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ESTIMATION OF VARIANCE BY A RECURSIVE EQUATION

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16. Abstract
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ESTIMATION OF VARIANCE BY A RECURSIVE EQUATION*

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SUMMARY

A recursive equation is presented for the purpose of estimation of the variance of a sequence of independent random numbers. The use of this recursive equation makes it possible to perform a running estimate of the variance as the samples are received sequentially. An analysis of the recursive equation is included to show that it gives an asymptotically unbiased estimate of the variance. The variance of the estimated variance is derived for the special case of random numbers with a Gaussian probability distribution.

INTRODUCTION

In an investigation of an adaptive binary detector, a need was discovered for a running estimate of the variance of a received sequence of random numbers. This estimate was required to be updated as each number was received. Therefore, the method of implementation of the estimation technique must be simple enough to keep the computation time to a minimum. It was also required that the quantity of received samples not be a factor in the estimation technique since the system may be required to operate on a very long sequence of random numbers in which the number of samples could approach infinity.

A search of the literature revealed no appropriate estimation techniques. J.C. Dale (ref. 1) discusses estimation of the variance by a sum of squares, but his method requires that the number of samples to be handled be known. Sliding "window" methods are also available, but the sliding "window" method requires that large numbers of previous samples be labeled and stored in a memory.

This report discusses a recursive equation that gives an asymptotically unbiased estimate (ref. 2) of the variance of a sequence of random numbers provided that the

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random numbers in the sequence are independent. The recursive equation is moderately simple to implement, has a potentially low computation time, requires a very small amount of data storage, and is not concerned with the number of samples to be processed.

An analysis of the method of estimation of the variance is included to show that the estimation is asymptotically unbiased. The accuracy of the estimation method is investigated by deriving the asymptotic value of the variance of the estimated variance for the special case of random numbers having a Gaussian probability distribution.

SYMBOLS

a	mean of function being sampled
A	factor used in estimation of mean
B	factor used in estimation of variance
C, C_2, M, P, Q	constants
$E\left\{ \right\}$	expectation, average, or mean of argument given within braces
$H(s)$	transfer function of system
i, j, k, n	dummy variables
m	general term for mean
s	complex frequency
s_k	partial sum
T	sampling period
t_c	time constant
v^2	general term for variance
$V\left\{ \right\}$	variance of argument within braces
x_k	kth input data sample

$\hat{x}_k$	kth estimate of mean
$\hat{X}(z)$	z-transform of $\hat{x}_k$
z	variable associated with z-transform, $z = e^{sT}$
α	estimation factor (from ref. 3)
σ^2	variance of function being sampled
$\hat{\sigma}_k^2$	kth estimate of variance
$p(x\|y)$	conditional probability of x with y given

ESTIMATION OF THE VARIANCE

A recursive equation can be used to estimate variance. The justification of the equation follows the discussion of the recursive equation. The equation used for estimation of the variance is

$$\hat{\sigma}_k^2 = B\hat{\sigma}_{k-1}^2 + \frac{1 - B}{C}\left(x_k - \hat{x}_k\right)^2 \tag{1}$$

where $0 < B < 1.0$. Equation (1) requires knowledge of the mean of the input data. If the mean is known, it is used in place of $\hat{x}_k$ in the equation. If the mean is not known, it must also be estimated. For the purposes of this investigation, it is assumed that the mean will also be estimated by a recursive equation. The equation used for $\hat{x}_k$ is

$$\hat{x}_k = A\hat{x}_{k-1} + (1 - A)x_k \tag{2}$$

where $0 < A \leqq 1.0$. Brown (ref. 3) has shown that equation (2) gives an asymptotically unbiased estimate of the mean of x_i. He has shown that the asymptotic variance of the estimated mean is

$$\lim_{k \to \infty} V\left\{\hat{x}_k\right\} = \frac{1 - A}{1 + A}\sigma^2 \tag{3}$$

where σ^2 is the actual variance of the input samples. An analysis of the recursive method of estimation of the mean is presented in appendix A.

Both equations (1) and (2) must have an initial guess, $\hat{\sigma}_o^2$ or $\hat{x}_o$, in order to begin operation. The equation then calculates the first estimate, $\hat{\sigma}_1^2$ or $\hat{x}_1$, from the initial guess and the value of the first input sample. The process is continued as more input samples are received. The constant C in equation (1) is required to remove the bias in this estimation technique.

If $\hat{x}_k$ is replaced by equation (2), a more usable form of equation (1) is obtained:

$$\hat{\sigma}_k{}^2 = B\hat{\sigma}_{k-1}{}^2 + \frac{1 - B}{C}\left(x_k - A\hat{x}_{k-1} - x_k + Ax_k\right)^2$$

$$= B\hat{\sigma}_{k-1}{}^2 + \frac{A^2(1 - B)}{C}\left(x_k - \hat{x}_{k-1}\right)^2 \tag{4}$$

This equation for $\hat{\sigma}_k{}^2$ leads to faster calculation than equation (1) since it employs $\hat{x}_{k-1}$ instead of $\hat{x}_k$. This calculation can be performed in parallel with the kth estimate of the mean instead of after $\hat{x}_k$ is calculated.

As was done in the case of the estimate of the mean, the mean and variance of the estimate of the variance are determined. The constant C is selected to force the mean of the estimate of the variance to converge to the actual variance of the data being sampled. The variance of the estimate serves as an indication of the average error of the estimate.

The mean of the estimated variance is calculated for several values of k. Enough terms are used to recognize the series being generated. The general expression is then written, and the limiting value is determined. Thus,

$$\hat{\sigma}_1{}^2 = B\hat{\sigma}_o{}^2 + \frac{A^2(1 - B)}{C}\left(x_1 - \hat{x}_o\right)^2$$

and

$$E\left\{\hat{\sigma}_1{}^2\right\} = B\hat{\sigma}_o{}^2 + \frac{A^2(1 - B)}{C}E\left\{x_1{}^2 - 2x_1\hat{x}_o + \hat{x}_o{}^2\right\}$$

$$= B\hat{\sigma}_o{}^2 + \frac{A^2(1 - B)}{C}\left(E\left\{x_1{}^2\right\} - 2\hat{x}_oE\left\{x_1\right\} + \hat{x}_o{}^2\right)$$

$$= B\hat{\sigma}_o{}^2 + \frac{A^2(1 - B)}{C}\left(a^2 + \sigma^2 - 2a\hat{x}_o + \hat{x}_o{}^2\right)$$

$$= B\hat{\sigma}_o{}^2 + \frac{A^2(1 - B)}{C}\left[\sigma^2 + \left(a - \hat{x}_o\right)^2\right]$$

The same techniques are used to calculate $E\left\{\hat{\sigma}_2^{\,2}\right\}$, which gives

$$\hat{\sigma}_2^{\,2} = B\hat{\sigma}_1^{\,2} + \frac{A^2(1 - B)}{C}\left(x_2 - \hat{x}_1\right)^2$$

$$= B\left[B\hat{\sigma}_o^{\,2} + \frac{A^2(1 - B)}{C}\left(x_1 - \hat{x}_o\right)^2\right] + \frac{A^2(1 - B)}{C}\left[x_2 - A\hat{x}_o - (1 - A)x_1\right]^2$$

$$= B^2\hat{\sigma}_o^{\,2} + \frac{A^2B(1 - B)}{C}\left(x_1^{\,2} - 2x_1\hat{x}_o + \hat{x}_o^{\,2}\right) + \frac{A^2(1 - B)}{C}\left[x_2^{\,2} + A^2\hat{x}_o^{\,2}\right.$$

$$\left. + (1 - A)^2x_1^{\,2} - 2Ax_2\hat{x}_o - 2(1 - A)x_1x_2 + 2A(1 - A)x_1\hat{x}_o\right]$$

and

$$E\left\{\hat{\sigma}_2^{\,2}\right\} = B^2\hat{\sigma}_o^{\,2} + \frac{A^2B(1 - B)}{C}\left(E\left\{x_1^{\,2}\right\} - 2\hat{x}_oE\left\{x_1\right\} + \hat{x}_o^{\,2}\right)$$

$$+ \frac{A^2(1 - B)}{C}\left[E\left\{x_2^{\,2}\right\} + A^2\hat{x}_o^{\,2} + (1 - A)^2E\left\{x_1^{\,2}\right\} - 2A\hat{x}_oE\left\{x_2\right\}\right.$$

$$\left. - 2(1 - A)E\left\{x_1x_2\right\} + 2A(1 - A)\hat{x}_oE\left\{x_1\right\}\right]$$

The data samples are considered to be independent; thus,

$$E\left\{x_ix_j\right\} = E\left\{x_i\right\}E\left\{x_j\right\} \qquad (i \neq j)$$

This relation yields

$$E\left\{\hat{\sigma}_2^{\,2}\right\} = B^2\hat{\sigma}_o^{\,2} + \frac{A^2B(1 - B)}{C}\left(a^2 + \sigma^2 - 2a\hat{x}_o + \hat{x}_o^{\,2}\right)$$

$$+ \frac{A^2(1 - B)}{C}\left[a^2 + \sigma^2 + A^2\hat{x}_o^{\,2} + (1 - A)^2\left(a^2 + \sigma^2\right)\right.$$

$$\left. - 2Aa\hat{x}_o - 2(1 - A)a^2 + 2A(1 - A)a\hat{x}_o\right]$$

$$= B^2\hat{\sigma}_o^{\,2} + \frac{A^2(1 - B)}{C}\left[(1 + B)\sigma^2 + (1 - A)^2\sigma^2 + \left(A^2 + B\right)\left(a - \hat{x}_o\right)^2\right]$$

If these same techniques are used, the mean value of $\hat{\sigma}_3^{\,2}$ and $\hat{\sigma}_4^{\,2}$ can be determined:

$$E\left\{\hat{\sigma}_3{}^2\right\} = B^3\hat{\sigma}_o{}^2 + \frac{A^2(1 - B)}{C}\left[\left(1 + B + B^2\right)\sigma^2 + \left(1 + A^2 + B\right)(1 - A)^2\sigma^2 + \left(A^4 + A^2B + B^2\right)\left(a - \hat{x}_o\right)^2\right]$$

$$E\left\{\hat{\sigma}_4{}^2\right\} = B^4\hat{\sigma}_o{}^2 + \frac{A^2(1 - B)}{C}\left\{\left(1 + B + B^2 + B^3\right)\sigma^2 + \left[\left(1 + A^2 + A^4\right) + B\left(1 + A^2\right) + B^2\right](1 - A)^2\sigma^2 + \left(A^6 + A^4B + A^2B^2 + B^3\right)\left(a - \hat{x}_o\right)^2\right\}$$

From these four mean values it is possible to recognize the general term of this series as

$$E\left\{\hat{\sigma}_k{}^2\right\} = B^k\hat{\sigma}_o{}^2 + \frac{A^2(1 - B)}{C}\left[\sigma^2 \sum_{i=0}^{k-1} B^i + \left(a - \hat{x}_o\right)^2 \sum_{i=0}^{k-1} A^{2i}B^{k-1-i} + (1 - A)^2\sigma^2 \sum_{j=0}^{k-2}\left(B^j \sum_{i=0}^{k-2-j} A^{2i}\right)\right] \tag{5}$$

The next problem is to find the value of $E\left\{\hat{\sigma}_k{}^2\right\}$ as k approaches infinity. Since $|B|$ is less than 1.0,

$$\lim_{k\to\infty} B^k\hat{\sigma}_o{}^2 = 0$$

and

$$\lim_{k\to\infty} \sigma^2 \sum_{i=0}^{k-1} B^i = \sigma^2\left(\frac{1}{1 - B}\right)$$

The limiting value of the second term within the brackets in equation (5) is found from

$$\left(a - \hat{x}_o\right)^2 \sum_{i=0}^{k-1} A^{2i}B^{k-1-i} = \left(a - \hat{x}_o\right)^2\left(B^{k-1} + A^2B^{k-2} + A^4B^{k-3} + \ldots + A^{2k-2}\right)$$

It is known that

$$0 < A < 1$$

$$0 < B < 1$$

Let

$$A < C_2 < 1$$

$$B < C_2 < 1$$

If C_2 is substituted for A and B in this series, the resulting series is greater term by term than the original series involving A and B. If the limiting value of the series of C_2 is shown to approach zero, the limiting value of the series of A and B must also approach zero. Thus,

$$\sum_{i=0}^{k-1} C_2^{2i} C_2^{k-1-i} = C_2^{k-1} \sum_{i=0}^{k-1} C_2^{i}$$

This series is a truncated geometric series whose partial sum s_k (ref. 4) is

$$s_k = C_2^{k-1} \frac{1 - C_2^{k}}{1 - C_2} = \frac{C_2^{k-1} - C_2^{2k-1}}{1 - C_2}$$

Since $C_2 < 1$,

$$\lim_{k \to \infty} s_k = 0$$

Therefore,

$$\lim_{k \to \infty} \left(a - \hat{x}_o\right)^2 \sum_{i=0}^{k-1} A^{2i} B^{k-1-i} = 0$$

The last term in equation (5) is

$$\begin{aligned}(1 - A)^2 \sigma^2 \sum_{j=0}^{k-2} \left(B^j \sum_{i=0}^{k-2-j} A^{2i}\right) = (1 - A)^2 \sigma^2 \Big[&\left(1 + A^2 + A^4 + \ldots + A^{2k-4}\right) \\ &+ B\left(1 + A^2 + A^4 + \ldots + A^{2k-6}\right) \\ &+ B^2\left(1 + A^2 + A^4 + \ldots + A^{2k-8}\right) + \ldots + B^{k-3}\left(1 + A^2\right) + B^{k-2}\Big]\end{aligned}$$

The limiting value is

$$\lim_{k\to\infty} (1 - A)^2\sigma^2 \sum_{j=0}^{k-2} \left(B^j \sum_{i=0}^{k-2-j} A^{2i}\right) = (1 - A)^2\sigma^2\left(1 + A^2 + A^4 + . . .\right)\left(1 + B + B^2 + . . .\right)$$

$$= \frac{(1 - A)^2\sigma^2}{\left(1 - A^2\right)(1 - B)}$$

$$= \frac{(1 - A)\sigma^2}{(1 + A)(1 - B)} .$$

These expressions are inserted into the equation for $E\left\{\hat{\sigma}_k{}^2\right\}$ (eq. (5)) in order to find the limit as k approaches infinity. Thus,

$$\lim_{k\to\infty} E\left\{\hat{\sigma}_k{}^2\right\} = \frac{A^2(1 - B)}{C}\left[\frac{\sigma^2}{1 - B} + \frac{(1 - A)\sigma^2}{(1 + A)(1 - B)}\right]$$

$$= \frac{A^2\sigma^2}{C}\left(1 + \frac{1 - A}{1 + A}\right)$$

$$= \frac{\sigma^2}{C}\left(\frac{2A^2}{1 + A}\right)$$

In order for this limit to converge to the actual variance σ^2 of the function being sampled, the relation

$$C = \frac{2A^2}{1 + A}$$

must hold. The value of C obtained is inserted into the estimation equation (eq. (4)) to give

$$\hat{\sigma}_k{}^2 = B\hat{\sigma}_{k-1}{}^2 + \frac{(1 - B)(1 + A)}{2}\left(x_k - \hat{x}_{k-1}\right)^2 \tag{6}$$

DERIVATION OF VARIANCE OF ESTIMATED VARIANCE

The variance of the estimated variance is also of interest since it gives an indication of the error of the estimate. Because of the complexity of the procedure of calculating the variance of the estimated variance for the general case, the derivation is performed here only for input data consisting of samples taken from a Gaussian distribution with mean of a and variance of σ^2. However, the technique of estimation described in the previous section is not limited to this case; it applies to any probability distribution whose mean and variance exist. If the moments of a variable are expressed in terms of the mean and variance of the variable, it is found that moments of order greater than two are dependent on the probability distribution of the variable. The variance of the estimated variance is a function of the probability distribution since it involves moments of order greater than two. The moments of a Gaussian variable are used for this derivation. If x is a probabilistic variable having a Gaussian probability distribution with mean of m and variance of v^2, the first four moments of x are (ref. 5, p. 162):

$$E\left\{x\right\} = m$$

$$E\left\{x^2\right\} = m^2 + v^2$$

$$E\left\{x^3\right\} = m^3 + 3mv^2$$

$$E\left\{x^4\right\} = m^4 + 6m^2v^2 + 3v^4$$

Since the equation used for estimation of the mean is a linear equation, the estimated mean has a Gaussian distribution if the data have a Gaussian distribution. The term $x_k - \hat{x}_{k-1}$ is the difference of two terms, each of which has a Gaussian probability distribution. The probability of the difference is also Gaussian. The square of this difference has a chi-square distribution with one degree of freedom (ref. 5, pp. 250-253).

In order to investigate the probability distribution of the estimated variance, it is necessary to examine several estimation steps by using

$$\hat{\sigma}_k{}^2 = B\hat{\sigma}_{k-1}{}^2 + \frac{(1 - B)(1 + A)}{2}\left(x_k - \hat{x}_{k-1}\right)^2$$

The initial guess $\hat{\sigma}_o{}^2$ has a delta function for a probability distribution since it can have only one value. The distribution of $\hat{\sigma}_1{}^2$ is the weighted convolution of a delta function and a chi-square distribution with its origin shifted. The equation for $\hat{\sigma}_2{}^2$ is a weighted sum of $\hat{\sigma}_1{}^2$ and $\left(x_2 - \hat{x}_1\right)^2$. Because of the estimation technique, $\hat{x}_1$ is not independent of $\hat{\sigma}_1{}^2$ and is fixed exactly when $\hat{\sigma}_1{}^2$ is determined. However, x_2 is independent of either $\hat{\sigma}_1{}^2$ or $\hat{x}_1$. The distribution of $\hat{\sigma}_2{}^2$ is a weighted convolution of the chi-square distribution representing $\hat{\sigma}_1{}^2$ and the distribution $p\left[\left(x_2 - \hat{x}_1\right)^2 \middle| \hat{\sigma}_1{}^2\right]$, which is a chi-square distribution with its mean a function of $\hat{\sigma}_1^2$. The probability distribution of any estimate of the variance by this recursive equation is a weighted convolution of the distribution of the previous estimate and a chi-square distribution whose mean is determined by the previous estimate of the variance. The probability distribution of the estimate of the variance is not determined since it is not practical to make a detailed calculation. Although the distribution of the estimate of the variance is not derived, its variance serves as an indication of the error of the estimate. The error decreases as the variance decreases.

By using the definition of variance, the variance of $\hat{\sigma}_k^2$ is

$$
\begin{aligned}
V\left\{\hat{\sigma}_k^2\right\} &= E\left\{\hat{\sigma}_k^4\right\} - E^2\left\{\hat{\sigma}_k^2\right\} \\
&= E\left\{\left[B\hat{\sigma}_{k-1}^2 + \frac{(1-B)(1+A)}{2}\left(x_k - \hat{x}_{k-1}\right)^2\right]^2\right\} - E^2\left\{B\hat{\sigma}_{k-1}^2 + \frac{(1-B)(1+A)}{2}\left(x_k - \hat{x}_{k-1}\right)^2\right\} \\
&= B^2E\left\{\hat{\sigma}_{k-1}^4\right\} + B(1-B)(1+A)E\left\{\left(x_k - \hat{x}_{k-1}\right)^2\hat{\sigma}_{k-1}^2\right\} + \frac{(1-B)^2(1+A)^2}{4}E\left\{\left(x_k - \hat{x}_{k-1}\right)^4\right\} \\
&\quad - B^2E^2\left\{\hat{\sigma}_{k-1}^2\right\} - B(1-B)(1+A)E\left\{\hat{\sigma}_{k-1}^2\right\}E\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\} \\
&\quad - \frac{(1-B)^2(1+A)^2}{4}E^2\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\} \\
&= B^2\left(E\left\{\hat{\sigma}_{k-1}^4\right\} - E^2\left\{\hat{\sigma}_{k-1}^2\right\}\right) + B(1-B)(1+A)\Bigg[E\left\{\left(x_k - \hat{x}_{k-1}\right)^2\hat{\sigma}_{k-1}^2\right\} \\
&\quad - E\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\}E\left\{\hat{\sigma}_{k-1}^2\right\}\Bigg] + \frac{(1-B)^2(1+A)^2}{4}\left[E\left\{\left(x_k - \hat{x}_{k-1}\right)^4\right\} - E^2\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\}\right] \\
&= B^2V\left\{\hat{\sigma}_{k-1}^2\right\} + B(1-B)(1+A)\Bigg[E\left\{x_k^2\hat{\sigma}_{k-1}^2 - 2x_k\hat{x}_{k-1}\hat{\sigma}_{k-1}^2 + \hat{x}_{k-1}^2\hat{\sigma}_{k-1}^2\right\} \\
&\quad - E\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\}E\left\{\hat{\sigma}_{k-1}^2\right\}\Bigg] + \frac{(1-B)^2(1+A)^2}{4}V\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\} \\
&= B^2V\left\{\hat{\sigma}_{k-1}^2\right\} + \frac{(1-B)^2(1+A)^2}{4}V\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\} + B(1-B)(1+A)\Bigg[\left(a^2 + \sigma^2\right)E\left\{\hat{\sigma}_{k-1}^2\right\} \\
&\quad - 2aE\left\{\hat{x}_{k-1}\hat{\sigma}_{k-1}^2\right\} + E\left\{x_{k-1}^2\hat{\sigma}_{k-1}^2\right\} - E\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\}E\left\{\hat{\sigma}_{k-1}^2\right\}\Bigg]
\end{aligned}
\tag{7}
$$

This last step can be made since $\hat{\sigma}_{k-1}^2$ and $\hat{x}_{k-1}$ are independent of x_k.

Let $k = n$ where n is large enough to allow all terms in the equation for $V\left\{\hat{\sigma}_k^{\,2}\right\}$ except $V\left\{\hat{\sigma}_{k-1}^{\,2}\right\}$ to become infinitesimally close to their limiting values. The convergence of each of these terms is shown by the derivation of their limiting values. See appendixes B, C, and D. In appendix B these limits are

$$\lim_{k\to\infty} V\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\} = \frac{8\sigma^4}{(1 + A)^2} \tag{8}$$

and

$$\lim_{k\to\infty} E\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\} = \frac{2\sigma^2}{1 + A} \tag{9}$$

Appendix C shows that

$$\lim_{k\to\infty} E\left\{\hat{x}_{k-1}\hat{\sigma}_{k-1}^{\,2}\right\} = a\sigma^2 \tag{10}$$

Appendix D shows that

$$\lim_{k\to\infty} E\left\{\hat{x}_{k-1}^{\,2}\hat{\sigma}_{k-1}^{\,2}\right\} = a^2\sigma^2 + \left[\frac{1 - A}{1 + A} + \frac{(1 - A)^2(1 - B)}{(1 + A)\left(1 - A^2B\right)}\right]\sigma^4 \tag{11}$$

By the choice of C in equation (4), it has been insured that

$$\lim_{k\to\infty} E\left\{\hat{\sigma}_{k-1}^{\,2}\right\} = \sigma^2 \tag{12}$$

Substitution of equations (8) to (12) into equation (9) yields

$$\begin{aligned}
V\left\{\hat{\sigma}_n^{\,2}\right\} &= B^2V\left\{\hat{\sigma}_{n-1}^{\,2}\right\} + \frac{(1 - B)^2(1 + A)^2}{4}\,\frac{8\sigma^4}{(1 + A)^2} + B(1 - B)(1 + A)\left\{\left(a^2 + \sigma^2\right)\sigma^2 - 2a\left(a\sigma^2\right) + a^2\sigma^2 + \left[\frac{1 - A}{1 + A} + \frac{(1 - A)^2(1 - B)}{(1 + A)\left(1 - A^2B\right)}\right]\sigma^4 - \frac{2\sigma^2}{1 + A}\,\sigma^2\right\} \\
&= B^2V\left\{\hat{\sigma}_{n-1}^{\,2}\right\} + 2(1 - B)^2\sigma^4 + B(1 - B)(1 + A)\left[a^2\sigma^2 + \sigma^4 - 2a^2\sigma^2 + a^2\sigma^2 + \left(\frac{1 - A}{1 + A}\right)\sigma^4 + \frac{(1 - A)^2(1 - B)}{(1 + A)\left(1 - A^2B\right)}\,\sigma^4 - \frac{2\sigma^4}{1 + A}\right] \\
&= B^2V\left\{\hat{\sigma}_{n-1}^{\,2}\right\} + 2(1 - B)^2\sigma^4 + B(1 - B)(1 + A)\left[\sigma^4 + \frac{1 - A}{1 + A}\,\sigma^4 + \frac{(1 - A)^2(1 - B)}{(1 + A)\left(1 - A^2B\right)}\,\sigma^4 - \frac{2\sigma^4}{1 + A}\right] \\
&= B^2V\left\{\hat{\sigma}_{n-1}^{\,2}\right\} + 2(1 - B)^2\sigma^4 + B(1 - B)(1 + A)\left[\frac{2\sigma^4}{1 + A} + \frac{(1 - A)^2(1 - B)}{(1 + A)\left(1 - A^2B\right)}\,\sigma^4 - \frac{2\sigma^4}{1 + A}\right] \\
&= B^2V\left\{\hat{\sigma}_{n-1}^{\,2}\right\} + 2(1 - B)^2\sigma^4 + \frac{B(1 - B)^2(1 - A)^2}{1 - A^2B}\,\sigma^4
\end{aligned} \tag{13}$$

The method of determining the limiting value of variance of $\hat{\sigma}_n^2$ is to insert some constant M for $V\left\{\hat{\sigma}_{n-1}^2\right\}$. Several terms are determined in order to recognize the series being generated. Thus,

$$V\left\{\hat{\sigma}_{n-1}^2\right\} = M$$

$$V\left\{\hat{\sigma}_n^2\right\} = B^2M + (1 - B)^2\left[\frac{B(1 - A)^2}{1 - A^2B} + 2\right]\sigma^4$$

$$V\left\{\hat{\sigma}_{n+1}^2\right\} = B^4M + \left(1 + B^2\right)(1 - B)^2\left[\frac{B(1 - A)^2}{1 - A^2B} + 2\right]\sigma^4$$

$$V\left\{\hat{\sigma}_{n+2}^2\right\} = B^6M + \left(1 + B^2 + B^4\right)(1 - B)^2\left[\frac{B(1 - A)^2}{1 - A^2B} + 2\right]\sigma^4$$

The general term is

$$V\left\{\hat{\sigma}_{n+j}^2\right\} = B^{2(j+1)}M + (1 - B)^2\left[\frac{B(1 - A)^2}{1 - A^2B} + 2\right]\sigma^4 \sum_{i=0}^{j} B^{2i} \tag{14}$$

The limiting value is

$$\lim_{j\to\infty} V\left\{\hat{\sigma}_{n+j}^2\right\} = M \lim_{j\to\infty} B^{2(j+1)} + (1 - B)^2\left[\frac{B(1 - A)^2}{1 - A^2B} + 2\right]\sigma^4 \lim_{j\to\infty} \sum_{i=0}^{j} B^{2i}$$

Since

$$\lim_{j\to\infty} \sum_{i=0}^{j} B^{2i} = \frac{1}{1 - B^2} \qquad (|B| < 1.0)$$

$$\lim_{j\to\infty} B^{2(j+1)} = 0 \qquad (|B| < 1.0)$$

$$\lim_{j\to\infty} V\left\{\hat{\sigma}_{n+j}^2\right\} = (1 - B)^2\left[\frac{B(1 - A)^2}{1 - A^2B} + 2\right]\left(\frac{1}{1 - B^2}\right)\sigma^4$$

$$\lim_{j\to\infty} V\left\{\hat{\sigma}_k^{\,2}\right\} = \lim_{j\to\infty} V\left\{\hat{\sigma}_{n+j}^{\,2}\right\}$$

$$= \left(\frac{1 - B}{1 + B}\right)\left[\frac{B(1 - A)^2}{1 - A^2B} + 2\right]\sigma^4 \qquad (15)$$

No attempt is made to apply the standard mathematical tests for convergence because of the complexity of the series. The method of derivation used shows the convergence of the series since the starting point has no effect on the limiting value of the sequence and the limiting value is determined.

A calculation which adds to the credibility of this derivation is that of the estimation of the variance when the mean is known exactly. For this case A is equal to 1.0 and the equation for the variance of the estimated variance reduces to

$$\lim_{j\to\infty} V\left\{\hat{\sigma}_j^{\,2}\right\} = \frac{1 - B}{1 + B}\left(2\sigma^4\right) \qquad (16)$$

This relation can be checked by actually calculating the mean and variance of the estimated variance with the mean known exactly. Thus,

$$\hat{\sigma}_k^{\,2} = B\hat{\sigma}_{k-1}^{\,2} + (1 - B)\left(x_k - a\right)^2 \qquad (17)$$

and

$$E\left\{\hat{\sigma}_o^{\,2}\right\} = \hat{\sigma}_o^{\,2}$$

$$E\left\{\hat{\sigma}_1^{\,2}\right\} = B\hat{\sigma}_o^{\,2} + (1 - B)\sigma^2$$

$$E\left\{\hat{\sigma}_2^{\,2}\right\} = B^2\hat{\sigma}_o^{\,2} + (1 + B)(1 - B)\sigma^2$$

$$E\left\{\hat{\sigma}_3^{\,2}\right\} = B^3\hat{\sigma}_o^{\,2} + \left(1 + B + B^2\right)(1 - B)\sigma^2$$

The general term is

$$E\left\{\hat{\sigma}_k^{\,2}\right\} = B^k\hat{\sigma}_o^{\,2} + (1 - B)\sigma^2 \sum_{j=0}^{k-1} B^j \qquad (18)$$

Since $B < 1.0$,

$$\lim_{k \to \infty} B^k \hat{\sigma}_o^2 = 0$$

and

$$\lim_{k \to \infty} \sum_{j=0}^{k-1} B^j = \frac{1}{1 - B}$$

Then,

$$\lim_{k \to \infty} E\left\{\hat{\sigma}_k^2\right\} = (1 - B)\sigma^2 \left(\frac{1}{1 - B}\right) = \sigma^2 \tag{19}$$

The variance of the estimation is determined by

$$V\left\{\sigma_o^2\right\} = 0$$

$$V\left\{\hat{\sigma}_1^2\right\} = E\left\{\hat{\sigma}_1^4\right\} - E^2\left\{\hat{\sigma}_1^2\right\}$$

The variances for several values of k are

$$V\left\{\hat{\sigma}_1^2\right\} = (1 - B)^2 \left(2\sigma^4\right)$$

$$V\left\{\hat{\sigma}_2^2\right\} = 2\left(1 + B^2\right)(1 - B)^2 \sigma^4$$

$$V\left\{\hat{\sigma}_3^2\right\} = 2\left(1 + B^2 + B^4\right)(1 - B)^2 \sigma^4$$

The general term is

$$V\left\{\hat{\sigma}_k^2\right\} = 2(1 - B)^2 \sigma^4 \sum_{j=0}^{k-1} B^{2j} \tag{20}$$

The limiting value is

$$\lim_{k \to \infty} V\left\{\hat{\sigma}_k^2\right\} = 2(1 - B)^2 \sigma^4 \left(\frac{1}{1 - B^2}\right) = \frac{2(1 - B)}{1 + B}\,\sigma^4 \tag{21}$$

This equation checks with that obtained by letting $A = 1.0$ in the general equation (15).

Equation (15) is plotted in figure 1 as a function of A and B. It can be seen from figure 1 that the asymptotic value of the variance of the estimated variance can be made as small as desired by making B closer to 1.0. The value of A is seen to have very little effect on the variance of the estimated variance. Although a mathematical proof is not made, it is suspected that as the estimation of the variance is made more accurate (A and B near 1.0), the time constant of the estimation is greatly increased.

CONCLUDING REMARKS

A recursive equation, which is capable of estimating variance, has been presented and analyzed. The constants used in the equation can be varied to control the accuracy of the estimation. Additional work is required in this area to determine the effective time constant of this estimation technique.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., June 29, 1969.

APPENDIX A

ANALYSIS OF THE EXPONENTIAL SMOOTHING TECHNIQUE

This appendix is a derivation of some of the properties of the estimation of the mean by the exponential smoothing technique. Although these results were published in reference 3 by R. G. Brown, they are derived here in a different manner.

$$\hat{x}_k = A\hat{x}_{k-1} + (1 - A)x_k \qquad (k = 1,2,3,\ldots) \qquad \text{(A1)}$$

where

$\hat{x}_k$ kth estimate of mean

x_k kth data sample

A recursive constant, $A < 1.0$

$\hat{x}_o$ initial guess of mean

This equation can be compared with that in reference 3 if $(1 - A)$ is set equal to α.

Derivation of Mean of Estimation

The general term of the estimation equation is rearranged by inserting an expression for $\hat{x}_{k-1}$ into the expression for $\hat{x}_k$, inserting an expression for $\hat{x}_{k-2}$, and continuing until $\hat{x}_o$ is reached. The resulting expression is

$$\hat{x}_k = A^k\hat{x}_o + (1 - A)\left(x_k + Ax_{k-1} + A^2x_{k-2} + \ldots + A^{k-1}x_1\right) \qquad \text{(A2)}$$

The mean value of $\hat{x}_k$ is

$$E\left\{\hat{x}_k\right\} = E\left\{A^k\hat{x}_o\right\} + (1 - A)\left(E\left\{x_k\right\} + AE\left\{x_{k-1}\right\} + \ldots + A^{k-1}E\left\{x_1\right\}\right)$$

The random function from which the samples x_i are taken is assumed to be stationary with a mean of a and a variance of σ^2 so that

$$E\left\{x_i\right\} = a$$

APPENDIX A

The mean of $\hat{x}_k$ can be rewritten to yield

$$E\left\{\hat{x}_k\right\} = E\left\{A^k\hat{x}_o\right\} + (1 - A)E\left\{x_i\right\}\left(1 + A + . . . + A^{k-1}\right)$$

$$= A^k\hat{x}_o + (1 - A)a\left(1 + A + A^2 + . . . + A^{k-1}\right) \tag{A3}$$

Since $|A| < 1.0$,

$$\lim_{k\to\infty} A^k\hat{x}_o = 0$$

and

$$\lim_{k\to\infty} \left(1 + A + A^2 + . . . + A^{k-1}\right) = \frac{1}{1 - A}$$

Therefore,

$$\lim_{k\to\infty} E\left\{\hat{x}_k\right\} = a(1 - A)\frac{1}{1 - A} = a \tag{A4}$$

Derivation of Variance of Estimation

The variance of $\hat{x}_k$ is calculated by using

$$V\left\{\hat{x}_k\right\} = E\left\{\hat{x}_k{}^2\right\} - E^2\left\{\hat{x}_k\right\} \tag{A5}$$

The first two terms are

$$V\left\{\hat{x}_o\right\} = E\left\{\hat{x}_o{}^2\right\} - E^2\left\{\hat{x}_o\right\} = \hat{x}_o{}^2 - \hat{x}_o{}^2 = 0 \tag{A6}$$

and

$$V\left\{\hat{x}_1\right\} = E\left\{A^2\hat{x}_o{}^2 + 2A(1 - A)x_1\hat{x}_o + (1 - A)^2x_1{}^2\right\} - \left[E\left\{A\hat{x}_o\right\} + E\left\{(1 - A)x_1\right\}\right]^2$$

$$= A^2\hat{x}_o{}^2 + 2A(1 - A)a\hat{x}_o + (1 - A)^2\left(a^2 + \sigma^2\right) - A^2\hat{x}_o{}^2 - 2A(1 - A)a\hat{x}_o - (1 - A)^2a^2$$

$$= (1 - A)^2\sigma^2 \tag{A7}$$

The variances for several values of k have been calculated by the same technique and are presented in the following table:

k	$V\left\{\hat{x}_k\right\}$
0	0
1	$(1 - A)^2\sigma^2$
2	$\left(1 + A^2\right)(1 - A)^2\sigma^2$
3	$\left(1 + A^2 + A^4\right)(1 - A)^2\sigma^2$
4	$\left(1 + A^2 + A^4 + A^6\right)(1 - A)^2\sigma^2$

The general expression of the $V\left\{\hat{x}_k\right\}$ is written from the table by inspection to yield

$$V\left\{\hat{x}_k\right\} = (1 - A)^2\sigma^2 \sum_{i=0}^{k-1} \left(A^2\right)^i \tag{A8}$$

As k approaches infinity,

$$\lim_{k\to\infty} V\left\{\hat{x}_k\right\} = (1 - A)^2\sigma^2 \frac{1}{1 - A^2}$$

$$\lim_{k\to\infty} V\left\{\hat{x}_k\right\} = \frac{1 - A}{1 + A}\sigma^2 \tag{A9}$$

Derivation of Time Constant of Estimation

Since the estimation equation must also react to step changes in the mean of the incoming data, it is desirable to determine the time required to respond to a step change. The estimation equation is analyzed as if it were a filter by the use of the z-transform method. (See ref. 6.) The impulse response of the following equation is found:

$$\hat{x}_k = A\hat{x}_{k-1} + (1 - A)x_k$$

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Let

$$\hat{x}_{-1} = 0$$

$$x_o = 1$$

$$x_i = 0 \qquad (i \neq 0)$$

This set of conditions determines the response of the estimation technique to an input of a unit impulse at $t = 0$. From the definition of the z-transform (ref. 6, p. 145)

$$\hat{X}(z) = \sum_{k=0}^{\infty} \hat{x}_k z^{-k}$$

$$\hat{X}(z) = (1 - A)z^0 + A(1 - A)z^{-1} + A^2(1 - A)z^{-2} + \dots$$

$$\hat{X}(z) = (1 - A)\left(1 + Az^{-1} + A^2z^{-2} + \dots\right)$$

$$\hat{X}(z) = (1 - A)\frac{1}{1 - Az^{-1}}$$

$$\hat{X}(z) = \frac{(1 - A)z}{z - A} \tag{A10}$$

The Laplace transform which corresponds to the z-transform is

$$H(s) = \frac{(1 - A)}{s - \frac{1}{T} \log_e A} \tag{A11}$$

The time constant associated with this function is

$$t_c = \frac{-T}{\log_e A} \tag{A12}$$

APPENDIX B

DERIVATION OF THE $\lim_{k\to\infty} V\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\}$

The first moment of $\left(x_k - \hat{x}_{k-1}\right)^2$ is

$$E\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\} = E\left\{x_k^2 - 2x_k\hat{x}_{k-1} + \hat{x}_{k-1}^2\right\}$$

$$= E\left\{x_k^2\right\} - 2E\left\{x_k\right\}E\left\{\hat{x}_{k-1}\right\} + E\left\{\hat{x}_{k-1}^2\right\}$$

This step can be made since $\hat{x}_{k-1}$ and x_k are independent.

$$\lim_{k\to\infty} E\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\} = a^2 + \sigma^2 - 2a \lim_{k\to\infty} E\left\{\hat{x}_{k-1}\right\} + \lim_{k\to\infty} E\left\{\hat{x}_{k-1}^2\right\}$$

$$= a^2 + \sigma^2 - 2a(a) + a^2 + \frac{1 - A}{1 + A}\sigma^2$$

$$= \left(1 + \frac{1 - A}{1 + A}\right)\sigma^2$$

$$= \frac{2\sigma^2}{1 + A}$$

The second moment is

$$E\left\{\left(x_k - \hat{x}_{k-1}\right)^4\right\} = E\left\{x_k^4 - 4x_k^3\hat{x}_{k-1} + 6x_k^2\hat{x}_{k-1}^2 - 4x_k\hat{x}_{k-1}^3 + \hat{x}_{k-1}^4\right\}$$

where x_k and $\hat{x}_{k-1}$ both have a Gaussian distribution with known mean and variance and are independent. Substitutions of the Gaussian moments yield

APPENDIX B

$$\lim_{k\to\infty} E\left\{\left(x_k - \hat{x}_{k-1}\right)^4\right\} = a^4 + 6a^2\sigma^2 + 3\sigma^4 - 4a^4 - 12a^2\sigma^2 + 6a^4 + 6a^2\sigma^2 + 6\,\frac{1 - A}{1 + A}\,a^2\sigma^2$$

$$+ 6\,\frac{1 - A}{1 + A}\,\sigma^4 - 4a^4 - 12\,\frac{1 - A}{1 + A}\,a^2\sigma^2 + a^4 + 6\,\frac{1 - A}{1 + A}\,a^2\sigma^2 + 3\left(\frac{1 - A}{1 + A}\right)^2\sigma^4$$

$$= 3\sigma^4 + 6\,\frac{1 - A}{1 + A}\,\sigma^4 + 3\left(\frac{1 - A}{1 + A}\right)^2\sigma^4$$

$$= 3\sigma^4\left(1 + \frac{1 - A}{1 + A}\right)^2$$

$$= \frac{12\sigma^4}{(1 + A)^2}$$

The limiting value is

$$\lim_{k\to\infty} V\left\{\left(x_k - \hat{x}_{k-1}\right)^2\right\} = \lim_{k\to\infty} E\left\{\left(x_k - \hat{x}_{k-1}\right)^4\right\} - \lim_{k\to\infty} E^2\left\{\left(x_k - x_{k-1}\right)^2\right\}$$

$$= \frac{12\sigma^4}{(1 + A)^2} - \frac{4\sigma^4}{(1 + A)^2}$$

$$= \frac{8\sigma^4}{(1 + A)^2}$$

APPENDIX C

DERIVATION OF $\lim_{k \to \infty} E\left\{\hat{x}_k \hat{\sigma}_k^2\right\}$

This appendix gives the derivation of $\lim_{k \to \infty} E\left\{\hat{x}_k \hat{\sigma}_k^2\right\}$

$$E\left\{\hat{x}_k \hat{\sigma}_k^2\right\} = E\left\{\left[A\hat{x}_{k-1} + (1 - A)x_k\right]\left[B\hat{\sigma}_{k-1}^2 + \frac{(1 + A)(1 - B)}{2}\left(x_k - \hat{x}_{k-1}\right)^2\right]\right\}$$

$$= ABE\left\{\hat{x}_{k-1}\hat{\sigma}_{k-1}^2\right\} + B(1 - A)E\left\{x_k\hat{\sigma}_{k-1}^2\right\}$$

$$+ \frac{A(1 + A)(1 - B)}{2}\left[E\left\{x_k^2\hat{x}_{k-1}\right\} - 2E\left\{x_k\hat{x}_{k-1}^2\right\} + E\left\{\hat{x}_{k-1}^3\right\}\right]$$

$$+ \frac{(1 - A)(1 + A)(1 - B)}{2}\left[E\left\{x_k^3\right\} - 2E\left\{x_k^2\hat{x}_{k-1}\right\} + E\left\{x_k\hat{x}_{k-1}^2\right\}\right] \tag{C1}$$

Since x_k is independent of $\hat{x}_{k-1}$ and $\hat{\sigma}_{k-1}^2$, the expected values of the product of these variables can be separated into the product of the expected values. By using the Gaussian moments, equation (C1) becomes

$$E\left\{\hat{x}_k\hat{\sigma}_k^2\right\} = ABE\left\{\hat{x}_{k-1}\hat{\sigma}_{k-1}^2\right\} + B(1 - A)aE\left\{\hat{\sigma}_{k-1}^2\right\} + \frac{(1 + A)(1 - B)}{2}\left[(1 - A)\left(a^3 + 3a\sigma^2\right)\right.$$

$$\left. + (3A - 2)\left(a^2 + \sigma^2\right)E\left\{\hat{x}_{k-1}\right\} + (1 - 3A)(a)E\left\{\hat{x}_{k-1}^2\right\} + AE\left\{\hat{x}_{k-1}^3\right\}\right] \tag{C2}$$

The technique for finding the limit as k approaches infinity of this recursive equation is the same as that used in the text of this paper for the variance of the estimated variance. The index k is set equal to n where n has a value large enough to permit all terms on the right-hand side of equation (C2) except $E\left\{\hat{x}_{k-1}\hat{\sigma}_{k-1}^2\right\}$ to become infinitesimally close to their limiting values. By using the Gaussian moments and equations (A4) and (A9), equation (C2) becomes

APPENDIX C

$$E\left\{\hat{x}_n\hat{\sigma}_n^{\,2}\right\} = ABE\left\{\hat{x}_{n-1}\hat{\sigma}_{n-1}^{\,2}\right\} + B(1 - A)a\sigma^2 + \frac{(1 + A)(1 - B)}{2}\left[(1 - A)\left(a^3 + 3a\sigma^2\right)\right.$$

$$\left. + (3A - 2)\left(a^2 + \sigma^2\right)a + (1 - 3A)(a)\left(a^2 + \frac{1 - A}{1 + A}\sigma^2\right) + A\left(a^3 + 3\frac{1 - A}{1 + A}a\sigma^2\right)\right]$$

$$= ABE\left\{\hat{x}_{n-1}\hat{\sigma}_{n-1}^{\,2}\right\} + (1 - AB)a\sigma^2 \qquad \text{(C3)}$$

Several terms are calculated in order to recognize the series being generated. Let

$$E\left\{\hat{x}_{n-1}\hat{\sigma}_{n-1}^{\,2}\right\} = P$$

Then

$$E\left\{\hat{x}_n\hat{\sigma}_n^{\,2}\right\} = AB(P) + (1 - AB)a\sigma^2$$

$$E\left\{\hat{x}_{n+1}\hat{\sigma}_{n+1}^{\,2}\right\} = A^2B^2(P) + (1 + AB)(1 - AB)a\sigma^2$$

$$E\left\{\hat{x}_{n+2}\hat{\sigma}_{n+2}^{\,2}\right\} = A^3B^3(P) + \left(1 + AB + A^2B^2\right)(1 - AB)a\sigma^2$$

and

$$E\left\{\hat{x}_{n+i}\hat{\sigma}_{n+i}\right\} = A^{i+1}B^{i+1}(P) + (1 - AB)a\sigma^2\sum_{j=0}^{i-1}(AB)^j \qquad \text{(C4)}$$

Since $A < 1$ and $B < 1$, equation (C4) becomes

$$\lim_{i\to\infty} E\left\{\hat{x}_{n+i}\hat{\sigma}_{n+i}^{\,2}\right\} = \lim_{k\to\infty} E\left\{\hat{x}_k\hat{\sigma}_k^{\,2}\right\} = \frac{(1 - AB)a\sigma^2}{1 - AB}$$

$$= a\sigma^2 \qquad \text{(C5)}$$

The convergence of this series has been verified by calculating the exact expression for the series for $k = 0, 1,$ and 2 but has not been included because of its length. From these expressions it is possible to recognize the general expression for the coefficients of all terms in the expression. It is found that the limit of coefficients of all terms approached zero as k approached infinity except for the coefficient of $a\sigma^2$. This coefficient is found to approach 1.0.

APPENDIX D

DERIVATION OF $\lim_{k \to \infty} E\left\{\hat{x}_k{}^2 \hat{\sigma}_k{}^2\right\}$

This appendix gives the derivation of $\lim_{k \to \infty} E\left\{\hat{x}_k{}^2 \hat{\sigma}_k{}^2\right\}$

$$E\left\{\hat{x}_k{}^2 \hat{\sigma}_k{}^2\right\} = E\left\{\left[A\hat{x}_{k-1} + (1 - A)x_k\right]^2 \left[B\hat{\sigma}_{k-1}{}^2 + \frac{(1 + A)(1 - B)}{2}\left(x_k - \hat{x}_{k-1}\right)^2\right]\right\}$$

$$= A^2BE\left\{\hat{x}_{k-1}{}^2 \hat{\sigma}_{k-1}{}^2\right\} + 2AB(1 - A)aE\left\{\hat{x}_{k-1}\hat{\sigma}_{k-1}{}^2\right\} + B(1 - A)^2\left(a^2 + \sigma^2\right)E\left\{\hat{\sigma}_{k-1}{}^2\right\}$$

$$+ \frac{(1 + A)(1 - B)}{2}\left[\left(a^4 + 6a^2\sigma^2 + 3\sigma^4\right)(1 - A)^2\right.$$

$$+ E\left\{\hat{x}_{k-1}\right\}\left(a^3 + 3a\sigma^2\right)\left(-2 + 6A - 4A^2\right) + E\left\{\hat{x}_{k-1}{}^2\right\}\left(a^2 + \sigma^2\right)\left(1 - 6A + 6A^2\right)$$

$$\left. + E\left\{\hat{x}_{k-1}{}^3\right\}(a)\left(2A - 4A^2\right) + E\left\{\hat{x}_{k-1}{}^4\right\}A^2\right] \tag{D1}$$

Since x_k is independent of $\hat{x}_{k-1}$ and $\hat{\sigma}_{k-1}{}^2$, the expected value of the product of these variables is separated into the product of the expected values in this expression.

The technique for finding the limit as k approaches infinity of this recursive equation is the same as that used in appendix C. The index k is set equal to n where n has a value large enough to insure that all terms on the right-hand side of equation (D1) except $E\left\{\hat{x}_{k-1}{}^2 \hat{\sigma}_{k-1}{}^2\right\}$ have become infinitesimally close to their limiting values. By using the Gaussian moments, equation (D1) becomes

$$E\left\{\hat{x}_n{}^2\hat{\sigma}_n{}^2\right\} = A^2BE\left\{\hat{x}_{n-1}{}^2\hat{\sigma}_{n-1}{}^2\right\} + 2AB(1 - A)a^2\sigma^2 + B(1 - A)^2\left(a^2 + \sigma^2\right)\sigma^2$$

$$+ \frac{(1 + A)(1 - B)}{2}\left\{\left(a^4 + 6a^2\sigma^2 + 3\sigma^4\right)(1 - A)^2 + \left(a^4 + 3a^2\sigma^2\right)\left(-2 + 6A - 4A^2\right)\right.$$

$$+ \left(a^2 + \frac{1 - A}{1 + A}\sigma^2\right)\left(a^2 + \sigma^2\right)\left(1 - 6A + 6A^2\right) + \left(a^3 + 3\,\frac{1 - A}{1 + A}\,a\sigma^2\right)(a)\left(2A - 4A^2\right)$$

$$\left. + \left[a^4 + 6\,\frac{1 - A}{1 + A}\,a^2\sigma^2 + 3\left(\frac{1 - A}{1 + A}\right)\sigma^4\right]A^2\right\}$$

$$= A^2BE\left\{\hat{x}_{n-1}{}^2\hat{\sigma}_{n-1}{}^2\right\} + \left(1 - A^2B\right)a^2\sigma^2$$

$$+ \frac{\sigma^4}{1 + A}\left[(1 - A)\left(1 - A^2B\right) + (1 - A)^2(1 - B)\right] \qquad \text{(D2)}$$

The term $E\left\{\hat{x}_{n-1}{}^2\hat{\sigma}_{n-1}{}^2\right\}$ is set equal to an arbitrary constant Q, and several terms are calculated in order to recognize the series being generated:

$$E\left\{\hat{x}_{n-1}{}^2\hat{\sigma}_{n-1}{}^2\right\} = Q$$

$$E\left\{\hat{x}_n{}^2\hat{\sigma}_n{}^2\right\} = A^2B(Q) + \left(1 - A^2B\right)a^2\sigma^2 + \frac{\sigma^4}{1 + A}\left[(1 - A)\left(1 - A^2B\right) + (1 - A)^2(1 - B)\right]$$

$$E\left\{\hat{x}_{n+1}{}^2\hat{\sigma}_{n+1}{}^2\right\} = A^4B^2(Q) + \left(1 + A^2B\right)\left\{\left(1 - A^2B\right)a^2\sigma^2\right.$$

$$\left. + \frac{\sigma^4}{1 + A}\left[(1 - A)\left(1 - A^2B\right) + (1 - A)^2(1 - B)\right]\right\}$$

The general term can be recognized to be

$$E\left\{\hat{x}_{n+j}{}^2\hat{\sigma}_{n+j}{}^2\right\} = A^{2(j+1)}B^{j+1}(Q) + \left\{\left(1 - A^2B\right)a^2\sigma^2 + \frac{\sigma^4}{1 + A}\left[(1 - A)\left(1 - A^2B\right)\right.\right.$$

$$\left.\left. + (1 - A)^2(1 - B)\right]\right\} \sum_{i=0}^{j} \left(A^2B\right)^i \tag{D3}$$

Since $A^2B < 1.0$,

$$\lim_{k\to\infty} E\left\{\hat{x}_k{}^2\hat{\sigma}_k{}^2\right\} = \lim_{j\to\infty} E\left\{\hat{x}_{n+j}{}^2\hat{\sigma}_{n+j}{}^2\right\}$$

$$= \frac{1}{1 - A^2B}\left\{\left(1 - A^2B\right)a^2\sigma^2 + \frac{\sigma^4}{1 + A}\left[(1 - A)\left(1 - A^2B\right) + (1 - A)^2(1 - B)\right]\right\}$$

$$= a^2\sigma^2 + \sigma^4\left[\frac{1 - A}{1 + A} + \frac{(1 - A)^2(1 - B)}{(1 + A)\left(1 - A^2B\right)}\right] \tag{D4}$$

The limit of this sequence has been verified by the method of verification discussed in appendix C.

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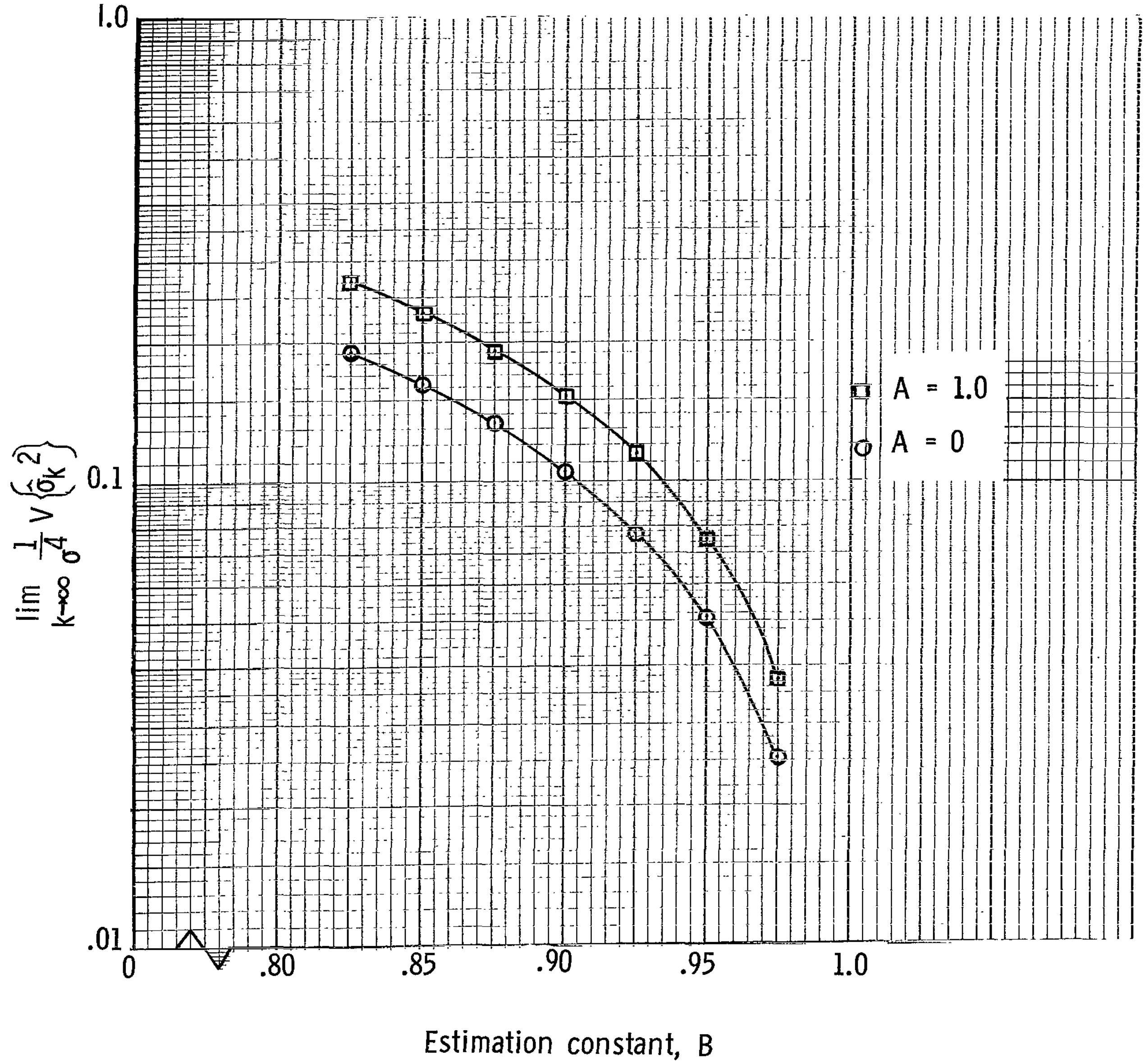

Figure 1.- Asymptotic variance of estimated variance.

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