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ANALYSIS OF MAXIMUM RANGE TRAJECTORIES FOR ROCKET-PROPELLED LUNAR FLYING VEHICLES IN A UNIFORM GRAVITATIONAL FIELD

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16. Abstract

The problem of obtaining maximal-range planar soft landing trajectories for a thrust-limited rocket having a prescribed amount of fuel and subject to constant gravitational acceleration and negligible aerodynamic forces has been analyzed in nondimensional form. Bounds on the thrust magnitude are treated parametrically. Nondimensional curves which allow the calculation of extremal trajectories for a wide range of vehicle designs are given. Least-square approximate equations linear in the ratio of final mass to initial mass are presented for control parameters which may be used to determine extremal control time histories completely. These equations could therefore be useful in analytical design studies, for example, those for which data are needed on maximum attitude angle, maximum attitude rates, and so forth. For very short range trajectories, relationships to impulsive solutions are noted and comparisons are made between solutions with variable thrust direction and solutions where the thrust direction is held constant during thrust periods. Dimensional examples with lunar flying vehicles are presented.

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ANALYSIS OF MAXIMUM RANGE TRAJECTORIES FOR ROCKET-PROPELLED LUNAR FLYING VEHICLES IN A UNIFORM GRAVITATIONAL FIELD

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SUMMARY

The problem of obtaining maximal-range soft landing trajectories for a thrust-limited rocket having a prescribed amount of fuel under the assumptions of planar flight subject to uniform gravitational acceleration and negligible aerodynamic forces is analyzed through the Pontryagin maximum principle and the Davidon method of minimization. Bounds on the rocket thrust magnitude are treated parametrically. Nondimensional curves are given for calculating extremal trajectories for a wide range of vehicle designs. Least-square approximate equations linear in the ratio of final mass to initial mass are presented for control parameters which may be used to determine extremal control time histories completely. These equations could therefore be useful in analytical design studies, for example, those for which data are needed on maximum attitude angle, maximum attitude rates, and so forth.

For zero lower thrust bound cases in which the extremal range is short because of the use of small amounts of fuel, the solutions approach impulsive results. Also, for zero lower thrust bound cases having short burn periods, it is shown that the results of the maximum principle for variable thrust direction may be closely approximated by use of a constant thrust direction over the thrusting intervals.

INTRODUCTION

Rocket flight may become a very practical method for traveling over the lunar surface. Numerous rocket-propelled flying vehicles have been proposed for transportation in the vicinity of the moon. A study of such vehicles is presented in reference 1. In the design and use of these vehicles, it is desirable to know the longest range attainable by a given rocket for a prescribed amount of fuel. Accordingly, this report analyzes thrust-limited rocket trajectories which maximize the range traveled for a given amount of fuel and negligible aerodynamic forces. Since men or fragile instruments may be onboard the rocket, a soft landing constraint is imposed on the trajectories; that is, the final velocity and altitude of the rocket must be close to zero. The study considers planar trajectories which have sufficiently low altitude and short range to insure that gravitational acceleration may be assumed to be constant and the curvature of the moon's surface may be neglected. The dynamic equations are nondimensionalized and the optimal thrust control law is determined by use of the Pontryagin maximum principle. Bounds on the thrust magnitude are treated parametrically.

Problem formulations such as described, that is, problems concerning a thrust-limited rocket with the simplifying assumptions of planar flight subject to constant gravitational acceleration and negligible aerodynamic forces, form a fundamental and important class in the study of trajectory analysis. Trajectory optimization problems – such as, thrust law for minimum fuel, maximum range, or minimum time – using this model have been frequently discussed in the literature. (See refs. 2 to 8.) However, the purpose of these discussions has, in general, been only to give qualitative results. The purpose of this report is to give quantitative results which will be useful in guidance studies.

The analysis presented may be considered a generalization of the similar quantitative study by Manci (ref. 9) who considers a minimum fuel-fixed range problem with the same dynamic model and the same assumptions as this study but with a zero lower thrust bound. The generalization follows from the parametric treatment of thrust bound (which allows zero lower thrust bound as a special case) and the duality that exists between the minimum fuel-fixed range problem and the fixed-fuel—maximum-range problem. The analysis presented also differs from reference 9 in that, for zero lower thrust bound cases with short thrusting periods, comparisons of optimal performance are made between solutions in which the thrust direction is held fixed over the thrusting intervals and solutions in which the thrust direction is allowed to vary.

SYMBOLS

$A = 1 + \alpha_2^2$

$\tilde{A} = Q\left[\bar{t}_1 - P\left(\overline{m}_o + \overline{m}_f\right)\right] + P\left(\overline{m}_f - \overline{m}_o + \bar{t}_1\right)$

$B = -2\left(\alpha_1 + \alpha_2\alpha_3\right)$

$\tilde{B} = PQ\left(P^2 - Q^2\right)$

$C = \alpha_1^2 + \alpha_3^2$

$\tilde{C} = P^3\left[Q\left(\overline{m}_f - \overline{m}_o\right) + \overline{m}_o - \overline{m}_f - \bar{t}_1\right] + P^2Q\bar{t}_1 + P\left[Q^3\left(\overline{m}_o - \overline{m}_f\right) + Q^2\left(\overline{m}_f - \overline{m}_o + \bar{t}_1\right)\right] - Q^3\bar{t}_1$

c effective exhaust speed

D,E,F,M,N arbitrary constants in equation (15)

$$f = w_1\left[\dot{\bar{x}}(\bar{t}_f)\right]^2 + w_2\left[\bar{y}(\bar{t}_f)\right]^2 + w_3\left[\dot{\bar{y}}(\bar{t}_f)\right]^2$$

$$f_s = p_2 \frac{\cos\theta}{\bar{\bar{m}}} + p_4 \frac{\sin\theta}{\bar{\bar{m}}} - p_5 \quad \text{(switching function)}$$

$f_s(t^-)$ value of f_s obtained from $\lim\limits_{\substack{x\to t\\ x<t}} f_s(x)$

g constant gravitational acceleration

$$H = -p_0\dot{\bar{x}} + p_1\dot{\bar{x}} + p_2\left(\frac{\bar{T}\cos\theta}{\bar{\bar{m}}}\right) + p_3\dot{\bar{y}} + p_4\left(\frac{\bar{T}\sin\theta}{\bar{\bar{m}}} - 1\right) - p_5\bar{T}$$

m mass

$$P = \log_e\left(\frac{\bar{\bar{m}}_o}{\bar{\bar{m}}_o - \bar{t}_1}\right)$$

p_i Pontryagin auxiliary variables (i = 0, 1, . . ., 5)

$$Q = \log_e\left(\frac{\bar{\bar{m}}_f}{\bar{\bar{m}}_o - \bar{t}_1}\right)$$

s dummy variable

T thrust

t time

t_1,t_2 switching times

W_f final weight

W_o initial weight

w_i weights for f (i = 1, . . ., 3)

x downrange distance

y altitude

$\alpha_i = \left(p_{i+1}(0)\right)_n \quad (i = 1, \ldots, 3)$

θ angle thrust vector makes with approaching horizon

θ_1, θ_2 constant attitude angles

$\varphi(s) = As^2 + Bs + C$

Subscripts:

f final

max maximum

min minimum

n normalized (divided by $-p_o$)

o initial

Operations:

$(\bar{\ })$ ( ) nondimensionalized

$(\dot{\ })$ ( ) differentiated with respect to time (that is, with respect to t or $\bar{t}$, whichever is appropriate)

$(\ddot{\ })$ ( ) twice differentiated with respect to dimensional or nondimensional time

DYNAMIC EQUATIONS

The coordinate system chosen for the problem is given in figure 1. Downrange distance is x, altitude is y, the thrust magnitude is T $\left(T_{min} \leqq T \leqq T_{max}\right)$, and the thrust angle is θ. The dynamic equations for the stated problem are based on the assumption that the rocket is to be launched from rest at $t = 0$ and is required to perform a soft landing at $t = t_f$, and are (ref. 10):

$$\ddot{x} = \frac{T \cos \theta}{m} \qquad \left(x(0) = 0; \quad \dot{x}(0) = 0; \quad \dot{x}(t_f) = 0\right)$$

$$\ddot{y} = \frac{T \sin \theta}{m} - g \qquad \left(y(0) = 0; \quad \dot{y}(0) = 0; \quad y(t_f) = 0; \quad \dot{y}(t_f) = 0\right) \tag{1}$$

$$\dot{m} = -\frac{T}{c} \qquad \left(m(0) = m_o; \quad m(t_f) = m_f\right)$$

where

m_o	known initial mass
m_f	known terminal mass
c	effective exhaust speed of rockets
g	constant gravitational acceleration

and $x(t_f)$ is to be maximized. As in reference 9, equation (1) may be nondimensionalized by

$$\overline{\text{Length}} = \text{Length}\left(\frac{g}{c^2}\right)$$

$$\overline{\text{Mass}} = \text{Mass}\left(\frac{g}{T_{max}}\right)$$

$$\overline{\text{Thrust}} = \frac{\text{Thrust}}{T_{max}}$$

$$\overline{\text{Time}} = \text{Time}\left(\frac{g}{c}\right)$$

The bars indicate nondimensional quantities. The nondimensional equations corresponding to equation (1) are

$$\ddot{\bar{x}} = \frac{\bar{T} \cos \theta}{\bar{m}} \qquad \left(\bar{x}(0) = 0; \quad \dot{\bar{x}}(0) = 0; \quad \dot{\bar{x}}(\bar{t}_f) = 0\right)$$

$$\ddot{\bar{y}} = \frac{\bar{T} \sin \theta}{\bar{m}} - 1 \qquad \left(\bar{y}(0) = 0; \quad \dot{\bar{y}}(0) = 0; \quad \bar{y}(\bar{t}_f) = 0; \quad \dot{\bar{y}}(\bar{t}_f) = 0\right) \tag{2}$$

$$\dot{\bar{m}} = -\bar{T} \qquad \left(\bar{m}(0) = \bar{m}_o; \quad \bar{m}(\bar{t}_f) = \bar{m}_f\right)$$

where

$$\frac{T_{min}}{T_{max}} \leqq \bar{T} \leqq 1$$

The ($\dot{}$) and ($\ddot{}$) notations are used for differentiation with respect to nondimensional time as well as with respect to dimensional time. Only solutions with positive altitude are considered.

OPTIMAL CONTROL

Equations of Optimal Motion

Pontryagin's maximum principle (ref. 11) was applied to the foregoing maximum range problem with the stated equations and boundary conditions. The maximum principle is a necessary condition which the control of a problem must satisfy in order to be optimal. The controls of the problem are the thrust magnitude $\bar{T}$ and attitude θ. The quantity to be optimized is the range $\bar{x}(\bar{t}_f)$. The maximum principle states that the controls which minimize the negative of the range (that is, maximize the range) will be the same as those which maximize

$$H = -p_0\dot{\bar{x}} + p_1\dot{\bar{x}} + p_2\left(\frac{\bar{T}\cos\theta}{\bar{m}}\right) + p_3\dot{\bar{y}} + p_4\left(\frac{\bar{T}\sin\theta}{\bar{m}} - 1\right) - p_5\bar{T}$$

for every $\bar{t}$ with the auxiliary variables p_i $(i = 0, 1, \ldots, 5)$ given by

$$\dot{p}_0 = 0 \qquad (p_0 \leqq 0) \tag{3}$$

$$\dot{p}_1 = 0 \qquad \left(p_1(\bar{t}_f) = 0\right) \tag{4}$$

$$\dot{p}_2 = p_0 - p_1 \tag{5}$$

$$\dot{p}_3 = 0 \tag{6}$$

$$\dot{p}_4 = -p_3 \tag{7}$$

$$\dot{p}_5 = \frac{\bar{T}}{\bar{m}^2}\left(p_2\cos\theta + p_4\sin\theta\right) \tag{8}$$

The principle also states that when H has been maximized, it will be zero for all $\bar{t}$.

The part of H that involves $\bar{T}$ is

$$\bar{T}\left(p_2\frac{\cos\theta}{\bar{m}} + p_4\frac{\sin\theta}{\bar{m}} - p_5\right) = \bar{T}f_s$$

where f_s is called a "switching function." It is clear that no matter what value θ assumes, the $\overline{T}$ which maximizes H will be (ref. 11)

$$\overline{T}(\bar{t}) = \begin{cases} 1 & f_s(\bar{t}) > 0 \\ \dfrac{T_{min}}{T_{max}} & f_s(\bar{t}) < 0 \end{cases} \tag{9}$$

For $\bar{t} > 0$ so that $f_s(\bar{t}) = 0$, $\overline{T}(\bar{t})$ takes the value given by equation (9) determined from the sign of $f_s(\bar{t}^-)$.

It will be subsequently demonstrated that $f_s(0) > 0$ and singular arcs (where $f_s(\bar{t}) \equiv 0$ over a nonzero time interval) cannot occur. Thus the maximum principle asserts that the engine thrust is either fully on $(T = T_{max})$ or at its lowest value $(T = T_{min})$. The zeros of $f_s(\bar{t})$ corresponding to transitions between $\overline{T} = 1$ $(T = T_{max})$ and $\overline{T} = T_{min}/T_{max}$ $(T = T_{min})$ are called "switching times."

In order to determine the thrust attitude, that part of H involving θ is examined. The terms containing θ are

$$\frac{\overline{T}}{m}\left(p_2 \cos \theta + p_4 \sin \theta\right)$$

The terms in parentheses may be considered as the inner product

$$\begin{pmatrix} p_2 \\ p_4 \end{pmatrix} \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

To maximize this product it is necessary to choose the unit vector $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ to have the direction of $\begin{pmatrix} p_2 \\ p_4 \end{pmatrix}$; thus,

$$\begin{pmatrix} \cos \theta \\ \\ \sin \theta \end{pmatrix} = \begin{pmatrix} \dfrac{p_2}{\sqrt{p_2^2 + p_4^2}} \\ \\ \dfrac{p_4}{\sqrt{p_2^2 + p_4^2}} \end{pmatrix} \tag{10}$$

From equations (4) to (7),

$$p_1(\bar{t}) = p_1(\bar{t}_f) = 0$$

$$p_2(\bar{t}) = p_2(0) + p_0 \bar{t}$$

$$p_3(\bar{t}) = p_3(0)$$

$$p_4(\bar{t}) = p_4(0) - p_3(0)\bar{t}$$

From equation (3) p_0 is a nonpositive constant. If p_0 were zero, then $p_2(\bar{t})$ would be constant and $\cos\theta$ would have the same sign for all $\bar{t}$. However, physically it is known that $\cos\theta$ must be positive along some portion of the trajectory (in order to achieve positive range) and negative along the terminal portion (in order to achieve a soft landing). Therefore, p_0 must not be zero and $p_2(0) > 0$. Since only positive altitude solutions are considered

$$\sin\theta(0) > 0$$

or

$$\frac{p_4(0)}{\sqrt{p_2^2(0) + p_4^2(0)}} > 0$$

whereby $p_4(0) > 0$. If $p_2(\bar{t})$, $p_3(\bar{t})$, and $p_4(\bar{t})$ are divided by $-p_0$, neither $\theta(\bar{t})$ nor the switching times are changed and the equations for $p_2(\bar{t})$ and $p_4(\bar{t})$ may be expressed in terms of three rather than four constants:

$$p_{2,n}(\bar{t}) = \alpha_1 - \bar{t} \tag{11}$$

$$p_{4,n}(\bar{t}) = \alpha_3 - \alpha_2\bar{t} \tag{12}$$

where

$$\alpha_1 = \frac{p_2(0)}{-p_0}$$

$$\alpha_2 = \frac{p_3(0)}{-p_0}$$

$$\alpha_3 = \frac{p_4(0)}{-p_0}$$

An n subscript means that the equations have been normalized by $-p_0$.

Next consider the determination of the switching times. Differentiating $f_{s,n}$ by using equations (2), (8), (10), (11), and (12) gives

$$\dot{f}_{s,n} = \frac{\frac{d}{d\bar{t}}\left(p_{2,n}\cos\theta + p_{4,n}\sin\theta\right)}{\bar{m}(\bar{t})} - \left(p_{2,n}\cos\theta + p_{4,n}\sin\theta\right)\frac{\dot{\bar{m}}}{\bar{m}^2} - \dot{p}_{5,n}$$

$$= \frac{\frac{d}{d\bar{t}}\sqrt{p_{2,n}^{\ \ 2} + p_{4,n}^{\ \ 2}}}{\bar{m}(\bar{t})}$$

$$= \frac{\frac{B}{2} - A\bar{t}}{\bar{m}(\bar{t})\sqrt{\varphi(\bar{t})}} \tag{13}$$

with

$$A = 1 + \alpha_2^{\ 2}$$

$$B = -2\left(\alpha_1 + \alpha_2\alpha_3\right)$$

$$C = \alpha_1^{\ 2} + \alpha_3^{\ 2}$$

and

$$\varphi(\bar{t}) = A\bar{t}^2 + B\bar{t} + C$$

Since the numerator of $\dot{f}_{s,n}(\bar{t})$ is linear in $\bar{t}$, $\dot{f}_{s,n}(\bar{t})$ can have at most one zero. By Rolle's theorem from elementary calculus, $f_{s,n}(\bar{t})$ can have at most two zeros. Since $A > 0$, singular arcs cannot be optimal.

The condition $H = 0$ at $\bar{t} = 0$ gives

$$\bar{T}(0)f_{s,n}(0) - \alpha_3 = 0$$

and implies

$$f_{s,n}(0) = \alpha_3 \tag{14}$$

Thus $f_{s,n}(\bar{t})$ is determined by the differential equation (13) with initial condition (14).

Suppose now that a set α_i (i = 1, 2, and 3) is found which yields a trajectory having positive altitude and satisfying the fuel constraint

$$\bar{m}(\bar{t}_f) = \bar{m}_f$$

A set α_i (i = 1, 2, and 3) satisfying the positive altitude condition may be found by either trial and error or by the use of impulsive results. The fuel constraint can be satisfied by using

$$\bar{m}(\bar{t}_f) = \bar{m}_f$$

to determine $\bar{t}_f$. The first switching time $\bar{t}_1$ (if it should occur) may be sought by use of differential equation (13) for the switching function $f_{s,n}(\bar{t})$. With $\bar{T} = 1$ and $\bar{m}(\bar{t}) = \bar{m}_o - \bar{t}$, equation (13) can be integrated in closed form over $[0, \bar{t}_1]$ by making use of the indefinite integral (ref. 12)

$$\int \frac{Ds^2 + Es + F}{(Ms + N)\sqrt{\varphi(s)}}\, ds = \frac{1}{M}\left\{\frac{D}{A}\sqrt{\varphi(s)} + \frac{2A\left(E - \frac{DN}{M}\right) - BD}{2A\sqrt{A}} \log_e\left[B + 2As + 2\sqrt{A\varphi(s)}\right]\right\}$$

$$+ \frac{FM^2 - NME + DN^2}{M^2\sqrt{AN^2 + CM^2 - BMN}} \log_e\left\{\frac{M\left[2CM - BN + (BM - 2AN)s - 2\sqrt{(AN^2 + CM^2 - BMN)\varphi(s)}\right]}{Ms + N}\right\} \qquad (M \neq 0) \cdot \qquad (15)$$

where $\varphi(s) = As^2 + Bs + C$. This operation yields

$$f_{s,n}(\bar{t}_1) = \alpha_3 + \frac{\frac{B}{2} + A\bar{m}_o}{\sqrt{\varphi(\bar{m}_o)}} \log_e\left[\frac{\dfrac{2C + B\bar{m}_o + (B + 2A\bar{m}_o)\bar{t}_1 + 2\sqrt{\varphi(\bar{m}_o)\varphi(\bar{t}_1)}}{\bar{m}_o - \bar{t}_1}}{\dfrac{2C + B\bar{m}_o + 2\sqrt{\varphi(\bar{m}_o)C}}{\bar{m}_o}}\right]$$

$$- \sqrt{A}\, \log_e\left[\frac{B + 2A\bar{t}_1 + 2\sqrt{A\varphi(\bar{t}_1)}}{B + 2\sqrt{AC}}\right] \qquad (16)$$

where $f_{s,n}(0) = \alpha_3$ has been employed. Then $\bar{t}_1$ is the first positive real root of $f_{s,n}(\bar{t}_1) = 0$ in the interval $[0, \bar{m}_o - \bar{m}_f]$ which may be found with Newton's method from numerical analysis. If no such root occurs in the interval $[0, \bar{m}_o - \bar{m}_f]$, the rocket continuously burns with $\bar{T} = 1$ until the allotted fuel is consumed.

When $\bar{t}_1$ occurs, a second root $\bar{t}_2 \geqq \bar{t}_1$ may exist. For the $T_{min} = 0$ case, $\bar{t}_2$ has a simple relation to $\bar{t}_1$. Since mass is constant over $(\bar{t}_1, \bar{t}_2]$, integration of equation (13) over $(\bar{t}_1, t_2]$ yields

$$f_{s,n}(\bar{t}_2) = f_{s,n}(\bar{t}_1) + \left. \frac{\sqrt{p_{2,n}^2(\bar{t}) + p_{4,n}^2(\bar{t})}}{\overline{m}(\bar{t}_1)} \right]_{\bar{t}_1}^{\bar{t}_2}$$

and $f_{s,n}(\bar{t}_1) = f_{s,n}(\bar{t}_2) = 0$ gives

$$p_{2,n}^2(\bar{t}_2) + p_{4,n}^2(\bar{t}_2) = p_{2,n}^2(\bar{t}_1) + p_{4,n}^2(\bar{t}_1)$$

Substituting for $p_{2,n}$ and $p_{4,n}$ from equations (11) and (12) gives, for $\bar{t}_2$ different from $\bar{t}_1$,

$$\bar{t}_2 = -\frac{B}{A} - \bar{t}_1 = \frac{2(\alpha_1 + \alpha_2\alpha_3)}{1 + \alpha_2^2} - \bar{t}_1 \tag{17}$$

From $\bar{t}_2 - \bar{t}_1 > 0$ and equation (17),

$$\bar{t}_1 < \frac{\alpha_1 + \alpha_2\alpha_3}{1 + \alpha_2^2}$$

This inequality is useful for applying Newton's method to equation (16). For $T_{min} \neq 0$, equation (15) may again be applied to integrate equation (14) $\left(\text{with } \overline{T} = T_{min}/T_{max}\right)$ with $\overline{m}(\bar{t}) = \overline{m}(\bar{t}_1) - \frac{T_{min}}{T_{max}}(\bar{t} - \bar{t}_1)$ over $(\bar{t}_1, \bar{t}_2]$ to obtain

$$f_{s,n}(\bar{t}_2) = f_{s,n}(\bar{t}_1) + \frac{BM - 2AN}{2M\sqrt{AN^2 + CM^2 - BMN}}$$

$$\times \log_e \left[\frac{\dfrac{2CM^2 - BMN + (BM^2 - 2AMN)\bar{t}_2 - 2M\sqrt{(AN^2 + CM^2 - BMN)\varphi(\bar{t}_2)}}{\overline{m}(\bar{t}_2)}}{\dfrac{2CM^2 - BMN + (BM^2 - 2AMN)\bar{t}_1 - 2M\sqrt{(AN^2 + CM^2 - BMN)\varphi(\bar{t}_1)}}{\overline{m}(\bar{t}_1)}} \right]$$

$$+ \frac{\sqrt{A}}{M} \log_e \left[\frac{B + 2A\bar{t}_2 + 2\sqrt{A\varphi(\bar{t}_2)}}{B + 2A\bar{t}_1 + 2\sqrt{A\varphi(\bar{t}_1)}} \right] \tag{18}$$

where

$$f_{s,n}(\bar{t}_1) = 0$$

$$N = (\overline{m}_o - \bar{t}_1) + \frac{T_{min}}{T_{max}} \bar{t}_1$$

$$M = - \frac{T_{min}}{T_{max}}$$

By Newton's method, $\bar{t}_2$ can be found as the real root, $\bar{t}_2 > \bar{t}_1$, of

$$f_{s,n}(\bar{t}_2) = 0$$

with $f_{s,n}(\bar{t}_2)$ given by equation (18). Trajectories in which $\bar{t}_1$ and $\bar{t}_2$ exist are composed of three subarcs – initial and final subarcs with $\overline{T} = 1$ and an intermediate subarc with $\overline{T} = T_{min}/T_{max}$.

For positive altitude trajectories satisfying the fuel constraint, a single root of equation (16) can occur only when $f_{s,n}(\bar{t})$ has a minimum at $\bar{t}_1$. In this event the trajectory is composed of one arc obtained by using $\overline{T}(\bar{t}) = 1$ for $\bar{t} \in [0, \bar{t}_f]$ where $\bar{t}_f = \overline{m}_o - \overline{m}_f$.

For trajectories composed of three arcs, $\overline{m}(\bar{t})$ has the form:

$$\overline{m}(\bar{t}) = \overline{m}_o - \bar{t} \qquad \left(\bar{t} \in [0, \bar{t}_1]\right) \tag{19}$$

$$\overline{m}(\bar{t}) = \overline{m}(\bar{t}_1) - \frac{T_{min}}{T_{max}}(\bar{t} - \bar{t}_1) \qquad \left(\bar{t} \in (\bar{t}_1, \bar{t}_2]\right) \tag{20}$$

$$\overline{m}(\bar{t}) = \overline{m}(\bar{t}_2) - (\bar{t} - \bar{t}_2) \qquad \left(\bar{t} \in (\bar{t}_2, \bar{t}_f]\right) \tag{21}$$

so that

$$\overline{m}_f = \overline{m}(\bar{t}_f) = \bar{t}_2 + (\overline{m}_o - \bar{t}_1) - \frac{T_{min}}{T_{max}}(\bar{t}_2 - \bar{t}_1) - \bar{t}_f$$

The final time is then a function of the known quantities $\overline{m}_o$, $\overline{m}_f$, $\bar{t}_1$, and $\bar{t}_2$ by the equation

$$\bar{t}_f = (\overline{m}_o - \overline{m}_f) + (\bar{t}_2 - \bar{t}_1) - \frac{T_{min}}{T_{max}}(\bar{t}_2 - \bar{t}_1) \tag{22}$$

With $\overline{m}(\bar{t})$ given by equations (19) to (21) and $p_{2,n}(\bar{t})$ and $p_{4,n}(\bar{t})$ given by equations (11) and (12), the equations of motion for $\bar{t} \in [0, \bar{t}_f]$ have solutions

$$\bar{x}(\bar{t}) = \bar{x}(\bar{t}_o) + (\bar{t} - \bar{t}_o)\dot{\bar{x}}(t_o) + \int_{\bar{t}_o}^{\bar{t}} \frac{(\bar{t} - s)\overline{T}(\alpha_1 - s)}{\overline{m}(s)\sqrt{\varphi(s)}}\, ds \tag{23}$$

$$\dot{\bar{x}}(\bar{t}) = \dot{\bar{x}}(\bar{t}_o) + \int_{\bar{t}_o}^{\bar{t}} \frac{\overline{T}(\alpha_1 - s)}{\overline{m}(s)\sqrt{\varphi(s)}}\, ds \tag{24}$$

$$\bar{y}(\bar{t}) = \bar{y}(\bar{t}_o) + (\bar{t} - \bar{t}_o)\dot{\bar{y}}(\bar{t}_o) + \int_{\bar{t}_o}^{\bar{t}} (\bar{t} - s)\left[\frac{\overline{T}(\alpha_3 - \alpha_2 s)}{\overline{m}(s)\sqrt{\varphi(s)}} - 1\right] ds \tag{25}$$

$$\dot{\bar{y}}(\bar{t}) = \dot{\bar{y}}(\bar{t}_o) + \int_{\bar{t}_o}^{\bar{t}} \frac{\overline{T}(\alpha_3 - \alpha_2 s)}{\overline{m}(s)\sqrt{\varphi(s)}} - 1\, ds \tag{26}$$

where

$$\left.\begin{aligned} \bar{t}_o &= 0 \\ \overline{T} &= 1 \\ \overline{m}(s) &= \overline{m}_o - s \end{aligned}\right\} \bar{t} \in \left[0, \bar{t}_1\right] \tag{27}$$

$$\left.\begin{aligned} \bar{t}_o &= \bar{t}_1 \\ \overline{T} &= \frac{T_{min}}{T_{max}} \\ \overline{m}(s) &= \overline{m}_o - \bar{t}_1 - \frac{T_{min}}{T_{max}}(s - t_1) \end{aligned}\right\} \bar{t} \in \left(\bar{t}_1, \bar{t}_2\right] \tag{28}$$

and

$$\left.\begin{aligned} \bar{t}_o &= \bar{t}_2 \\ \overline{T} &= 1 \\ \overline{m}(s) &= \overline{m}_o - \bar{t}_1 - \frac{T_{min}}{T_{max}}\left(\bar{t}_2 - \bar{t}_1\right) - \left(s - \bar{t}_2\right) \end{aligned}\right\} \bar{t} \in \left(\bar{t}_2, \bar{t}_f\right] \tag{29}$$

For a trajectory possessing three arcs, closed-form solutions of equations (23) to (26) can be obtained by the successive application of equation (15). For a single maximal thrust arc, only equation (15) and equations (23) to (26) with

$$\bar{m}(\bar{t}) = \bar{m}_o - \bar{t} \qquad \left(\bar{t}\in\left[0,\bar{m}_o - \bar{m}_f\right]\right)$$

are needed.

The Optimal Control Law

The optimal control law for maneuvering the vehicle described by equation (1) from launch to a soft landing while using a given amount of fuel and maximizing the range is given by equations (9) and (10). Of course, synthesis of the control law requires equations (11) to (29). For positive altitude, the extremal trajectories either are composed of one maximal thrust arc or three arcs generated by the thrust sequence:

(1) A maximum thrust period from $\bar{t} = 0$ to $\bar{t} = \bar{t}_1$ (the first switching time)

(2) A minimal thrust period from $\bar{t} = \bar{t}_1$ to $\bar{t} = \bar{t}_2$ (the second switching time)

(3) A maximal thrust period from $\bar{t} = \bar{t}_2$ to the final time $\bar{t}_f$.

Once a set of constants α_i (i = 1, 2, and 3) which generates a trajectory having positive altitude and satisfying the fuel constraint has been found, a procedure has been indicated for constructing the corresponding trajectory. Zeros of equations (16) and (18) (obtained numerically) provide switching times. Either equation (22) or $\bar{t}_f = \bar{m}_o - \bar{m}_f$ determines the final time. Once this information has been given, closed-form solutions of equations (23) to (26) which completely specify the trajectory can be obtained with the aid of equation (15). There is however no guarantee that the given trajectory will satisfy the soft landing conditions. In general, it should only possess positive altitude and satisfy the fuel constraint. The next section details a procedure for correcting the given constants α_i (i = 1, 2, and 3) to meet the soft landing conditions.

Determination of Corrected Constants

The integrated form of equations (24) to (26) evaluated at $\bar{t}_f$ yields equations for $\bar{y}(\bar{t}_f)$, $\dot{\bar{x}}(\bar{t}_f)$, and $\dot{\bar{y}}(\bar{t}_f)$ explicitly in terms of the α_i (i = 1, 2, and 3). Setting these equations equal to zero gives a system of equations in α_i which satisfy the soft landing conditions. This system can be solved by minimizing to zero with respect to the α_i (i = 1, 2, and 3) the function

$$f(\alpha_1,\alpha_2,\alpha_3) = w_1\left[\dot{\bar{x}}(\bar{t}_f)\right]^2 + w_2\left[\bar{y}(\bar{t}_f)\right]^2 + w_3\left[\dot{\bar{y}}(\bar{t}_f)\right]^2 \qquad (30)$$

where w_1, w_2, and w_3 are positive weights. Constants α_i (i = 1, 2, and 3) which cause $f(\alpha_1,\alpha_2,\alpha_3)$ to vanish and generate trajectories satisfying the fuel constraint are the type desired.

In the sections to follow, examples of the results obtainable from the foregoing analysis are presented. In all cases it was found that an initial set of α_i (i = 1, 2, and 3) satisfying the altitude and fuel constraint could be easily obtained. The value of $f(\alpha_1,\alpha_2,\alpha_3)$ given by equation (30), with weights $w_1 = 10^{-3}$, $w_2 = 10^3$, and $w_3 = 10^{-3}$, was iteratively minimized by use of the Langley Research Center Control Data 6600 digital computer and the Stewart modification of the Davidon method of minimization. (See refs. 13, 14, and 15.) Rapid convergence was obtained to a set of constants α_i (i = 1, 2, and 3) satisfying all the required conditions. A representative computing time is 20 seconds. A listing of the program employed is available from the authors at the Langley Research Center.

RESULTS

The results obtained are of two types: nondimensional and dimensional. Nondimensional results are completely general with respect to all g, c, and T_{max}. Solutions for a range of payload fractions $\overline{m}_f/\overline{m}_o$ with initial mass as a parameter are found for $T_{min} = 0$ and $T_{min} = 0.1T_{max}$. Although both one arc and three subarc trajectories may be extremal, no one arc trajectories were found which satisfied all the necessary conditions; only three subarc trajectories are presented.

Nondimensional Results

Nondimensional results are given in figures 2 to 8. Least-squares approximations are made for the data points of figures 2 to 5. The equations for these approximations are presented in appendix A.

Figure 2(a) shows the nondimensional range achievable with an extremal trajectory as a function of payload fraction for three values of nondimensional initial mass (planet take-off weight divided by maximum thrust) when $T_{min} = 0$. The figure may also be used to give the fuel required by a minimum fuel trajectory for a given range and initial mass. The fraction of initial rocket mass used as fuel is $1 - \frac{\overline{m}_f}{\overline{m}_o}$. Note that the range increases with increasing initial thrust-weight ratio. The approximating curves for figure 2(a) are quadratic.

Figure 2(b) is the same as figure 2(a) except that $T_{min} = 0.1T_{max}$. The nondimensional range is less for given values of $\overline{m}_o$ and $\overline{m}_f/\overline{m}_o$ than when $T_{min} = 0$. Theoretically, this result is implied from the necessary conditions since if $T_{min} = 0$, then $T = 0$ for $f_s < 0$ satisfies the maximum principle whereas $T = 0.1T_{max}$ for $f_s < 0$ does not satisfy the maximum principle. Physically, gravity is apparently more efficient than rocket thrust for the intermediate period of this type of trajectory. The lessening of nondimensional range is greater for lower $\overline{m}_o$ values. This result is reasonable since a high thrust-weight ratio (lightweight) vehicle will "waste" a greater percentage of its mass during the nonextremal (with respect to the "unconstrained" $T_{min} = 0$ problem) intermediate period. The approximating curves for figure 2(b) are quadratic.

Figure 3(a) shows $\bar{t}_1$ as a function of $\overline{m}_f/\overline{m}_o$ for $T_{min} = 0$. If $\bar{t}_1$ is known, $\bar{t}_2$ and $\bar{t}_f$ may be computed from equations (17) and (22). From these results and recalling that $\overline{m}_o$ is the inverse of the initial thrust-weight ratio, it is seen that a larger thrust-weight ratio will not only produce a longer range but also a shorter (in time) trajectory with a longer coast period for the same fuel fraction. The approximating curves for figure 3(a) are linear.

Figure 3(b) gives $\bar{t}_1$ and $\bar{t}_2$ as functions of $\overline{m}_f/\overline{m}_o$ for $T_{min} = 0.1T_{max}$. Comparison with figure 3(a) reveals that for $T_{min} = 0.1T_{max}$, the intermediate period $(\bar{t}_1,\bar{t}_2]$ is longer, but the entire period of flight $[0,\bar{t}_f]$ is shorter. When $T_{min} = 0.1T_{max}$, fuel is burned during the intermediate period and thus the initial and final periods are forced to be shorter. It seems reasonable to state that if T_{min} is close enough to zero, the time lengths of the entire trajectory and of the intermediate period would be close to those for $T_{min} = 0$. Thus, if the burn periods are shorter for some value of T_{min}, and if this value of T_{min} is sufficiently near zero, the intermediate period should be longer than when $T_{min} = 0$. Also, for T_{min} sufficiently near zero, the entire trajectory should be shorter in time since all the periods (initial, intermediate, and final) would be of about the same length as those for $T_{min} = 0$, but fuel would be depleted more rapidly since $T_{min} \neq 0$. The approximating curves for figure 3(b) are linear.

Figures 4(a), 4(b), and 4(c) show, respectively, α_1, α_2, and α_3 as functions of $\overline{m}_f/\overline{m}_o$ for $T_{min} = 0$. The approximating curves for figure 4 are linear. By using these curves and $\tan\theta(\bar{t}) = \dfrac{\alpha_3 - \alpha_2\bar{t}}{\alpha_1 - \bar{t}}$, the approximate attitude may be calculated for any $\bar{t}$. The parameters α_1, α_2, and α_3 increase with increasing fuel. Since α_1 increases more rapidly than does α_3, the initial thrust angle becomes smaller for larger amounts of fuel (and longer ranges). Since α_2 increases, for longer range problems θ will change more rapidly with time along the extremal trajectory. The parameters α_1 and α_2 increase with $\overline{m}_o$, but α_3 decreases with increasing $\overline{m}_o$ values. Thus, θ is smaller and changes more rapidly when the rocket vehicle is heavier or has a relatively small thrust.

Figures 5(a), 5(b), and 5(c) give, respectively, α_1, α_2, and α_3 as functions of $\overline{m}_f/\overline{m}_o$ with $\overline{m}_o$ as a parameter for $T_{min} = 0.1T_{max}$. A comparison with figures 4(a) to 4(c) reveals that when $T_{min} = 0.1T_{max}$, α_1 is greater, α_2 remains about the same, and α_3 is less. These changes should make $\cos\theta$ larger and $\sin\theta$ smaller for all values of $\bar{t}$. Thus, the values of θ (or after $\theta = 90^{\circ}$, the supplement of θ) for $T_{min} = 0.1T_{max}$ should be smaller than those for $T_{min} = 0$ for all values of $\bar{t}$ for which $\theta \neq 90^{\circ}$ once a time translation has been made to make the two 90° points coincide. Therefore, the rocket is depending more on the thrust and less on gravity for obtaining range than in the $T_{min} = 0$ case. This statement is reasonable since the rocket can no longer gain as much potential energy from its shorter initial burn period or dissipate as much kinetic energy with its shorter final burn period. The approximating curves for figure 5 are linear.

For high thrust-weight ratio $\left(\overline{m}_o \text{ small}\right)$ and large values of $\overline{m}_f/\overline{m}_o$ (between 0.95 and 1) trajectories are produced $\left(\text{for } T_{min} = 0\right)$ which have short ranges and whose properties approach those of impulsive cases. Figure 6 (variable attitude) shows nondimensional range as a function of $\overline{m}_f/\overline{m}_o$ for $\overline{m}_o = 0.2$. It is a magnification of the high $\overline{m}_f/\overline{m}_o$ portion of the curve in figure 2(a). Figure 7 shows for $\overline{m}_o = 0.2$ and $\overline{m}_f/\overline{m}_o$ varying between 0.95 and 1 that maximum altitude increases linearly with range. The slope of the line is about 0.225 compared with the analytically determined slope of 0.250 for the impulsive $\left(\overline{m}_o = 0\right)$ case. (See ref. 9.) Other studies with $\overline{m}_o > 0.2$ indicate that the slope decreases with increasing $\overline{m}_o$. Figure 8 (variable attitude) is a graph of $\bar{t}_1$, $\bar{t}_2$, and $\bar{t}_f$ as functions of $\overline{m}_f/\overline{m}_o$ for $\overline{m}_o = 0.2$. The coast $\left(\overline{T} = 0\right)$ portion is seen to be the predominant part of the trajectory. As $\overline{m}_o$ is further reduced, the results approach the impulsive case in which the trajectory consists entirely of a coast period with initial and terminal thrust impulses.

Also, for short-range problems, the rocket attitude changes little during the burn periods. Since this condition was noted, a study was performed in which the attitude was held constant over each burn period. In the study, the times $\bar{t}_1$, $\bar{t}_2$, and $\bar{t}_f$ were redetermined by taking into account the constant-attitude constraint. Appendix B summarizes the analysis.

Figures 6 and 8 (constant attitude), respectively, give nondimensional range and $\bar{t}_1$, $\bar{t}_2$, and $\bar{t}_f$ as functions of $\overline{m}_f/\overline{m}_o$ for constant attitude and $\overline{m}_o = 0.2$. These results, for $\overline{m}_f/\overline{m}_o$ between 0.95 and 0.995, lie so close to those for the variable-attitude case that any differences shown on the graphs might be attributable to an error in meeting the soft landing conditions in the variable-attitude solutions. The constant-attitude solutions satisfy these conditions exactly. Raw data indicate, however, that for constant attitude the range is less. No clear trend is obtained for $\bar{t}_1$, $\bar{t}_2$, and $\bar{t}_f$ from these cases.

Dimensional Results

Dimensional results obtained from the computer are presented for two vehicles on the lunar surface. The lunar surface gravitational acceleration constant g is approximately 5.315 ft/sec^2. The two vehicles are:

(1) A two-man vehicle modeled after a configuration for the lunar-module ascent stage with

$$T_{max} = 3507.7 \text{ lb} \quad (15\ 603.0 \text{ N})$$

$$c = 9853.2 \text{ ft/sec} \quad (3003.3 \text{ m/s})$$

$$W_o \approx 10\ 500 \text{ lb} \quad (46\ 706.3 \text{ N}) \qquad \left(\bar{m}_o = 0.49\right)$$

$$W_f \approx 8600 \text{ lb} \quad (38\ 254.7 \text{ N}) \qquad \left(\frac{\bar{m}_f}{\bar{m}_o} = 0.82\right)$$

(2) A single-man vehicle proposed in reference 1 with

$$T_{max} = 275 \text{ lb} \quad (1223.3 \text{ N})$$

$$c = 9660 \text{ ft/sec} \quad (2944.4 \text{ m/s})$$

$$W_o \approx 550 \text{ lb} \quad (2446.5 \text{ N}) \qquad \left(\bar{m}_o = 0.33\right)$$

$$W_f \approx 450 \text{ lb} \quad (2001.7 \text{ N}) \qquad \left(\frac{\bar{m}_f}{\bar{m}_o} = 0.82\right)$$

Two-man vehicle.- Figure 9 illustrates dimensional results with $T_{min} = 0$. The times t_1, t_2, and t_f may be dimensionalized from the values of $\bar{t}_1$, $\bar{t}_2$, and $\bar{t}_f$ of figure 3(a), by using the before-given values of $\bar{m}_o$ and $\bar{m}_f/\bar{m}_o$, to give the thrust history. The first burn period is of longer duration than the second. This difference is explained by recognizing that the average mass of the rocket during the second burn period is less than it was during the first burn period.

Figure 9(a) shows the θ time history for extremal control. For $T_{min} = 0$, θ is undefined over the coast arc. Points of interest in figure 9(a) may be approximated through the nondimensional results. The attitude of the rocket with respect to the nearest horizon is $180^o - \theta$ for θ greater than 90^o. The attitude with respect to the nearest horizon is seen to be approximately symmetrical for the two burn periods.

Figures 9(b), 9(c), and 9(d) give the trajectory achieved by the extremal control. Any point on the trajectory may be found with equations (23) to (26). The extremal range $x(t_f)$ and maximum altitude are about 21 and $3\frac{1}{3}$ nautical miles, respectively. The range can also be obtained from figure 2.

One-man vehicle.- Figure 10 illustrates and compares dimensional results for $T_{min} = 0$ and $T_{min} = 0.1T_{max}$. The intermediate period is longer, the two burn periods are shorter, and the trajectory is shorter (in time) when $T_{min} = 0.1T_{max}$ than when $T_{min} = 0$.

Figure 10(a) shows the θ time history for extremal control. For $T_{min} = 0.1T_{max}$, θ is given by the bilinear tangent formula. For $T_{min} = 0$, θ is undefined over the intermediate period, but for comparative purposes with $T_{min} = 0.1T_{max}$, is plotted according to the bilinear formula for the entire period of flight.

Figures 10(b), 10(c), and 10(d) show comparisons for $T_{min} = 0$ and $T_{min} = 0.1T_{max}$ for the extremal trajectories. The range is shorter and the maximum altitude is less when $T_{min} = 0.1T_{max}$. The range for $T_{min} = 0$ is about 23 nautical miles whereas for $T_{min} = 0.1T_{max}$, it is about 22 nautical miles. The maximum altitude for $T_{min} = 0$ is about $4\frac{1}{2}$ nautical miles whereas for $T_{min} = 0.1T_{max}$ it is about $3\frac{1}{2}$ nautical miles. For $T_{min} = 0$, the horizontal velocity during the intermediate period is constant, but when $T_{min} = 0.1T_{max}$, the horizontal velocity climbs to a peak and then falls to a value at t_2 comparable with its value at t_1. The vehicle reaches a smaller maximum and larger minimum vertical velocity during the trajectory when the rocket engines are burning during the intermediate period.

CONCLUDING REMARKS

The problem of obtaining maximal range soft landing trajectories for a thrust-limited rocket having a prescribed amount of fuel under the assumptions of planar flight subject to constant gravitational acceleration and negligible aerodynamic forces has been analyzed through the Pontryagin maximum principle and the Davidon method of minimization. The problem was solved in nondimensional form with the bounds on thrust magnitude treated parametrically. Nondimensional plots were given to allow calculation of extremal trajectories for a wide range of vehicle designs. Least-square approximate equations linear in the ratio of final mass to initial mass are presented for control parameters which may be used to determine extremal control completely. These equations could therefore be useful in analytical design studies, for example, those for which data are needed on maximum attitude angle, maximum attitude rates, and so forth. Dimensional results were given for lunar flights of a rocket vehicle modeled after the ascent stage of the lunar module and a single-man vehicle proposed in NASA CR-365.

All "best extremal" solutions were found to have similar characteristics. Each solution had an initial and terminal maximal thrust period separated by a single period at minimal thrust. Thrusting was initially at some positive angle and the vehicle pitched up as the initial thrust period continued. After the attitude angle had increased to 90° for nonzero lower thrust bounds or after the coast period for zero lower thrust bounds, the supplement of the attitude angle descended a path nearly symmetric to that taken previously by the attitude angle. For zero lower thrust bound cases in which the extremal range was short because of the use of small amounts of fuel, the solutions approached impulsive results. Also for zero lower thrust bound cases having short thrusting periods, it was shown that the performance of extremal solutions with variable thrust attitude may be closely approximated by solutions in which the thrust attitude is held fixed over the burn portions of the trajectory.

Langley Research Center,
National Aeronautics and Space Administration,
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APPENDIX A

LEAST-SQUARES APPROXIMATION

Least-squares polynomial approximations were made to the data points of figures 2 to 5. Quadratic fits were made to the data of figures 2(a) and 2(b) (range as a function of $\overline{m}_f/\overline{m}_o$) since these results showed an obvious quadratic tendency. Linear fits were used for the data of figures 3 to 5 since either the results were close to being linear or their variations from linearity took on no definite pattern so that a linear fit was "as good as any." The data of figures 3 to 5 may be used to determine extremal control completely so that the "best" linear fits might be useful in the analytic design of a simple suboptimal system. The equations for the approximating polynomials are:

For figure 2(a):

$$\overline{\text{Range}} = 0.30256 - 0.61506\frac{\overline{m}_f}{\overline{m}_o} + 0.31284\left(\frac{\overline{m}_f}{\overline{m}_o}\right)^2 \qquad (\overline{m}_o = 0.2)$$

$$\overline{\text{Range}} = 0.27241 - 0.55611\frac{\overline{m}_f}{\overline{m}_o} + 0.28413\left(\frac{\overline{m}_f}{\overline{m}_o}\right)^2 \qquad (\overline{m}_o = 0.4)$$

$$\overline{\text{Range}} = 0.24394 - 0.50028\frac{\overline{m}_f}{\overline{m}_o} + 0.25688\left(\frac{\overline{m}_f}{\overline{m}_o}\right)^2 \qquad (\overline{m}_o = 0.6)$$

For figure 2(b):

$$\overline{\text{Range}} = 0.26854 - 0.54531\frac{\overline{m}_f}{\overline{m}_o} + 0.27705\left(\frac{\overline{m}_f}{\overline{m}_o}\right)^2 \qquad (\overline{m}_o = 0.2)$$

$$\overline{\text{Range}} = 0.26440 - 0.53946\frac{\overline{m}_f}{\overline{m}_o} + 0.27548\left(\frac{\overline{m}_f}{\overline{m}_o}\right)^2 \qquad (\overline{m}_o = 0.4)$$

$$\overline{\text{Range}} = 0.24195 - 0.49606\frac{\overline{m}_f}{\overline{m}_o} + 0.25463\left(\frac{\overline{m}_f}{\overline{m}_o}\right)^2 \qquad (\overline{m}_o = 0.6)$$

For figure 3(a):

$$\bar{t}_1 = 0.10634 - 0.10666\frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.2)$$

$$\bar{t}_1 = 0.21646 - 0.21739\frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.4)$$

$$\bar{t}_1 = 0.33599 - 0.33820 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.6\right)$$

For figure 3(b):

$$\bar{t}_1 = 0.07348 - 0.07358 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.2\right)$$

$$\bar{t}_1 = 0.19178 - 0.19242 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.4\right)$$

$$\bar{t}_1 = 0.32021 - 0.32205 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.6\right)$$

$$\bar{t}_2 = 0.72527 - 0.72961 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.2\right)$$

$$\bar{t}_2 = 0.67959 - 0.68580 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.4\right)$$

$$\bar{t}_2 = 0.62877 - 0.63763 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.6\right)$$

For figure 4(a):

$$\alpha_1 = 0.41055 - 0.41279 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.2\right)$$

$$\alpha_1 = 0.43161 - 0.43463 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.4\right)$$

$$\alpha_1 = 0.45551 - 0.45965 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.6\right)$$

For figure 4(b):

$$\alpha_2 = 0.04901 - 0.04994 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.2\right)$$

$$\alpha_2 = 0.11675 - 0.11761 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.4\right)$$

$$\alpha_2 = 0.25434 - 0.25323 \frac{\overline{m}_f}{\overline{m}_o} \qquad \left(\overline{m}_o = 0.6\right)$$

For figure 4(c):

$$\alpha_3 = 0.38492 - 0.38721 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.2)$$

$$\alpha_3 = 0.37924 - 0.38253 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.4)$$

$$\alpha_3 = 0.37934 - 0.38444 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.6)$$

For figure 5(a):

$$\alpha_1 = 0.39958 - 0.40178 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.2)$$

$$\alpha_1 = 0.42820 - 0.43120 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.4)$$

$$\alpha_1 = 0.45401 - 0.45809 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.6)$$

For figure 5(b):

$$\alpha_2 = 0.08132 - 0.08458 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.2)$$

$$\alpha_2 = 0.11943 - 0.12043 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.4)$$

$$\alpha_2 = 0.25498 - 0.25390 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.6)$$

For figure 5(c):

$$\alpha_3 = 0.29823 - 0.29957 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.2)$$

$$\alpha_3 = 0.35315 - 0.35593 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.4)$$

$$\alpha_3 = 0.37116 - 0.37594 \frac{\overline{m}_f}{\overline{m}_o} \qquad (\overline{m}_o = 0.6)$$

APPENDIX B

CONSTANT-ATTITUDE STUDY

This section outlines an analysis of the optimal trajectory problem of this report with the added assumptions that the optimal trajectory is composed of two maximal thrust arcs separated by one coast arc and that the thrust attitude θ is held constant over the thrusting periods. By defining θ_1 and θ_2 to be, respectively, the fixed attitudes over the initial and terminal burn periods, equations (2) may be integrated and the conditions

$$\dot{\bar{x}}\left(\bar{t}_f\right) = 0$$

$$\bar{y}\left(\bar{t}_f\right) = 0$$

$$\dot{\bar{y}}\left(\bar{t}_f\right) = 0$$

$$\bar{m}\left(\bar{t}_f\right) = \bar{m}_f$$

applied to determine the unknowns $\bar{t}_2$, $\bar{t}_f$, θ_1, and θ_2 in terms of the single unknown $\bar{t}_1$. This procedure yields

$$\bar{t}_f = -\frac{\tilde{B} \pm \sqrt{\tilde{B}^2 - 4\tilde{A}\tilde{C}}}{2\tilde{A}} \tag{B1}$$

$$\bar{t}_2 = \bar{t}_f - \left(\bar{m}_o - \bar{m}_f\right) + \bar{t}_1 \tag{B2}$$

$$\sin\theta_1 = \frac{P^2 - Q^2 - \bar{t}_f^{\,2}}{2Q\bar{t}_f} \tag{B3}$$

$$\sin\theta_2 = \frac{P^2 - Q^2 + \bar{t}_f^{\,2}}{2P\bar{t}_f} \tag{B4}$$

where

$$P = \log_e \frac{\bar{m}_o}{\bar{m}_o - \bar{t}_1}$$

$$Q = \log_e \frac{\bar{m}_f}{\bar{m}_o - \bar{t}_1}$$

and

$$\tilde{A} = Q\left[\bar{t}_1 - P\left(\overline{m}_o + \overline{m}_f\right)\right] + P\left(\overline{m}_f - \overline{m}_o + \bar{t}_1\right)$$

$$\tilde{B} = PQ\left(P^2 - Q^2\right)$$

$$\tilde{C} = P^3\left[Q\left(\overline{m}_f - \overline{m}_o\right) + \overline{m}_o - \overline{m}_f - \bar{t}_1\right] + P^2Q\bar{t}_1 + P\left[Q^3\left(\overline{m}_o - \overline{m}_f\right) + Q^2\left(\overline{m}_f - \overline{m}_o + \bar{t}_1\right)\right] - Q^3\bar{t}_1$$

The value of $\bar{t}_1$ which maximizes $\bar{x}(\bar{t}_f)$, if equations (B1) to (B4) are known, can then be found by scanning the allowable values of $\bar{t}_1$ $\left(0 < \bar{t}_1 < \overline{m}_o - \overline{m}_f\right)$ and observing $\bar{x}(\bar{t}_f)$. After determining $\bar{t}_1$, the complete trajectory may be constructed by use of equations (B1) to (B4) and the solutions of equations (2).

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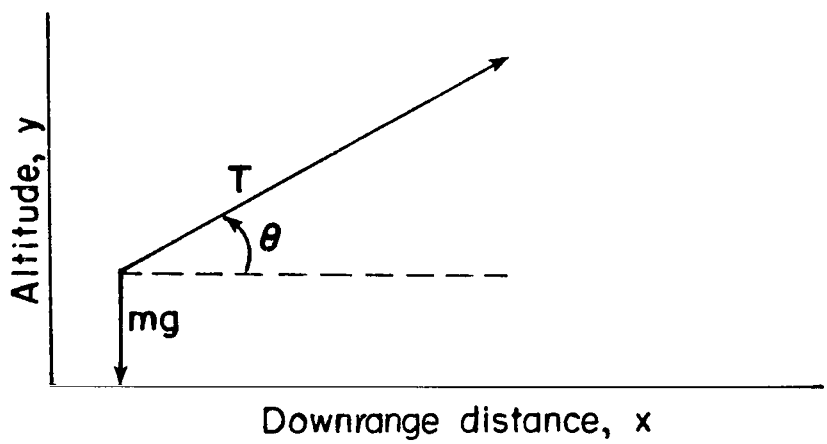


Figure 1.- Coordinate system for "flat" surface maximum range problem with thrust components, magnitude T and attitude θ.

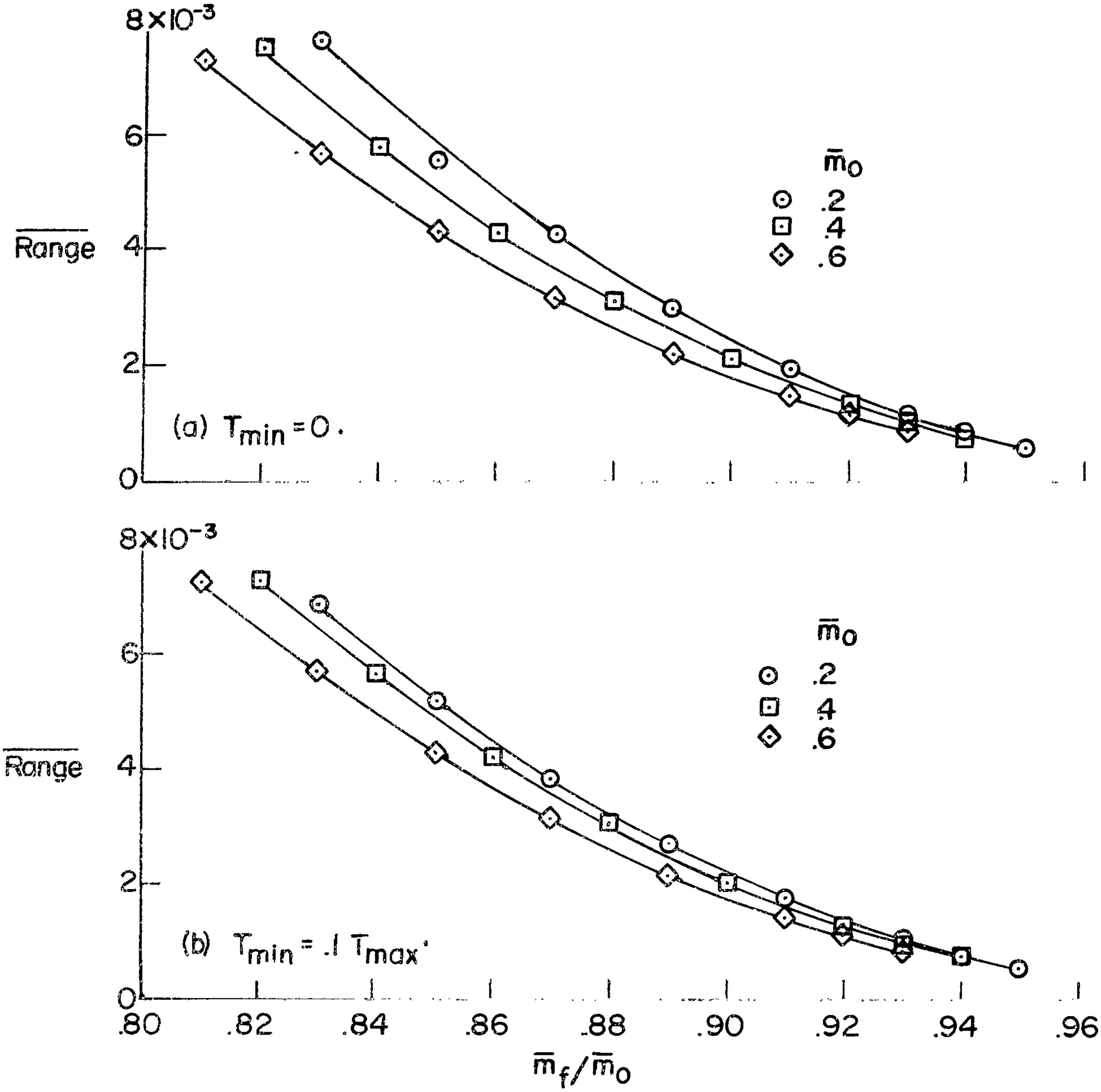
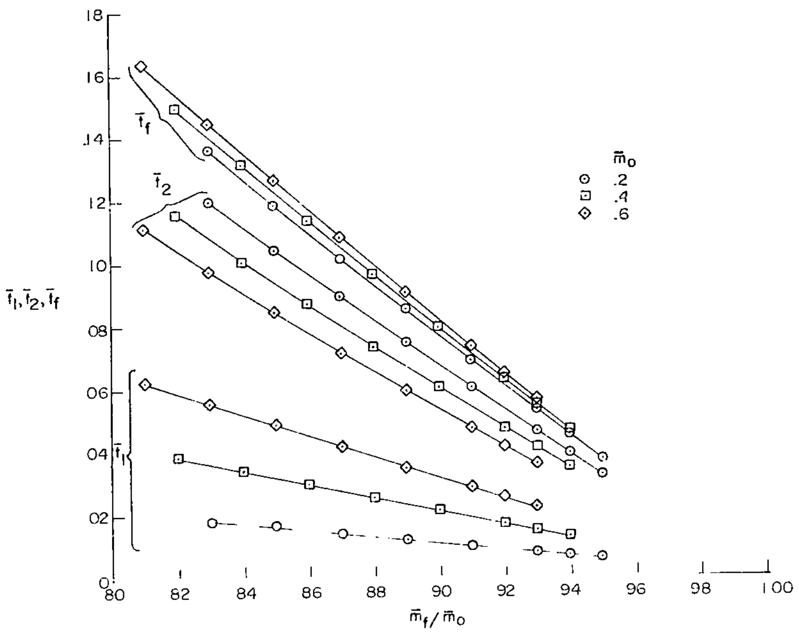
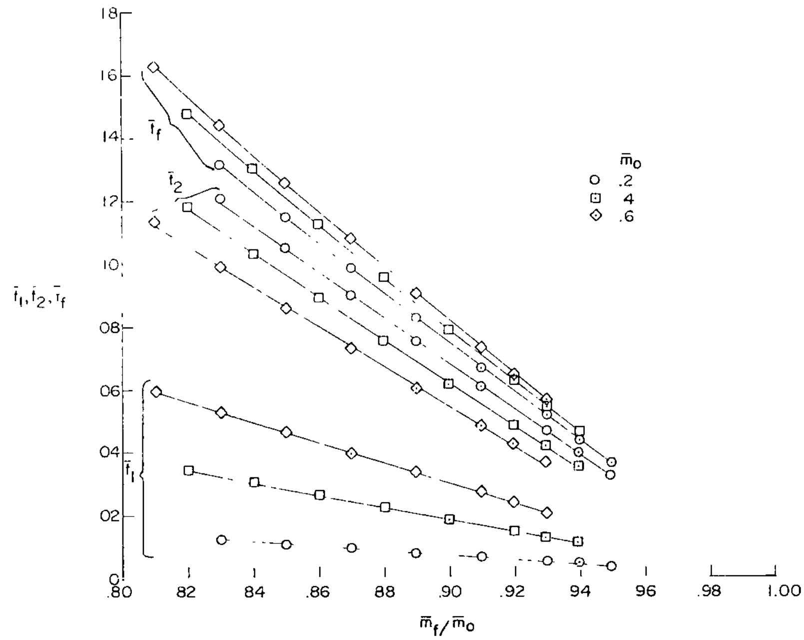


Figure 2.- Nondimensional range as a function of the ratio of final mass to initial mass $\bar{m}_f/\bar{m}_0$ with nondimensional initial mass $\bar{m}_0$ as a parameter.

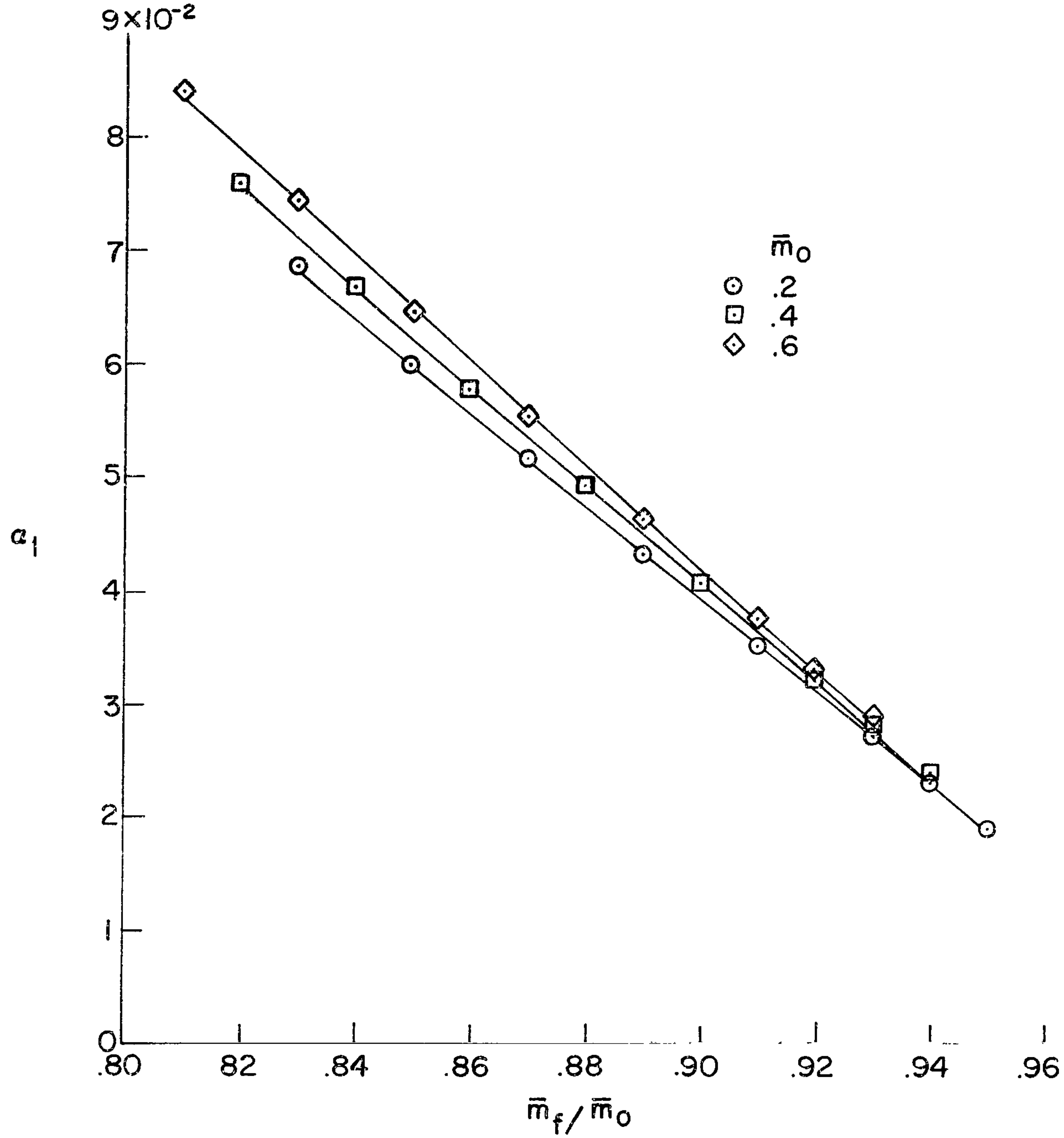


(a) $T_{min} = 0$.



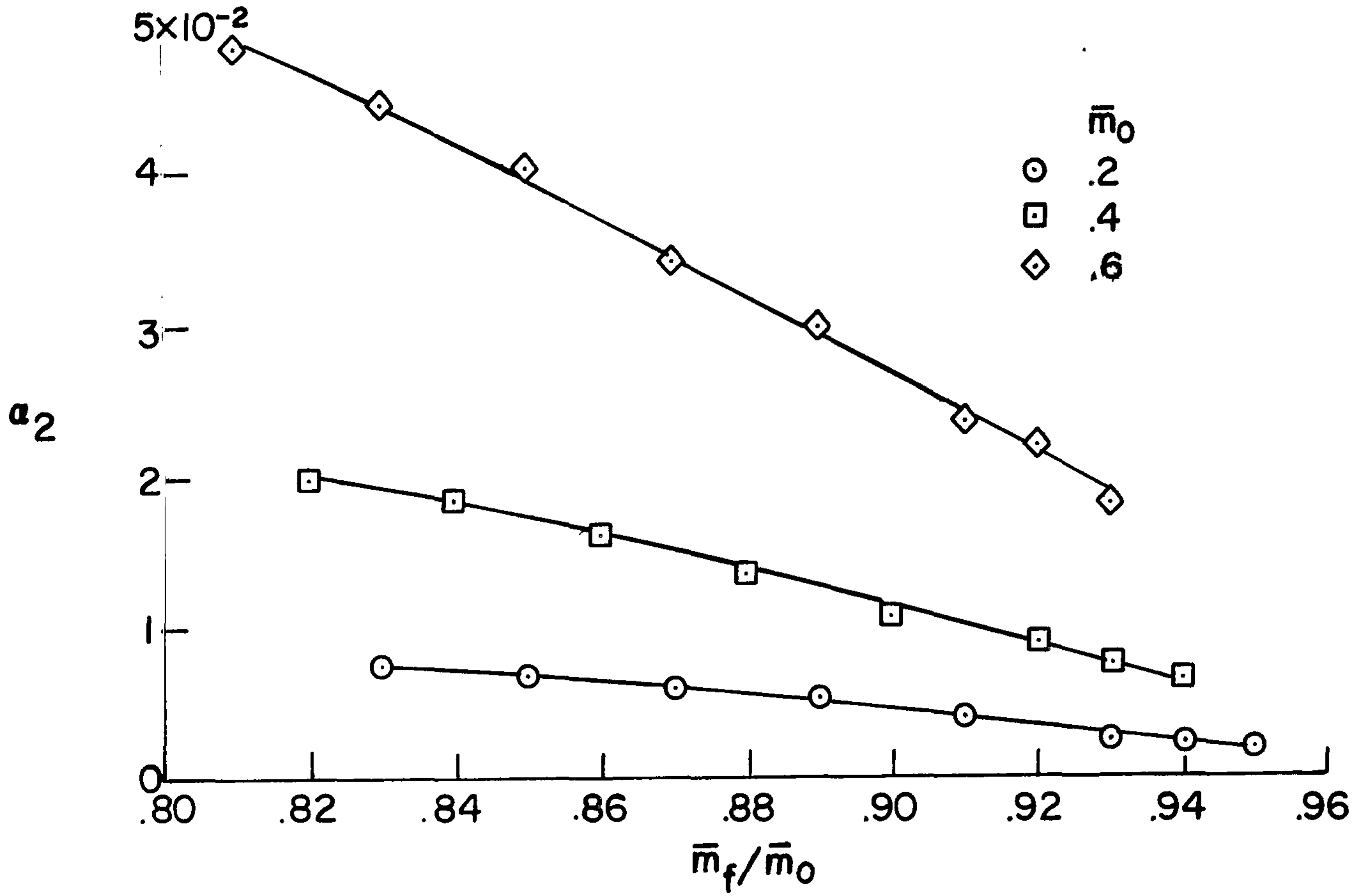
(b) $T_{min} = 0.1T_{max}$.

Figure 3.- First switching time $\bar{t}_1$, second switching time $\bar{t}_2$, and final time $\bar{t}_f$ as functions of the ratio of final mass to initial mass $\bar{m}_f/\bar{m}_0$ with nondimensional initial mass $\bar{m}_0$ as a parameter.



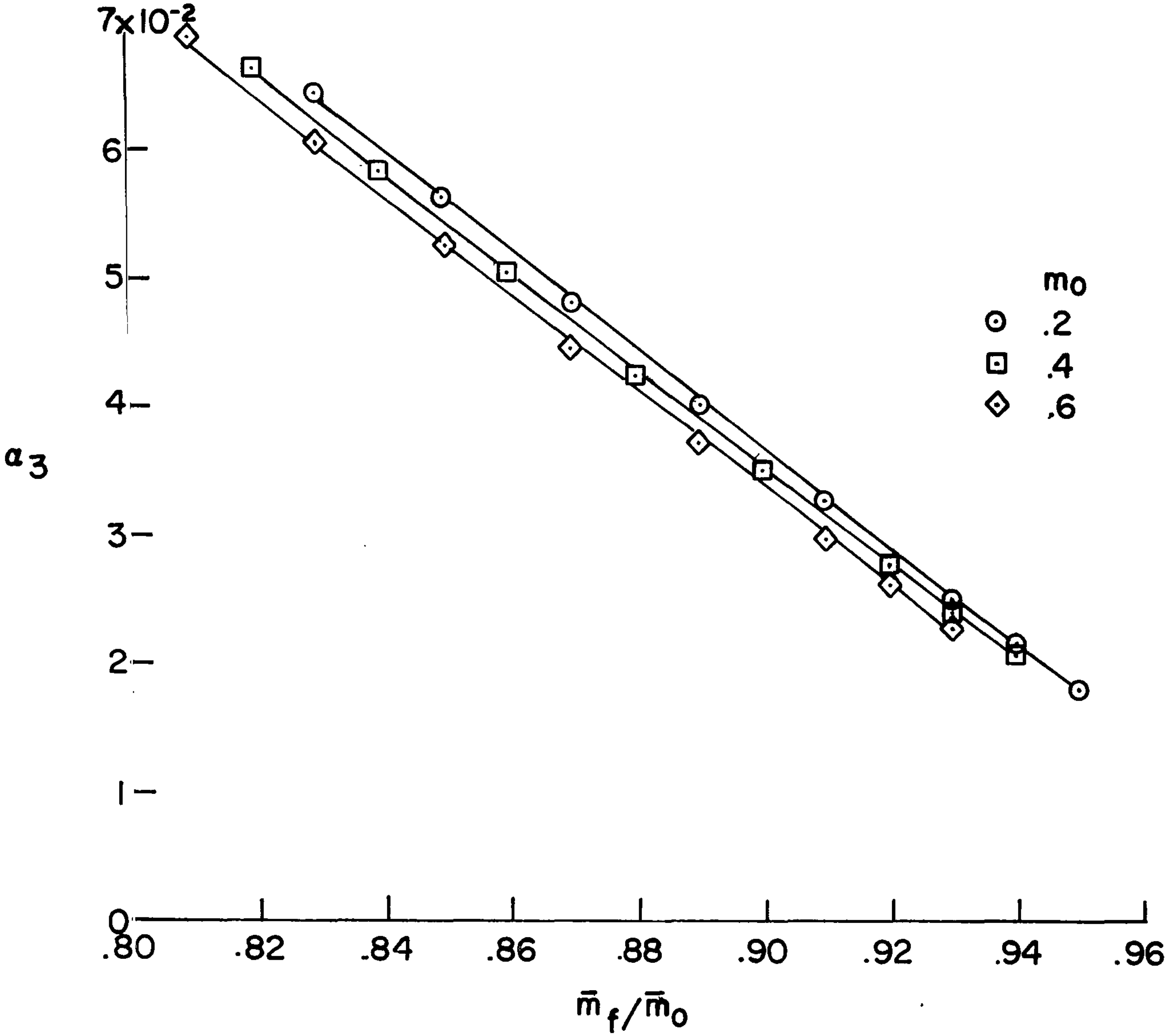
(a) α_1.

Figure 4.- Unknown parameters α_1, α_2, and α_3 as functions of ratio of final mass to initial mass $\bar{m}_f/\bar{m}_0$ with nondimensional initial mass $\bar{m}_0$ as a parameter for $T_{min} = 0$.



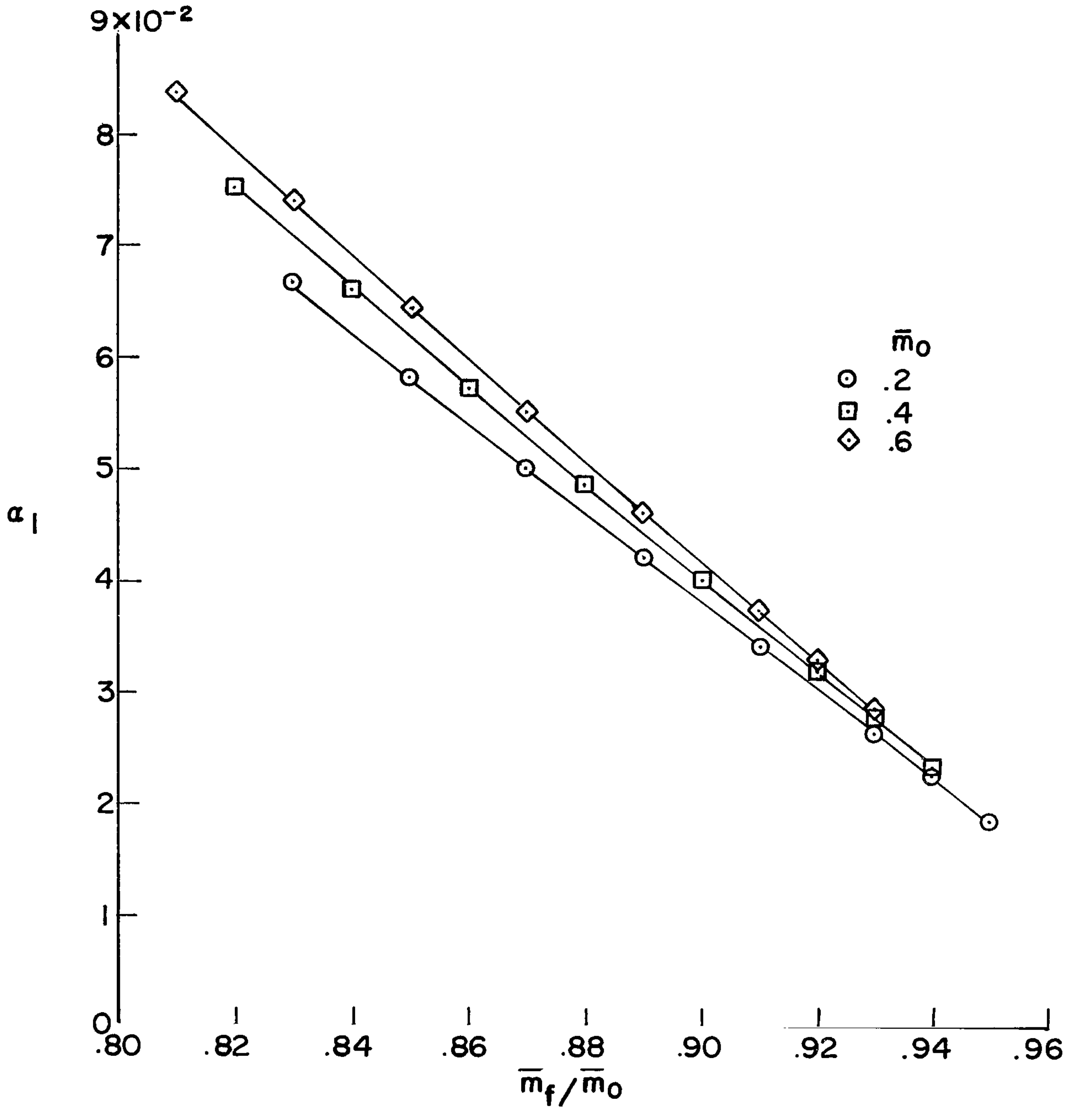
(b) α_2.

Figure 4.- Continued.



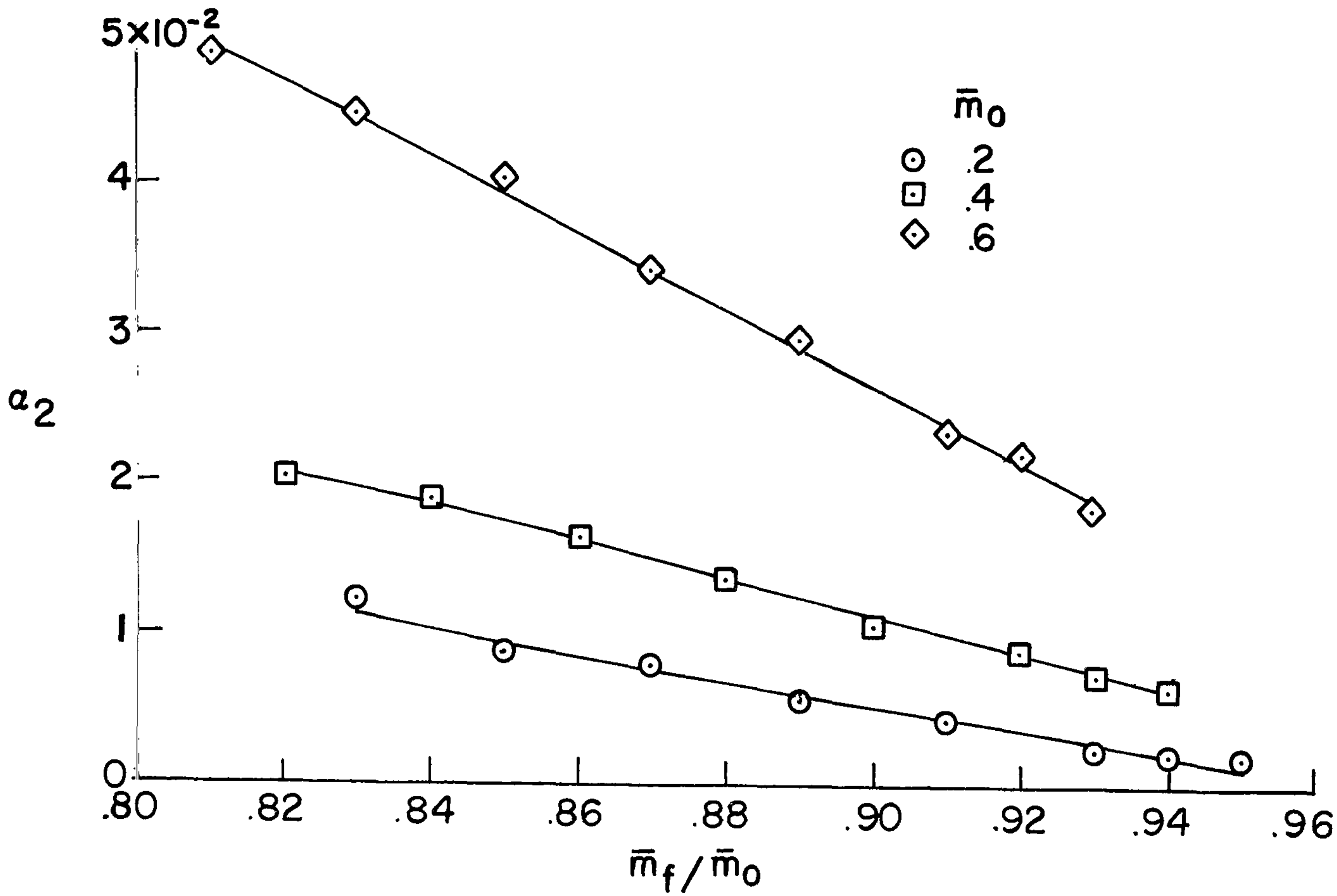
(c) α_3.

Figure 4.- Concluded.



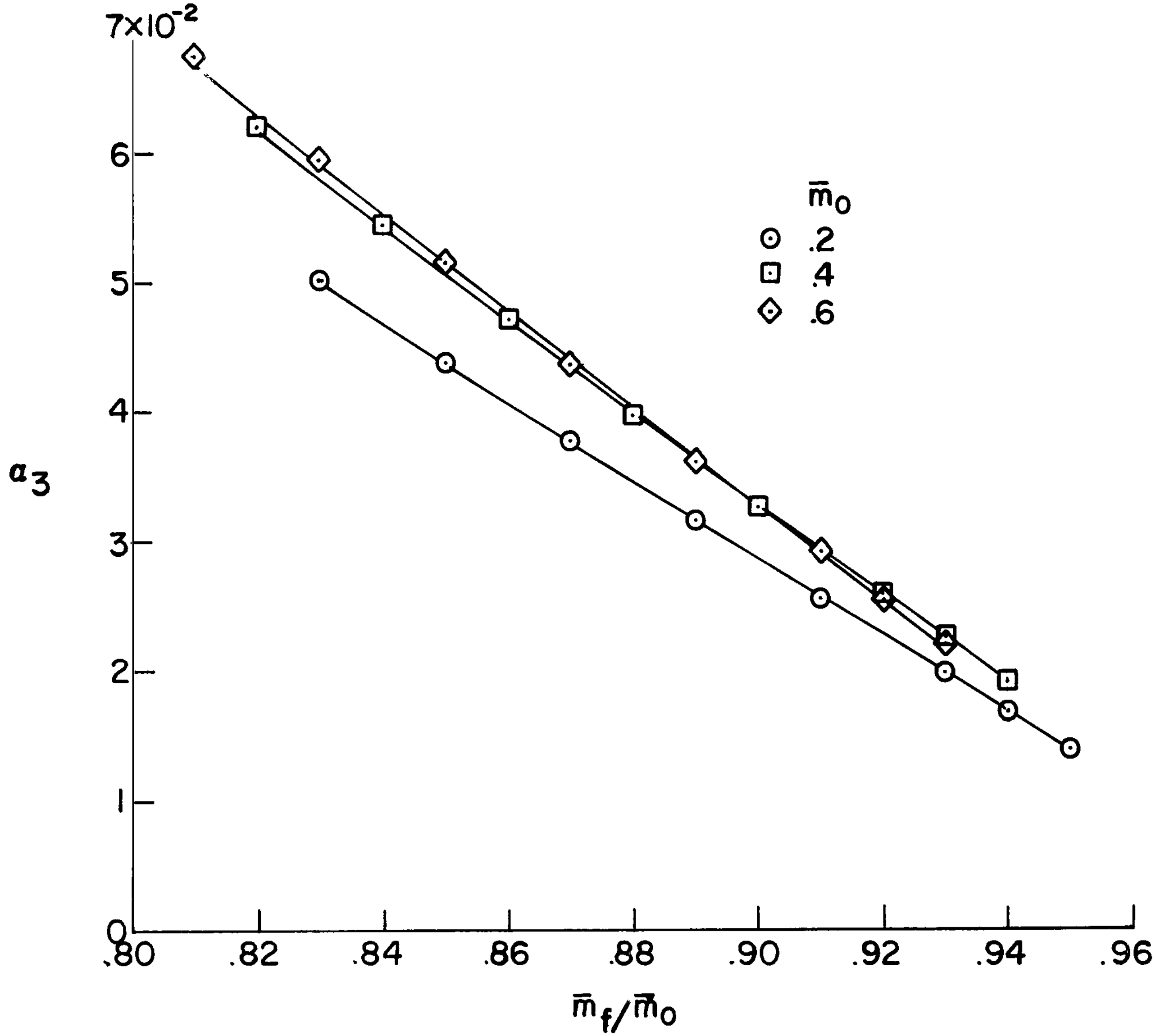
(a) α_1.

Figure 5.- Unknown parameters α_1, α_2, and α_3 as functions of ratio of final mass to initial mass $\overline{m}_f/\overline{m}_0$ with nondimensional initial mass $\overline{m}_0$ as a parameter for $T_{min} = 0.1T_{max}$.



(b) α_2.

Figure 5.- Continued.



(c) a_3.

Figure 5.- Concluded.

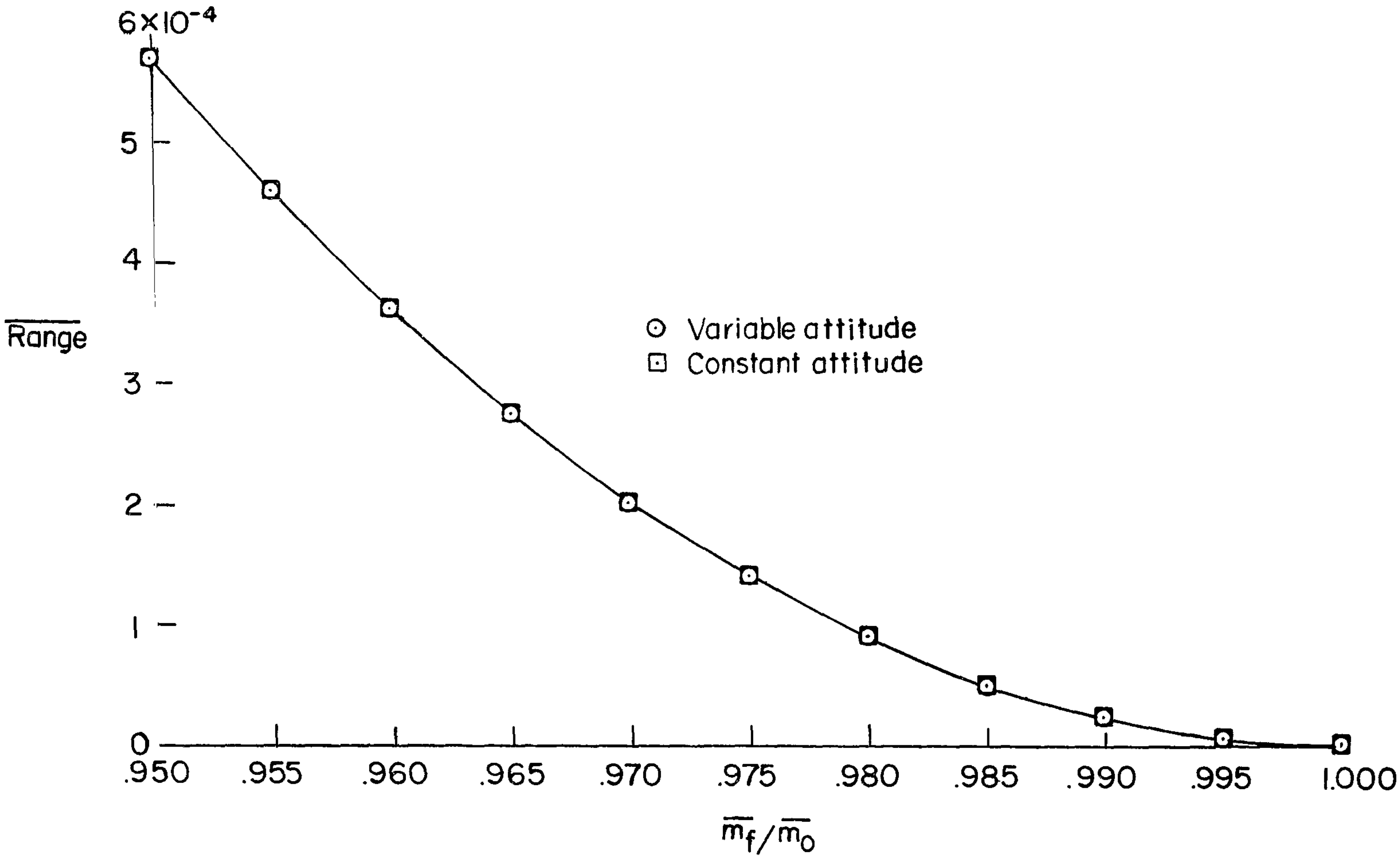


Figure 6.- Nondimensional range as a function of ratio of final mass to initial mass $\overline{m}_f/\overline{m}_0$ for very short range trajectories using variable and constant attitude. $\overline{m}_0 = 0.2$; $T_{min} = 0$.

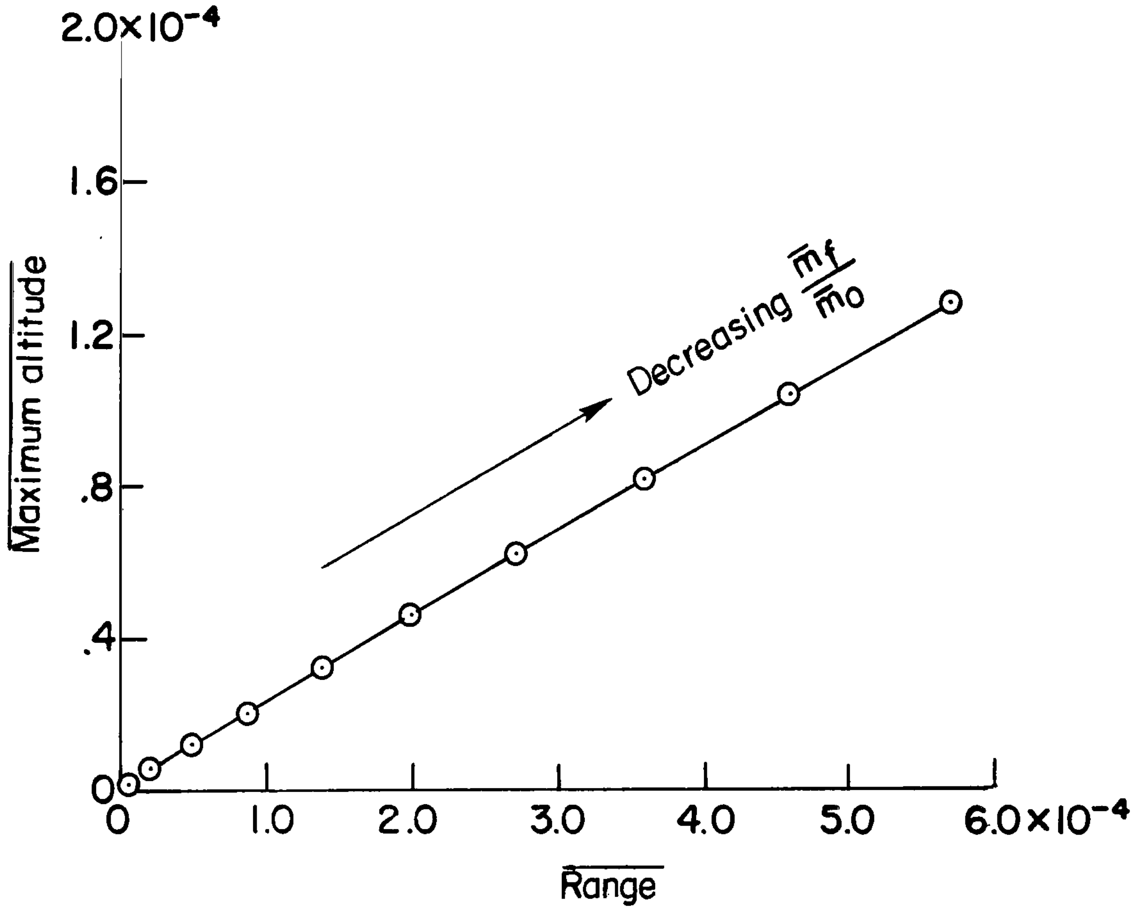


Figure 7.- Nondimensional maximum altitude as a function of nondimensional range. $\overline{m}_0 = 0.2$; $T_{min} = 0$; and $\overline{m}_f/\overline{m}_0$ from 0.95 to 1.

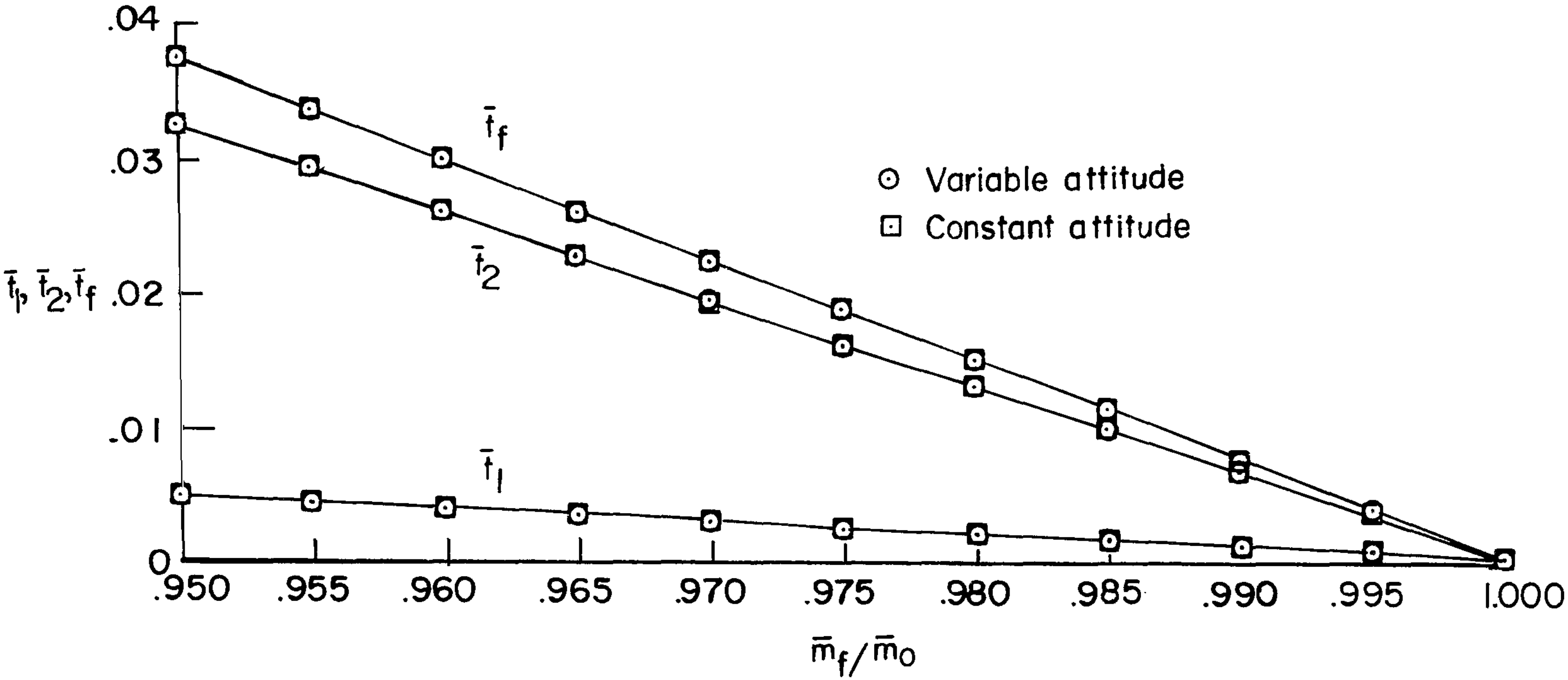
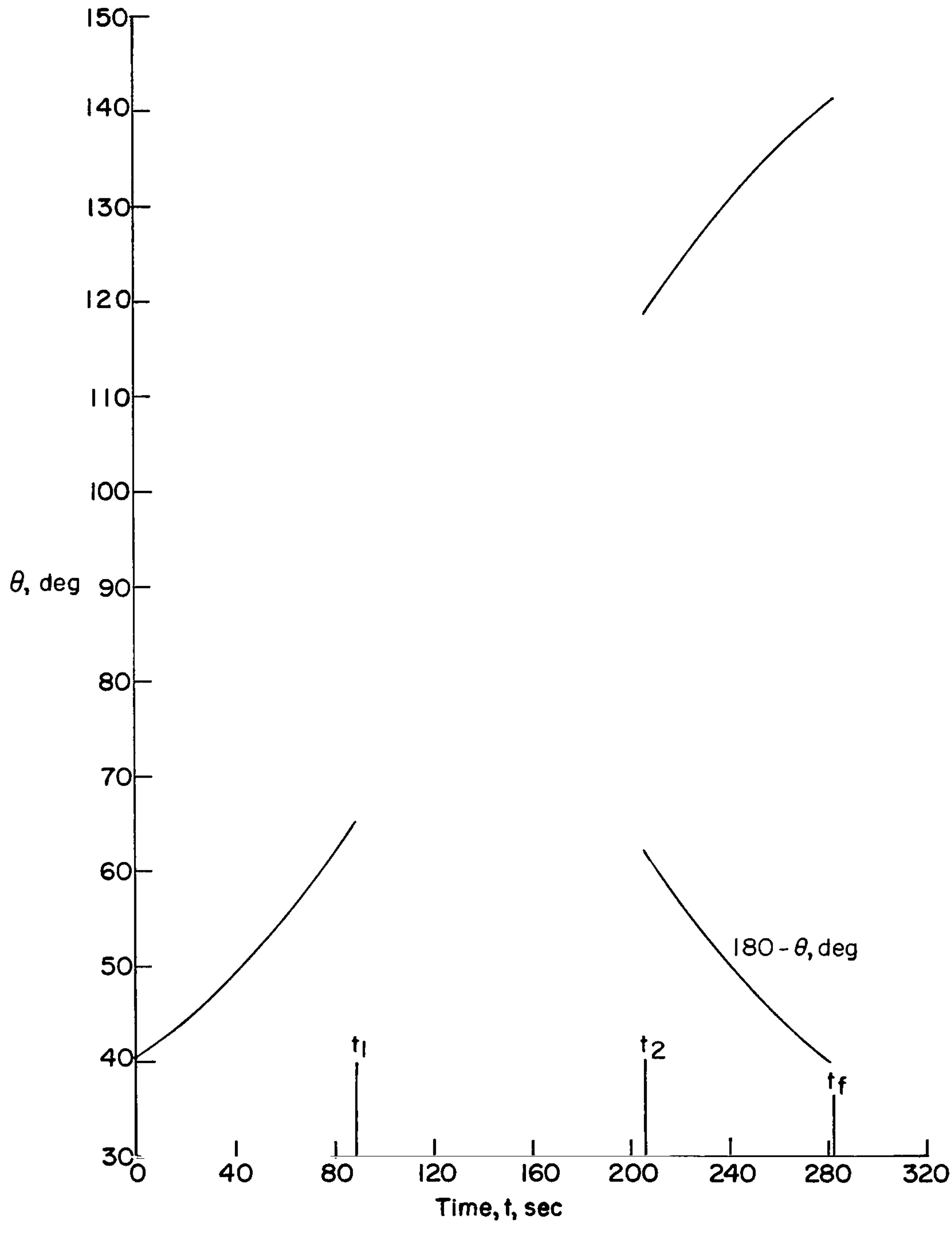
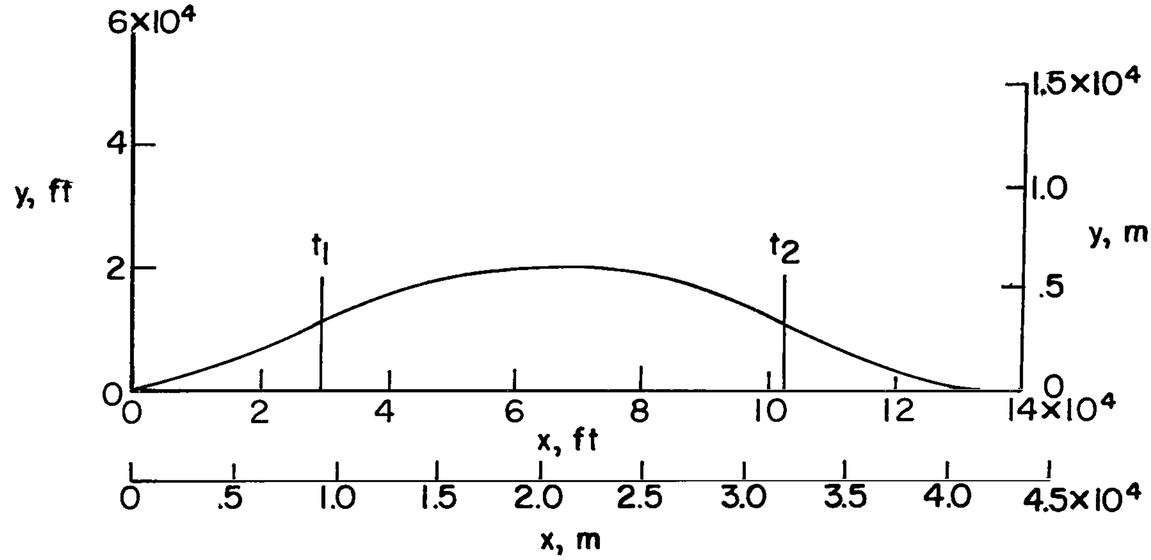


Figure 8.- First switching time $\bar{t}_1$, second switching time $\bar{t}_2$, and final time $\bar{t}_f$ as functions of ratio of final mass to initial mass $\bar{m}_f/\bar{m}_0$ for very short range trajectories using variable and constant attitude. $\bar{m}_0 = 0.2$; $T_{min} = 0$.

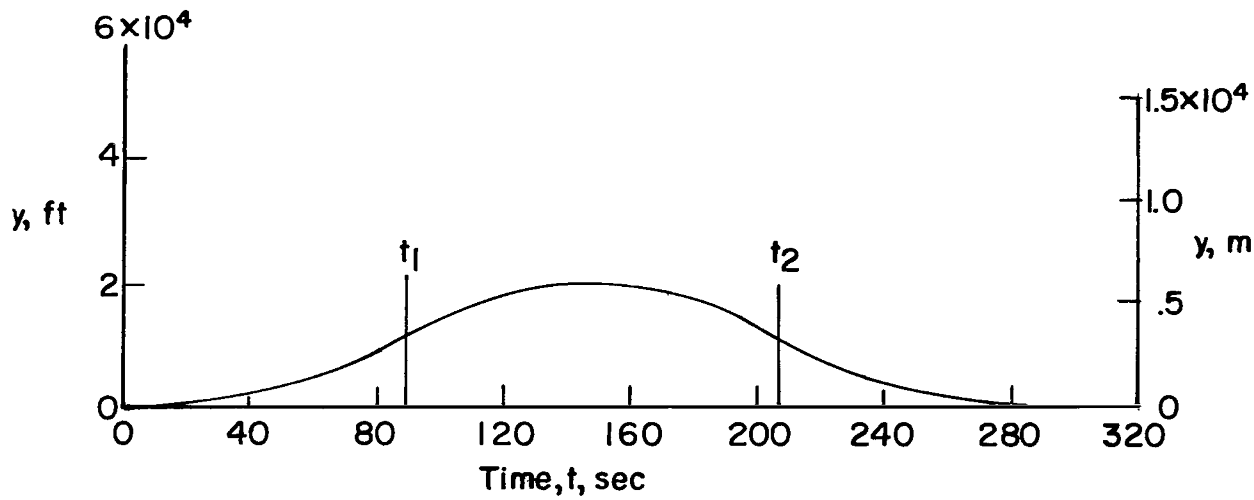


(a) Variation of thrust attitude θ with time t.

Figure 9.- Optimal control and trajectories. $W_0 \approx$ 10 500 lb (46 706.3 N); $W_f \approx$ 8 600 lb (38 254.7 N); T_{max} = 3 507.7 lb (15 603.0 N); T_{min} = 0 lb (0 N); c = 9 853.2 ft/sec (3 003.3 m/s).

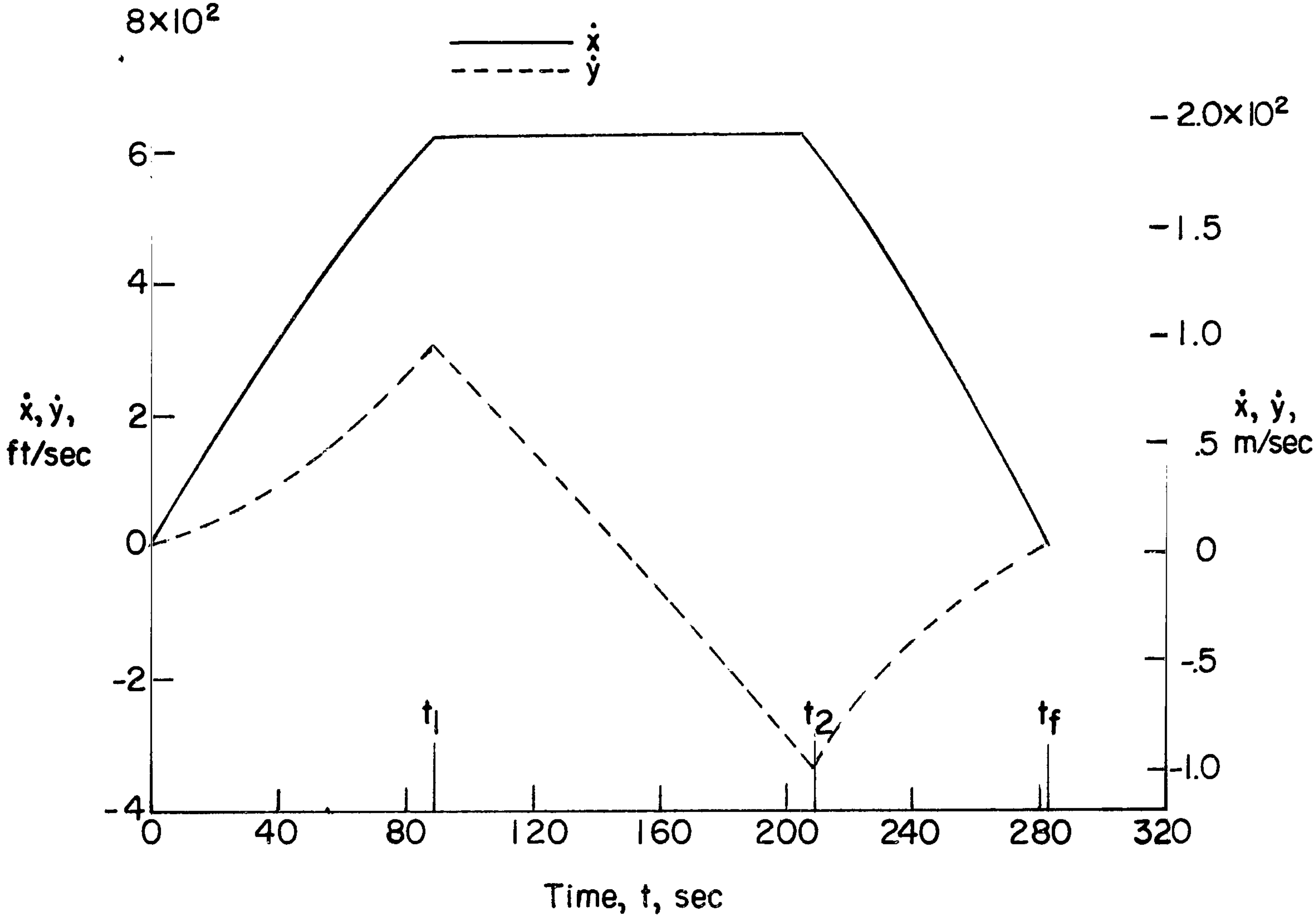


(b) Variation of altitude y with downrange distance x.



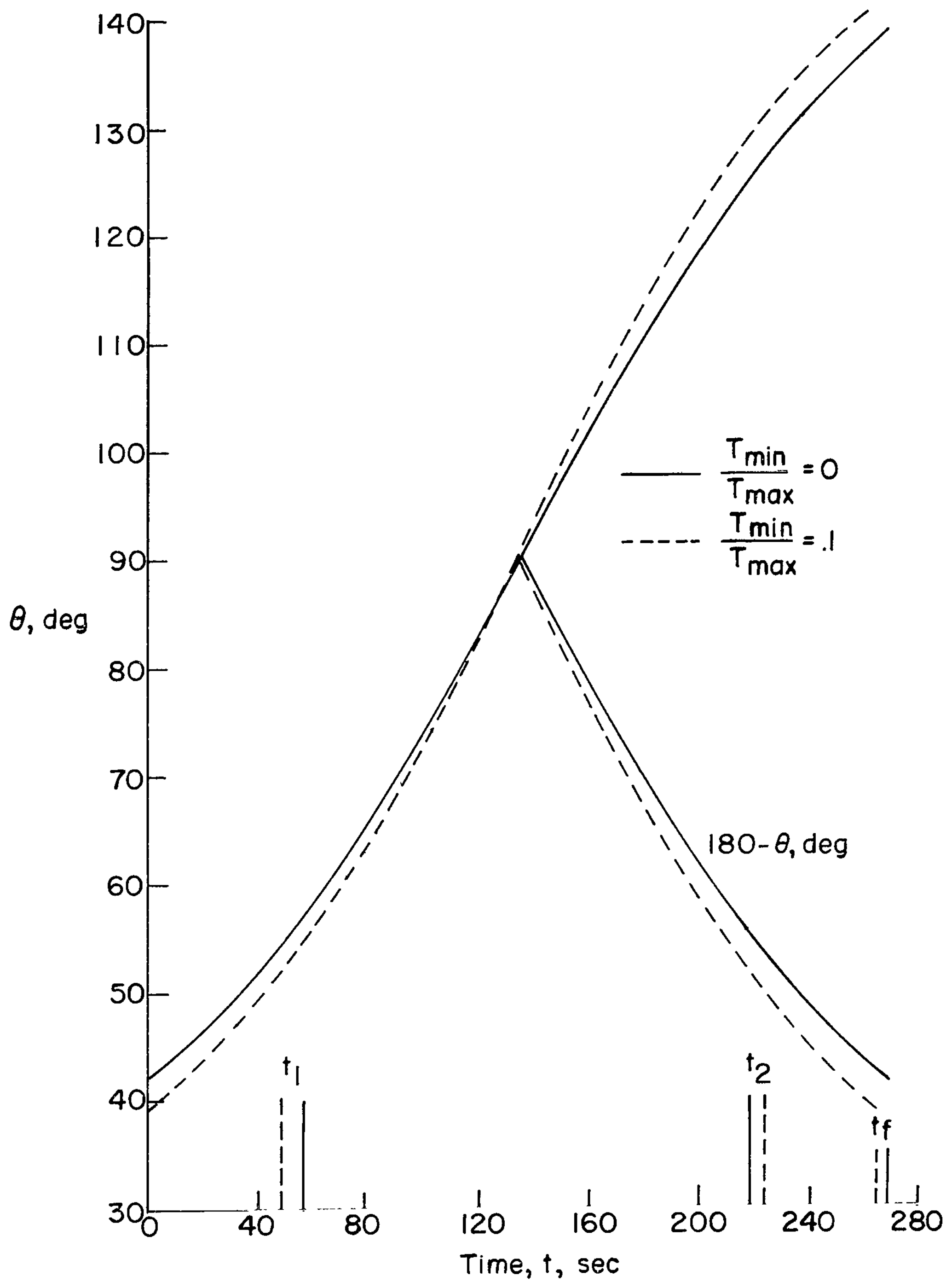
(c) Variation of altitude y with time t.

Figure 9.- Continued.



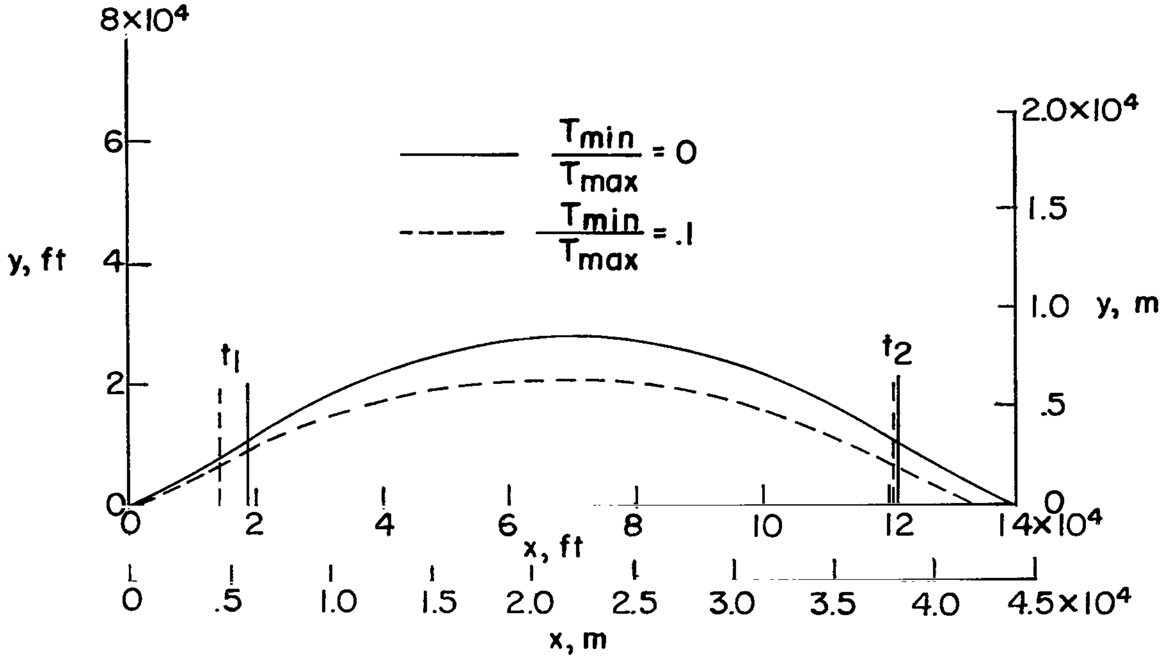
(d) Variation of horizontal velocity $\dot{x}$ and vertical velocity $\dot{y}$ with time t.

Figure 9.- Concluded.

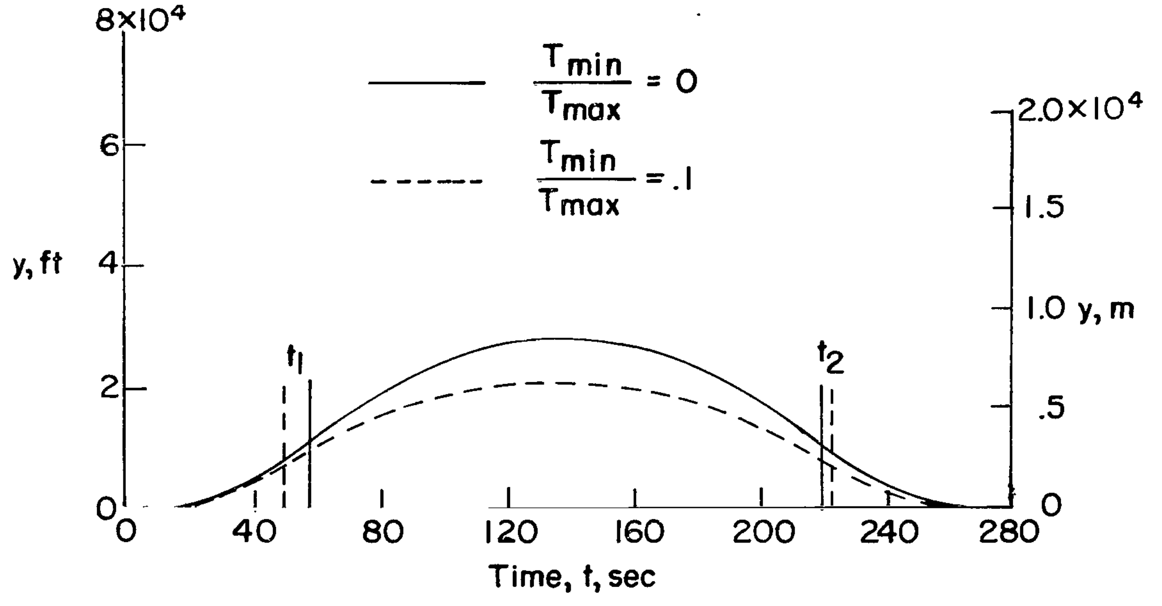


(a) Variation of thrust attitude θ with time t.

Figure 10.- Optimal control and trajectories. $W_0 \approx 550$ lb (2446.5 N); $W_f = 450$ lb (2001.7 N); $T_{max} = 275$ lb (1223.3 N); c = 9660 ft/sec (2944.4 m/s).

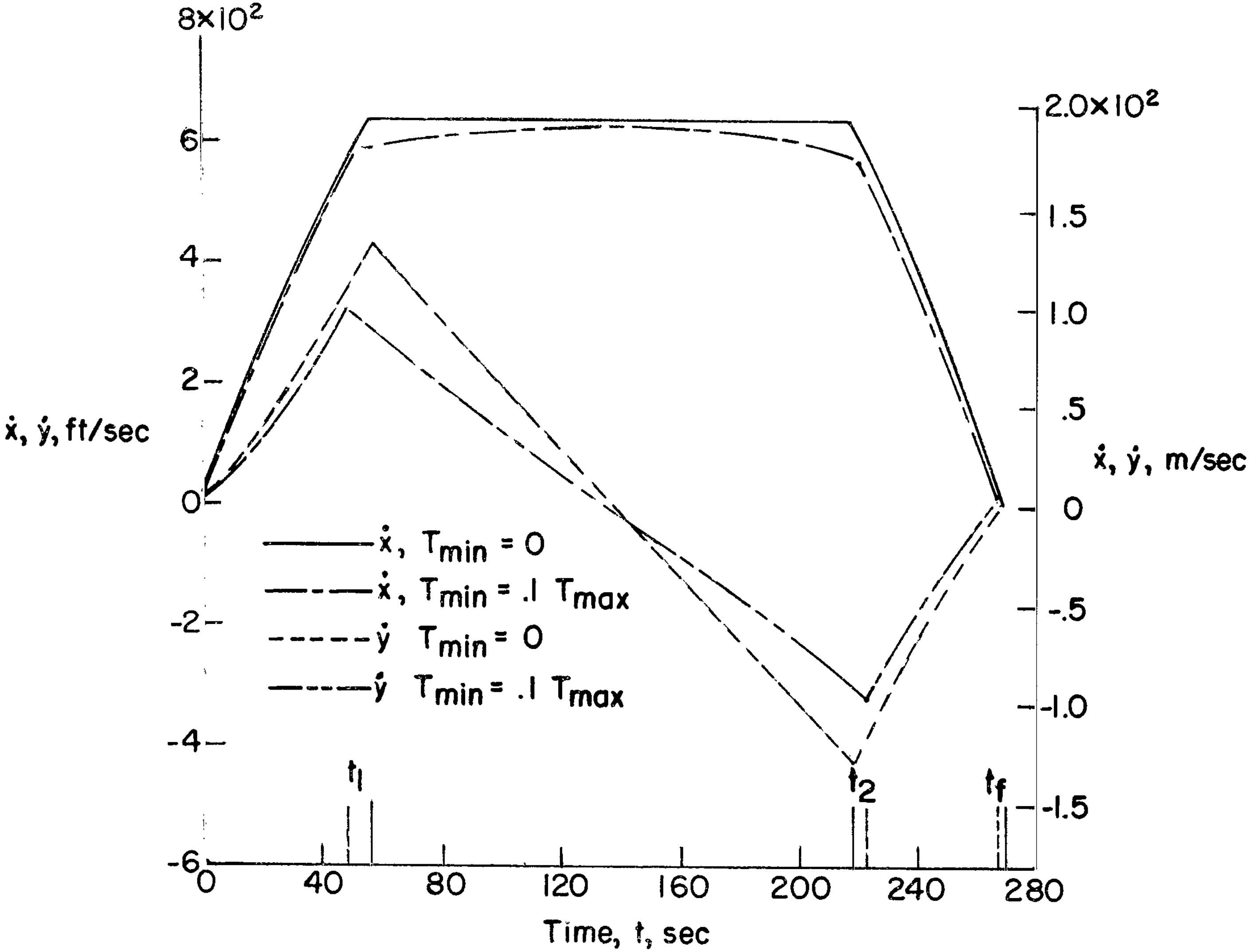


(b) Variation of altitude y with downrange distance x.



(c) Variation of altitude y with time t.

Figure 10.- Continued.



(d) Variation of horizontal velocity $\dot{x}$ and vertical velocity $\dot{y}$ with time t.

Figure 10.- Concluded.

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