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The Hamilton-Jacobi Theory of Dynamics

by

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1. General statement of the question. In the previous chapter the principles of mechanics in their most general form as well as the equations of motion which arise from them were stated and discussed. Following this the next natural question is, how do we go about the actual integration of these equations and if in particular we cannot draw important conclusions from their character as differential equations of mechanics. This is indeed to a large degree the case, especially with problems for which a kinetic potential exists (cf. Chap. 2, No. 10).

For this mainly the theory of integration was developed systematicly by Jacobi[1] and Hamilton[2]. It is of very great importance on the one hand for celestial mechanics and on the other for the atom; this is because, for both, at least as long as one disregards the periods viz. reaction forces of radiation, there are neither bonds nor non-conservative forces.

Its development will take place in three steps. First, we will attempt to get the simplest possible form for the differential equations. This leads us to the canonical equations

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[1] G. C. Jacobi, Vorlesungen über Dynamik, Werke Supplementband, 2. Aufl., Berlin 1888.

[2] W. A. Hamilton, Brit. Ass. Rep. 1834, S. 513; Phil. Trans. 1835, S. 95.

of mechanics. Second, we can question the general laws of transformations of these differential equations in which they retain their form. This leads us to the canonical transformations and the theory of their most important invariants. Third, we will present the actual theory of integration of the canonical equation systems which consists of the setting up and integration of the Hamiltonian partial differential equation.

The limitation already introduced above to systems with a kinetic potential is the same which makes the Hamiltonian principle an actual variation principle. Therefore, the application of the methods of the calculus of variations leads us to expect a very great facilitation, and also the deeper significance of the singular Hamilton-Jacobi integration process is only disclosed by it. We will return to this at the conclusion.[1]

As an up-to-date presentation we mention above all the book by Whittaker.[2] The first systematic development which was also of fundamental significance to the point was given by Jacobi[3]

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[1]The following presentations are similar in many respects, especially in the application of the calculus of variation, to those which one of us heard (Nordheim) in lectures by Hilbert. Also here we would like to thank sincerely Mr. (Privy Councillor) Hilbert for permission to use them.

[2]E. T. A. Whittaker, Analytical Dynamics, 2. Aufl., Cambridge 1917, Deutsche Übersetzung von F. u. K. Mittelsten-Scheid, Berlin: Julius Springer 1924.

[3]Siehe Anm. 1 von S. 91.

in his famous lecture on dynamics. Many important relationships, especially regarding the theory of canonical transformations, are contained in the investigations of Lie.[1)]

Our starting point is the Hamiltonian principle. We assume therefore, that a kinetic potential (cf. Chap. 2, No 10) exists, which is a function of the coordinates and velocities L (q_k, $\dot{q}_k$, t), and the movements of the system should satisfy the Hamiltonian principle (see Chap. 2, No. 22).

$$\int_{t_1}^{t_2} L(q_k, \dot{q}_k, t)\,dt = \text{Extremum} \tag{1}$$

According to the calculus of variations, they are stated as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0. \qquad (k = 1, 2, \dots f) \tag{2}$$

L thereby can be of the most general form, and can therefore contain even the time t. Likewise forces are also admitted which depend on the velocities in the sense of Chap. 2, No. 10. For a single electron, for example, the Lagrange function in the most general case, that is, with regard to the special theory of relativity and under the influence of any given number of electrical and magnetic fields, which arise from the potentials and A, is

$$L = m_0 c^2\left(1 - \sqrt{1 - \frac{\mathfrak{v}^2}{c^2}}\right) + \frac{e}{c}\,\mathfrak{A}\mathfrak{v} - e\varphi. \tag{3}$$

The expression to the left in (2) one calls the variation derivative of L according to q_k. For the sake of brevity we

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1) S. Lie, Theorie der Transformationsgruppen, Bd. I-III, Leipzig 1888-1890, insbesondere Bd. II.

will designate it with the abbreviation $[L]_{qk}$:

$$L_{,q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}. \tag{4}$$

2. Reduction of the problem to the canonical form. We now take the first step and attempt to find a new simpler form for the variation problem. In formula (1) of No. 1, L is a function of q_k, $\dot{q}_k$ and possibly even t. Obviously we would get a simpler problem in a certain respect if we could eliminate the derivatives $\dot{q}_k$ as new variables to vary independently by placing

$$\dot{q}_k - k_k = 0 \tag{1}$$

The variation problem is then expressed

$$\int_{t_1}^{t_2} L(q_k, k_k, t)\,dt = \text{Extremum}, \tag{2}$$

whereby now, to be sure, the equations (1) are to be added as side conditions.

The latter can be treated in the known manner with the Lagrange factor method[1]. We multiply them with the still-to-be-determined factors and solve the absolute variations problem now with 3f unknowns:

$$\int_{t_1}^{t_2} \left\{L + \sum_k \lambda_k(\dot{q}_k - k_k)\right\} dt = \text{Extremum}. \tag{3}$$

Here one can determine the from the requirement that the

[1] In the present problem naturally also the neighboring curves should satisfy the side conditions. One has to make use of them even before the variation in contrast to the usual non-holonomis side conditions, in which the neighboring curves do not satisfy the side conditions as in Chapter 2, No. 20 and 27.

variation derivatives according to the new variable k_i

$$\left[L + \sum_k \lambda_k (\dot{q}_k - k_k)\right]_{k_l} = 0$$

must disappear. Since within the brackets the k_2 does not occur, the equations can be reduced to $\frac{\partial L}{\partial k_l} - \lambda_l = 0; \quad \lambda_l = \frac{\partial L}{\partial k_l}.$

Thus the λ_k are determined. One can introduce their value and obtain a free variation problem with 2f unknown functions

$$\int_{t_1}^{t_2} \left\{ L(q_k, k_k, t) + \sum_k \frac{\partial L}{\partial k_k} (\dot{q}_k - k_k) \right\} dt = \text{Extremum}. \tag{4}$$

Hereby the extremum is to be selected among all functions $q_k(t)$ and $k_k(t)$ whereby, however, no marginal conditions may be ascribed to the k_k since their derivatives do not enter into the integral and also (1) of No. 2 contains no conditions for the $\dot{q}_k$. That the requirement (4) is actually fully equivalent to (1) of No. 1 can be seen from the following. The conditions for the desired functions are

$$\left[L + \sum_k \frac{\partial L}{\partial k_k} (\dot{q}_k - k_k)\right]_{q_k} = 0,$$

$$\left[L + \sum_k \frac{\partial L}{\partial k_k} (\dot{q}_k - k_k)\right]_{k_k} = - \frac{\partial \left\{ L + \sum_k \frac{\partial L}{\partial k_k} (\dot{q}_k - k_k) \right\}}{\partial k_k} = - \frac{\partial^2 L}{\partial k_k^2} (\dot{q}_k - k_k) = 0.$$

Here the second line says that, aside from the singular cases $^2L/ k_k^2 = 0$ to be excluded here, $\dot{q}_k = k_k$. If we insert this into the first line then we return to the original form (1) of No. 2.

This proof of equality is necessary since in and of itself (3) viz (4) does not at all completely agree with (1) of No. 1. This is because the extremum in (1) of No. 1 is to be sought among all values which arise by inserting all arbitrary functions $q_k(t)$ in L. The q_k thereby are naturally included. In (3) on the other hand, the k_k are still to be taken as arbitrary functions. Correspondingly the region in which the extremum must be sought is a much broader one. Actually it can be shown as that in case the actual path curve makes the integral (1) of No. 1 atrue minimum, this with (4) can not be the case, but that then this integral assumes a saddle value in such a way that it, with at first fixed but arbitrarily chosen $q_k(t)$, is to be made a maximum with respect to $k_k(t)$ and only after condition the $q_k(t)$ are to be chosen so that then the integral becomes a minimum with respect to its variations. This has been shown by Hilbert in his lectures.

For the purposes of mechanics, however, the character of the extremum, that is, whether maximum, minimum or (as here) saddle value, is of no consequence. It is only important that the variation derivatives for the various forms of the variation problem become identical and therewith the curves, which make the integral an extreme-value, that is, the desired path curves. Therefore we will not go into this matter any further here, but will only mention that for sufficiently small regions the Hamiltonian integral (2) for the true motion becomes an actual minimum[1].

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[1] Siehe z. B. das in Anm. 1 von S. 92 zitierte Buch von Whittaker, Analytische Dynamik, S. 265.

Form (4) will be dealt with later. Here we will first go one step further by introducing in place of k_k as a new unknown the generalized momentum (see Chap. 2, No. 11).

$$p_k = \frac{\partial L(q_l, k_l)}{\partial k_k} = \frac{\partial L(q_l, \dot{q}_l)}{\partial \dot{q}_k} \tag{5}$$

By means of (5), the k_k become functions of p_k, q_k and possibly of t, and (4) receives the form $\int_{t_1}^{t_2}\left\{\sum_k p_k\dot{q}_k - H(p_k, q_k, t)\right\} dt = \text{Extremum},$

(6)

whereby $$H = -L + \sum_k k_k \frac{\partial L}{\partial k_k} = -L + \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}$$

(7)

signifies the so-called Hamiltonian function. Thereby in H the k_k are to be thought of as expressions of p_k, q_k, t. Equation (6) has now the simplest form which an absolute variation problem can assume in that only the derivatives of the one series of variables occur and there occur only linearly and multiplied with the other variables themselves. It is therefore called canonical. Correspondingly one calls the q_k and p_k also canonical variables and especially the p_k the canonically conjugated momenta to the q_k. A proof of equivalency of (4) with (6) is obtained by a direct transformation from (4).

From the variables p_k, q_k one moreover returns easily to the variables k_k (viz $\dot{q}_k$), q_k. For this we differentiate H partially according to p_k

$$\frac{\partial H}{\partial p_k} = \frac{\partial}{\partial p_k}\left(-L + \sum_l k_l p_l\right) = -\sum_l \frac{\partial L}{\partial k_l}\frac{\partial k_l}{\partial p_k} + \sum_l \frac{\partial k_l}{\partial p_k} p_l + k_k = k_k. \tag{8a}$$

From this we further get

$$H = -L + \sum_k k_k p_k = -L + \sum_k \frac{\partial H}{\partial p_k} p_k, \qquad L = -H + \sum_k \frac{\partial H}{\partial p_k} p_k. \tag{8b}$$

Therefore the change from L to H is of the same form as the reverse from H to L. One designates it as Legendre transformation which also plays an important role in many other areas of mathematics and physics. It produces, for example, in thermodynamics the transformation between the various thermodynamic potentials.

In the new variables the differential equation of the variation problem, that is the equations of motion of the system, are especially simple form. They are at first

$$\left[\sum_l p_l \dot{q}_l - H\right]_{p_k} = 0, \qquad \left[\sum_l p_l \dot{q}_l - H\right]_{q_k} = 0$$

and can be reduced, as one can see, to

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \qquad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}. \tag{9}$$

These are the so-called canonical equations of mechanics which are the starting point for most of the studies of higher dynamics. In place of the system of 2. order of the f Lagrange differential equations (2) of No. 1 for the q_k, they form a system of 1. order of 2f differential equations for the q_k and p_k. They are according to their derivation completely equivalent to the former.

One can perform the transformation of the differential equations of a mechanical system to the canonical form even if all side conditions are not eliminated but are carried along separately. If these side conditions are $\varphi_r(q_k, t) = 0$,

then the corresponding Hamiltonian equations are

$$\left.\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k}, \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k} + \sum_r \lambda_r \frac{\partial \varphi_r}{\partial q_k}. \end{aligned}\right\} \qquad (10)$$

If the conditions are of the non-holonomic form

$$\sum_r a_{rk} \delta q_k = 0,$$

then in place of the second row in (10) we place

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} + \sum_r \lambda_r a_{rk}. \qquad (10a)$$

However, the use of these equations gives us no advantage since their symmetry has been lost[1].

We now ask concerning the mechanical significance of quantity H. If, as is usually the case, the kinetic energy T is a homogeneous quadratic function of $\dot{q}_k$, then according to the Euler law for homogeneous functions the following is true:

$$T = \frac{1}{2} \sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k. \qquad (11)$$

And then, since L = T-U according to our assumptions:

$$\sum p_k \dot{q}_k = \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k = \sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T,$$

in case the potential energy V does not depend on the velocities. Accordingly then under the given limitations

[1] Siehe hierzu T. Pöschl, C. R. Bd. 156, S. 1829. 1913; S. Dautheville, S. M. T. Bull. Bd. 37, S. 120. 1909.

$$H = -L + \sum_k p_k \dot{q}_k = -T + U - 2T - T + U \quad (12)$$

is the total energy of the system.

The recipe for setting up the canonical equations is therefore as simple as can be imagined. One needs to know only the energy as a function of the coordinates and momenta in order to be able to write them down immediately. Acording to (12) it should be noted, however, that this simple mechanical meaning of H is only valid under the conditions of (11). For other cases, for example, with reference to a rotating coordinate system, H is no longer the energy, and one has to go back to equation (7) to determine the Hamiltonian function[1)]

A first integral of the motion equations is obtained immediately, if the Hamiltonian function does not explicitely contain the time. If one multiplies the canonical equations (9) with $\dot{q}_k$ or, as the case might be, with p_k, then from them

$$\frac{dH}{dt} = \sum_k \frac{\partial H}{\partial p_k}\dot{p}_k + \sum_k \frac{\partial H}{\partial q_k}\dot{q}_k = \sum_k \dot{q}_k\dot{p}_k - \sum_k \dot{p}_k\dot{q}_k = 0. \quad (13)$$

$$H = \text{const.} = W$$

is therefore an integral of the canonical equations. In the just mentioned simplest case, this is nothing but conservation of energy.

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1)About the Hamiltonian function and integration theory in relativistic mechanics see Chapter 10 of the volume of Handbook. Also see J. Frenkel, Lehrbuch der Elektrodynamik, Chapter 10, pp. 330 ff. Berlin 1926.

If, further, the Hamiltonian function does not explicitly contain a coordinate, i.e. q_1, then it follows immediately

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = 0, \qquad p_1 = \text{konst.} \tag{14}$$

We therefore again have an integral of the canonical equations. In the same manner, for example, the conservation law p = const. follows in the case of the Kepler motion, whose Hamiltonian function is written in plane polar coordinates r,

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2} p_\varphi^2\right) - \frac{a}{r}. \tag{15}$$

Certainly in connection with this example, in which has the meaning of the azimuth in the path plane, one calls such coordinates, of which the Hamiltonian function is independent, cyclic variables. This case always occurs when the energy of the random value of one coordinate is not affected, therefore, for example, is not changed when the entire system is translated or rotated. One obtains thus for free systems, for example, the center of gravity and surfaces. We will return to this in No. 9 and 11. (Cf. also No. 11 of the previous Chapter 2).

3. Canonical Transformations. We now turn to our second question and investigate what kinds of transformations of the variables can be made while preserving the canonical form of the motion equations.

We therefore look for substitutions

$$\left.\begin{aligned} q_i &= q_i(Q_k, P_k, t), \\ p_i &= p_i(Q_k, P_k, t), \end{aligned}\right\} \tag{1}$$

which changes the variation problem (6), No. 2 into an equivalent one with a new Hamiltonian function K

$$\int_{t_1}^{t_2}\Big\{\sum_k P_k\dot{Q}_k - K(P_k, Q_k, t)\Big\}\,dt = \text{Extremum} \tag{2}$$

Thereby it is not required that the two integrals themselves become identical but only that they assume their extremum at the same time; that means, when the integral (6) of No. 2 for the functions $q_k(t)$, $p_k(t)$ assumes its extreme-value, then the integral (2) for those functions $Q_k(t)$, $P_k(t)$ should do it also; these functions result from q_k and p_k by means of the substitution inverse to (1).

This is then and only then guaranteed when the two integrands differ only by the complete derivative of an otherwise arbitrary function (Q_k, P_k, t) according to t. For such a one the integral is independent of the path and produces in all cases with fixed integration limits a constant amount, which influences the occurance of an extremum in no way. The condition, which Q_k and P_k must fulfill, is stated thusly

$$\sum_k p_k\dot{q}_k - H = \sum_k P_k\dot{Q}_k - K + \frac{d\Phi}{dt}(P, Q, t). \tag{3}$$

This condition must naturally be true also for all non-mechanical, varied integration paths in the p, q, t space. Since now between the q_k no kinematic conditions are supposed to exist, then one can write more clearly for (3)

$$\sum_k p_k\Delta q_k - H\Delta t = \sum_k P_k\Delta Q_k - K\Delta t + \Delta\Phi \tag{4}$$

which condition must be fullfilled for any arbitrary choice of differentials Δq_k, ΔQ_k, Δt. Here $\Delta\Phi$ is explained by

$$\Delta\Phi = \sum_k \frac{\partial\Phi}{\partial Q_k}\Delta Q_k + \sum_k \frac{\partial\Phi}{\partial P_k}\Delta P_k + \frac{\partial\Phi}{\partial t}\Delta t$$

in which ΔP_k is already defered by Δq_k, ΔQ_k, Δ t since (with a definite Δ t) obviously between the 4f differentials Δq_k, Δp_k ΔQ_k, ΔP_k always the 2f relationships

$$\Delta q_k = \sum_l \frac{\partial q_k}{\partial Q_l}\Delta Q_l + \sum_l \frac{\partial q_k}{\partial P_l}\Delta P_l + \frac{\partial q_k}{\partial t}\Delta t,$$

$$\Delta p_k = \sum_l \frac{\partial p_k}{\partial Q_l}\Delta Q_l + \sum_l \frac{\partial p_k}{\partial P_l}\Delta P_l + \frac{\partial p_k}{\partial t}\Delta t$$

exist. The fundamental determinant of transformation (1) we naturally assume here to be $\neq 0$.

In order to obtain from (4) real conditions for the transformation equations (1), we introduce in Φ in place of P_k the q_k in that we think of the relationship

$$q_k = q_k(P_l, Q_l, t)$$

according to P_k as being solved for:

$$P_k = P_k(q_l, Q_l, t)$$

We assume that this solution is possible. Φ thereby changes into a function $V(q_k, Q_k, t)$. Then out of (4) we get

$$\sum_k p_k \Delta q_k - H(p_k, q_k, t)\Delta t = \sum_k P_k \Delta Q_k - K(P_k, Q_k, t)\Delta t + \Delta V(q_k, Q_k, t)$$

with mit

$$\Delta V = \sum_k \frac{\partial V}{\partial q_k}\Delta q_k + \sum_k \frac{\partial V}{\partial Q_k}\Delta Q_k + \frac{\partial V}{\partial t}\Delta t. \qquad (4a)$$

So that equation (4a) is satisfied identically, the factors of Δq_k, ΔQ_k, Δ t must be equal on both sides:

$$\left.\begin{aligned} p_k &= \frac{\partial V}{\partial q_k}, \\ P_k &= -\frac{\partial V}{\partial Q_k}, \\ K &= H + \frac{\partial V}{\partial t}. \end{aligned}\right\} \qquad (5)$$

Since one can calculate from the equations of the second line in general the q_k from those of the first and then the p_k as functions of P_k, Q_k, the equations (5) with an arbitrary choce of function $V(q_k, Q_k, t)$ always produce a canonical transformation, whereby the new Hamiltonian function K is given by the third line. The function V is called the generator of the transformation. The new canonical transformations are:

$$\frac{dP_k}{dt} = -\frac{\partial K}{\partial Q_k}, \quad \frac{dQ_k}{dt} = \frac{\partial K}{\partial P_k}; \quad K = H + \frac{\partial V}{dt}.$$

It especially V does not contain the time explicitly, then simply

$$K = H.$$

It is very remarkable that the canonical transformations are independent of the special mechanical problems. The property of a transformation to be canonical does not depend, therefore, at all on the nature of the considered problem, but is peculiar to it itself.

We have just favoured the variables q_k, Q_k in the generator V. We could just as well have taken any f of the variables q_k, p_k and f of the Q_k, P_k. The general result can then be expressed[1]: If $V(x_k, X_k, t)$ is an arbitrary function of 2f + 1 variables x_k, X_k, t whereby the X_k (K = 1, ... f) are any of the variables q_k, p_k, the X_k are any of the Q_k, P_k, then

[1] Siehe M. Born, Vorlesungen über Atommechanik, S. 35. Berlin 1925; vgl. außerdem die Einzelausführungen im folgenden Kap. 4, Ziff. 3, ds. Bd. des Handbuchs.

$$\left.\begin{aligned} y_k &= \pm \frac{\partial V}{\partial x_k}, \\ Y_k &= \mp \frac{\partial V}{\partial X_k}, \\ K &= H + \frac{\partial V}{\partial t} \end{aligned}\right\} \tag{6}$$

is a canonical transformation. Thereby y_k is conjugated with x_k, Y_k with X_k and the upper symbol is valid when differentiating according to the coordinates, the lower when differentiating according to momentum. Very often the canonical transformation is used the form

$$\left.\begin{aligned} V &= V(q_k, P_k, t), \\ p_k &= +\frac{\partial V}{\partial q_k}, \\ Q_k &= +\frac{\partial V}{\partial P_k}. \end{aligned}\right\} \tag{5a}$$

Each one of the transformations of the position coordinates above

$$q^K = q^K(Q_l, t)$$

which is designated as a point transformation, since it changes each point in the locus of q_k into such a one, is also canonical. One needs only to take as a transformation function

$$V = -\sum q_k(Q_l) p_k \tag{7}$$

and then according to (6)

$$q_k = -\frac{\partial V}{\partial p_k} = q_k(Q_l).$$

The identical transformation is contained within

$$V = -\sum_k Q_k p_k \tag{8}$$

Above and beyond this the theory of canonical transformations permits the introduction of general dynamic coordinates in such an extraordinarily free manner that their choice can be adapted very exactly to each problem. With the general transformations

(6) naturally the character of the variables $Q_k P_k$ as location and momenta coordinates is lost. Only in their totality do they give a picture of location and motion states of the system under consideration. Because of their mathematical relationship with the tangential transformations of geormetry, these transformations are frequently given the name tangential transformations.

One can also list other canonical transformations which fulfill certain side conditions if the latter can be brought into the form of a relationship between the old and the new coordinates

$$\Omega_r(q_k, Q_k, t) = 0 \tag{9}$$

These can simply be listed with the identity (4) with Lagrange multiplicators and one obtains then as determiner equations of the corresponding canonical transformations

$$\left.\begin{aligned} P_k &= \frac{\partial V}{\partial Q_k} + \sum_r \lambda_r \frac{\partial \Omega_r}{\partial Q_k}, \\ p_k &= -\frac{\partial V}{\partial q_k} - \sum_r \lambda_r \frac{\partial \Omega_r}{\partial q_k}, \\ K &= H + \frac{\partial V}{\partial t} + \sum_r \lambda_r \frac{\partial \Omega_r}{\partial t}, \end{aligned}\right\} \tag{10}$$

which together with relationships (9) are sufficient to determine the quantities q_k, p_k, λ_r as functions of Q_k, P_k. A special case of this is, for example, the existence of a side condition

$$\varphi(q_k, t) = 0$$

for the original coordinates.

Finally one would have been able to multiply the left side of (3) also with a constant factor λ without destroying the property of the transformation to be canonical. That leads us, for example, to transformations of the type

$$P_k = p_k, \quad Q_k = \lambda q_k, \quad K = \lambda H, \tag{11}$$

which are often used. On the other hand the general form of the tangential transformation usual to geometry-- λ an arbitrary function of the variables--is not applicable here.

The canonical transformations are, as stated, independent of the choice of the special Hamiltonian function. If one therefore wants to have only the conditions for the transformations of p_k, q_k into P_k, Q_k, then one can limite himself in (4) to the variations with $\Delta t = 0$, that is, treat t as a constant parameter. If we designate these variations for the sake of distinction with a δ , then one can write the conditions for canonical transformations also in the form

$$\sum_k p_k \delta q_k = \sum_k P_k \delta Q_k + \delta \Phi(P_k, Q_k, t) \tag{12}$$

in which no reference is made at all to the special mechanical problem. The variations Δ and δ are thereby in each case explained by [1)]

$$\left.\begin{aligned} \Delta F(p_k, q_k, t) &= \sum_k \frac{\partial F}{\partial q_k} \Delta q_k + \sum_k \frac{\partial F}{\partial p_k} \Delta p_k + \frac{\partial F}{\partial t} \Delta t, \\ \delta F(p_k, q_k, t) &= \sum_k \frac{\partial F}{\partial q_k} \delta q_k + \sum_k \frac{\partial F}{\partial p_k} \delta p_k \end{aligned}\right\} \tag{13}$$

Equation (12) has thereby for the characterization of the transformation the same degree of generality as (4) and one needs

[1)] The symbols Δ and δ are chosen in analogy with the general and virtual shifts in Chapter 2, No. 23. The difference is merely that now also p_k can be varied since it also appears as a variable in the variation problem.

the latter form only to designate the new Hamilton function. Naturally one can introduce also in ϕ as previously in place of P the q and obtain the explicit transformation equations (5) with the help of the function $V(q_k, Q_k, t)$.

With the introduction of the canonical transformations the most important step in the integration theory of mechanical equations is already taken. This will be presented in No. 12ff. In order to understand them a knowledge of Nos. 4 to 11, which give further explanation about the properties of canonical transformations, is not absolutely necessary. These can therefore be skipped during preliminary study.

4. Introduction of time as canonical variable. Above and beyond the canonical variation problem one can arrive at a still more systematic form of the general variation principle of mechanics by divesting time of its special role. First one can formally eliminate from the integral in equation (6), No. 2 the Hamiltonian function $H(p, q, t)$ still remaining there by adding a side condition and requiring

$$\int\Big(\sum_k p_k\dot{q}_k - W\Big)dt = \text{Extremum} \tag{1}$$

among the side conditions

$$W = H(p, q, t)$$

If we now introduce in place of t a new parameter τ, $t = t(\tau)$, the arc of the path curve or in the theory of relativity the Eigenzut (proper time) then we get the form

$$\int\left\{\sum_k p_k \frac{dq_k}{d\tau} - W\frac{dt}{d\tau}\right\} d\tau = \text{Extremum},$$

with (2)

$$W = H(p, q, t)$$

as side condition. This form suggests that we introduce t itself as a new canonical variable $\mathfrak{q}$, to which the $\mathfrak{p}$ = -W is conjugated as momentum. Through this we obtain the absolutely symetrical form

$$\int\{\sum p_k q'_k + \mathfrak{p}\mathfrak{q}'\} d\tau = \text{Extremum}, \tag{3}$$

while

$$F(p_k, q_k, \mathfrak{p}, \mathfrak{q}) = H + \mathfrak{p} = H - W = 0,$$

(4)

In this the dash characterizes the derivative according to .
The mechanical system is then no longer characterized by a function, the Hamiltonian function, but an equation, namely

$$F(p_k, q_k, \mathfrak{p}, \mathfrak{q}) = H - W = 0 \tag{4}$$

between the 2f + 2 canonical variables and momenta. This form of the variation problem can also be applied, for example, to the theory of relativity. In general, in place of F $\equiv$ H-W an arbitrary function F(p, q, W, t) = 0 can appear, but through solving according to W always the canonical form (4) can be obtained.

The general motion equations become according to the multiplicator rule of No. 2

$$\left.\begin{aligned} \frac{dq_k}{d\tau} &= +\lambda\frac{\partial F}{\partial p_k}, & \frac{dt}{d\tau} &= +\lambda\frac{\partial F}{\partial \mathfrak{p}} \equiv -\lambda\frac{\partial F}{\partial W}, \\ \frac{dp_k}{d\tau} &= -\lambda\frac{\partial F}{\partial q_k}, & \frac{d\mathfrak{p}}{d\tau} &\equiv -\frac{dW}{d\tau} = -\lambda\frac{\partial F}{\partial t}, \end{aligned}\right\} \tag{5}$$

which for the canonical form F = H-W because

$$\frac{dt}{d\tau} = -\lambda\frac{\partial F}{\partial W} = -\lambda\frac{\partial(H-W)}{\partial W} = \lambda$$

can be reduced to the usual canonical equations

$$\left.\begin{aligned} \frac{dq_k}{d\tau}\frac{d\tau}{dt} &= \frac{\partial(H-W)}{\partial p_k} = \frac{\partial H}{\partial p_k}, & \frac{dW}{d\tau}\frac{d\tau}{dt} &= \frac{\partial(H-W)}{\partial t} = \frac{\partial H}{\partial t}, \\ \frac{dp_k}{d\tau}\frac{d\tau}{dt} &= -\frac{\partial(H-W)}{\partial q_k} = -\frac{\partial H}{\partial q_k}, & \frac{dt}{d\tau} &= \lambda \end{aligned}\right\} \qquad (6)$$

Also the canonical equations can be generalized so that they include time. For this the necessary and sufficient condition obviously that the differential form

$$\sum_k p_k \Delta p_k + \mathfrak{p}\Delta t,$$

with which the variables p_k, q_k, $\mathfrak{p}$, t are still joined by the side condition $H + \mathfrak{p} = 0$

(7)

should change to a differential form $\sum P_k \Delta Q_k + \mathfrak{P}\Delta T + \Delta\Phi$

the variables of which are joined by a corresponding side condition: $K + \mathfrak{P} = 0$

Thus each arbitrary canonical transformation of 2f + 2 variables q_k, p_k, t, $\mathfrak{p}$ renders into Q_k, P_k, T, $\mathfrak{P}$, which therefore are generated by an arbitrary function $V^*(q_k, Q_k, t, T)$. Thereby the function is to be so designated that one undertakes the transformation in equation (7), solves the thus obtained relationship and finds $\mathfrak{P} = -K(Q_k, P_k, T)$

If especially t is not to be transformed, that is, if t changes to T, then V* has the form $V^* = \mathfrak{P}t + V(q_k, Q_k, t)$,

since then according to No. 3, equation (6)

that is
$$T = \frac{\partial V^*}{\partial \mathfrak{P}} = t, \qquad \mathfrak{p} = -W = \frac{\partial V^*}{\partial t} = \mathfrak{P} + \frac{\partial V}{\partial t},$$
$$-\mathfrak{P} = K(Q_k, P_k, t) = W + \frac{\partial V}{\partial t} = H + \frac{\partial V}{\partial t}$$

One thus returns to the formulas of No. 3.

5. Integral invariants. As with every transformation, the question as to the invariants is of great importance also with the canonical transformations, that is, the question as to functions which do not change their value in the transformation. One can give a series of such invariants of all canonical transformations. We will discuss next the integral invariants first considered by Poincaré[1].

The integral
$$J_1 = \iint \sum_k dp_k dq_k, \tag{1}$$

extended over an arbitrary two dimensional area of the 2f--dimensional phase space of the p_k and q_k is an invariant of the canonical transformations. To prove that, we set up this two dimensional area so that we give p_k and q_k as function of two

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1) H. Poincaré, Les méthodes nouvelles de la mécanique céleste, Bd. III, Kap. 22/24. Paris 1899. Beweis nach E. Brody, ZS. f. Phys. Bd. 6, S. 224, 1921.

parameters u and v.

$$J_1 = \iint \sum_k \begin{vmatrix} \frac{\partial p_k}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial p_k}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix} du\, dv. \tag{2}$$

The canonical transformations are taken in the form

$$\left.\begin{aligned} p_k &= \frac{\partial V(q_k, P_k, t)}{\partial q_k}, \\ Q_k &= \frac{\partial V(q_k, P_k, t)}{\partial P_i} \end{aligned}\right\} \tag{3}$$

and by means of q the equations of the first line, change the p_k as functions of the q_k, P_k into J_1, whereby the value of t in (3) remains constant, therefore t is to be treated as a constant parameter. Then

$$\sum_k \begin{vmatrix} \frac{\partial p_k}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial p_k}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix} = \sum_k \begin{vmatrix} \sum_l \frac{\partial^2 V}{\partial q_k \partial P_l} \frac{\partial P_l}{\partial u} & \frac{\partial q_k}{\partial u} \\ \sum_l \frac{\partial^2 V}{\partial q_k \partial P_l} \frac{\partial P_l}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix} = \sum_{kl} \frac{\partial^2 V}{\partial q_k \partial P_l} \begin{vmatrix} \frac{\partial P_l}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial P_l}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix}$$

By exchanging the indicator we get for this

$$\sum_{lk} \frac{\partial^2 V}{\partial q_l \partial P_k} \begin{vmatrix} \frac{\partial P_k}{\partial u} & \frac{\partial q_l}{\partial u} \\ \frac{\partial P_k}{\partial v} & \frac{\partial q_l}{\partial v} \end{vmatrix}.$$

If we now with the help of the seconf row of equations (3) change the q_k, P_k into Q_k, P_k, then we get

$$\sum_k \begin{vmatrix} \frac{\partial P_k}{\partial u} & \sum_l \frac{\partial^2 V}{\partial P_k \partial q_l} \frac{\partial q_l}{\partial u} \\ \frac{\partial P_k}{\partial v} & \sum_l \frac{\partial^2 V}{\partial P_k \partial q_l} \frac{\partial q_l}{\partial u} \end{vmatrix} = \sum_k \begin{vmatrix} \frac{\partial P_k}{\partial u} & \frac{\partial Q_k}{\partial u} \\ \frac{\partial P_k}{\partial v} & \frac{\partial Q_k}{\partial v} \end{vmatrix}.$$

And then it finally becomes

$$\sum_k \begin{vmatrix} \frac{\partial p_k}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial p_k}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix} = \sum_k \begin{vmatrix} \frac{\partial P_k}{\partial u} & \frac{\partial Q_k}{\partial u} \\ \frac{\partial P_k}{\partial v} & \frac{\partial Q_k}{\partial v} \end{vmatrix}, \tag{4}$$

with which the invariance of the integral (1) is proven.

Analoguusly we can prove that invariance of

$$J_2 = \iiiint \sum_{kl} dp_k\, dp_l\, dq_k\, dq_l \tag{5}$$

and generally that of

$$J_n = \int \overset{2n}{\cdots} \int \sum_{k_1 \ldots k_n} dp_{k_1} \ldots dp_{k_n} dq_{k_1} \ldots dq_{k_n} \tag{6}$$

The last integral of this series is the volume in the phase space of p_k and q_k.

$$J_f = \int \overset{2f}{\cdots} \int dp_1 \ldots dp_f dq_1 \ldots dq_f, \tag{7}$$

which is therefore also an invariant with respect to canonical transformations. Thus it is simultaneously demonstrated that the fundamental determinant of a canonical transformation is equal to (1).

As we will show later (No. 9), the time change of the coordinates and momenta of a mechanical system can also be regarded as a canonical transformation of the same. All invariants of canonical transformations are therefore also motion invariants. This is so to be understood, that the points of the corresponding 2n-dimensional areas in the phase space are to be thought of as image points of a corresponding multiplicity of similar mechanical systems with somewhat different initial positions. Through the motion of these systems the original value region of the p, q over which we are to integrate is changed into a different arc which, according to our law, has the same volume. In the pqt-space therefore the world line of these systems forms a tube of constant diameter. For J_f this is the Liouville principle fundamental to statistical mechanics.

The integral invariants (1) and (6) to (7) are called absolute, because in them over the integral region no prerequisites are made. They can be changed with the help of the multi-dimensional generalizations of the Stoke principle into relative, that is, over the closed integral region into extending integral invariants, whose order, that is, number of integrations, is lower. For example, in place of (1) comes the invariance of the integral to be taken over a closed curve of the pq-space (which would have to be in the pqt-space on a plane t = const.).

$$J_1 = \oint \sum_k p_k dq_k . \qquad (8)$$

From the existence of the integral invariant (8) viz (2) for a system of transformation equations

$$\left.\begin{array}{l} q_i = q_i(Q_k P_k t), \\ p_i = p_i(Q_k P_k t) \end{array}\right\} \qquad (9)$$

it follows inversely, as will be shown in No. 6, that they can be brought into form No. 3, equation (6), and that therefore the used transformation is canonical.

If one choses as integration region in (1) that of two infinitesimal vectors of the pq-space whose components are dq_k, dp_k, viz. δq_k, δp_k, a stretched parallelogram, then the invariance of the bilinear covariants which belong to the differential form $\sum p_k dq_k$ follows

$$\sum_k (\delta p_k dq_k - dp_k \delta q_k). \qquad (10)$$

Also their invariance is, according to what has been said, sufficient for the canonical nature of a transformation. Moreover, the invariance of (10) according to that which we said about equation (3) is only true when either V is independent of t or the two small vectors together with their images in the PΩt-space lie on plane t = const., that is, when they are δ - variations in the sense of No. 3. Otherwise (10) is not invariant, but the covariant belonging to the differential form

$\sum p_k \, dq_k$ - Hdt

$$\sum_k (\Delta p_k dq_k - dp_k \Delta q_k) - (\Delta H dt - dH \Delta t). \tag{11}$$

6. <u>The conditions for canonical transformations, expressed by means of the Lagrange and the Poisson-Jacobi bracket symbols.</u>

One designates the expressions in No. 5(4)

$$\left.\begin{aligned} [u, v] &= \sum_k \left(\frac{\partial q_k}{\partial u}\frac{\partial p_k}{\partial v} - \frac{\partial p_k}{\partial u}\frac{\partial q_k}{\partial v}\right) \\ &= -\sum_k \begin{vmatrix} \frac{\partial p_k}{\partial u} & \frac{\partial q_k}{\partial u} \\ \frac{\partial p_k}{\partial v} & \frac{\partial q_k}{\partial v} \end{vmatrix} \end{aligned}\right\} \tag{1}$$

as Lagrange bracket expressions. They are, as we saw there, invariant with respect to canonical transformations. Under u and v were understood in No. 5 any parameters coordinate with the coordinate values of a two-dimensional sections of the pq-space. As such the coordinate values themselves may naturally serve. This leads to the equations

$$\left.\begin{aligned} [p_i, p_k] &= [q_i, q_k] = 0, \\ [q_i, p_k] &= \delta_{ik} = \begin{vmatrix} 0 & \text{für} & i \neq k \\ 1 & \text{für} & i = k \end{vmatrix}. \end{aligned}\right\} \tag{2}$$

Their invariance signifies the correctness also of the equations

$$\left.\begin{aligned} [P_i, P_k] &= [Q_i, Q_k] = 0, \\ [Q_i, P_k] &= \delta_{ik}, \end{aligned}\right\} \tag{3}$$

whenever the transformation (p, q) $\longrightarrow$ (P, Q) is canonical. Conversely equations (3) are sufficient to assure the canonical character of the transformations as we will soon show. They are therefore the characteristic differential equations which the p, q as functions of P, Q must satisfy, so that the transformation is canonical. The proof is stated as follows:

Equations (3) in their complete form are

$$[Q_k, P_j] = \sum_l \left(\frac{\partial q_l}{\partial Q_k}\frac{\partial p_l}{\partial P_j} - \frac{\partial p_l}{\partial Q_k}\frac{\partial q_l}{\partial P_j}\right) = \delta_{jk},$$
$$[Q_k, Q_j] = \sum_l \left(\frac{\partial q_l}{\partial Q_k}\frac{\partial p_l}{\partial Q_j} - \frac{\partial p_l}{\partial Q_k}\frac{\partial q_l}{\partial Q_j}\right) = 0,$$
$$[P_k, P_j] = \sum_l \left(\frac{\partial q_l}{\partial P_k}\frac{\partial p_l}{\partial P_j} - \frac{\partial p_l}{\partial P_k}\frac{\partial q_l}{\partial P_j}\right) = 0.$$

They can be rewritten as follows

$$\frac{\partial}{\partial P_j}\left(\sum_l p_l \frac{\partial q_l}{\partial Q_k} - P_k\right) - \frac{\partial}{\partial Q_k}\left(\sum_l p_l \frac{\partial q_l}{\partial P_j}\right) = 0,$$
$$\frac{\partial}{\partial Q_j}\left(\sum_l p_l \frac{\partial q_l}{\partial Q_k} - P_k\right) - \frac{\partial}{\partial Q_k}\left(\sum_l p_l \frac{\partial q_l}{\partial Q_j} - P_j\right) = 0,$$
$$\frac{\partial}{\partial P_j}\left(\sum_l p_l \frac{\partial q_l}{\partial P_k}\right) - \frac{\partial}{\partial P_k}\left(\sum_l p_l \frac{\partial q_l}{\partial P_j}\right) = 0.$$

These equations mean, however, that a function $\Phi(Q_k, P_k, t)$ exists for which

$$\sum_l p_l \frac{\partial q_l}{\partial Q_k} - P_k = \frac{\partial \Phi}{\partial Q_k}$$

and

$$\sum_l p_l \frac{\partial q_l}{\partial P_k} = \frac{\partial \Phi}{\partial P_k}.$$

If one now forms the δ-variations of Φ

$$\delta\Phi = \sum_k \frac{\partial \Phi}{\partial Q_k}\delta Q_k + \sum_k \frac{\partial \Phi}{\partial P_k}\delta P_k.$$
$$= \sum_{kl} p_l \frac{\partial q_l}{\partial Q_k}\delta Q_k + \sum_{kl} p_l \frac{\partial q_l}{\partial P_k}\delta P_k - \sum_k P_k \delta Q_k,$$

and considers

$$\delta q_l = \sum_k \frac{\partial q_l}{\partial Q_k}\delta Q_k + \sum_k \frac{\partial q_l}{\partial P_k}\delta P_k,$$

then one gets $\delta\Phi = \sum_k p_l \delta q_l - \sum_k P_k \delta Q_k .$

Therefore for the transformation formula

$$q_k = q_k(Q_l, P_l, t), \qquad p_k = p_k(Q_l, P_l, t) \tag{4}$$

the relationship (12), No. 3:

$$\sum_k p_k \delta q_k = \sum_k P_k \delta Q_k + \delta\Phi(P, Q, t).$$

With other words, transformation (4) is canonical.

With this we have demonstrated the proof for the claim stated earlier in cannection with No. 5 (8) that the existence of the invariants No. 5 (8) or No. 5 (2) is sufficient to assure the canonical character of transformation (4). Those invariants are derived from equations (3).

Closely related to the Lagrange bracket expressions are the symbols named after Poisson or Jacobi.

$$(u, v) = \sum_k \left(\frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} \right). \tag{5}$$

The relationship of the two exists in the fact that for any 2f independent functions $u_1, \ldots . u_{2f}$ of the p_k, q_k these equations are valid:

$$\sum_{t=1}^{2f} (u_t, u_r)\,[u_t, u_s] = \delta_{rs}. \tag{6}$$

They can be confirmed immediately by direct calculation taking into account that the sums $\sum_{t=1}^{2f} \frac{\partial u_t}{\partial x} \frac{\partial y}{\partial u_t}$

only then can differ from zero and are equal to one when x and y mean the same as the quantities p_k, q_k.

Equations (3) and (6) produce as further necessary and sufficient properties of a canonical transformation the system

$$(P_i, P_k) = 0, \quad (Q_i, Q_k) = 0, \\ (Q_i, P_k) = \delta_{ik}, \tag{7}$$

by taking for the u_t the P_k and Q_k themselves. They produce the differential equations which the new variables P, Q must fulfill as functions of the original p, q (therefore inverse formula of the transformation) so that this is canonical. Equations (7) are synonomous with the unvariance of the corresponding special bracket symbols. With the help of (6), however, the invariance of the Poisson bracket (u, v) is proven for any two functions u and v of $q_k p_k$ from the invariance of [u, v].

7. Further properties of the bracket symbols; the laws (principles) of Poisson and Lagrange. The Poisson bracket symbols have recently attained special significance as the result of their introduction into quantum mechanics [1]. Therefore a few further calculating rules and laws related to them will be listed here.

First, according to definition (5) of No. 6 the identities hold

$$\left.\begin{aligned} &(u, u) = 0, \qquad (u, v) = -(v, u), \\ &\frac{\partial u}{\partial q_j} = (u, p_j) = -(p_j, u), \quad \frac{\partial u}{\partial p_j} = (q_j, u) = -(u, q_j). \end{aligned}\right\} \tag{1}$$

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[1]) Vgl. besonders die Arbeiten von P. A. M. Dirac in den Proc, Roy, Soc. London (A), Bd. 109, S. 642. 1925; 110, S. 561, 1926; 111, S. 281, 405, 1926.

$$(u, (v, w)) + (v, (w, u)) + (w, (u, v)) = 0 \qquad (2)$$

The left side is obviously linear and homogeneous in the second derivatives of u, v, w. We now take together the members which contain the second derivatives of u. The first member of (2) certainly contains only first derivatives. The second and third can be written according to (1) in the form

$$(v, (w, u)) + (w, (u, v)) = (v, (w, u)) - (w, (v, u))$$

If we introduce the differential operators

$$D_1(f) = (v, f), \quad D_2(f) = (w, f)$$

then the members which can contain the second derivatives can be brought together in the form

$$(D_1D_2 - D_2D_1)\, u$$

Such a combination of two linear differential operators never contains two derivatives. If for example

$$D_1 = \sum_k \xi_k \frac{\partial}{\partial x_k}, \qquad D_2 = \sum_k \eta_k \frac{\partial}{\partial x_k},$$

then

$$D_1 D_2 = \sum_{kl} \xi_k \eta_l \frac{\partial^2}{\partial x_k \partial x_l} + \sum_{kl} \xi_k \frac{\partial \eta_l}{\partial x_k} \frac{\partial}{\partial x_l},$$

$$D_2 D_1 = \sum_{kl} \eta_k \xi_l \frac{\partial^2}{\partial x_k \partial x_l} + \sum_{kl} \eta_k \frac{\partial \xi_l}{\partial x_k} \frac{\partial}{\partial x_l}.$$

Therefore

$$D_1 D_2 - D_2 D_1 = \sum_l \left[\sum_k \left(\xi_k \frac{\partial \eta_l}{\partial x_k} - \eta_k \frac{\partial \xi_l}{\partial x_k} \right) \right] \frac{\partial}{\partial x_l}$$

is also an operator which contains only first derivatives. It follows that in (2) absolutely no members with the second derivatives of u can enter, and since the same must apply for v and w, then the entire expression must disappear identically. Equation (2) is the so-called Jacobi identify.

As the result of (1) it is possible to give the canonical motion equations [cf. No. 2 (9)]

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}, \qquad \dot{q}_k = \frac{\partial H}{\partial p_k} \tag{3}$$

in the form

$$\dot{p}_k = (p_k, H), \qquad \dot{q}_k = (q_k, H) \tag{4}$$

which is used in quantum mechanics in an obvious transcription.

If one considers (3) then one sees further that for every integral F(q, p) = a of motion which does not contain t explicitly

$$(F, H) = 0 \tag{5}$$

This statment means, namely, only that the gradient of the hypersurface F(q, p) = a in the 2f-dimensional pq-space on the phase path element

$$dq_k = \dot{q}_k dt = \frac{\partial H}{\partial p_k} dt$$
$$dp_k = \dot{p}_k dt = -\frac{\partial H}{\partial q_k} dt$$

stands vertical. The element thus lies entirely in the surface.

Finally we will derive still another unusual and important principle from Poisson which, however, was first recognized by Jacobi for its complete significance. He made it possible in a few cases to find new integrals of the mechanical equations. He said: If F = const. and G = const. are two time - independent integrals ofthe canonical equations (3), then their Poisson bracket expression

$$(F, G) = \sum_k \left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right) = \text{konst.}, \tag{6}$$

is also an integral.

The proof follows directly from (2) if one remembers that according to (5)

$$(H, F) = 0 \quad \text{and} \quad (H, G) = 0.$$

It follows that

$$(H, (F, G)) = 0. \tag{7}$$

that is, also (F, G) = const, is an integral of the canonical equations.

Naturally through this process one does not always get new integrals since there is only a limited number of them, but one the other hand, one gets quite often only a trivial one or one which is a function of the two first F, G.

Also for the Lagrange brackets there is an analog to theorum (6). If we use the already-mentioned theorum, which we will prove later, that the coordinate change of a mechanical system in the course of its motion can be regarded as the development of a canonical transformation, then one receives from the invariance of the brackets the theorum of Lagrange. It says that for any two-dimensional solution grouping

$$q_j = q_j(a, b, t), \qquad p_j = p_j(a, b, t)$$

of the canonical equations where therefore a and b are arbitrary integration constants, for all times, that is, along the entire mechanical path the corresponding Lagrange brackets are

$$[a, b] = \text{const}. \tag{8}$$

of the canonical equations where therefore a and b are arbitrary

All above theorums can be generalized easily to systems, viz integrals which contain time explicitly by regarding according to No. 4 time also as a canonical variable. As definition for the Poisson brackets, which we now write with braces to distinguish them, one now has

$$\left.\begin{aligned}\{u, v\} &= (u, v) - \frac{\partial u}{\partial t}\frac{\partial v}{\partial W} + \frac{\partial u}{\partial W}\frac{\partial v}{\partial t} \\ &= (u, v) + \frac{\partial u}{\partial t}\frac{\partial v}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial v}{\partial t}\,.\end{aligned}\right\} \tag{9}$$

Correspondingly one can also extend the Lagrange brackets. The considerations of this No. and of No. 6 can then be literally transfered, only that instead of H we must write H - W viz H + .

Form (4) of the canonical equations therefore now is

$$\left.\begin{aligned}\dot p_k &= \{p_k, (H - W)\} = -\frac{\partial(H-W)}{\partial q_k} = -\frac{\partial H}{\partial q_k}, \\ \dot q_k &= \{q_k, (H - W)\} = \frac{\partial(H-W)}{\partial p_k} = \frac{\partial H}{\partial p_k}, \\ \dot t &= \{t, (H - W)\} = 1, \\ \dot W &= \{W, (H - W)\} = \frac{\partial(H-W)}{\partial t} = \frac{\partial H}{\partial t}.\end{aligned}\right| \tag{10}$$

From them it follows for arbitrary functions $F(p_k, q_k, W, t)$

$$\dot F = \sum_k \left(\frac{\partial F}{\partial q_k}\dot q_k + \frac{\partial F}{\partial p_k}\dot p_k\right) + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial W}\dot W = \{F, (H - W)\}. \tag{11}$$

Each integral of the motion equations fulfills thus the condition analogous to (5)

$$\{F, (H - W)\} = 0, \tag{12}$$

which for integrals independent of W reduces to

$$(F, H) + \frac{\partial F}{\partial t} = 0 \tag{13}$$

The Poisson theorum says now that with F = const, and G = const. also

$$\{F, G\} = \text{konst.} \tag{14}$$

is an integral of the canonical equations (10). From equation (14) there follows a simple form (6) if only F and G are both independent of W. The limitation to time-independent integrals is therefore not essential for (6).

8. Continuous transformation groups. The question as to the significance which the integrals of the canonical equations have for the problem of variation, can be treated in a very elegant manner with the help of the theory of transformation groups. On this subject we must state in advance a few theorums.

We change the mechanical system of a transformation to the form[1)]

$$\left.\begin{aligned} P_k &= P_k(p_l, q_l, \alpha) = p_k + \sum_{n=1}^{\infty} \alpha^n p_k^{(n)}(p_l, q_l), \\ Q_k &= Q_k(p_l, q_l, \alpha) = q_k + \sum_{n=1}^{\infty} \alpha^n q_k^{(n)}(p_l, q_l). \end{aligned}\right\} \quad (1)$$

This transformation thus contains still another parameter according to which it can be developed in a power series and changes for $\alpha = 0$ into the identical transformation. If α is very small, then we have a transformation in the vicinity of the identical. One calls it then an infinitesimal transformation. For every value of x we have a definite transformation. Through (1), therefore, a whole set of transformations is determined.

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1)Here it makes no difference whether or not one takes the p_k, q_k or the P_k, Q_k as the original variables. For the sake of convenience in No. 9 we use the above form which agrees with the solution of a transformation $p_k = p_k(P, Q)$, $q_k = q_k(P, Q)$.

We want to require of these transformations first that they form a set, that is, that two of the transformations with any values α_1, α_2 stated one after the other, again produce a transformation of the set. Lie[1] has shown that the linear members of the development (1), which we will designate with $\mathfrak{p}_k$, $\mathfrak{q}_k$, on the basis of this requirement determine completely all following members also, and thus alone are characteristic of the transformation. To a set of such members belong therefore only one group. A proof of this would take us too far afield. We limit ourselves to listing the transformations, and thus to showing how one obtains the higher members from those of the first order.

One form with the help of $\mathfrak{p}_k$, $\mathfrak{q}_k$ the following differential operator:

$$D = \sum_k \mathfrak{p}_k \frac{\partial}{\partial p_k} + \sum_k \mathfrak{q}_k \frac{\partial}{\partial q_k}, \tag{2}$$

which one designates as generating symbol of the set. With $\mathfrak{p}_k$, $\mathfrak{q}_k$ thus also D is given. One can define in three different ways the transformations forming the set, which naturally lead to identical results.

a) One forms the series

$$\left.\begin{aligned} P_k = [p_k] \equiv p_k + \alpha D p_k + \frac{\alpha^2}{2} D^2 p_k + \cdots = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} D^n p_k, \\ Q_k = [q_k] = q_k + \alpha D q_k + \frac{\alpha^2}{2} D^2 q_k + \cdots = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} D^n q_k; \end{aligned}\right| \tag{3}$$

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1) S. Lie, Theorie der Transformationsgruppen, Bd. I, S. 51 ff., Leipzig 1888.

whereby the D^n are operators which arise through n applications of D. For the sake of abbreviation we introduce the symbol

$$[F] = \sum_{n=0} \frac{\alpha^n}{n!} D^n F \qquad (4)$$

The series (5) are therefore only definable through differentiation and multiplication with the help of ρ_k, q_k, as we can easily show, even for sufficiently small α convergent. For an arbitrary function $F(p_k, q_k)$ it is true also that

$$F(P_k, Q_k) = F([p_k], [q_k]) = [F(p_k, q_k)]. \qquad (5)$$

From the representation of (3) one can also see that the general transformation (1) can be built up by continuous repetition of the linear (infinitesimal) transformation

$$P_k = p_k + \alpha \rho_k, \qquad Q_k = q_k + \alpha q_k$$

(b) One forms the partial differential equation for the function F of 2f + 1 variables p_k, q_k,

$$\frac{\partial F}{\partial \alpha} = DF = \sum_k p_k \frac{\partial F}{\partial p_k} + \sum_k q_k \frac{\partial F}{\partial q_k}, \qquad (6)$$

and looks for thos integrals $F(p_k, q_k, \alpha)$ which for $\alpha = 0$ change themselves into the variables p_k, q_k. Then the integrals designated thus 2f $P_k(\alpha, p_1, q_1)$ $Q_k(\alpha, p_1, q_1)$ are exactly the desired transformation functions. That this definition agrees with the first can be seen from the definition (4) according to which for every function [F] it follows that

$$D[F] = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} D^{n+1} F,$$

$$\frac{\partial}{\partial \alpha}[F] = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} D^{n+1} F$$

Every function [F] satisfies by itself the differential equation (6). Therefore the functions P_k (p_1, q_1), Q_k (p_1, q_1) defined in both ways must agree also for $\alpha = 0$ which clearly defines them together with differential equation (6).

(c) The functions describing the transformation are also the solutions of the system of 2f ordinary differential equations

$$\left.\begin{aligned} \frac{dP_k}{d\alpha} &= \mathfrak{p}_k(P_l, Q_l), \\ \frac{dQ_k}{d\alpha} &= \mathfrak{q}_k(P_l, Q_l), \end{aligned}\right\} \qquad (7)$$

which for $= 0$ assume the values p_k, q_k. Here on the right side the new variables are to be thought of as being introduced by (3) while the old variables appear as integration constants of system (7). That this definition agrees also with the first and thus agrees also with the second can be recognized with the help of the series development (3) and definitions (2), (4) and (5); then one has one after the other, for example

$$\begin{aligned} \frac{dP_k}{d\alpha} = \frac{d[p_k]}{d\alpha} &= [Dp_k] = [\mathfrak{p}_k] \\ &= \mathfrak{p}_k([p_l], [q_l]) = \mathfrak{p}_k(P_l, Q_l). \end{aligned}$$

The relationship between the various transformations of the group is likewise a very simple one as can be shown with the help of presentation (2). If namely $f_1, f_2 \ldots\ldots f_f$ are solutions of a linear homogeneous partial differential equation such as (6), then it is obviously also an arbitrary function $F(f_1 \ldots f_f)$. Since now, for example, $[p_k]_{\alpha = \alpha_1}$ is a solution of (6), then it is also $\left[[p_k]_{\alpha 1}\right]_{\alpha = \alpha_2}$, and since $[p_k]_0$ is the identical transformation then

$[p_k]_o]_{\alpha_2} = [p_k]_{\alpha 2}$. This property of $\alpha_1 = 0$ to become equal to $[p_k]_{\alpha 2}$ has also the solution $[p_k]_{\alpha 1 + \alpha 2}$, but since there is only one solution of the partial differential equation which for $\alpha_1 = 0$ is equal to $[p_k]_\alpha$, then

$$[[p_k]_{\alpha_1}]_{\alpha_2} = [p_k]_{\alpha_1 + \alpha_2}$$

that is, the transformations with the parameter α_1 and α_2 introduced one after the other, produce the transformation with the parameter $\alpha_1 + \alpha_2$. With this we have also demonstrated that our transformations really form a set.

If one considers now a function $f(P_k, Q_k)$ and applies to it transformation (3), then it goes over into

$$f(P_k, Q_k) = [f(p_k, q_k)] = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} D^n f(p_k, q_k).$$

If here f goes over into itself, then one calls a function of this type an invariant of the group. For this obviously to be necessary and sufficient

$$Df\ (p_k, q_k) = 0$$

becomes identical in the p_k, q_k since then all higher members of the exponent development disappear and only the zero member, i.e. the unity operator remains. The invariants of the group satisfy therefore the partial differential equation

$$Df \equiv \sum_k p_k \frac{\partial f}{\partial p_k} + \sum_k q_k \frac{\partial f}{\partial q_k} = 0. \tag{8}$$

9. <u>The meaning of the integrals of the canonical equations</u>. After this preparatory discussion let us return to mechanics and

and ask when is such a transformation group canonical, and therefore contains only canonical transformations. We will limit ourselves for the sake of simplicity to the case that the independent variable t does not appear in the Hamiltonian function of the system. Otherwise as in No. 4, t would have to be treated likewise as canonical variable and also be transformed.

The condition for canonical transformations was [equation (12), No. 3]

$$\sum_k p_k \delta q_k = \sum P_k \delta Q_k + \delta \Phi, \tag{1}$$

where operation was defined by $\overline{\delta} f(p_k q_k) = \sum_k \frac{\partial f}{\partial p_k} \delta p_k + \sum_k \frac{\partial f}{\partial q_k} \delta q_k$

If we introduce into this the formulas (3) of No. 8, then by taking equation (2) of No. 8 into consideration

$$\left.\begin{aligned} \sum_k p_k \delta q_k = \sum_k \left(p_k + \alpha \mathfrak{p}_k + \frac{\alpha^2}{2!} D \mathfrak{p}_k + \cdots\right)\left(\delta q_k + \alpha \delta \mathfrak{q}_k + \frac{\alpha^2}{2!} \delta D \mathfrak{q}_k + \cdots\right) \\ + \sum_{n=0}^{\infty} \alpha^n \delta \Phi_n, \end{aligned}\right\} (2) \tag{2}$$

where Φ is also added as power series in

$$\Phi = \sum_{n=0}^{\infty} \alpha^n \Phi_n .$$

In order that relationship (2) is fulfilled identically, all powers of α must have equal coefficients on both sides. Therefore at first Φ_0 = 0. The linear members produce

$$\sum_k \mathfrak{p}_k \delta q_k + \sum_k p_k \delta \mathfrak{q}_k = \delta \Phi_1 \tag{3}$$

identical in the p_k, q_k. If one has chosen $\mathfrak{p}_k$, $\mathfrak{q}_k$ so that this relationship is fulfilled, then the higher powers are produced through the appropriate repeated application of operator

D to this first relationship, and one sees very easily that equation (2) is generally fulfilled if one places

$$\Phi = \alpha \Phi_1 + \frac{\alpha^2}{2!} D \Phi_1 + \frac{\alpha^3}{3!} D^2 \Phi_1 + \cdots$$

If we now instead of Φ_1 introduce the function

$$-\Psi(p_k, q_k) = \Phi_1 - \sum_k p_k q_k,$$
$$-\delta\Psi = \delta\Phi_1 - \sum_k p_k \delta q_k - \sum_k q_k \delta p_k$$

then (3) changes into the condition

$$\sum_k p_k \delta q_k - \sum_k q_k \delta p_k = -\delta\Psi \tag{4}$$

It is then and only then identically fulfilled in the p_k, q_k, if

$$p_k = -\frac{\partial\Psi}{\partial q_k}, \quad q_k = +\frac{\partial\Psi}{\partial p_k}$$

$\Psi(p_k, q_k)$ is itself to be chosen absolutely arbitrarily and one thus obtains the most general group of canonical transformations by means of the operators

$$D = \sum_k \frac{\partial\Psi}{\partial p_k}\frac{\partial}{\partial q_k} - \sum_k \frac{\partial\Psi}{\partial q_k}\frac{\partial}{\partial p_k} \tag{5}$$

whereby according to equation (2) and (3) of No. 8 the transformation formulae themselves are given by

$$\left.\begin{aligned} P_k &= p_k - \alpha\frac{\partial\Psi}{\partial q_k} + \frac{\alpha^2}{2!} D\frac{\partial\Psi}{\partial q_k} - \cdots \\ Q_k &= q_k + \alpha\frac{\partial\Psi}{\partial p_k} + \frac{\alpha^2}{2!} D\frac{\partial\Psi}{\partial p_k} + \cdots \end{aligned}\right\} \tag{6}$$

These transformation functions are according to the results of No. 9 simultaneously the solutions of the partial differential equation

$$\frac{\partial F}{\partial \alpha} = DF, \tag{7}$$

which for $\alpha = 0$ change relatively in p_k, q_k. Furthermore they are those solutions of the system of differential equations

$$\frac{dP_k}{d\alpha} = -\frac{\partial\Psi}{\partial Q_k}, \quad \frac{dQ_k}{d\alpha} = \frac{\partial\Psi}{\partial P_k}, \tag{8}$$

which for $\alpha = 0$ assume the values q_k, p_k. The canonical groups depend in agreement with No. 3 on one single arbitrary function, namely Ψ, which is designated as the generating function of the group.

By means of the canonical group in general naturally the Hamiltonian function of the mechanical problem changes into another function. We now ask--that is the essential point of the following investigation--if there are groups which convert the problem into themselves, that is with respect to which H is invariant. For this according to equation (8) of No. 8 it is necessary that H satisfies the partial differential equation

$$DH = \sum_k \left(\frac{\partial \Psi}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial \Psi}{\partial p_k} \frac{\partial H}{\partial q_k} \right) \quad (\Psi, H) = 0 \qquad (9)$$

where (Ψ, H) signifies the Poisson bracket symbol (see No. 6).

If we therefore want to designate for a previously given Hamiltonian function H the transformation groups with respect to which it is invariant, then we must select the respective functions which satisfy partial equation (9). These are then the generating functions of the group. There are thus as many canonical transformations of the problem within itself, as there are integrals of this differential equation.

According to No. 7 (5) equation (9) means that Ψ is an integral of the motion equations. We have thus reached the fundamental statement that the generating functions of those canonical transformation groups, which let H be invariant, are integrals of the canonical equations. Inversely it is obvious that each one of these integrals generates a group of canonical transformations of the problem within themselves. The knowledge of the transformation groups of the system is therefore equivalent to the knowledge of the integrals.

As one gathers from (8) the formulae which produce a transformation group have the form of canonical equations. These therefore can be interpreted inversely also as a canonical transformation, with which t plays the role of the parameter

and H itself forms the generating function. This transformation adjoins to every value system $p_k^{(0)}$, $q_k^{(0)}$ at a definite time to $_{(t)}$ that value system $p_k^{(t)}$, $q_k^{(t)}$ in which the mechanical system would find itself through the course of motion from the starting state $p_k^{(0)}$, $q_k^{(0)}$ to the time $t \cdot t_0$. One can therefore conceive of the course of motion of the mechanical system as the development of a canonical transformation. This statement we have used already in Nos. 5 and 7.

The simplest special case is that of the cyclical coordinates (cf. Chap. 2, No. 11). If for example q is cyclic and therefore does not appear in the Hamiltonian function, then is a transformation of the system in and of itself and

$$p = \text{const.}$$

the corresponding integral of the canonical equations.

With the help of the general theory of transformation groups one sees immediately the significance of the ten general integrals of the systems of free mass points[1], for these systems are the displacements, Galileo transformations and turning transformations of the system within itself, which do not change the energy. To them correspond the principles of the center of gravity, conservation of linear and angular momentum. To the conservation of energy correspond the transformation T = t + const., which also transfers the system within itself but contains time in addition.

If, for example x_n, y_n, z_n are the x, y, z coordinates of the n-th mass point, then the first group of the transformations are

$$\begin{aligned} x_n &= X_n + x_n, & p_{x_n} &= P_{x_n}, \\ y_n &= Y_n, & p_{y_n} &= P_{y_n}, \\ z_n &= Z_n. & p_{z_n} &= P_{z_n}; \end{aligned}$$

It means a simple displacement of the system in the x-direction. The corresponding symbol of the group is according to (5) and (6)

$$\Psi = \sum_n p_{x_n}, \qquad D = \sum_n \frac{\partial}{\partial x_n}.$$

The corresponding integral therefore is $\sum_n p_{x_n} = \text{konst.}$

This is the first center of gravity integral however. Likewise one finds the two others $\sum_n p_{y_n} = \text{konst.}$, $\sum_n p_{z_n} = \text{konst.}$

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[1] Siehe Kap. 7, Ziff. 24 ds. Bd. des Handb. Man bgl. auch: F. Engel, Über die zehn allegemeinen Integrale der klassischen Mechanik. Göttinger Nachr, 1916 u. 1917.

The second group of the center of gravity integrals

$$\sum_n m_n x_n = t \sum_n p_{x_n} + \text{konst.}$$

contains time explicitely. To treat them therefore the previous considerations about transformations, which contain time, would have to be expanded.

To the angular momentun conservation laws belong the group of rotations

$$\begin{aligned} X_n &= x_n \cos\alpha + y_n \sin\alpha, \\ Y_n &= -x_n \sin\alpha + y_n \cos\alpha, \\ P_{x_n} &= p_{x_n} \cos\alpha + p_{y_n} \sin\alpha, \\ P_{y_n} &= -p_{x_n} \sin\alpha + p_{y_n} \cos\alpha. \end{aligned}$$

The corresponding symbol is as can be demonstrated easily by expansion according to $D = \sum_n \left(y_n \frac{\partial}{\partial x_n} - x_n \frac{\partial}{\partial y_n} + p_{x_n} \frac{\partial}{\partial p_{y_n}} - p_{y_n} \frac{\partial}{\partial p_{x_n}} \right).$

To this belongs the integral $\Psi = \sum_n (p_{y_n} x_n - p_{x_n} y_n) = \text{konst.},$

and this is the conservation of angular momentun about the z-axis. Similar ones are applicable for the x and y axis.

10. Reduction of the order with the help of known integrals. The canonical transformations make it possible for us to utilize any previous knowledge we might have of integrals of the canonical equations and thus to reduce the order of the differential equation system. In very many cases there exist, for example, the energy integral corresponding to conservation of energy and the center of gravity and surface integrals corresponding to conservation of angular momentum. In the problem of the three bodies with their

help one reduces from the 18th to the 6th order[1]. In general one can with the help of the known integral eliminate a canonical pair and therefore each time reduce the number of variables by two.

Let therefore an integral

$$G\ (p_k,\ q_k) = \text{const.} = g$$

be known. The task is therefore transformation to reach a suitable new variable so that a pair, for example P_1, Q_1, drops out of the Hamiltonian integral $\int_{t_1}^{t_2} \sum_k (P_k \dot{Q}_k - K)\,dt = \text{Extremum}$

This is accomplished obviously when we are successful in making the new variable

$$P_1 = G(p_k,\ q_k) = g \tag{1}$$

Then P_1 becomes constant; thus $\dot{P}_1 = 0$ is an integral of the transformation problem; and because

$$\dot{P}_1 = -\frac{\partial K}{\partial Q_1} = 0, \quad \dot{Q}_1 = \frac{\partial K}{\partial P_1}$$

Q_1 and K must then drop out while P_1 now only plays the role of a constant parameter. The variables Q_l, P_l (l = 2, ... f) form thus by themselves a canonical system with the Hamiltonian function K.

So that (1) is now true, the transformation function V, which should generate the desired canonical transformation, according to No. 3, equation (5) must satisfy the condition

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[1]Vgl. Kap. 7, Ziff. 24, 27 und 28 ds. Bd. ds. Handb.

$$P_1 = -\frac{\partial V}{\partial Q_1} = G\left(\frac{\partial V}{\partial q_k}, q_k\right) \qquad (2)$$

This is a partial differential equation which possesses corresponding integrals with which the possibility of reduction is shown. It can be carried out without really looking for a solution to the partial differential equation. If one has namely first determined V according to (2), then with the corresponding canonical transformation, Q_1 falls out of K by itself. One can therefore give Q_1 for the purpose of the transformation any arbitrary value, especially the value zero, and must nevertheless still come to the correct function K. Therefore one does not need to know the dependency of function V on Q_1; moreover it is sufficient to have its value $V(q_k, 0, Q_2, \ldots Q_f)$ for $Q_1 = 0$. This is, however, entirely arbitrary, for according to the existence principle for partial differential equations, one can always give an integral of (2) which for $Q_1 = 0$ changes into an arbitrary given function $V(q_k, Q_2, \ldots Q_f)$.

We can therefore proceed as follows. We take an arbitrary (except for one limitation which we will give soon) function $V(q_r, Q_2, \ldots Q_f)$ of the 2f - 1 variables $q_1, \ldots q_f, Q_2, \ldots Q_f$ and possibly even of t and express first the p_k by means of the equation

$$p_k = \frac{\partial V}{\partial q_k} = p_k(q_1, \ldots q_f, Q_2, \ldots Q_f) \qquad (3)$$

as functions of q_k and Q_k. These values we insert into the side condition (1) so that we get

$$G\left(q_1, \ldots q_f, \frac{\partial V}{\partial q_1}, \ldots \frac{\partial V}{\partial q_f}\right) = G(q_1, \ldots q_f, Q_2, \ldots Q_f) = g = P_1. \tag{4}$$

This equation we take in place of $P_1 = \frac{V}{Q}$ which according to the above boundary condition is permissible. We now set

$$P_l = -\frac{\partial V}{\partial Q_l} = P_l(q_1, \ldots q_f, Q_2, \ldots Q_f), \qquad (l = 2, \ldots f) \tag{5}$$

then (3), (4), and (5) are together the desired transformation formulae for p, q into P, Q. V thereby undergoes only the limitation that the equations (3) and (4) must be solvable for q_k. The new Hamiltonian function then is the usual

$$K = H + \frac{\partial V}{\partial t}$$

and does not contain the variable Q_1, however $P_1 = g$ is to be considered as the constant parameter.

The simplest special case is again the cyclic coordinates. If for example, q_1 is cyclic and therefore does not appear in L and thus also not in H, but it does in $\dot{q}_1$ viz p_1, then

$$\frac{\partial L}{\partial \dot{q}_1} = p_1 = \text{konst.} = c$$

if the integral and the canonical problem already has the form we are seeking. We therefore can simply suppress p_1 and q_1 so that we as variation problem get

$$\int_A^B \left\{\sum_l p_l \dot{q} - K(p_l, q_l, c)\right\} dt = \text{Extremum}, \qquad (l = 2, \ldots f)$$

where $H(p_1, p_L, q_L) = K(c, p_L, g_L)$. The whole procedure of this section (Nr_1) means that one with the help of an integral can make a variable into a cyclic one.

11. <u>The relationship between the various integral principles</u>. The just discussed ideas make it possible for us to explain the

relationship between the various integral principles in a very instructive manner by evaluating the energy equator as a side condition. These concepts which are closely related to those of the previous chapter are inserted and discussed here since only now do we have the necessary mathematical apparatus available.

First, we must return from the canonical variation problem to the Hamiltonian. We assume, thereby, that we would have eliminated in the former, the side conditions by the introduction of cyclical variables, as in the previous no. and apply now the Legendre transformation. No. 2 equator (8b), In this manner, the new Lagrange function--let q_1 be cyclical--becomes

$$L^* = \sum_l p_l \frac{\partial K}{\partial p_l} - K. \qquad (l = 2, \ldots f)$$

On the other hand

$$L = p_1 \frac{\partial H}{\partial p_1} + \sum_l p_l \frac{\partial H}{\partial p_l} - H = \sum_l p_l \frac{\partial K}{\partial p_l} + c\dot{q}_1 - K.$$

and therefore,

$$L^* = L - c\dot{q}_1,$$

and the variation problem contains the form:

$$\int_A^B \{L(q_1, \dot{q}_1) - c\dot{q}_1\} \, dt = \text{Extrenum}. \qquad (1)$$

In this the quantity q_1, which does not even appear, in contrast to the other coordinates, is not subjected to any limiting conditions and $\dot{q}_1$, therefore, is a completely arbitrary function. One can thus think of the problem as if no longer contained an unknown $\dot{q}_1$ whose derivative does not occur and whose corresponding Lagrange equation, therefore, is

$$\frac{\partial L}{\partial \dot{q}_1} - c = 0 \qquad (2)$$

while the other Lagrange equations are not changed and thus give the same extreme-values. Since (2) must always be satisfied, therefore, one can require this relationship as a side condition and then treat it as in No. 2. Obviously, the result is that (1) is equivalent to the relative minimal principle

(3)

with equation (2) as a side condition.

Finally, one can eliminate now, $\dot{q}_1$ entirely by solving (2) for $\dot{q}_1$ and placing it in (1). Then we actually obtain, again, a simple minimal principle

$$\int_A^B F(c, p_l, q_l)\,dt = \text{Extremum}, \qquad (l = 2, \dots f) \tag{3a}$$

only with one less desired function.

One can, as has already been said, use these ideas in order to go from the Hamiltonian principle to the other integral principle by applying them to the energy laws. This procedure, however, is only valid for conservative systems. In this case t itself, is cyclical since it does not appear in the kinetic potential. In order to be able to apply the above method, we must further introduce (no. 4) as before, a parameter which places t equal to the other variables. Let us assume all values as functions of an auxilliary parameter τ:

$$t = t(\tau), \qquad q_k = q_k(\tau),$$

so that $t(\tau_1) = t_1$, $t(\tau_2) = t_2$ and designate the derivative according to τ by a dash, then we have

$$\dot{q}_k = \frac{\dot{q}_k}{t'}$$

and thus the kinetic energy T, which we presuppose to be a homogenous quadratic function of $\dot{q}_k$

$$T(\dot{q}_k) = \frac{1}{t'^2} T(\dot{q}_k).$$

The Hamiltonian principle, therefore, changes to

$$\int_{\tau_1}^{\tau_2} \left\{ \frac{1}{t'} T(q_k') - U(q_k) t' \right\} d\tau = \text{Extremum}$$

whereby as alimiting condition, it is required that for $\tau = \tau_1$, viz. $\tau = \tau_2$ the q_k and t change to definite values $q_k^{(1)}$ and $t^{(1)}$ viz. $q_k^{(2)}$, $t^{(2)}$. Now t is no longer distinguished and we can, therefore, apply the previous concepts. Thus t takes the place of q_1 and τ takes the place of t while:

$$L = \frac{1}{t'} T - U t'$$

An integral of this variation problem becomes

$$\frac{\partial L}{\partial t'} = -\frac{1}{t'^2} T(q_k') - U = -E, \tag{4}$$

therefore of course the energy integral. With its help one obtains as equivalent with the Hamiltonian principle the form (1) which here is

$$\int_{\tau_1}^{\tau_2} \left\{ \frac{1}{t'} T(q_k') - U t' + E t' \right\} d\tau = \text{Extremum} \tag{5}$$

where thus the limiting value of t is no longer designated. If we again inversely introduce to as variable, then we get

$$\int_A^B (T - U + E)\, dt = \text{Extremum}. \tag{6}$$

This is a new principle of mechanics equivalent to the Hamiltonian which is still unknown in the literature and which should be called the Hilbert principle. It says:

A POINT SYSTEM MOVES SO THAT WITH ALL MOTIONS WHICH WITH ANY PASSAGE OF TIME LEAD FROM THE STARTING POINT A WITH THE COORDINATES $q_k = q_k^{(1)}$ TO THE END POINT B WITH THE COORDINATES $q_k = q_k^{(2)}$, THE ACTUALLY OCCURING MOTION MAKES THE INTEGRAL (6) AN EXTREME-VALUE WHERE E IS THE VALUE OF THE TOTAL ENERGY GIVEN AT THE STARTING POINT.

From the principle follows naturally the energy law since it does not appear explicitly in the integrand. It does not require it however as side condition and it stands correspondingly in the middle between the Hamiltonian principle and the principle of least effect.

Since E is the constant, for (6) one can write

$$\int_A^B (T-U)\,dt + E(t_2 - t_1) = \text{Extremum},$$

where $t_2 - t_1$ is the still unknown time which the system needs for its path. One arrives back at the Hamiltonian principle when one gives the time $t_2 - t_1$ to the motion.

To the principle of least effect we arrive by adding the energy law T + U = E which follows from (6) as a side condition. One thus comes to form (3) which because of (4) assumes the form

$$2\int_A^B T\,dt = \text{Extremum} \qquad \text{while} \qquad T + U = E$$

therefore exactly the principle of least effect (see Chap. 2, No. 25).

The extreme-value is to be sought among all functions which in any time at all lead from starting to end point.

Finally one can still eliminate t entirely and therefore obtain form (3a). To do this one again uses the pramater presentation properly. This is procedure which led in Chapter 2, No. 26 to the Jacobi principle which can therefore find a place in these discussions.

12. The Hamilton-Jacobi partial differential equation. We turn now to the integration theory of canonical motion equations

$$H = H(q_k, p_k, t), \qquad \dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} \tag{1}$$

We have come across parts of these several times already (in No. 2, 7, 9, and 10), but the most important thing is still lacking: a systematic procedure, which will be described in the following. In this we will make extensive use of the canonical transformations.

According to No. 3(5) the new Hamiltonian function with a canonical transformation of problem (1) becomes

$$K = H + \frac{\partial V}{\partial t}.$$

We ask if it is possible through a suitable choice of function V to cause the new Hamiltonian function K of the system to disappear. Then in a certain respect the mechanical problem is transformed into an equilibrium problem. The function which does this we will designate with R to distinguish it from other generators.

Now R is a function of q_k, Q_k and t and it becomes

$$p_k = \frac{\partial R}{\partial q_k}, \quad P_k = -\frac{\partial R}{\partial Q_k}, \quad K = H + \frac{\partial R}{\partial t} \tag{2}$$

The condition which R must fulfill so that K disappears is therefore $\frac{\partial}{\partial t} R(q_k, Q_k, t) + H(q_k, p_k, t) = 0$

or according to (2) $\frac{\partial R}{\partial t} + H\left(q_k, \frac{\partial R}{\partial q_k}, t\right) = 0.$

(3)

This is a partial differential equation of the first order for R which was discovered first by Hamilton. It arises by replacing in the Hamiltonian function H the p_k by the derivatives of R according to the corresponding $_K$. Since (3) for all arbitrary values of Q_k must stand, then they play the role of integration constants.

The significance of the partial differential equation (3) lies in the following. Let us assume we had found an integral of (3) containing f arbitrary constants

$$R(q_1 \ldots q_f, \alpha_1, \ldots \alpha_f, t) = 0$$

therefore a function which for all values of the integration constants satisfied the differential equation. This is naturally not the most general solution of the partial differential equation which would have to contain certainly an arbitrary function but a so-called complete integral. We can then introduce these constants α_k as new variables since R should be a function of the old and new position parameters. The transformation formulae

(5) of No. 3 produce in this case

$$\left.\begin{aligned} p_k &= \frac{\partial R}{\partial q_k}, \\ P_k &= -\frac{\partial R}{\partial \alpha_k} = +\beta_k, \\ K &= 0. \end{aligned}\right\} \quad (4)$$

and the new canonical equations become as a result of the third line simply $\frac{dQ_k}{dt} = \frac{d\alpha_k}{dt} = 0, \quad \frac{dP_k}{dt} = \frac{d\beta_k}{dt} = 0.$

Therefore both the α_k and the β_k are constant quantities for the mechanical system to which arbitrary values can be given. They are called the canonically conjugated constants. With this the integration of the differential equations of the mechanical problem is completely carried out; this is because the equations (4) produce the original coordinates of the system as functions of time and of the 2f arbitrary constants α_k and β_k.

The integration of the canonical equations is therefore reduced to the discovery of an integral of the partial differential equation (3) which contains f constants. At first not much seems to be gained by this since partial differential equations as a rule are more difficult to handle than usual ones. But in mechanics it has been shown that for many important cases the partial differential equation assumes relatively simple forms so that their introduction actually means a big step forward.

Only one single step will be developed here. If the Hamiltonian function H does not contain time explicitly then the differential equation (3) can be somewhat simplified. If we for

R make the following condition:

$$R = S(q_k, \alpha_1, \ldots \alpha_f) - \alpha_1 t, \tag{5}$$

where S no longer depends on t, and if we enter this condition in (3), then

$$\alpha_1 = H\left(q_k, \frac{\partial S}{\partial q_k}\right) = \ldots \tag{6}$$

whereby time ti is eliminated. α_1 thereby in general becomes the energy constant and as such is designated by W. If we now have found an integral S of the partial equation (6) which aside from α_1 still depends on f - 1 further independent constants, then the solutions of the motion equations are

$$p_k = \frac{\partial S}{\partial q_k}, \quad \beta_l = -\frac{\partial S}{\partial \alpha_l}, \quad t - \beta_1 = \frac{\partial S}{\partial \alpha_1}. \qquad (l = 2, \ldots f) \tag{7}$$

The equations (3) and (6) are the simplest forms of the Hamiltonian partial differential equation. Formulas (4) and (7) contain the solutions of the motion problem in the obvious form. But from a practical point of view many variations of the described procedure are used. Thus one in place of (3) can also require that the new Hamiltonian function K, instead of disappearing, becomes an arbitrary time function f(t). One has to take as generator of the canonical transformation the solution of the differential equation

$$\frac{\partial T}{\partial t} + H\left(q_k, \frac{\partial T}{\partial q_k}\right) = f(t) \tag{8}$$

R is then related to T through

$$R = T - \int f(t)\,dt \tag{9}$$

For example, one can require that $f(t) = \text{konst.} = \alpha_1$

(This is obvious when it is a question of a small disturbance coming from outside of an otherwise closed system which without it contains a constant amount of energy.) For this from equations (8) and (9) we get

$$\frac{\partial T}{\partial t} + H\left(q_k, \frac{\partial T}{\partial q_k}\right) = \alpha_1, \tag{8a}$$

$$R = T - \alpha_1 t. \tag{9a}$$

If especially H does not depend explicitly on t, then one can assume T to be independent of t and thus return to (6) viz (5).

Furthermore in the case of a closed system it is for example simplest but not always most practical to choose the energy constant itself as one of the integration constants of the complete integral S. From normalization reasons in the theory of stipulated periodical systems (cf. Chapter 4) and their applications in quantum theory other integration constants are chosen--we will call them J_k--in which the new Hamiltonian function is written

$$\alpha_1 = K(J_1 \cdots J_f) \tag{10}$$

One can however easily transform with a generator of the form

$$V = \sum_k \alpha_k (J_1 \ldots J_f) \beta_k$$

the variables of α_k, β_k to the new constants J_k and the variables conjugated canonically with them. The latter are because of (10) and $\dot{w}_k = \frac{\partial K}{\partial J_k}$ = const. linear functions of time.

In all cases for the drawing up of the Hamiltonian partial differential equation the view point remains standing that one change to new variables of which a set are motion constants of which conjugated set therefore do not occur in K. With other words: one looks for a generator of a canonical transformation in cyclical variables to the discovery of which the Hamiltonian partial differential equation leads. Let it be mentioned parenthetically that form (1) of the Hamilton'an differential equation corresponds formally entirely to form (6) when one according to No. 4 treats time likewise as a canonical variable.

13. <u>The simplest cases of the integration</u>. The solution to the motion problem No. 12 (1) is now reduced to the integration of the partial differential equation No. 12 (3) or (6). We must look for a complete integral of the same provided with f integration constants α_k. A procedure which always leads to this goal can not be given. Let us discuss here only two simple cases of the treatment of No. 12.

The first case that permits a simple integration is before us when all variables with the exception of one single one (9) are cyclic. One knows then the f - 1 first integrals

$$p_k = \frac{\partial S}{\partial q_k} = \alpha_k \qquad (k = 2, \ldots f)$$

and finds

$$S = \sum_{k=2}^{f} \alpha_k q_k + S_1(q_1, \alpha_1, \alpha_2, \ldots \alpha_f).$$

The differential equation No. 12 (6) can be reduced, since H is independent of the cyclic variables $q_2 \ldots q_f$, to an ordinary

$$H\left(\frac{\partial S_1}{\partial q_1}, q_1, \alpha_2, \ldots \alpha_f\right) = W = \alpha_1,$$

from which then S_1 can be obtained by squaring.

The other case that permits a simple integration occurs when the differential equation No. 12 (6) can be separated into the variables p_k, q_k. This means that with the problem

$$S = \sum_k S_k(q_k, \alpha_1, \dots \alpha_f)$$

$$p_k = \frac{\partial S}{\partial q_k} = \frac{\partial S_k(q_k)}{\partial q_k}$$

that is, when S is given as the sum of functions which individually depend only on one coordinate q_k -- the differential equation No. 12 (6) separates into f different differential equations for the S_k. For this it is necessary that already within the equation

$$H(p_1, \dots p_f, q_1, \dots q_f) = W$$

each momentum p_k can be conceived of as function of the pertinent coordinates q_k above and therefore this equation separates into f separate ones

$$H_k(p_k, q_k) = A_k(\alpha_1, \dots \alpha_f)$$

The f separate differential equations for the S_k then are

$$H_k\left(\frac{\partial S_k}{\partial q_k}, q_k\right) = A_k.$$

They make possible the calculation of the S_k by mere squaring.

The condition that H can be separated into the used coordinates can be written according to Levi-Civita[1)]

$$\begin{vmatrix} 0 & \frac{\partial H}{\partial q_j} & \frac{\partial H}{\partial p_j} \\ \frac{\partial H}{\partial q_k} & \frac{\partial^2 H}{\partial q_j \partial q_k} & \frac{\partial^2 H}{\partial p_j \partial q_k} \\ \frac{\partial H}{\partial p_k} & \frac{\partial^2 H}{\partial q_j \partial p_k} & \frac{\partial^2 H}{\partial p_j \partial p_k} \end{vmatrix} = 0 \quad \text{für} \quad \begin{cases} j, k = 1, 2, \dots f \\ j \neq k. \end{cases}$$

1) T. Levi-Civita, Math. Ann. Bd. 59, S. 383. 1904; F. A. Dall'Acqua, ebenda Bd. 66, S. 398. 1908; H. Kneser, ebenda Bd. 84, S. 277. 1921.

Usually the separability is to be looked for in function H. It is dependent upon the coordinate system and generally is in need of the introduction of special separation coordinates in order to accomplish the desired separation. In many cases the separation system distinguished physically by the boundaries of the path region. However, this is not always so[1]; indeed, Burgers[2] has shown that with the motion of an electrically charged oscillator in the magnetic field, the separation system can be introduced only by a tangential transformation.

Examples for the integration by separation are among others each central motion [as can be seen from No. 2 (15)], and the two-center problem which, as Jacobi has shown, can be separated into elliptical coordinates with the two fixed centers as foci.[3] Weinacht succeeded also for the case of a single mass point in a conservative force field in finding all systems which can be separated by point transformation[4]. The important result is that the most general position coordinates coming into consideration for the separation of the variables in this case are those of the ellipsoid with three axies (including their degenerate forms). Also the related functions for the potential energy can be

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[1] E. Fues, ZS. f. Phys. Bd. 34, S. 788. 1925.

[2] J. M. Burgers, Het Atoommodel van Rutherford-Bohr, Leiden 1918.

[3] T. N. Hamilton in his writings used the term "Characteristic Function" where Nordheim and Fues use "Eikonal."

[4] J. Weinacht, Math. Ann. Bd. 91, S. 279. 1924.

listed and are obvious generalizations of the above mentioned cases. In addition, each small oscillation of an arbitrarily constructed system around a stable point of equilibrium makes possible a separation according to the method of separate oscillations. For the motion of a rigid body the cases of the most general force-free gyroscope (possibly also with built-in fly-wheel) and that of the symmetrical gyroscope in a gravitational field are separable[4].

14. The independency law of the calculus of variation; Characteristic Function. At the close of the chapter on Hamilton-Jacobi mechanics we still want to try to give an insight into the profound thought processes which led the creators of this theory and which recently in the works of de Broglie, Schrödinger and others have brought about a fundamental broadening of mechanics. In order to understand this real kernel of the Hamilton-Jacobi theory it is useful to mention again a few theorems of the calculus of variations. For this we start with form (4) of No. 2 of the variation problem

$$\int_{t_1}^{t_2}\left\{L+\sum_k \frac{\partial L}{\partial k_k}(\dot q_k - k_k)\right\}dt = \text{Extremum} \tag{1}$$

The integral here has the simple form

$$J=\int_{t_1}^{t_2}\left(A+\sum_k B_k \frac{dq_k}{dt}\right)dt \tag{2}$$

with

[1]Vgl. G. Kolossoff, Math. Ann. Bd. 60, S. 232. 1905; F. Reiche, Phys. ZS. Bd. 19, S. 394. 1918; P. S. Epstein, Verh. d. Phys. Ges. Bd. 17, S. 398. 1916; Phys. ZS. Bd. 20, S. 289. 1919; H. A. Kramers, ZS. f. Phys. Bd. 3, S. 343. 1923.

with $$A = L - \sum_k \frac{\partial L}{\partial k_k} k_k, \qquad B_k = \frac{\partial L}{\partial k_k}.$$

The integrand is thus a linear expression in the derivatives $\dot{q}_k$ of q_k. In addition the functions k_h which are to vary independently of the q_k appear, but not their derivatives. This form reminds one of the complete derivative

$$\sum \frac{\partial \Phi}{\partial q_k} \dot{q}_k + \frac{\partial \Phi}{\partial t}$$

of a function Φ according to time. It suggests the question as to whether or not it is possible with a special choice of k_k as functions of q_k and t to make the integral (2) indepedent of the path in the qt-space so that it keeps the same value for all possible functions $q_k(t)$ and therefore from a function of a function in the sense of the calculus of variation it degenerates into a pure position function of integration limits. The values of the k_k then form a kind of proof of the qt-space to the extent that to each point is given a definite value of the k_k. One calls such a proof a field and the question is whether or not there are proofs in which the integral (2) becomes independent of the path. Necessary and sufficient for this is that B_k and A appear as partial derivatives of the function $\Phi(q_k, t)$:

$$A = \frac{\partial \Phi}{\partial t}, \quad B_k = \frac{\partial \Phi}{\partial q_k}.$$

Then the integral

$$J = \int_{t_1}^{t_2} \left(A + \sum_k B_k \dot{q}_k\right) dt = \int_{t_1}^{t_2} \left(\frac{\partial \Phi}{\partial t} + \sum_k \frac{\partial \Phi}{\partial q_k} \dot{q}_k\right) dt = \Phi(t_2, q_k^{(2)}) - \Phi(t_1, q_k^{(1)})$$

becomes a pure function of the integration limits in the qt-space.

For this A and B_k must fulfill the conditions of integratibility

$$\frac{\partial A}{\partial q_k} = \frac{\partial B_k}{\partial t}, \qquad \frac{\partial B_k}{\partial q_l} = \frac{\partial B_l}{\partial q_k}$$

The general answer how one must choose the k-field so that these conditions will be satisfied is given by the independency law of Hilbert:

The integral (2) becomes independent of the path when one takes any system of intermediary integrals.

$$\frac{dq_k}{dt} = \dot{q}_k(q_1, \ldots q_f, t)$$

of the Lagrange differential equations:

$$[L]_{q_k} = 0 \tag{3}$$

and for every point $q_1, \ldots . q_f$, t then chooses the corresponding $\dot{q}_k$.

We will prove this law here, only for systems with only one single degree of freedom, that only one pair p, q viz. k. Then there exists only one single condition of integratability, namely:

$$\frac{\partial}{\partial q}\left(L - k\frac{\partial L}{\partial k}\right) = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial k}\right). \tag{5}$$

If we differentiate, then we get as a condition for the independence of the integral (1), a partial differential equation of the first order for k(q, t)

$$\frac{\partial L}{\partial q} + \frac{\partial L}{\partial k}\frac{\partial k}{\partial q} - \frac{\partial L}{\partial k}\frac{\partial k}{\partial q} - k\left(\frac{\partial^2 L}{\partial k\,\partial q} + \frac{\partial^2 L}{\partial k^2}\frac{\partial k}{\partial q}\right) = \frac{\partial^2 L}{\partial t\,\partial k} + \frac{\partial^2 L}{\partial k^2}\frac{\partial k}{\partial t}$$

or

$$\frac{\partial^2 L}{\partial k^2}\left(\frac{\partial k}{\partial t} + k\frac{\partial k}{\partial q}\right) + k\frac{\partial^2 L}{\partial k\,\partial q} + \frac{\partial^2 L}{\partial k\,\partial t} - \frac{\partial L}{\partial q} = 0, \tag{6}$$

which is called the partial differential equation adjoined to the variation problem. This differential equation is now,--that is the claim--satisfied then and only then, when k(q, t) is an intermediary integral of the Lagrange differential equation

$$[L]_q \equiv \frac{\partial^2 L}{\partial \dot{q}^2}\ddot{q} + \frac{\partial^2 L}{\partial \dot{q}\partial q}\dot{q} + \frac{\partial^2 L}{\partial \dot{q}\partial t} - \frac{\partial L}{\partial q} = 0 \tag{7}$$

If namely $\dot{q}$ = k(q, t) is such an integral of (7), that is, if (7) is satisfied identically when one inserts for q, the general solution

$$q = q(t, \alpha) \tag{8}$$

of the differential equation q = k(q, t), which still contains the constant α, then $\ddot{q} = \frac{\partial k}{\partial t} + \frac{\partial k}{\partial q}\dot{q}$,

is valid identically in t and α. If we place this in (7),

$$\frac{\partial^2 L}{\partial \dot{q}^2}\left(\frac{\partial k}{\partial t} + \frac{\partial k}{\partial q}\dot{q}\right) + \frac{\partial^2 L}{\partial \dot{q}\partial q}\dot{q} + \frac{\partial^2 L}{\partial \dot{q}\partial t} - \frac{\partial L}{\partial q} = 0,$$

and again write k for $\dot{q}$, then one gets a relationship which formally looks exactly like the adjoined partial differential equation (6), but at first represents an ordinary equation in t and α, which must be satisfied identically for all values of t and α. If one α with q by means of (8), then it must also be identically true in t and q, that is, all intermediary integrals $\dot{q}$ = k(q, t) of the Lagrange differential equation satisfy also the adjoined partial differential equation.

If conversely, k(q, t) is a solution of the adjoined partial differential equation (6) and if q(t) satisfies the equation $\dot{q}$ = k(q, t), then we can insert $\partial k/\partial t + k \cdot \partial k/\partial q = \ddot{q}$ and arrive back with, if we again write $\dot{q}$ for k, at the Lagrange differential equation 7 and thus our law is completely proven. For several degrees of freedom then, the law can be generalized by taking it back to this special case.

The solutions to a variation problem, that is the curves which satisfy the Lagrange differential equations, are usually designated as extreme-values. With the help of a set of extreme-values of f parameters an independence field can always be produced. In order to carry this out in the most general manner, that is, to give to each value system $q_1 \ldots q_f$, t a value system $k_1 \ldots k_f$, and thus to fulfill the condition of the independency integral, one proceeds as follows. We choose entirely arbitrarily any function $F(q_k, t)$ which placed equal to zero, produces an f-dimensional hypersurface in the space of q_k, t:

$$F(q_1, \ldots . q_f, t) = 0, \qquad (9)$$

and determine next the k_k for all points of the surface from the requirement that for them, the integrand of the independency integral

$$L + \sum_k \frac{\partial L}{\partial k_k}(\dot{q}_k - k_k)$$

disappears. We accomplish this calculating the f values of k_k from the f equations

$$\left(L - \sum_k \frac{\partial L}{\partial k_k} k_k\right) : \frac{\partial L}{\partial k_1} : \frac{\partial L}{\partial k_2} : \cdots \frac{\partial L}{\partial k_f} = \frac{\partial F}{\partial t} : \frac{\partial F}{\partial q_1} : \frac{\partial F}{\partial q_2} : \cdots \frac{\partial F}{\partial q_f} \qquad (10)$$

since then the integrand except for a negligible factor, becomes equal to $\frac{\partial F}{\partial t} + \sum_k \frac{\partial F}{\partial q_k}\dot{q}_k = \frac{dF}{dt}$

and thus indeed, disappears for the surface. Then from each point of the surface, we let such a curve $q_k = q_k(t)$ go out whose direction factors $\dot{q}_k$ are there exactly equal to the just determined k_k, and in their further course, satisfy the Lagrange differential equations (3). This is always possible since always at a given point with given direction for an arbitrary differential equation of second order, such an integral curve can be found. This simply means that we take the integral curve which stands transversal to the surface which condition is usually identical with an orthogonality in the ordinary sense.

Since the sufrace F = 0 itself, is f-dimensional, we have disignated an f-parameter curve set which fills the f + 1-dimensional q t-space everywhere completely, since in general, aside from occasional singular points, a curve goes through every space point. The values of the k_k at a random point we determine simply from the tangent direction of the extreme value going through it and we set, therefore,

$$k_k = \dot{q}_k.$$

This k-field according to the independency law, makes the integral a pure place function.

the significance of the independency integral can now be recognized as follows. We imagine to ourselves that in the field all transversal surfaces are drawn in, that is, all surfaces F = consant, which satisfy conditions (10). The integral J, reaching between any two points of such a surface, is obviously equal to qero. We now solve it further for the pa h which leads from the starting point A of the actual motion to the end point B. Because of the independency from the path, we can choose the latter as appropriately as possible. We next go to the transversal surface on which the starting point lies forewards to point C, to which the extreme-value joins, which also goes through the end point B and then on to this extreme-value. The first part AC gives no sum to the integral. For the second part, CB everywhere the $k_k = \dot{q}_k$ and $\mathfrak{J}$ are reduced to $\int_C^B L(q_k, \dot{q}_k, t)dt$, since the $\dot{q}_k = k_k(t)$ were so designated that they satisfy the Lagrange differential equations. $\mathfrak{J}$ is therefore the extreme-value of the integral of the Hamiltonian principle between the two transversal surfaces which go through starting point and end point. Since $\mathfrak{J}$ disappears for paths on these surfaces, they are, therefore also surfaces of constant value difference of the Hamiltonian integral between corresponding points, i.e. points which lie on the same extreme-value. The quantity $\mathfrak{J}$ which for a given extreme-value field is a function of the starting point and end point, has for many branches of mathematics and physics, great importance and is usually called by the name, characteristic function.

Naturally, there are many kinds of characteristic functions, since they depend on an arbitrary function, namely the starting surface $F = 0$. Among all possible starting surfaces, there are especially those which have degenerated into a point, namely the starting point of the integration path. Also, from it one gets a field which covers the entire space by taking all extreme-values which pass through it as generators of the field. The characteristic function for a point which is reached from the starting point in the course of the motion is thus obviously equal to the extreme-value of the Hamiltonian integral itself, taken over the actual path curve.

15. Application in mechanics; the meaning of the Hamilton-Jacobi differential equation. For all possible characteristic functions, a partial differential equation can be set up. From definition (2) of No. 14 we see immediately that the derivatives of J are given by

$$\left.\begin{aligned} \frac{\partial J}{\partial t} &= L - \sum_k \frac{\partial L}{\partial k_k} k_k, \\ \frac{\partial J}{\partial q_k} &= \frac{\partial L}{\partial k_k}. \end{aligned}\right| \qquad (1)$$

The right sides are still functions of the k_k, that is, of the chosen field. From these f + 1 relationships the f values of k_k can be eliminated and there remains over then a condition between the derivatives of J , that is, a partial differential equation. This elimination can be carried out directly with the Legendre transformation, therefore the transition to canonical coordinates. We had set in (5) and (7), No. 2

$$p_k = \frac{\partial L}{\partial k_k} = \frac{\partial L}{\partial \dot{q}_k},$$
$$H = \sum_k \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - L$$

and we received from (1) by eliminating the p_k

$$\frac{\partial J}{\partial t} + H\left(q_k, \frac{\partial J}{\partial q_k}, t\right) = 0 \qquad (2)$$

as partial differential equation for the characteristic function. This is however, the Hamilton-Jacobi differential equation (3) of No. 12. <u>Through this fundamental relationship, the significance of the integral of the Hamilton-Jacobi differential equation as a value of the Hamiltonian integral between the transversal surfaces of the field is disclosed.</u>

With the help of these findings, the main law of No. 12 can be derived in a new manner. Let us suppose that in differential equation No. 14 (9) of the starting surface f parameters are introduced so that we in all have an f-parameter set of surfaces of which one is our initial surface. To every other surface of this set there is likewise, an independency field determined by our construction so that also have an f-parameter set of such fields. This means we take for our definition of the field, a set of intermediary integrals of the Lagrange equations which contains f integration constants $k_k = \dot{q}_k(q_l, \alpha_l, t)$.

To every value system of α_k belongs then a characteristic function and the totality of these characteristic functions can be summed up in a single function $J(\alpha_k)$ which depends on the f parameters in addition to the starting and end points:

$$J = \int_A^B \left\{ L(q_l, k_l(q_l, \alpha_l, t), t) + \sum_k \frac{\partial L}{\partial k_k} (\dot{q}_k - k_k) \right\} dt.$$

With however, the derivatives according to the parameters must become pure location functions and we get, because of

$$\frac{\partial L}{\partial \alpha_l} = \sum_k \frac{\partial L}{\partial k_k} \frac{\partial k_k}{\partial \alpha_l}$$

simply

$$\frac{\partial J}{\partial \alpha_l} = \int_A^B \sum_k (\dot{q}_k - k_k) \frac{\partial^2 L}{\partial k_k \partial \alpha_l} dt. \tag{3}$$

The integrals on the right side disappear as we move along integral curves since for these always $\dot{q}_k = k_k$, that is, $\partial J / \partial \alpha_l$ produce functions of q_k and t which are constant, themselves, on the integral curve. They must, therefore, be placed equal to constants - β_1,

$$\frac{\partial J}{\partial \alpha_l} = -\beta_l, \tag{4}$$

be integrals of the Lagrange differential equations, which was to be demonstrated.

By reversing this law, one also gets an important mechanical theorum. If we know half of the integral of a mechanical system, then the other half can be found by squaring. Indeed of f functions

$\varphi_l(\dot{q}_k, q_k, t, \alpha_1, \ldots \alpha_f) = 0$ viz. $\varphi_l(p_k, q_k, t, \alpha_1, \ldots \alpha_f) = 0$

are known then by solving according to $\dot{q}_k$ one can find these as functions of q_k, t and of the f first integration constants α_f, hence also an f-parameter extreme-value field $k_k = \dot{q}_k(q_l, t, \alpha_l)$.

$$J = \int_A^B \left(\sum_k p_k \dot{q}_k - H\right) dt. \tag{5}$$

We form, therefore, according to our assumption

$$dJ = \sum_k p_k dq_k - H dt \tag{6}$$

a complete differential.

According to the principle just proven, every mechanical problem with one degree of freedom can be solved by squaring, for example, if it possesses the energy integral, and every problem with two degrees of freedom when in addition to the energy integral one additional integral is known.

Also integral S of Hamilton-Jacobi partial differential equation (6) of No. 12 integrated according to time has a simple significance for systems which do not contain time explicitly. It is namely the extreme-value of the integral of least action, therefore, the action function, and thus also of the integral of the Jacobi principle identical with it for conservative systems. We have, since we postulate the law of conservation of energy

$$2T = T - U + T + U = T - U + \alpha_1,$$

where α_1 is the energy constant. Consequently, according to (5) of No. 12

$$2\int_A^B T\,dt = \int_A^B (T - U)\,dt + \alpha_1 t = J + \alpha_1 t = S; \quad (7)$$

i.e. S is related to the principle of least action in the Jacobi form in the same way as J to the Hamiltonian principle.

The concepts of this section show that the integration of a partial differential equation of the Hamilton-Jacobi form, which means no essential restrictionof generality, is equivalent to the integration of the corresponding canonical equation. This is nothing but the Jacobi integration method of the partial differential equations of first order and the extreme-value curves of the Hamiltonian principle, thus the mechanical path curves, represent the characteristics of the partial differential equation. Indeed when the canonical equations are solved and thus all extreme-values are found, for every function $F(q_k, t) = 0$ one can find a solution of the partial differential equation which for $t = t_1$, $q_k = q_k^{(1)}$

change over to F. Actually one proceeds, as indicated, conversely by integrating with the help of integrals of the partial differential equation (2) the Lagrange or canonical equations.

This was the starting point which led Jacobi to his theory. The other discoverer of these relationships, Hamilton, started from the geometrical meaning of the characte4istic function, which to be sure, is very remarkable. If we go from the presentation of the characteristic function in No. 14 (description in the q t-space) over to a construction in the f-dimensional q-space above, then we get a system of moving characteristic function surfaces. and in general, also extreme-values (path curves) found in flux as their trajectories. The latter lie firmly in the case discussed above [equation (7)] of a time-independent Hamiltonian function. The characteristic function surfaces according to $\mathcal{J} = S - W_1 t$ expand then beyond the fixed surfaces S = constant to the extent that they always coincide with a new S-surface. The picture is that of the emission of a series of waves as one usually thinks of it, for example, in optical processes.

If we take the initial surface F = 0 as the excitation surface of an optical process, the extreme-values are the light rays in the sense of geometrical optics and the expanding characteristic function surfaces are surfaces of like phase, therefore a kind of wave surface in the sense of the Huygens principle. The principle of least action then agrees exactly with the Fermat principle of shortest light path when we assume the refraction index in the q-space to be proportional to the root of the kinetic energy, which is equal to W - U, therefore, also a pure position function. Thus the solution of the mechanical problem is related to that of the corresponding optical problem. The path curves fall together with the rays of optics. The Hamilton-Jacobi theory corresponds thus with geometrical optics. These ideas recently have become the basis for the further development of quanten mechanics by Schrodinger[1], which is based on the concept that from the mechanics of atoms one does not go directly to that of wave optics, but an extension in the sense of the actual wave must lie as the basis[2].

[1] E. Schrodinger, Leipzig 1927.

[2] "Optik und Mechanik", von A. Lande in Bd. XX ds. Handb.