N69-38181
NASA CR-61302

NASA CONTRACTOR
REPORT

Report No. 61302

SAMPLE REDUCTION TECHNIQUES FOR MINIMIZING QUANTIZATION ERRORS

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August 10, 1969

Prepared for

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER
Marshall Space Flight Center, Alabama 35812

TECHNICAL REPORT STANDARD TITLE PAGE

1. REPORT NO. NASA CR-61302	2. GOVERNMENT ACCESSION NO.	3. RECIPIENT'S CATALOG NO.
4. TITLE AND SUBTITLE SAMPLE REDUCTION TECHNIQUES FOR MINIMIZING QUANTIZATION ERRORS		5. REPORT DATE August 10, 1969
		6. PERFORMING ORGANIZATION CODE
7. AUTHOR(S) Ming-Yang Su		8. PERFORMING ORGANIZATION REPORT # PR-794-9-504F
9. PERFORMING ORGANIZATION NAME AND ADDRESS Northrop-Huntsville, Northrop Corporation, P. O. Box 1484, Huntsville, Alabama 35807		10. WORK UNIT NO.
		11. CONTRACT OR GRANT NO. NAS8-30518
		13. TYPE OF REPORT & PERIOD COVERED Contractor Report
12. SPONSORING AGENCY NAME AND ADDRESS NASA-George C. Marshall Space Flight Center, Marshall Space Flight Center, Alabama, 35812		14. SPONSORING AGENCY CODE

15. SUPPLEMENTARY NOTES

This study was made by Northrop Electro-Mechanical Division for Marshall Space Flight Center with Dr. F. R. Krause as the Project monitor.

16. ABSTRACT

Quantization errors resulting from the analog-to-digital conversion of random time histories have been studied. An existing analysis of the quantization error of the time history itself was reviewed and further extended to treat the quantization error associated with the first derivative of the time history. An analytical theory on reducing the quantization error by means of a sample reduction method is also presented. A practical criterion for the amount of reduction in the quantization error by this method is also discussed.

17. KEY WORDS	18. DISTRIBUTION STATEMENT PUBLIC RELEASE E. D. Geissler E. D. GEISSLER, Director Aero-Astrodynamics Lab, MSFC

19. SECURITY CLASSIF. (of this report)	20. SECURITY CLASSIF. (of this page)	21. NO. OF PAGES	22. PRICE
U	U		

MSFC - Form 3292 (May 1969)

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DEFINITION OF SYMBOLS

Symbol	Definition
$E_d(f)$	Relative quantization error for the derivative of the frequency component (f)
$\overline{E}_d(f)$	Average quantization error for the derivative of the frequency component (f)
$\overline{E}_{d,atm}$	Average quantization error for atmospheric crossed-beam data
$\overline{E}_{d,w}$	Average quantization error for band-limited white noise
f	Frequency
f_l	Lower frequency
f_u	Upper frequency
Int()	Integer of the quantity in the parentheses
m	(2m+1) representing the number of digital samples used in the sample reduction method
N_b	Number of quantization bits in a digital converter
N_s	Total number of scale units in a digital converter
R_{xy}	Cross-correlation between x(t) and y(t)
$S(f)$	Spectral density function
t	Time
Δt	Increment in time
$x(t)$	Time history
$x(t,f)$	Frequency component of x(t) with frequency f
$\overline{x}(t,f)$	An equivalent to x(t,f) defined by equation (22)
$\hat{\overline{x}}(t,f_u,m)$	An estimated value of $\overline{x}(t,f_u)$ by the sample reduction method using (2m+1) samples
$x_s(t)$	Time history x(t) measured in scale units
$y(t)$	Time history

Greek Symbols	Definition
$\bar{\varepsilon}(m)$	Average percentage of overall reduction of quantization error using (2m+1) samples for any spectral density s(f)
$\bar{\varepsilon}(m, f_u)$	Same as for $\bar{\varepsilon}(m)$ except for frequency component (f_u)
$\varepsilon_{min}(m, f_u)$	The minimum percentage of reduction of quantization error for frequency component f_u using (2m+1) samples.
θ	Angle between $\bar{x}(t;\ f_u)$ and the abscissa
$\sigma_{x,q}$	rms of quantization error in x(t)
$\mu_{x,q}$	mean of quantization error in x(t)

Section I

INTRODUCTION

Quantization errors occur in the process of converting a continuous analog time history to a discrete digital time history by an electronic device. Two factors cause quantization errors. The first is the imperfection of electronic devices which read the analog time history and subsequently process the sample into digital form. In this analysis we shall ignore this factor by assuming that we have an ideal analog-to-digital converter with respect to the electronics. The second factor is the finite size of the scale unit which is pre-fixed as the smallest increment for recording digital samples. The problem of finite size of a scale unit is usually attacked by making the scale unit as small as possible, i.e., increasing the number of digits in quantization. This requires expensive equipment, and the maximum number of digital bits is limited for any digitizer.

In the digital analysis of atmospheric crossed-beam data for remote wind speed and turbulence detection, (ref. 1), it was shown that the signal time (first) derivatives rather than the signals themselves are needed for calculating the cross-correlation function. Furthermore, the signal-to-noise ratio of the crossed-beam data is usually very low; say 0.3 or less (ref. 2 and 3). It follows that the signal-to-noise ratio in the signal-time derivatives, which are evaluated by a finite difference scheme, will be even smaller. The quantization error for a digital time history has been discussed by Bendat and Piersol (ref. 4).

The purpose of this report is two-fold. First, Section II presents an estimate of the quantization error for the first derivative of digital time histories, which is made based on the results in reference 4. A specific application to crossed-beam data will be given. Based on this estimate, the minimum required bits of quantization of the time history can be determined. Secondly, we shall deal with the problem concerning how the quantization error can be reduced further without increasing the number of digital bits. One solution is to employ the sample reduction method. Using sample reduction, a given number, say m, of consecutive digital samples of a time history is added to form a new time history with only 1/m of the original number of digital samples. In contrast with the usual central moving average, the sample reduction may be considered as a jumping average.

In Section III we shall develop a theoretical basis and then establish some practical criteria for employing the sample reduction method to reduce quantization error.

Section II

QUANTIZATION ERRORS FROM ANALOG/DIGITAL CONVERSION

In this section we shall deal mainly with the quantization error in the first derivative of the digital time history. However, before we can do this, we must know the quantization error in the digital time history, since the latter error will be carried over to the former.

2.1 QUANTIZATION ERROR IN THE DIGITAL TIME HISTORY

Consider an analog time history x(t), which is plotted in Figure 2-1, with time as the abscissa and prefixed scale units as the ordinate. Now consider taking a digital sample of the time history at an arbitrary time t. The exact value of the sample is indicated graphically by point A on the curve. This point happens to lie just below the midpoint line between two consecutive scale units (j+1) and (j+2). Hence, it is registered by the digital converter as having (j+1) scale units. The difference between the exact analog value $\overline{AC}$ and the approximate, rounded digital value $\overline{BC}$, as indicated by the segment $\overline{AB}$, is called the quantization error (due to any analog/digital conversion). This error may vary from zero to a maximum of one-half scale unit.

Here we are interested in random time histories, so the mean and rms of the error are of primary interest. To obtain these two average quantities, the statistical distribution of point A must be known. For most time histories it would seem safe to assume that point A will be found, with equal probability, between two consecutive scale units. Thus, we shall make the assumption that the probability density of the time history x(t) over any scale unit is uniform. It is also recalled that any point laying above (or below) the midpoint between two scale units is considered to belong to the upper (or lower) scale unit. These two statements can be expressed mathematically as

$$\text{Prob}\ (x) = 1, \qquad -0.5 \le x \le 0.5 \text{ scale unit and}$$

$$\text{and} \quad \text{Prob}\ (x) = 0, \qquad \text{otherwise} \tag{1}$$

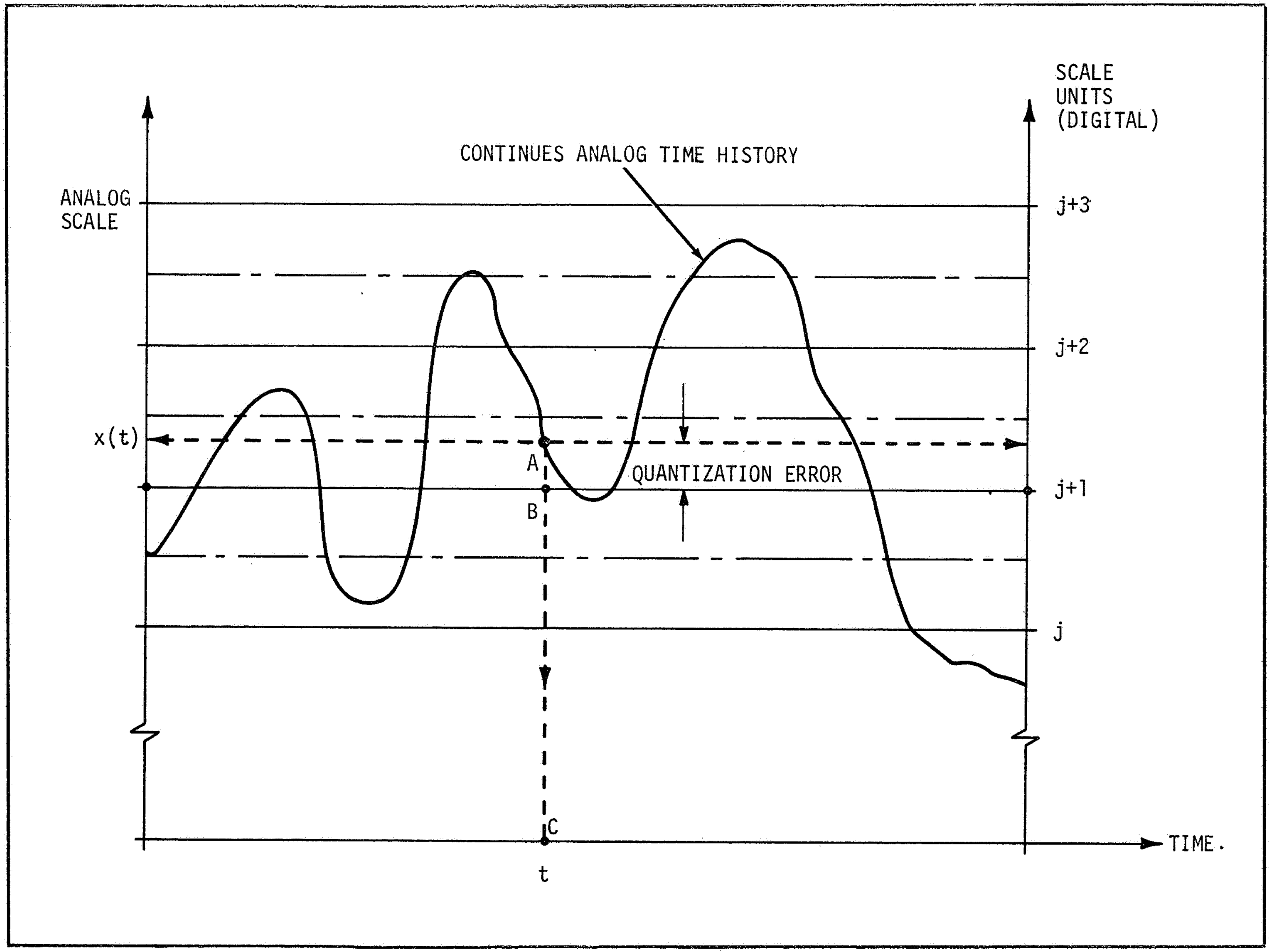

Figure 2-1. QUANTIZATION ERROR OF A DIGITAL TIME HISTORY

With this probability, the mean value of the quantization error $\mu_{x,q}$ will be

$$\mu_{x,q} = \int_{-\infty}^{\infty} \text{Prob}\ (x)\ dx = 0 \quad , \tag{2A}$$

since the probability density is symmetric about x = 0. The variance is given by

$$\sigma_{x,q}^2 = \int_{-\infty}^{\infty} (x - \mu_{x,q})^2\ \text{Prob}\ (x)\ dx = \frac{1}{12}\ (\text{scale unit}) \tag{2B}$$

So, the standard deviation (or rms) of the quantization error for this digital time history is

$$\sigma_{x,q} = \sqrt{\frac{1}{12}} \cong 0.29 \quad \text{scale unit} \tag{3}$$

This might be considered as rms noise added to the time history during the analog/digital conversion.

2.2 QUANTIZATION ERROR FOR FIRST DERIVATIVE OF DIGITAL TIME HISTORY

We shall now consider the error in the calculation of the first derivative of the digital time history with the presence of the rms quantization error as given by equation (3). Let the time interval in the data sampling be Δt. Consider two consecutive sampling times, t and $t + \Delta t$ (see Figure 2-2). The rms quantization errors for x(t) and $x(t + \Delta t)$ will both be $\sigma_{x,q} = 0.29$ scale unit. Without these quantization errors the correct first derivative (within the accuracy of the finite difference) would be given by

$$\left(\frac{dx(t)}{dt}\right)_{cor} = \frac{x(t + \Delta t) - x(t)}{(t + \Delta t) - t} = \frac{\Delta x}{\Delta t} \tag{4}$$

With the quantization errors, the derivative would be, in the rms sense,

$$\left(\frac{dx(t)}{dt}\right)_{qe} = \frac{[x(t + \Delta t) \pm \sigma_{x,q}] - [x(t) \pm \sigma_{x,q}]}{(t + \Delta t) - t} = \frac{\Delta x}{\Delta t} \pm \frac{2\sigma_{x,q}}{\Delta t} \tag{5}$$

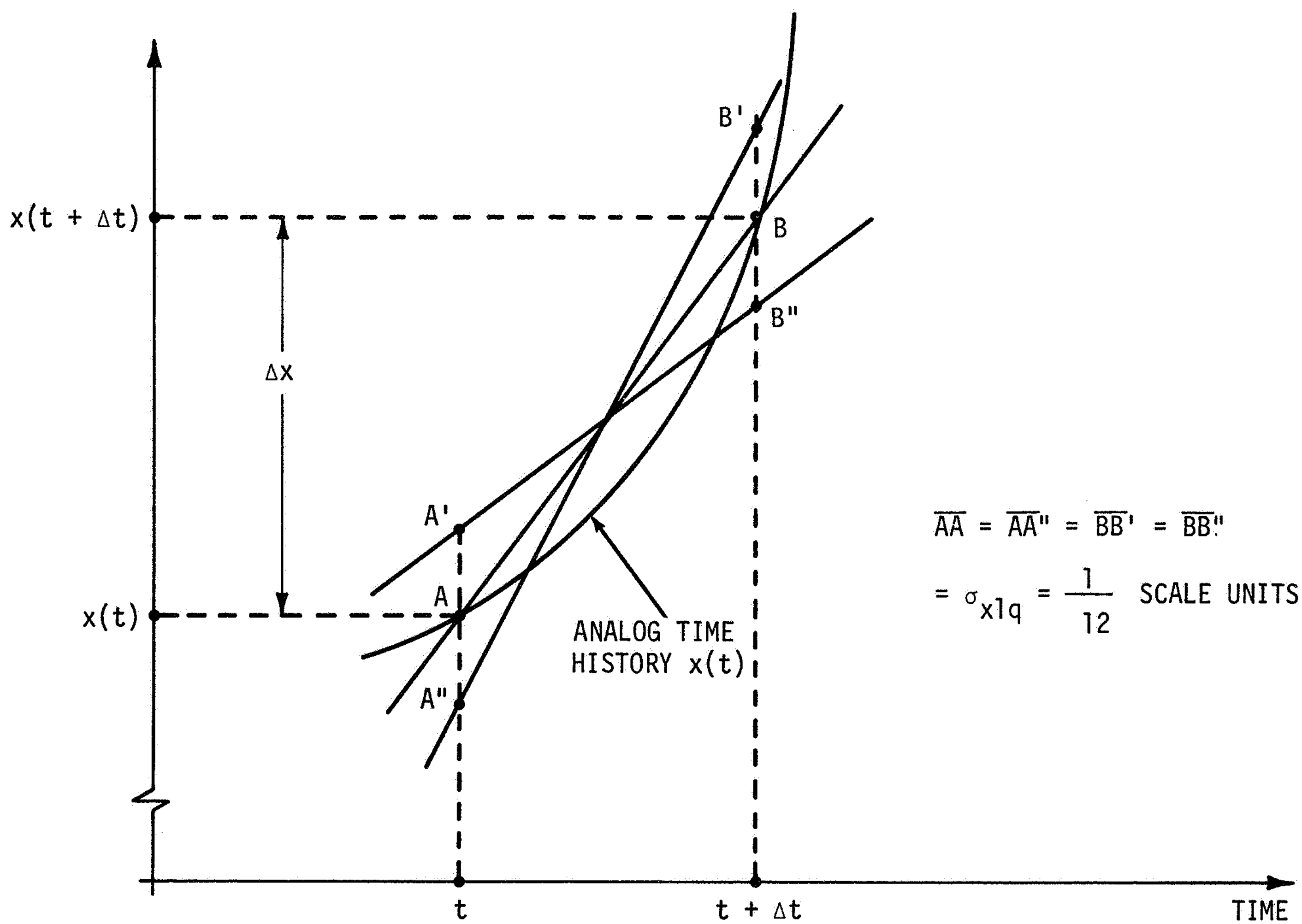

Figure 2-2. QUANTIZATION ERROR OF THE FIRST DERIVATIVE OF A DIGITAL TIME HISTORY

It is logical, then, to define a measure for relative quantization error for the derivative of the time history E_d as follows:

$$E_d = \left| \frac{\left(\frac{dx(t)}{dt}\right)_{qe} - \left(\frac{dx(t)}{dt}\right)_{cor}}{\left(\frac{dx(t)}{dt}\right)_{cor}} \right| \tag{6}$$

That is, E_d is the absolute value of the percentage error with respect to the correct derivative of the time history.

Substitution of equations (4) and (5) into (6) yields

$$E_d = \frac{2\sigma_{x,q}}{x} = \frac{2\sigma_{x,q}}{t} \frac{1}{\left(\frac{dx(t)}{dt}\right)_{cor}} \tag{7}$$

Let the total number of scale units of the A/D converter be denoted by N_S. Now convert the time history x(t) to a new signal $x_s(t)$ measured in scale units, i.e.,

$$x(t) = N_S x_s(t) \tag{8}$$

Clearly,

$$x_s(t) \leq 1$$

With the above transformation, equation (7) can be rewritten as

$$E_d = \frac{2\sigma_{x,q}}{N_S \ \Delta t} \left(\frac{dx_s(t)}{dt}\right)^{-1} \tag{9}$$

This shows that the quantization error is inversely proportional to the sampling interval Δt. The larger the sampling interval, the smaller the error. However, the maximum allowable sampling interval is related to the upper cut-off frequency f_u in the time history by Nyquist's criterion or Shannon's sampling theorem (ref. 4),

$$(\Delta t)_{max} = \frac{1}{2\ f_u} \tag{10}$$

However, in practical application, the maximum sampling interval will be one half of the above value. Thus, using the maximum sampling interval, equation (9) becomes

$$E_d = \frac{8\ f_u\ \sigma_{x,q}}{N_S} \left(\frac{dx_s(t)}{dt} \right)^{-1} , \tag{11}$$

which is the minimum percentage error possible.

In the above expression, the value of the derivative $\frac{dx_s(t)}{dt}$ is a function of time and may range from $-\infty$ to $+\infty$. Thus, the above error expression is useless unless some kind of average is taken over the value of the derivative. Such an averaging may be accomplished as follows:

The time history $x_s(t)$ can be expressed by a Fourier series expansion. Consider Fourier sine (or cosine) components of $x_s(t)$ at frequency f, i.e.,

$$x_s(t,f) = \sin(2\pi\ ft) \tag{12}$$

The time derivative of this component is

$$\frac{dx_s(t,f)}{dt} = 2\ f \cos(2\pi\ ft)$$

The average of the absolute value of the derivative over the entire time history is equal to

$$\overline{\left| \frac{dx_s(t,f)}{dt} \right|} = \lim_{T\to\infty} \frac{1}{T} \int_o^T \left| \frac{dx_s(t,f)}{dt} \right| dt$$

$$= \frac{1}{\frac{1}{4f}} \int_o^{\frac{1}{4f}} 2\ f \cos(2\ ft)\ dt = 4f\ \frac{\text{scale unit}}{\text{time unit}} \tag{13}$$

Thus, an "average" quantization error of the derivative of the Fourier series component at frequency f is given by

$$E_d(f) = \frac{8\, f_u\, \sigma_{x,q}}{N_S \left| \dfrac{dx_s(t,f)}{dt} \right|} \approx \frac{2\, f_u\, \sigma_{x,q}}{f\, N_S} \tag{14}$$

The above estimate of quantization error is valid for a particular frequency component only. Hence, the next logical step is to obtain an estimate of error averaged over the frequency range of the time history of interest.

If the time history is a bandwidth-limited white noise, then the "average" quantization error for the first derivatives of signals over the frequency range is

$$\overline{\overline{E}}_{d,w} = \frac{1}{f_u - f_1} \int_{f_1}^{f_u} E_d(f)\, df = \frac{f_u\, \sigma_{x,q}}{(f_u - f_1)\, N_S} \log \left(\frac{f_u}{f_1} \right) \tag{15}$$

Actually, the power spectrum function of atmospheric, turbulent-wind fluctuations closely follows a minus five-thirds power law in the frequency range of interest for crossed-beam data. If one takes this actual power spectrum function into consideration, then a more realistic and accurate average quantization error of the first derivative of atmospheric data would be

$$\overline{\overline{E}}_{d,atm} = \frac{\int_{f_1}^{f_u} \left(f^{-5/3}\right)^{1/2} E_d(f)\, df}{\int_{f_1}^{f_u} \left(f^{-5/3}\right)^{1/2} df}$$

$$= \frac{2\sigma_{x,q}\, f_u}{5\, N_S} \left(\frac{f_1^{-5/6} - f_u^{-5/6}}{f_u^{1/6} - f_1^{1/6}} \right) \tag{16}$$

The reason for using the "square root" of the power spectrum function as a weighting factor instead of the power spectrum function itself is due to the fact that the power spectrum is proportional to the mean "square" value of the time history.

Equations (15) and (16) indicate that the mean quantization error of the first derivative of the time history is related to the upper and lower cutoff frequencies, the rms value of the quantization error of signals, the power spectrum function of the time history, and the total number of scale units in the digital converter. The first two parameters, f_u and f_1, are set according to the expected spectrum function of signals, and can be regarded as known parameters. The third parameter, $\sigma_{x,q}$, is already given by equation (3). Hence, if one assigns a desired tolerable percentage error for $\overline{\overline{E}}_{d,atm}$ or $\overline{\overline{E}}_{d,w}$, then equation (16) or equation (15) can be used to estimate a minimum total number of scale units, $(N_S)_{min}$. A required minimum number of bits for quantization of the time history, N_b, could in turn be determined by the use of $(N_S)_{min}$ by the following relation

$$2^{N_b - 1} \leq (N_S)_{min} \leq 2^{N_b} \tag{17}$$

It should be noted that in the above analysis the time history x(t) is assumed to be pure signal without any noise, other than that due to quantization. Therefore, equation (16) cannot be applied without some modifications to crossed-beam data where the noise-to-signal ratio is quite high. The parameter in equation (16) which needs to be modified is N_S, which should be scaled down by the ratio of the square root of the peak cross-correlation of time histories x(t) and y(t), $\max(R_{xy})$, to the total range of quantization, Δx. This may be written as

$$(N_S)_{modified} = N_S \frac{\max(R_{xy})}{\Delta x} = N_S \sqrt{\frac{\max(R_{xy})}{\sigma_x \sigma_y}} \sqrt{\frac{\sigma_x \sigma_y}{\Delta x}} , \tag{18}$$

where σ_x and σ_y are rms values of the time histories x(t) and y(t), respectively. Substituting the modified N_S for the original one in equation (16) yields

$$E_{d,atm} = \frac{2\sigma_{x,q}}{5\ N_S} \sqrt{\frac{\sigma_x\ \sigma_y}{\max(R_{xy})}}\ \frac{\Delta x}{\sqrt{\sigma_x\ \sigma_y}}\ f_u \left(\frac{f_1^{-5/6} - f_u^{-5/6}}{f_u^{1/6} - f_1^{1/6}} \right) \tag{19}$$

The above equation could now be used to determine the minimum total number of scale units $(N_S)_{min}$, for atmospheric crossed-beam data.

Example

The upper and lower cutoff frequencies are set to be

$$f_u = 4 \text{ cps}$$

and

$$f_1 = 0.01 \text{ cps}.$$

The ratio

$$\frac{\Delta x}{\sqrt{\sigma_x\ \sigma_y}} = \frac{\Delta x}{\sigma x},$$

assuming $\sigma_x = \sigma_y$. This ratio is usually called the dynamic range of a recording system. Assume this ratio has a value of four (convervative). The ratio $\sqrt{\frac{\sigma_x\ \sigma_y}{\max(R_{xy})}}$ for the atmospheric crossed-beam data is usually no less than 5, (ref. 1). Finally, let the average quantization error $\bar{\bar{E}}_{d,atm}$ be less than 10 percent. Substituting the above numerical values into equation (19) and solving for N_S, one obtains the required minimum total number of scale units as follows:

$$(N_S)_{min} = \frac{2\ \cdot\ 0.29}{5\ \cdot\ 0.1}\ \cdot \sqrt{5}\ \cdot\ 4\ \cdot\ 4\ \frac{(0.01^{-5/6} - 4^{-5/6})}{(4^{1/6} - 0.01^{1/6})} = 2450$$

By use of equation (17) and the above estimated $(N_S)_{min}$ one finds that the minimum required number of bits of quantization is

$$N_b = 12. \tag{20}$$

Section III

REDUCTION OF QUANTIZATION ERROR BY THE SAMPLE REDUCTION METHOD

Section II has shown that quantization errors are always associated with digital time histories. The errors are directly proportional to the size of scale unit used, or inversely proportional to the total number of scale units in the digital converter. Since any digital converter has a maximum number of scale units, the minimum quantization error is fixed for any digital converter.

In this section we shall develop a theory to further reduce this minimum quantization error by means of the sample reduction method.

Consider an analog time history of a random process $x(t)$, and let f_l and f_u be the lower and upper frequencies of interest, respectively. The usual sampling criterion, such as Nyquist's criteria (ref. 4), depends on the upper frequency of the time history. The same consideration will be followed here, and for the time being we shall concentrate attention on the upper frequency component of $x(t)$ only. This component will be denoted by $x(t, f_u)$. Later in the analysis, the whole spectrum density function of $x(t)$ will be taken into consideration. Specifically, $x(t, f_u)$ may be expressed by a sine wave with frequency f_u as

$$x(t, f_u) = \sin (2\pi f_u t) \tag{21}$$

The time derivative of this component will fluctuate between -1 and +1. Since it is the absolute value of the derivative which affects quantization errors, and since there is interest only in the average effect of quantization errors in the evaluation of time derivatives, we shall replace $\bar{x}(t, f_u)$ by an equivalent straight line, $\bar{x}(t, f_u)$; the slope of which is equal to the average of the absolute value of the time derivatives of $x(t, f_u)$. That is

$$\bar{x}(t, f_u) = \overline{\left|\frac{dx(t, f_u)}{dt}\right|} \cdot t \tag{22}$$

where the over bar stands for temporal average. Now evaluate

$$\overline{\left|\frac{dx(t, f_u)}{dt}\right|} = \lim_{T\to\infty} \int_o^T \left|\frac{dx(t, f_u)}{dt}\right| dt$$

$$= \frac{1}{\frac{1}{4 f_u}} \int_o^{\frac{1}{4 f_u}} 2\pi f_u \cos (2\pi f_u t) dt = 4 f_u \tag{23}$$

Hence,

$$\bar{x}(t, f_u) = 4 f_u t$$

A segment of this line is plotted, in Figure 3-1, over a time interval that will be regarded as the analog time history of interest. We will now convert the continuous $\bar{x}(t, f_u)$ into digital samples and then show how the sample reduction method may reduce quantization errors involved.

Consider taking a digital sample at time t (see Figure 3-1). The exact value of $\bar{x}(t, f_u)$ is indicated by Point A on the straight line. In this figure, j-1, j, j+1, etc., are the prefixed digital scale unit marks. For convenience of discussion the mid-point mark between two consecuitive scale units are indicated as j-1/2, j+1/2, etc. Since Point A is shown between the mid-point mark j+1/2 and j scale unit, its value will be recorded in digital form as j scale units. The quantization error due to the finite size of scale unit is then given by the fraction of a scale unit, i.e., $\overline{AI}/\overline{IJ}$.

To apply the sample reduction method to reduce the error $\overline{AI}/\overline{IJ}$, more digital samples in the vicinity of the sampling time t are needed. Denote the points of intersection between $\bar{x}(t, f_u)$, j, j+1/2, and j+1, respectively, by C, F, and B. Also denote the points E and D on each side of Point A along $\bar{x}(t, f_u)$ at a distance $\overline{FB}$ from Point A. Finally, denote the point G on $\bar{x}(t, f_u)$ such that $\overline{AG} = \overline{AF}$. The times corresponding to points A to G are

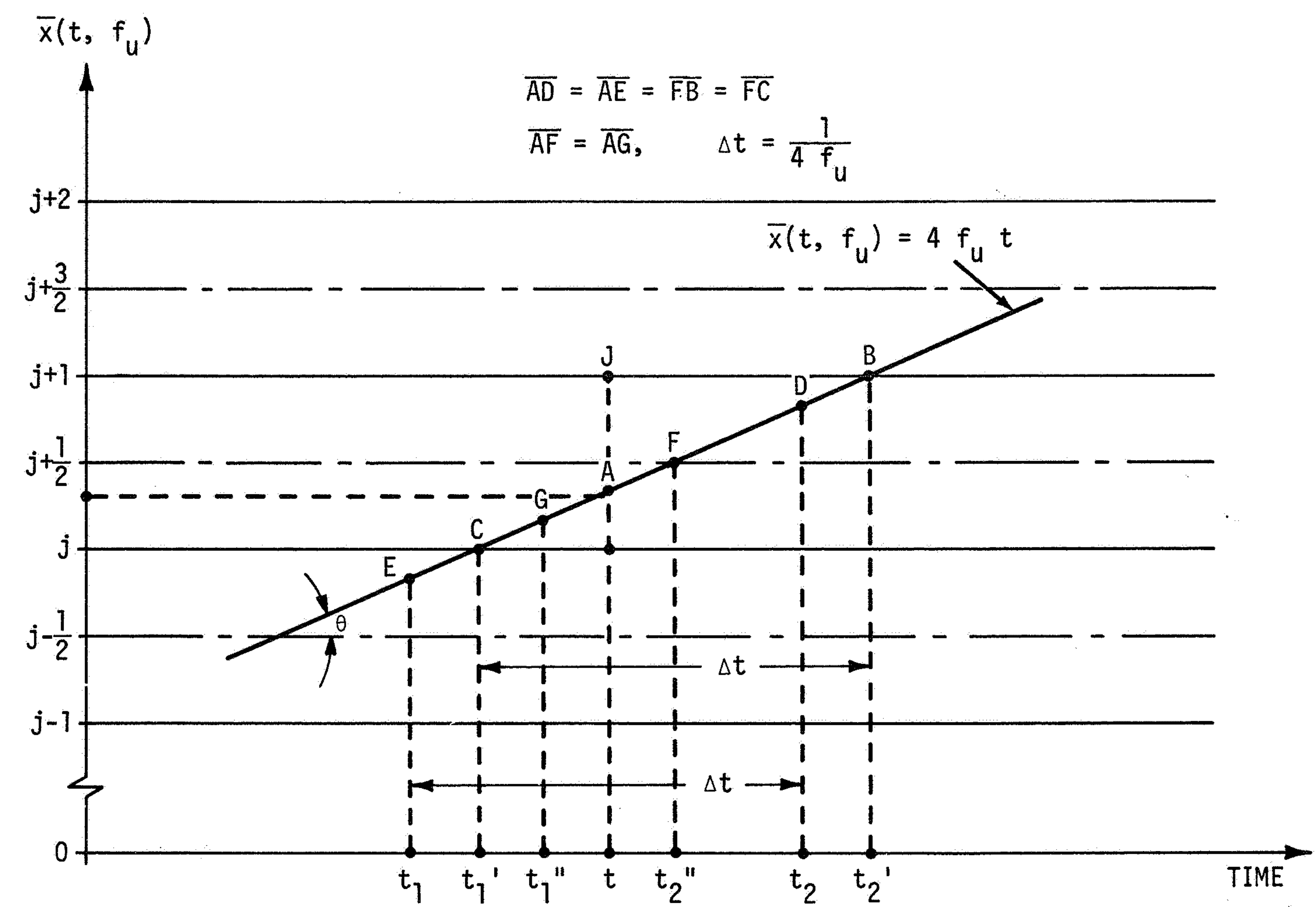

Figure 3-1. REDUCTION OF THE QUANTIZATION ERROR BY THE SAMPLE REDUCTION METHOD

specified in Figure 3-1. Note first that the largest time interval around t for applying the sample reduction can not exceed the largest permissible sampling interval $(\Delta t)_{max}$, which, in turn, is determined by the upper frequency f_u through the following relationship.

$$(\Delta t)_{max} = \frac{1}{4\ f_u} \tag{25}$$

Before proving that $(\Delta t)_{max}$ is also the only suitable time interval to apply sample reduction, observe from Figure 3-1 how a shorter time interval than $(\Delta t)_{max}$ may result. Suppose many digital samples between t_1'' and t_2'' are taken. Since exact values of all these samples in analog form lie between j and j+1/2, they will all be recorded as j scale units. Thus, application of the sample reduction method over the interval between t_1'' and t_2'' apparently does not reduce quantization errors. This observation serves to indicate that the time interval for applying the sample reduction can not be too small either.

We shall now show that the sample reduction method applied over the time interval $\Delta t = t_2 - t_1$ will yield a consistent estimate of the exact value of Point A, i.e., $\bar{x}(t,\ f_u)$. Referring to Figure 3-1, we have, by definition,

$$\Delta t = t_2 - t_1 = t_2' - t_1'$$

and

$$t - t_1 = t_2 - t = \frac{1}{2}\ \Delta t. \tag{26}$$

Since the slope of $\bar{x}(t,\ f_u)$ is 4 f_u (see equation (23)),

$$\tan\ \theta = \frac{(j+1) - j}{t_2' - t_1'} = \frac{1}{\Delta t} = 4\ f_u$$

or

$$\Delta t = \frac{1}{4\ f_u} \equiv (\Delta t)_{max} \tag{27}$$

This last equation simply states that the time interval we intend to use for applying the sample reduction happens to be the maximum permissible sampling interval. Next, take (2m+1) equally spaced digital samples between t_1 and t_2. Specifically, these samples are taken at times

$$t_i = t + (i-m-1) \frac{\Delta t}{2m} \tag{28}$$

where i = 1, 2, ..., 2m, 2m+1. Clearly, every digital sample within the interval $t_1 \leq t_i \leq t_2''$ will be recorded as having j scale units. On the other hand, every sample within the interval $t_2'' \leq t_i \leq t_2$ will be recorded as having j+1 scale units. Therefore, the estimated value of $\bar{x}(t, f_u)$ by the sample reduction method, denoted by $\bar{x}(t, f_u, m)$, will be given by

$$\hat{\bar{x}}(t, f_u, m) = \frac{1}{2m+1} \left\{ \left[\mathrm{Int} \left(\frac{t_2'' - t_1}{\frac{\Delta t}{2m}} \right) + 1 \right] j + \left[\mathrm{Int} \left(\frac{t_2 - t_2''}{\frac{\Delta t}{2m}} \right) + 1 \right] (j+1) \right\} \text{ scale units,} \tag{29}$$

where the symbol Int ( ) is employed to indicate that only the integer part of the number within parentheses has been used. As the number of digital samples taken in the interval (t_1, t_2) becomes infinitely large, i.e., $(2m+1) \to \infty$, the ratio between these two terms

$$\left[\mathrm{Int} \left(\frac{t_2'' - t_1}{\frac{\Delta t}{2m}} \right) + 1 \right] \text{and} \left[\mathrm{Int} \left(\frac{t_2 - t_2''}{\frac{\Delta t}{2m}} \right) \right]$$

approaches asymptotically the ratio between the two distances, $\overline{FE}$ and $\overline{FD}$. Thus as $m \to \infty$, equation (29) can be written as

$$\hat{\bar{x}}(t, f_u, m \to \infty) = \frac{1}{\overline{ED}} [\overline{FE}\, j + \overline{FD}\, (j+1)] \text{ scale units} \tag{30}$$

Since, by definition,

$$\overline{AD} = \overline{AE} = \overline{FB} = \overline{FC}, \tag{31}$$

we have

$$\overline{FE} = \overline{AB},\ \overline{FD} = \overline{AC},\ \text{and}\ \overline{ED} = \overline{BC}. \tag{32}$$

Substituting equations (32) into (30) yields

$$\hat{\bar{x}}(t,\ f_u,\ m\to\infty) = \frac{1}{\overline{BC}}\left[\overline{AB}\ j + \overline{AC}\ (j+1)\right] \quad \text{scale units.} \tag{33}$$

On the other hand, the exact value of the analog time history at time t, $\bar{x}(t,\ f_u)$, can also be expressed in terms of fraction of scale units as

$$\begin{aligned} \bar{x}(t,\ f_u) &= j + \frac{\overline{AC}}{\overline{BC}}\left[(j+1) - j\right] \\ &= \left(1 - \frac{\overline{AC}}{\overline{BC}}\right) j + \frac{\overline{AC}}{\overline{BC}}\ (j+1) \\ &= \frac{1}{\overline{BC}}\left[\overline{AB}\ j + \overline{AC}\ (j+1)\right] \quad \text{scale units} \end{aligned} \tag{34}$$

Comparing equations (33) and (34) then shows that

$$\hat{\bar{x}}(t,\ f_u,\ m\to\infty) = \bar{x}(t,\ f_u) \tag{35}$$

This means that if we apply the sample reduction method by taking an infinitely large number of equally spaced digital samples over a time interval to $1/(4\ f_u)$, then the quantization error may be reduced as desired. In statistical terminology, the estimator (defined by equation (29)) is then said to be consistent.

Next, we will show that the time interval $1/(4 f_u)$ used in the above analysis is also the only one that will yield a consistent estimate of $\bar{x}(t, f_u)$. We have observed that applying the sample reduction method does not reduce any quantization error if the time interval $\Delta t = (t_1'', t_2'')$ is used. It remains now to consider a larger time interval, i.e., $(t_2'' - t_1'') < \Delta t \leq (\Delta t)_{max}$. Let us express the time interval within this range as

$$\Delta t = (\Delta t)_{max} - 2 \Delta^2 t, \tag{36}$$

where $0 < \Delta^2 t < \frac{1}{2} (t_2 - t_2'')$. Using equation (36), one obtains

$$\hat{\bar{x}}(t, f_u, m\to\infty) = \frac{1}{\overline{BC} - 2 \Delta^2 t \csc \theta} [(AB - \Delta^2 t \csc \theta) j$$

$$+ (\overline{AC} - \Delta^2 t \csc \theta) (j+1)]$$

$$= \frac{1}{\overline{BC} - 2 \Delta^2 t \csc \theta} \left\{ \overline{AB} j + \overline{AC} (j+1) - (2j+1) \Delta^2 t \csc \theta \right\}$$

$$\begin{cases} \neq \bar{x}(t, f_u) \text{ in general} \\ = \bar{x}(t, f_u) \text{ only when } \bar{x}(t, f_u) = j + \frac{1}{2} \end{cases}, \tag{37}$$

where $\theta = \tan^{-1}(4 f_u)$. The above equation shows that no matter how large m may be, the estimated value by the sample reduction method, in general, never approaches the exact value $\bar{x}(t, f_u)$ when $\Delta^2 t \neq 0$. Nevertheless, it is important to see that the estimated value $\bar{x}(t, f_u, m)$ will be no less accurate than the original digital sample at time t even for $\Delta t < (\Delta t)_{max}$.

In actual applications, the number of digital samples (2m+1), taken over $(\Delta t)_{max}$, will always be finite, so quantization errors can be reduced only partially by the sample reduction method. We would like to estimate next how much quantization error is eliminated by the estimator (equation (29)) for a given m using $\Delta t = (\Delta t)_{max}$. As a matter of fact, the only error involved in the estimator when compared to the exact value of $\bar{x}(t, f_u)$ is due to truncation of decimal parts by the integer operator Int ( ). The maximum uncertainty $(\Delta\hat{x})_{max}$ in equation (29) can be found for the simplest case by setting j=0, i.e.,

$$\hat{x}(t, f_u, m) = \frac{1}{2m+1}\left[\text{Int}\left(\frac{t_2 - t_2''}{\frac{\Delta t}{2m}}\right) + 1\right] , \tag{38}$$

since the value of j is immaterial. Thus,

$$\begin{aligned}(\Delta\hat{x})_{max} &= \max\,[x(t, f_u) - \hat{x}(t, f_u, m)]\\ &= \frac{1}{2m+1}\max\left[\left(\frac{t_2 - t_2''}{\frac{\Delta t}{2m}}\right) - \text{Int}\left(\frac{t_2 - t_2''}{\frac{\Delta t}{2m}}\right)\right]\\ &= \frac{1}{2m+1}\ \text{scale unit} ,\end{aligned} \tag{39}$$

while the rms quantization error in the original digital sample is $1/\sqrt{12}$ scale unit (equation (3)). Hence, the minimum percentage of reduction of quantization errors by the sample reduction method using (2m+1) equally spaced digital samples with $\Delta t = \frac{1}{4\,f_u}$ is given by

$$\varepsilon_{min}(m, f_u) = \left(1 - \frac{\frac{1}{2m+1}}{\frac{1}{\sqrt{12}}}\right) \text{x } 100\% = \left(1 - \frac{\sqrt{12}}{2m+1}\right) \text{ x } 100\% \tag{40}$$

Actually, a more accurate estimate of the amount of quantization error reduction by the sample reduction method can be made by assuming that the probability density of Point A is uniform between two consecutive scale units. With this assumption, which is indeed quite realistic, the rms value of truncation in equation (29),

$$[A - \text{Int}\ (A)]\ ,$$

will be equal to $1/\sqrt{3}$ instead of 1 as shown in equation (39). So the "average" percentage of reduction of quantization errors by the sample reduction method, denoted by $\bar{\varepsilon}(m, f_u)$, will be

$$\bar{\varepsilon}(m, f_u) = \left(1 - \frac{\frac{1}{2m+1} \quad \frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{12}}}\right) \times 100\% = \frac{2m-1}{2m+1} \times 100\%\ . \tag{41}$$

It should be recalled that so far all the estimates were derived based only on the upper frequency component of the actual analog time history. Therefore, we need to take into consideration the power spectral density function S(f) of x(t) as a weighting function. The correct amount of weighting required for equation (41) may be derived as follows. Without loss of generality, let us regard the time history x(t) as having N discrete frequency components, i.e.,

$$x(t) = \sum_{i=1}^{N} x_i(t)\quad , \tag{42}$$

with

$$x_i(t) = a_i \sin(2\ f_i\ t + \phi_i),\ f \le f_i \le f_{i+1} \le f_u,$$

where ϕ_i is the random phase shift associated with i^{th} frequency component. The corresponding power spectrum density is given by

$$S(f) = \frac{1}{2} \sum_{i=1}^{N} a_i^{\ 2}\ \delta(f - f_i), \tag{43}$$

where $\delta(f)$ is Dirac's delta function. Recall that it is the average of the absolute value of the slope of each frequency component that will affect the application of the sample reduction method. Hence, consider the average

$$\overline{\frac{dx_i(t)}{dt}} = 4\ a_i\ f_i \tag{44}$$

Furthermore, from equation (43), we see that

$$\int_{f_i - \frac{\Delta f}{2}}^{f_i + \frac{\Delta f}{2}} S(f)\ df = \frac{1}{2}\ a_i^{\ 2} \quad ,$$

which implies that $a_i = c\sqrt{S(f_i)}$; c being some constant. Thus, equation (44) becomes

$$\overline{\frac{dx_i(t)}{dt}} = 4\ c\ f_i \sqrt{S(f_i)} \tag{45}$$

Therefore, the suitable weighting function should be $f\sqrt{S(f)}$. The "average" percentage of "overall" reduction of quantization errors $\bar{\varepsilon}(m)$ in the original analog time history x(t) using the sample reduction method (using 2m+1 samples over $\Delta t = 1/4\ f_u$) will be given by

$$\bar{\varepsilon}(m) = \frac{\int_{f_1}^{f_u} f\sqrt{S(f)}\ \bar{\varepsilon}(m,\ f_u)\ df}{f_u \int_{f_1}^{f_u} \sqrt{S(f)}\ df} \times 100\%$$

$$= \frac{2m-1}{2m+1}\ \frac{\int_{f_1}^{f_u} f\sqrt{S(f)}\ df}{f_u \int_{f_1}^{f_u} \sqrt{S(f)}\ df} \times 100\% \tag{46}$$

In particular, for a bandwidth-limited white noise; i.e., S(f) = constant for $f_1 \leq f \leq f_u$; equation (22) reduces to

$$\bar{\varepsilon}(m) = \frac{2m-1}{2m+1}\left(1 + \frac{f_1}{f_u}\right) \times 50\% \qquad (47)$$

For a time history with a spectrum of the form $S(f) = \text{constant} \times f^{-2}$, equation (42) becomes

$$\bar{\varepsilon}(m) = \frac{2m-1}{2m+1} \frac{\log\left(\frac{f_u}{f_1}\right)}{\left(\frac{f_u}{f_1} - 1\right)} \times 100\% \qquad (48)$$

Finally, for a spectrum with extremely narrow bandwidth equation (42) reduces to approximately

$$\bar{\varepsilon}(m) \cong \frac{2m-1}{2m+1} \times 100\% \quad , \qquad (49)$$

as would be expected by considering equation (21).

Numerical examples: For $f_u/f_1 = 10$, we have from equation (48)

$$\bar{\varepsilon}(m) = 21\% \qquad \text{for } m = 5$$
$$= 23\% \qquad \text{for } m \to \infty; \qquad (50)$$

for $f_u/f = 5$, we have

$$\bar{\varepsilon}(m) = 32.8\% \qquad \text{for } m = 5$$
$$= 36\% \qquad \text{for } m \to \infty. \qquad (51)$$

These examples serve to show that little benefit can be obtained for $m > 5$. Therefore, for practical applications, we need only sample about ten times higher than normally required (i.e., 40 f_u samples/sec) to use the sample reduction method.

It is interesting to compare the efficiency of the sample reduction method with that of increasing digital bits to reduce quantization errors. For an increase of two digital bits, a scale unit is reduced in size to one-fourth. Hence, the quantization error will be reduced by (1 - 1/4) or 75 percent, regardless of the spectrum function of the analog time history. Comparing this efficiency with 32.8 percent from equation (51), the method of increasing digital bits is more effective than the sample reduction method. However, the sample reduction method is not intended to replace any possible increase of quantization bits, but rather devised for further reduction of quantization bits, but rather devised for further reduction of quantization errors when the highest possible number of digital bits of a given digitizer is used.

Section IV

CONCLUSIONS AND RECOMMENDATIONS

An estimate of the average quantization error of the first derivative of the time history has been derived. This estimate of error is further specified for atmospheric crossed-beam data. The analysis indicates that 12-bit quantization of the atmospheric crossed-beam data might be necessary in order to obtain an average quantization error of the first derivative of the crossed-beam data below 10 percent.

A theory has been developed on the application of the sample reduction method to reduce quantization errors resulting from digitization of random time histories. Quantization errors occur because of the finite size of scale units used.

This analysis shows that the sample reduction method will give a statistically consistent estimate of the exact value of the associated analog time history if the time history is of a single frequency f and if the time interval used for averaging of digital samples is set to be $1/(4\ f)$. For a time history with a spectrum density $S(f)$ within the frequency range from f_l to f_u and using $(2m+1)$ equally spaced digital samples taken over a time interval of $1/4\ f_u$, the average percentage of overall reduction in quantization errors by the sample reduction method is given by equation (46).

From the above estimate, it is suggested for actual applications of the sample reduction method that the analog time history of interest should be digitized at a sampling interval of $1/40\ f_u$ and then averaged over every ten, consecutive, digital samples. By doing so, an approximate 30 percent reduction in quantization errors may be expected for the commonly encountered spectrum density of the form f^{-2}.

Section V

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