emitted from the origin such that $x_1 = x'_1 = y_1 = y'_1 = z_1 = z'_1 = 0$ of equations (II-1) and (II-2). Suppose that each observer has set up photoelectric cells and time recorders at various points fixed in their respective systems to follow the progress of the wave pulse as it expands in all directions from the source. Now drop the subscript 2 in equations (II-1) and (II-2) and note that the equation of the wavefront has the same form to each observer in his system of reference. The K observer will say that

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{III-4}$$

is the equation of the wavefront, while the K' observer will say

$$(x')^2 + (y')^2 + (z')^2 - c^2(t')^2 = 0. \tag{III-5}$$

Let us now insert equations (III-2) and (III-3) along with the relationships $y' = y$, $z' = z$, into equation (III-5):

$$(\gamma^2 - B^2c^2)\, x^2 + y^2 + z^2 + (\gamma^2 v^2 - A^2 c^2)\, t^2 - 2(ABc^2 + \gamma^2 v)xt = 0. \tag{III-6}$$

By comparing equations (III-6) with (III-4), one sees that

$$\gamma^2 - B^2 c^2 = 1,\ A^2 c^2 - \gamma^2 v^2 = c^2,\ ABc^2 + \gamma^2 v = 0. \tag{III-7}$$

These three equations suffice to determine A, B, and γ in terms of v; the result is

$$\gamma = A = \left(1 - \frac{v^2}{c^2}\right)^{-1/2},\quad B = -\frac{v\gamma}{c^2} \tag{III-8}$$

If one calls $\beta = \dfrac{v}{c}$, the transformation equations are

$$x' = \gamma(x - \beta ct), \tag{III-9}$$

$$y' = y, \tag{III-10}$$

$$z' = z, \tag{III-11}$$

$$t' = \gamma\left(t - \frac{\beta x}{c}\right) \quad . \tag{III-12}$$

The above linear transformation of coordinates and time is a special form of a Lorentz transformation where uniform rectilinear motion is restricted to the x (x') direction of each system. The inverse of the above transformation is found to be

$$x = \gamma \, (x' + \beta c t') \tag{III-13}$$

$$y = y', \tag{III-14}$$

$$z = z', \tag{III-15}$$

$$t = \gamma \left(t' + \frac{\beta x'}{c}\right) \tag{III-16}$$

Suppose that in a certain inertial reference system there are clocks that are moving with constant speed relative to an observer in an arbitrary direction. In particular, suppose an observation is made from the K system on the K' system of clocks rigidly at rest and synchronized in the K' system (Fig. 1). Also imagine a number of clocks rigidly at rest and synchronized in the K-rest frame.

In the course of an infinitestimal time interval dt, read by a clock in the K rest frame, the moving clocks go a distance (dx). What time interval, dt', is indicated for this period by the moving clocks? Of course in a system of coordinates linked to the moving clocks, the latter are at rest, i.e., $dx' = dy' = dz' = 0$. Because of the invariance of the intervals,

$$ds^2 = c^2 \, dt^2 - dx^2 = c^2 \, (dt')^2, \tag{III-17}$$

from which

$$dt' = \frac{ds}{c} = \frac{1}{c} \sqrt{c^2 \, dt^2 - (dx)^2}, \tag{III-18}$$

or else

$$dt' = dt \sqrt{1 - \frac{dx^2}{c^2\, dt^2}} \quad . \qquad \text{(III-19)}$$

But

$$\frac{dx^2}{dt^2} = v^2 \quad , \qquad \text{(III-20)}$$

where v is the velocity of the moving clocks; therefore,

$$dt' = \frac{ds}{c} = dt \sqrt{1 - \frac{v^2}{c^2}} \quad . \qquad \text{(III-21)}$$

Integrating this expression, we can obtain the time interval indicated by the moving clocks when the elapsed time according to a clock at rest is $t_2 - t_1$; hence,

$$t_2' - t_1' = (t_2 - t_1) \sqrt{1 - \frac{v^2}{c^2}} = \frac{t_2 - t_1}{\gamma} \quad . \qquad \text{(III-22)}$$

The time read by a clock moving with a given object is called the proper time for this object. Equations (III-17) and (III-22) express the proper time in terms of the time for a system of reference from which the motion is observed.

For uniform rectilinear motion along the x axis we may use equation (III-16) to calculate the result of equation (III-22) for the same spatial location x' in the K' system. The two observers in the K and K' systems, respectively, therefore disagree on the length of the time interval. By use of equation (III-12), an observer in K' examining clocks rigidly attached to K sees the result

$$t_2 - t_1 = \frac{t'_2 - t'_1}{\gamma} \quad . \qquad \text{(III-23)}$$

Hence, we conclude that each observer will find that a clock in the other's system is running at a slower rate than his own. This effect is called the special relativistic time dilatation, and has been verified experimentally in many various ways.

The results just derived may appear somewhat paradoxical at first glance, for how can an observer at rest find a moving observer's clocks running slower than his own? The resolution of this apparent paradox lies in the realization that the experiments by which these observers make these comparisons are symmetrical but not identical. This is because the term "simultaneous" no longer has the universal significance that was assumed in writing the Galilean transformation.

Another puzzling aspect of the time dilatation is the detailed mechanism by which the rate of a clock changes. A clock is, after all, a physical device whose motions are governed by the laws of nature. It seems strange indeed that one could conclude that the rates are the same for all clocks, no matter what their design. Why should an hourglass behave in just the same way as a pendulum clock or as an oscillating electrical circuit? The answer is to be found in the form taken by the laws of nature rather than in a detailed analysis of each possible kind of clock. For if the laws which govern the operation of clocks can be written in a form that is covariant with respect to the Lorentz transformation, then it will automatically be true that the rates of all clocks governed by these laws will change in the same way under the Lorentz transformation. Since all ordinary clocks obey the laws of mechanics and electrodynamics, all that is required is that these laws are covariant with respect to the Lorentz transformation. Although easily proven, it will not be done here.

Consider a clock composed of an electric-flash tube and a photoelectric cell, arranged as in Figure 2.

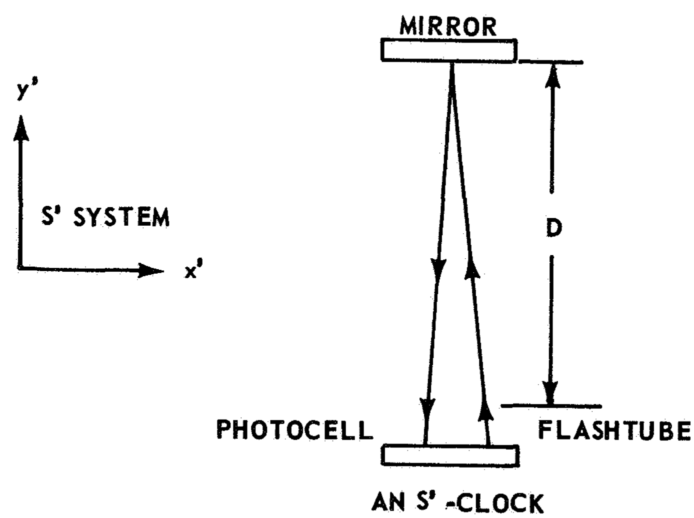

FIGURE 2. A PULSE LIGHT CLOCK AT REST

The flash tube and photocell are side by side with a light baffle between them, and a circuit is so arranged that, when the photocell receives a pulse of light, it causes the flash tube to emit another pulse of light with a negligible delay. The "period" of this clock is then equal to the time required for a light pulse to travel from the flash tube to the mirror and back to the photocell, i.e.,

$$T' = \frac{2D}{c} . \tag{III-24}$$

Now consider the operation of this clock as viewed by an observer moving to the left with speed **v**. The observer sees the clock as moving to the right at speed v and analyzes its operation as shown in Figure 3.

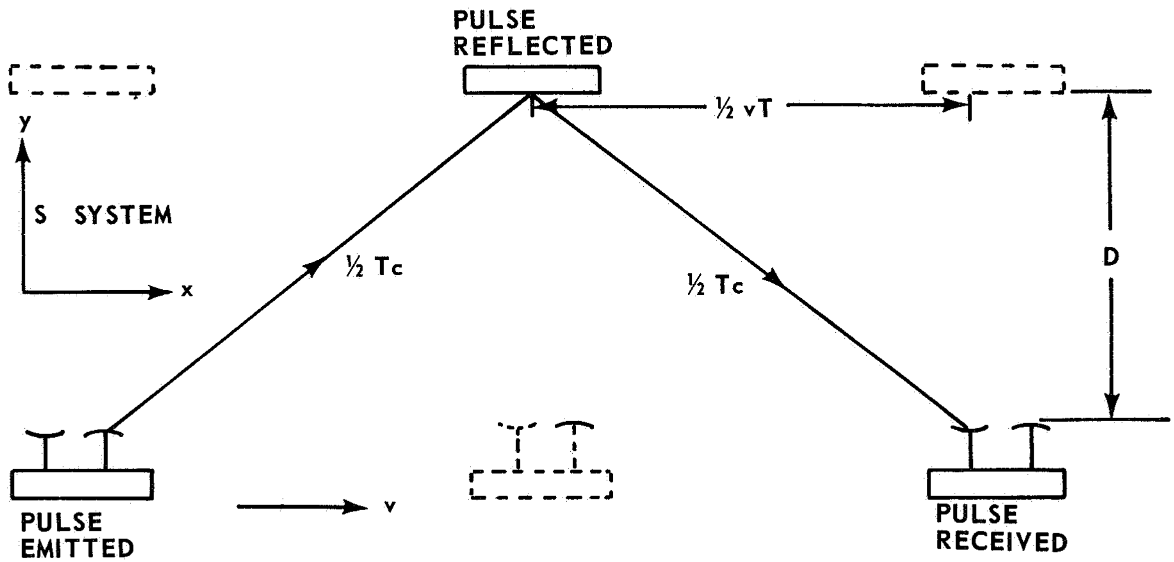

FIGURE 3. A UNIFORMLY MOVING PULSE LIGHT CLOCK

Because the clock is moving, the light must proceed in a diagonal path from the flash tube to the mirror and thence to the photocell. By the second postulate, the light travels on this diagonal path at the same speed c as it travels directly back and forth in S'. It must, therefore, take longer for the light to return to the photocell, as seen by S than as seen by S'. Thus,

$$cT = 2\sqrt{(1/2\ vT)^2 + (D)^2} , \tag{III-25}$$

$$cT = \sqrt{(vT)^2 + (2D)^2} ,$$

$$c^2T^2 = (vT)^2 + (2D)^2 ,$$

$$T = \frac{2D}{c\sqrt{1 - \frac{v^2}{c^2}}} = T'\gamma \ . \tag{III-26}$$

The Lorentz transformation for arbitrary uniform rectilinear motion may be obtained by rotating the K reference system through a set of Eulerian angles while the time is imagined to be held at some fixed value for an instant.

IV. PROPERTIES OF NONINERTIAL REFERENCE SYSTEMS, THE SPACE-TIME METRIC, AND DISTANCES AND TIME INTERVALS

In an inertial system of reference, in Cartesian coordinates, the interval ds is given by

$$ds^2 = c^2\, dt^2 - (dx^2 + dy^2 + dz^2), \tag{IV-1}$$

and upon transforming to any other inertial reference system under a Lorentz transformation, the interval retains the same form. However, if we transform to a noninertial system of reference, $(ds)^2$ will no longer be a sum of squares of the four coordinate differentials.

For example, let us transform to a uniformly rotating system of coordinates (Fig. 4):

$$x = x' \cos(\omega t) - y' \sin(\omega t), \tag{IV-2}$$

$$y = x' \sin(\omega t) + y' \cos(\omega t), \tag{IV-3}$$

$$z = z', \tag{IV-4}$$

where ω is the angular velocity of the rotation directed along the z axis. The interval takes on the form

$$ds^2 = [c^2 - \omega^2 (x'^2 + y'^2)]\, dt^2 - (dx'^2 + dy'^2 + dz'^2) + 2\omega\, y'\, dx'\, dt - 2\omega\, x'\, dy'\, dt. \tag{IV-5}$$

No matter what the law of transformation of the time coordinate, this expression cannot be represented as a sum of squares of the coordinate differentials.

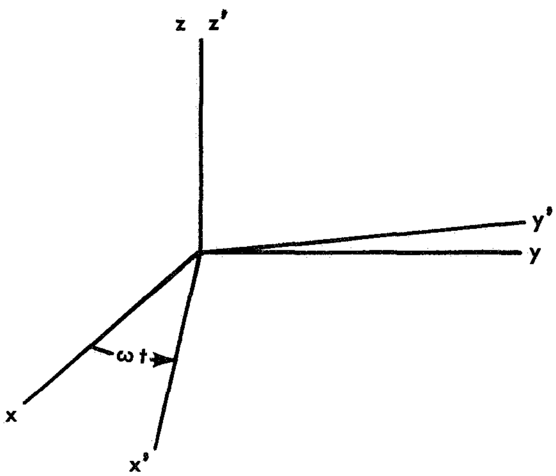

FIGURE 4. NONINERTIAL, UNIFORMLY ROTATING COORDINATE SYSTEM

Thus in a noninertial system of reference the square of an interval appears as a quadratic form of general type in the coordinate differentials, that is, it has the form

$$ds^2 = g_{ik}\, dx^i\, dx^k \qquad \text{(IV-6)}$$

where the g_{ik} are certain functions of the space coordinates x^1, x^2, x^3 and the time coordinate x^0. Thus, when we use a noninertial system, the four-dimensional coordinate system x^0, x^1, x^2, x^3 is curvilinear. The quantities g_{ik}, determining all the geometric properties in each curvilinear system of coordinates, represent the space-time metric. The g_{ik} are symmetric in the indices i and k ($g_{ik} = g_{ki}$). In an inertial system of reference when we use Cartesian space coordinates $x^{1,2,3} = x, y, z$, and the time $x^0 = ct$, the quantities g_{ik} are

$$g_{11} = g_{22} = g_{33} = -1;\ g_{00} = 1;\ g_{ik} = 0 \text{ for } i \neq k. \qquad \text{(IV-7)}$$

We call a four-dimensional system of coordinates with these values of g_{ik} Galilean.

When the values of the g_{ik} become non-Galilean and in general functions of the space coordinates and time, they represent functional variations from which one can determine certain and specific force fields. The natural gravitational field we will see is represented by a metric of space-time, as determined by the quantities g_{ik}. The important thing to remember is the geometrical properties of space-time are determined by physical phenomena, and are not intrinsic properties of space and time.

The theory of force fields, constructed on the basis of the theory of relativity, is called the general theory of relativity. It was established by

Einstein (and finally formulated by him in 1916), and represents probably the most beautiful of all existing theories. It is remarkable that it was developed by Einstein in a purely deductive manner and only later was substantiated by astronomical observations.

In the general theory of relativity, it is generally impossible to have a system of bodies that are fixed or move uniformly relative to one another. This result essentially changes the very concept of a system of reference in the general theory of relativity, as compared to its meaning in the special theory. In the latter, a reference system is a set of noninteracting bodies at rest or moving uniformly relative to one another. Such systems of bodies do not exist in the presence of a variable gravitational field, and for the exact determination of the position of a particle in space we must, strictly speaking, have an infinite number of bodies that fill all the space like some sort of "medium." The specific appearances of physical phenomena, including the properties of the motion of bodies, become different in all systems of reference.

In general relativity, the choice of a coordinate system is not limited in any way; the triplet of space coordinates x^1, x^2, x^3 can be any set of quantities defining the position of bodies in space, and the time coordinate x^0 can be defined by an arbitrarily running clock. The question arises of how, in terms of the values of the quantities x^1, x^2, x^3, x^0, one can determine actual distances and time intervals.

First one must find the relation of the proper time (the time read by a clock moving with a given object), which from now on shall be denoted by τ, to the coordinate x^0. To do this, one considers two infinitesimally separated events, occurring at one and the same place in spatial position. Then the interval ds between the two events is just $c d\tau$, where $d\tau$ is the proper time interval between the two events. Setting $dx^1 = dx^2 = dx^3 = 0$ in the general expression $ds^2 = g_{ik}\, dx^i\, dx^k$, one finds that

$$ds^2 = c^2\, d\tau^2 = g_{00}\, (dx^0)^2, \tag{IV-8}$$

from which

$$d\tau = \frac{1}{c} \sqrt{g_{00}}\; dx^0, \tag{IV-9}$$

or else, for the time between any two events occurring at the same point in space (or spatial position),

$$\tau = \frac{1}{c} \int \sqrt{g_{00}}\, dx^0 \; . \qquad \text{(IV-10)}$$

This relation determines the proper or actual time interval for a change of the coordinate x^0 at a given place in space.

We now determine the element dl of spatial distance. In the special theory of relativity, we can define dl as the interval between two infinitesimally separated events occurring at the same time. In the general theory of relativity, it is usually impossible to do this, i.e., it is impossible to determine dl by simply setting $dx^0 = 0$ in ds. This is related to the fact that in a force field, the proper time at different points in space has a different dependence on the coordinate x^0.

To find dl one now proceeds as follows.

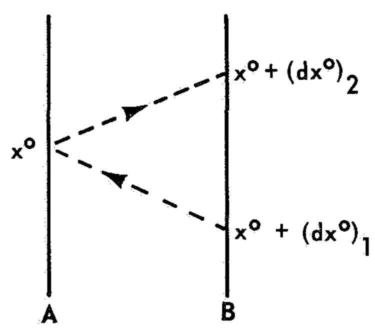

FIGURE 5. LINES REPRESENTING WORLD POINTS

Suppose, as in Figure 5, a light signal is directed from some point B in space (with coordinates $x^\alpha + dx^\alpha$) to a point A infinitely near to it (and having coordinates x^α) and then back over the same path. The time (as observed from point B) required for this, when multiplied by c, is twice the distance between the two points.

One now writes the interval, denoting the space and time coordinates:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta + 2\, g_{0\alpha}\, dx^0\, dx^\alpha + g_{00}\, (dx^0)^2, \qquad \text{(IV-11)}$$

where it is understood that one sums over repeated Greek indices from 1 to 3. The path of an electromagnetic signal from one point in space to another point in space constitutes what is commonly referred to as a null geodesic. The interval, ds, is zero by definition. Hence, the interval between the events corresponding to the departure and arrival of the signal from one point to the other is equal to zero. Setting $(ds)^2 = 0$, we find two roots:

$$(dx^0)_1 = -\frac{1}{g_{00}} \left[g_{0\alpha}\, dx^\alpha - \sqrt{(g_{0\alpha}\, g_{0\beta} - g_{\alpha\beta}\, g_{00})\, dx^\alpha\, dx^\beta} \right], \qquad \text{(IV-12.1)}$$

$$(dx^0)_2 = -\frac{1}{g_{00}}\left[g_{0\alpha}\, dx^\alpha + \sqrt{(g_{0\alpha}\, g_{0\beta} - g_{\alpha\beta}\, g_{00})\, dx^\alpha\, dx^\beta}\,\right], \qquad \text{(IV-12.2)}$$

corresponding to the propagation of the signal in the two directions between A and B. If x^0 is the moment of arrival of the signal at A the times when it left B and when it will return to B are $x^0 + (dx^0)_1$ and $x^0 + (dx^0)_2$, respectively. In Figure 5, the solid lines are the world lines corresponding to the given coordinates x^α and $x^\alpha + dx^\alpha$, while the dashed lines are the world lines of the signals. The interval of "time" between the departure of the signal and its return to the original point is equal to

$$(dx^0)_2 - (dx^0)_1 = -\frac{2}{g_{00}}\sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}\, g_{00})\, dx^\alpha dx^\beta}\,. \qquad \text{(IV-13)}$$

Now by using equation (IV-9), one obtains the proper time interval:

$$(d\tau)_{21} = \frac{-2}{c}\sqrt{-g_{\alpha\beta} + \frac{g_{0\alpha}\, g_{0\beta}}{g_{00}}\; dx^\alpha\, dx^\beta}\;. \qquad \text{(IV-14)}$$

Evidently, the spatial distance dl between the two points can be obtained from

$$2\, dl = c\,(d\tau)_{21}\,, \qquad \text{(IV-15)}$$

and comparison with equation (IV-14) yields

$$dl^2 = \left(-g_{\alpha\beta} + \frac{g_{0\alpha}\, g_{0\beta}}{g_{00}}\right) dx^\alpha\, dx^\beta\,. \qquad \text{(IV-16)}$$

One defines

$$\gamma_{\alpha\beta} = \left(-g_{\alpha\beta} + \frac{g_{0\alpha}\, g_{0\beta}}{g_{00}}\right), \qquad \text{(IV-17)}$$

and writes the expression defining the infinitesimal spatial distance in terms of the space coordinates as

$$dl^2 = \gamma_{\alpha\beta}\, dx \quad dx^\beta\,. \qquad \text{(IV-18)}$$

The expression $\gamma_{\alpha\beta}$ is the three-dimensional metric tensor determining the metric, or the geometric properties of real space. It must be remembered that the g_{ik} generally depend on x^0 so that the space metric $\gamma_{\alpha\beta}$ also changes with time. For this reason, it is meaningless to integrate dl; such an integral would depend on the world line chosen between the two given space points. Thus, as a rule, in the general theory of relativity, the concept of a definite distance between bodies loses its meaning, remaining valid only for infinitesimal distances. The only case where the distance can be defined also over a finite domain is that in which the g_{ik} do not depend on the time, so that the integral of dl along a space curve is path dependent and has a definite meaning physically.

The definition of simultaneity in the general theory of relativity will now be discussed, i.e., the question of synchronizing clocks located at different points in space, or the setting up of a correspondence between the readings of these clocks will now be discussed. Such a synchronization must be achieved by means of an exchange of electromagnetic signals between two points. Considering the process of propagation of signals between two infinitely near points A and B (as in Figure 5), one should regard as simultaneous with the "time" x^0 at the point A that reading of the clock at point B which is halfway between the moments of departure and return of the signal to that point, i.e., the moment

$$x^0 + (\Delta x^0) = x^0 + 1/2 \ [(dx^0)_2 + (dx^0)_1] \ . \qquad \text{(IV-19)}$$

Using equations (IV-12.1) and (IV-12.2), one finds that the difference in the values of the "time" x^0 for two simultaneous events occurring at infinitely near points is given by

$$\Delta x^0 = - \frac{g_{0\alpha}}{g_{00}} \ dx^{\alpha} \ . \qquad \text{(IV-20)}$$

This relation enables us to synchronize clocks in any infinitesimal region of space. Multiplying equation (IV-20) by g_{00} and bringing both terms to one side, one can state the condition for synchronization in the form $dx_0 = g_{0i} \ dx^i = 0$; the "covariant differential" dx_0 between two infinitely near simultaneous events must be equal to zero.

Suppose there exists an initial point D on an open curve in space where a clock is located. An identical clock is located at some arbitrary distance away at a point E on the open curve in space. By summing over "infinitely many" infinitesimal distances of space one can synchronize clocks along any open curve:

$$\delta x^0 = \sum_{i=1}^{n\to\infty} (\Delta x^0)_i = -\int_D^E \frac{g_{0\alpha}}{g_{00}} dx^\alpha , \tag{IV-21}$$

where δx^0 is the difference between the values of world time for two simultaneous events occurring at different points in space.

As a rule, synchronization of clocks along a closed contour turns out to be impossible. In fact, starting out along the contour and returning to the initial point, one would obtain for δx^0 a value different from zero, i.e.,

$$\delta x^0 = - \oint \frac{g_{0\alpha}}{g_{00}} dx^\alpha \neq 0 , \tag{IV-22}$$

in general. Thus, it is a fortiori impossible to synchronize clocks over all space. The exceptional cases are those reference systems in which all the components $g_{0\alpha}$ are equal to zero. One should also assign to this class those cases where the $g_{0\alpha}$ can be made equal to zero by a simple transformation of the time coordinate, which does not involve any choice of the system of objects serving for the definition of the space coordinates. Indeed, the integral in equation (IV-22) is identically zero if the sum $g_{0\alpha}\, dx^\alpha / g_{00}$ is an exact differential of some function of the space coordinates. But such a case would simply mean that we are actually dealing with a static field, and that all the $g_{0\alpha}$ could be made equal to zero by a transformation of the form $x^0 \to x^0 + \phi\,(x^\alpha)$.

V. CURVILINEAR COORDINATES, BASIC PROPERTIES OF TENSORS, AND COVARIANT DIFFERENTIATION

In studying force fields, it is necessary to consider phenomena in curvilinear coordinates and develop four-dimensional geometry in arbitray curvilinear coordinates.

Consider the transformation from one curvilinear coordinate system, x^0, x^1, x^2, x^3, to another, x'^0, x'^1, x'^2, x'^3:

$$x^i = f^i\,(x'^0,\ x'^1,\ x'^2,\ x'^3), \tag{V-1}$$

where the f^i are certain functions (endowed with all the mathematical properties of existence, continuity, differentiability, etc.). The coordinate differentials transform according to

$$dx^i = \frac{\partial x^i}{\partial x'^k}\ dx'^k \quad . \tag{V-2}$$

A summation convention is used such that when the same latin index appears as both a superscript and a subscript a summation is made from zero to three. Every set of four quantities, A^i, which under a transformation of coordinates transform like coordinate differentials is called a <u>contravariant</u> four-vector, i.e.,

$$A^i = \frac{\partial x^i}{\partial x'^k}\ A'^k \quad . \tag{V-3}$$

Contravariant vectors are designated by a superscript.

Let φ be some function of the space and time coordinates. Under a coordinate transformation the four quantities $\frac{\partial \varphi}{\partial x^i}$ transform according to the formula

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial x'^k}\ \frac{\partial x'^k}{\partial x^i} \quad . \tag{V-4}$$

Define $A_i = \frac{\partial \varphi}{\partial x^i}$; $A'_k = \frac{\partial \varphi}{\partial x'^k}$. Every set of four quantities A_i which, under a coordinate transformation, transform like the derivatives of a scalar, is called a <u>covariant</u> four-vector. Thus, under a coordinate transformation,

$$A_i = \frac{\partial x'^k}{\partial x^i}\ A'_k , \tag{V-5}$$

and the components of a covariant vector are designated by a subscript.

In curvilinear coordinates two types of vectors and three types of tensors of the second rank occur. A contravariant tensor of rank two, A^{ik}, is a set of sixteen quantities that transform like the products of the components of two contravariant vectors, i.e.,

$$A^{ik} = \frac{\partial x^i}{\partial x'^{\ell}} \frac{\partial x^k}{\partial x'^m} A'^{\ell m} . \tag{V-6}$$

Similarly, a covariant tensor transforms according to the formula

$$A_{ik} = \frac{\partial x'^{\ell}}{\partial x^i} \frac{\partial x'^m}{\partial x^k} A'_{\ell m} , \tag{V-7}$$

and a mixed tensor transforms like

$$A^i_k = \frac{\partial x^i}{\partial x'^{\ell}} \frac{\partial x'^m}{\partial x^k} A'^{\ell}_m . \tag{V-8}$$

Tensors of higher rank transform in a completely analogous fashion. For example,

$$A^m_{ik\ell} = \frac{\partial x'^p}{\partial x^i} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^s}{\partial x^{\ell}} \frac{\partial x^m}{\partial x'^t} A'^t_{pns} . \tag{V-9}$$

A tensor that is symmetric or antisymmetric in any pair of indices (both of which are covariant, or contravariant) remains so for any coordinate system. For a mixed tensor, the concept of symmetry or antisymmetry has no meaning, since to the different indices there correspond different laws of transformation, so that when we transform from one coordinate system to another, the symmetry changes. If a tensor is zero (every component) in one coordinate system, then it is zero in every other system. The sum of two tensors of the same co- or contravariant character is a tensor of the same character. The product of components of the vectors A_i and B_k is a tensor of the form A_{ik}, and of the vectors A_i and B^k is a tensor of the form A^k_i. The product of the vector A_{ℓ} and the tensor A^{ik} is a tensor of the form A^{ik}_{ℓ}, etc.

The formation of the scalar product from two vectors is a special case of the following law of <u>contraction</u> of tensors: if one has a tensor A^{--i--}_{--k--}, then the expression A^{-i-}_{-i-} is a tensor lower in rank by two than the tensor A^{--i--}_{--k--}. For example from the tensor A^k_i one can form the scalar A^i_i :

$$A^i_i = \frac{\partial x^i}{\partial x'^{\ell}} \frac{\partial x'^m}{\partial x^i} A'^{\ell}_m = A'^{\ell}_{\ell} = \frac{\partial x'^m}{\partial x'^{\ell}} A'^{\ell}_m . \qquad \text{(V-10)}$$

A^i_i is actually an invariant. Similarly, the expression A^{ik}_{ik}, $\left(A^k_i B^i_k\right)$ are scalars, etc. The expression $A^i_{k\ell i}$ is a covariant tensor of the second rank; $\left(A^i_k B^k\right)$ is a contravariant vector, etc. Quantities having two or more superscripts or subscripts identified by the same index are not defined or used here.

In curvilinear coordinates the unit tensor is defined as

$$\delta^i_k = 0 \quad \text{for } i \neq k; \qquad \delta^i_k = 1 \text{ for } i = k. \qquad \text{(V-11)}$$

The unit mixed tensor has, as one of its properties, the reproduction of a vector A^k when A^k is multiplied by δ^i_k :

$$\delta^i_k A^k = A^i . \qquad \text{(V-12)}$$

δ^i_k was introduced briefly in the previous section when the square of the line element, ds^2, was shown to be of quadratic form in the differentials dx^i:

$$ds^2 = g_{ik}\, dx^i\, dx^k. \qquad \text{(V-13)}$$

The g_{ik} were indicated to be functions of the coordinates with symmetry properties such that

$$g_{ik} = g_{ki} \; . \tag{V-14}$$

Since the contracted product of g_{ik} with the contravariant tensor $dx^i \, dx^k$ is a scalar, g_{ik} is a covariant tensor; it is called the metric tensor. Two tensors are said to be reciprocal to each other if ,

$$A_{ik} \, B^{k\ell} = \delta_i^\ell \; . \tag{V-15}$$

In particular, the contravariant metric tensor is the tensor g^{ik} reciprocal to the tensor g_{ik}, that is,

$$g_{ik} \, g^{k\ell} = \delta_i^\ell \; . \tag{V-16}$$

In a Cartesian system of coordinates (four-dimensional) void of force producing phenomena, the g_{ik} components are defined as

$$g_{ik} = \delta_{ik}, \tag{V-17}$$

where $\delta_{ik} = 1$ for $i = k$; $\delta_{ik} = 0$ for $i \neq k$.

Using equations (V-16) and (V-17), is is easily seen that

$$g^{ik} = \delta^{ik} \tag{V-18}$$

in a Cartesian system of coordinates, where δ^{ik} is defined similarly to δ_{ik}. Since the g_{ik} determine the physical properties of space, we are led to the conclusion that in a Cartesian system of reference there is no difference between a covariant and contravariant vector representing a physical quantity. In the physical space concerned here, it should be clear that the only quantities that can determine the relation between co- and contravariant components are the components of the metric tensor. Therefore,

$$A^i = g^{ik} \, A_k, \tag{V-19}$$

and inversely

$$A_i = g_{ik} A^k . \tag{V-20}$$

It is evident, then, that in a Cartesian system of reference $A_i = A^i$. However, in curvilinear coordinates the relationships in equations (V-19) and (V-20) are required to change from a contravariant to a covariant representation of physical quantities represented by vectors. However, it should be noted that in a Galilean system, unlike a Cartesian system, there is no complete identity between the co- and contravariant forms of vectors; they differ at most by a sign (see previous section).

All that has been said above in changing from co- to contravariant forms, and vice versa, applies also to tensors. Every tensor in a Cartesian system can, on transformation to curvilinear coordinates, be presented in several forms, with different co- and contravariant character. The transformation between the different forms of the tensor is accomplished in a manner similar to that for vectors. Thus

$$A^i_{k\ell} = g_{\ell m} A^{im}_k ; \; A^{ik} = g^{i\ell} g^{km} A_{\ell m}, \text{ etc.} \tag{V-21}$$

Note that if a tensor of the second rank is not symmetric, then we must distinguish between A^i_k and A^k_i, i.e., the position from which the index was raised. In a Cartesian system of coordinates, the square of the absolute value of a vector is equal to the sum of the squares of its components. In curvilinear coordinates, the square of the absolute value of a vector is the scalar

$$A_i A^i = g_{ik} A^i A^k = g^{ik} A_{ik} . \tag{V-22}$$

Indices over which summation occurs in a product of tensors ("dummy" indices) have a certain freedom of movement . Thus, for example,

$$A^{ik} B_{ik} = A_{ik} B^{ik} ; \; A_{ik} B^{\ell k} = A^k_i B^\ell_k, \text{ etc.} \tag{V-23}$$

An index can be raised in one of the factors provided the same index is lowered in the other.

The volume in four-dimensional space in Cartesian coordinates is defined as

$$d\Omega = dx^0\, dx^1\, dx^2\, dx^3\ . \tag{V-24}$$

When transforming to curvilinear coordinates, g is defined as the determinant of the tensor array g_{ik} and it is found that the volume element is

$$d\Omega = \sqrt{-g'}\, d\Omega' \tag{V-25}$$

where the prime symbolizes the curvilinear system of coordinates. Thus, in curvilinear coordinates, when integrating over any region of four-space, $\sqrt{-g}\; d\Omega$ behaves like an invariant.

If φ is a scalar, then the quantity $\sqrt{-g}\,\varphi$, (which, upon integration over $d\Omega$, gives an invariant), is sometimes called a scalar density. Similarly, one speaks of <u>vector</u> and <u>tensor densities</u> $\sqrt{-g}A^i$, $\sqrt{-g}A^{ik}$, etc. These quantities give vectors or tensors when multiplied by the four-volume element $d\Omega$. The integral

$$\int A^i \sqrt{-g}\; d\Omega \tag{V-26}$$

over a finite region, generally speaking, cannot be a vector, since, as we shall see shortly, the laws of transformation of the vector A^i are different for different points.

In general, a hypersurface is a three-dimensional manifold (three-dimensional volume). Analogous to the theorems of Gauss and Stokes for three-dimensional integrals, there are theorems which enable us to transform four-dimensional integrals. An integral over a closed hypersurface can be converted into an integral over the four-volume enclosed by it by defining the element of integration, dS^i, by the operator

$$dS^i = d\Omega\, \frac{\partial}{\partial x^i}\, . \tag{V-27}$$

For example, for the integral of the vector A_i,

$$\oint A_i\, dS^i = \int \frac{\partial A_i}{\partial x^i}\; d\Omega\ , \tag{V-28}$$

which is a generalization of Gauss' theorem.

An integral over an ordinary surface is transformed into an integral over the hypersurface "spanning" it by defining the element of integration by the operator

$$df^{ik} = \left(dS^i \frac{\partial}{\partial x^k} - dS^k \frac{\partial}{\partial x^i} \right). \tag{V-29}$$

For example, for the integral of the antisymmetric tensor A_{ik},

$$\frac{1}{2} \int A_{ik}\, df^{ik} = \frac{1}{2} \int \left(dS^i \frac{\partial A_{ik}}{\partial x^k} - dS^k \frac{\partial A_{ik}}{\partial x^i} \right) = \int \frac{\partial A_{ik}}{\partial x^k}\, dS^i. \tag{V-30}$$

The rule for transformation of an integral over a closed four-dimensional curve into an integral over a surface spanning it is included for completeness; it consists of defining

$$dx^i = df^{ik} \frac{\partial}{\partial x^k}. \tag{V-31}$$

For example, for the integral of a vector,

$$\oint A_i dx^i = \int df^{ki} \frac{\partial A_i}{\partial x^k} = \frac{1}{2} \int df^{ik} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \right), \tag{V-32}$$

which is a generalization of Stokes' theorem.

In Cartesian coordinates A_i, dA_i, and the $\frac{\partial A_i}{\partial x^k}$ are tensor components. In curvilinear coordinates dA_i is not a vector and $\frac{\partial A_i}{\partial x^k}$ is not a tensor. This is due to the fact that dA_i is the difference of vectors located at different (infinitesimally separated) points in space. Vectors transform differently at different points in space, since the coefficients in the transformation formulas are functions of the coordinates. The transformation formula for a covariant vector is

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k \;; \qquad (V\text{-}33)$$

therefore,

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \, d\left(\frac{\partial x'^k}{\partial x^i}\right), \qquad (V\text{-}34)$$

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^\ell} dx^\ell . \qquad (V\text{-}35)$$

The same also applies to the differential of a contravariant vector. Only if the x'^k are linear functions of the x^k, i.e., $\frac{\partial^2 x'^k}{\partial x^i \partial x^\ell} = 0$, do the dA_i transform like a vector.

The definition of a tensor in curvilinear coordinates which plays the same role as $\frac{\partial A_i}{\partial x^k}$ in Cartesian coordinates will now be constructed. In other words $\frac{\partial A_i}{\partial x^k}$ must be transformed from Cartesian to curvilinear coordinates

In curvilinear coordinates two vectors are required to be located at the same point in space so that they can be added or subtracted from one another. In other words, one of the vectors must somehow be "translated" to the point where the second is located, after which is determined the sum or difference of two vectors which now refer to the same point in space. The operation of translation itself must be defined so that in Cartesian coordinates the sum or difference shall coincide with the ordinary differential dA_i. And dA_i is just the sum or difference of the components of two infinitesimally separated vectors; this means that when Cartesian coordinates are used the components of the vector should not change as a result of the translation operation. But such a translation is precisely the translation of a vector parallel to itself. Under a <u>parallel translation</u> of a vector, its components in Cartesian coordinates do not change. If, on the other hand, we use curvilinear coordinates, then, in general, the components of the vector will change under such a translation. Therefore, in curvilinear coordinates, the sum or difference in the components of the two vectors after translating one of them to the point where the other is located will not coincide with their sum or difference before the translation (i.e., with the differential dA_i).

Consider an arbitrary contravariant vector: If its value at the point x^i is A^i, then at the neighboring point $x^i + dx^i$, it is equal to $A^i + dA^i$. If the vector A^i is subjected to an infinitesimal parallel displacement to the point $x^i + dx^i$; the change in the resulting vector is denoted by δA^i. Then the difference DA^i between the two vectors now located at the same point is

$$DA^i = dA^i - \delta A^i . \qquad (V-36)$$

The change δA^i in the components of a vector under an infinitesimal parallel displacement depends on the values of the components themselves, where the dependence must clearly be linear. This follows directly from the fact that the sum or difference of two vectors must transform according to the same law as each of the constituents. Thus δA^i has the same form

$$\delta A^i = - \Gamma^i_{k\ell} A^k dx^\ell , \qquad (V-37)$$

where the $\Gamma^i_{k\ell}$ are certain functions of the coordinates. Their form depends, of course, on the coordinate system; for a Cartesian system, $\Gamma^i_{k\ell} = 0$.

Clearly the $\Gamma^i_{k\ell}$ are not tensors, for a tensor equal to zero in one coordinate system is equal to zero in every other coordinate system. It is generally impossible to make the $\Gamma^i_{k\ell}$ vanish everywhere in non-Euclidean space. However, it is possible to choose a coordinate system in an infinitesimal region of non-Euclidean space for which the $\Gamma^i_{k\ell}$ become zero. The quantities $\Gamma^i_{k\ell}$ are called the Christoffel symbols. One defines

$$\Gamma_{i,k\ell} = g_{im} \Gamma^m_{k\ell} , \qquad (V-38)$$

and, of course, conversely

$$\Gamma^i_{k\ell} = g^{im} \Gamma_{m,k\ell} . \tag{V-39}$$

The relationship governing the change in a covariant vector under a parallel displacement will now be found. Let A_i and B^i be any co- and contravariant vectors. Their product, $A_i B^i$, is a scalar, and, under a parallel displacement, a scalar is unchanged. Hence,

$$\delta\,(A_i B^i) = 0, \tag{V-40}$$

$$B^i\,\delta A_i = -\,A_i\,\delta B^i = \Gamma^i_{k\ell}\,A_i\,B^k\,dx^\ell . \tag{V-41}$$

We are permitted to change indices:

$$B^i\,\delta A_i = \Gamma^k_{i\ell}\,A_k\,B^i\,dx^\ell . \tag{V-42}$$

The B^i are arbitrary, so

$$\delta A_i = \Gamma^k_{i\ell}\,A_k\,dx^\ell . \tag{V-43}$$

This determines the change in a covariant vector under a parallel displacement. Note that

$$dA^i = \frac{\partial A^i}{\partial x^\ell}\,dx^\ell . \tag{V-44}$$

Upon substitution of equations (V-37) and (V-44) into equation (V-36), one obtains

$$DA^i = \left(\frac{\partial A^i}{\partial x^\ell} + \Gamma^i_{k\ell}\,A^k\right)dx^\ell , \tag{V-45}$$

and, similarly, for a covariant vector

$$DA_i = \left(\frac{\partial A_i}{\partial x^\ell} - \Gamma^k_{i\ell}\,A_k\right)dx^\ell . \tag{V-46}$$

The expressions in parentheses in equations (V-45) and (V-46) are tensors, which, when multiplied by the vector dx^ℓ give a vector. These are the tensors which in curvilinear coordinates play the same role as the tensor $\partial A^i / \partial x^k$ in Cartesian coordinates. These tensors are called the covariant derivatives of the vectors A^i and A_i, respectively. We shall denote them by $A^i{}_{;\ell}$ and $A_{i;\ell}$. Thus

$$DA^i = A^i{}_{;\ell}\, dx^\ell , \quad DA_i = A_{i;\ell}\, dx^\ell , \tag{V-47}$$

where

$$A^i{}_{;\ell} = \frac{\partial A^i}{\partial x^\ell} + \Gamma^i_{k\ell} A^k \tag{V-48}$$

$$A_{i;\ell} = \frac{\partial A_i}{\partial x^\ell} - \Gamma^k_{i\ell} A_k . \tag{V-49}$$

Note the reduction to Cartesian coordinates when $\Gamma^i_{k\ell} = \Gamma^k_{i\ell} = 0$.

Let us now calculate the covariant derivative of a tensor under an infinitesimal parallel displacement. Consider a contravariant tensor expressed as a product of two contravariant vectors $A^i B^k$. Under a parallel displacement

$$\delta\left(A^i B^k\right) = A^i \delta B^k + B^k \delta A^i = - A^i \Gamma^k_{\ell m} B^\ell dx^m - B^k \Gamma^i_{\ell m} A^\ell dx^m . \tag{V-50}$$

By virtue of the linearity of this transformation there must also be, for an arbitrary tensor A^{ik},

$$\delta A^{ik} = - \left(A^{im} \Gamma^k_{m\ell} + A^{mk} \Gamma^i_{m\ell}\right) dx^\ell . \tag{V-51}$$

Substituting equation (V-51) into

$$DA^{ik} = dA^{ik} - \delta A^{ik} \equiv A^{ik}{}_{;\ell}\, dx^\ell , \tag{V-52}$$

obtains the covariant derivative of the tensor A^{ik} in the form

$$A^{ik}{}_{;\ell} = \frac{\partial A^{ik}}{\partial x^{\ell}} + \Gamma^{i}_{m\ell} A^{mk} + \Gamma^{k}_{m\ell} A^{im} . \qquad \text{(V-53)}$$

In similar fashion, one obtains the covariant derivative of the mixed tensor A^{i}_{k} and the covariant tensor A_{ik} in the form

$$A^{i}_{k;\ell} = \frac{\partial A^{i}_{k}}{\partial x^{\ell}} - \Gamma^{m}_{k\ell} A^{i}_{m} + \Gamma^{i}_{m\ell} A^{m}_{k} , \qquad \text{(V-54)}$$

$$A_{ik;\ell} = \frac{\partial A_{ik}}{\partial x^{\ell}} - \Gamma^{m}_{i\ell} A_{mk} - \Gamma^{m}_{k\ell} A_{im} . \qquad \text{(V-55)}$$

One can similarly determine the covariant derivative of a tensor of arbitrary rank. In doing so one finds the following rule of covariant differentiation: to obtain the covariant derivative of the tensor $A^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot}$ with respect to ∂x^{ℓ} for each covariant index $i(A^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot})$, one adds to the ordinary derivative $\partial A^{\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot}/\partial x^{\ell}$ a term $- \Gamma^{k}_{i\ell} A^{\cdot\,\cdot\,\cdot}_{\cdot\, k\, \cdot}$, and for each contravariant index $i(A^{\cdot\, i\, \cdot}_{\cdot\,\cdot\,\cdot})$ a term $+ \Gamma^{i}_{k\ell} A^{\cdot\, k\, \cdot}_{\cdot\,\cdot\,\cdot}$. [9]

One can easily verify that the covariant derivative of a product is found by the same rule as for ordinary differentiation of products. Consider the covariant derivative of a scalar φ as an ordinary derivative, that is, as the covariant vector $\varphi_k = \frac{\partial \varphi}{\partial x^k}$, in accordance with the fact that for a scalar $\delta\varphi = 0$, and therefore $D\varphi = d\varphi$. The covariant derivative of the product $A_i B_k$ is

$$\left(A_i B_k\right)_{;\ell} = A_{i;\ell} B_k + A_i B_{k;\ell} . \qquad \text{(V-56)}$$

If, in a covariant derivative, one raises the index signifying the differentiation one obtains the so-called contravariant derivative. Thus,

$$A_i{}^{;k} = g^{k\ell} A_{i;\ell} \; ; \; A^{i;k} = g^{k\ell} A^{i}{}_{;\ell} . \qquad \text{(V-57)}$$

The Christoffel symbols $\Gamma^i_{k\ell}$ are symmetric in the subscripts. Since the covariant derivative of a vector $A_{i;k}$ is a tensor, the difference $A_{i;k} - A_{k;i}$ is also a tensor. Let the vector A_i be the gradient of a scalar, that is, $A_i = \frac{\partial \varphi}{\partial x^i}$. Since $\partial A_i / \partial x^k = \frac{\partial^2 \varphi}{\partial x^k \partial x^i} = \frac{\partial A_k}{\partial x^i}$, with the help of equation (V-49),

$$A_{k;i} - A_{i;k} = \left(\Gamma^\ell_{ik} - \Gamma^\ell_{ki}\right) \frac{\partial \varphi}{\partial x^\ell} \quad . \qquad \text{(V-58)}$$

In Cartesian coordinates the left side equation (V-58) is zero. Since the left side is a tensor, it must be zero in all coordinate systems. Therefore,

$$\Gamma^i_{k\ell} = \Gamma^i_{\ell k} \quad . \qquad \text{(V-59)}$$

In a similar fashion it can be shown that

$$\Gamma_{i,k\ell} = \Gamma_{i,\ell k} \quad . \qquad \text{(V-60)}$$

So, in general, there are forty different quantities $\Gamma^i_{k\ell}$ or $\Gamma_{i,\ell k}$.

In speaking of the Christoffel symbols mention should be made of the formulas for transforming the Christoffel symbols from one coordinate system to another. Compare the laws of transformation of the two sides of the equations defining the covariant derivatives, and require that these laws be the same for both sides. The result is

$$\Gamma^i_{k\ell} = \Gamma'^m_{np} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^p}{\partial x^\ell} + \frac{\partial^2 x'^m}{\partial x^k \partial x^\ell} \frac{\partial x^i}{\partial x'^m} \quad . \qquad \text{(V-61)}$$

It is clear that the $\Gamma^i_{k\ell}$ behaves like a tensor only under linear transformations (meaning the second term in equation (V-61) is zero). Equation (V-61) shows that it is possible for a coordinate system in which all the $\Gamma^i_{k\ell}$ become zero at a preassigned point (locally inertial, locally geodesic).

Let a given point be chosen as the origin of coordinates, and let the values of the $\Gamma^i_{k\ell}$ at that point be initially (in the coordinate x^ℓ) equal to $\left(\Gamma^i_{k\ell}\right)_0$. In the neighborhood of this point, the transformation is made:

$$x'^i = x^i + \frac{1}{2}\left(\Gamma^i_{k\ell}\right)_0 x^k x^\ell . \tag{V-62}$$

Then,

$$\left(\frac{\partial^2 x'^m}{\partial x^k \partial x^\ell} \quad \frac{\partial x^i}{\partial x'^m}\right)_0 = \left(\Gamma^i_{k\ell}\right)_0 , \tag{V-63}$$

and according to equation (V-61), all of the Γ'^m_{np} become equal to zero. Note also for the transformation in equation (V-62),

$$\left(\frac{\partial x'^i}{\partial x^k}\right) = \delta^i_k , \tag{V-64}$$

so that it does not change the value of any tensor, including the tensor g_{ik}, at the given point. Hence, the Christoffel symbols vanish at the same time that the g_{ik} is brought to Galilean form.

The relation of the Christoffel symbols to the metric tensor will now be calculated. First, it will be shown that the covariant derivative of the metric tensor g_{ik} is zero. Note that

$$DA_i = g_{ik}\, DA^k \tag{V-65}$$

is valid for the vector DA_i, as for any other vector. On the other hand, $A_i = g_{ik}A^k$. Therefore,

$$DA_i = D\left(g_{ik}A^k\right) = g_{ik}DA^k + A^k Dg_{ik} . \tag{V-66}$$

Since A^k is arbitrary, then

$$Dg_{ik} = 0; \tag{V-67}$$

therefore, the covariant derivative

$$g_{ik;\ell} = 0 , \tag{V-68}$$

and g_{ik} may be considered constant during covariant differentiation. Equation (V-68) can be used to express the Christoffel symbols $\Gamma^i_{k\ell}$ in terms of the metric tensor g_{ik}. Using equation (V-55) for the covariant derivative of a tensor one writes for equation (V-68):

$$g_{ik;\ell} = \frac{\partial g_{ik}}{\partial x^\ell} - g_{mk}\,\Gamma^m_{i\ell} - g_{im}\,\Gamma^m_{k\ell} ,$$

$$g_{ik;\ell} = \frac{\partial g_{ik}}{\partial x^\ell} - \Gamma_{k,i\ell} - \Gamma_{i;k\ell} = 0 . \tag{V-69}$$

Thus, the derivatives of g_{ik} are expressed in terms of the Christoffel symbols. One writes the values of the derivatives of g_{ik}, permuting the indices i, k, ℓ:

$$\frac{\partial g_{ik}}{\partial x^\ell} = \Gamma_{k,i\ell} + \Gamma_{i,k\ell} , \tag{V-70}$$

$$\frac{\partial g_{\ell i}}{\partial x^k} = \Gamma_{i,k\ell} + \Gamma_{\ell,ik} , \tag{V-71}$$

$$- \frac{\partial g_{k\ell}}{\partial x^i} = - \Gamma_{\ell,ki} - \Gamma_{k,\ell i} . \tag{V-72}$$

If one recalls that $\Gamma_{i,k\ell} = \Gamma_{i,\ell k}$ and takes one half the sum of equations (V-70), (V-71), and (V-72) one gets

$$\Gamma_{i,k\ell} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^{\ell}} + \frac{\partial g_{i\ell}}{\partial x^{k}} - \frac{\partial g_{k\ell}}{\partial x^{i}}\right) . \tag{V-73}$$

Recalling that $\Gamma^{i}_{k\ell} = g^{im}\,\Gamma_{m,k\ell},$ one gets from equation (V-73)

$$\Gamma^{i}_{k\ell} = \frac{1}{2}\, g^{im}\left(\frac{\partial g_{mk}}{\partial x^{\ell}} + \frac{\partial g_{m\ell}}{\partial x^{k}} - \frac{\partial g_{k\ell}}{\partial x^{m}}\right) . \tag{V-74}$$

Equations (V-73) and (V-74) are the required expressions for the Christoffel symbols in terms of the metric tensor.

An expression for the contracted Christoffel symbol Γ^{i}_{ki} , which will be important later on, will now be derived. The differential, dg, of the determinant, g, made up from the components of the tensor g_{ik} is calculated; dg can be obtained by taking the differential of each component of the tensor g_{ik} and multiplying it by its coefficient in the determinant, i.e., by the corresponding minor. The components of the tensor g^{ik} reciprocal to g_{ik} are equal to the minors of the determinant of the g_{ik}, divided by the determinant. Hence, the minors of the determinant g are equal to the minors of the determinant of the g_{ik}, divided by the determinant, and the minors of the determinant g are equal to gg^{ik}. Thus,

$$dg = gg^{ik}\, dg_{ik} = -\, gg_{ik}\, dg^{ik} , \tag{V-75}$$

since $g_{ik}\, g^{ik} = \delta^{i}_{i} = 4,\ g^{ik}\, dg_{ik} = -\, g_{ik}\, dg^{ik}.$

From equation (V-74) one has

$$\Gamma^{i}_{ki} = \frac{1}{2}\, g^{im}\left(\frac{\partial g_{mk}}{\partial x^{k}} + \frac{\partial g_{mi}}{\partial x^{k}} - \frac{\partial g_{ki}}{\partial x^{m}}\right) . \tag{V-76}$$

If the positions of the indices m and i are changed in the third and first terms in parentheses, the two terms cancel each other so that

$$\Gamma^i_{ki} = \frac{1}{2} g^{im} \frac{\partial g_{im}}{\partial x^k} , \qquad (V\text{-}77)$$

and by equation (V-75) one writes

$$\Gamma^i_{ki} = \frac{1}{2g} \frac{\partial g}{\partial x^k} = \frac{\partial \ln \sqrt{-g}}{\partial x^k} . \qquad (V\text{-}78)$$

Note also the expression for the quantity $g^{k\ell} \Gamma^i_{k\ell}$

$$g^{k\ell} \Gamma^i_{k\ell} = \frac{1}{2} g^{k\ell} g^{im} \left(\frac{\partial g_{mk}}{\partial x^\ell} + \frac{\partial g_{\ell m}}{\partial x^k} - \frac{\partial g_{k\ell}}{\partial x^m} \right) , \qquad (V\text{-}79)$$

$$g^{k\ell} \Gamma^i_{k\ell} = g^{k\ell} g^{im} \left(\frac{\partial g_{mk}}{\partial x^\ell} - \frac{1}{2} \frac{\partial g_{k\ell}}{\partial x^m} \right) . \qquad (V\text{-}80)$$

One may write equation (V-80) as

$$g^{k\ell} \Gamma^i_{k\ell} = g^{k\ell} g^{im} \frac{\partial g_{mk}}{\partial x^\ell} - \frac{1}{2} g^{im} g^{k\ell} \frac{\partial g_{k\ell}}{\partial x^m} . \qquad (V\text{-}81)$$

One uses equation (V-75) to write equation (V-81) as

$$g^{k\ell} \Gamma^i_{k\ell} = g^{k\ell} g^{im} \frac{\partial g_{mk}}{\partial x^\ell} - \frac{1}{2} \frac{g_{im}}{g} \frac{\partial g}{\partial x^m} . \qquad (V\text{-}82)$$

Recall that

$$g_{i\ell}\, g^{\ell k} = \delta^k_i . \qquad (V\text{-}83)$$

Differentiation with respect to x^m yields

$$g_{i\ell} \frac{\partial g^{\ell k}}{\partial x^m} = - g^{\ell k} \frac{\partial g_{i\ell}}{\partial x^m} . \qquad (V\text{-}84)$$

Interchange m and ℓ and change sign:

$$-g_{im} \frac{\partial g^{mk}}{\partial x^{\ell}} = g^{mk} \frac{\partial g_{im}}{\partial x^{\ell}} \quad . \tag{V-85}$$

Then interchange k and i:

$$-g_{km} \frac{\partial g^{mi}}{\partial x^{\ell}} = g^{mi} \frac{\partial g_{km}}{\partial x^{\ell}} \quad , \tag{V-86}$$

$$-g_{mk} \frac{\partial g^{im}}{\partial x^{\ell}} = g^{im} \frac{\partial g_{mk}}{\partial x^{\ell}} \quad . \tag{V-87}$$

Substitute equation (V-87) into equation V-85 to obtain

$$-g^{k\ell} g_{mk} \frac{\partial g^{im}}{\partial x^{\ell}} - \frac{1}{2} \frac{g^{im}}{g} \frac{\partial g}{\partial x^{m}} = g^{k\ell} \Gamma^{i}_{k\ell} \quad . \tag{V-88}$$

In view of equations (V-83) and (V-88) one has

$$g^{k\ell} \Gamma^{i}_{k\ell} = - \frac{\partial g^{im}}{\partial x^{m}} - \frac{1}{2} \frac{g^{im}}{g} \frac{\partial g}{\partial x^{m}} \quad . \tag{V-89}$$

Since m is a dummy index, it may be replaced by k:

$$g^{k\ell} \Gamma^{i}_{k\ell} = - \frac{\partial g^{ik}}{\partial x^{k}} - \frac{1}{2} \frac{g^{ik}}{g} \frac{\partial g}{\partial x^{k}} \quad , \tag{V-90}$$

which may be written as

$$g^{k\ell} \Gamma^{i}_{k\ell} = - \frac{1}{\sqrt{-g}} \frac{\partial \left(\sqrt{-g}\, g^{ik}\right)}{\partial x^{k}} \quad . \tag{V-91}$$

Finally, it need only be pointed out that similar calculations for the contravariant metric tensor, g^{ik}, yields

$$g^{ik}{}_{;\ell} = 0 , \tag{V-92}$$

from which it follows

$$\frac{\partial g^{ik}}{\partial x^{\ell}} = - \Gamma^{i}_{m\ell}\, g^{mk} - \Gamma^{k}_{m\ell}\, g^{im} . \tag{V-93}$$

Now, recalling equation (V-48) and equation (V-78), one can compute the expression for $A^{i}{}_{;i}$, the generalized divergence of a vector in curvilinear coordinates in the convenient form

$$A^{i}{}_{;i} = \frac{\partial A^{i}}{\partial x^{i}} + \Gamma^{i}_{\ell i} A^{\ell} = \frac{\partial A^{i}}{\partial x^{i}} + A^{\ell} \frac{\partial \ln\sqrt{-g}}{\partial x^{\ell}} , \tag{V-94}$$

which may be written as

$$A^{i}{}_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\, A^{i})}{\partial x^{i}} \tag{V-95}$$

One can derive an analogous expression for the divergence of an antisymmetric tensor A^{ik}. Recall equation (V-53):

$$A^{ik}{}_{;k} = \frac{\partial A^{ik}}{\partial x^{k}} + \Gamma^{i}_{mk} A^{mk} + \Gamma^{k}_{mk} A^{im} . \tag{V-96}$$

But, since $A^{mk} = -A^{km}$, then

$$\Gamma^{i}_{mk} A^{mk} = -\Gamma^{i}_{km} A^{km} = 0 . \tag{V-97}$$

So, using equation (V-78) again one obtains

$$A^{ik}{}_{;k} = \frac{1}{\sqrt{-g}} \frac{\partial\left(\sqrt{-g}\, A^{ik}\right)}{\partial x^{k}} , \tag{V-98}$$

for the divergence of an antisymmetric tensor.

Now suppose A^{ik} is a symmetric tensor; one calculates the expression $A^k_{i;k}$ for its mixed components. Recalling equation (V-54), we have

$$A^k_{i;k} = \frac{\partial A^k_i}{\partial x^k} + \Gamma^k_{\ell k} A^\ell_i - \Gamma^\ell_{ik} A^k_\ell = \frac{1}{\sqrt{-g}} \frac{\partial\left(\sqrt{-g}\, A^k_i\right)}{\partial x^k} - \Gamma^\ell_{ki} A^k_\ell \,. \tag{V-99}$$

The last term here is equal to

$$-\frac{1}{2}\left(\frac{\partial g_{i\ell}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^\ell}\right) A^{k\ell} \,. \tag{V-100}$$

Because of the symmetry of $A^{k\ell}$, two of the terms in parentheses cancel each other, leaving

$$A^k_{i;k} = \frac{1}{\sqrt{-g}} \frac{\partial\left(\sqrt{-g}\, A^k_i\right)}{\partial x^k} - \frac{1}{2} \frac{\partial g_{k\ell}}{\partial x^i} A^{k\ell} \,. \tag{V-101}$$

In Cartesian coordinates, $\dfrac{\partial A_i}{\partial x^k} - \dfrac{\partial A_k}{\partial x^i}$ is an antisymmetric tensor. In curvilinear coordinates, the tensor is $A_{i;k} - A_{k;i}$. With the help of equation (V-49) for $A_{i;k}$, and since $\Gamma^i_{k\ell} = \Gamma^i_{\ell k}$, one has

$$A_{i;k} - A_{k;i} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} \,. \tag{V-102}$$

Finally, we transform to curvilinear coordinates the sum $\dfrac{\partial^2 \varphi}{\partial x^{i\,2}}$ of the second derivative of a scalar φ. In curvilinear coordinates this sum goes over to $\varphi^{;i}_{;i}$. But $\varphi_{;i} = \dfrac{\partial \varphi}{\partial x^i}$, since covariant differentiation of a scalar reduces to ordinary differentiation. Raising the index i one obtains

$$\varphi^{;i} = g^{ik} \frac{\partial \varphi}{\partial x^k} \,, \tag{V-103}$$

and then by equation (V-95) one writes

$$\varphi^{;i}_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left(\sqrt{-g}\, g^{ik} \frac{\partial \varphi}{\partial x^k} \right) . \tag{V-104}$$

One can also use equation (V-95) to write Gauss' theorem for the transformation of the integral of a vector over a hypersurface into an integral over a four-volume:

$$\oint A^i \sqrt{-g}\, dS_i = \int A^i_{\;;i} \sqrt{-g}\, d\Omega . \tag{V-105}$$

VI. THE RIEMANN CURVATURE TENSOR, ITS PROPERTIES, AND THE EMPTY SPACE GRAVITATIONAL FIELD EQUATIONS

In the following development of the Riemann curvature tensor and the empty space gravitational field equations, the authors briefly recall some of the pertinent facts that were developed in the previous section. This is done not as a matter of redundancy but to help elucidate the following material.

The authors evaluate the variation of a contravariant vector field $A^i(x^i)$ in coordinate values x^i to $x^i + dx^i$ in affined space. Comparison of the value $A^i(x^i + dx^i)$ with the value of the vector $A^{i*}(x^i + dx^i)$ obtained from $A^i(x^i)$ by vector transplantation from x^i to $x^i + dx^i$ will be made. At the point $(x^i + dx^i)$ the vector difference $A^i(x^i + dx^i) - A^{i*}(x^i + dx^i)$ has the value

$$\begin{aligned} & A^i (x^i + dx^i) - A^i(x^i) - \left[A^{i*}(x^i + dx^i) - A^i (x^i) \right] \\ &= \frac{\partial A^i}{\partial x^k} dx^k + 0\,(dx^k)^2 + \Gamma^i_{k\ell} A^\ell dx^k \\ &= \left(\frac{\partial A^i}{\partial x^k} + \Gamma^i_{k\ell} A^\ell \right) dx^k + 0\,(dx^k)^2 . \end{aligned} \tag{VI-1}$$

By analogy, to the first-order term in a Taylor's expansion the quantity

$$A^i_{;k} = \frac{\partial A^i}{\partial x^k} + \Gamma^i_{k\ell} A^\ell \tag{VI-2}$$

is interpreted as a "derivative," and is by definition the covariant derivative of the contravariant vector field. It is directly demonstrated from the tensor transformation laws that $A^i_{;k}$ is a tensor, and to contrast the covariant differentiation with ordinary differentiation it is written

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{k\ell} A^\ell . \tag{VI-3}$$

If equation (VI-3) is specialized to a Riemann space, the Christoffel symbols $\Gamma^i_{k\ell}$ are replaced by the symbols $\left\{ {i \atop k\ \ell} \right\}$:[1 *]

$$A^i_{;k} = A^i_{,k} + \left\{ {i \atop k\ \ell} \right\} A^\ell . \tag{VI-4}$$

For a <u>covariant</u> vector field B_m the covariant derivative is defined by

$$B_{m;\ell} = B_{m,\ell} - \left\{ {r \atop m\ \ell} \right\} B_r . \tag{VI-5}$$

For a tensor field $T^{ij}{}_k$ the covariant derivative $T^{ij}{}_{k;\ell}$ is defined by

$$T^{ij}{}_{k;\ell} = T^{ij}{}_{k,\ell} + \left\{ {i \atop m\ \ell} \right\} T^{mj}{}_k + \left\{ {j \atop m\ \ell} \right\} T^{im}{}_k - \left\{ {m \atop k\ \ell} \right\} T^{ij}{}_m . \tag{VI-6}$$

The generalization to a tensor of any order can be inferred from equation (VI-6).

The product formula for differentiation is valid for covariant differentiation:

$$\left(T^i_{jk}\, S^m_{np}\right)_{;\ell} = T^i_{jk;\ell}\, S^m_{np} + T^i_{jk}\, S^m_{np;\ell} . \tag{VI-7}$$

---

1 * Instead of $\Gamma^i_{k\ell}$ and $\Gamma_{i,k\ell}$, the symbols $\left\{ {i \atop k\ \ell} \right\}$ and $[i, k\ell]$ are sometimes used. The interchange is well known and can be used without ambiguity according to personal choice.

The Einstein gravitational field equations are based on the Riemann curvature tensor. For this reason more space will be given to the development of the Riemann curvature tensor.

First it will be recalled that the Lorentz metric has the property that in the coordinates of special relativity its components are constant over all of space:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{VI-8}$$

This is equivalent to the statement that a system of <u>geodesic coordinates</u> exists <u>in the large.</u> <u>All</u> Christoffel symbols are zero in this system, i.e., for any vector

$$A^i_{;k} = A^i_{,k} \tag{VI-9}$$

Hence,

$$A^i_{;k;j} = A^i_{,k,j} \quad , \tag{VI-10}$$

and

$$A^i_{;k;j} - A^i_{;j;k} = 0 \ . \tag{VI-11}$$

This is a tensor equation, i.e., since it is true in one system of coordinates, it is true in all systems, not just in the geodesic system. Thus when a space admits a Lorentz metric, equation (VI-11) holds. Define a tensor

$$T^i_j = A^i_{;j} = A^i_{,j} + \begin{Bmatrix} i \\ j\ k \end{Bmatrix} A^k \ . \tag{VI-12}$$

Then

$$T^i_{j;\ell} = A^i_{;j;\ell} = T^i_{j,\ell} + \begin{Bmatrix} i \\ m\ \ell \end{Bmatrix} T^m_j - \begin{Bmatrix} n \\ j\ \ell \end{Bmatrix} T^i_n \ . \tag{VI-13}$$

The substitution of equation (VI-12) into equation (VI-13) yields

$$A^{i}_{;j;\ell} = A^{i}_{,j,\ell} + \begin{Bmatrix} i \\ j\,k \end{Bmatrix}_{,\ell} A^{k} + \begin{Bmatrix} i \\ j\,k \end{Bmatrix} A^{k}_{,\ell} + \begin{Bmatrix} i \\ m\,\ell \end{Bmatrix} A^{m}_{,j}$$

$$+ \begin{Bmatrix} i \\ m\,\ell \end{Bmatrix} \begin{Bmatrix} m \\ j\,k \end{Bmatrix} A^{k} - \begin{Bmatrix} n \\ j\,\ell \end{Bmatrix} T^{i}_{n} \quad . \qquad \text{(VI-14)}$$

If j and ℓ are interchanged, then

$$A^{i}_{;\ell;j} = A^{i}_{,\ell,j} + \begin{Bmatrix} i \\ \ell\,k \end{Bmatrix}_{,j} A^{k} + \begin{Bmatrix} i \\ \ell\,k \end{Bmatrix} A^{k}_{,j}$$

$$+ \begin{Bmatrix} i \\ m\,j \end{Bmatrix} A^{m}_{,\ell} + \begin{Bmatrix} i \\ m\,j \end{Bmatrix} \begin{Bmatrix} m \\ \ell\,k \end{Bmatrix} A^{k} - \begin{Bmatrix} n \\ \ell\,j \end{Bmatrix} T^{i}_{n} \quad . \qquad \text{(VI-15)}$$

Then

$$A^{i}_{;j;\ell} - A^{i}_{;\ell;j} = \begin{Bmatrix} i \\ j\,k \end{Bmatrix}_{,\ell} A^{k} - \begin{Bmatrix} i \\ \ell\,k \end{Bmatrix}_{,j} A^{k}$$

$$+ \begin{Bmatrix} i \\ m\,\ell \end{Bmatrix} \begin{Bmatrix} m \\ j\,k \end{Bmatrix} A^{k} - \begin{Bmatrix} i \\ m\,j \end{Bmatrix} \begin{Bmatrix} m \\ \ell\,k \end{Bmatrix} A^{k} \; ,$$

$$= \left[\begin{Bmatrix} i \\ j\,k \end{Bmatrix}_{,\ell} - \begin{Bmatrix} i \\ \ell\,k \end{Bmatrix}_{,j} + \begin{Bmatrix} i \\ m\,\ell \end{Bmatrix} \begin{Bmatrix} m \\ j\,k \end{Bmatrix} - \begin{Bmatrix} i \\ m\,j \end{Bmatrix} \begin{Bmatrix} m \\ \ell\,k \end{Bmatrix} \right] A^{k} \quad . \qquad \text{(VI-16)}$$

The object within the brackets is a tensor, by the quotient theorem. It is known as the Riemann curvature tensor and plays a central role in the geometric structure of Riemann space. It is denoted by

$$R^{i}_{jk\ell} = \begin{Bmatrix} i \\ k\,j \end{Bmatrix}_{,\ell} - \begin{Bmatrix} i \\ j\,\ell \end{Bmatrix}_{,k} + \begin{Bmatrix} i \\ m\,\ell \end{Bmatrix} \begin{Bmatrix} m \\ k\,j \end{Bmatrix} - \begin{Bmatrix} i \\ m\,k \end{Bmatrix} \begin{Bmatrix} m \\ \ell\,j \end{Bmatrix} . \qquad \text{(VI-17)}$$

The necessary condition that a Riemann space admits a Lorentz metric can be written

$$R^{i}_{jk\ell} A^{j} = A^{i}_{;k;\ell} - A^{i}_{;\ell;k} = 0 \quad . \qquad \text{(VI-18)}$$

Since A^j is an arbitrary vector, the necessary condition that a Riemann space admits a Lorentz metric may be written

$$R^i_{\ jk\ell} = 0 \ . \tag{VI-19}$$

A space is said to be flat if its Riemann curvature tensor vanishes. Equation (VI-19) states that if a space has a Lorentz metric, then it is a flat space. The converse statement that a physically acceptable space has a Lorentz metric if it is flat can be proved. Hence, a necessary and sufficient condition that a space has a Lorentz metric is given by equation (VI-19).

The generalization of equation (VI-18) to tensors of higher rank is a straight-forward process. For example, for a second rank tensor,

$$T^{ij}_{\ \ ;k;\ell} - T^{ij}_{\ \ ;\ell;k} = R^i_{\ mk\ell}\, T^{mj} + R^j_{\ mk\ell}\, T^{im} \ . \tag{VI-20}$$

If the index i in equation (VI-18) is lowered, the result is

$$A_{i;k;\ell} - A_{i;\ell;k} = R_{imk\ell}\, A^m \ . \tag{VI-21}$$

This formula is useful in covariant differentiation of covariant vectors. This formula requires commitment to a metric space. If the same calculations for the interchange of derivations of covariant vectors are made as for contravariant vectors, the result is

$$A_{i;j;k} - A_{i;k;j} = R^\ell_{\ ijk}\, A_\ell \ . \tag{VI-22}$$

This formula does not require a commitment to a metric space and is valid in the general affine case.

The Riemann curvature tensor has $4^4 = 256$ components. The number of independent components is much smaller because of symmetry relations. An inspection of equation (VI-17) shows that the Riemann curvature tensor is anti-symmetric in the third and fourth indices k and ℓ . Hence, the $k\ell$ sub-block has only six, instead of 16, independent components. In combination with the 16 components of the ij block, this reduces the number of independent component to a maximum of 96.

The Riemann curvature tensor $R^i_{\ jk\ell}$ is a mixed tensor which, for convenience in stating its symmetry properties, will be changed into the covariant form by lowering the index i :

$$R_{ijk\ell} = g_{im} R^{m}{}_{jk\ell} \; . \qquad \text{(VI-23)}$$

Then the symmetry properties

$$R_{ijk\ell} = - R_{ij\ell k} \; ,$$

$$R_{ijk\ell} = - R_{ji\ell k} \; ,$$

$$R_{ijk\ell} = - R_{\ell kij} \; , \qquad \text{(VI-24)}$$

are stated without proof. Also,

$$R_{ijk\ell} + R_{kj\ell i} + R_{\ell jik} = 0 \; ,$$

$$R_{jik\ell} + R_{ki\ell j} + R_{\ell ijk} = 0 \; . \qquad \text{(VI-25)}$$

Between the components R_{0123} , R_{0231} , R_{0312} , there exists the relation

$$R_{1023} + R_{2031} + R_{3012} = 0 \; . \qquad \text{(VI-26)}$$

The relation

$$R^{i}{}_{jk\ell;m} + R^{i}{}_{jmk;\ell} + R^{i}{}_{j\ell m;k} = 0 \qquad \text{(VI-27)}$$

is valid for the Riemann curvature tensor. It is known as the Bianchi Identity.

By the definition of the parallel displacement of a vector the change in the component A^i under parallel displacement is given by

$$dA^i = - \left\{ \begin{matrix} i \\ j\;k \end{matrix} \right\} A^j \, dx^k \; . \qquad \text{(VI-28)}$$

If a curve S_{12} between two points P_1 and P_2 is given and the value of A^i is given at P_1, then the value of A^i at P_2 can be computed by using equation (VI-28):

$$J^i_{12} = \int_{S_{12}} dA^i \tag{VI-29}$$

where J^i_{12} is the change in A^i along the curve S_{12} . If a different curve S'_{12} is used to connect P_1 and P_2 then, in general,

$$J^i_{12} \neq J^{i'}_{12} \,, \tag{VI-30}$$

i.e., the change in A^i is <u>path dependent.</u> If S is a closed curve starting and terminating at P_1, then

$$J^i(S) = \int_S dA^i \tag{VI-31}$$

is not necessarily zero.

A parallel displacement of A^i from the point P_i to the point P_f along the path $P_i P_1 P_f$ will be compared with the parallel displacement of A^i along the path $P_i P_2 P_f$, according to Figure 6 :

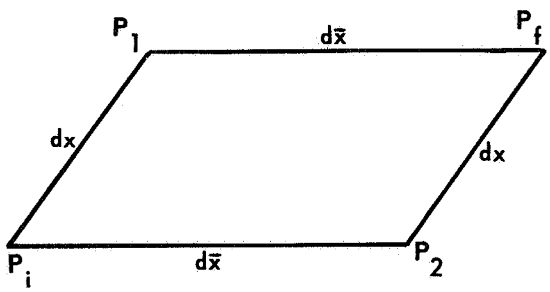

FIGURE 6. PARALLEL DIS-
PLACEMENT PATHS

These displacements and values of A^i are described by the following equations:

$$dA^i\,(P_i P_1) = -\left\{ \begin{matrix} i \\ m\;n \end{matrix} \right\} A^n\, dx^m \,, \tag{VI-32}$$

$$A^i\,(P_1) = A^i(P_i) - \left\{ \begin{matrix} i \\ m\;n \end{matrix} \right\} A^n\, dx^m \,, \tag{VI-33}$$

$$\left\{ \begin{matrix} i \\ j\;k \end{matrix} \right\}_{P_1} = \left\{ \begin{matrix} i \\ j\;k \end{matrix} \right\}_{P_i} + \left\{ \begin{matrix} i \\ j\;k \end{matrix} \right\}_{,m} dx^m \,, \tag{VI-34}$$

$$dA^i\,(P_1 P_f) = -\left[\left\{ \begin{matrix} i \\ j\;k \end{matrix} \right\} + \left\{ \begin{matrix} i \\ j\;k \end{matrix} \right\}_{,n} dx^n \right] \left[A^k - \left\{ \begin{matrix} k \\ m\;n \end{matrix} \right\} A^n dx^m \right] d\bar{x}^j . \tag{VI-35}$$

By a rearrangement of terms, equation (VI-35) may be written in the form

$$dA^i\,(P_1P_f) = -\left\{ {i \atop j\,k} \right\} A^k\, d\bar{x}^j - \left\{ {i \atop j\,k} \right\}_{,\,n} A^k\, dx^n\, d\bar{x}^j$$

$$+ \left\{ {i \atop j\,k} \right\} \left\{ {k \atop m\,n} \right\} A^n\, dx^m\, d\bar{x}^j \quad . \tag{VI-36}$$

Hence,

$$A^i\,(P_iP_1P_f) = A^i - \left\{ {i \atop j\,k} \right\} A^k\, d\bar{x}^j - \left\{ {i \atop m\,k} \right\} A^k\, dx^m$$

$$- \left\{ {i \atop j\,k} \right\}_{,\,m} A^k\, d\bar{x}^j\, dx^m + \left\{ {i \atop j\,\ell} \right\} \left\{ {\ell \atop m\,k} \right\} A^k\, dx^m\, d\bar{x}^j \tag{VI-37}$$

By interchanging dx and $d\bar{x}$,

$$A^i\,(P_iP_2P_f) = A^i - \left\{ {i \atop j\,k} \right\} A^k\, d\bar{x}^j - \left\{ {i \atop m\,k} \right\} A^k\, dx^m$$

$$- \left\{ {i \atop m\,k} \right\}_{,\,j} A^k\, d\bar{x}^j\, dx^m + \left\{ {i \atop m\,\ell} \right\} \left\{ {\ell \atop j\,k} \right\} A^k\, d\bar{x}^j\, dx^m \, . \tag{VI-38}$$

The difference between the parallel displacements along the infinitesimal paths $P_iP_1P_f$ and $P_iP_2P_f$ is

$$\Delta A^i = \left\{ {i \atop m\,k} \right\}_{,\,j} A^k\, d\bar{x}^j\, dx^m - \left\{ {i \atop j\,k} \right\}_{,\,m} A^k\, d\bar{x}^j\, dx^m$$

$$+ \left\{ {i \atop j\,\ell} \right\} \left\{ {\ell \atop m\,k} \right\} A^k dx^m\, d\bar{x}^j - \left\{ {i \atop m\,\ell} \right\} \left\{ {\ell \atop j\,k} \right\} A^k\, dx^m\, d\bar{x}^j \, . \tag{VI-39}$$

By the definition of the Riemann curvature tensor, this is exactly

$$\Delta A^i = R^i_{\ kmj}\, A^k dx^m\, d\bar{x}^j \quad . \tag{VI-40}$$

Thus the value of A^i at the nearby point is independent of the path if and only if $R^i_{kmj} = 0$, and for a non-zero Riemann curvature tensor the difference in final values is given by equation (VI-40).

The meaning of the above discussion is that a vector field $A^i(x)$ can be established by parallel displacement of an arbitrary vector A^i from some initial point P to all points in the neighborhood of P in a Riemannian space if and only if the Riemann curvature tensor of the space is identically zero. That is, the system of differential equations

$$A^i_{,j} = -\left\{ {i \atop k\ j} \right\} A^k \tag{VI-41}$$

is integrable if and only if the Riemann curvature tensor is identically 0, i.e., the space is flat.

These Einstein field equations will now be introduced by an heuristic argument. It has been pointed out in preceding paragraphs that a gravity-free space with a Lorentz metric is correctly described by the equation

$$R_{ijk\ell} = 0 . \tag{VI-42}$$

The complete field equations might be expected to be some generalization of equation (VI-42), i.e., a weakening of equation (VI-42). This weakening is suggested by a consideration of Laplace's equation for the classical gravitational potential:

$$\sum_{i=1}^{3} \frac{\partial^2 \varphi}{\partial x^{i_2}} = \sum_{i=1}^{3} \varphi_{,i,i} = 0 . \tag{VI-43}$$

If Newton's second law of motion and the geodesic equation of motion are to yield approximately the same trajectories for slowly moving particles in a weak gravitational field, the g_{00} component of the metric tensor must be approximately given by

$$g_{00} = 1 + \frac{2\varphi}{c^2} . \tag{VI-44}$$

Hence,

$$\varphi = \frac{c^2}{2} (g_{00} - 1) , \tag{VI-45}$$

and Laplace's equation becomes

$$\sum_{i=1}^{3} g_{00,i,i} = 0 . \tag{VI-46}$$

This equation must be an approximate form of the relativistic field equations and involves second derivatives of the metric tensor with a summation over the repeated index i. In a covariant tensor equation, the analogue of such a summation is a contraction. This suggests a contraction of the Riemann tensor. A contraction between i and j or between k and ℓ yields a null tensor since $R_{ijk\ell}$ is antisymmetric in these pairs. Contraction between i and k, between i and ℓ, and between j and k differ only in sign. Hence, the only meaningful contraction on $R_{ijk\ell} = 0$ is

$$R^{i}_{\ ji\ell} = R_{j\ell} = 0 , \tag{VI-47}$$

where $R_{j\ell}$ is the contracted Riemann curvature tensor. This is the equation Einstein adopted to describe the gravitational field in free space.

The contracted Riemann curvature tensor is symmetric:

$$R_{j\ell} = R_{\ell j} . \tag{VI-48}$$

Hence, the contracted Riemann curvature tensor has 10 independent components.

In their expanded form, the free-space gravitational equations are

$$R_{j\ell} = \left\{ {i \atop j\ i} \right\}_{,\ell} - \left\{ {i \atop j\ \ell} \right\}_{,i} + \left\{ {i \atop m\ \ell} \right\} \left\{ {m \atop j\ i} \right\} - \left\{ {i \atop m\ i} \right\} \left\{ {m \atop j\ \ell} \right\} = 0 . \tag{VI-49}$$

If the Riemann curvature tensor $R^{i}_{\ .k\ell m}$ is contracted, a tensor of rank 2 is obtained satisfying the conditions:

$$R_{ik} = R^m_{.imk} = -R^m_{.ikm} = -R^m_{i.mk} = R^m_{mk.i} = R^m_{.kmi} \quad . \qquad (VI-50)$$

These will be consequences of the symmetry relation in equation (VI-24) and equation (VI-23). The symmetry relation in equation (VI-47) will be seen repeated in equation (VI-50).

By a further contraction the curvature scalar R is obtained:

$$R = R^i_i = g^{ik} R_{ik} \quad . \qquad (VI-51)$$

If the Bianchi identity in equation (VI-27) is contracted relative to the indices i and k, the following equations are obtained:

$$R_{j\ell;m} + R^i_{.j\ell m;i} - R_{jm;\ell} = 0 \, ,$$

or

$$R^j_{\ell;m} - R^{jk}_{..\ell m;k} - R^j_{m;\ell} = 0 \, . \qquad (VI-52)$$

Further contraction relative to j and ℓ yields the result

$$R_{;m} - 2 R^k_{m;k} = 0 \, , \qquad (VI-53)$$

or

$$\left(R^{ij} - \frac{1}{2} g^{ij} R \right)_{;k} = 0 \, , \qquad (VI-54)$$

i.e., the covariant divergence of the tensor

$$R^{ij} - \frac{1}{2} g^{ij} R \qquad (VI-55)$$

is zero. Because of the symmetry properties this tensor has only ten independent components.

If R = 0, then

$$G^{ij} = R^{ij} - \frac{1}{2} g^{ij} R = 0 \, . \qquad (VI-56)$$

This is the Ricci tensor. Hence if the Riemann curvature tensor satisfies the free-space equations $R^{ij} = 0$, then the Ricci tensor is zero. Conversely, if $G^{ij} = 0$, then

$$G^i_j = R^i_j - \frac{1}{2} \delta^i_j R = 0 . \tag{VI-57}$$

The contraction of this equation yields

$$G^i_i = R - \frac{1}{2} \delta^i_i R = R - 2R = 0 . \tag{VI-58}$$

Thus,

$$R^i_j = G^i_j + \frac{1}{2} \delta^i_j R = 0 . \tag{VI-59}$$

Therefore, G^i_j is zero if and only if $R^i_j = 0$. Hence, the Einstein field equation for free space may be written in terms of the zero-divergence Ricci tensor:

$$G^i_j = R^i_j - \frac{1}{2} \delta^i_j R = 0 . \tag{VI-60}$$

This form of the field equations for free space is extremely useful in discussing conservation laws in relativistic physics.

VII. THE CONSEQUENTIAL SPACE-TIME METRICS FROM THE GRAVITATIONAL FIELD EQUATIONS

The motion of a particle or a wave in a force field characterized by a space-time metric manifests itself in a change in the expression for ds and the g_{ik} in terms of the dx^i. Particles or waves move along an extremal or a geodesic line in four-space x^0, x^1, x^2, x^3. The geodesic equation of motion is

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{k\ell} \left(\frac{dx^k}{ds}\right)\left(\frac{dx^\ell}{ds}\right) = 0 , \tag{VII-1}$$

where by (V-74)

$$\Gamma^i_{k\ell} = \frac{1}{2} g^{im} \left(\frac{\partial g_{km}}{\partial x^\ell} + \frac{\partial g_{\ell m}}{\partial x^k} - \frac{\partial g_{k\ell}}{\partial x^m} \right) , \qquad \text{(VII-2)}$$

and the g^{im} (so-called contravariant components of the metric tensor) are calculated from the relationship

$$g^{im} g_{mn} = \delta^i_n , \quad \delta^i_n = 0 \text{ for } i \neq n,$$

$$\delta^i_n = 1 \text{ for } i = 0. \qquad \text{(VII-3)}$$

The motion of a particle in a force field is determined by the quantities $\Gamma^i_{k\ell}$. The derivative $(d^2 x^i / ds^2)$ is the four-acceleration of the particle. If m is the particle mass, the quantity may then be called $- m \Gamma^i_{k\ell} (dx^k/ds) (dx^\ell/ds)$, the "four-force" acting on the particle in the force field. Here the tensor g_{ik} plays the role of the "potential" of the force field; its derivatives determine the field "intensity" $\Gamma^i_{k\ell}$.

Equation (VII-1) is not applicable to the propagation of an electromagnetic signal, since along the world line of the propagation of an electromagnetic signal the interval ds is zero, so that all the terms in equation (VII-1) become infinite. However, from the theory of geodesics, there can be introduced a four-dimensional wave vector in the form

$$k^i = \frac{dx^i}{d\lambda} , \qquad \text{(VII-4)}$$

where λ is a parameter that varies along the ray path. In a force field the geodesic equation for the path of an electromagnetic wave is

$$\frac{dk^i}{d\lambda} + \Gamma^i_{k\ell} k^k k^\ell = 0 . \qquad \text{(VII-5)}$$

These equations also determine the parameter λ.

To solve the geodesic equation of motion of a particle or an electromagnetic propagation, it is necessary to know the components of the metric tensor for the space-time under investigation and observation. Quite often one can derive the desired knowledge of the particular physical situation from components of the metric tensor without solving the geodesic equation of motion.

The systems of equations VI-47 and VI-60 connect the time derivatives of the components g_{ij} of the metric tensor with the components and their space derivatives. Mathematically, those equations lead to the problem: given the metric tensor g_{ij} and all of its first derivatives at a given moment x^0 in the entire three-dimensional space of the remaining three variables x^α, to compute its value for all future time. This problem will be recognized as a typical Cauchy problem in partial differential equations. Without loss of generality, the three-dimensional space may be taken as a three-dimensional hypersurface S oriented in space and described by the equation $x^0 = 0$. The normal to this hypersurface is oriented in time so that $g_{00} > 0$. The components of the metric tensor g_{ij} and their first derivatives are prescribed in S. It is to be noticed that the prescription of the g_{ij} in S allows the computation of the first derivatives $g_{ij,\,k}$, i.e., all first derivatives not involving differentiation relative to the time. Hence, it is sufficient to prescribe the following initial values on S:

$$g_{ij}\ ,\ g_{ij,k} \tag{VII-6}$$

i.e., metric potentials and their normal derivatives. For further discussion of the mathematical structure of the Einstein field equations for free space (VI-47 and VI-60) the reader is referred to Adler, Bazin, and Schiffer, 1965 (Chap. 7, p. 210).

The gravitational field equations VI-47 and VI-60 are nonlinear and extremely difficult to solve. However, in certain special cases, symmetry conditions greatly simplify the field equations. An extremely important case is the time-independent and spherically symmetric line element. Schwarzschild (1915) solved the associated field equations in 1916 and found the empty space static solution for a mass to have the following expressions for the components of the metric tensor:

$$g_{00} = 1 - \frac{2\,GM}{c^2\,r} \quad , \tag{VII-7}$$

$$g_{0\alpha} = 0 \,, \tag{VII-8}$$

$$g_{\alpha\beta} = -\,\delta^{\alpha}_{\beta} - \frac{2\,GM}{c^2 r^3 g_{00}}\; x^{\alpha}\, x^{\beta} \,, \tag{VII-9}$$

where M is the mass of the spherically symmetric mass distribution and r is the magnitude of the position vector locating the point mass.

Schwarzschild's solution is significant because it is the only solution of the field equations in empty space which is static, which has spherical symmetry, and which goes over into the flat metric at infinity. Therefore, if one considers a concentration of matter of finite dimensions which is spherically symmetric, one knows that the gravitational field outside the region filled with matter must be Schwarzschild's field.

For convenience, define

$$K = -\,\frac{2\,GM}{c^2\,r} \tag{VII-10}$$

and write equations (VII-7), (VII-8), and (VII-9) as

$$g_{00} = 1 + K \,, \tag{VII-11}$$

$$g_{0\alpha} = 0 \,, \tag{VII-12}$$

$$g_{\alpha\beta} = -\,\delta^{\alpha}_{\beta} + \left(\frac{K}{1+K}\right)\left(\frac{x^{\alpha}\, x^{\beta}}{r^2}\right) . \tag{VII-13}$$

Lense and Thirring (1918) made a significant extension to the static solution of Schwarzschild. Now imagine that the centrally symmetric mass distribution is rotating uniformly. Then, associated with the central body are components of the angular momentum per unit mass which will be designated $L_{\alpha\beta}$. Lense and Thirring found, by writing the appropriate values for the energy-momentum tensor for a uniformly rotating spherical mass distribution and making linear approximations, that the $g_{0\alpha}$ are not zero under these conditions but

$$g_{0\alpha} = - \frac{K}{cr^2} L_{\alpha\beta} x^\beta . \qquad \text{(VII-14)}$$

Equation (VII-14) is valid when one considers the gravitational field to be weak at all distances. Also, without further consideration, the solutions for the components of g_{ik} can be considered to be valid only outside of the radius, r_0, of the centrally symmetric mass distribution.

VIII. COMBINED EFFECTS OF UNIFORM ROTATION AND THE GRAVITATIONAL FIELD ON PROPER TIME

Although, in general, gravitational fields cannot be superimposed upon one another, there is no reason why force fields cannot be superimposed as long as there is no physical evidence to suggest that they are in some manner coupled together.

One is motivated at this point to examine the combined effects of a uniformly rotating coordinate system and a gravitational field, because of an intense interest in the scientific community to measure the theoretically predicted frequency difference between identical atomic oscillators — one placed at a point on the earth's surface near the equatorial plane and the other positioned in a synchronous orbit above the earth-based atomic oscillator.

An earth-based atomic oscillator is by definition at rest in a coordinate system that rotates uniformly in the equatorial plane relative to a coordinate system originating at the center of mass of the earth and fixed in space relative to the so-called "fixed" stars. An atomic oscillator in synchronous earth orbit positioned above the earth-based oscillator will be characterized by an inclination angle to the equatorial plane and an orbit eccentricity that in general will cause it to move in a figure eight relative to the earth-based atomic oscillator.

To compare the theoretical frequency difference of the two identical atomic oscillators described above, it is necessary to transmit electromagnetic signals between the two atomic oscillators. To a first approximation, one can use the Schwarzschild solution for free space superimposed upon a uniformly rotating coordinate system to calculate the proper time of the atomic oscillators in the uniformly rotating coordinate system. This, of course, considers the earth to be perfectly spherically symmetric and ignores the presence of other masses in our solar system. The perturbing effects of these bodies will be discussed later. Terms in the resulting metric of order greater than $(K)^{3/2}$ will be considered negligible.

Consider a right-handed xyz coordinate system originating at the center of mass of the earth, the positive z axis passing through the North Pole and the xy plane coinciding with the equatorial plane of the earth. Let the x and y axes be orthogonal to one another, at rest relative to a fixed point on the equator, and therefore rotating uniformly relative to the "fixed" stars. The spin of the earth is by definition nonexistent in such a coordinate system, and, therefore there is no contribution to the $g_{0\alpha}$ components of the metric tensor from the spin of the earth.

If one linearly superimpose the results of equations (VII-11), (VII-12), and (VII-13) for the Schwarzschild solution to the interval equation (IV-5) for uniform rotation (dropping the prime notation) one obtains to an order $(K)^{3/2}$ the result

$$(ds)^2 = \left(1 - \frac{\omega^2}{c^2} X^2 + K\right) (cdt)^2 - 2 \frac{\omega}{c} (\bar{k} \cdot \bar{r} \times d\bar{r}) (cdt) - (d\bar{r} \cdot d\bar{r}) + K \left(\frac{\bar{r} \cdot d\bar{r}}{r}\right)^2 , \qquad \text{(VIII-1)}$$

where $\bar{k}$ is a unit vector along the z axis and

$$\bar{r} = \bar{i}\, x + \bar{j}\, y + \bar{k}\, z , \qquad \text{(VIII-2)}$$

$$d\bar{r} = \bar{i}\, (dx) + \bar{j}\, (dy) + \bar{k}\, (dz) , \qquad \text{(VIII-3)}$$

with $\bar{i}$ and $\bar{j}$ being unit vectors along the x and y axes, respectively. Also,

$$X^2 = x^2 + y^2, \qquad \text{(VIII-4)}$$

$$r = (x^2 + y^2 + z^2)^{1/2} \tag{VIII-5}$$

The identification $dx^0 = cdt$, $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dz$ was made from the metric written in the form

$$ds^2 = g_{ik}\, dx^i\, dx^k . \tag{VIII-6}$$

The components of the metric tensor are

$$g_{00} = \left(1 - \frac{\omega^2}{c^2} X^2 + K\right), \tag{VIII-7}$$

$$g_{01} = g_{10} = \frac{y\omega}{c} , \tag{VIII-8}$$

$$g_{02} = g_{20} = - \frac{x\omega}{c} , \tag{VIII-9}$$

$$g_{03} = g_{30} = 0 , \tag{VIII-10}$$

$$g_{11} = -1 + \frac{Kx^2}{r^2} , \tag{VIII-11}$$

$$g_{22} = -1 + \frac{Ky^2}{r^2} , \tag{VIII-12}$$

$$g_{33} = -1 + \frac{Kz^2}{r^2} \tag{VIII-13}$$

$$g_{12} = g_{21} = \frac{Kxy}{r^2} , \tag{VIII-14}$$

$$g_{13} = g_{31} = \frac{Kxz}{r^2} , \tag{VIII-15}$$

$$g_{32} = g_{23} = \frac{Kyz}{r^2} . \tag{VIII-16}$$

Let us at this time introduce the velocity

$$v = \frac{d\ell}{d\tau} = \frac{c d\ell}{\sqrt{g_{00}}\, dx^0} , \qquad \text{(VIII-17)}$$

measured in terms of the proper time, that is, by an observer located at the given point. Equations (IV-17) and (IV-18) must be used to calculate the spatial element $d\ell$. Using equation (IV-17), components of the spatial metric tensor are obtained:

$$\gamma_{11} = 1 - K \frac{x^2}{r^2} + \frac{\omega^2 y^2}{c^2} , \qquad \text{(VIII-18)}$$

$$\gamma_{12} = - K \frac{xy}{r^2} - \frac{\omega^2}{c^2} xy , \qquad \text{(VIII-19)}$$

$$\gamma_{13} = - K \frac{xz}{r^2} , \qquad \text{(VIII-20)}$$

$$\gamma_{23} = - K \frac{yz}{r^2} , \qquad \text{(VIII-21)}$$

$$\gamma_{22} = 1 - K \frac{y^2}{r^2} + \frac{\omega^2 x^2}{c^2} , \qquad \text{(VIII-22)}$$

$$\gamma_{33} = 1 - K \frac{z^2}{r^2} . \qquad \text{(VIII-23)}$$

Substituting the above relationships into equation (IV-18) yields

$$d\ell^2 = (d\bar{r} \cdot d\bar{r}) - K \left(\frac{\bar{r} \cdot d\bar{r}}{r} \right)^2 + \frac{\omega^2}{c^2} (\bar{k} \cdot \bar{r}\, x d\, \bar{r})^2 . \qquad \text{(VIII-24)}$$

Using the results of equations (VIII-7) and (VIII-24) by placing them in equation (VIII-17) results in

$$v = \frac{\left[(d\bar{r} \cdot d\bar{r}) - K \left(\frac{\bar{r} \cdot d\bar{r}}{r} \right)^2 + \frac{\omega^2}{c^2} (\bar{k} \cdot \bar{r}\, x d\, \bar{r})^2 \right]^{1/2}}{\left[1 - \frac{\omega^2}{c^2} X^2 + K \right]^{1/2} dt} \qquad \text{(VIII-25)}$$

Upon squaring equation (VIII-25) it is written as

$$-(d\bar{r} \cdot d\bar{r}) + K\left(\frac{\bar{r} \cdot d\bar{r}}{r}\right)^2 = \left[1 - \frac{\omega^2}{c^2} X^2 + K\right] v^2 dt^2 + \frac{\omega^2}{c^2} (k \cdot \bar{r} \, x d \, \bar{r})^2 . \qquad \text{(VIII-26)}$$

We now substitute equation (VIII-26) into equation (VIII-1):

$$ds^2 = \left(1 - \frac{\omega^2}{c^2} X^2 + K\right)\left(1 - \frac{v^2}{c^2}\right)(cdt)^2 - 2\frac{\omega}{c^2} (\bar{k} \cdot \bar{r} \times \dot{\bar{r}}) + \frac{\omega^2}{c^4} (\bar{k} \cdot \bar{r} \times \dot{\bar{r}})^2 (cdt)^2 , \qquad \text{(VIII-27)}$$

where $\dot{\bar{r}} = (d\bar{r}/dt)$. The last term can be ignored for it is of the order $(K)^2$. Terms of an order greater than $(K)^{3/2}$ can be neglected. Hence,

$$\left(1 - \frac{\omega^2}{c^2} X^2 + K\right)\left[2 \frac{\omega}{c^2} (\bar{k} \cdot \bar{r} \times \dot{\bar{r}})\right] = 2 \frac{\omega}{c^2} (\bar{k} \cdot \bar{r} \times \dot{\bar{r}}) , \qquad \text{(VIII-28)}$$

note that the order of accuracy of the calculations is not lost by replacing

$$\dot{\bar{r}} = \bar{v} . \qquad \text{(VIII-29)}$$

The assumptions in equations (VIII-28) and (VIII-29) then enables equation (VIII-27) to be written as

$$\left(\frac{ds}{c}\right) = \left(1 - \frac{\omega^2}{c^2} X^2 + K\right)^{1/2} \left[1 - 2\frac{\omega}{c^2} \bar{k} \cdot \bar{r} \times \bar{v} - \frac{v^2}{c^2}\right]^{1/2} dt. \qquad \text{(VIII-30)}$$

The result, equation (VIII-30), may be considered as a more generalized value of the proper time of a point, whether it is at rest or moving in the coordinate system in which the physical action is being described. The result, equation (VIII-30), has the property of reducing to equation (IV-9), namely

$$d\tau = \frac{1}{c} \sqrt{g_{00}} \, dx^0 ,$$

for cases where the observations are made at (1) a point at rest in the reference coordinate system or (2) by a co-moving clock. In both cases, the relative velocity, $\bar{v}$, is zero and the generalized proper time for this system reduces to the value one gets by substitution for g_{00} and dx^0.

In the next section, a method of measuring the frequency difference of the two indentical atomic oscillators will be described.

IX. MEASUREMENT OF THE FREQUENCY DIFFERENCE

As mentioned in the last section, there is much interest in the scientific community to measure the frequency difference between two identical atomic oscillators — one based rigidly on the earth near the equatorial plane and the other one in synchronous orbit above the earth-based atomic oscillator. To perform this measurement, it is necessary to describe the position and motion of the immediate domains of the atomic oscillators. With the exception of the perturbations involved, equation (VII-30) in section VII does this approximately. The immediate problem is how to conveniently transfer the information from one domain to the other domain without seriously disturbing the physical processes involved. Evidently, the way to proceed is through the exchange of electromagnetic signals. The most serious drawback to the use of electromagnetic signals is the Doppler effect. An attempt will be made to devise a physical scheme to eliminate at least the first order Doppler effect.

First consider equation (VIII-25) written in the form

$$vdt \mp \left[\left(1 + \frac{\omega}{c^2} X^2 - K\right)(\vec{dr} \cdot \vec{dr}) - K\left(\frac{\vec{r} \cdot \vec{dr}}{r}\right) + \frac{\omega^2}{c^2}(\vec{k} \cdot \vec{r} x \vec{dr})^2\right]^{1/2} \tag{IX-1}$$

to the order $(K)^{3/2}$. Then define

$$dS = \left[\left(1 + \frac{\omega^2}{c^2} X^2 - K\right)(\vec{dr} \cdot \vec{dr}) - K\left(\frac{\vec{r} \cdot \vec{dr}}{r}\right)^2 + \frac{\omega^2}{c^2}(\vec{k} \cdot \vec{r} x \vec{dr})^2\right]^{1/2} \tag{IX-2}$$

An electromagnetic wave travels with the speed of light, c; so, substituting c for v and using equations (IX-1) and (IX-2), one writes for the geodesic motion of an electromagnetic wave for this system of reference

$$dt = \frac{1}{c} dS . \tag{IX-3}$$

For the emission and reception of an electromagnetic wave from $\vec{r}_0$ to $\vec{r}$,

$$t - t_0 = \frac{1}{c} \int_{\vec{r}_0}^{\vec{r}} dS \ , \tag{IX-4}$$

where t_0 corresponds to the time of emission of the electromagnetic wave from a point, $\vec{r}_0$, at rest in our coordinate system and t corresponds to the reception of the electromagnetic wave at $\vec{r}$. Suppose $\vec{r}$ changes to $\vec{r} + \vec{dr}$ while t changes to t + dt and t_0 changes to $t_0 + dt_0$. $\vec{r}_0$ remains fixed. Then rewrite (IX-4) as

$$(t + dt) - (t_0 + dt_0) = \frac{1}{c} \int_{\vec{r}_0}^{\vec{r} + \vec{dr}} dS = \frac{1}{c} \int_{\vec{r}_0}^{\vec{r}} dS + \frac{1}{c} \int_{\vec{r}}^{\vec{r} + \vec{dr}} dS \tag{IX-5}$$

Subtracting equation (IX-4) from (IX-5) yields

$$dt - dt_0 = \frac{1}{c} \int_{\vec{r}}^{\vec{r} + \vec{dr}} dS = \frac{v \cos \theta}{c} \, dt \ , \tag{IX-6}$$

or

$$dt_0 = \left(1 - \frac{v \cos \theta}{c}\right) dt, \text{ where } v = \left| \frac{d\bar{r}}{dt} \right| , \tag{IX-7}$$

and θ is the angle between the direction of the electromagnetic ray and the infinitesimal element $d\bar{r}$.*

The frequency of an atomic oscillator can be characterized by a certian number of events, say dN, per unit proper time. For the frequency of an atomic oscillator located at $\vec{r}_0$ one uses equation (VIII-30) for the generalized proper time to define

$$\nu = c \left(\frac{dN}{ds}\right)_0 \ . \tag{IX-8}$$

Suppose an observer, moving with speed v between $\vec{r}$ and $\vec{r} + \vec{dr}$, receives the wave. The frequency received by the observer will be

$$\nu + \Delta \nu = c \left(\frac{dN}{ds}\right)_v \tag{IX-9}$$

---

* This result is to an accuracy of slightly less than 1 part in 10^{14}.

Since the number of events, dN, is an invariant, the comparison is made by taking the ratio of equations (IX-9) to (IX-8) to get

$$1 + \frac{\Delta\nu}{\nu} = \frac{\left(\frac{ds}{c}\right)_0}{\left(\frac{ds}{c}\right)_v} = \frac{\left(1 - \frac{\omega^2}{c^2} X_0^2 + K_0\right)^{1/2} dt_0}{\left(1 - \frac{\omega^2}{c^2} X^2 + K\right)^{1/2} \left(1 - 2\frac{\omega}{c^2} \vec{k} \cdot \vec{r} x \overleftrightarrow{v} - \frac{v^2}{c^2}\right)^{1/2} dt} . \tag{IX-10}$$

Upon substituting equation (IX-7) into equation (IX-10), one obtains

$$1 + \frac{\Delta\nu}{\nu} = \frac{\left(1 - \frac{\omega^2}{c^2} X_0^2 + K_0\right)^{1/2} \left(1 - \frac{v}{c}\right)}{\left(1 - \frac{\omega^2}{c^2} X^2 + K\right)^{1/2} \left(1 - 2\frac{\omega}{c^2} \vec{k} \cdot \vec{r} x \vec{v} - \frac{v^2}{c^2}\right)^{1/2}} . \tag{IX-11}$$

After all expansions this yields (to orders of magnitude of $\left(\frac{v^3}{c^3}\right)$)

$$1 + \frac{\Delta\nu}{\nu} = \left[1 + \frac{1}{2}\frac{\omega^2}{c^2}(X^2 - X_0^2) + \frac{(K_0 - K)}{2} - \frac{v\cos\theta}{c} + \frac{1}{2}\frac{v^2}{c^2}\right.$$
$$+ \frac{\omega}{c^2}\vec{k}\cdot\vec{r}x\vec{v} - \frac{v\cos\theta}{2c}\left(\frac{\omega^2}{c^2}\right)(X^2 - X_0^2) - \frac{v\cos\theta}{2c}(K_0 - K)$$
$$\left. - \frac{(v\cos\theta)}{c^3}\omega\vec{k}\cdot\vec{r}x\vec{v} - \frac{1}{2}\frac{v^3}{c^3}\cos\theta\right] . \tag{IX-12}$$

Equation (IX-12) was developed for the reception of a wave, which was emitted by a source at rest in our rotating coordinate system, by a moving source. For convenience one defines

$$\Delta = \frac{(K_0 - K)}{2} + \frac{1}{2}\frac{\omega^2}{c^2}(X^2 - X_0^2) + \frac{\omega}{c^2}\vec{k}\cdot\vec{r}x\vec{v} + \frac{1}{2}\frac{v^2}{c^2} , \tag{IX-13}$$

$$\mu = \frac{v\cos\theta}{c}(\Delta) , \tag{IX-14}$$

$$\epsilon = -\frac{v\cos\theta}{c} \tag{IX-15}$$

and writes equation (IX-12) as

$$1+\frac{\Delta\nu}{\nu} = (1+\epsilon+\Delta-\mu) \ . \tag{IX-16}$$

One should now look for an expression for the wave emitted by a moving source and received by a fixed source in this coordinate system. The variables for this particular happening will be denoted by primes. Suppose the emission of an electromagnetic wave takes place at the position $\vec{r}\,'$ and is received at the fixed position $\vec{r}_0\,'$.

Then one writes analogous to equation (IX-4) the expression

$$t_0' - t' = -\frac{1}{c}\int_{\vec{r}_0\,'}^{\vec{r}_0\,'} dS' \ . \tag{IX-17}$$

The minus sign on the right-hand side of equation (IX-17) is for path reversal of the electromagnetic wave. Again one supposes that $\vec{r}\,' \rightarrow \vec{r}\,' + d\vec{r}\,'$ while $t_0' \rightarrow t_0' + dt_0'$ and $t' \rightarrow t' + dt'$; then

$$(t_0' + dt_0') - (t' + dt') = -\frac{1}{c}\int_{\vec{r}+d\vec{r}\,'}^{\vec{r}_0\,'} dS' = \frac{1}{c}\int_{\vec{r}_0\,'}^{\vec{r}\,'+d\vec{r}\,'} dS' \ . \tag{IX-18}$$

Subtracting equation (IX-17) from equation (IX-18) there remains

$$dt_0' - dt' = \frac{1}{c}\int_{\vec{r}\,'}^{\vec{r}\,'+d\vec{r}\,'} dS' = \frac{v'}{c}\cos\theta'\, dt' \ , \tag{IX-19}$$

or

$$dt_0' = \left(1+\frac{v'}{c}\cos\theta'\right) dt' \tag{IX-20}$$

where again θ' is the angle between the directions of the electromagnetic ray and the infinitesimal element $d\vec{r}\,'$.

The frequency of the emitted wave of the moving source will be defined similarly to equation (IX-8) to be

$$\nu' = c\left(\frac{dN}{ds'}\right)_{v'} , \qquad \text{(IX-21)}$$

and the frequency received or observed by the observer at rest will be defined similarly to equation (IX-9) to be

$$\nu' + \Delta\nu' = c\left(\frac{dN}{ds'}\right)_0 . \qquad \text{(IX-22)}$$

Again the two frequencies are compared by taking the ratio of equation (IX-22) to

$$1 + \frac{\Delta\nu'}{\nu'} = \frac{\left(1 - \frac{\omega^2}{c^2} X'^2 + K'\right)^{1/2}\left(1 - 2\frac{\omega}{c^2}\vec{k}\cdot\vec{r}'x\vec{v}' - \frac{v'^2}{c^2}\right)^{1/2} dt'}{\left(1 - \frac{\omega^2}{c^2} X_0'^2 + K_0'\right)^{1/2} dt_0'} . \qquad \text{(IX-23)}$$

The result, equation (IX-20), is used to express equation (IX-23) as

$$1 + \frac{\Delta\nu'}{\nu'} = \frac{\left(1 - \frac{\omega^2}{c^2} X'^2 + K'\right)^{1/2}\left(1 - 2\frac{\omega}{c^2}\vec{k}\cdot\vec{r}'x\vec{v}' - \frac{v'^2}{c^2}\right)^{1/2}}{\left(1 - \frac{\omega^2}{c^2} X_0'^2 + K_0'\right)^{1/2}\left(1 + \frac{v'}{c}\cos\theta'\right)} . \qquad \text{(IX-24)}$$

Expansion of equation (IX-24) yields

$$1 + \frac{\Delta\nu'}{\nu'} = \left(1 + \epsilon' - \Delta' + \mu' + \frac{v'^2}{c^2}\cos^2\theta' - \frac{v'^3}{c^3}\cos^3\theta'\right) , \qquad \text{(IX-25)}$$

where Δ , μ , and ϵ are defined by equations (IX-13), (IX-14), and (IX-15).

Figure 7 depicts the electromagnetic signal being sent from the point of rest in the uniformly rotating coordinate system to a moving point at $\vec{r}$. Imagine that the signal received by the moving point is reprocessed to be rebroadcast back to the point at rest. Suppose also there is a signal to be broadcast simultaneously from the moving source whose phase is to be compared with the reprocessed and rebroadcasted signal upon being received at the ground. We assume that the rebroadcasted signal is not changed in any way whatsoever while being reprocessed from the value it had upon being received by the moving source. However, it is assumed that while the received signal at the moving source is being reprocessed to be rebroadcast that the moving source might traverse a

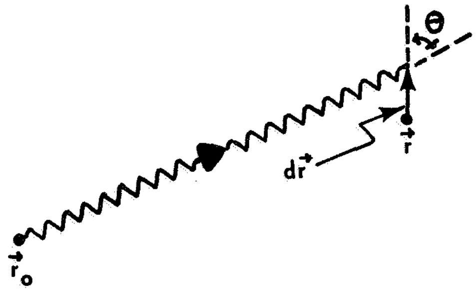

FIGURE 7. SIGNAL PASSAGE UP

distance $\delta \vec{r}$. Then after these two signals have been emitted simultaneously from the moving source and received by the point at rest, the moving source will in general have traveled a distance $d\vec{r}'$ (Figure 8).

In the case of the signal that is rebroadcast, equation (IX-16) is used to write for ν'

$$\nu' = \nu + \Delta\nu = \nu(1+\epsilon+\Delta+\mu) \ . \tag{IX-26}$$

Then from equation (IX-25) one writes

$$\nu' + \Delta\nu' = \nu'\left(1+\epsilon' - \Delta' + \mu' + \frac{v'^2}{c^2}\cos^2\theta' - \frac{v'^3}{c^3}\cos^3\theta'\right) \tag{IX-27}$$

and substituting equation (IX-26) into equation (IX-27):

$$\nu' + \Delta\nu' = \nu(1+\epsilon+\Delta-\mu)\left(1+\epsilon' - \Delta' + \mu' + \frac{v'^2}{c^2}\cos^2\theta' - \frac{v'^3}{c^3}\cos^3\theta'\right) \ . \tag{IX-28}$$

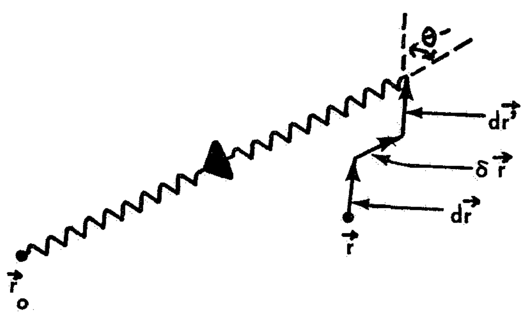

FIGURE 8. SIGNAL PASSAGE DOWN

Let us now assume that the reprocess time is negligibly zero such that $\delta\vec{r} = 0$. Further, assume that $d\vec{r} = d\vec{r}'$ to within the error limit of this calculation.* Upon dropping the prime notation,

$$\nu + \Delta\nu = \nu(1+\epsilon+\Delta-\mu)\left(1+\epsilon-\Delta+\mu + \frac{v^2}{c^2}\cos^2\theta - \frac{v^3}{c^3}\cos^3\theta\right) \tag{IX-29}$$

One may write equation (IX-29) as

$$\nu + \Delta\nu = \nu\left[(1+\epsilon)^2 - (\Delta-\mu)^2 + (1+\epsilon)\left(\frac{v^2}{c^2}\cos^2\theta - \frac{v^3}{c^3}\cos^3\theta\right)\right] \ . \tag{IX-30}$$

---

* It is to be noted that effects of aberration and retardation have not been treated here. They will be treated in a later paper.

Now $(\Delta + \mu)^2$ is several orders of magnitude below the accuracy requirements concerned here. Hence, it is dropped and one writes for equation (IX-30)

$$\nu + \Delta\nu = \nu\left[1 + 2\epsilon + \epsilon^2 + (1+\epsilon)\left(\frac{v^2}{c^2}\cos^2\theta - \frac{v^3}{c^3}\cos^3\theta\right)\right] \quad . \quad \text{(IX-31)}$$

To the required order of accuracy:

$$2\epsilon = -2\frac{v}{c}\cos\theta\ , \quad \epsilon^2 = \frac{v^2}{c^2}\cos^2\theta\ , \qquad \text{(IX-32)}$$

$$(1+\epsilon)\left(\frac{v^2}{c^2}\cos^2\theta - \frac{v^3}{c^3}\cos^3\theta\right) = \frac{v^2}{c^2}\cos^2\theta - 2\frac{v^3}{c^3}\cos^3\theta \quad . \quad \text{(IX-33)}$$

So equation (IX-31) becomes

$$\nu + \Delta\nu = \nu\left(1 - 2\frac{v}{c}\cos\theta + 2\frac{v^2}{c^2}\cos^2\theta - 2\frac{v^3}{c^3}\cos^3\theta\right) \quad . \quad \text{(IX-34)}$$

For the signal that was broadcasted simultaneously with the reprocessed and rebroadcasted signal, equation (IX-25) is used, dropping the primes and denoting $\Delta\nu'$ by $\delta\nu$ (to distinguish it from $\Delta\nu$ in equation (IX-34) to write

$$\nu + \delta\nu = \nu\left[1 - (\Delta - \mu) - \frac{v}{c}\cos\theta + \frac{v^2}{c^2}\cos^2\theta - \frac{v^3}{c^3}\cos^3\theta\right] . \quad \text{(IX-35)}$$

Now, put the equation (IX-34) through a phase divider such that its phase is divided by two. Hence, one writes

$$\nu + \frac{\Delta\nu}{2} = \nu\left(1 - \frac{v}{c}\cos\theta + \frac{v^2}{c^2}\cos^2\theta - \frac{v^3}{c^3}\cos^3\theta\right) . \qquad \text{(IX-36)}$$

Comparing equation (IX-36) with equation (IX-35),

$$(D\nu)_0 = \nu + \delta\nu - \left(\nu + \frac{\Delta\nu}{2}\right) = \delta\nu - \frac{\Delta\nu}{2} \qquad , \qquad \text{(IX-37)}$$

or

$$(D\nu)_0 = -(\Delta - \mu)\nu \ . \qquad \text{(IX-38)}$$

Now use the definitions for Δ and μ, equations (IX-13) and (IX-14), respectively and write out the physical quantities:

$$
(D\nu)_0 = \left[\frac{K - K_0}{2} - \frac{1}{2}\frac{\omega^2}{c^2}(X^2 - X_0^2) - \frac{\omega}{c^2}\vec{k}\cdot\vec{r}\times\vec{v} - \frac{1}{2}\frac{v^2}{c^2} \right.
$$

$$
+ \frac{v}{2c}\cos\theta\left(\frac{\omega^2}{c^2}\right)(X^2 - X_0^2) + \frac{v\cos\theta\,\omega}{c^3}\vec{k}\cdot\vec{r}\times\vec{v} \qquad \nu
$$

$$
\left. - \frac{v}{2c}\cos\theta\,(K - K_0) + \frac{1}{2}\frac{v^3}{c^3}\cos\theta \quad . \right] \tag{IX-39}
$$

Recall that $K \equiv -\frac{2GM}{c^2 r}$. Then one defines

$$
(\Delta\nu)_G = \left(\frac{K - K_0}{2}\right)\nu \tag{IX-40}
$$

$$
(\Delta\nu)_v = \frac{\nu}{2c^2}\left[\omega^2(X^2 - X_0^2) + 2\,\omega\,\vec{k}\cdot\vec{r}\times\vec{v} + v^2\right] . \tag{IX-41}
$$

It is easily seen that in view of equations (IX-38) and (IX-39) one may write equation (IX-37) as

$$
(D\nu)_0 = (\Delta\nu)_G - (\Delta\nu)_v - \frac{v}{c}\cos\theta\left[(\Delta\nu)_G - (\Delta\nu)_v\right] . \tag{IX-42}
$$

The first term on the right involves an observed change in the atomic oscillator frequency caused only by a change in the gravitational potential. This term will be measured in a forthcoming space experiment. The second term involves shifts in the atomic oscillator frequency caused by the motion of the oscillators in a coordinate system maintaining a rigid orientation relative to the "fixed" stars. The third term is a possible third-order effect arising from a coupling between the first-order Doppler effect and the first two second-order terms mentioned above. Note that by using such a technique as described in this section the first-order Doppler effect has been theoretically eliminated. If at the moment one does not consider the possible errors that may arise from perturbations the terms in equation (IX-37) are the terms to third order in the frequency shift that involve only the pure, spherically symmetric, gravitational potential difference and the relative motion between the two atomic oscillators.

In general, the medium through which the electromagnetic signal is propagated is not homogeneous and uniform because of atmospheric variations, etc. It is possible that the electromagnetic path length is not the same for a signal going from the ground station to the satellite (the "up" signal) as it is for a signal going from the satellite to the ground station (the "down" signal). A possible way to determine the errors involved in this path length difference is to measure the frequency difference of the two oscillators at the satellite in synchronous orbit. The result should be just the negative of the equation (IX-40), i.e.,

$$(D\nu)_S = -(D\nu)_0 = -\left\{(\Delta\nu)_G - (\Delta\nu)_V - \frac{v}{c}\cos\theta\left[(\Delta\nu)_G - (\Delta\nu)_V\right]\right\}. \tag{IX-43}$$

Any difference in the absolute values of equations (IX-42) and (IX-43) would correspond to electromagnetic path length difference between the "up" and "down" signals. Possible electromagnetic path length differences could exist from consistent upward thermal air flow, which is common in the tropics.

A final word will now be said concerning errors arising from perturbations, although such errors will be treated more extensively in a later paper. Sources of error to be considered are:

1. $\delta r = 0$. This assumes that there is no time elapse between rebroadcast of a received signal, either at the ground oscillator or satellite oscillator. This approximation may be valid but needs to be investigated.

2. $d\vec{r} = d\vec{r}'$. This may be valid but certainly needs to be checked for the quarter of a second round trip of the "up" and "down" signals.

3. Changes in the rate of rotation of the earth. ω is constant until the fourth significant figure. The moon and other solar system bodies cause a slight change in its value.

4. Changes in K from the spherically symmetric distribution. This is thought to be a sensitive source of perturbations. The four chief contributing factors are the earth's oblateness, the moon, the sun, and drag.

5. Environmental factors such as changes in refractive index, magnetic and electric fields, solar flares, large air pockets, etc.

6. Equipment errors and limitations.

7. Motion of the earth, moon, satellite system around the sun.

These errors will be discussed in subsequent papers.

It should also be pointed out that the results (IX-7) and (IX-20) were not extended to third order. This too shall be done in a later paper.

BIBLIOGRAPHY

Adler, R., Bazin, M., Schiffer, M.: Introduction to General Relativity, McGraw-Hill Book Company, New York, 1965.

Bedessa, R. G., Kent, R. L., Nowell, J. C., and Searle, C. L.: A Doppler-Cancellation Technique for Determining the Altitude Dependence of Gravitational Red Shift in an Earth Satellite. Proc. IRE, Vol. 48, April 1960, pp. 758-764.

Bergmann, P. G.: Introduction to the Theory of Relativity. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1942.

Fock, V.: The Theory of Space, Time and Gravitation. Second Revised Edition. (Translated from the Russian by N. Kemmer). The Macmillan Company, New York, New York, 1964.

Landau, L. D., and Lifshitz, E. M.: The Classical Theory of Fields. Revised Second Edition. (Translated from the Russian by Morton Hamermesh). Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962.

Leighton, Robert B.: Principles of Modern Physics. McGraw-Hill Book Company, Inc., New York, New York, 1959.

McVittie, G. C.: General Relativity and Cosmology. The University of Illinois Press, Urbana, Illinois, 1965.

Møller, C.: The Theory of Relativity. First Edition. Oxford University Press, London and New York, 1952.

Newton, R. R.: Applications of Doppler Measurements to Problems in Relativity, Space Probe Tracking, and Geodesy. Proc. IRE, Vol. 48, April 1960, pp. 754-758.

Schwarzschild, K.: Das Gravitationsfeld eines Massenpunktes Nach der Einstein'schen Theorie. Sitzungsher, Berlin, 1916, p. 189.

Thirring, H. and Lense, J.: Uber den Einfluss der Eigenrotation der Zentralhörper auf die Bewegung der Planeten und Monde Nach der Einsteinschen Gravitationstheorie. Phys. Zs., Vol. 19, 1918, pp. 156-163.

Tolmann, R. C.: Relativity, Thermodynamics, and Cosmology. First Eddition. Oxford University Press, London and New York, 1934.

BIBLIOGRAPHY (Concluded)

Vessot, R.: Research Leading to the Development of an Atomic Hydrogen Maser for Space Vehicle Application. Monthly Progress Report Nos. 12 and 13, NASW-1337, Hewlett-Packard, Beverly, Massachusetts, February 1, 1967 through March, 1967.

APPROVAL

TM X-53802

RELATIVISTIC PHYSICS OF CLOCKS AND THE SYNCHRONOUS ORBIT CLOCK EXPERIMENT

By Nat Edmonson, Jr., and Fred D. Wills

The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

Henry E. Stern

HENRY E. STERN
Acting Chief, Nuclear & Plasma Physics Division

Gerhard B. Heller

GERHARD B. HELLER
Director, Space Sciences Laboratory

MSFC—RSA, Ala

DISTRIBUTION

NASA TM X-53802

INTERNAL

DIR
Dr. von Braun

AD-S
Dr. Stuhlinger

S&E-DIR
Mr. Weidner

S&E-R
Dr. Johnson
Mr. Chase

PD-DIR
Dr. Lucas

PD-MP-P
Mr. Dudley
Mr. Breazeale
Mr. Potter

PD-PS-T
Mr. Tidd

PD-SA-O
Mr. Stafford

S&E-ASTR-DIR
Mr. Moore
Mr. Horton

S&E-ASTR-B
Mr. Weber
Mr. Currie
Mr. White
Mr. Shearer

S&E-ASTR-I
Mr. Powell
Mr. Saunders

S&E-ASTR-IR
Mr. Harper

S&E-ASTR-S
Mr. Mack
Mr. Gregory

S&E-AERO-DIR
Dr. Geissler

S&E-AERO-T
Dr. Krause

S&E-AERO-Y
Mr. Smith

CSE-DIR
Dr. Haeussermann
Mr. Hoberg
Mr. Palaoro

CSE-ADD
Mr. Denton

S&E-QUAL-DIR
Mr. Grau

S&E-QUAL-PC
Mr. Absher

S&E-SSL-DIR
Mr. Heller
Mr. Hembree

DISTRIBUTION (Continued)

INTERNAL (Continued)

S&E-SSL-X
- Dr. Decher
- Dr. Weber
- Mr. Weathers
- Dr. Dozier

S&E-SSL-C
- Reserve (15)

S&E-SSL-N
- Mr. Stern
- Dr. Hale
- Dr. Costes
- Mr. Wells

S&E-SSL-NA
- Dr. Edmonson (5)
- Mr. Peasley
- Mr. Wills (5)
- Dr. Eby
- Dr. deLoach
- Mr. Derrickson

S&E-SSL-NP
- Dr. Dalins
- Mr. Urban
- Mr. Lacy
- Dr. Peters
- Mr. Stone
- Dr. Shukla
- Dr. Gregory
- Dr. Rehmann

S&E-SSL-NR
- Mr. Burrell
- Mr. Byrd
- Dr. Parnell
- Dr. Oran
- Dr. Guenther

S&E-SSL-P
- Mr. Naumann
- Mr. Chisholm

S&E-SSL-PA
- Mr. Holland
- Mr. Darbro
- Mr. Parker

S&E-SSL-PO
- Dr. Frary
- Dr. Edwards
- Mr. Williams
- Mr. Moore
- Mr. Tashbar
- Mr. Kurtz

S&E-SSL-PM
- Mr. Shriver
- Mr. Espy
- Mr. Klingman
- Mr. Clifton
- Mr. McGuire

S&E-SSL-S
- Dr. Sieber
- Mr. Katz
- Dr. Hudson
- Mr. Bensko

DISTRIBUTION (Continued)

INTERNAL (Continued)

S&E-SSL-SF
Mr. Jones
Mr. Parker

S&E-SSL-SP
Mr. Yarbrough
Mr. Robinson

S&E-SSL-T
Mr. Snoddy
Dr. Schocken

S&E-SSL-TE
Mr. Miller
Mr. Atkins
Mr. Duncan
Mr. Fields
Mr. J. Fountain
Mr. W. Fountain

S&E-SSL-TR
Mr. Arnett
Mr. Bannister
Dr. Kroes
Mr. Zwiener
Dr. Yernini
Dr. Kulshreshtha

S&E-SSL-T
Mr. Jones
Mr. Gates
Mr. Watkins
Dr. Hagyard
Mr. Calvert
Dr. Gary

DEP-T

PAT
Mr. L.D. Wofford, Jr.

I-RM-M

A&TS-MS-IP (2)

A&TS-MS-IL (8)

A&TS-MS-T (6)

A&TS-MS-H

EXTERNAL

National Aeronautics and Space Administration
Washington, D. C. 20546
ATTN: Mr. Ernest Ott, Code SG

Scientific & Technical Information Facility (25)
P. O. Box 33
College Park, Maryland 20740
Attn: NASA Representative (S-AK/RKT)

Dr. Norman Ramsey
Department of Physics
Harvard University
Cambridge, Massachusetts 02138

Dr. Daniel Kleppner
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

Dr. Robert Vessot
Hewlett-Packard
Beverly, Massachusetts 02114