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The Existence and Stability of Nonlinear Wave Equations

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1. Introduction

The object of the present paper is to establish some criteria for the existence, the uniqueness and the stability or asymptotic stability of a solution of the second order wave equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} = Lu + f(x,u,\frac{\partial u}{\partial x_i},\frac{\partial u}{\partial t})$$
$$\equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + c(x)u + f(x,u,\frac{\partial u}{\partial x_i},\frac{\partial u}{\partial t}) \quad (x \in \Omega) \tag{1-1}$$

with the boundary condition

$$u(t,x)\Big|_{\partial\Omega} = h(x') \qquad x' \in \partial\Omega \tag{1-2}$$

where $a \geqq 0$, Ω is a bounded domain in R^n, $\partial\Omega$ is the boundary of Ω and f is a (nonlinear) function defined on some suitable space. By choosing a suitable function g(x) defined on $\bar{\Omega}$ such that $g(x')=h(x')$ on $\partial\Omega$ and replacing u by u-g, equation (1-1) remains the same from with a homogeneous boundary condition except with a different function f. Thus we shall assume that the boundary condition (1-2) is homogeneous. Let $u_1=u$, $u_2 = \dot{u} = \frac{\partial u}{\partial t}$, then (1-1) is reduced to a system of equations of the form

$$\frac{\partial \underline{u}}{\partial t} = \begin{bmatrix} 0 & 1 \\ L & -a \end{bmatrix} \underline{u} + \underline{f}(\underline{u}) \tag{1-3}$$

where

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \underline{f}(\underline{u}) = \begin{bmatrix} 0 \\ f(x,u_1,\frac{\partial u_1}{\partial x_i},u_2) \end{bmatrix} .$$

By a suitable choice of a function space, equation (1-3) together with the zero boundary condition may be considered as an abstract operator differential equation of the form

$$\frac{d\underline{u}(t)}{dt} = A\,\underline{u}(t) + \underline{f}(\underline{u}(t)) \qquad t \in R^+ = [0,\infty) \tag{1-4}$$

where A is an abstract linear operator from some function space into itself. In this paper, we shall formulate A as a linear operator mapping part of the Hilbert space $H \equiv H_o^1(\Omega) \times L^2(\Omega)$ into itself and $\underline{f}$ is defined on H into H, and then apply the results developed for operator differential equations in [1] and [2] (which are based on the work by Kato [3]) to establish criteria on the coefficients of the partial differential operator L and on the function f so that the existence and uniqueness of a solution and the stability or asymptotic stability of an equilibrium solution (or any unperturbed solution) can be ensured. We shall first discuss the equation (1-1) with $f\equiv 0$ and then consider the more general nonlinear problem which is closely related to the linear form.

2. Background

<u>Definition 2.1.</u> By a solution $\underline{u}(t)$ of (1-4) with initial condition $\underline{u}(0) = \underline{u}_o \in D(A)$ in a Hilbert space H, we mean the following:

(a) $\underline{u}(t)$ is uniformly Lipschitz continuous in t for each $t \geq 0$ with $\underline{u}(0) = \underline{u}_o$;

(b) $\underline{u}(t) \in D(A)$ for all $t \geq 0$ and the strong derivative of $\underline{u}(t)$ exists for almost all values of $t \geq 0$ and satisfies (1-4) a.e. in R^+.

<u>Definition 2.2.</u> An equilibrium solution of (1-1) is a solution $\underline{u}_e$ such that for any solution $\underline{u}(t)$ of (1-1) with $\underline{u}(0) = \underline{u}_e$

$$||\underline{u}(t)-\underline{u}_e|| = 0 \qquad \text{for all } t \geqq 0.$$

It can be shown that if u(t) is a solution of (1-4) then it is an equilibrium solution if and only if $A\underline{u}(t) + f(u(t)) = 0$ for all $t \geqq 0$ (cf [1]).

Definition 2.3. An equilibrium solution (or any unperturbed solution) $\underline{u}_e$ of (1-4) is said to be stable (with respect to initial perturbations) if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$||\underline{u}-\underline{u}_e|| < \delta \text{ implies } ||\underline{u}(t)-\underline{u}_e|| < \varepsilon \qquad \text{for all } t \geqq 0;$$

$\underline{u}_e$ is said to be asymptotically stable if

(i) it is stable; and

(ii) $\lim_{t\to\infty} ||\underline{u}(t)-\underline{u}_e|| = 0$

where $\underline{u}(t)$ is any solution of (1-4) with $\underline{u}(0)=\underline{u}\varepsilon D(A)$. If there exist positive constants M and β such that

$$\text{(ii)}' \quad ||\underline{u}(t)-\underline{u}_e|| \leqq Me^{-\beta t} ||\underline{u}-\underline{u}_e|| \qquad \text{for all } t \geqq 0$$

then $\underline{u}_e$ is called exponentially asymptotically stable.

Definition 2.4. Let u(t) be a solution to (1-4) with $\underline{u}(0) = \underline{u}$. A subset D of H is said to be a stability region of the equilibrium solution u_e (or any unperturbed solution) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\underline{u} \in D \text{ and } ||\underline{u}-\underline{u}_e|| < \delta \text{ imply } ||\underline{u}(t)-\underline{u}_e|| < \varepsilon \qquad \text{for all } t \geqq 0.$$

Definition 2.5. Two inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_e$ defined on the same vector space H are said to be equivalent if there exist some positive constants δ, γ, such that

$$\delta||\underline{u}|| \leqq ||\underline{u}||_e \leqq \gamma||\underline{u}|| \qquad \text{for all } \underline{u} \in H. \tag{2-1}$$

The following two theorems are from [1] and [2] by the authors.

Theorem 2.1. Let A be a linear operator with domain $D(A)$ and range $R(A)$ both contained in a Hilbert space H such that $D(A)$ is dense in H and $R(I-A)=H$. If A satisfies

$$(u,Au) \leqq -\beta||u||^2 \qquad (\beta \geqq 0) \qquad \text{for all } u \in D(A) \tag{2-2}$$

and f is defined on all of H and satisfies

$$||f(u) - f(v)|| \leqq k||u-v|| \qquad u, v \in H \tag{2-3}$$

for some $k \leqq \beta$, then (a) for any $u_o \in D(A)$ there exists a unique solution $u(t)$ of (1-4) with $u(0) = u_o$; (b) any unperturbed solution (e.g., equilibrium solution or periodic solution) is asymptotically stable if $k < \beta$ and is stable if $k = \beta$; (c) a stability region is $D(A)$ which can be extended to the whole space H.

Remark: (a) An operator A satisfying (2-2) is called strictly dissipative if $\beta>0$ and is called dissipative if $\beta=0$. The number β is called a dissipative constant of A. (b) Weaker condition on f can be found in [2].

Theorem 2.2. In theorem 2.1, if A is strictly dissipative with a dissipative constant β with respect to an equivalent inner product $(\cdot,\cdot)_e$ and (2-3) is replaced by

$$||f(u)-f(v)||_e \leqq k||u-v||_e \qquad u, v \in H \tag{2-4}$$

for some $k \leqq \beta$, then all the results in theorem 2.1 hold.

3. Formulation of Abstract Operators

Throughout this paper, the following conventional notations will be used: $C^m(\Omega)$ denotes the class of all m-times $(0 \leqq m \leqq \infty)$ continuously differentiable real-valued functions on Ω; the subset $C_o^m(\Omega)$ of $C^m(\Omega)$ consists of functions in $C^m(\Omega)$ with compact support in Ω; we denote

$$(u,v)_o = \int_\Omega u(x)\ v(x)\ dx \qquad\qquad u,v \in C^o(\Omega)$$

$$(u,v)_1 = \int_\Omega (u(x)v(x) + \sum_{i=1}^{n} \frac{\partial u(x)}{\partial x_i} \cdot \frac{\partial v(x)}{\partial x_i})dx \qquad u,v \in C^1(\Omega)$$

$$||u||_o = (u,u)_o^{1/2}\ ,\ ||u||_1 = (u,u)_1^{1/2}$$

where $dx = dx_1 \cdots dx_n$ is Lebesque measure in R^n; for $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, and $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $C^1(\Omega) \times C^o(\Omega)$, we define

$$(\underline{u}, \underline{v})_o = (u_1, v_1)_o + (u_2, u_2)_o;$$

$$(\underline{u}, v)_H = (u_1, v_1)_1 + (u_2, v_2)_o;$$

$$||u||_H = (\underline{u}, \underline{u})_H^{1/2}\ .$$

The linear space $C_o^1(\Omega)$ equipped with the inner product $(u, v)_1$ is an inner product space. The closure in the norm $||\cdot||_1$ of $C_o^1(\Omega)$ is a Hilbert space and is denoted by $H_o^1(\Omega)$. The product space $H = H_o^1(\Omega) \times L^2(\Omega)$, where $L^2(\Omega)$ is the class of Lebesque square-integrable functions with its usual inner product, is a linear space with addition and scalar multiplication defined by

$$\underline{u} + \underline{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix},\ \alpha\ \underline{u} = \begin{bmatrix} \alpha\ u_1 \\ \alpha\ u_2 \end{bmatrix}$$

and it is a Hilbert space with the inner product $(\underline{u}, \underline{v})_H$.

<u>Definition 3.1.</u> Let Ω_o be an open domain in the Euclidean space R^n. The operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x) \quad (x \in \Omega_o)$$

is called a formal partial differential operator if $a_{ij}(x) = a_{ji}(x)$

and together with $b_i(x)$, $c(x)$ are all infinitely differentiable functions in Ω_o; and L is called strongly elliptic if there exists a positive constant α such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\ \xi_i \xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2 \qquad (x \in \Omega_o) \tag{3-1}$$

on R^n for any real vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$.

Consider the strongly elliptic formal partial differential operator in the divergence form

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) + c(x) \qquad x \in \Omega_o. \tag{3-2}$$

Let Ω be a bounded subdomain whose closure $\bar{\Omega}$ is contained in Ω_o and whose boundary is a smooth surface $\partial\Omega$ with no point in $\partial\Omega$ interior to $\bar{\Omega}$. Assume that $c(x) < 0$ for $x \in \bar{\Omega}$ and write $c_m = \min_{x\in\bar{\Omega}}(-c(x))$ and $c_M = \max_{x\in\bar{\Omega}}(-c(x))$. Define T_o as a linear operator from $L^2(\Omega)$ into itself by the equations:

$$\begin{aligned} D(T_o) &= \{u \in C^\infty(\bar{\Omega});\ u(x) = 0 \quad \text{for} \quad x \in \partial\Omega\} \\ T_o u &= Lu \qquad u \in D(T_o) \end{aligned} \tag{3-3}$$

then T_o is a linear operator with domain $D(T_o)$ dense in $L^2(\Omega)$ since $D(T_o) \supset C_o^\infty(\Omega)$ which is dense in $L^2(\Omega)$. We denote the closure of T_o (i.e., the smallest closed extension of T_o) by T.

Lemma 3.1. Let T_o be defined in (3-3). Then for any $u,v \in D(T_o)$ $(T_o u, v) = (u, T_o v)$. Moreover, the closure T of T_o is strictly dissipative with respect to the inner product $(\cdot,\cdot)_o$.

Proof. For any u, $v \in D(T_o)$, integration by parts twice and notice that $a_{ij}(x) = a_{ji}(x)$, it can easily be shown that $(T_o u, v) = (u, T_o v)$. Integration by parts once and using the strong ellipticity of L, we have for any $u \in D(T_o)$

$$(u, T_o u)_o = \int_\Omega [-\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + c(x)u^2] dx$$

$$\leqq - \int_\Omega [\alpha \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i})^2 - c(x) u^2] dx \leqq -c_m ||u||_o^2$$

which shows that T_o is strictly dissipative. For any $u \varepsilon D(T)$, there exists a sequence $\{u_n\}$ in $D(T_o)$ such that $u_n \to u$ and $T_o u_n \to T u$ since T is the closure of T_o. It follows by the continuity of inner product that

$$(u, Tu)_o = \lim_{n\to\infty} (u_n, T_o u_n)_o \leqq \lim_{n\to\infty} (-c_m ||u_n||_o^2) = -c_m ||u||_o^2$$

which proves the strict dissipativity of T.

Lemma 3.2. T is a linear operator with $D(T) \subset H_o^1(\Omega)$ and $R(T) \subset L^2(\Omega)$, and for any $\alpha > 0$, $(\alpha I - T)^{-1}$ is an everywhere defined continuous operator on $L^2(\Omega)$ into itself.

Proof. By the definition of T, $D(T) \subset H_o^1(\Omega)$ and the spectrum $\sigma(T)$ of T consists of a countable discrete set of points with no finite limit point (cf. Dunford and Schwartz [4] theorem XIV. 6.23). Hence there exists a real number $\alpha_o > 0$ such that $\alpha_o \notin \sigma(T)$ which implies that the resolvent $R(\alpha_o; T) = (\alpha_o I - T)^{-1}$ is an everywhere defined continuous linear operator on $L^2(\Omega)$. Moreover the dissipativity of T implies that $(\alpha I - T)^{-1}$ exists and is continuous for every $\alpha > 0$ and together with the condition $D((\alpha_o I - T)^{-1}) = L^2(\Omega)$ for some $\alpha_o > 0$, we have $D((\alpha I - T)^{-1}) = L^2(\Omega)$ for every $\alpha > 0$ (cf. Kato [5]).

In order to formulate the partial differential equation (1-3) as an operator differential equation of the form (1-4), we define the operator A by the following equations:

$$D(A) = D(T) \times H_o^1(\Omega) \quad \text{and} \quad A = \begin{pmatrix} 0 & I \\ T & -aI \end{pmatrix} \tag{3-4}$$

where $a \geqq 0$ is the constant appearing in (1-1).

Lemma 3.3. The operator A is a linear operator with domain $D(A)$ dense in $H = H_o^1(\Omega) \times L^2(\Omega)$ and range $R(A)$ in H. Moreover, $R(I-A) = H$.

Proof. It is obvious that A is linear with domain $D(A)$ dense in H since $D(T) \supset D(T_o) \supset C_o^\infty(\Omega)$ which is dense in $H_o^1(\Omega)$, and $H_o^1(\Omega)$, when considered as a subset of $L^2(\Omega)$, is dense in $L^2(\Omega)$. It is also clear that $R(A) \subset H$ since $R(T) \subset L^2(\Omega)$. To show $R(I-A) = H$, let $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in H$ and show that there exists an $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in D(A)$ such that $(I-A)\,\underline{u} = w$, which is equivalent to find an $u_1 \in D(T)$ and $u_2 \in H_o^1(\Omega)$ satisfying the system of equations

$$u_1 - u_2 = w_1 \quad \text{and} \quad -Tu_1 + (1+a)u_2 = w_2.$$

By substituting $u_2 = u_1 - w_1$ into the second equation yields

$$((1+a)I - T)u_1 = (1 + a)\, w_1 + w_2.$$

By lemma 3.2, $R(\alpha I-T) = L^2(\Omega)$ for every $\alpha > 0$ which insures the existence of an $u_1 \in D(T)$ satisfying the above equality since $(1+a)w_1 + w_2$ is in $L^2(\Omega)$. The fact that $D(T) \subset H_o^1(\Omega)$ implies $u_2 = u_1 - w_1 \in H_o^1(\Omega)$ which completes the proof of the lemma.

4. Equivalent Inner Product

In order to prove our main results, we shall introduce an equivalent inner product on H with respect to which the operator A is dissipative or strictly dissipative. Following the same idea used by Buis (cf. [6]), we introduce a linear operator S defined by:

$$D(S) = D(T_o) \times L^2(\Omega)$$

$$S\,\underline{u} = \begin{pmatrix} -2T_o + a^2 I & aI \\ aI & 2I \end{pmatrix} \underline{u} \qquad \underline{u} \in D(S) \tag{4-1}$$

and show the following lemmas.

Lemma 4.1. The functional $V(\underline{u}, \underline{v})$ defined by

$$V(\underline{u}, \underline{v}) = (\underline{u}, S\,\underline{v})_o \quad \underline{u}, \underline{v} \in D(S)$$

is a continuous bilinear functional on D(S) (in the topology of H).

Proof. It is easily seen that $V(\underline{u}, \underline{v})$ is bilinear and that for any $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in D(S)$

$$V(\underline{u},\underline{v}) = -2(u_1, T_o v_1)_o + a^2(u_1, v_1)_o + a(u_1, v_2)_o + a(u_2, v_1)_o + 2(u_2, v_2)_o.$$

Integration by parts of the first term on the right and using the Schwartz inequality, we have

$$|-2(u_1, T_o v_1)_o| = 2\left| \int_\Omega [\sum_{i,j=1}^{n} a_{ij} \frac{\partial u_1}{\partial x_i}\frac{\partial v_1}{\partial x_j} - c(x)u_1 v_1]dx \right|$$

$$\leqq 2M[(\int_\Omega (\sum_{i=1}^{n} |\frac{\partial u_1}{\partial x_i}|)^2 dx)^{1/2} (\int_\Omega (\sum_{j=1}^{n} |\frac{\partial v_1}{\partial x_j}|^2 dx)^{1/2}] + 2c_M||u_1||_o||v_1||_o$$

$$\leqq 2Mn[(\sum_{i=1}^{n} \int_\Omega (\frac{\partial u_1}{\partial x_i})^2 dx)^{1/2} (\sum_{j=1}^{n} \int_\Omega (\frac{\partial v_1}{\partial x_j})^2 dx)^{1/2}] + 2\,c_M||u_1||_o||v_1||_o$$

where $M = \max_{i,j}(\max_{x\in\bar{\Omega}}|a_{ij}(x)|)$ and $c_M = \max_{x\in\bar{\Omega}}(-c(s))$. It follows that

$$|V(\underline{u},\underline{v})| \leqq k(||u_1||_1\,||v_1||_1 + ||u_1||_o\,||v_2||_o + ||u_2||_o\,||v_1||_o +$$
$$+ ||u_2||_o\,||v_2||_o)$$

$$\leqq k\,(||u_1||_1 + ||u_2||_o)(||v_1||_1 + ||v_2||_o)$$

where $k = \max\,(2Mn,\ 2C_M + a^2,\ 2)$. On squaring both sides of the above inequality and notice that $(\alpha+\beta)^2 \leqq 2(\alpha^2 + \beta^2)$ for any real numbers α, β we have

$$|V(\underline{u},\underline{v})|^2 \leqq 4k^2(||u_1||_1^2 + ||u_2||_o^2)(||v_1||_1^2 + ||v_2||_o^2)$$

which is equivalent to

$$|V(\underline{u},\underline{v})| \leqq k_1||\underline{u}||_H||\underline{v}||_H \tag{4-2}$$

where

$$k_1 = 2 \max (2Mn, 2c_M + a^2, 2). \qquad (4\text{-}3)$$

Thus $V(\underline{u}, \underline{v})$ is continuous in the topology of H which completes the proof of the lemma.

Lemma 4.2. The bilinear functional $V(\underline{u}, \underline{v})$ defines an equivalent inner product on D(S) in H and the extension $\bar{V}(\underline{u}, \underline{v})$ of $V(\underline{u}, \underline{v})$ to H defines an equivalent inner product on the whole space H.

Proof. Define $(\underline{u}, \underline{v})_s = V(\underline{u}, \underline{v})$, then $(\underline{u}, \underline{v})_s$ is bilinear and for any $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \varepsilon$ D(S) we have by lemma 3.1

$$\begin{aligned} (\underline{u}, \underline{v})_s &= -2(u_1, T_o v_1)_o + a^2(u_1, v_1)_o + a(u_1, v_2)_o + a(u_2, v_1)_o + 2(u_2, v_2)_o \\ &= -2(v_1, T_o u_1)_o + a^2(v_1, u_1)_o + a(v_2, u_1)_o + a(v_1, u_2)_o + 2(v_2, u_2)_o \\ &= (\underline{v}, \underline{u})_s \end{aligned}$$

which shows that $(\underline{u}, \underline{v})_s$ is symmetric. Integration by parts and using the strong ellipticity condition of L, we have

$$\begin{aligned} (\underline{u}, \underline{u})_s &= \int_\Omega [2 \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u_1}{\partial x_i} \frac{\partial u_1}{\partial x_j} - 2c(x)\, u_1^2 + (au_1 + u_2)^2 + u_2^2]dx \\ &\geqq \int_\Omega [2\, \alpha \sum_{i=1}^{n} (\frac{\partial u_1}{\partial x_i})^2 + 2c_m\, u_1^2 + u_2^2]dx \geqq k_2 ||\underline{u}||_H^2 \end{aligned} \qquad (4\text{-}4)$$

where

$$k_2 = \min (2\alpha, 2c_m, 1). \qquad (4\text{-}5)$$

Hence $V(\underline{u}, \underline{v})$ is positive definite on D(S) and that $(\underline{u}, \underline{u})_s \neq 0$ iff $\underline{u} \neq 0$. It follows that $(\underline{u}, \underline{v})_s = V(\underline{u}, \underline{v})$ defines an inner product on D(S). From (4-2) and (4-4), the inner product $(\underline{u}, \underline{v})_s$ is equivalent to $(\underline{u}, \underline{v})_H$ on D(S). Let $\bar{V}(\underline{u}, \underline{v})$ be the extension of $V(\underline{u}, \underline{v})$ from D(S) to H and define

$$(\underline{u}, \underline{v})_e = \bar{V}(\underline{u}, \underline{v}) \qquad \text{for } \underline{u}, \underline{v} \in H \qquad (4\text{-}6)$$

then by the continuity of $V(\underline{u}, \underline{v})$ on D(S), $\bar{V}(\underline{u}, \underline{v})$ possesses all the properties of bilinearity, symmetry, boundedness and positivity on H.

Moreover, from (4-2) and (4-4)

$$k_2\|u\|_H^2 \leq \|\underline{u}\|_e^2 \leq k_1\|u\|_H^2 \qquad u \in H. \tag{4-7}$$

Note that $(\underline{u},\underline{v})_e$ coincides with $(\underline{u},\underline{v})_s$ for u, $v \in D(S)$. Therefore $(\cdot,\cdot)_e$ is an equivalent inner product of $(\cdot,\cdot)_H$ which completes the proof.

Lemma 4.3. The operator A is strictly dissipative with respect to $(\cdot,\cdot)_e$ if $a > 0$ and is dissipative if $a = 0$ where a is the constant appearing in (1-1).

Proof. We first show that A is (strictly) dissipative on $D(T_o)xD(T_o)$. For any $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in D(T_o) \times D(T_o)$

$$A\,\underline{u} = \begin{bmatrix} u_2 \\ Tu_1 - au_2 \end{bmatrix} = \begin{bmatrix} u_2 \\ T_o u_1 - au_2 \end{bmatrix} \in D(T_o) \times L^2(\Omega) = D(S).$$

Since $\overline{V}(\underline{u},\underline{v})$ is an extension of $V(\underline{u},\underline{v})$ we have, by the definition of S and by lemma 3.1,

$$(\underline{u},A\underline{u})_e = (\underline{u},S\,A\underline{u})_o = -2(u_1,T_o u_2)_o + a(u_1,T_o u_1)_o + 2(u_2,T_o u_1)_o -$$

$$-a(u_2,u_2)_o = a(u_1,T_o u_1)_o - a(u_2,u_2)_o.$$

It follows by the strong ellipticity of L and the relation (4-7) that

$$(\underline{u},A\underline{u})_e = a\int_\Omega [-\sum_{i,j=1}^n a_{ij}(x)\frac{\partial u_1}{\partial x_i}\frac{\partial u_1}{\partial x_j} + c(x)u_1^2 - u_2^2]dx$$

$$\leq -a\int_\Omega [\alpha\sum_{i=1}^n (\frac{\partial u_1}{\partial x_i})^2 + c_m u_1^2 + u_2^2]dx \leq -a\lambda\|\underline{u}\|_H^2 \leq -\beta\|\underline{u}\|_e^2 \tag{4-8}$$

where

$$\lambda = \min(\alpha, c_m, 1), \quad \beta = \frac{a\lambda}{k_1}. \tag{4-9}$$

Thus A is dissipative on $D(T_o) \times D(T_o)$ if $a = 0$ and is strictly dissipative if $a > 0$. Next we show that A is (strictly) dissipative on $D(T)\times D(T_o)$. Let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \varepsilon\ D(T) \times D(T_o)$ then $A\,\underline{u} = \begin{bmatrix} u_2 \\ Tu_1 - au_2 \end{bmatrix} \varepsilon\ D(S)$. Hence

$$(\underline{u},\ A\ \underline{u})_e = (\underline{u}, S\ A\underline{u})_o = -2(u_1, T_o u_2)_o + a(u_1, Tu_1)_o + 2(u_2, Tu_1)_o - a(u_2, u_2)_o.$$

Since T is the closure of T_o in $L^2(\Omega)$, there exists a sequence $\{v_n\}$ in $D(T_o)$ such that $v_n \to u_1$ and $T_o v_n \to Tu_1$ in the topology of $L^2(\Omega)$ which implies by lemma 3.1 that

$$(u_2, Tu_1)_o = \lim(u_2, T_o v_n)_o = \lim(T_o u_2, v_n)_o = (T_o u_2, u_1)_o.$$

Moreover by Garding's Inequality, there exist constant c_1, c_2 such that

$$||v_n||_1^2 \leqq c_1(Tv_n, v_n)_o + c_2||v_n||_o^2 \leqq c_1||Tv_n||_o||v_n||_o + c_2||v_n||_o^2$$

we have $v_n \to u_1$ in $H_o^1(\Omega)$. Let $\underline{u}_n = \begin{bmatrix} v_n \\ u_2 \end{bmatrix}$ then $\underline{u}_n \to \underline{u}$ in H and thus $\underline{u}_n \to \underline{u}$ with respect to $||\cdot||_e$. It follows from (4-8) that

$$(\underline{u},\ A\ \underline{u})_e = a(u_1, Tu_1)_o - a(u_2, u_2)_o = \lim_{n\to\infty}[a(v_n, T_o v_n)_o - a(u_2, u_2)_o]$$

$$\leqq \lim_{n\to\infty} [-\beta||\underline{u}_n||_e^2] = -\beta||\underline{u}||_e^2 \qquad \underline{u}\ \varepsilon\ D(T)\times D(T_o). \qquad (4\text{-}10)$$

To show the dissipativity of A on $D(A)$, let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \varepsilon\ D(A)$, that is, $u_1\ \varepsilon\ D(T)$ and $u_2\ \varepsilon\ H_o^1(\Omega)$. Then there exists a sequence $\{w_n\}$ in $D(T_o)$ such that $w_n \to u_2$ in the topology of $H_o^1(\Omega)$ since $D(T_o) \supset C_o^\infty(\Omega)$ which is dense in $H_o^1(\Omega)$. Let $\underline{u}_n = \begin{bmatrix} u_1 \\ w_n \end{bmatrix} \varepsilon\ D(T)\times D(T_o)$ then $\lim_{n\to\infty} \underline{u}_n = \underline{u}$ and

$$\lim_{n\to\infty} A\ \underline{u}_n = \lim_{n\to\infty} \begin{bmatrix} w_n \\ Tu_1 - aw_n \end{bmatrix} = \begin{bmatrix} u_2 \\ Tu_1 - au_2 \end{bmatrix} = A\ \underline{u}$$

where the convergence of both $\{\underline{u}_n\}$ and $\{A\underline{u}_n\}$ hold in the topology of H. Hence by the equivalence between $||\cdot||_H$ and $||\cdot||_e$, $\underline{u}_n \to \underline{u}$ and $A\underline{u}_n \to A\underline{u}$ in the topology of H_e. It follows from (4-10) that

$$(\underline{u}, A\underline{u})_e = \lim_{n\to\infty}(\underline{u}_n, A\underline{u}_n)_e \leqq -\beta||\underline{u}||_e^2 \qquad \underline{u}\ \varepsilon\ D(A) \qquad (4\text{-}11)$$

which shows that A is dissipative if $\beta=0$ (i.e., $a=0$) and is strictly dissipative if $\beta>0$ (i.e. $a>0$).

5. The Main Results

Theorem 5.1. Let the operator L in (1-1) be a strongly elliptic formal partial differential operator with $c(x)<0$ for $x \in \bar{\Omega}$ where Ω is a bounded domain in R^n with sufficiently smooth surface $\partial\Omega$. Then for any initial data $u_o(x)$ in D(T) and $v_o(x)$ in $H_o^1(\Omega)$ there exists a unique strong solution $u(t,x)$ (in the sense of definition 2.1 with $u(t,x)$ strongly differentiable in t for all $t \geqq 0$) of the linear equation (1-1) (i.e., $f\equiv 0$) and the homogeneous boundary condition such that $u(0,x)=u_o(x)$ and $\dot{u}(0,x)=v_o(x)$. Moreover, the null solution $u(t,x)\equiv 0$ is stable if $a=0$ and is asymptotically stable if $a>0$.

Proof. By lemma 3.3 A is a linear operator with D(A) dense in H and R(A) contained in H such that $R(I-A)=H$. By lemma 4.3, A is strictly dissipative (resp., dissipative) with a dissipative constant β with respect to the equivalent inner product $(\cdot,\cdot)_e$. It follows by applying theorem 2.2 with $f\equiv 0$ that all the results stated in the theorem hold, where the underlying Hilbert space is H. The strong differentiability of $u(t,x)$ for all $t \geqq 0$ follows from a theorem due to Lumer and Phillips [7] (see also [3]).

Theorem 5.2. Let the operator L in (1-1) be the same as in theorem 5.1. Assume that f is a function defined on all of H into $L^2(\Omega)$. If there exists a number $k \geqq 0$ such that for any $u,v \in H_o^1(\Omega)$ and $u', v' \in L^2(\Omega)$

$$||f(x,u,u_{x_i},u')-f(x,v,v_{x_i},v')||_o \leqq k(||u-v||_o^2+\sum_{i=1}^n ||u_{x_i}-v_{x_i}||_o^2+||u'-v'||_o^2)^{1/2} \tag{5-1}$$

Then for any initial element $u_o(x)$ in $D(T)$ and $v_o(x)$ in $H_o^1(\Omega)$ there exists a unique solution $u(t,x)$ of (1-1) satisfying the homogeneous boundary condition such that $u(0,x)=u_o(x)$ and $u_t(0,x)=v_o(x)$. Moreover, any unperturbed solution such as equilibrium solution or periodic solution (if any), is stable if $k=(k_2/k_1)^{1/2}\beta$ and is exponentially asymptotically stable if $k<(k_2/k_1)^{1/2}\beta$ where k_1, k_2 are given by (4-3), (4-5) respectively and β is given by (4-9).

Remark. (a) The initial elements $u_o(x)$ and $v_o(x)$ in theorems 5.1 and 5.2 can be in the spaces $H_o^1(\Omega)$ and $L^2(\Omega)$ respectively since the initial element in $D(A)=D(T)xH_o^1(\Omega)$ can be extended to $\overline{D(A)}$, the closure of $D(A)$, which is equal to H; (b) The condition (5-1) can be weakened to some extent (see [2]); (c) For $k>(k_2/k_1)^{1/2}\beta$, the solution in theorem 5.2 is a weak solution.

Proof. It suffices to show that $\underset{\sim}{f}(u)$ satisfies the conditions in theorem 2.2 in the equivalent Hilbert space H_e since by hypotheses all the assumptions on A are satisfied as is shown in theorem 5.1. By hypothesis, $\underset{\sim}{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}$ is defined on all of $H=H_o^1(\Omega)xL^2(\Omega)$ into H. By the relation (5-1) for any $\underset{\sim}{u} = \begin{pmatrix} u \\ u' \end{pmatrix}$, $\underset{\sim}{v} = \begin{pmatrix} v \\ v' \end{pmatrix}$ in H, we have

$$||\underset{\sim}{f}(\underset{\sim}{u})-\underset{\sim}{f}(\underset{\sim}{v})||_e \leqq k_1^{1/2} \; ||\underset{\sim}{f}(\underset{\sim}{u})-\underset{\sim}{f}(\underset{\sim}{v})||_H = k_1^{1/2}||f(x,u,u_{x_i},u')-f(x,v,v_{x_i},v')||_o$$

$$\leqq k_1^{1/2} \; k(||u-v||_1^2 + ||u'-v'||_o^2)^{1/2} \leqq (k_1/k_2)^{1/2}k||\underset{\sim}{u}-\underset{\sim}{v}||_e \qquad (5\text{-}2)$$

where we have used the equivalence relation (4-7). Hence if $(k_1/k_2)^{1/2}\; k\leqq\beta$, that is, $k \leqq (k_2/k_1)^{1/2}\beta$, all the results stated in the theorem follows directly from theorem 2.2. In case $k > (k_2/k_1)^{1/2}\beta$, the existence and uniqueness of a solution in H_e satisfying the properties in the theorem follow from [8] due to Browder. It has been shown in [1] that if $u(t,x)$ is a weak solution in an equivalent Hilbert space H_e, it is also a weak solution in the original Hilbert space H. Therefore the existence and uniqueness of a weak solution is proved for any finite value of k which completes the proof of the theorem.

Corollary. Let the operator L in (1-1) be the same as in theorem 5.1. If $f(x,u,u_{x_i}, u_t) = f(x)$ is in $L^2(\Omega)$ then all the results in theorem 3.2 hold.

The condition (5-1) implies that f is Lipschitz continuous on H. Conversely, if f is continuous on H, not necessarily Lipschitz continuous, we can weaken the condition (5-1) to some extent as in the following theorem which can be proved in a straight forward way.

Theorem 5.3. Let the operator L in (1-1) be the same as in theorem 5.1. If f is defined and continuous on H such that condition (5-1) holds for any dense subset D of H (e.g., $D=C_o^\infty(\Omega)\times C_o^\infty(\Omega)$), then all the results in theorem 5.2 are valid.

References

1. Pao, C. V., The Existence and Stability of Solutions to Nonlinear Operator Differential Equation. Arch. Rational Mech. Anal. (to appear).

2. Pao, C. V. and Vogt, W. G., On the Stability of Nonlinear Operator Differential Equations, and Applications. Arch. Rational Mech. Anal. (to appear).

3. Yosida, K., Functional Analysis. Berlin: Springer-Verlag, 1966.

4. Dunford, N. and Schwartz, J., Linear Operators, Vol. 2. New York: Interscience, 1963.

5. Kato, T., Nonlinear Semi-groups and Evolution Equations. J. Math. Soc. Japan, 19 (1967) 508-520.

6. Buis, G. R., Lyapunov Stability for Partial Differential Equations, Part I, NASA CR-1100, 1968.

7. Lumer, G. and Phillips, R. S. Dissipative Operators in a Banach Space. Pacific J. Math., 11 (1961), 679-698.

8. Browder, F. E., Nonlinear Equations of Evolution and Nonlinear Accretive Operators in Banach Spaces. Bull. Amer. Math Soc., 73, (1967), 867-874.