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APPLICATION OF MODERN NETWORK THEORY TO ANALYSIS OF COMPLEX SYSTEMS

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16. Abstract

Present day systems have grown to exceed the intuitive grasp of systems analysts, thus negating the value of techniques which depend upon intuition as a substitute for mathematical rigor. A methodology based on the Modern Network Theory of electrical engineering is presented which, when considered in the light of digital computer technology, shows up as a capable and powerful tool for the analysis of complex systems. The examples presented to demonstrate the various techniques show that Modern Network Theory is not limited to electrical systems but appears to be applicable to systems in general.

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SUMMARY

The large and complex systems made possible by present day technology require a straightforward and rigorous mathematical methodology for determining their performance. Existing techniques rely to a large degree on intuition to handle the multidisciplinary nature of such systems. The techniques of Modern Network Theory can be applied to these systems in a consistent and reliable manner, thus allowing for the complete analysis of systems in terms of the variables associated with the system components.

The examples presented indicate the ease with which fairly complex systems can be studied by use of these techniques, and further show that any system should respond to the methodology in a straightforward manner.

INTRODUCTION

Present day technology has brought with it both the desire and the ability to conceive and construct large and extremely complex systems. Inherent in the concept of any system is the need for techniques to analyze the operation of the system in terms of the variables associated with the system components. In the past, it has been possible to analyze most systems with monodisciplinary techniques backed up with a bit of intuition. Unfortunately, this approach is becoming impractical, mainly because of the extreme size and complexity of new systems which rapidly exceed the intuitive grasp of the systems analyst. Thus, there exists a need for techniques which can completely encompass an entire complex system while, at the same time, allowing rigorous mathematical analysis.

With the widespread availability of high-speed digital computers has come a reevaluation of the various analytical techniques available to the engineer. In the electrical engineering community there exists a body of theory known as Modern Network Theory (hereinafter referred to as MNT) which, when considered in the light of digital computer technology, shows up as a most capable and powerful analytical tool (refs. 1 to 4). The growth of MNT to its present status follows quite closely the growth of digital computer technology and has been formulated to take advantage of this technology. The principal advantages provided by MNT are as follows:

(1) MNT is a clear and simple methodology for the generation of the system equations of an interconnected finite set of components.

(2) MNT has straightforward techniques for obtaining a suitable equation format consistent with the requirements for digital computer solutions.

(3) MNT contains rather extensive techniques for synthesis of systems having desired characteristics.

Although MNT has been exclusively a tool of the electrical engineering community, it is now clear to the author and to others (e.g., ref. 5) that there is no unique feature of electrical engineering that would limit the use of the MNT techniques to that field. In fact, it is now the contention of the author that, in principle, the MNT systems analysis methods can be applied to any system. The implications of this are quite profound, and although there are problems standing in the way of a complete working methodology for systems in general, none of these problems appears insurmountable. Indeed, some of the more formidable ones have already yielded to the efforts of this investigation.

MODERN NETWORK THEORY

A discussion of Modern Network Theory must really be preceded by a description of Graph Theory, which forms the basis for the establishment of the system equations associated with a given network.

Graph Theory

Graph Theory, as the name implies, is concerned with the properties of linear oriented graphs. Linear refers to the fact that such graphs are made up of lines, and oriented refers to a "sense of direction" associated with the lines.

A few definitions of some of the labels used in Graph Theory (essentially as found in ref. 4) now follow as an aid to the understanding of the examples which are to be presented in the section APPLIED NETWORK ANALYSIS.

Definition 1: Element - An element is a line segment and its vertices, always one vertex on each end of the line segment.

Definition 2: Vertex - A vertex is a dot. A vertex is the only significant physical joining or ending of line segments (elements) in a graph.

Vertices are illustrated either as small circles or solid dots. The intersection of line segments at X in figure 1 is not a vertex.

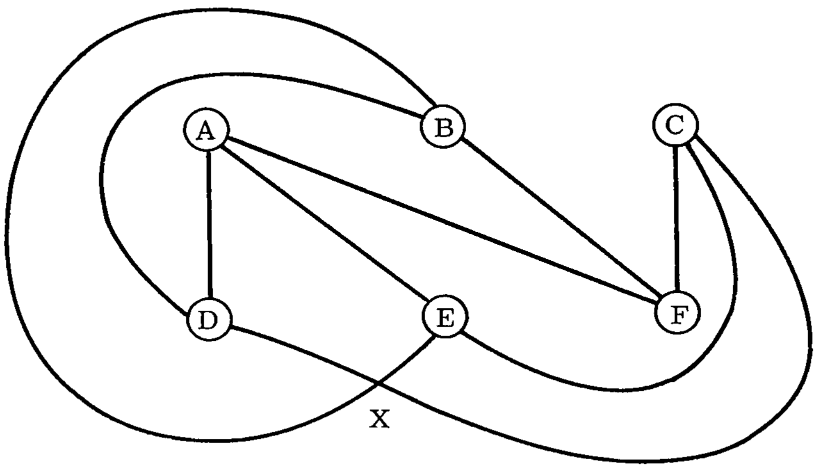


Figure 1. - Example of graph showing vertices.

Definition 3: Graph - A graph is a finite set of elements and associated vertices.

The letters e and v are used to denote the number of elements and vertices that are contained in a graph. The notation $G(e,v)$ is used to mean that the graph G contains e elements and v vertices.

The following definitions refer to elements and vertices in general.

Definition 4: Incidence - An element is incident to a vertex and a vertex is incident to an element if the vertex is a vertex of the element.

In figure 2, elements 1, 2, and 3 are incident to vertex A, and vertex A is incident to elements 1, 2, and 3.

Definition 5: Degree of vertex - The degree of a vertex is the number of elements incident to the vertex.

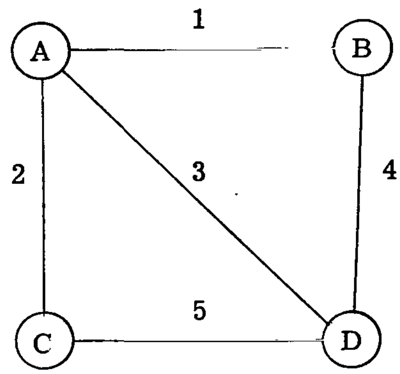


Figure 2. - Example of graph showing incidence.

Vertices A and D of figure 2 are of degree 3. Vertices B and C are of second degree.

Definition 6: Adjacent elements - Two elements are adjacent if the elements are incident to the same vertex.

Definition 7: Adjacent vertices - Two vertices are adjacent if the vertices are incident to the same element.

Elements 2 and 3 in figure 2 are adjacent. In the same graph, vertices A and C are adjacent, but vertices B and C are not. In figure 3 each vertex is adjacent to every other vertex in the graph. A graph of this type is given the special name "complete graph."

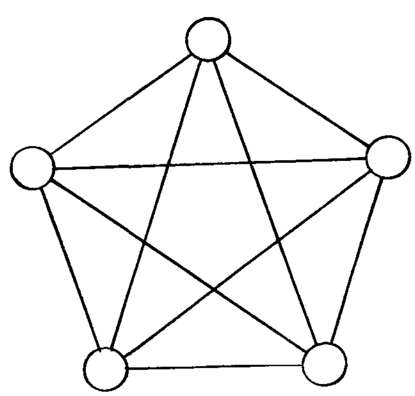


Figure 3. - Complete graph.

Definition 8: End vertex - An end vertex is a vertex of degree 1.

Vertices A, C, and E of figure 4 are end vertices.

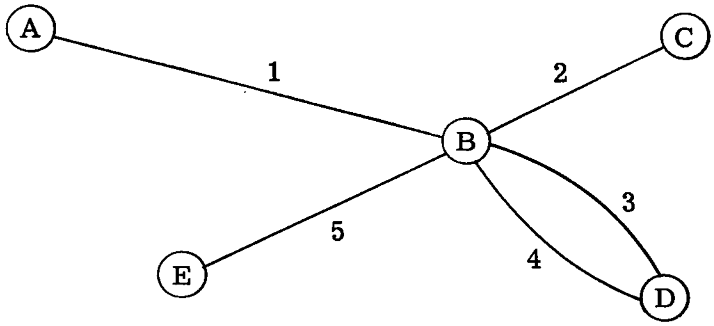


Figure 4. - Example of graph showing end and interior vertices and elements.

Definition 9: End element - An end element is an element incident to at least one end vertex.

Elements 1, 2, and 5 of figure 4 are end elements.

Definition 10: Interior vertex - A vertex of degree greater than 1 is an interior vertex.

Vertices B and D of figure 4 are interior vertices.

Definition 11: Interior element - If both vertices of an element are of degree greater than 1, the element is an interior element.

Elements 3 and 4 of figure 4 are interior elements.

In most instances the important characteristics relate to something less than the totality of a graph. Thus, the following definitions are appropriate:

Definition 12: Subgraph - A subgraph G_s of a graph G is a subset of the elements of G.

By definition 3, a subgraph is also a graph in its own right. On occasion it is convenient, as well as necessary, to consider a subset of the elements of a subgraph (i.e., a subgraph of a subgraph).

Definition 13: Proper subgraph - A proper subgraph of a graph $G(e, v)$ is a subgraph of G containing at least one but fewer than e elements.

Definition 14: Complement - With respect to a graph G, the complement G'_s of a subgraph G_s of G is the set of elements of G not contained in G_s; G_s and G'_s are complementary subgraphs with respect to G.

The various subgraphs of a graph may or may not share vertices or elements with one another. The following definitions then apply:

Definition 15: Disjoint (Vertex disjoint) - Two subgraphs are disjoint (vertex disjoint) if the subgraphs have no vertices in common.

Definition 16: Joint (Vertex joint) - Two subgraphs are vertex joint if the subgraphs share at least one common vertex.

Definition 17: Element disjoint - Two subgraphs are element disjoint if the subgraphs have no elements in common.

Complementary subgraphs G_s and G'_s are always element disjoint.

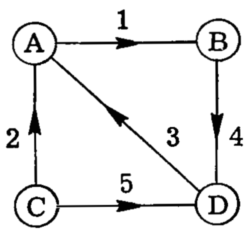
Definition 18: Element joint - Two subgraphs are element joint if the subgraphs share at least one common element.

To provide a sense of orientation to a graph the following definition is appropriate:

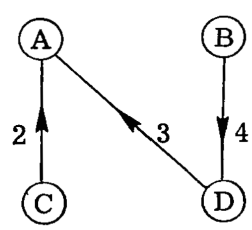
Definition 19: Directed, or oriented, graph - A directed, or oriented, graph is a graph in which every element is arbitrarily marked with some symbol that denotes direction.

It is very significant that the method of choosing the direction of the symbol for each element can be entirely arbitrary and should not depend on such ideas as the direction in which something appears to flow, or on any other similar idea. This idea often proves to be most difficult to grasp, and is important because apparent flow directions and similar orientation ideas can be quite misleading.

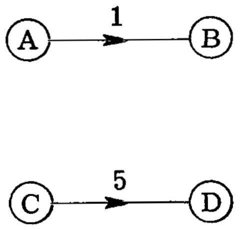
A commonly used technique in graph theory is to mark each element of a directed graph with an arrowhead. The graph of figure 5(a) is an example of a directed version of the graph of figure 2.



(a) Graph.



(b) Tree of graph.



(c) Cotree of graph.

Figure 5. - Directed version of graph shown in figure 2.

Particular subgraphs of graphs are sufficiently important to the study of graph theory to be given names. One such subgraph is called a "tree" and is described as follows:

Definition 20: Tree - A tree is any subgraph of a graph $G(e, v)$ which has the following properties:

(1) Contains all vertices of G

(2) Contains $(v - 1)$ elements of G

(3) Is connected

(4) Contains no circuits

A definition of "connected" has not been given, but for the purpose of this discussion it should be sufficient to say that a connected graph (or subgraph) is one in which there exists between every pair of vertices at least one route, over one or more elements of the graph, by which to travel from one vertex of a pair to the other. If no element along this "route" is traversed more than once, the route is given the special name "path."

A circuit is defined as follows:

Definition 21: Circuit - A circuit is a connected subgraph in which all vertices of the subgraph are of second degree.

It is interesting to note that any three of the four properties in the definition of a tree are sufficient to imply the remaining condition. In fact, the second and fourth conditions are alone sufficient to imply the other two. The subgraph of figure 5(b) is one of the trees of the graph in 5(a).

Another type of subgraph, the cotree, follows directly from the tree subgraph.

Definition 22: Cotree - A cotree is a subgraph which is the complement of a tree.

Figure 5(c) depicts the cotree complement of the tree in figure 5(b). The elements of the tree and cotree are also given names as an aid to discussion. Single elements of a tree are called "branches," and single elements of a cotree are called "chords."

Another useful subgraph is the "seg," which is best defined by the following discussion. If the vertices of a graph are segregated into n mutually exclusive and all inclusive sets, all those elements of the graph that have one vertex in one set and the other vertex in a different set form an n-seg. For brevity, a 2-seg is simply called a seg.

An example of a seg of the graph of figure 5 is shown in figure 6. This seg is the result of segregating vertices A, C, and D in one set and vertex B into another. Elements 2, 3, and 5 of the graph do not appear in this seg since both vertices of each of these elements are in the same set. Note that vertex C does not appear in the seg since no elements in this seg are incident to it.

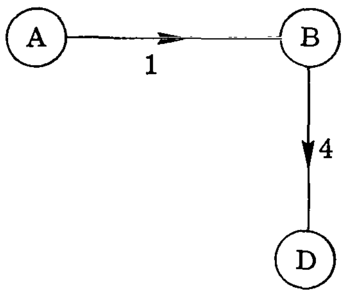
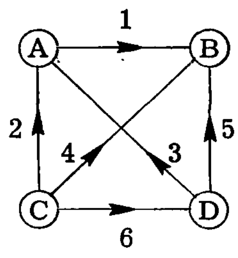


Figure 6. - Seg of graph shown in figure 5.

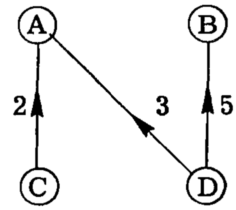
Matrix Notation

The listing of the various subgraphs of a particular graph is easily accomplished through the use of a matrix notation technique. The number of columns in the matrix for a given graph will be equal to the number of elements contained in the graph. In fact, each column is labeled with one element of the graph. The rows of the graph then are vector representations of the subgraphs. The appearance of an element in a subgraph is indicated by a "1" entry in the row representing that subgraph and under the column representing that element. Elements not present in a given subgraph are entered as 0 (zero) in the proper row and column position.

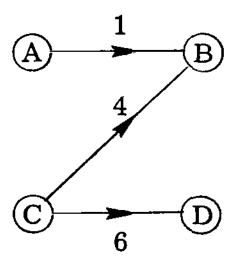
If the graph is directed, the 1 entries are modified by the inclusion of either a plus sign or a minus sign to indicate the alinement of the element with respect to a sense of direction assigned to the subgraph. The types of subgraphs which concern us here are segs and circuits. For segs, the positive sense of direction of the subgraph is that followed in going from one set of vertices which define the seg, to the other. Which direction is taken to be positive is not really important. In circuits, the positive sense is assigned to one or the other of the two possible modes of circumnavigating the circuit. Again the direction taken to be positive is not important. In the case of "fundamental" seg and circuit subgraphs, the direction of the subgraph is usually based on that of the element used to form the particular subgraph.



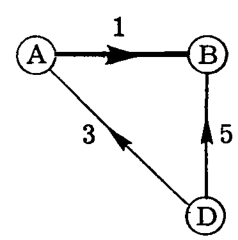
(a) Graph.



(b) One tree of graph.



(c) Complementary cotree.



(d) Fundamental circuit based on chord number 1.

$$
\begin{array}{r c}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{array}{r} \text{Tree} \\ \text{Cotree} \\ \text{Circuit based on 1} \\ \text{Circuit based on 4} \\ \text{Circuit based on 6} \end{array} &
\left\{ \begin{array}{cccccc}
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & 1 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 & 0 & 1
\end{array} \right\}
\end{array}
$$

(e) Matrix representation.

Figure 7. - A graph, one of its trees, the complementary cotree, a circuit, and a matrix representation of these and two other subgraphs.

A fundamental seg is formed by considering a tree and then noting the natural segregation of vertices that occurs when any one branch is removed. The seg that results from that particular vertex segregation will, of course, contain the one branch upon which it was based (but no other branches of that tree), and the sign given to the seg is such that the branch would be positive.

Fundamental circuits are based on chords of a cotree and are formed simply by considering one chord and a path within the complementary tree which connects to both vertices of the chord. One such path will always exist and together with the chord will always form exactly one circuit. Figure 7 shows such a circuit. The sign of this type of circuit is usually assigned so that the chord direction is positive. Continuing around the circuit in the direction assigned by the chord will allow the signs for the other circuit elements to be determined.

Figure 7 also shows a directed graph, one of its trees, the complementary cotree, and a matrix listing of all of the fundamental circuits based on the cotree. The first two rows of the matrix represent the tree and the cotree with no orientation associated with these two subgraphs.

Variables in Modern Network Theory

The variables used to quantitatively describe the behavior of systems are as follows: Each element has associated with it variables in pairs. Each pair consists of one through variable, and one across variable. Also, each element has associated with it exactly one element characteristics equation (ECE).

In simple cases, the ECE associated with an element is a relation between one pair of variables associated with that same element. But generally, the ECE associated with an element may involve any or all of the variables of the graph. The variables of MNT are usually functions of time, and the ECE's commonly involve differential operators.

Certain restrictions are placed upon the variables of MNT. First, for practical purposes, it is required that one, but not necessarily both, of the variables of a pair associated with an element must be calculable in terms of measurable quantities.

This requirement really concerns the scientific application of MNT and is not important to the theory, per se. It guarantees that the MNT model of the system is testable. There are two really fundamental requirements which must be satisfied by the variables used in MNT:

(1) The sum of the through variables for the elements of any n-seg must vanish.

(2) The sum of the across variables for the elements of any circuit must vanish.

In the field of electrical engineering, these requirements are known as Kirchhoff's laws. (In statics these requirements are used, but in a rather unconscious way: in fact, the second law is never mentioned explicitly because of its obviousness (the across variable is position). In hydraulics, both laws are explicitly stated. Outside of these fields, the applicability of network analysis has not been recognized. This is clear from the fact that the very concept of through and across variables and the "laws" pertaining to them have no explicit formulation.) It should be noted, however, that the conformity of the electrical variables (current and voltage) to the first and second laws is actually the consequence of the methods used in defining the numerical scales on meters used to measure the variables.

The important point to make here is that any subsystem should respond to similar treatment if the "meters" that are used to define the variables can also be calibrated (or defined) in such a way that the requirements are satisfied.

These requirements can be restated in a form that again makes use of matrix notation in the following way. If S represents a matrix of segs generated in the way previously described and Y represents a column matrix of the through variables associated with the elements of the graph, the following must be true:

$$SY = 0 \tag{1}$$

where the right side is a column matrix of zeros. Likewise, for the other requirement, if B is a matrix of circuits and X is a column matrix of across variables,

$$BX = 0 \tag{2}$$

Of course, it is necessary that the matrices be arranged so that they are conformable for multiplication, and the ordering of the variables in the rows of the column matrices must be the same as the ordering of the columns in the seg and circuit matrices in order for the two statements to have any significance. From equation (1) it can be concluded that the value of all through variables associated with end elements must always be identically zero.

A direct consequence of the properties of graphs (which are based on the previous definitions) is that the following is true:

$$SB^t = 0 \tag{3}$$

where superscript t means the transpose of the matrix which it modifies, and S and B are <u>any</u> seg and circuit matrices that are dimensionally conformable for multiplication.

Theorems

At this point it is appropriate to introduce a few interesting theorems, based on the properties of graphs, that have immediate value in the analysis of real-world problems.

Theorem 1: The maximum number of specified across variables in a network is numerically equal to $(v - 1)$, and the elements with a specified across variable must all be contained in some tree of the corresponding graph.

Translated to an electrical engineering situation (where the variables already satisfy the requirements), the meaning is that only $(v - 1)$ of the electrical components may have an arbitrarily specified value of voltage; and further, all such specified voltage components must be contained in some corresponding tree of the graph which represents the electrical network involved. Note that the theorem does not mention voltage but only the general idea of the across variable. It would thus be applicable to any network that was described in terms of a satisfactory across variable.

The following theorem concerns through variables in general:

Theorem 2: The maximum number of specified through variables in a network is numerically equal to $(e - v + 1)$, and all elements having specified through variables must be contained in the cotree of a tree that contains all elements that have specified across variables if any such elements exist.

No rigorous proofs are given in this work. Nevertheless, it seems appropriate to indicate the nature of the proofs and to show plausibility where possible.

Regarding theorem 1, it is obvious that every graph contains a tree. Now trees contain no circuits, and so $(v - 1)$ across variables can be specified. If another across variable is specified, however, the corresponding element must always complete a circuit, and the new across variable is already determined by the second requirement (Kirchhoff's law for across variables) on the variables of MNT.

To see the truth of theorem 2, we need the help of a microtheorem, namely, "No cotree contains a seg." A proof by contradiction runs as follows: If a cotree contained a seg, it would contain all of the elements which join the two sets of vertices that define the seg. The complement of the cotree, then, would <u>not</u> contain any element joining the one set with the other. But, the complement of a cotree is a tree, which <u>does</u> join all vertices.

Therefore, no violation of the first requirement imposed on variables (Kirchhoff's law for through variables) can be encountered if $(e - v + 1)$ through variables on a cotree are specified. If another through variable is added to the list of specified through variables, a seg will be included in the set of elements, namely, that defined by the removal of the element from the tree.

Theorems 1 and 2 are important in themselves, of course, but they also serve to clearly show the plausibility of the following theorems which are central to MNT:

Theorem 3: There are exactly (e - v + 1) linearly independent circuit vectors.

Theorem 4: There are exactly (v - 1) linearly independent seg vectors.

Theorem 3 follows directly from theorem 1. For if the maximum number of across variables which may be specified is (v - 1), then by using Kirchhoff's laws for across variables the information can be "spread" to the remaining (e - v + 1) elements. Thus, there must be at least (e - v + 1) linearly independent circuit equations. On the other hand, if there were more, (v - 1) across variables could not be arbitrarily specified. Theorem 4 follows from theorem 2 by the same sort of argument.

Theorems 3 and 4 indicate the number of circuit and seg equations required to ensure conformity with the Kirchhoff requirements, but do not indicate any systematic plan for forming these equations. A complete set of circuit or seg equations is given the name "basis set" and in matrix notation is written B_b for circuits and S_b for segs. A fundamental set of circuits B_f (i.e., one for each and every chord of a cotree) is always a B_b. Similarly, a fundamental set of segs S_f (i.e., one for each and every branch of a tree) is always an S_b.

APPLIED NETWORK ANALYSIS

The material in the preceding section has been intended only as cursory background material to aid in the understanding of the various analytical techniques which are to be presented. These presentations will take the form of analyses which are applied to sample systems or subsystems.

With theorems 1 and 2, it is possible to consider the analysis of certain types of systems problems of which the following is an example.

Hydraulics Problem

The problem to be considered involves the hypothetical hydraulic network of figure 8. The question to be answered is: "Are the numbers and locations of the various pumps consistent with the physical laws?"

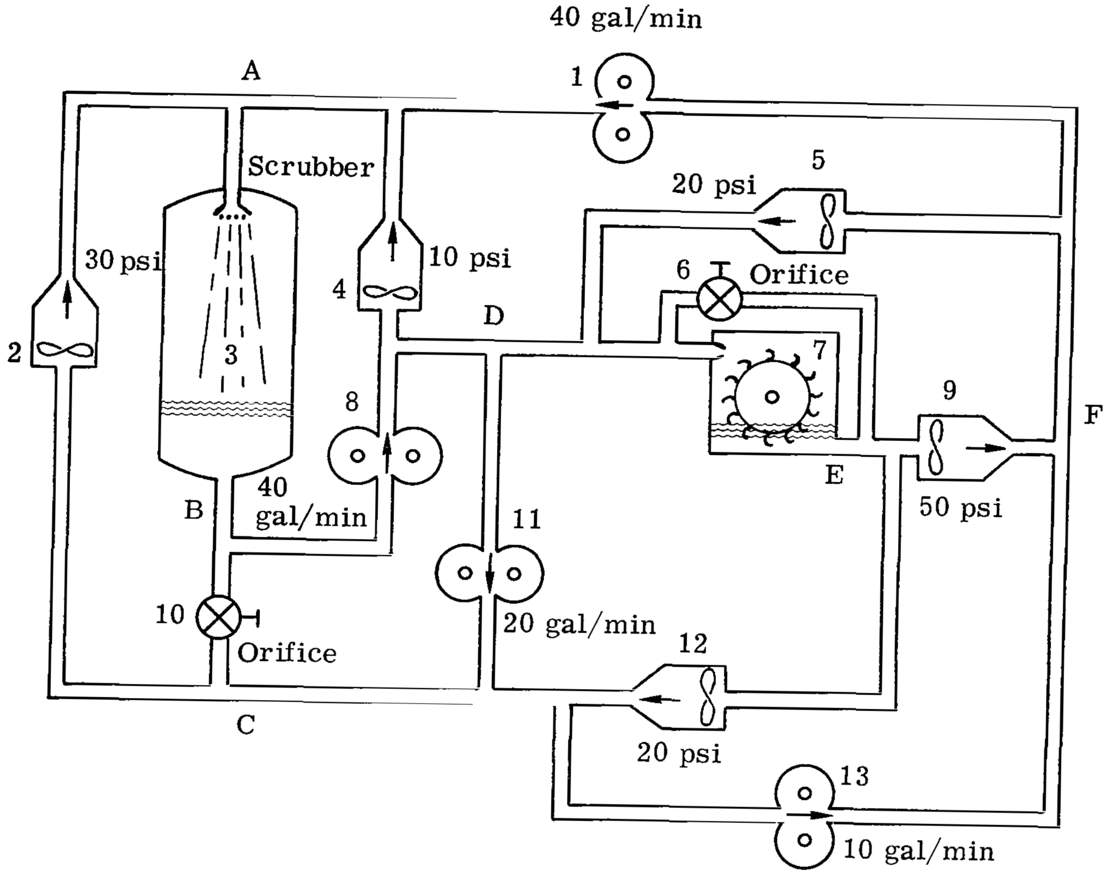


Figure 8. - Hypothetical hydraulic network.

Note that there are two types of pumps used in the network: One type provides a constant flow rate (gear, piston, diaphragm, etc.), and the other provides a constant pressure rise between its inlet and outlet ports (turbine, centrifugal, etc.).

The other devices that are depicted in the network (e.g., the scrubber, orifices, and water wheel) are essentially simple resistive loads, with the exception of the pipes that interconnect the various devices. For this example, the pipes are considered as being "loss free", although pipe losses could also be included in a more rigorous treatment of the problem.

To apply the techniques of MNT analysis to this problem, the first step is to construct a graph of the network under study. From the graph it is then possible to study the interrelation of the various elements (devices) which arises from the particular manner in which they are interconnected.

To simplify the construction of the graph, the devices have been coded with numbers 1 to 13, without regard to order. The connection points have been coded with the letters A to F, again without regard to any order (see fig. 8).

The graph is then constructed as is shown in figure 9. The vertices (A to F) and the elements (1 to 13) have been arranged so that they approximate their particular alinement in the network. This is done in the interest of clarity and is not at all necessary to the solution of the problem.

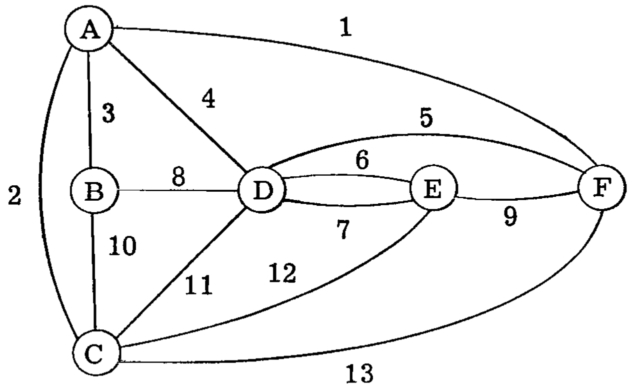


Figure 9. - Graph of hydraulic network shown in figure 8.

The theorem relating to the across variables states: The maximum number of specified "across variables" (pressure in this case) is equal to one less than the number of vertices (v - 1), and the location of these specified elements must correspond to "branches" of a common "tree".

Theorem 2, which will also be used in analyzing this problem, states: The maximum number of specified through variables (in this case flow rate) is equal to (e - v + 1), where e is the number of elements in the graph and v the number of vertices. It is further necessary that all of the elements which have a specified flow rate be chords of the particular cotree that complements the tree containing all of the specified pressure elements.

Note that the sum of elements in a tree and in its complementary cotree indeed sum to the total number of elements in the graph, that is,

$$\underset{\text{Tree}}{(v - 1)} + \underset{\text{Cotree}}{(e - v + 1)} = \underset{\text{Graph}}{e}$$

In the graph of the hydraulics network, there are 13 elements (e = 13) and six vertices (v = 6). Thus, from the first theorem we find that the maximum allowable number of constant pressure pumps is

$$v - 1 = 6 - 1 = 5$$

Those elements of the graph which correspond to constant pressure pumps are 2, 4, 5, 9, and 12 (i.e., five such elements). Thus, the number of these pumps is consistent.

The maximum allowable number of constant flow pumps is

$$e - v + 1 = 13 - 6 + 1 = 8$$

And since there are only four such devices (elements 1, 8, 11, and 13), there is no conflict here.

To analyze the consistency of the location of the pumps in the network, it is well to construct a subgraph which corresponds to the constant pressure pumps and then see if a tree can be found which includes all of these elements. Figure 10 is a graph of the constant pressure elements. It is immediately clear that these elements are not consistent with the first theorem since the subgraph contains a circuit. Because no tree can contain a circuit, it is not possible to find a tree which will contain all of these elements.

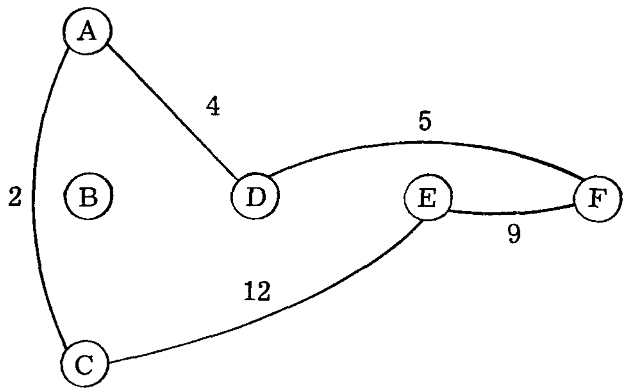


Figure 10. - Subgraph of constant pressure elements, hydraulic system example.

Thus the question is answered: "The number of pumps is consistent with the physical laws, but their location in the network is not." If the hydraulics network is to function, some change in the system must be made. A repeat of the previous analysis will allow us to evaluate the success of any proposed change.

Consider, for example, the possibility of switching the inlet of pump (2) from below to above orifice (10), dotted connection in figure 8. The number of pumps remains the same so this consistency is maintained. The graph of the constant pressure elements is now as shown in figure 11. Note that this subgraph has all of the properties of a tree just

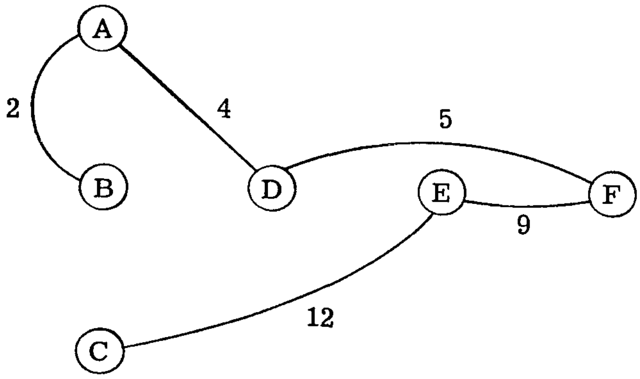


Figure 11. - Graph of reconnected constant pressure elements, hydraulic system example.

as it stands and is therefore a tree. Thus, the location of the constant pressure pumps is now consistent.

Since the tree of figure 11 does not contain any elements corresponding to constant flow devices, all of these elements must be in the complement of this tree (i.e., in the cotree (fig. 12)). Thus, the location of the constant flow pumps is also consistent.

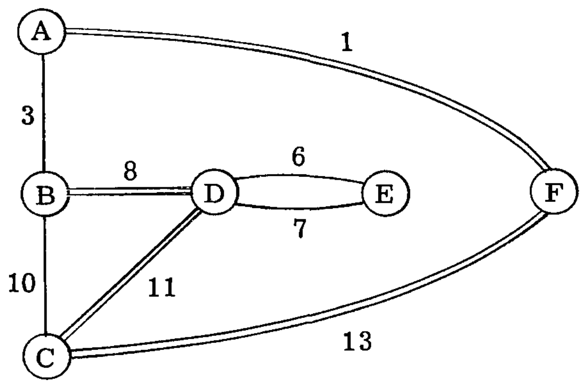


Figure 12. - Cotree of tree shown in figure 11. Constant flow rate elements are depicted with double lines.

The answer to the question is now: "Yes, the number and location of the pumps in the new network are consistent with the physical laws."

General Applicability of Modern Network Theory

Modern Network Theory is applicable to a wide variety of systems. In the previous example, a hydraulic system, theorems of MNT were used to decide whether the system was overspecified. It should be apparent that the conclusions did not in any way depend on the system being a hydraulic system.

Much more can be done with MNT than merely determining whether a system is overspecified. The analysis of a system implies the ability to follow the variation with time of the values of the through and across variables for each element. Bear in mind, however, that the process of analysis is applied not to the real-world system but to a symbolic representation thereof. Whether the results of the analysis apply to the real-world system depends on the model or the graph embodying all the significant features of the real system.

Given a graph with elements whose characteristics are expressible in terms of sets of variable pairs, MNT can be applied to analyze it. The problem then is under what circumstances an arbitrary real-world system can be represented by such a graph. The first requirement is that the performance of the system - that is, the time variation of some observable quantities or parameters - must be expressible in terms of sets of variable pairs. Suitable variable pairs must be found or invented. The second requirement is that the finite set of components which comprise the system be representable by a finite set of elements connected in a known way. It is not necessary that there be a one-to-one correspondence between the real system components and the elements of the graph, a single physical component may be best represented by a multiplicity of elements. Finally, the characteristics of each of the elements must be describable in terms of functional relations among the variables. There must be one relation known for each variable pair used.

Many complex systems may be considered as functioning sets of subsystems, where a subsystem is an interconnected set of similar components (e.g., electrical, mechanical, etc.). The analysis of such a subsystem will usually require a lesser number of variable pairs than would the analysis of the complete system as a whole.

That MNT can be readily applied to hydraulic systems and to electrical systems is perhaps obvious. But the reader may have some difficulty in imagining how this type of analysis can apply to a dynamic mechanical system; say for instance, a spring-mass-dashpot system. In this system, the components themselves are in motion, and there is no "real" thing which flows through the components. Nevertheless, MNT can be applied to systems of this type, and a rather detailed treatment of a typical problem will prove doubly beneficial. First, mechanical systems are interesting in themselves. Secondly, analysis of such a system will illustrate two of the general problems of applying MNT: (1) how to make a proper selection of variable pairs, and (2) how to meter these variables.

Metering

The voltage-current pair is a useful one since meters are available for the measurement of each. Charge meters can also be constructed, but they are not nearly as simple and as convenient as current meters. The third condition is also satisfied by the choice of the voltage-current pair since the three basic types of electrical components (i.e., resistors, inductors, and capacitors) can all be described in terms of voltage and current relations. Note that again the voltage-charge pair also satisfies the third condition.

The two requirements on the variables as given by equations (1) and (2) imply that numbers which represent the variable values must be measured or determined in a very particular manner. This applies to the determination of both magnitude and arithmetic sign. This last point places definite restrictions on the characteristics of the metering system that is to be used for any particular variable pair. In the case of electrical subsystems, both voltage and current are conventionally measured in such a way that all requirements for MNT analysis are met. This is not the case, for example, in linear mechanical subsystems, where length and force are usually the variables of interest. Length, an across variable, is conventionally measured by means of a meter (e.g., a ruler) that, in general, does not fix a sign to go with the magnitude of this variable. Also, while the usual force meter may provide a sign, it refers to ideas such as tension or compression, which unfortunately are not consistent with the requirements.

An attempt at reinterpreting the response of the length and force meters has resulted in a simple and consistent set of techniques which will produce variable values that completely satisfy all requirements and thus allow for the unrestricted use of MNT in the analysis of finite linear mechanical subsystems.

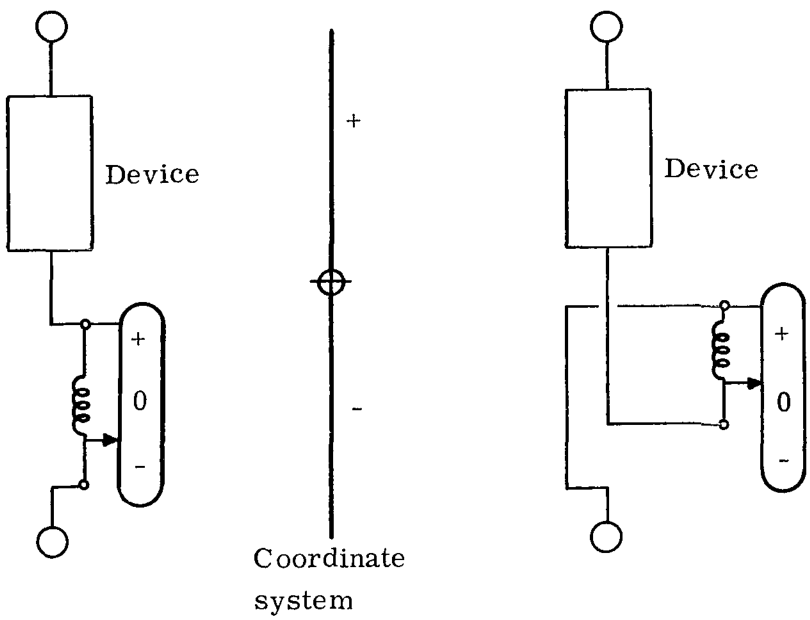
The principal requirement is the provision of a coordinate system upon which all measurements of force and length (or velocity, acceleration, etc.) can be based. In the analysis of a typical linear mechanical subsystem, a Cartesian coordinate system appears to be the most useful reference. The coordinate system belongs to the actual physical network and not to the graph which is drawn to represent the network.

In some cases it appears advantageous to consider a graph which is a union of three disjoint subgraphs, each representing one component of the three-dimensional coordinate system. This, of course, would result in three variable pairs (six variables) associated with each component of the network. The summation of the across variable would then be zero around any circuit, and the length variable would be a satisfactory MNT variable.

The force variable measurement is not quite as easily adapted for use. The procedure requires that conventional force-measuring meters be used and interpreted in a nonconventional manner.

For example, consider a spring-type force meter. The scale on this device may carry the idea of plus or minus along with the magnitude, but these signs refer to ideas such as compression and tension which, again, are not consistent with the MNT requirements. It is a consequence of the requirements that reversal of the meter connections must result in reversal of the sign of the measurement, but this is not the case for force meters used in the usual manner. This point constitutes a major failing of previous work which has attempted to apply the MNT techniques to mechanical subsystems (e.g., ref. 5). It is, however, possible to obtain the proper response in the following way.

First of all, the meter is alined with the same coordinate system used in the length-measuring operation. In other words, the length changes within the force-measuring device that are interpreted as force are measured in the same manner as any other length in the network. Reversal of the meter is interpreted as a switch of the connections to the meter from one end to the other without physically turning the meter end for end. (Fig. 13 should clarify this operation.)



(a) Normal.

(b) With meter reversed.

Figure 13. - Force measurement.

In this manner, the magnitude and the sign of the force variables will be consistent with the MNT requirements.

Once a consistent variable set is defined, the MNT techniques provide a methodology for the establishment of the defining equation for the subsystem. This system equation will be in the form of a coefficient matrix times a column matrix of all subsystem variables, the product of which is equal to another column matrix that contains all zeros and constants. The equation as written in matrix notation is

$$CV = K$$

where C is the coefficient matrix, V represents the column matrix containing all of the variables, and K is the column matrix containing all zeros and constants.

Mechanical Subsystems

An example of a linear one-dimensional mechanical system (fig. 14) is used to

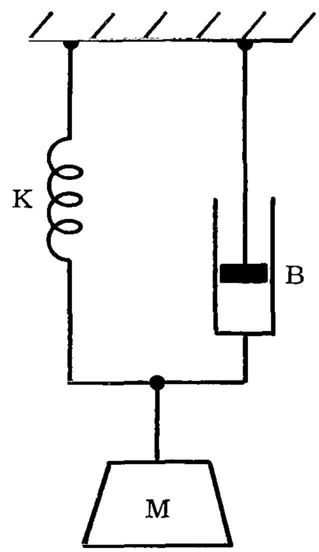


Figure 14. - Linear one-dimensional mechanical system.

demonstrate the manner in which the system equation is actually formed. The K element is a spring, the M element is a mass, and the B element is a dashpot or motion-damping device. It is assumed that we are interested in an analysis of the forces and displacements of this arrangement as a function of time, and it is further assumed that motion and displacement will take place only in the vertical direction (i.e., up and down only).

Before proceeding with the generation of the system equation, it is necessary to compose a representative graph of the network that will correspond to the variables of interest. Since the elements of graphs are always two-terminal elements, it is necessary to depict each of the network components in terms of two-terminal equivalents. For the spring and dashpot, this is a trivial matter since these devices already have two terminals. It can be considered that the characteristics of a mass are measured with respect to some reference point, and this point may be considered as the other terminal of this apparently one-terminal device. This reference point may be any point that is stationary with respect to the coordinate system used for measuring the problem variables. For convenience, the reference is chosen to be the support to which the spring and dashpot elements are physically connected.

A graph of the network may then be drawn as shown in figure 15, where the elements 1, 2, and 3 represent the spring, dashpot, and mass in that order, and where the arrows on the elements are all arbitrarily directed downward.

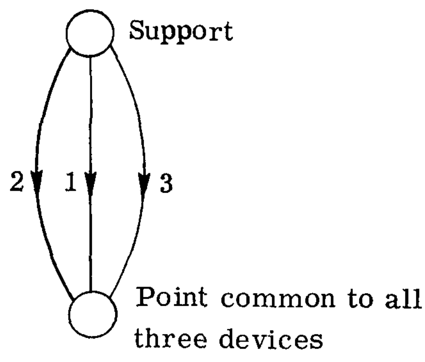


Figure 15. - Graph of linear one-dimensional mechanical system.

The format for the system equation is, in general, as follows:

$$\begin{bmatrix} S_b & 0 \\ 0 & B_b \\ R_1 & R_2 \end{bmatrix} \times \begin{bmatrix} Y(t) \\ X(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ C \end{bmatrix}$$

where S_b is a submatrix that defines a basis set of segs, B_b is a basis set of circuits, R_1 and R_2 describe the terminal characteristics of the individual network components in the general isolated condition, and C is a submatrix of constants associated with the R_1 R_2 . For this example an S_f and a B_f will be used for the S_b and B_b.

As was described in the section Matrix Notation, a fundamental set of segs can be based on the set of branches of some tree of the graph and will thus contain (v - 1) segs. Since the graph of this network contains two vertices, a set of fundamental segs will consist of one seg.

The set of fundamental circuits can be based on the cotree that is the complement of the tree used to define the S_f submatrix and will have (e - v + 1), or two, circuits contained in it.

Since a tree of this graph consists of simply one element and the graph contains three elements, any one of these may be selected as the tree used to define the S_f and B_f submatrices. Element 1 shall be chosen for this example. The cotree thus contains elements 2 and 3. The S_f then is

$$\text{Based on 1} \quad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

where the direction of the seg is taken to be such that the branch upon which the seg is based is entered as a plus one. The B_f to be used in the analysis is

$$\begin{array}{l} \text{Based on 2} \\ \\ \text{Based on 3} \end{array} \quad \begin{bmatrix} -1 & 1 & 0 \\ \\ -1 & 0 & 1 \end{bmatrix}$$

The submatrix describing the terminal equations is generated in the following manner. The relation for the spring device is

$$y_1(t) = Kx_1(t) - KL_o \qquad L_o < 0$$

where $y_1(t)$ is the time-dependent "through" variable (i.e., force), $x_1(t)$ is the time-dependent "across" variable (in this case distance or length) as measured in the appropriate manner described in the previous section, K is the "spring constant," and L_o the length of the spring at t = 0.

For the purpose of making self-consistent measurements, the coordinate system origin is placed at the "tail-of-the-arrow" terminal of each device, and the length measurement is noted at the point where the other terminal meets the coordinate axis. For this example, "up" is arbitrarily defined as the positive direction. Thus, the L_o for the spring is less than zero (i.e., negative), as indicated in the terminal equation for this device. (Fig. 16 may help to clarify this point.) Similar arguments are applied to the measurement of the force variables.

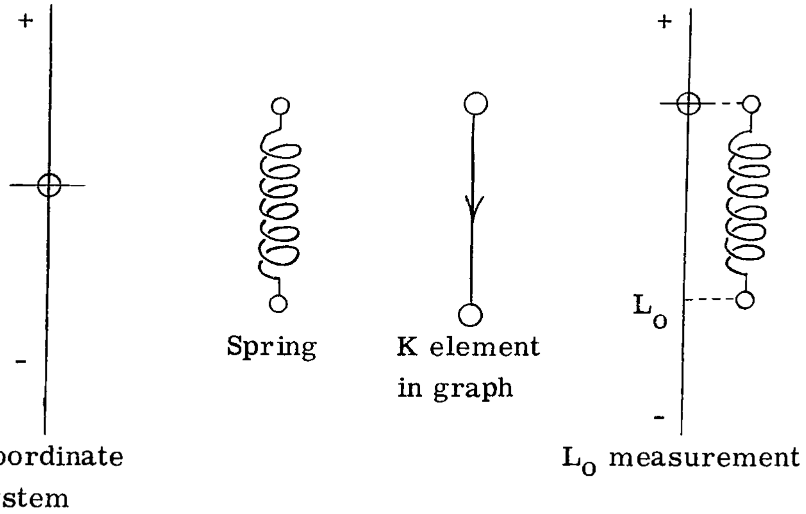


Figure 16. - Details of measurement of spring parameters.

The terminal equations for the dashpot and the mass are

Dashpot:

$$y_2(t) = B\dot{x}_2(t)$$

Mass:

$$y_3(t) = M\ddot{x}_3(t) + W \qquad W > 0$$

where B is the dashpot constant, M the mass of the mass device, $\dot{x}_2(t)$ the first derivative with respect to time of the length of the dashpot, $\ddot{x}_3(t)$ the second derivative of $x_3(t)$ with respect to time, and W the weight of the mass device. Note that W represents the value of force for no acceleration of the body.

The terminal equations are then put into a consistent matrix form as follows:

$$y_1(t) - Kx_1(t) = -KL_0$$

$$y_2(t) - B\dot{x}_2(t) = 0$$

$$y_3(t) - M\ddot{x}_3(t) = W$$

and then

$$
\begin{bmatrix} 1 & 0 & 0 & -K & 0 & 0 \\ 0 & 1 & 0 & 0 & -B\frac{d}{dt} & 0 \\ 0 & 0 & 1 & 0 & 0 & -M\frac{d^2}{dt^2} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} -KL_o \\ 0 \\ W \end{bmatrix}
$$

where d/dt and d^2/dt^2 are differential operators representing the first and second derivatives.

All parts of the system equation are now available in the proper form, so the equation can be assembled as follows:

$$
\begin{bmatrix} S_f & 0 \\ 0 & B_f \\ R_1 & R_2 \end{bmatrix} \times \begin{bmatrix} Y \\ X \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ C \end{bmatrix}
$$

where

$$
S_f = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}
$$

$$
B_f = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}
$$

$$
R_1 \; R_2 = \begin{bmatrix} 1 & 0 & 0 & -K & 0 & 0 \\ 0 & 1 & 0 & 0 & -B\frac{d}{dt} & 0 \\ 0 & 0 & 1 & 0 & 0 & -M\frac{d^2}{dt^2} \end{bmatrix}
$$

$$\begin{bmatrix} Y \\ X \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

and

$$\begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} -KL_o \\ 0 \\ W \end{bmatrix}$$

Thus, the system equation becomes

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -K & 0 & 0 \\ 0 & 1 & 0 & 0 & -B\frac{d}{dt} & 0 \\ 0 & 0 & 1 & 0 & 0 & -M\frac{d^2}{dt^2} \end{bmatrix} \times \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -KL_o \\ 0 \\ W \end{bmatrix}$$

This particular form of the system equation is called the primary system equation, and can be reduced to a more compact form called the secondary system of equations by a series of steps that depend somewhat on the particular form of the R_1R_2 submatrix.

For the arrangement given in this problem, the following represents the secondary equation formulation:

$$S_f(-R_2)S_f' x_3(t) = -S_f C$$

where S_f' means the transpose of S_f. Thus,

$$\begin{bmatrix}1 & 1 & 1\end{bmatrix} \begin{bmatrix} K & 0 & 0 \\ 0 & B\frac{d}{dt} & 0 \\ 0 & 0 & M\frac{d^2}{dt^2} \end{bmatrix} \begin{bmatrix}1 \\ 1 \\ 1\end{bmatrix} x_3(t) = -\begin{bmatrix}1 & 1 & 1\end{bmatrix} \begin{bmatrix} -KL_o \\ 0 \\ W \end{bmatrix}$$

which produces

$$M\ddot{x}_3(t) + B\dot{x}_3(t) + Kx_3(t) = KL_o - W$$

where

$$M > 0, \qquad B > 0, \qquad K > 0, \qquad L_o < 0, \qquad W > 0$$

This is the well-known result expected for the mechanical system under analysis. Similar results have been obtained for rotational, hydraulic, and thermal subsystems.

A most significant result of this phase of the work has been a complete clarification of the requirements for MNT analysis of any subsystem, and the development of a reasonable methodology for satisfying the requirements. The value of this result is most appreciated when attempting to extend MNT analysis techniques to man as a subsystem.

Another very helpful result of the investigation has been the realization that the quantities represented by the variable pairs of a subsystem need not really exist as entities but need only represent useful ideas that can be correlated to the observable characteristics of the subsystem. Thus, a choice of variables for a biological subsystem can be based simply on the phenomenological appearance of this subsystem to the remainder of the system.

Mixed Systems

Almost without exception, complex systems are composed of a mixture of subsystems of different types. If the MNT techniques are to be useful as a systems analysis tool, it is necessary that they can be made to handle the multiple sets of variable pairs associated with such mixed systems.

It can be shown that none of the requirements of the MNT methodology precludes the analysis of systems with more than one variable pair, provided that it is possible to meet the summation requirements on the across and through variables. This can be accomplished if the graph of a mixed system is chosen in such a way that it is never necessary to sum variables of different types.

If the graph of a system network is drawn as a union of disjoint subgraphs, each subgraph representing elements associated with each variable pair, no circuit or seg will contain elements associated with more than one variable pair. Thus, it will never be necessary to include, for example, both voltage and length in the same summation.

It should be pointed out, however, that voltage and length variables associated with a circuit must still sum to zero provided that both variables satisfy the general requirements. It is clear that the sum of voltages around a circuit of the girders of a bridge

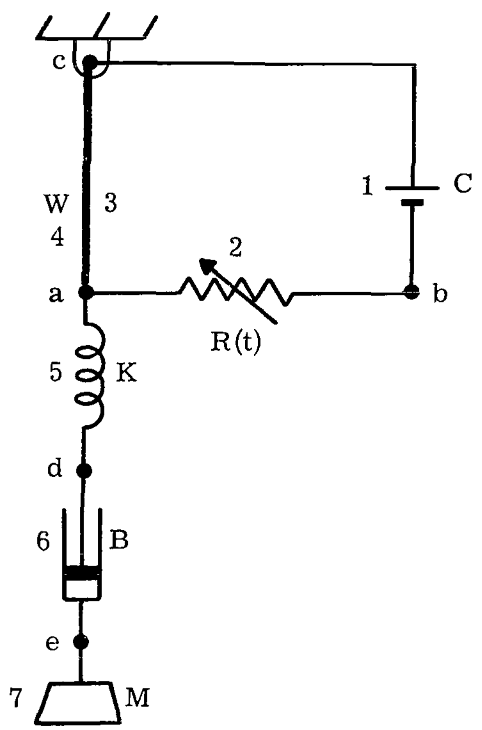


Figure 17. - Electromechanical system.

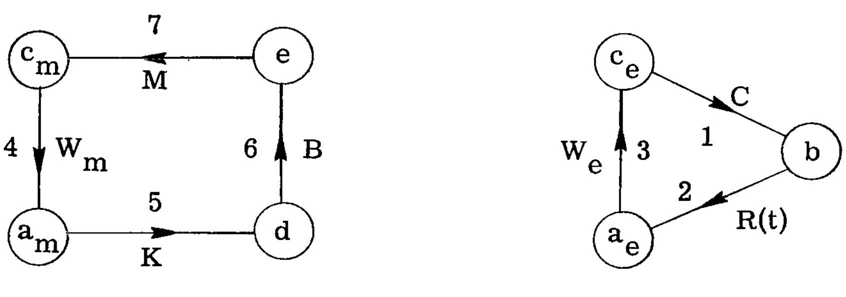
must be zero, just as is the sum of forces at a junction of electrical components. But in the interest of simplicity the disjoint union of subsystem graphs is used in the following example.

Consider the electromechanical system of figure 17. Note that the W-element has both electrical and mechanical properties that are significant to the system operation. The problem is to determine the value of all of the system variables.

The establishment of the system graph is now made on the basis of the method discussed for mixed systems. Since there are two type of variable pairs (voltage-current and length-force) associated with the network, the graph is made up as a disjoint union of two subgraphs, one based on the electrical subsystem and the other on the mechanical subsystem. Figure 18 is the graph chosen for the analysis of this example. Note that the W device appears twice in the system graph. In general, a network component will have one element in the graph for each variable pair associated with it. Again, the arrows that provide a basis for establishing directedness of the measurements are assigned arbitrarily.

Next the S_b and B_b matrices are formed. The following represents the choice made for this problem:

$$
S_b = \begin{array}{c} \\ c_e \\ b \\ c_m \\ d \\ e \end{array}
\begin{array}{c}
\begin{array}{ccc|cccc} 3 & 2 & 1 & 4 & 5 & 6 & 7 \end{array} \\
\left[\begin{array}{ccc|cccc}
-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1
\end{array}\right]
\end{array}
\begin{array}{l} \text{(Electrical)} \\ \\ \text{(Mechanical)} \end{array}
$$



(a) Mechanical. (b) Electrical.

Figure 18. - Graph of electromechanical system.

$$
B_b = \begin{array}{c} \left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \end{array}
\begin{array}{l} \text{(Electrical)} \\ \text{(Mechanical)} \end{array}
$$

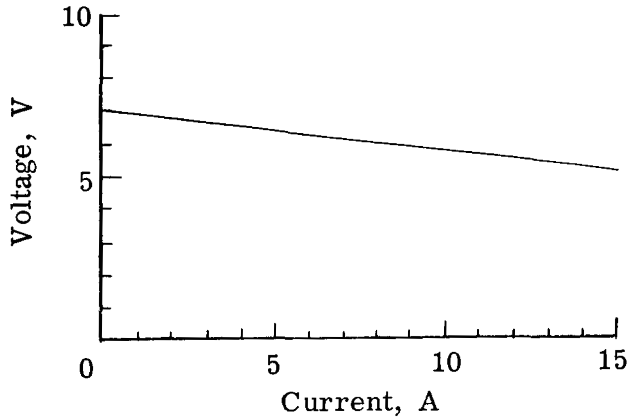


Figure 19. - Characteristics of C device (battery).

The next step is to determine the matrix that describes the individual component characteristics. The variable relations are to be obtained from the performance plots given for each device. For example, the C device (battery) has the laboratory-measured terminal characteristics shown in figure 19. The variable relation for this device can be stated as follows:

$$x_1(t) = \frac{2}{15} y_1(t) + 7$$

where x and y refer to the across and through variables, respectively. The mathematical signs ascribed to the terms of the relation are based on the orientation of the arrow assigned to element 1. The more standard form of relations for a battery would be as follows:

$$\text{Terminal voltage} = 7 - \frac{2}{15} \text{ Load current}$$

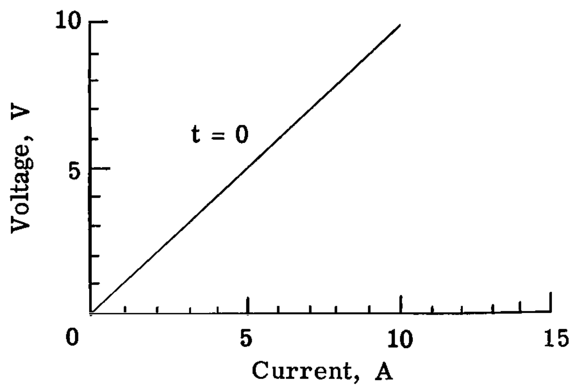
or

$$x(t) = 7 - \frac{2}{15} y(t)$$

but this form would not be consistent with the rules that have been laid out in the preceding discussion and would not be compatible with the MNT analysis technique.

The same technique is applied to the remaining components of the network as follows:

The R device (element 2) is described by two plots, as shown in figure 20, which results in



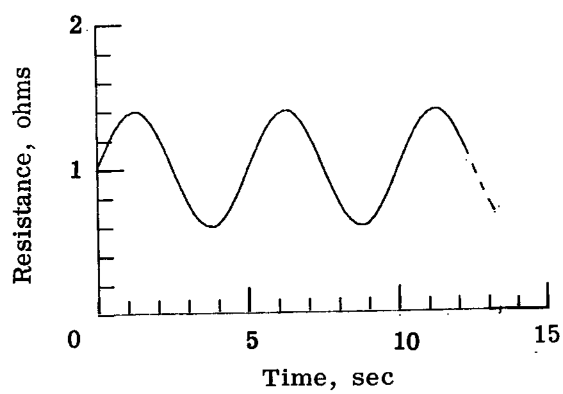


Figure 20. - Characteristics of R device.

$$x_2(t) = R(t)y_2(t)$$

$$x_2(t) = \left[1 + 0.4 \sin \frac{2\pi}{5} t\right] y_2(t)$$

The B device or dashpot (element 6) is described graphically in figure 21, from which is derived the relation

$$y_6(t) = \frac{100}{12} \dot{x}_6(t) \qquad 10 < x_6 < 25$$

where the dot over the x refers to the derivative with respect to time.

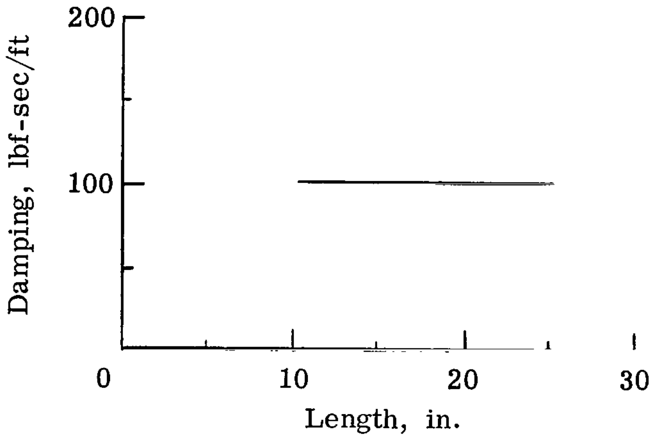


Figure 21. - Characteristics of B device (dashpot).

The W device (elements 3 and 4) is described by three plots (fig. 22); one electrical, one mechanical, and one mixed. The mixed plot can be interpreted as describing the "coupling" between the electrical and the mechanical subsystems.

First, the electrical characteristics, as described in figure 22(a), result in

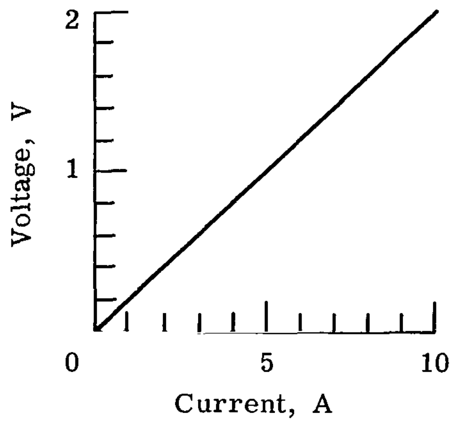
$$x_3(t) = 0.2\ y_3(t)$$

and the mechanical characteristics, as described in figure 22(b), result in

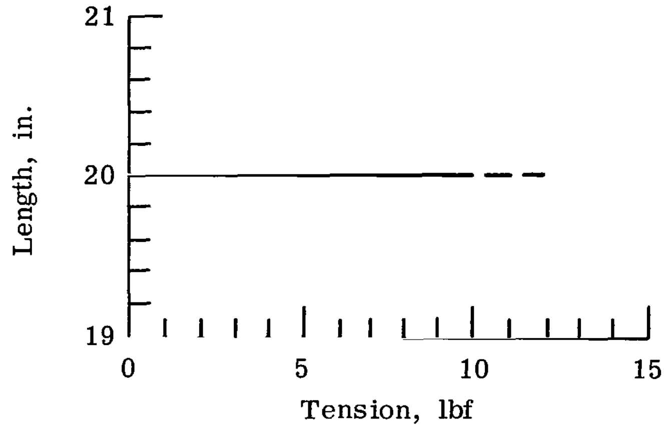
$$x_4(t) = -20 \neq f(y_4)$$

The coupling plot (fig. 22(c)) modifies the $x_4(t)$ relation, which then results in

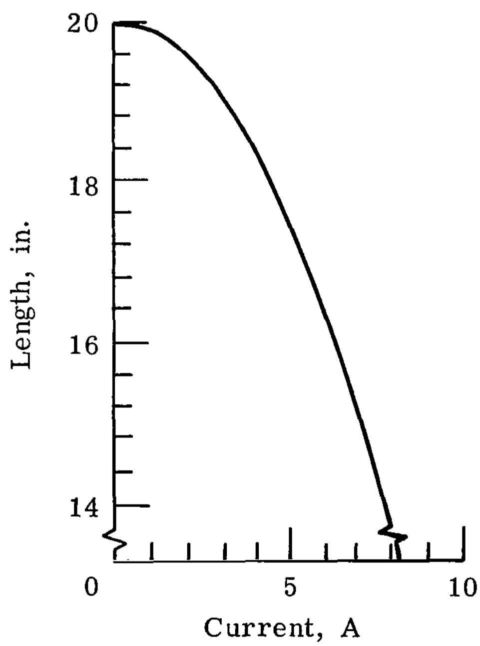
$$x_4(t) = 0.1 \left[y_3(t)\right]^2 - 20$$



(a) Electrical characteristics.



(b) Mechanical characteristics.



(c) Coupling between electrical and mechanical subsystems.

Figure 22. - Characteristics of W device.

The mass (element 7) is assumed to weigh 5 pounds, so its variable relation is simply

$$y_7(t) = \frac{5}{32.2}\ddot{x}_7(t) - 5$$

where 32.2 is the gravitational constant.

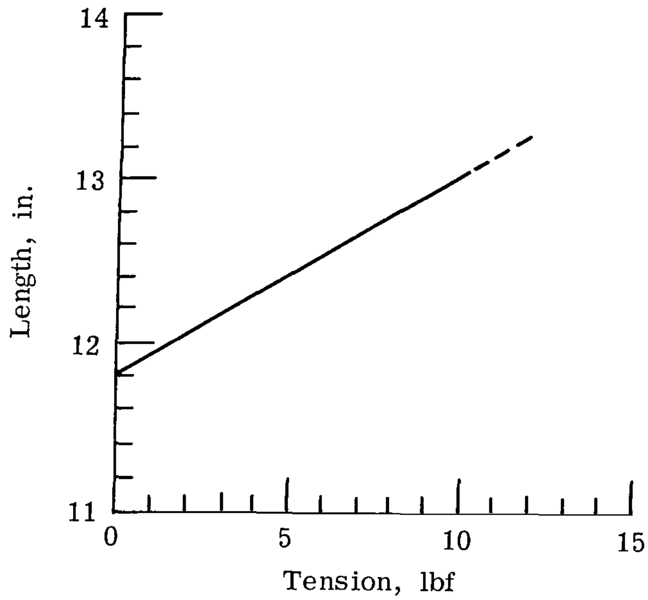


Figure 23. - Characteristics of K element (spring).

The spring (element 5) is described in figure 23, which produces

$$x_5(t) = 0.12\ y_5(t) - 11.8$$

As was done in the preceding examples, the terminal relation for each system component is arranged in a form consistent with matrix notation. There is a definite advantage to either an $x(t)$ or $y(t)$ explicit arrangement, but for the set of terminal relations derived for this system neither of the two arrangements can be conveniently formed. The matrix formulation will thus be in implicit form. The terminal relations submatrix is then formed:

$$
\left[\begin{array}{ccccccc|ccccccc}
0.2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & R_2(t) & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{15} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0.1()^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.12 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{100}{12} D & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{32} D^2
\end{array}\right]
$$

$$
\begin{bmatrix} Y(t) \\ X(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -7 \\ 20 \\ 11.8 \\ 0 \\ 5 \end{bmatrix}
$$

where D is the time derivative operator.

The primary system equation is then assembled in the same manner as in the preceding all-mechanical example. This equation is as follows:

$$
\begin{array}{c}
\begin{array}{cccccccccccccc} 3 & 2 & 1 & 4 & 5 & 6 & 7 & 3 & 2 & 1 & 4 & 5 & 6 & 7 \end{array} \\
\left[\begin{array}{ccc|cccc|ccccccc}
-1 & 0 & 0 & & & 0 & & & & & & & & \\
0 & 1 & -1 & & & & & & & & & & & \\
\hline
 & & & 1 & 0 & 0 & -1 & & & & 0 & & & \\
 & 0 & & 0 & -1 & 1 & 0 & & & & & & & \\
 & & & 0 & 0 & -1 & 1 & & & & & & & \\
\hline
 & & & & & & & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & & & 0 & & & & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\hline
0.2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & R_2(t) & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{15} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0.1()^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.12 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{100}{12}D & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{32}D^2
\end{array}\right]
\end{array}
\times
\begin{bmatrix} y_3(t) \\ y_2(t) \\ y_1(t) \\ y_4(t) \\ y_5(t) \\ y_6(t) \\ y_7(t) \\ x_3(t) \\ x_2(t) \\ x_1(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \\ x_7(t) \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -7 \\ 20 \\ 11.8 \\ 0 \\ 5 \end{bmatrix}
$$

The solution of this matrix equation in terms of a set of initial conditions can be accomplished by a number of different methods. In the case of this example, a digital computer program was written and appears in the appendix, along with some of the results for two sets of initial conditions. Figure 24 shows a plot of just one of the system variables (position of the mass $x_7(t)$) as a function of time. The two initial condition sets differ only in that in case 2 the mass has an initial downward velocity.

It is seen in the preceding example that the modified MNT techniques provide a completely straightforward method for obtaining a mathematical system equation for this mixed system. The author is not aware of any other technique that could duplicate the simplicity of this method. It should be clear from this example that mixed systems in general will respond to the MNT analysis technique in a straightforward and reasonable manner.

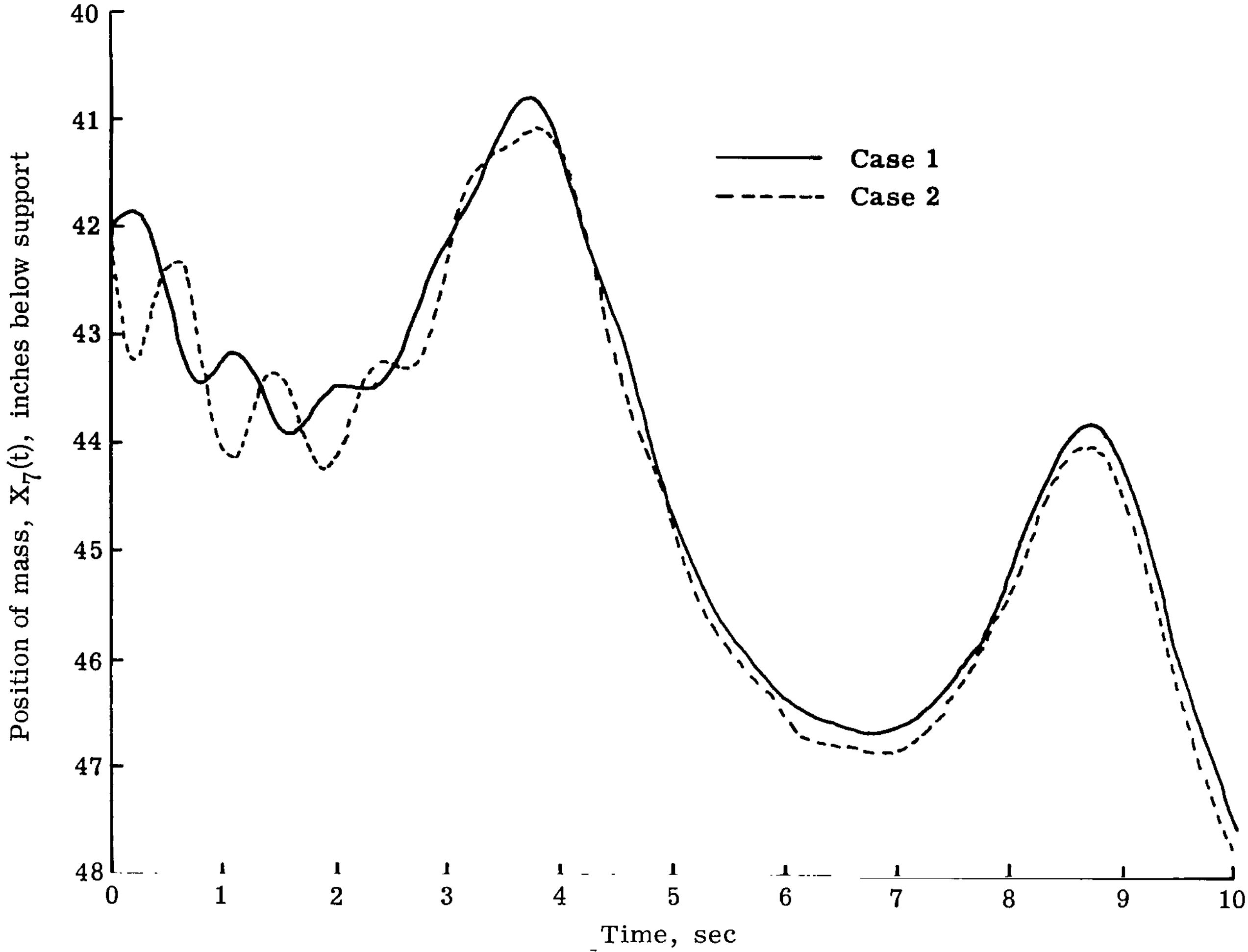


Figure 24. - Position of mass as function of time for two sets of initial conditions.

CONCLUSIONS

In view of the results obtained in the analyses of the examples presented in this report, and after consideration of the techniques used to obtain these results, certain significant conclusions can be stated:

1. The body of knowledge called Modern Network Theory is not at all limited to problems in the electrical engineering area, but may be considered applicable to the analysis of systems problems in general under these conditions:

(a) The systems are made up of (or can be considered to be made up of) elements or devices whose phenomenological characteristics are definable in terms of mathematical relations of variable pairs.

(b) The elements or devices are interconnected in a describable manner.

2. There is no requirement that the system variables represent tangible system entities. It is, however, necessary that they satisfy certain general requirements on variables and variable pairs.

3. There is no requirement that interconnections between elements or devices be physical. It is only necessary that interrelations between such elements or devices can be described in terms of the defining system variables.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, August 18, 1969,
120-27.

APPENDIX - COMPUTER PROGRAM AND RESULTS
FOR ELECTROMECHANICAL MIXED SYSTEM

Solutions for the electromechanical mixed system sample problem given in the main text can be obtained with the digital computer program presented in this appendix. The program is written in FORTRAN IV and includes ample comment statements as an aid to understanding the various steps used to arrive at the solutions.

A sample input data listing is also included for two different sets of conditions. In the first case the mass begins at rest, while in the second case the mass has an initial velocity.

A tabulation of the computer output for the two cases follows the listing of the computer program. Appropriate headings (as supplied by the computer) appear preceding each set of output so as to properly identify the case to which the output applies.

Note that the output values are to be interpreted in terms of the orientation of the appropriate elements (i.e., the arrows) to which they belong. It should also be pointed out that the values displayed in the output listing have been truncated to allow printing of all 10 columns on one page line. A small change in format statement 105 to provide at least four decimal places in the output values is recommended for serious use of this program.

```
PROGRAM FOR THE SOLUTION OF THE MECHANICAL/ELECTRICAL MIXED SYSTEM
LANGUAGE IS FORTRAN IV

$IBFTC ELMECH  DECK
      DIMENSION SOL(9),Y(3),YS(3),PK(4,3)
      INTEGER CASE
      PI= 3.14159265
      G= 32.2
    2 READ(5,106) CASE
C     TEST FOR END OF DATA INPUT
      IF(CASE - 99) 4,60,4
C     XK = WIRE CONSTANT, X60 = INITIAL LENGTH OF DASHPOT
C     X70 = INITIAL DISPLACEMENT OF MASS, X7PO = INITIAL VELOCITY OF MASS
    4 READ(5,101) XK,X60,X70,X7PO
C     N = TOTAL NUMBER OF SOLUTION POINTS,  H = DELTA TIME
      READ(5,100)N
      READ(5,101)H
      SOL(1)= -21.0/4.0
      Y(1)= X60
      Y(2)= X70
      Y(3)= X7PO
      SOL(2)= (31.8-XK*SOL(1)*SOL(1)-X60-X70)/0.12
      SOL(3)= 0.2*SOL(1)
      SOL(4)= SOL(1)
      SOL(5)= -1.2*SOL(1)
      SOL(6)= XK*SOL(1)*SOL(1) - 20.0
      SOL(7)= 0.12*SOL(2)-11.8
      WRITE(6,107)CASE
      WRITE(6,102)
      WRITE(6,103)XK,H,N
```

```
      WRITE(6,108)X60,X70,X7P0
      WRITE(6,104)
      T= 0.
      WRITE(6,105) T,  (SOL(J),J=1,7),Y(1),Y(2)
      DO 20 I=1,N
      T1= T
      DO 5 J=1,3
    5 YS(J)= Y(J)
C     RUNGE-KUTTA FROM HERE TO STATEMENT 14
      DO 10 K=1,4
      RK= (K+1)/2
      R2= 1.0+0.4*SIN(2.0*PI*T/5.0)
      Y3= -21.0/(3.0*R2+1.0)
      FT= 31.8-XK*Y3*Y3
      PK(K,1)= H*(FT-YS(1)-YS(2))
      PK(K,2)= H*YS(3)
      PK(K,3)= 161.0/3.0*PK(K,1)+H*G
      IF(K-4) 6,12,12
    6 DO 8 J=1,3
    8 YS(J)= Y(J)+(PK(K,J)/2.0)*RK
   10 T= T1+(H/2.0)*RK
   12 DO 14 J=1,3
   14 Y(J)= Y(J)+(PK(1,J)+2.*PK(2,J)+2.*PK(3,J)+PK(4,J))/6.0
      SOL(1)= Y3
      SOL(2)= (FT-Y(1)-Y(2))/0.12
      SOL(3)= 0.2*SOL(1)
      SOL(4)= R2*SOL(1)
      SOL(5)= -(SOL(3)+SOL(4))
      SOL(6)= XK*SOL(1)*SOL(1) - 20.0
      SOL(7)= 0.12*SOL(2)-11.8
   20 WRITE(6,105) T,  (SOL(J),J=1,7),Y(1),Y(2)
      GO TO 2
   60 CALL EXIT
  100 FORMAT(I4)
  101 FORMAT(4F10.5)
  102 FORMAT(1H0,36H SOLUTION   Y1=Y2=Y3 AND Y4=Y5=Y6=Y7)
  103 FORMAT(5H0  K=,F5.2,7X,9H DELTA T=,F5.2,27H NUMBER OF SOLUTION POI
     1NTS=,I4/)
  104 FORMAT(1H ,2H T,6X,2HY3,6X,2HY7,6X,2HX3,6X,2HX2,6X,2HX1,7X,2HX4,
     3 7X,2HX5,7X,2HX6,7X,2HX7,//)
  105 FORMAT (1H ,F3.1,3X,F5.2,3X,F5.2,3X,F5.2,3X,F5.2,3X,F5.2,3X,F6.2,
     4 3X, F6.2,3X,F6.2,3X,F6.2)
  106 FORMAT(I10)
  107 FORMAT(1H1,5H CASE,I3//)
  108 FORMAT(26H0INITIAL CONDITIONS X6(0)=,F8.3,3X,7HX7(0) =,F8.3,3X,
     5 8HDX7/DT =,F6.3//)
      STOP
      END


$DATA
         1
  0.100    -12.0      42.0       0.0
 100
  0.100
         2
  0.100    -12.0       42.0       10.0
 100
   0.100
        99
```

CASE 1

SOLUTION Y1=Y2=Y3 AND Y4=Y5=Y6=Y7
K= 0.10 DELTA T= 0.10 NUMBER OF SOLUTION POINTS= 100

INITIAL CONDITIONS X6(0)= -12.000 X7(0) = 42.000 DX7/DT = 0.

T	Y3	Y7	X3	X2	X1	X4	X5	X6	X7
0.	-5.25	-7.97	-1.05	-5.25	6.30	-17.24	-12.76	-12.00	42.00
0.1	-5.06	-5.13	-1.01	-5.31	6.33	-17.44	-12.42	-12.08	41.93
0.2	-4.89	-2.68	-0.98	-5.37	6.35	-17.61	-12.12	-12.12	41.86
0.3	-4.73	-1.75	-0.95	-5.42	6.37	-17.76	-12.01	-12.15	41.92
0.4	-4.59	-2.65	-0.92	-5.47	6.39	-17.90	-12.12	-12.17	42.19
0.5	-4.46	-4.74	-0.89	-5.51	6.40	-18.01	-12.37	-12.22	42.60
0.6	-4.36	-6.89	-0.87	-5.55	6.42	-18.10	-12.63	-12.29	43.02
0.7	-4.26	-8.04	-0.85	-5.58	6.43	-18.18	-12.77	-12.38	43.33
0.8	-4.19	-7.71	-0.84	-5.60	6.44	-18.25	-12.73	-12.48	43.45
0.9	-4.13	-6.20	-0.83	-5.62	6.45	-18.30	-12.54	-12.56	43.40
1.0	-4.08	-4.35	-0.82	-5.64	6.46	-18.33	-12.32	-12.62	43.28
1.1	-4.06	-3.09	-0.81	-5.65	6.46	-18.36	-12.17	-12.67	43.19
1.2	-4.04	-2.97	-0.81	-5.65	6.46	-18.37	-12.16	-12.70	43.23
1.3	-4.04	-3.93	-0.81	-5.65	6.46	-18.37	-12.27	-12.74	43.38
1.4	-4.06	-5.42	-0.81	-5.65	6.46	-18.36	-12.45	-12.80	43.60
1.5	-4.08	-6.65	-0.82	-5.64	6.46	-18.33	-12.60	-12.87	43.80
1.6	-4.13	-7.05	-0.83	-5.62	6.45	-18.30	-12.65	-12.95	43.90
1.7	-4.19	-6.52	-0.84	-5.60	6.44	-18.25	-12.58	-13.04	43.86
1.8	-4.26	-5.41	-0.85	-5.58	6.43	-18.18	-12.45	-13.11	43.74
1.9	-4.36	-4.32	-0.87	-5.55	6.42	-18.10	-12.32	-13.17	43.59
2.0	-4.46	-3.78	-0.89	-5.51	6.40	-18.01	-12.25	-13.21	43.48
2.1	-4.59	-4.03	-0.92	-5.47	6.39	-17.90	-12.28	-13.26	43.44
2.2	-4.73	-4.86	-0.95	-5.42	6.37	-17.76	-12.38	-13.31	43.46
2.3	-4.89	-5.84	-0.98	-5.37	6.35	-17.61	-12.50	-13.38	43.49
2.4	-5.06	-6.48	-1.01	-5.31	6.33	-17.44	-12.58	-13.45	43.47
2.5	-5.25	-6.51	-1.05	-5.25	6.30	-17.24	-12.58	-13.53	43.36
2.6	-5.46	-5.99	-1.09	-5.18	6.27	-17.02	-12.52	-13.61	43.15
2.7	-5.67	-5.21	-1.13	-5.11	6.24	-16.78	-12.42	-13.67	42.88
2.8	-5.90	-4.57	-1.18	-5.03	6.21	-16.52	-12.35	-13.73	42.60
2.9	-6.14	-4.34	-1.23	-4.95	6.18	-16.23	-12.32	-13.78	42.34
3.0	-6.37	-4.55	-1.27	-4.88	6.15	-15.94	-12.35	-13.84	42.12
3.1	-6.61	-5.01	-1.32	-4.80	6.12	-15.64	-12.40	-13.89	41.93
3.2	-6.83	-5.38	-1.37	-4.72	6.09	-15.34	-12.45	-13.96	41.74
3.3	-7.03	-5.40	-1.41	-4.66	6.06	-15.06	-12.45	-14.02	41.53
3.4	-7.21	-4.99	-1.44	-4.60	6.04	-14.81	-12.40	-14.09	41.29
3.5	-7.35	-4.27	-1.47	-4.55	6.02	-14.60	-12.31	-14.14	41.06
3.6	-7.44	-3.53	-1.49	-4.52	6.01	-14.46	-12.22	-14.19	40.87
3.7	-7.49	-3.05	-1.50	-4.50	6.00	-14.38	-12.17	-14.23	40.78
3.8	-7.49	-2.99	-1.50	-4.50	6.00	-14.38	-12.16	-14.26	40.81
3.9	-7.44	-3.35	-1.49	-4.52	6.01	-14.46	-12.20	-14.30	40.96
4.0	-7.35	-3.95	-1.47	-4.55	6.02	-14.60	-12.27	-14.34	41.22
4.1	-7.21	-4.56	-1.44	-4.60	6.04	-14.81	-12.35	-14.40	41.55
4.2	-7.03	-4.97	-1.41	-4.66	6.06	-15.06	-12.40	-14.45	41.91
4.3	-6.83	-5.13	-1.37	-4.72	6.09	-15.34	-12.42	-14.51	42.27
4.4	-6.61	-5.07	-1.32	-4.80	6.12	-15.64	-12.41	-14.57	42.62
4.5	-6.37	-4.96	-1.27	-4.88	6.15	-15.94	-12.39	-14.64	42.97
4.6	-6.14	-4.92	-1.23	-4.95	6.18	-16.23	-12.39	-14.69	43.32

CASE 1 CONCLUDED

T	Y3	Y7	X3	X2	X1	X4	X5	X6	X7
4.7	-5.90	-5.02	-1.18	-5.03	6.21	-16.52	-12.40	-14.75	43.67
4.8	-5.67	-5.25	-1.13	-5.11	6.24	-16.78	-12.43	-14.82	44.03
4.9	-5.46	-5.50	-1.09	-5.18	6.27	-17.02	-12.46	-14.88	44.36
5.0	-5.25	-5.65	-1.05	-5.25	6.30	-17.24	-12.48	-14.95	44.67
5.1	-5.06	-5.66	-1.01	-5.31	6.33	-17.44	-12.48	-15.01	44.93
5.2	-4.89	-5.51	-0.98	-5.37	6.35	-17.61	-12.46	-15.08	45.16
5.3	-4.73	-5.30	-0.95	-5.42	6.37	-17.76	-12.44	-15.15	45.35
5.4	-4.59	-5.11	-0.92	-5.47	6.39	-17.90	-12.41	-15.21	45.52
5.5	-4.46	-5.03	-0.89	-5.51	6.40	-18.01	-12.40	-15.27	45.68
5.6	-4.36	-5.08	-0.87	-5.55	6.42	-18.10	-12.41	-15.33	45.84
5.7	-4.26	-5.20	-0.85	-5.58	6.43	-18.18	-12.42	-15.39	46.00
5.8	-4.19	-5.33	-0.84	-5.60	6.44	-18.25	-12.44	-15.46	46.14
5.9	-4.13	-5.39	-0.83	-5.62	6.45	-18.30	-12.45	-15.52	46.26
6.0	-4.08	-5.35	-0.82	-5.64	6.46	-18.33	-12.44	-15.58	46.36
6.1	-4.06	-5.25	-0.81	-5.65	6.46	-18.36	-12.43	-15.65	46.43
6.2	-4.04	-5.12	-0.81	-5.65	6.46	-18.37	-12.41	-15.71	46.49
6.3	-4.04	-5.04	-0.81	-5.65	6.46	-18.37	-12.40	-15.77	46.54
6.4	-4.06	-5.04	-0.81	-5.65	6.46	-18.36	-12.40	-15.83	46.59
6.5	-4.08	-5.11	-0.82	-5.64	6.46	-18.33	-12.41	-15.89	46.64
6.6	-4.13	-5.22	-0.83	-5.62	6.45	-18.30	-12.43	-15.95	46.67
6.7	-4.19	-5.31	-0.84	-5.60	6.44	-18.25	-12.44	-16.02	46.70
6.8	-4.26	-5.35	-0.85	-5.58	6.43	-18.18	-12.44	-16.08	46.70
6.9	-4.36	-5.33	-0.87	-5.55	6.42	-18.10	-12.44	-16.15	46.69
7.0	-4.46	-5.27	-0.89	-5.51	6.40	-18.01	-12.43	-16.21	46.65
7.1	-4.59	-5.22	-0.92	-5.47	6.39	-17.90	-12.43	-16.27	46.59
7.2	-4.73	-5.21	-0.95	-5.42	6.37	-17.76	-12.43	-16.33	46.52
7.3	-4.89	-5.26	-0.98	-5.37	6.35	-17.61	-12.43	-16.40	46.44
7.4	-5.06	-5.33	-1.01	-5.31	6.33	-17.44	-12.44	-16.46	46.34
7.5	-5.25	-5.42	-1.05	-5.25	6.30	-17.24	-12.45	-16.53	46.22
7.6	-5.46	-5.47	-1.09	-5.18	6.27	-17.02	-12.46	-16.59	46.07
7.7	-5.67	-5.46	-1.13	-5.11	6.24	-16.78	-12.46	-16.66	45.89
7.8	-5.90	-5.39	-1.18	-5.03	6.21	-16.52	-12.45	-16.72	45.69
7.9	-6.14	-5.29	-1.23	-4.95	6.18	-16.23	-12.43	-16.79	45.45
8.0	-6.37	-5.15	-1.27	-4.88	6.15	-15.94	-12.42	-16.85	45.20
8.1	-6.61	-5.00	-1.32	-4.80	6.12	-15.64	-12.40	-16.91	44.94
8.2	-6.83	-4.82	-1.37	-4.72	6.09	-15.34	-12.38	-16.97	44.68
8.3	-7.03	-4.62	-1.41	-4.66	6.06	-15.06	-12.35	-17.02	44.44
8.4	-7.21	-4.39	-1.44	-4.60	6.04	-14.81	-12.33	-17.08	44.21
8.5	-7.35	-4.12	-1.47	-4.55	6.02	-14.60	-12.29	-17.13	44.03
8.6	-7.44	-3.86	-1.49	-4.52	6.01	-14.46	-12.26	-17.18	43.90
8.7	-7.49	-3.65	-1.50	-4.50	6.00	-14.38	-12.24	-17.22	43.85
8.8	-7.49	-3.55	-1.50	-4.50	6.00	-14.38	-12.23	-17.27	43.88
8.9	-7.44	-3.60	-1.49	-4.52	6.01	-14.46	-12.23	-17.31	44.00
9.0	-7.35	-3.79	-1.47	-4.55	6.02	-14.60	-12.26	-17.35	44.21
9.1	-7.21	-4.11	-1.44	-4.60	6.04	-14.81	-12.29	-17.40	44.50
9.2	-7.03	-4.49	-1.41	-4.66	6.06	-15.06	-12.34	-17.45	44.85
9.3	-6.83	-4.84	-1.37	-4.72	6.09	-15.34	-12.38	-17.51	45.23
9.4	-6.61	-5.10	-1.32	-4.80	6.12	-15.64	-12.41	-17.57	45.62
9.5	-6.37	-5.26	-1.27	-4.88	6.15	-15.94	-12.43	-17.63	46.00
9.6	-6.14	-5.32	-1.23	-4.95	6.18	-16.23	-12.44	-17.69	46.37
9.7	-5.90	-5.32	-1.18	-5.03	6.21	-16.52	-12.44	-17.76	46.71
9.8	-5.67	-5.30	-1.13	-5.11	6.24	-16.78	-12.44	-17.82	47.04
9.9	-5.46	-5.31	-1.09	-5.18	6.27	-17.02	-12.44	-17.88	47.35
10.0	-5.25	-5.34	-1.05	-5.25	6.30	-17.24	-12.44	-17.95	47.63

CASE 2

SOLUTION Y1=Y2=Y3 AND Y4=Y5=Y6=Y7
K= 0.10 DELTA T= 0.10 NUMBER OF SOLUTION POINTS= 100

INITIAL CONDITIONS X6(0)= -12.000 X7(0) = 42.000 DX7/DT =10.000

T	Y3	Y7	X3	X2	X1	X4	X5	X6	X7
0.	-5.25	-7.97	-1.05	-5.25	6.30	-17.24	-12.76	-12.00	42.00
0.1	-5.06	12.35	-1.01	-5.31	6.33	-17.44	-13.28	-12.13	42.85
0.2	-4.89	12.92	-0.98	-5.37	6.35	-17.61	-13.35	-12.28	43.24
0.3	-4.73	-9.75	-0.95	-5.42	6.37	-17.76	-12.97	-12.42	43.15
0.4	-4.59	-4.73	-0.92	-5.47	6.39	-17.90	-12.37	-12.51	42.77
0.5	-4.46	-0.46	-0.89	-5.51	6.40	-18.01	-11.86	-12.54	42.40
0.6	-4.36	1.05	-0.87	-5.55	6.42	-18.10	-11.67	-12.53	42.31
0.7	-4.26	-0.64	-0.85	-5.58	6.43	-18.18	-11.88	-12.52	42.58
0.8	-4.19	-4.40	-0.84	-5.60	6.44	-18.25	-12.33	-12.55	43.13
0.9	-4.13	-8.20	-0.83	-5.62	6.45	-18.30	-12.78	-12.63	43.71
1.0	-4.08	10.17	-0.82	-5.64	6.46	-18.33	-13.02	-12.74	44.09
1.1	-4.06	-9.54	-0.81	-5.65	6.46	-18.36	-12.95	-12.86	44.16
1.2	-4.04	-6.86	-0.81	-5.65	6.46	-18.37	-12.62	-12.96	43.95
1.3	-4.04	-3.62	-0.81	-5.65	6.46	-18.37	-12.23	-13.03	43.63
1.4	-4.06	-1.45	-0.81	-5.65	6.46	-18.36	-11.97	-13.05	43.38
1.5	-4.08	-1.31	-0.82	-5.64	6.46	-18.33	-11.96	-13.07	43.36
1.6	-4.13	-3.07	-0.83	-5.62	6.45	-18.30	-12.17	-13.09	43.56
1.7	-4.19	-5.70	-0.84	-5.60	6.44	-18.25	-12.48	-13.15	43.87
1.8	-4.26	-7.84	-0.85	-5.58	6.43	-18.18	-12.74	-13.23	44.15
1.9	-4.36	-8.51	-0.87	-5.55	6.42	-18.10	-12.82	-13.33	44.25
2.0	-4.46	-7.53	-0.89	-5.51	6.40	-18.01	-12.70	-13.43	44.14
2.1	-4.59	-5.55	-0.92	-5.47	6.39	-17.90	-12.47	-13.50	43.87
2.2	-4.73	-3.63	-0.95	-5.42	6.37	-17.76	-12.24	-13.56	43.56
2.3	-4.89	-2.72	-0.98	-5.37	6.35	-17.61	-12.13	-13.60	43.34
2.4	-5.06	-3.16	-1.01	-5.31	6.33	-17.44	-12.18	-13.63	43.25
2.5	-5.25	-4.63	-1.05	-5.25	6.30	-17.24	-12.36	-13.68	43.28
2.6	-5.46	-6.31	-1.09	-5.18	6.27	-17.02	-12.56	-13.74	43.32
2.7	-5.67	-7.37	-1.13	-5.11	6.24	-16.78	-12.68	-13.82	43.29
2.8	-5.90	-7.34	-1.18	-5.03	6.21	-16.52	-12.68	-13.91	43.11
2.9	-6.14	-6.31	-1.23	-4.95	6.18	-16.23	-12.56	-14.00	42.79
3.0	-6.37	-4.85	-1.27	-4.88	6.15	-15.94	-12.38	-14.06	42.38
3.1	-6.61	-3.64	-1.32	-4.80	6.12	-15.64	-12.24	-14.11	41.99
3.2	-6.83	-3.17	-1.37	-4.72	6.09	-15.34	-12.18	-14.15	41.67
3.3	-7.03	-3.51	-1.41	-4.66	6.06	-15.06	-12.22	-14.19	41.47
3.4	-7.21	-4.30	-1.44	-4.60	6.04	-14.81	-12.32	-14.24	41.36
3.5	-7.35	-5.00	-1.47	-4.55	6.02	-14.60	-12.40	-14.30	41.30
3.6	-7.44	-5.19	-1.49	-4.52	6.01	-14.46	-12.42	-14.36	41.24
3.7	-7.49	-4.74	-1.50	-4.50	6.00	-14.38	-12.37	-14.42	41.17
3.8	-7.49	-3.89	-1.50	-4.50	6.00	-14.38	-12.27	-14.47	41.12
3.9	-7.44	-3.10	-1.49	-4.52	6.01	-14.46	-12.17	-14.51	41.14
4.0	-7.35	-2.78	-1.47	-4.55	6.02	-14.60	-12.13	-14.55	41.28
4.1	-7.21	-3.12	-1.44	-4.60	6.04	-14.81	-12.17	-14.58	41.56
4.2	-7.03	-4.00	-1.41	-4.66	6.06	-15.06	-12.28	-14.62	41.96
4.3	-6.83	-5.04	-1.37	-4.72	6.09	-15.34	-12.40	-14.68	42.42
4.4	-6.61	-5.84	-1.32	-4.80	6.12	-15.64	-12.50	-14.74	42.88
4.5	-6.37	-6.12	-1.27	-4.88	6.15	-15.94	-12.53	-14.82	43.29
4.6	-6.14	-5.87	-1.23	-4.95	6.18	-16.23	-12.50	-14.89	43.63

CASE 2 CONCLUDED

T	Y3	Y7	X3	X2	X1	X4	X5	X6	X7
4.7	-5.90	-5.33	-1.18	-5.03	6.21	-16.52	-12.44	-14.96	43.91
4.8	-5.67	-4.82	-1.13	-5.11	6.24	-16.78	-12.38	-15.02	44.18
4.9	-5.46	-4.61	-1.09	-5.18	6.27	-17.02	-12.35	-15.07	44.45
5.0	-5.25	-4.79	-1.05	-5.25	6.30	-17.24	-12.37	-15.13	44.75
5.1	-5.06	-5.23	-1.01	-5.31	6.33	-17.44	-12.43	-15.19	45.06
5.2	-4.89	-5.69	-0.98	-5.37	6.35	-17.61	-12.48	-15.25	45.35
5.3	-4.73	-5.93	-0.95	-5.42	6.37	-17.76	-12.51	-15.32	45.60
5.4	-4.59	-5.86	-0.92	-5.47	6.39	-17.90	-12.50	-15.40	45.79
5.5	-4.46	-5.51	-0.89	-5.51	6.40	-18.01	-12.46	-15.46	45.93
5.6	-4.36	-5.09	-0.87	-5.55	6.42	-18.10	-12.41	-15.53	46.04
5.7	-4.26	-4.78	-0.85	-5.58	6.43	-18.18	-12.37	-15.59	46.14
5.8	-4.19	-4.72	-0.84	-5.60	6.44	-18.25	-12.37	-15.64	46.26
5.9	-4.13	-4.91	-0.83	-5.62	6.45	-18.30	-12.39	-15.70	46.39
6.0	-4.08	-5.23	-0.82	-5.64	6.46	-18.33	-12.43	-15.76	46.52
6.1	-4.06	-5.49	-0.81	-5.65	6.46	-18.36	-12.46	-15.83	46.64
6.2	-4.04	-5.59	-0.81	-5.65	6.46	-18.37	-12.47	-15.89	46.73
6.3	-4.04	-5.48	-0.81	-5.65	6.46	-18.37	-12.46	-15.96	46.78
6.4	-4.06	-5.24	-0.81	-5.65	6.46	-18.36	-12.43	-16.02	46.81
6.5	-4.08	-5.00	-0.82	-5.64	6.46	-18.33	-12.40	-16.09	46.82
6.6	-4.13	-4.87	-0.83	-5.62	6.45	-18.30	-12.38	-16.14	46.82
6.7	-4.19	-4.93	-0.84	-5.60	6.44	-18.25	-12.39	-16.20	46.84
6.8	-4.26	-5.11	-0.85	-5.58	6.43	-18.18	-12.41	-16.26	46.86
6.9	-4.36	-5.34	-0.87	-5.55	6.42	-18.10	-12.44	-16.33	46.87
7.0	-4.46	-5.50	-0.89	-5.51	6.40	-18.01	-12.46	-16.39	46.86
7.1	-4.59	-5.54	-0.92	-5.47	6.39	-17.90	-12.46	-16.46	46.82
7.2	-4.73	-5.45	-0.95	-5.42	6.37	-17.76	-12.45	-16.52	46.74
7.3	-4.89	-5.31	-0.98	-5.37	6.35	-17.61	-12.44	-16.59	46.64
7.4	-5.06	-5.19	-1.01	-5.31	6.33	-17.44	-12.42	-16.65	46.51
7.5	-5.25	-5.17	-1.05	-5.25	6.30	-17.24	-12.42	-16.71	46.38
7.6	-5.46	-5.24	-1.09	-5.18	6.27	-17.02	-12.43	-16.78	46.23
7.7	-5.67	-5.37	-1.13	-5.11	6.24	-16.78	-12.44	-16.84	46.06
7.8	-5.90	-5.47	-1.18	-5.03	6.21	-16.52	-12.46	-16.90	45.88
7.9	-6.14	-5.47	-1.23	-4.95	6.18	-16.23	-12.46	-16.97	45.66
8.0	-6.37	-5.35	-1.27	-4.88	6.15	-15.94	-12.44	-17.03	45.41
8.1	-6.61	-5.11	-1.32	-4.80	6.12	-15.64	-12.41	-17.10	45.15
8.2	-6.83	-4.81	-1.37	-4.72	6.09	-15.34	-12.38	-17.16	44.87
8.3	-7.03	-4.50	-1.41	-4.66	6.06	-15.06	-12.34	-17.21	44.61
8.4	-7.21	-4.22	-1.44	-4.60	6.04	-14.81	-12.31	-17.27	44.38
8.5	-7.35	-4.01	-1.47	-4.55	6.02	-14.60	-12.28	-17.31	44.20
8.6	-7.44	-3.84	-1.49	-4.52	6.01	-14.46	-12.26	-17.36	44.08
8.7	-7.49	-3.73	-1.50	-4.50	6.00	-14.38	-12.25	-17.41	44.04
8.8	-7.49	-3.68	-1.50	-4.50	6.00	-14.38	-12.24	-17.45	44.08
8.9	-7.44	-3.71	-1.49	-4.52	6.01	-14.46	-12.25	-17.50	44.20
9.0	-7.35	-3.84	-1.47	-4.55	6.02	-14.60	-12.26	-17.54	44.41
9.1	-7.21	-4.07	-1.44	-4.60	6.04	-14.81	-12.29	-17.59	44.68
9.2	-7.03	-4.39	-1.41	-4.66	6.06	-15.06	-12.33	-17.64	45.02
9.3	-6.83	-4.74	-1.37	-4.72	6.09	-15.34	-12.37	-17.69	45.40
9.4	-6.61	-5.05	-1.32	-4.80	6.12	-15.64	-12.41	-17.75	45.79
9.5	-6.37	-5.28	-1.27	-4.88	6.15	-15.94	-12.43	-17.81	46.19
9.6	-6.14	-5.39	-1.23	-4.95	6.18	-16.23	-12.45	-17.88	46.56
9.7	-5.90	-5.41	-1.18	-5.03	6.21	-16.52	-12.45	-17.94	46.91
9.8	-5.67	-5.36	-1.13	-5.11	6.24	-16.78	-12.44	-18.01	47.23
9.9	-5.46	-5.31	-1.09	-5.18	6.27	-17.02	-12.44	-18.07	47.53
10.0	-5.25	-5.29	-1.05	-5.25	6.30	-17.24	-12.43	-18.14	47.81

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