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**THE RADIATION RESISTANCE  
OF CYLINDRICAL SHELLS**

*by Charles J. Runkle and Franklin D. Hart*

*Prepared by*

NORTH CAROLINA STATE UNIVERSITY

Raleigh, N. C.

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## THE RADIATION RESISTANCE OF CYLINDRICAL SHELLS

By Charles J. Runkle and Franklin D. Hart

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## SUMMARY

A study is made of the radiation resistance of long cylindrical shells in contact with an ideal compressible acoustic medium of infinite extent. The goal of the study is the development of useful engineering results in graphical form which are valid over a wide frequency range for broad types of shell materials and for various acoustic media.

The problem is formulated mathematically in terms of two descriptive differential equations: one for the cylindrical shell and the other for the acoustic medium. The equation of motion for the shell which is excited by a source in the interior is developed in terms of the radial displacement,  $w$ , of the surface subject to the hypotheses that the shell, composed of isotropic elastic material, obeys thin shell equations of deformation. The equation of motion of the acoustic medium is the well known wave equation. The solution to these equations is obtained by imposition of a boundary condition establishing velocity compatibility at the shell-fluid interface and by the requirement that the results satisfy the radiation condition in the limit at large distances from the surface of the shell. The solution which is written in terms of the acoustic velocity potential is employed to obtain the total power radiated and the mean-square surface velocity of the surface of the shell. This information is used to calculate the radiation resistance. For convenience and generality, the results are obtained in

terms of dimensionless series that are numerically evaluated for realistic ranges of the dimensionless parameters involved.

Although the problem is examined only in terms of axisymmetric and lobar mode shapes, the results agree with experimental results in the literature both in terms of the peak at the critical frequency and the asymptotic approach to the radiation resistance of a flat plate of equal area at large values of the dimensionless frequency parameter. The analytical results are presented in a form amenable to utilization of a digital computer with the actual numerical computation being carried out for both representative shell materials and fluids. The numerical results are presented in terms of graphs and tables to facilitate usefulness.

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## 1. INTRODUCTION

A rigorous study of the vibration of a structure in contact with a fluid requires the inclusion of the effects of the presence of the fluid on the motion of the structure. Even in less rigorous analyses, however, at least three situations require the consideration of structure-fluid interaction.

The first situation is the case of large fluid density. In this case the fluid is usually a liquid and the relative magnitude of the forces due to the presence of the fluid as compared to the magnitude of other forces acting on the structure preclude any meaningful analysis when the structure-fluid interaction is neglected.

A second instance arises when it is desirable to be able to predict and control the amount of vibratory power transmitted to the fluid by the structure. This is particularly important in mechanical design where the fluid is air and the vibratory power is unwanted sound that can adversely effect both men and sensitive equipment in the surrounding area.

The third situation occurs when a structure is excited by an intense acoustic power level in the surrounding fluid. In this case, which is essentially the inverse of the second case, the fluid is air and the acoustic power spectral components occur predominantly in the low-audible to sub-audible frequency ranges. The need to understand the structure-fluid interaction phenomenon in this case is also due to the desirability of reducing noise transmission as well as structural vibration.

Radiation resistance is a parameter which is used to quantify the structure-fluid interaction phenomena discussed briefly in the three preceding paragraphs. In the study reported herein, the radiation resistance for unit elements of infinite cylindrical shells is developed. Results are presented in the form of graphs for convenience as well as conciseness.

Chapter 4 provides a discussion of the development of the differential equations of motion of the shell for the two cases considered. A brief development of the wave equation for the acoustic medium and the boundary conditions at the shell-fluid interface and at large distances from the shell surface are included. In the last two sections of the chapter, an analytical solution for the case of axisymmetric mode shapes of the cylinder is presented along with a solution for the case of lobar mode shapes.

Chapter 5 presents the methods employed and results of numerical evaluation of the analytical expressions obtained in Chapter 4. The effects and relative importance of shell material properties and fluid properties are discussed.

Chapter 6 contains an evaluation of the results and a comparison of the results with previous work.

Several appendices are included for a more detailed explanation of one application of radiation resistance and procedures employed in the main body of this report. The first appendix is actually a paper, "Response Analysis of a Randomly Excited Rigid Piston in an Infinite Baffle," presented by the author at the 76th Meeting of the Acoustical Society of America and represents a detailed illustration of an

application of radiation resistance. The second appendix gives the detailed verification that the analytical solutions employed in this work satisfy the radiation condition. The last appendix gives the details of the computer programs utilized in the evaluation of the analytical results.

## 2. GENERAL CONCEPT OF RADIATION RESISTANCE

An analysis of acoustically excited vibration of structures leads to a consideration of the coupling between the sound pressure waves and the induced vibration of the structure. In a similar fashion, the control of the intensity of sound emitted from a vibrating structure requires a consideration of the coupling between the vibration of the structure and the induced pressure variations in the surrounding air.

This coupling depends on the radiation resistance and can be characterized by a quantity  $\mu$ , the resistance ratio, which is a measure of the amount of power radiated from the vibrating body as compared to the total power dissipated both by radiation to the surrounding fluid and by mechanical losses within the structure. Figure 2.1 illustrates the meaning of the resistance ratio with reference to the power dissipation that occurs in the steady-state vibration of a structure. Since  $W_i$  is the input power,  $\mu W_i$  is the energy dissipated to the surrounding fluid, and  $(1-\mu)W_i$  is the energy dissipated within the structure. In other words, from the definition

$$\mu = W_r / W_o , \quad (2.1)$$

where  $W_r$  is the radiated power and  $W_o$  is the radiated power plus the power dissipated in the structure.

### 2.1 Radiation Resistance

A more useful form of equation (2.1) can be developed in terms of the radiation resistance,  $R_{rad}$ , which is the central topic of this study. For generality without complexity, consider the

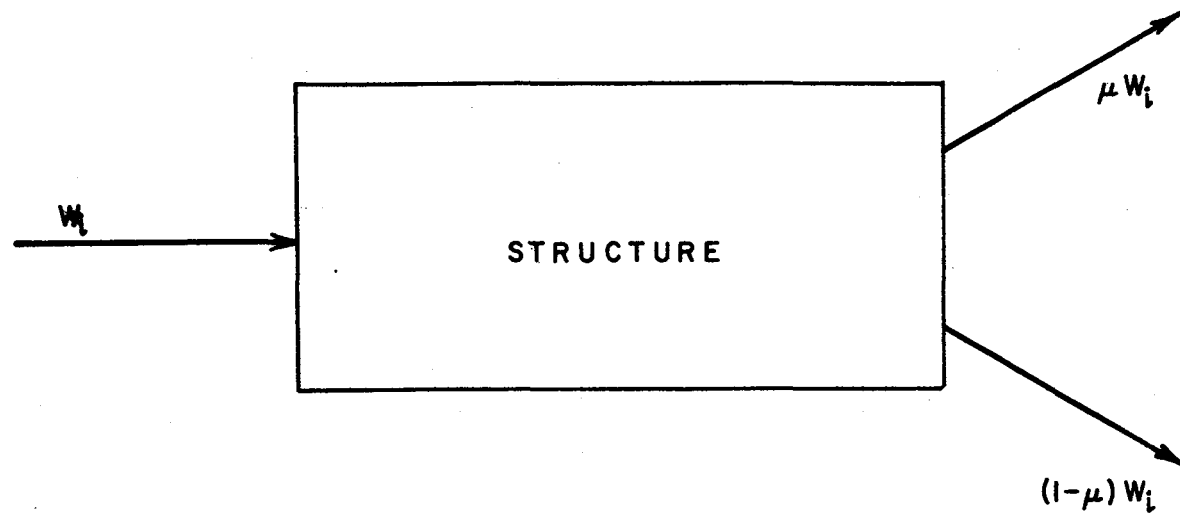


FIGURE 2.1 SCHEMATIC ILLUSTRATION OF ENERGY FLOW FOR VIBRATING STRUCTURE IN CONTACT WITH AN ACOUSTIC MEDIUM

single-degree-of-freedom system shown in Figure 2.2 where the acoustic damping effects are represented in terms of the radiation resistance,  $R_{rad}$ . The differential equation of motion for this system is

$$\frac{d^2x}{dt^2} + \frac{(R_m + R_{rad})}{M} \frac{dx}{dt} + \frac{S}{M} x = \frac{F_o}{M} e^{i\omega t}, \quad (2.2)$$

or

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \frac{S}{M} x = \frac{F_o}{M} e^{i\omega t}, \quad (2.3)$$

where

$$2\zeta\omega_n = (R_{rad} + R_m)/M. \quad (2.4)$$

The solution of equation (2.3) is a complex quantity the real part of which will hereafter be interpreted as being physically significant: the solution is of the form  $x(t) = Ae^{i\omega t}$  where A is evaluated as,

$$A = \frac{F_o/M}{S/M + i2\zeta\omega_n\omega - \omega^2}, \quad (2.5)$$

or using equation (2.4)

$$x(t) = \frac{-i \frac{F_o}{\omega} e^{i\omega t}}{(R_m + R_{rad}) + i(M\omega - \frac{S}{\omega})}. \quad (2.6)$$

But  $R = (R_m + R_{rad})$  is the total resistance, and  $X = (M\omega - \frac{S}{\omega})$  is the mechanical reactance; so, the impedance can be written as

$$Z = Z_{max} e^{i\phi}, \text{ where } Z_{max} = \sqrt{R^2 + X^2} \text{ and } \phi = \tan^{-1} \frac{X}{R}. \quad (2.7)$$



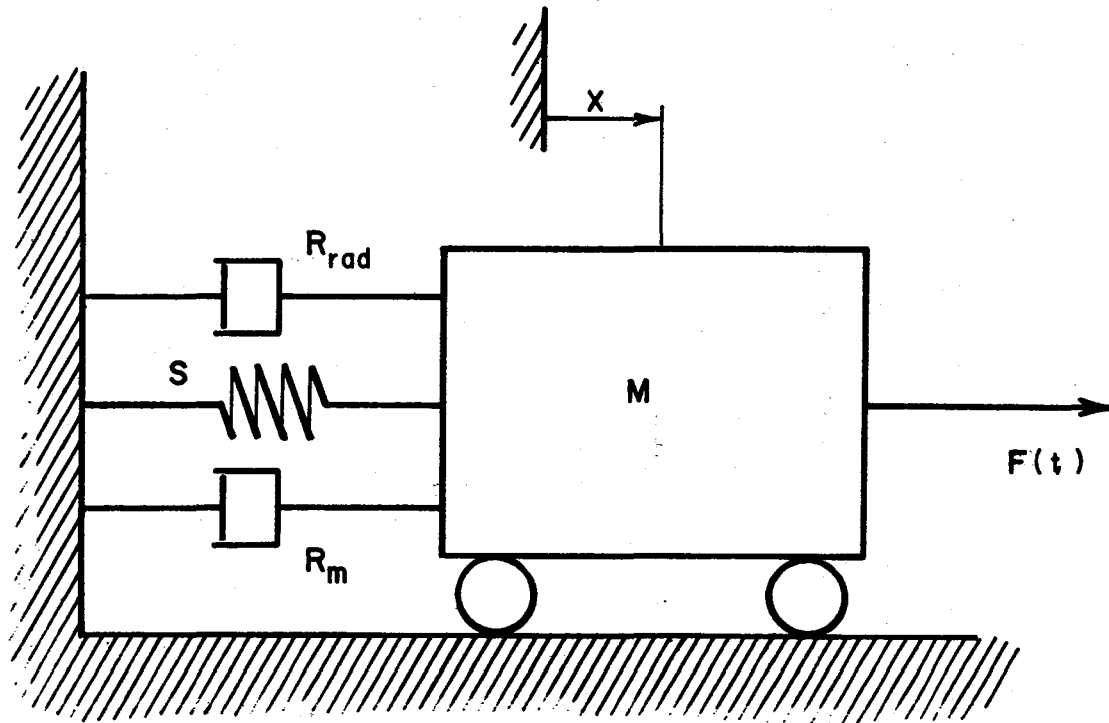


FIGURE 2.2 MATHEMATIC MODEL OF SINGLE - DEGREE - OF - FREEDOM SYSTEM WITH ACOUSTIC DAMPING INCLUDED

Equation (2.7) illustrates that the radiation resistance contributes to the real part of the total mechanical impedance. Using equation (2.7), equation (2.6) can be written as

$$x(t) = \frac{-iF_o}{\omega Z_{\max}} e^{i(\omega t - \phi)} \quad (2.8)$$

The instantaneous input power can now be calculated by multiplying the real part of the time derivative of equation (2.8) by the real part of the exciting force,  $F_o \cos \omega t$ . Hence, the input power, averaged over an interval of time corresponding to an integral number of cycles of the exciting force, is

$$W_i = \frac{1}{T} \int_0^T \text{Re}\{\dot{x}(t)\} F_o \cos \omega t \, dt. \quad (2.9)$$

Substitution and evaluation gives,

$$W_i = \frac{F_o^2 \cos \phi}{2 Z_{\max}},$$

but

$$\cos \phi = \frac{R}{Z_{\max}}$$

so that

$$W_i = \frac{F_o^2 R}{2 Z_{\max}^2}. \quad (2.10)$$

The average power dissipated by the total resistance is computed in a similar manner and is given by

$$W_i = \frac{1}{2} \frac{F_o^2 R}{Z_{\max}^2}, \quad (2.11)$$

while the average power dissipated through the radiation resistance term, alone, is

$$W_r = \frac{1}{2} \frac{F_o^2 R_{\text{rad}}}{Z_{\max}^2} \quad (2.12)$$

Equation (2.1) can now be written

$$\mu = \frac{R_{\text{rad}}}{R} = \frac{R_{\text{rad}}}{R_m + R_{\text{rad}}} \quad (2.13)$$

Thus, the necessity for knowledge about  $R_{\text{rad}}$  in order to determine  $\mu$  is demonstrated.

## 2.2 Applications of Radiation Resistance

The radiation resistance and the resistance ratio have greater value than simply that of indicating the ratio of radiated power to total dissipated power. Knowledge of  $\mu$  is necessary in establishing a quantitative relationship between the vibration of a structure and associated acoustic vibration. In relating structural vibration with acoustic pressure excitation, the ratio of the acceleration power spectral density to the pressure spectral density was presented by Lyon and Maidanik (1962), Smith (1962), and Maidanik (1962) as

$$S_a(\omega) S_p(\omega) = \Gamma(\omega) \mu(\omega), \quad (2.14)$$

where  $S_a(\omega)$  is the acceleration spectral density,  $S_p(\omega)$  is the

pressure spectral density,  $\Gamma(\omega) = [2\pi^2 n_s(\omega)/M] (c_o/\rho_o)$  and  $\mu(\omega)$  is, of course, the resistance ratio. The symbols,  $n_s(\omega)$ ,  $M$ ,  $c_o$ , and  $\rho_o$  represent the modal density of the structure, the lumped mass of the system, the speed of sound in the acoustic medium, and the ambient density of the acoustic medium respectively. The radiation resistance and  $\Gamma(\omega)$  likewise play a key role in usage of this method of vibration analysis. Detailed examination of equation (2.14) indicates that knowledge about either  $S_a$  or  $S_p$  can be translated into knowledge about the other provided  $\Gamma(\omega)$  and  $\mu(\omega)$  can be determined.

The quantity,  $\Gamma(\omega)$ , depends on the variables,  $c_o$ ,  $\rho_o$ ,  $M$ , and  $n_s(\omega)$ , the modal density of the structure. The modal density of structures has been studied by a number of investigators, including Heckl, 1962; Bolotin, 1963; Smith and Lyon, 1965; Miller and Hart, 1967; Hart and Desai, 1967;<sup>1</sup> and Miller, 1969. Hence, it is possible to determine  $\Gamma(\omega)$  except in very special cases.

The quantity  $\mu(\omega)$  can be determined easily if  $R_{rad}$  can be computed analytically. The mechanical resistance,  $R_m$ , can be determined experimentally by measuring the reverberation time,  $T_s$ , of the structure (see Morse, 1948; Hueter and Bolt, 1955; Kinsler and Frey, 1962; and Morse and Ingard, 1968).

If  $R_{rad}$  is not known analytically, then the unknown spectral density ( $S_a$  or  $S_p$ ) must be experimentally determined. Experimental determination of spectral densities is neither as simple nor as

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<sup>1</sup>Hart, F. D. and V. D. Desai (Department of Mechanical and Aerospace Engineering, North Carolina State University at Raleigh, North Carolina). 1967. Additive properties of modal density for complete structures. Presented at the 74th Meeting of the Acoustical Society of America, Miami, Florida, Paper No. DD11.

inexpensive to perform as is determination of the reverberation time. Therefore, the ability to predict  $R_{\text{rad}}$  analytically is of considerable value.

Equations (2.2) and (2.12) indicate, of course, that the radiation resistance is required in order to include acoustic damping force terms in the equations of motion of a structure and to calculate the total acoustic power radiated by a structure, respectively. The radiation resistance is also used in the study of random vibration of a structure in contact with a fluid. Appendix 10.1 provides a detailed analysis of a randomly excited rigid piston in an infinite baffle as an illustration of this type of application.

### 3. REVIEW OF LITERATURE

The radiation resistance of cylindrical shells has not been studied in either great depth or with broad generality. In fact, the need to do so in relationship to acoustics problems in air has only recently materialized in conjunction with the development of statistical energy methods of vibration analysis (Lyon and Maidanik, 1962; Smith, 1962; Maidanik, 1962; and Ungar, 1966).

Prior to this time, work in the area was done because of interests in underwater vibrations and sound transmission, and because of interests in transducer and wave guide design. Morse (1948) considered the problem of a long cylindrical shell vibrating with uniform surface velocity, and generated an expression for the total power radiated per unit of length. He also developed expressions for the power radiated from a long element of a long cylinder, and also for a cylinder vibrating with a velocity which is a function of polar angle only. The radiation resistance can be readily calculated using this information, but the results in each case depend upon an assumed velocity distribution over the surface of the shell.

Junger (1952a) developed acoustic resistance and reactance ratios for partial waves emitted by long cylindrical sources whose dynamic configuration is expressible by an infinite series which is a function of polar angle only. The resistance and reactance ratios were subsequently employed to determine the sound power radiated. The results of Junger's work differ from those of Morse (1948) basically in the added generality of the motional behavior of the cylindrical shell as a

function of the polar angle. Although the pronounced purpose of Junger's work in this case was to study the natural frequencies and forced vibrations of submerged shells, the general character of the mathematical problem formulation has bearing on this research.

Later, Junger (1952b) published work dealing with the vibration of a thin elastic cylindrical shell freely suspended in a compressible fluid medium. The problem was analyzed by means of the classical methods of vibration theory employing the Lagrange equations for the system with the fluid reaction being introduced in terms of generalized forces. The fact that the normal shell deflection is equal to the normal fluid-particle displacement at the shell-fluid interface allows determination of the radial and tangential deflections of the shell as well as total radiated power. The formulation and solution of this particular problem as opposed to that of the previous one (Junger, 1952a) differs in that the specific influence of shell geometry and material properties are included. Hence, the correlation between theory and physical situation is better. The Lagrange method of attacking the problem will not be used in this work but formulation of the problem in such a manner as to include shell geometry and material parameters will be desirable.

All previous work has considered the case of motional behavior of the cylindrical shell being a function of only polar angle. Junger (1953) broadened the problem by considering the case of an infinite shell which exhibited a dynamic configuration periodic in both polar angle and the axial coordinate. The additional degree of complexity considered exposes the phenomenon of all axial-coordinate-dependent

modes being non-radiating below certain "cut-off" frequencies. In other matters, this part of Junger's work compares directly with some of his earlier efforts (Junger, 1952a).

Further work with the vibration of an infinitely long cylindrical shell in contact with an acoustic medium was done by Bleich and Baron (1954). Their work amounted to a simultaneous solution of the wave equation and appropriate equations describing the motional behavior of the shell. The approach used by Bleich and Baron differs from Junger (1953) in the manner in which the shell is handled. Although the dynamic configuration of the shell in both cases is a function of polar angle and the axial coordinate, Bleich and Baron use the invacuo modes of vibration of the shell as generalized coordinates. This feature of the analysis, as well as the use of simultaneous differential equations to describe the system, yields a solution containing general emphasis on the structural response of the shell, while the work of Junger (1953) disclosed the properties of the acoustic field but did not permit determination of the response of the shell without further analysis.

Kolotikhina (1957) considers also the problem of an infinite cylindrical shell in contact with an ideal compressible fluid. In this study the motional behavior of the shell is independent of the polar angle but is periodic with respect to the axial coordinate. The partial differential equation which describes the radial motion of the shell was developed by combining the three usual equations for radial, tangential, and axial motion. The resulting equation in the radial displacement was solved simultaneously with the wave equation in the acoustic medium and represented by an infinite series. The physical



characteristics of the solution compare, of course, with those of Junger (1953) and Bleich and Baron (1954).

Previous work has been restricted to the case of thin cylindrical shells; however, Greenspon (1961) considered the same problem but employed three dimensional elasticity theory for the shell. The results of the thick shell approach were compared to thin shell results and it was noted that the thin shell (approximate) theory gave excellent results as long as the ratio of inside to outside radius was 0.9 or greater. Both the natural frequency and radial displacement were also given accurately by the thin shell method for this general radius ratio, but for smaller values of the ratio, first the displacement, then the natural frequency proved to be in significant error.

The finite cylinder problem and its base in the literature will be mentioned here although its relationship to the research reported herein is indirect rather than direct. Manning and Maidanik (1964) developed a theoretical method for estimating the radiation efficiency of a cylindrical shell. The radiation efficiency was estimated by utilizing simple physical arguments based on considerations of the shape of typical modal patterns. The modes of the cylinder were divided into groups according to the magnitude of the bending-wave speed and the phase speeds in the directions of the panel edges as compared to the speed of sound in the surrounding fluid. Then modal radiation efficiencies were estimated for each group by examining the volume-velocity cancellation between adjacent cells of the modal vibration pattern. Although the procedure involved in utilizing this

technique is complex, the comparison of theoretical results and experiment was reported to indicate good correlation.

A more rigorous analysis was carried out on the finite cylinder problem by Williams et al. (1964). The technique employed was a numerical one that employed a least-squares finite series approximation of an infinite series expansion of boundary conditions. Expressions for the far-field pressure and the radiation resistance were developed. Sherman and Moran (1966) also examined the problem from this point of view, but their work extended previous effort by developing a low frequency approximation and by performing a more extensive computer analysis. A variety of boundary conditions and cylinder height to diameter ratios as well as both uniform and parabolic velocity distributions on the ends of the cylinder were examined.

The physical situation of a long cylindrical shell in contact with an ideal compressible fluid has been analyzed from several points-of-view utilizing several techniques. In this study, the shell-acoustic medium system is modeled mathematically in terms of simultaneous differential equations--one describing the motion of the shell and the other, the motion of the fluid. The solution is obtained by employing separation-of-variables methods. The results are used to obtain the power radiated and the radiation resistance. The value and originality inherent in this approach is the rigor and conciseness of the mathematical problem formulation as well as that the result exhibits clearly the relative importance of shell material properties compared to acoustic-medium properties. The resulting expressions for the

radiation resistance are, therefore, not only more rigorously developed, but are more general than previous work also.

#### 4. ANALYTICAL DEVELOPMENT

The problem of an infinite cylindrical shell in contact with an ideal compressible acoustic medium is examined. The mathematical formulation of the problem consists of one differential equation describing the behavior of the cylindrical shell and another equation describing the behavior of the acoustic medium. These two differential equations are solved simultaneously subject to a compatibility boundary condition at the shell-acoustic medium interface and another condition at the boundary at large distances from the surface of the shell. The solution of these equations is employed to compute the total radiated power which is used to determine the radiation resistance. The detailed description of the analytical solution of this problem is divided into a section for the equations of motion of the shell, a section for the wave equation in the acoustic medium and a section for the detailed solution of these equations in the two cases of mode shapes that are considered. The first of these deals with the descriptive equations for the shell.

##### 4.1 Equations of Motion for the Cylindrical Shell

Any point on the surface of the cylinder will vibrate upon excitation with an amplitude which has components that are radial, axial, and tangential to the surface at the point of interest. The radial vibrations correspond, in a manner analogous to the transverse vibrations of beams, to a predominately bending or flexing action. The energy associated with such flexure is directly representable in terms of the mean-square radial velocity of the structure surface; hence, there is a

direct link between the bending vibrations of a structure and the radiated energy. The axial and tangential vibrations are principally due to extensional deformation. The energy due to these components of vibration is represented in terms of axial and tangential velocity components which transmit energy to the surrounding fluid by means of a shearing action instead of by means of compression and rarefaction as is the case for bending vibration. For this investigation, it will be assumed that the acoustic medium is a perfect gas with a viscosity of zero, hence the contribution of the shear waves in transmitting energy away from the cylinder will be zero. In truth some small amount of energy is radiated to the fluid by means of shear waves, but it is negligible compared to the amount radiated by compression-rarefaction waves.

The equations of motion for the flexural vibration of the shell will be developed for two types of mode shapes. In the first case, the mode shape is described in terms of the axial coordinate,  $z$ , only and will hereafter be termed an axisymmetrical mode, while in the second case, the mode shape is described in terms of the polar angle,  $\theta$ , only and will hereafter be termed a lobar mode. Figure 4.1 illustrates an element of the cylindrical shell, the cylindrical coordinate system to be employed in the problem and the  $u$ ,  $v$ , and  $w$  components of the displacement of the shell surface at a general point of that surface,  $a$ ,  $\theta$ ,  $z$ . According to Vlasov (1949), the general differential equations relating the displacements to the like components of the applied forces are:

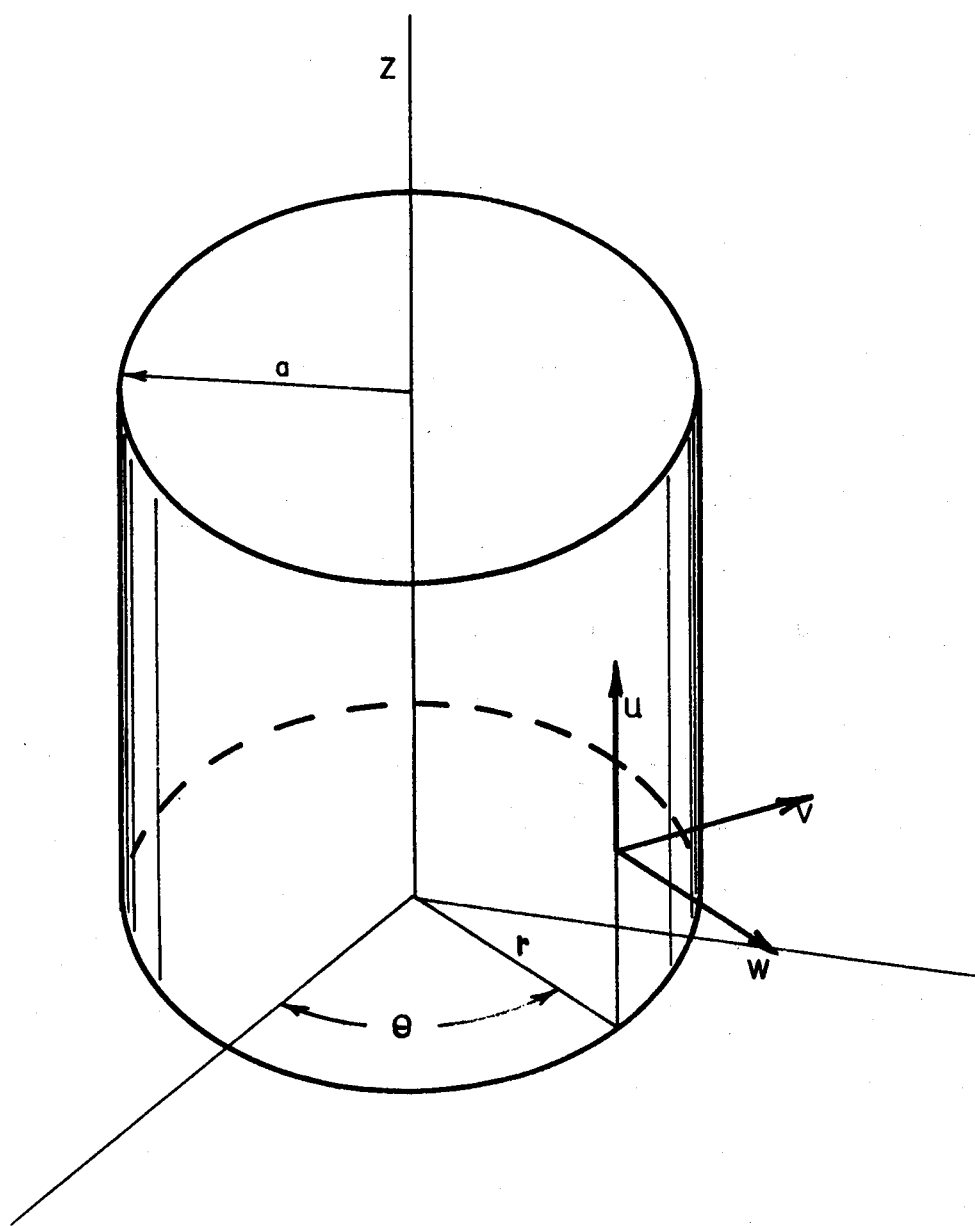


FIGURE 4.1 THE COORDINATE SYSTEM AND DISPLACEMENT COMPONENTS AT A POINT ON AN ELEMENT OF THE CYLINDRICAL SHELL

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{1-\nu}{2} \frac{\partial^2 u}{\partial \beta^2} + \frac{1+\nu}{2} \frac{\partial^2 v}{\partial \alpha \partial \beta} + \nu \frac{\partial w}{\partial \alpha} - c^2 \left( \frac{\partial^3 w}{\partial \alpha^3} - \frac{1-\nu}{2} \frac{\partial^3 w}{\partial \alpha \partial \beta^2} \right) + \frac{1-\nu^2}{Eh} a^2 F_u = 0, \quad (4.1a)$$

$$\frac{1+\nu}{2} \frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 v}{\partial \beta^2} + \frac{1-\nu}{2} \frac{\partial^2 v}{\partial \alpha^2} + \frac{\partial w}{\partial \beta} - \frac{3-\nu}{2} c^2 \frac{\partial^3 w}{\partial \alpha^2 \partial \beta} + \frac{1-\nu^2}{Eh} a^2 F_v = 0, \quad (4.1b)$$

and

$$\nu \frac{\partial u}{\partial \alpha} - c^2 \left( \frac{\partial^3 u}{\partial \alpha^3} - \frac{1-\nu}{2} \frac{\partial^3 u}{\partial \alpha \partial \beta^2} \right) + \frac{\partial v}{\partial \beta} - \frac{3-\nu}{2} c^2 \frac{\partial^3 v}{\partial \alpha^2 \partial \beta} + c^2 \left( \nabla^2 \nabla^2 w + 2 \frac{\partial^2 w}{\partial \beta^2} + w \right) + w - \frac{1-\nu^2}{Eh} a^2 F_w = 0; \quad (4.1c)$$

where  $\alpha = \frac{z}{a}$ ,  $\beta = \theta$ , and  $c^2 = h^2/(12a^2)$ .

The principal hypotheses involved in the development of these equations are that a straight-line element of the shell normal to the middle surface remains a straight line and normal to the surface after deformation and retains its length, and that the material of the shell is isotropic and obeys Hooke's elastic law. Note that  $F_u$ ,  $F_v$ , and  $F_w$  are components of the external surface load. For the situation of the vibrating cylinder in contact with an ideal acoustic medium,  $F_u$  and  $F_v$  are inertia forces while  $F_w$  is an inertia force plus a resisting force due to the presence of the acoustic medium. Since the concern of the work will be the particulars of shell - acoustic medium interaction, equation (4.1c) will be employed as the "principal" equation. In other

words, the flexural vibration is chiefly responsible for the acoustic vibration, hence equations (4.1a) and (4.1b) will be incorporated into (4.1c) in order to describe the shell-acoustic medium problem in terms of  $w$ , the radial displacement. Equations (4.1) can be further simplified for thin shells to give (Vlasov, 1949, p. 360)

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{1-\nu}{2} \frac{\partial^2 u}{\partial \beta^2} + \frac{1+\nu}{2} \frac{\partial^2 v}{\partial \alpha \partial \beta} + \nu \frac{\partial w}{\partial \alpha} = - \frac{(1-\nu^2)}{Eh} a^2 F_u, \quad (4.2a)$$

$$\frac{1+\nu}{2} \frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 v}{\partial \beta^2} + \frac{1-\nu}{2} \frac{\partial^2 v}{\partial \alpha^2} + \frac{\partial w}{\partial \beta} = - \frac{(1-\nu^2)}{Eh} a^2 F_v, \quad (4.2b)$$

and

$$\nu \frac{\partial u}{\partial \alpha} + \frac{\partial v}{\partial \beta} + c^2 \nabla^2 \nabla^2 w + w = \frac{(1-\nu^2)}{Eh} a^2 F_w. \quad (4.2c)$$

Equations (4.2) are derived from (4.1) based on the hypothesis that thin shells are characterized by a maximum value of the ratio  $h/a$  which can be neglected in comparison to unity. At least one author, Novozhilov (1964), interprets this to mean

$$\frac{h}{a} \leq \frac{1}{20}$$

In any case, for  $h/a$  much less than one, certain terms in equations (4.1) involving the coefficient  $c^2$  can be neglected to yield equations (4.2). These equations (4.2) will now be simplified and combined for axisymmetric and lobar mode shapes.



#### 4.1.1 Axisymmetric Mode Shapes

For this case, the tangential shell displacements are always zero, and the  $n^{\text{th}}$  order partial derivatives of any quantity with respect to  $\theta$  will also be zero; hence, equations (4.2) become

$$\frac{\partial^2 u}{\partial \alpha^2} + \nu \frac{\partial w}{\partial \alpha} = - \frac{(1-\nu^2)}{Eh} a^2 F_u, \quad (4.3a)$$

and

$$\nu \frac{\partial u}{\partial \alpha} + c^2 \nabla^2 \nabla^2 w + w = \frac{(1-\nu^2)}{Eh} a^2 F_w. \quad (4.3b)$$

By differentiating equation (4.3b) with respect to  $\alpha$  and substituting equation (4.3a), the result is

$$- \nu \frac{(1-\nu^2)}{Eh} a^2 F_u - \nu^2 \frac{\partial w}{\partial \alpha} + c^2 \frac{\partial^5 w}{\partial \alpha^5} + \frac{\partial w}{\partial \alpha} = \frac{(1-\nu^2)}{Eh} a^2 \frac{\partial F_w}{\partial \alpha}. \quad (4.4)$$

For the purposes of this study, the inertia force term involving  $F_u$  will be neglected in comparison to other terms of equation (4.4), since  $F_u$  is directly proportional to the axial deflection,  $u$ , at any given frequency and  $u$  is of negligible magnitude compared to the radial deflection,  $w$ . Integrating equation (4.4) with respect to  $\alpha$  and rearranging gives

$$\frac{D}{a^4} \left[ \frac{\partial^4 w}{\partial \alpha^4} + \frac{Eha^2}{D} w \right] = F_w. \quad (4.5)$$

The quantity  $F_w$ , the external radial surface load, will be considered to be composed of three components: the first is the inertia forces, the second is the acoustic resisting forces, and the third is the

applied surface load,  $q$ , which is due to a source within the cylinder and will be considered to be a function of both time,  $t$ , and the axial coordinate,  $\alpha$ . The forces due to the presence of the acoustic medium will be expressed in terms of an acoustic velocity potential,  $\phi$ . In this case, therefore,

$$F_w = -m_s \frac{\partial^2 w}{\partial t^2} + q(\alpha, t) + \rho_o \left[ \frac{\partial \phi}{\partial t} (r, \alpha, t) \right] . \quad (4.6)$$

Combining equation (4.6) with equation (4.5) and dividing by  $m_s$  gives the desired result.

$$\frac{\partial^2 w}{\partial t^2} + \frac{D}{m_s a^4} \left[ \frac{\partial^4 w}{\partial \alpha^4} + \frac{Eha^2}{D} w \right] = \frac{q(\alpha, t)}{m_s} + \frac{\rho_o}{m_s} \left[ \frac{\partial \phi}{\partial t} (r, \alpha, t) \right]_{r=a} . \quad (4.7)$$

Equation (4.7) is an equation of motion for the shell in terms of the radial displacement,  $w$ , alone. This equation will be employed to completely describe the cylindrical shell for axisymmetrical mode shapes, and will be solved simultaneously with the wave equation to obtain the desired expressions for the radiation resistance.

#### 4.1.2 Lobar Mode Shapes

This mode shape is characterized by the fact that the axial displacement of the shell surface is zero at all points, and the  $n^{\text{th}}$  order partial derivatives of any quantity with respect to  $\alpha$  will also be zero; hence equations (4.2) become

$$\frac{\partial^2 v}{\partial \beta^2} + \frac{\partial w}{\partial \beta} = - \frac{(1-\nu^2)}{Eh} a^2 F_v , \quad (4.8a)$$

and

$$\frac{\partial v}{\partial \beta} + c^2 \nabla^2 \nabla^2 w + w = \frac{(1-\nu^2)}{Eh} a^2 F_w . \quad (4.8b)$$

Differentiating equation (4.8b) with respect to  $\beta$  and substituting equation (4.8a) gives

$$- \frac{(1-\nu^2)}{Eh} a^2 F_v + c^2 \frac{\partial^5 w}{\partial \beta^5} = \frac{(1-\nu^2)}{Eh} a^2 \frac{\partial F_w}{\partial \beta} . \quad (4.9)$$

The term involving  $F_v$  is an inertia force term due to the tangential motion of the surface of the shell. Compared to the other terms of equation (4.9), the term involving  $F_v$  is small and can be neglected, since  $F_v$  is directly proportional to the tangential deflection,  $v$ , at any given frequency and  $v$  is of negligible magnitude compared to the radial deflection,  $w$ . By integrating the resulting equation with respect to  $\beta$  equation (4.10) results:

$$c^2 \frac{\partial^4 w}{\partial \beta^4} = \frac{(1-\nu^2) a^2}{Eh} F_w . \quad (4.10)$$

For this case, the external load,  $F_w$ , will be considered to be composed of a component due to the inertia forces, one due to the acoustic resisting forces, and one due to the applied surface load,  $q$ , generated in the interior of the cylindrical shell. Hence

$$F_w = -m_s \frac{\partial^2 w}{\partial t^2} + q(\beta, t) + \rho_o \left[ \frac{\partial \phi}{\partial t} (r, \beta, t) \right]_{r=a} . \quad (4.11)$$

Combining equation (4.11) and equation (4.10) yields

$$\frac{\partial^2 w}{\partial t^2} + \frac{D}{m_s a^4} \frac{\partial^4 w}{\partial \beta^4} = \frac{q(\beta, t)}{m_s} + \frac{\rho_o}{m_s} \left[ \frac{\partial \phi}{\partial t} (r, \beta, t) \right]_{r=a} . \quad (4.12)$$

Equation (4.12) is the desired equation of motion for the shell in the case of lobar mode shapes. This equation describes the dynamic behavior of the cylindrical shell in terms of the radial displacement of the surface and will be solved simultaneously with the wave equation to obtain the needed expressions for the radiation resistance.

#### 4.2 The Wave Equation in the Acoustic Medium

The differential equations of motion for the cylindrical shell were developed in section 4.1. This section will present a brief development of the wave equation - the differential equation of motion for the acoustic medium. Several sources (Morse, 1948; Morse and Feshbach, 1953; Kinsler and Frey, 1962; Rschevkin, 1963; Stephens and Bate, 1966; Morse and Ingard, 1968) develop the acoustic wave equation in depth; hence, there is no need for anything but brief coverage of that development here.

The basic principles characterized by the equation of continuity, the equation of state of the acoustic medium, and the equation for Newton's Second Law are combined into the single partial differential wave equation. The hypotheses inherent in the derivation of the equation are first that the fluctuations of the variables characterizing the sound are small such that only linear variations in these quantities need be considered, secondly that the process of sound propagation is adiabatic, and thirdly that the acoustic medium is an inviscid gas obeying the perfect gas law.

The derivation can be accomplished applying Newton's Second Law to a "particle" of gas which is transmitting a sound wave. The presence of the sound wave establishes pressure variations throughout the gas. The net pressure gradient tends to accelerate the gas "particle" in a direction which will neutralize the pressure differences, the resulting descriptive equation is

$$\text{grad } p = -\rho_0 \ddot{\vec{u}} , \quad (4.13)$$

where  $\ddot{\vec{u}}$  is the "particle" acceleration.

Applying the concept of continuity of mass to a control volume fixed in space relative to a passing sound wave, a second relationship can be developed. The flow of mass across the boundaries of the region represents the only way in which the total amount of mass within the region can change: the time rate of change of mass within the control volume will be equal to the net flow of mass into or from the volume.

Hence

$$\frac{\partial \rho}{\partial t} = -\text{div}(\rho \vec{u}) . \quad (4.14)$$

The fluid density is, in general, definitely a function of position, but a great simplification in the mathematics can be achieved at small expense in accuracy by representing  $\rho$  in terms of the ambient density,  $\rho_0$ ; hence

$$\frac{\partial \rho}{\partial t} = -\rho_0 \text{div}(\vec{u}) . \quad (4.15)$$

Employing the equation of state of the fluid gives information for three equations in three unknowns ( $p$ ,  $\vec{u}$ ,  $\rho$ ). The perfect gas law

utilized in conjunction with the information provided by the adiabatic process hypothesis gives

$$p(x_j, t) = c_o^2 \rho(x_j, t) , \quad (4.16)$$

where

$$c_o^2 = P_o \gamma / \rho_o, \quad \text{and} \quad \gamma = c_p / c_v .$$

Combining equations (4.13), (4.15), and (4.16), the wave equation results:

$$\nabla^2 p = \frac{1}{c_o^2} \frac{\partial^2 p}{\partial t^2} , \quad (4.17)$$

where  $\nabla^2$  is a symbolic operator known as the Laplacian operator. In cylindrical coordinates

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} .$$

For the purposes of this research, it is convenient to write equation (4.17) in terms of the velocity potential,  $\phi$ . Thus the wave equation becomes

$$\nabla^2 \phi = \frac{1}{c_o^2} \frac{\partial^2 \phi}{\partial t^2} , \quad (4.18)$$

and the pressure and the radial fluid particle velocity are then given by (Morse and Ingard, 1968)

$$p = \rho_o \frac{\partial \phi}{\partial t} , \quad \text{and} \quad u_r = - \frac{\partial \phi}{\partial r} . \quad (4.19)$$

Hence the wave equation in cylindrical coordinates is known and will be solved simultaneously with the equation of motion for the shell.

#### 4.3 Solution for Axisymmetric Mode Shapes

Descriptive equations were developed in the previous sections for both the cylindrical shell and the acoustic medium. This section indicates the boundary conditions employed and the solution obtained for the case of an axisymmetric mode shape.

Two boundary conditions are required for the simultaneous solution of equation (4.7) and the wave equation written in terms of the velocity potential,  $\phi$ . One applies at  $r = a$ ; the condition being that at the shell-fluid interface, the radial velocity of the shell surface is equal to the radial velocity of the fluid particles in contact with the shell. The other boundary condition applies at large distances from the surface of the shell as  $r$  approaches infinity where the intent of the condition is that only solutions of the acoustic wave equation which represent outgoing waves are admitted as pertinent in this case. The physical implications of the last boundary condition are that no reflection or other physical disturbance occur at the far boundaries of the acoustic medium.

At the shell-fluid interface, the boundary condition can be expressed mathematically as

$$\frac{\partial w}{\partial t}(\alpha, t) = \frac{\partial \phi}{\partial r}(r, \alpha, t) \Big|_{r=a}, \quad (4.20)$$

and at large distances from the surface of the cylindrical shell,

$$\lim_{r \rightarrow \infty} \sqrt{r} \left[ \frac{\partial \phi}{\partial r} - i k_r \phi \right] = 0 \quad (4.21)$$

for  $k_r > 0$ , and

$$\lim_{r \rightarrow \infty} \sqrt{r} \left[ \frac{\partial \phi}{\partial r} - k_r \phi \right] = 0 \quad (4.22)$$

for  $k_r = i k_r > 0$ , where  $k_r$  is the separation constant that appears in the separated ordinary differential equation in the spacial variable,  $r$ . The boundary condition expressed in terms of equation (4.21) and (4.22) is termed the radiation condition and is treated in detail in the literature by Sommerfeld (1949).

The solution of equations (4.7) and (4.18) subject to the boundary conditions (4.20), (4.21), and (4.22) is accomplished by utilizing separation of variables; i.e., let

$$\phi(r, \alpha, t) = R(r)G(\alpha)T'(t) . \quad (4.23)$$

Substituting equation (4.23) into equation (4.18) and for  $\phi \neq 0$ , dividing by  $\phi$  yields

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{a^2} \frac{G''}{G} = \frac{1}{c_o^2} \frac{T'''}{T'} . \quad (4.24)$$

Breaking this equation down into three differential equations in the three variables  $r$ ,  $\alpha$ , and  $t$  yields

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = -k_r^2 , \quad (4.25a)$$

$$\frac{1}{a^2} \frac{G''}{G} = -k_z^2 , \quad (4.25b)$$



$$\frac{1}{c_0^2} \frac{T'''}{T'} = -k^2, \quad \text{and} \quad (4.25c)$$

$$k_r^2 + k_z^2 = k^2. \quad (4.26)$$

Equation (4.25a) can be rewritten as

$$R'' + \frac{1}{r} R' + k_r^2 R = 0 \quad (4.27)$$

which is easily recognizable as a Bessel differential equation. For  $k_r > 0$  equation (4.27) has the solution

$$R(r) = C^{(1)} H_0^{(1)}(k_r r) + C^{(2)} H_0^{(2)}(k_r r), \quad (4.28)$$

where  $H_0^{(1)}$  and  $H_0^{(2)}$  are Hankel functions of the first and second kind of order one respectively. In the case such that  $k_r = i k_r > 0$ , the solution is again written in terms of Hankel functions of the first and second kind, but the arguments of the Hankel functions are imaginary rather than real as is the case in equation (4.28). Hence for  $k_r > 0$

$$R(r) = B^{(1)} H_0^{(1)}(i k_r r) + B^{(2)} H_0^{(2)}(i k_r r). \quad (4.29)$$

After rearranging, equation (4.25b) becomes

$$G''(\alpha) + k_z^2 a^2 G(\alpha) = 0. \quad (4.30)$$

The solution to equation (4.30) is a sinusoid; hence, the motion of the shell is periodic in the  $\alpha$ -direction according to  $A_n \sin \frac{n\pi a}{L} \alpha$  so

$$G(\alpha) = G_n(\alpha) = A_n \sin \frac{n\pi a}{L} \alpha, \quad (4.31)$$

and substitution into equation (4.30) yields

$$\left[-\left(\frac{n\pi a}{L}\right)^2 + k_z^2 a^2\right] A_n \sin \frac{n\pi a}{L} \alpha = 0 . \quad (4.32)$$

Consequently

$$k_z^2 = \left(\frac{n\pi}{L}\right)^2 ; \quad n = 1, 2, 3, \dots . \quad (4.33)$$

The solution to equation (4.25c) can, of course, be written as

$$T(t) = e^{i\omega t} , \quad (4.34)$$

where  $k^2 = \omega^2/c_o^2$ . Therefore the velocity potential is known in terms of equations (4.28), (4.29), (4.31), and (4.34). At this point, the application of the radiation condition simplifies the expression for  $\phi$  and verifies that the expression corresponds to an outgoing acoustic wave. The radiation condition (equations (4.21) and (4.22)) can now be written as

$$\lim_{r \rightarrow \infty} \sqrt{r} [R'(r) - ik_r R(r)] = 0 \quad (4.35)$$

for  $k_r > 0$ , and as

$$\lim_{r \rightarrow \infty} \sqrt{r} [R'(r) - \bar{k}_r R(r)] = 0 \quad (4.36)$$

for  $\bar{k}_r = ik_r > 0$ . Application of these conditions to the expressions for  $R(r)$  in Appendix 10.2 shows that the pertinent results are

$$R(r) = c^{(1)} H_o^{(1)}(k_r r) \quad \text{for } k_r > 0 ,$$

and

$$R(r) = B^{(1)} H_0^{(1)}(i\bar{k}_r r) \quad \text{for } \bar{k}_r = ik_r > 0 ;$$

where the constants  $C^{(1)}$  and  $B^{(1)}$  will be set equal to one in the expressions for  $\phi$  leaving  $A_n$  as the sole undetermined constant. The velocity potential is

$$\phi(r, \alpha, t) = i\omega \sum_{n=1} A_n \sin \frac{n\pi\alpha}{L} \propto H_0^{(1)}(k_r r) e^{i\omega t} \quad (4.37)$$

for  $k_r > 0$ , and

$$\phi(r, \alpha, t) = i\omega \sum_{n=1} A_n \sin \frac{n\pi\alpha}{L} \propto H_0^{(1)}(i\bar{k}_r r) e^{i\omega t} \quad (4.38)$$

for  $\bar{k}_r = ik_r > 0$ . The velocity potential is thus determined with the exception of the constant  $A_n$ . The procedure for determining  $A_n$ , of course, requires the use of equation (4.7) and the boundary condition equation (4.20).

Expanding the applied load on the surface of the cylinder as follows:

$$q(\alpha, t) = \sum_{n=1} Q_n \sin \frac{n\pi\alpha}{L} \propto e^{i\omega t} ; \quad (4.39)$$

and utilizing the boundary condition (4.20) to develop an expression for  $w(\alpha, t)$  yields for a given  $n$  value

$$w(\alpha, t) = R'(a) G(\alpha) T(t) . \quad (4.40)$$

Substitution of equation (4.39) and equation (4.40) and its derivatives into equation (4.7) produces

$$\begin{aligned}
& R'(a) G(\alpha) T''(t) + \epsilon^2 [R'(a) G^{IV}(\alpha) T(t)] \\
& + \epsilon^2 \frac{Eha^2}{D} [R'(a) G(\alpha) T(t)] = \frac{1}{m_s} \sum_{n=1} Q_n \sin \frac{n\pi a}{L} \alpha e^{i\omega t} \\
& + \frac{\rho_o}{m_s} R(a) G(\alpha) T''(t) .
\end{aligned} \tag{4.41}$$

Rearranging, noting that

$$G^{IV}(\alpha) = G_n^{IV}(\alpha) = A_n \left(\frac{n\pi a}{L}\right)^4 \sin \frac{n\pi a}{L} \alpha = \left(\frac{n\pi a}{L}\right)^4 G_n(\alpha) ,$$

and summing gives

$$\begin{aligned}
& \sum_{n=1} [R'(a) - \frac{\rho_o}{m_s} R(a)] G_n(\alpha) T''(t) + \epsilon^2 \sum_{n=1} \left[\left(\frac{n\pi a}{L}\right)^4 \right. \\
& \left. + \frac{Eha^2}{D}\right] G_n(\alpha) R'(a) T(t) = \frac{1}{m_s} \sum_{n=1} Q_n \sin \frac{n\pi a}{L} \alpha e^{i\omega t} .
\end{aligned} \tag{4.42}$$

Combining this result with equation (4.34) and dropping the common summation sign yields

$$\left\{ -[R'(a) - \frac{\rho_o}{m_s} R(a)] \omega^2 + \epsilon^2 \left[ \left(\frac{n\pi a}{L}\right)^4 + \frac{Eha^2}{D} \right] R'(a) \right\} A_n = \frac{Q_n}{m_s} . \tag{4.43}$$

For the case  $k_r > 0$  and  $A_n$  a real constant,  $R(r) = H_o^{(1)}(k_r r)$  and

$$R'(r) = \frac{d}{dr} H_o^{(1)}(k_r r) = -k_r H_1^{(1)}(k_r r) . \tag{4.44}$$

Hence equation (4.43) can be written

$$\begin{aligned} & \{ [-k_r H_1^{(1)}(k_r a) - \frac{\rho_o}{m_s} H_o^{(1)}(k_r a)] \omega^2 + \epsilon^2 \left[ \left( \frac{n\pi a}{L} \right)^4 \right. \\ & \left. + \frac{Eha^2}{D} \right] (-k_r H_1^{(1)}(k_r a)) \} A_n = \frac{Q_n}{m_s} . \end{aligned} \quad (4.45)$$

However  $H_m^{(1)}(x) = J_m(x) + i Y_m(x)$  consequently substitution of this expression into equation (4.45) yields a real and an imaginary equation as,

$$\begin{aligned} & \{ [k_r J_1(k_r a) + \frac{\rho_o}{m_s} J_o(k_r a)] \omega^2 - \epsilon^2 \left[ \left( \frac{n\pi a}{L} \right)^4 \right. \\ & \left. + \frac{Eha^2}{D} \right] k_r J_1(k_r a) \} A_n = \frac{Q_n}{m_s} , \end{aligned} \quad (4.46)$$

and

$$\begin{aligned} & \{ [k_r Y_1(k_r a) + \frac{\rho_o}{m_s} Y_o(k_r a)] \omega^2 - \epsilon^2 \left[ \left( \frac{n\pi a}{L} \right)^4 \right. \\ & \left. + \frac{Eha^2}{D} \right] k_r Y_1(k_r a) \} A_n = 0 . \end{aligned} \quad (4.47)$$

Equation (4.46) gives the desired relationship for  $A_n$  while equation (4.47) provides information about the natural frequencies of the immersed shell. Equation (4.46) yields an expression for  $A_n$  given by,

$$A_n = \frac{Q_n}{m_s \{ [k_r J_1(k_r a) + \frac{\rho_o}{m_s} J_o(k_r a)] \omega^2 - \chi_n^2 k_r J_1(k_r a) \}} , \quad (4.48)$$

where

$$\chi_n^2 = \epsilon^2 \left[ \left( \frac{n\pi a}{L} \right)^4 + \frac{Eha^2}{D} \right] . \quad (4.49)$$

Defining

$$\nu_n^2 = \frac{\chi_n^2}{1 + \frac{\rho_o}{m_s} \frac{J_0^2(k_r a)}{k_r J_1(k_r a)}} \quad (4.50)$$

and rewriting equation (4.48) produces

$$A_n = \frac{Q_n \nu_n^2}{m_s k_r J_1(k_r a) \chi_n^2 (\omega^2 - \nu_n^2)} . \quad (4.51)$$

Consequently, the velocity potential can be written as

$$\phi(r, \alpha, t) = \frac{i\omega}{m_s} \sum_{n=1}^{\infty} \frac{Q_n \nu_n^2}{\chi_n^2 (\omega^2 - \nu_n^2)} \sin \frac{n\pi a}{L} \propto \frac{H_0^{(1)}(k_r r) e^{i\omega t}}{k_r J_1(k_r a)} \quad (4.52)$$

for the case of  $k_r > 0$ .

If  $k_r$  is imaginary such that  $\bar{k}_r = ik_r > 0$ , the real constant  $A_n$  can be determined in the same manner as for  $k_r$  real. A detailed determination shows that for this case,

$$\phi(r, \alpha, t) = 0 . \quad (4.53)$$

At this point, the equations for the velocity potential will be written in terms of a non-dimensional series. The purpose for doing so being that numerical evaluation of the resulting expressions for the radiation resistance is desirable and worthwhile. Hence, the series will be expressed in terms of the dimensionless parameters:

$$\eta = \frac{\omega a}{c_o}, \quad \xi = \frac{v_n a}{c_o}, \quad \text{and} \quad x_r = k_r a. \quad (4.54)$$

Introduction, as well, of the notation

$$S_n(\alpha) = \frac{\frac{Q_n}{Q_o} \left(\frac{v_n}{x_n}\right)^2 \sin \frac{n\pi a}{L} \alpha}{\left[\left(\frac{\omega a}{c_o}\right)^2 - \left(\frac{v_n a}{c_o}\right)^2\right] k_r a J_1(k_r a)}, \quad (4.55)$$

or

$$S_n(\alpha) = \frac{\frac{Q_n}{Q_o} \left(\frac{v_n}{x_n}\right)^2 \sin \frac{n\pi a}{L} \alpha}{[\eta^2 - \xi^2] x_r J_1(x_r)},$$

yields the following dimensionless series for the velocity potential,

$$\phi(r, \alpha, t) = \frac{i\omega Q_o}{m_s} \left(\frac{a^3}{c_o^2}\right) \sum_{n=1} S_n(\alpha) H_o^{(1)}(k_r r) e^{i\omega t}, \quad (4.56)$$

for  $k_r > 0$  and  $\phi = 0$  for  $\bar{k}_r = ik_r > 0$ .

The velocity potential has thus been determined. The final objective of calculating the radiation resistance can now be undertaken. This task can be approached in at least two different ways. In the first instance the pressure acting on the vibrating surface can be evaluated; the velocity distribution of the vibrating surface, ascertained; and from these two results the radiation impedance,  $Z_r$ , determined. The radiation resistance will, of course, be the real portion of  $Z_r$ . Alternatively the acoustic pressure in the surrounding

medium can be determined and that pressure employed to find the radiated power,  $W_r$ . Next, utilizing the relationship between the radiated power and the mean square surface velocity of the vibrating surface, the radiation resistance can be determined by calculating the mean square surface velocity.

The results of the first technique is an instantaneous, local radiation resistance, while the second technique yields an average value for the radiation resistance both in time and spacial coordinates. The average value is the most useful from an engineering point of view; hence, the second approach will be employed in this research.

At large distances from the surface of the cylindrical shell, the acoustic pressure will be determined by noting that  $p = \rho_o \frac{\partial \phi}{\partial t}$  and that the Hankel function can be asymptotically represented in terms of an exponential function as,

$$H_o^{(1)}(k_r r) = \sqrt{\frac{2}{\pi k_r r}} e^{i(k_r r - \frac{\pi}{4})} \quad (4.57)$$

The acoustic pressure in the far-field is thus

$$p = - \frac{\omega^2 \rho_o Q_o}{m_s} \left( \frac{a^3}{c_o^2} \right) \sum_{n=1} S_n(\alpha) \sqrt{\frac{2}{\pi k_r r}} e^{i(\omega t + k_r r - \frac{\pi}{4})} \quad (4.58)$$

The radial fluid particle velocity is obtained by noting that at large distances from the surface of the shell, the cylindrical wave front behavior approaches that of a plane wave front in any small increment of polar angle. Consequently the plane wave relationship between  $p$  and  $u_r$  will be employed to generate



$$u_r = \frac{p}{\rho_o c_o} = - \frac{\omega^2 Q_o}{m_s} \left(\frac{a}{c_o}\right)^3 \sum_{n=1} S_n(\alpha) \sqrt{\frac{2}{\pi k_r r}} e^{i(\omega t + k_r r - \frac{\pi}{4})} . \quad (4.59)$$

The product of the real part of the acoustic pressure and the real part of the fluid velocity averaged over time is the intensity or radiated power per unit of acoustic field area. Then

$$\begin{aligned} \text{Re}\{p\}\text{Re}\{u_r\} &= \frac{\rho_o \omega^4 Q_o^2}{m_s^2} \left(\frac{a}{c_o}\right)^6 \\ &\times \sum_{n=1} S_n(\alpha) \sqrt{\frac{2}{\pi k_r r}} \cos(\omega t + k_r r - \frac{\pi}{4}) \\ &\times \sum_{m=1} S_m(\alpha) \sqrt{\frac{2}{\pi k_r r}} \cos(\omega t + k_r r - \frac{\pi}{4}) . \end{aligned} \quad (4.60)$$

But

$$\cos(\omega t + k_r r - \frac{\pi}{4}) = \cos \omega t \cos(k_r r - \frac{\pi}{4})$$

$$- \sin \omega t \sin(k_r r - \frac{\pi}{4}) = D_1 \cos \omega t - D_2 \sin \omega t ,$$

so

$$\begin{aligned}
\text{Re}\{p\}\text{Re}\{u_r\} &= \frac{\rho_o \omega^4 Q_o^2}{2m_s^2} \left(\frac{a}{c_o}\right)^6 \left\{ \sum_{n=1} S_n(\alpha) \sqrt{\frac{2}{\pi k_r r}} D_1 \sum_{m=1} S_m(\alpha) \sqrt{\frac{2}{\pi k_r r}} D_1 \right. \\
&\quad \times \cos^2 \omega t - \sum_{n=1} S_n(\alpha) \sqrt{\frac{2}{\pi k_r r}} D_1 \sum_{m=1} S_m(\alpha) \sqrt{\frac{2}{\pi k_r r}} D_2 \cos \omega t \sin \omega t \\
&\quad - \sum_{n=1} S_n(\alpha) \sqrt{\frac{2}{\pi k_r r}} D_2 \sum_{m=1} S_m(\alpha) \sqrt{\frac{2}{\pi k_r r}} D_1 \sin \omega t \cos \omega t \\
&\quad \left. + \sum_{n=1} S_n(\alpha) \sqrt{\frac{2}{\pi k_r r}} D_2 \sum_{m=1} S_m(\alpha) \sqrt{\frac{2}{\pi k_r r}} D_2 \sin^2 \omega t \right\} . \quad (4.61)
\end{aligned}$$

Averaging equation (4.61) over a number of full cycles of excitation gives the intensity

$$I_\alpha = \frac{1}{T} \int_0^T \text{Re}\{p\}\text{Re}\{u_r\} dt , \quad (4.62)$$

or

$$I_\alpha = \frac{\rho_o \omega^4 Q_o^2}{2m_s^2} \left(\frac{a}{c_o}\right)^6 \left\{ \sum_{n=1} S_n(\alpha) \sqrt{\frac{2}{\pi k_r r}} \sum_{m=1} S_m(\alpha) \sqrt{\frac{2}{\pi k_r r}} (D_1^2 + D_2^2) \right\} .$$

But  $D_1^2 + D_2^2 = \cos^2(k_r r - \frac{\pi}{4}) + \sin^2(k_r r - \frac{\pi}{4}) = 1$  , hence

$$I_\alpha = \frac{\rho_o \omega^4 Q_o^2}{2m_s^2} \left(\frac{a}{c_o}\right)^6 \sum_{n=1} S_n(\alpha) \sqrt{\frac{2}{\pi k_r r}} \sum_{m=1} S_m(\alpha) \sqrt{\frac{2}{\pi k_r r}} . \quad (4.63)$$

The total power radiated will now be computed as

$$W_r = \int_0^{\frac{L}{a}} 2\pi r \cdot I_\alpha \cdot d\alpha .$$

Therefore substitution of equation (4.63) into the above expression yields

$$W_r = \frac{\pi r \rho_o \omega^4 Q_o^2}{m_s^2} \left( \frac{a}{c_o} \right)^6 \int_0^{\frac{L}{a}} \left\{ \sum_{n=1} L_n \sin \frac{n\pi a}{L} \alpha \right. \\ \left. \times \sum_{m=1} L_m \sin \frac{m\pi a}{L} \alpha \right\} d\alpha \quad (4.64)$$

where

$$L_n = \frac{\frac{Q_n}{Q_o} \left( \frac{v_n}{x_n} \right)^2 \sqrt{\frac{2}{\pi k_r r}}}{[\eta^2 - \xi^2] x_r J_1(x_r)} \quad (4.65)$$

Two types of terms result when the series in equation (4.64) are multiplied; however, only one of these terms has a non-zero integral due to the orthogonality of the sine function. Thus

$$\int_0^{\frac{L}{a}} L_m^2 \sin^2 \frac{m\pi a}{L} \alpha d\alpha = \frac{L_m^2 L}{2a}$$

for  $n = m$  while for  $n \neq m$

$$\int_0^{\frac{L}{a}} L_n \sin \frac{n\pi a}{L} \alpha L_m \sin \frac{m\pi a}{L} \alpha d\alpha = 0.$$

Hence,

$$W_r = \frac{\pi r \rho_o \omega^4 Q_o^2}{m_s^2} \left( \frac{a}{c_o} \right)^6 \sum_{n=1} \frac{L_n^2 L}{2a},$$

or

$$W_r = \frac{\rho_o Q_o^2 a^2 L}{2 m_s c_o} \left( \frac{\omega a}{c_o} \right)^4 \sum_{m=1} \left\{ \frac{\frac{Q_m}{Q_o} \left( \frac{v_m}{\chi_m} \right)^2}{x_r^{\frac{1}{2}} (\eta - \xi^2)} \right\}^2 \left[ \frac{1}{x_r J_1(x_r)} \right]^2 . \quad (4.66)$$

The mean square surface velocity of the vibrating surface is now required, and will be calculated through the relation,

$$u_r \Big|_{r=a} = - \frac{\partial \phi}{\partial r} \Big|_{r=a} = u_a . \quad (4.67)$$

Thus

$$u_a = - \frac{i \omega Q_o}{m_s} \left( \frac{a^3}{c_o^2} \right) \sum_{n=1} S_n(\alpha) \frac{dH_o^{(1)}}{dr} (k_r r) \Big|_{r=a} e^{i \omega t} ,$$

or

$$u_a = \frac{i \omega Q_o}{m_s} \left( \frac{a^3}{c_o^2} \right) \sum_{n=1} S_n(\alpha) k_r H_1^{(1)}(k_r a) e^{i \omega t} , \quad (4.68)$$

where  $u_a$  is the instantaneous velocity on the vibrating surface and is a function of both time,  $t$ , and position,  $\alpha$ . The real part of this quantity will now be squared to yield  $U_a^2$ , a quantity which is to be averaged over time and space to produce the mean square surface velocity,  $U^2$ .

$$\begin{aligned}
U_a^2 = [\text{Re}(u_a)]^2 &= \frac{\omega^2 Q_o^2}{m_s^2} \left(\frac{a^6}{c_o^4}\right) \left\{ \sum_{n=1} S_n(\alpha) k_{rY_1} \sum_{m=1} S_m(\alpha) k_{rY_1} \cos^2 \omega t \right. \\
&+ \sum_{n=1} S_n(\alpha) k_{rY_1} \sum_{m=1} S_m(\alpha) k_{rJ_1} \cos \omega t \sin \omega t \\
&+ \sum_{n=1} S_n(\alpha) k_{rJ_1} \sum_{m=1} S_m(\alpha) k_{rY_1} \sin \omega t \cos \omega t \\
&\left. + \sum_{n=1} S_n(\alpha) k_{rJ_1} \sum_{m=1} S_m(\alpha) k_{rJ_1} \sin^2 \omega t \right\} , \quad (4.69)
\end{aligned}$$

Averaging  $U_a^2$  over time,  $t$ , generates

$$U_\alpha^2 = \frac{1}{T} \int_0^T U_a^2 dt ,$$

or

$$\begin{aligned}
U_\alpha^2 &= \frac{\omega^2 Q_o^2}{2m_s^2} \left(\frac{a^6}{c_o^4}\right) \left\{ \sum_{n=1} S_n(\alpha) k_{rY_1} \sum_{m=1} S_m(\alpha) k_{rY_1} \right. \\
&\left. + \sum_{n=1} S_n(\alpha) k_{rJ_1} \sum_{m=1} S_m(\alpha) k_{rJ_1} \right\} . \quad (4.70)
\end{aligned}$$

The mean square surface velocity,  $U^2$ , evolves after averaging with respect to variation in  $\alpha$ ; i.e.,

$$\begin{aligned}
U^2 &= \frac{\omega^2 Q_o^2}{2m_s^2} \left(\frac{a^6}{c_o^4}\right) \frac{a}{L} \int_0^{\frac{L}{a}} \left\{ \sum_{n=1} S_n(\alpha) k_{rY_1} \sum_{m=1} S_m(\alpha) k_{rY_1} \right. \\
&\left. + \sum_{n=1} S_n(\alpha) k_{rJ_1} \sum_{m=1} S_m(\alpha) k_{rJ_1} \right\} d\alpha . \quad (4.71)
\end{aligned}$$

Again, as in the case of the expression for power radiated, the

integration of a product of series gives the results as two characteristic types of terms. And because of the orthogonality of the sine function appearing in the quantity,  $S_n(\alpha)$ , one of these terms is always zero, while the other terms produce the result

$$U^2 = \left( \frac{Q_o a}{2m_s c_o} \right)^2 \left( \frac{\omega a}{c_o} \right)^2 \sum_{m=1} \left[ \frac{Q_m}{Q_o} \left( \frac{\nu_m}{X_m} \right)^2 \right]^2 \left[ 1 + \frac{Y_1^2(x_r)}{J_1^2(x_r)} \right]. \quad (4.72)$$

The radiation resistance per characteristic length of the shell is thus

$$R_{\text{rad}} = 2W_r/U^2 \quad \text{or}$$

$$R_{\text{rad}} = 4\rho_o c_o L \left( \frac{\omega a}{c_o} \right)^2 \frac{\sum_{m=1} \left[ \frac{Q_m/Q_o}{\eta^2 - \xi^2} \left( \frac{\nu_m}{X_m} \right)^2 \right]^2 \left[ \frac{1}{x_r^3 J_1^2(x_r)} \right]}{\sum_{m=1} \left[ \frac{Q_m/Q_o}{\eta^2 - \xi^2} \left( \frac{\nu_m}{X_m} \right)^2 \right]^2 \left[ 1 + \frac{Y_1^2(x_r)}{J_1^2(x_r)} \right]}. \quad (4.73)$$

Equation (4.73) represents the desired result for  $k_r > 0$  since for  $\bar{k}_r > 0$ ,  $R_{\text{rad}} = 0$ . This result is expressed in terms of a dimensionless series and is to be numerically evaluated for realistic values of the dimensionless parameters which determine its magnitude.

This completes the analytical development for axisymmetric mode shapes. The next section deals with the analytical development of the solution for the lobar mode shapes.

#### 4.4 Solution for Lobar Mode Shapes

Descriptive equations were developed in sections (4.1) and (4.2) for both the cylindrical shell and the unbounded acoustic medium. This section explains the boundary conditions employed and the solution obtained for the case of a lobar mode shape (i.e., a shape which can be described in terms of the polar angle,  $\theta$ , only).

As in the axisymmetric case, two boundary conditions are required for the solution of equation (4.12) and equation (4.18). One applies at the shell-fluid interface where  $r = a$  and the other, at large distances from the surface of the shell as  $r$  approaches infinity. At  $r = a$ , the condition means that the radial velocity of the shell surface is equal to the radial velocity of the fluid particles in contact with the shell; while the other boundary condition, applicable at large distances from the surface of shell, requires that the solution to the wave equation represent an outgoing rather than an incoming wave motion. Acoustically speaking, this boundary condition corresponds to a free or an anechoic simulation of a free field.

At the shell-fluid interface, the mathematical statement of the boundary condition is

$$\frac{\partial w}{\partial t}(\theta, t) = \frac{\partial \phi}{\partial r}(r, \theta, t) \Big|_{r=a}; \quad (4.74)$$

and at large distances from the surface of the shell,

$$\lim_{r \rightarrow \infty} \sqrt{r} \left[ \frac{\partial \phi}{\partial r} - i \frac{\omega}{c_0} \phi \right] = 0, \quad (4.75)$$

where  $\omega/c_0$  is the quantity analogous to  $k_r$  in the boundary condition for the axisymmetric case.

The solution of equations (4.12) and (4.18) subject to the boundary conditions (4.74) and (4.75) is accomplished by use of the method of separation of variables; i.e., let

$$\phi(r, \theta, t) = R(r) \Theta(\theta) T'(t) \quad (4.76)$$

Substituting equation (4.76) into equation (4.18), assuming that  $\phi \neq 0$ , and dividing by the expression for  $\phi$  yields

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} = \frac{1}{c_o^2} \frac{T'''}{T'} \quad (4.77)$$

But recalling that the quantity,  $T$ , was simply an exponential in time,  $t$ , in the previously discussed axisymmetric case, that analogy can be employed now and equation (4.77) can be easily rewritten as

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} = - \left( \frac{\omega}{c_o} \right)^2 \quad (4.78)$$

This equation can be separated into two ordinary differential equations in the variables  $r$  and  $\theta$  as follows:

$$r^2 \frac{R''}{R} + r \frac{R'}{R} + r^2 \left( \frac{\omega}{c_o} \right)^2 = m^2, \quad (4.79)$$

and

$$\frac{\Theta''}{\Theta} = -m^2 \quad (4.80)$$

Equation (4.79) can be rewritten in a more conventional form as

$$R'' + \frac{1}{r} R' + \left[ \left( \frac{\omega}{c_o} \right)^2 - \left( \frac{m}{r} \right)^2 \right] R = 0 \quad (4.81)$$

which is readily recognizable as a Bessel differential equation. For



$\omega/c_0 > 0$ , the solution to equation (4.81) is

$$R(r) = C_m^{(1)} H_m^{(1)} \left( \frac{\omega r}{c_0} \right) + C_m^{(2)} H_m^{(2)} \left( \frac{\omega r}{c_0} \right), \quad (4.82)$$

where  $H_m^{(1)}$  and  $H_m^{(2)}$  are Hankel functions of the first and second kind of order  $m$  respectively. This solution differs from the solution in the previous section on axisymmetric modes in that the order of the Hankel functions is now  $m$  compared to the order of one in the previous case. And because  $\omega/c_0$  is always real, equation (4.82) represents the only solution to equation (4.81).

Rearrangement gives equation (4.80) the form

$$\Theta'' + m^2 \Theta = 0. \quad (4.83)$$

The solution to this equation is a sinusoidally varying function in terms of  $\Theta$ ; hence, the motion of the shell is periodic in the  $\Theta$ -direction according to  $A_m \cos m\Theta$  yielding

$$\Theta(\Theta) = \Theta_m(\Theta) = A_m \cos m\Theta. \quad (4.84)$$

Knowledge of  $R$  and  $\Theta$ , of course, gives information about the velocity potential,  $\phi$ . Applying the radiation condition verifies which term of equation (4.82) corresponds to an outgoing wave and simplifies the expression for  $\phi$  as a result. The radiation condition (equation (4.75)) can be written as

$$\lim_{r \rightarrow \infty} \sqrt{r} [R'(r) - i \frac{\omega}{c_0} R(r)] = 0. \quad (4.85)$$

Substitution of equation (4.82) into equation (4.85) shows that the desired solution is (Appendix 10.2 gives the details of this

procedure for the axisymmetric case)

$$R(r) = C_m^{(1)} H_m^{(1)} \left( \frac{\omega r}{c_0} \right) \quad (4.86)$$

where the constant  $C_m^{(1)}$  will be set equal to unity in subsequent expressions for  $\phi$  leaving  $A_m$  as the single undetermined constant. The velocity potential, in fact, is

$$\phi(r, \theta, t) = i\omega \sum_{m=0} A_m \cos m\theta R(r) e^{i\omega t} \quad (4.87)$$

The procedure for determining  $A_m$ , of course, requires the use of equation (4.12) and the boundary condition equation (4.74).

The boundary condition (4.74) is utilized to express the shell deflection,  $w(\theta, t)$ , in terms of the known velocity potential,  $\phi$ , for any single  $m$  value

$$w(\theta, t) = R'(a) \Theta(\theta) T(t) \quad (4.88)$$

The derivatives of  $w$  are also known in terms of the expression for  $\phi$ ; hence, by expanding the applied load on the surface of the cylinder as

$$q(\theta, t) = \sum_{m=0} Q_m \cos m\theta e^{i\omega t}, \quad (4.89)$$

the equation of motion of the shell becomes

$$\begin{aligned} & -\omega^2 R'(a) \Theta(\theta) e^{i\omega t} + \frac{D}{m_s a^4} R'(a) \Theta^{IV}(\theta) e^{i\omega t} \\ & = \frac{1}{m_s} \sum_{m=0} Q_m \cos m\theta e^{i\omega t} - \frac{\rho_0 \omega^2}{m_s} R(a) \Theta(\theta) e^{i\omega t}. \end{aligned} \quad (4.90)$$

Rewriting, observing that

$$\Theta^{IV}(\Theta) = m^4 \Theta(\Theta) , \quad (4.91)$$

and summing gives

$$\begin{aligned} & -\omega^2 \sum_{m=0} A_m R'(a) \cos m\Theta e^{i\omega t} + \frac{D}{m_s a^4} \sum_{m=0} m^4 A_m R'(a) \cos m\Theta e^{i\omega t} \\ & = \frac{1}{m_s} \sum_{m=0} Q_m \cos m\Theta e^{i\omega t} - \frac{\omega^2 \rho_o}{m_s} \sum_{m=0} A_m R(a) \cos m\Theta e^{i\omega t} . \end{aligned} \quad (4.92)$$

Dropping the common summation sign on the above result yields

$$\left[ -\omega^2 + \frac{Dm^4}{m_s a^4} \right] A_m R'(a) + \frac{\omega^2 \rho_o}{m_s} A_m R(a) = \frac{Q_m}{m_s} . \quad (4.93)$$

But for  $\omega/c_o > 0$  , and  $A_m$  a real constant,

$$R(r) = H_m^{(1)} \left( \frac{\omega r}{c_o} \right) ,$$

and

$$R'(r) = \frac{d}{dr} H_m^{(1)} \left( \frac{\omega r}{c_o} \right) = - \frac{\omega}{c_o} H_{m+1}^{(1)} \left( \frac{\omega r}{c_o} \right) + \frac{m}{r} H_m^{(1)} \left( \frac{\omega r}{c_o} \right) . \quad (4.94)$$

Consequently equation (4.93) can be written

$$\left[ -\omega^2 + \frac{Dm^4}{m_s a^4} \right] A_m \frac{d}{dr} H_m^{(1)} \left( \frac{\omega a}{c_o} \right) + \frac{\omega^2 \rho_o}{m_s} A_m H_m^{(1)} \left( \frac{\omega a}{c_o} \right) = \frac{Q_m}{m_s} . \quad (4.95)$$

Defining

$$\chi_m^2 = \frac{Dm^4}{m_s a^4} , \quad (4.96)$$

and breaking the Hankel function down into its real and imaginary parts-- $J_m$ , a Bessel function of the first kind of order  $m$ , and  $Y_m$ , a Bessel function of the second kind of order  $m$ , respectively--yields a real and an imaginary equation as

$$\begin{aligned} [-\omega^2 + \chi_m^2] \frac{A_m}{a} [mJ_m \left(\frac{\omega a}{c_o}\right) - \left(\frac{\omega a}{c_o}\right) J_{m+1} \left(\frac{\omega a}{c_o}\right)] \\ + \frac{\omega^2 \rho_o A_m}{m_s} J_m \left(\frac{\omega a}{c_o}\right) = \frac{Q_m}{m_s} , \end{aligned} \quad (4.97)$$

and

$$\begin{aligned} [-\omega^2 + \chi_m^2] \frac{A_m}{a} [mY_m \left(\frac{\omega a}{c_o}\right) - \left(\frac{\omega a}{c_o}\right) Y_{m+1} \left(\frac{\omega a}{c_o}\right)] \\ + \frac{\omega^2 \rho_o A_m}{m_s} Y_m \left(\frac{\omega a}{c_o}\right) = 0 . \end{aligned} \quad (4.98)$$

Equation (4.97) gives the desired relationship for  $A_m$  while equation (4.98) provides information about the natural frequencies of the immersed shell. Equation (4.97) yields an expression for  $A_m$  given by

$$A_m = \frac{Q_m}{m_s (mJ_m \left(\frac{\omega a}{c_o}\right) - \frac{\omega a}{c_o} J_{m+1} \left(\frac{\omega a}{c_o}\right))} \cdot \quad (4.99)$$

$$\left\{ \frac{\chi_m^2}{a} - \frac{\omega^2}{a} \left[ 1 - \frac{\rho_o a}{m_s} \left( \frac{J_m \left(\frac{\omega a}{c_o}\right)}{mJ_m \left(\frac{\omega a}{c_o}\right) - \frac{\omega a}{c_o} J_{m+1} \left(\frac{\omega a}{c_o}\right)} \right) \right] \right\}$$

Defining

$$v_m^2 = \frac{\chi_m^2}{1 - \frac{\rho_o a}{m_s} \left( \frac{J_m \left( \frac{\omega a}{c_o} \right)}{m J_m \left( \frac{\omega a}{c_o} \right) - \frac{\omega a}{c_o} J_{m+1} \left( \frac{\omega a}{c_o} \right)} \right)}, \quad (4.100)$$

and substituting this expression into the formula for  $A_m$  produces the result

$$A_m = \frac{Q_m v_m^2}{m_s \left( m J_m \left( \frac{\omega a}{c_o} \right) - \frac{\omega a}{c_o} J_{m+1} \left( \frac{\omega a}{c_o} \right) \right) \chi_m^2 (v_m^2 - \omega^2)}. \quad (4.101)$$

Consequently, the velocity potential can be written as

$$\phi(r, \theta, t) = \frac{i\omega}{m_s} \sum_{m=0}^{\infty} \frac{Q_m v_m^2 \cos m\theta H_m^{(1)} \left( \frac{\omega r}{c_o} \right) e^{i\omega t}}{\chi_m^2 (v_m^2 - \omega^2) [m J_m(\eta) - \eta J_{m+1}(\eta)]} \quad (4.102)$$

for the lobar mode shapes.

At this point, the equation for the velocity potential will be written in terms of a non-dimensional series. The purpose for doing so being that numerical evaluation of the resulting expressions for the radiation resistance is desirable and worthwhile. Hence, the series will be expressed in terms of the dimensionless parameters:

$$\eta = \frac{\omega a}{c_o} \quad \text{and} \quad \xi = \frac{v_m a}{c_o}. \quad (4.103)$$

Introduction, as well, of the notation

$$S_m(\theta) = \frac{\frac{Q_m}{Q_o} \left(\frac{v_m}{X_m}\right)^2 \cos m\theta}{\left[\left(\frac{v_m a}{c_o}\right)^2 - \left(\frac{\omega a}{c_o}\right)^2\right] [mJ_m(\eta) - \eta J_{m+1}(\eta)]}, \quad (4.104)$$

or

$$S_m(\theta) = \frac{\frac{Q_m}{Q_o} \left(\frac{v_m}{X_m}\right)^2 \cos m\theta}{[\xi^2 - \eta^2] [mJ_m(\eta) - \eta J_{m+1}(\eta)]},$$

yields the following dimensionless series for the velocity potential,  $\phi$ :

$$\phi(r, \theta, t) = \frac{i\omega Q_o}{m_s} \left(\frac{a}{2}\right)^3 \sum_{m=0}^{\infty} S_m(\theta) H_m^{(1)}\left(\frac{\omega r}{c_o}\right) e^{i\omega t}, \quad (4.105)$$

for lobar mode shapes and  $\omega/c_o > 0$ .

The velocity potential has thus been determined, and the final objective of calculating the radiation resistance can now be undertaken. As mentioned in section 4.3, the approach which is employed here requires the determination of the acoustic pressure in the surrounding medium and from it, the radiated power,  $W_r$ . The radiation resistance is then determined by calculating the mean square velocity of the surface of the vibrating shell and employing the relationship between the radiated power and the mean square surface velocity. The result is a time- and space-averaged radiation resistance.

In the far acoustic field of the vibrating cylindrical shell, the pressure can be ascertained by noting that  $p = \rho_o \frac{\partial \phi}{\partial t}$  and that the Hankel functions in the expression for  $\phi$  can be asymptotically

represented in terms of an exponential function as

$$H_m^{(1)}\left(\frac{\omega r}{c_o}\right) = \sqrt{\frac{2c_o}{\pi\omega r}} e^{ikr - \frac{1}{2}\pi i(m + \frac{1}{2})} \quad (4.106)$$

where  $k = \omega/c_o$ . The acoustic pressure in the far-field is thus

$$p = \frac{-\omega^2 \rho_o Q_o}{m_s} \left(\frac{a}{c_o}\right)^3 \sum_{m=0} S_m(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} e^{i[\omega t + kr - \frac{1}{2}\pi(m + \frac{1}{2})]} \quad (4.107)$$

The radial fluid particle velocity, which is a quantity needed to determine the radiated power, is obtained by noting that cylindrical wave front behavior approaches that of a plane wave front in the far-field over any small increment of polar angle. Consequently the plane wave relationship between  $p$  and  $u_r$  will be used to generate

$$u_r = \frac{p}{\rho_o c_o} = \frac{-\omega^2 Q_o}{m_s} \left(\frac{a}{c_o}\right)^3 \sum_{m=0} S_m(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} e^{i[\omega t + kr - \frac{1}{2}\pi(m + \frac{1}{2})]} \quad (4.108)$$

The product of the real part of the acoustic pressure and the real part of the fluid velocity averaged over time is the intensity or radiated power per unit of acoustic field area. Then

$$\begin{aligned} \text{Re}\{p\}\text{Re}\{u_r\} &= \frac{\rho_o \omega^4 Q_o^2}{m_s^2} \left(\frac{a}{c_o}\right)^6 \\ &\times \sum_{m=0} S_m(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} \cos [\omega t + kr - \frac{1}{2}\pi(m + \frac{1}{2})] \\ &\times \sum_{n=0} S_n(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} \cos [\omega t + kr - \frac{1}{2}\pi(n + \frac{1}{2})] \quad (4.109) \end{aligned}$$

But

$$\cos[\omega t + kr - \frac{1}{2}\pi(m + \frac{1}{2})] = \cos \omega t \cos[kr - \frac{1}{2}\pi(m + \frac{1}{2})] - \sin \omega t \sin[kr - \frac{1}{2}\pi(m + \frac{1}{2})] = B_m \cos \omega t - c_m \sin \omega t ,$$

so

$$\begin{aligned} \text{Re}\{p\}\text{Re}\{u_r\} &= \frac{\rho_o \omega^4 Q_o^2}{m_s^2} \left(\frac{a}{c_o}\right)^6 \left\{ \sum_{m=0} S_m(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} B_m \sum_{n=0} S_n(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} B_n \right. \\ &\quad \times \cos^2 \omega t - \sum_{m=0} S_m(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} B_m \sum_{n=0} S_n(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} c_n \cos \omega t \sin \omega t \\ &\quad - \sum_{m=0} S_m(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} c_m \sum_{n=0} S_n(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} B_n \sin \omega t \cos \omega t \\ &\quad \left. + \sum_{m=0} S_m(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} c_m \sum_{n=0} S_n(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} c_n \sin^2 \omega t \right\} . \quad (4.110) \end{aligned}$$

Averaging equation (4.110) over a number of full cycles of excitation gives the intensity

$$I_\theta = \frac{1}{T} \int_0^T \text{Re}\{p\}\text{Re}\{u_r\} dt , \quad (4.111)$$

or



$$I_{\theta} = \frac{\rho_o \omega^4 Q_o^2}{2m_s^2} \left(\frac{a}{c_o}\right)^6 \left\{ \sum_{m=0} S_m(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} B_m \sum_{n=0} S_n(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} B_n \right. \\ \left. + \sum_{m=0} S_m(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} c_m \sum_{n=0} S_n(\theta) \sqrt{\frac{2c_o}{\pi\omega r}} c_n \right\} \quad (4.112)$$

As indicated in Figure 4.1, the total power radiated per unit of cylinder length is obtained by multiplying  $I_{\theta}$  by an element of area on an enveloping constant coordinate surface perpendicular to the direction of energy propagation and by then integrating over that surface.

Hence

$$W_r = \int_0^{2\pi} l \cdot r d\theta \cdot I_{\theta} \quad .$$

Substituting (4.112) into the above equation yields

$$W_r = \frac{\rho_o \omega^3 Q_o^2}{2m_s^2} \left(\frac{a}{c_o}\right)^6 \int_0^{2\pi} \left\{ \sum_{m=0} S_m(\theta) B_m \sum_{n=0} S_n(\theta) B_n \right. \\ \left. + \sum_{m=0} S_m(\theta) c_m \sum_{n=0} S_n(\theta) c_n \right\} d\theta \quad (4.113)$$

The product of the two series results in two types of terms:

$$B_m B_n \int_0^{2\pi} \frac{Q_m/Q_o \left(\frac{v_m}{\chi_m}\right)^2 \cos m\theta \cdot Q_n/Q_o \left(\frac{v_n}{\chi_n}\right)^2 \cos n\theta d\theta}{[\xi^2 - \eta^2][mJ_m(\eta) - \eta J_{m+1}(\eta)][\xi^2 - \eta^2][nJ_n(\eta) - \eta J_{n+1}(\eta)]} \quad ,$$

when  $n \neq m$  and

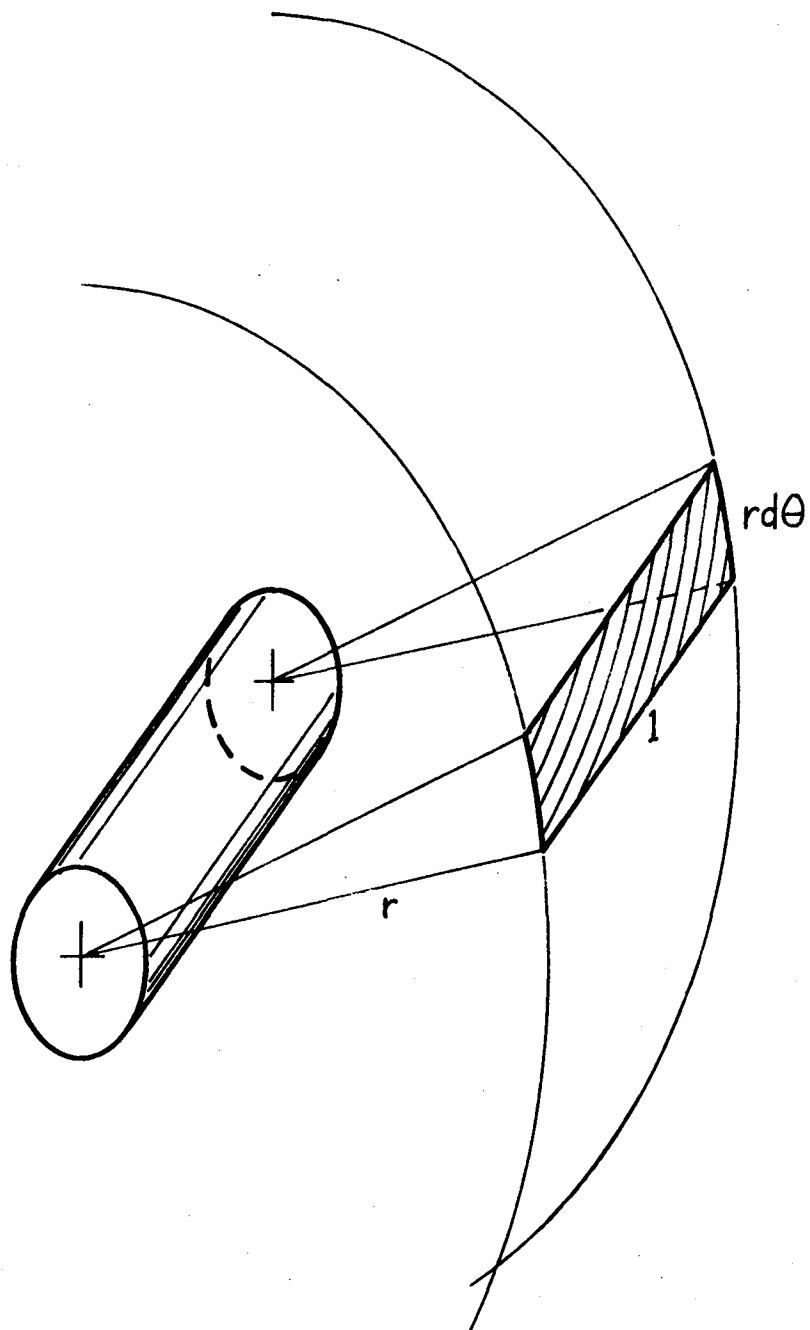


FIGURE 4.2 SCHEMATIC OF ACOUSTIC FIELD AREA  
 INTO WHICH VIBRATORY ENERGY IS  
 RADIATED BY THE CYLINDER

$$B_m^2 \int_0^{2\pi} \left[ \frac{Q_m/Q_0 \left(\frac{v_m}{\chi_m}\right)^2}{(\xi^2 - \eta^2) mJ_m(\eta) - \eta J_{m+1}(\eta)} \right]^2 \cos^2 m\theta d\theta$$

when  $n = m$ . From the orthogonality properties of the cosine function, the first type of term is equal to zero in all cases, while the second type of term is non-zero, so:

$$W_r = \frac{\rho_o \omega^3 Q_o^2}{m_s^2} \left(\frac{a}{c_o}\right)^6 \sum_{m=0} \left[ \frac{Q_m/Q_0 \left(\frac{v_m}{\chi_m}\right)^2}{(\xi^2 - \eta^2) (mJ_m(\eta) - \eta J_{m+1}(\eta))} \right]^2$$

$$\times [B_m^2 + C_m^2]$$

or utilizing equations expressing  $B_m$  and  $C_m$  in terms of sine and cosine as

$$B_m^2 + C_m^2 = \cos^2 \left[ kr - \frac{\pi}{2} (m + \frac{1}{2}) \right] + \sin^2 \left[ kr - \frac{\pi}{2} (m + \frac{1}{2}) \right] = 1 ;$$

hence,

$$W_r = \frac{\rho_o Q_o^2 a^3}{m_s^2 c_o} \left(\frac{a}{c_o}\right)^2 \sum_{m=0} \left[ \frac{Q_m/Q_0 \left(\frac{v_m}{\chi_m}\right)^2}{\xi^2 - \eta^2} \right]^2 \left[ \frac{1}{mJ_m(\eta) - \eta J_{m+1}(\eta)} \right]^2 .$$

(4.114)

The mean square surface velocity of the vibrating surface is now required, and will be calculated through the relation

$$u_r \Big|_{r=a} = - \frac{\partial \phi}{\partial r} \Big|_{r=a} = u_a .$$

(4.115)

Thus

$$u_a = \frac{-i\omega Q_o}{m_s} \left(\frac{a^3}{c_o^2}\right) \sum_{m=0} S_m(\theta) \frac{d}{dr} H_m^{(1)}\left(\frac{\omega r}{c_o}\right) \Big|_{r=a} e^{i\omega t},$$

or

$$u_a = \frac{i\omega Q_o}{m_s} \left(\frac{a^2}{c_o^2}\right) \sum_{m=0} S_m(\theta) [\bar{J}_m + i\bar{Y}_m] e^{i\omega t}, \quad (4.116)$$

where  $u_a$  is the instantaneous velocity on the vibrating surface and is a function of both time,  $t$ , and position,  $\theta$ . Note that  $\bar{J}_m$  and  $\bar{Y}_m$  are the real and imaginary parts of the derivative of  $H_m^{(1)}$  with respect to  $r$  respectively. The real part of this quantity,  $u_a$ , will now be squared to yield  $U_a^2$ , a quantity which is to be averaged over time and space to produce the mean square surface velocity,  $U^2$ .

$$\begin{aligned} U_a^2 = [\text{Re}(u_a)]^2 &= \frac{\omega^2 Q_o^2}{m_s^2} \left(\frac{a^4}{c_o^4}\right) \left\{ \sum_{m=0} S_m(\theta) \bar{Y}_m \sum_{n=0} S_n(\theta) \bar{Y}_n \cos^2 \omega t \right. \\ &+ 2 \sum_{m=0} S_m(\theta) \bar{Y}_m \sum_{n=0} S_n(\theta) \bar{J}_n \cos \omega t \sin \omega t \\ &\left. + \sum_{m=0} S_m(\theta) \bar{J}_m \sum_{n=0} S_n(\theta) \bar{J}_n \sin^2 \omega t \right\}. \end{aligned} \quad (4.117)$$

Averaging  $U_a^2$  over time,  $t$ , generates

$$U_\theta^2 = \frac{1}{T} \int_0^T U_a^2 dt,$$

or

$$U_{\theta}^2 = \frac{\omega^2 Q_o^2}{2m_s^2} \left(\frac{a}{c_o}\right)^4 \left\{ \sum_{m=0} S_m(\theta) \bar{Y}_m \sum_{n=0} S_n(\theta) \bar{Y}_n + \sum_{m=0} S_m(\theta) \bar{J}_m \sum_{n=0} S_n(\theta) \bar{J}_n \right\} . \quad (4.118)$$

The mean square surface velocity,  $U^2$ , evolves after averaging with respect to variation in  $\theta$ ; i.e.,

$$U^2 = \frac{\omega^2 Q_o^2}{4\pi m_s^2} \left(\frac{a}{c_o}\right)^4 \int_0^{2\pi} \left\{ \sum_{m=0} S_m(\theta) \bar{Y}_m \sum_{n=0} S_n(\theta) \bar{Y}_n + \sum_{m=0} S_m(\theta) \bar{J}_m \sum_{n=0} S_n(\theta) \bar{J}_n \right\} d\theta . \quad (4.119)$$

Again, as in the case of the expression for power radiated, the integration of a product of series gives the results as two characteristic types of terms. And because of the orthogonality of the cosine function appearing in the quantity,  $S_m(\theta)$ , one of these terms is always zero, while the other terms produce the result

$$U^2 = \left(\frac{Q_o a}{2m_s c_o}\right)^2 \left(\frac{\omega a}{c_o}\right)^2 \sum_{m=0} \left[ \frac{Q_m/Q_o \left(\frac{v_m}{\chi_m}\right)^2}{\xi^2 - \eta^2} \right]^2 \times \left[ 1 + \frac{(mY_m(\eta) - \eta Y_{m+1}(\eta))^2}{(mJ_m(\eta) - \eta J_{m+1}(\eta))^2} \right] . \quad (4.120)$$

The radiation resistance per unit of length of the shell is thus

$$R_{\text{rad}} = 2W_r/U^2 \quad \text{or}$$

$$R_{\text{rad}} = 8\rho_o c_o a \left(\frac{\omega a}{c_o}\right) \frac{\sum_{m=0}^{\infty} \left[ \frac{Q_m/Q_o \left(\frac{\nu_m}{X_m}\right)^2}{\xi^2 - \eta^2} \right]^2 \left[ \frac{1}{mJ_m(\eta) - \eta J_{m+1}(\eta)} \right]^2}{\sum_{m=0}^{\infty} \left[ \frac{Q_m/Q_o \left(\frac{\nu_m}{X_m}\right)^2}{\xi^2 - \eta^2} \right]^2 \left[ 1 + \frac{(mY_m(\eta) - \eta Y_{m+1}(\eta))^2}{(mJ_m(\eta) - \eta J_{m+1}(\eta))^2} \right]} \quad (4.121)$$

Equation (4.121) is the desired result in terms of a dimensionless series and is to be numerically evaluated for realistic values of the dimensionless parameters which determine its magnitude.

This completes the analytical development for lobar mode shapes. The next chapter deals with the numerical evaluation of the analytical expressions obtained in Chapter 4.

## 5. NUMERICAL EVALUATION

The radiation resistance expressions developed in Chapter 4 were written in terms of dimensionless parameters to facilitate subsequent numerical evaluation of the results. In this chapter, the numerical evaluation scheme is presented and the results of numerical evaluation included in the form of graphs and tables.

### 5.1 Numerical Evaluation for Axisymmetric Mode Shapes

Numerical evaluation of the expression (4.73) for the radiation resistance in the case of axisymmetric mode shapes can be accomplished by noting that  $x_r$ ,  $\xi$ , and  $v_n/\chi_n$  can be written in terms of more basic parameters which are related to the geometry of the shell and the physical properties of the shell material and fluid.

Equation (4.73) is written in terms of the dimensionless parameters,  $\eta$ ,  $\xi$ , and  $x_r$ . It is desirable to consider  $\xi$  and  $x_r$  as dependent variables and write them in terms of  $\eta$ , the independent variable. This approach is physically meaningful due to the fact that  $\eta = (\omega a)/c_0$  where  $\omega$  is the as yet unspecified and therefore arbitrary forcing frequency of the applied surface load,  $q(\alpha, t)$ . Thus from equation (4.26) and equation (4.33)

$$k_r^2 = k^2 - \left(\frac{n\pi}{L}\right)^2, \quad (5.1)$$

or

$$(k_r a)^2 = (ka)^2 - \left(\frac{n\pi a}{L}\right)^2. \quad (5.2)$$

Consequently from equation (4.54) and the fact that  $k = \omega/c_0$ ,

$$x_r = [\eta^2 - (\frac{n\pi a}{L})^2]^{\frac{1}{2}} \quad (5.3)$$

The quantity,  $x_r$ , is thus expressed in terms of  $\eta$  and  $a/L$ , a shell geometry parameter, as desired. From equation (4.50), it is possible to write  $v_n/\chi_n$  indirectly in terms of  $\eta$  also. Obviously

$$\left(\frac{v_n}{\chi_n}\right)^2 = \frac{1}{1 + \frac{\rho_o}{m_s} \frac{J_o(k_r a)}{k_r J_1(k_r a)}} ,$$

but by noting that  $m_s$  is the mass per unit area of the shell surface and is, hence, equal to the shell material density,  $\rho_s$ , multiplied by the shell thickness,  $h$ ; the above expression can be written

$$\left(\frac{v_n}{\chi_n}\right)^2 = \frac{1}{1 + \left(\frac{\rho_o}{\rho_s}\right) \left(\frac{a}{L}\right) \left(\frac{h}{L}\right)^{-1} \frac{J_o(x_r)}{x_r J_1(x_r)}} \quad (5.4)$$

Equation (5.4) is expressed in terms of the desired quantities as well as a new geometric parameter,  $h/L$ , the ratio of shell thickness to characteristic length.

Considering equations (4.54), the parameter  $\xi$  can be written as

$$\xi^2 = \left(\frac{v_n a}{c_o}\right)^2 = \left(\frac{x_n a}{c_o}\right)^2 \left(\frac{v_n}{\chi_n}\right)^2 \quad (5.5)$$

Employing equation (4.49) and the information that

$$\epsilon^2 = \frac{D}{a^4 m_s} \quad (5.6)$$

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad (5.7)$$



and that the longitudinal plate velocity

$$c_L^2 = \frac{E}{\rho_s (1-\nu^2)} , \quad (5.8)$$

permits the development of the expression

$$\xi^2 = \left( \frac{\nu_n}{\chi_n} \right)^2 \left[ \left( \frac{c_L}{c_o} \right)^2 \left\{ \frac{(m\pi)^4}{12} \left( \frac{h}{L} \right)^2 \left( \frac{a}{L} \right)^2 + (1-\nu^2) \right\} \right] . \quad (5.9)$$

Consequently all quantities in equation (4.73) for the radiation resistance are now represented in terms of the forcing frequency parameter,  $\eta$ ; the shell geometry parameters,  $a/L$ , and  $h/L$ ; and the material properties,  $\rho_o/\rho_s$ ,  $c_L/c_o$ , and  $\nu$  with  $Q_m/Q_o = 1/n$ .

The numerical analysis, per se, is a parameter study of the problem in terms of the previously mentioned parameters: the ranges of values for the shell geometry parameters and for  $\eta$  are approximately

$$10^{-2} \leq \eta \leq 10^3 ,$$

$$10^{-3} \leq a/L \leq 1 ,$$

and

$$10^{-5} \leq h/L \leq 10^{-1}$$

In the case of shell material and fluid property parameters, the materials and fluids considered in this work are indicated in Table 5.1. The shell materials and fluids considered here are representative of the kind encountered in structure-fluid interaction environments and of the range of physical properties exhibited by such materials and fluids.

Table 5.1 Physical properties of typical shell materials and fluids

| Shell<br>Materials             | Young's<br>Modulus<br>(lbf/in. <sup>2</sup> ) | Density<br>(lbm/in. <sup>3</sup> ) | Poisson<br>Ratio |
|--------------------------------|-----------------------------------------------|------------------------------------|------------------|
| 7075 Aluminum                  | 10.4 (10 <sup>6</sup> )                       | .101                               | .333             |
| 316 and 317<br>Stainless Steel | 28 (10 <sup>6</sup> )                         | .288                               | .176             |
| Titanium<br>(Ti-7AL-4Mo)       | 16.8 (10 <sup>6</sup> )                       | .162                               | .292             |
| Beryllium                      | 43.0 (10 <sup>6</sup> )                       | .067                               | .030             |
| Magnesium                      | 6.5 (10 <sup>6</sup> )                        | .064                               | .35              |
| Acoustic<br>Fluid              | Velocity<br>Sound (ft/sec)                    | Density<br>(lbm/in. <sup>3</sup> ) |                  |
| Air                            | 1117                                          | 4.4276 (10 <sup>-5</sup> )         |                  |
| Water                          | 4859                                          | 3.605 (10 <sup>-2</sup> )          |                  |

The details of the computer program employed in the parameter study are given in Appendix 10.3.1. It should be noted that the series in equation (4.73) are, in fact, finite series because of the fact that  $k_r > 0$ . The program incorporates this information and terminates summation for any value of  $n$  greater than  $\eta/(\pi a/L)$ . The program also terminates summation in the case of convergence to a stable value for each series.

The numerical results are shown in Figures 5.1 through 5.5 as well as in Table 5.2. Figure 5.1 presents the dimensionless  $R_{rad}$  for small  $\eta$ : the acoustic medium is air. Because this result peaks at each resonance of the shell, an averaged or smoothed curve would be more useful in octave band analysis work. Hence the computer (see Appendix 10.3.2 for the program for averaging  $R_{rad}$ ) is employed to average the theoretical curve to obtain the averaged curve also shown in Figure 5.1. Figure 5.2 depicts the same information for an acoustic medium of water instead of air. Figure 5.3 is essentially a comparison of the results for small  $\eta$  for acoustic environments of either air or water. In Figure 5.4, the averaged radiation resistance for a shell in contact with water is indicated for a wide range of  $\eta$  values. Finally in Table 5.2, the radiation resistance of a cylinder in air is shown for various values of the geometry parameter,  $h/L$ .

Table 5.3 presents a comparison of the radiation resistance of a cylinder in air for various choices of cylinder material. The results are obviously only weakly dependent on material properties.

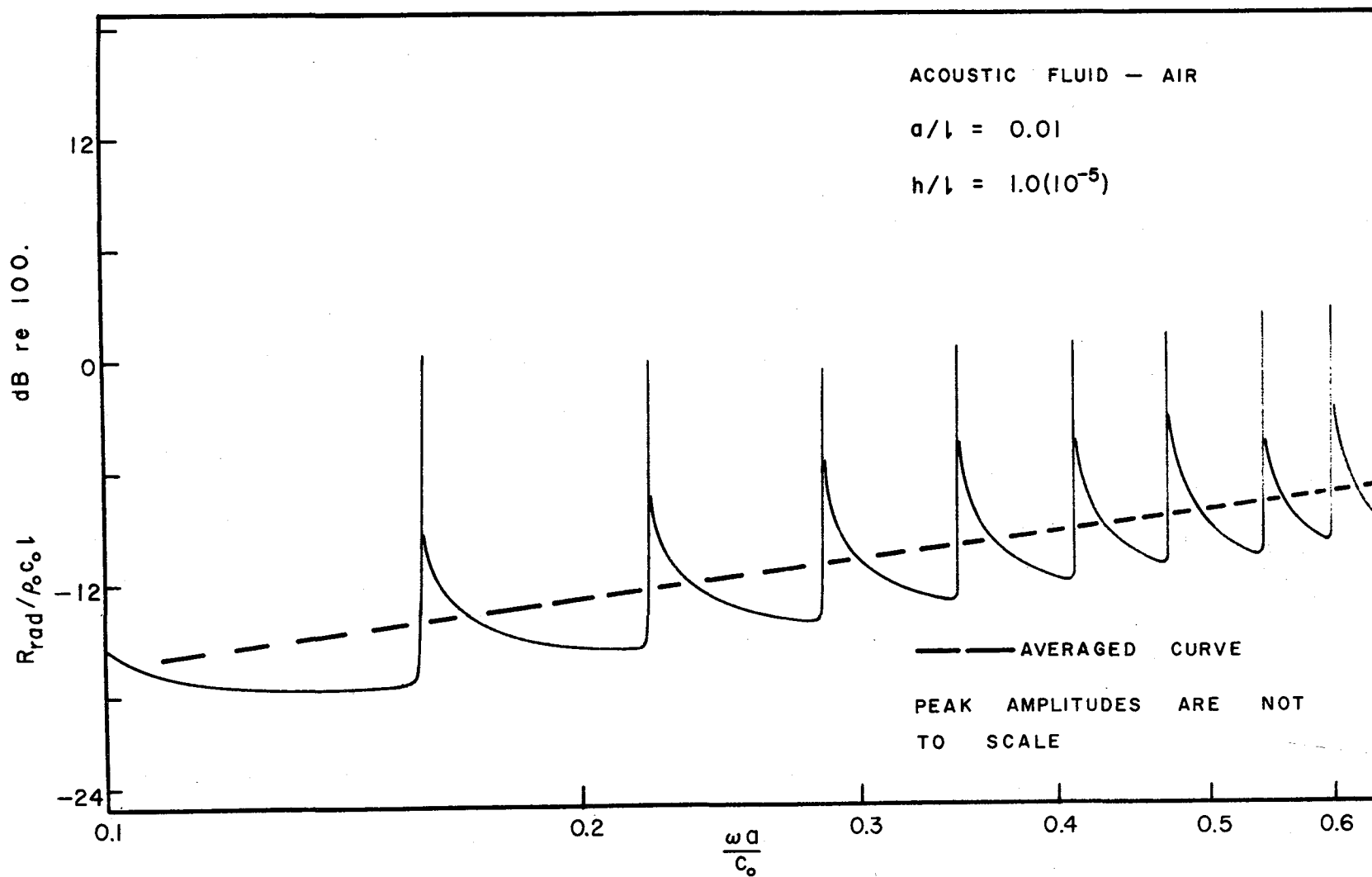


FIGURE 5.1 DIMENSIONLESS RADIATION RESISTANCE VS  $(\frac{\omega a}{c_0})$

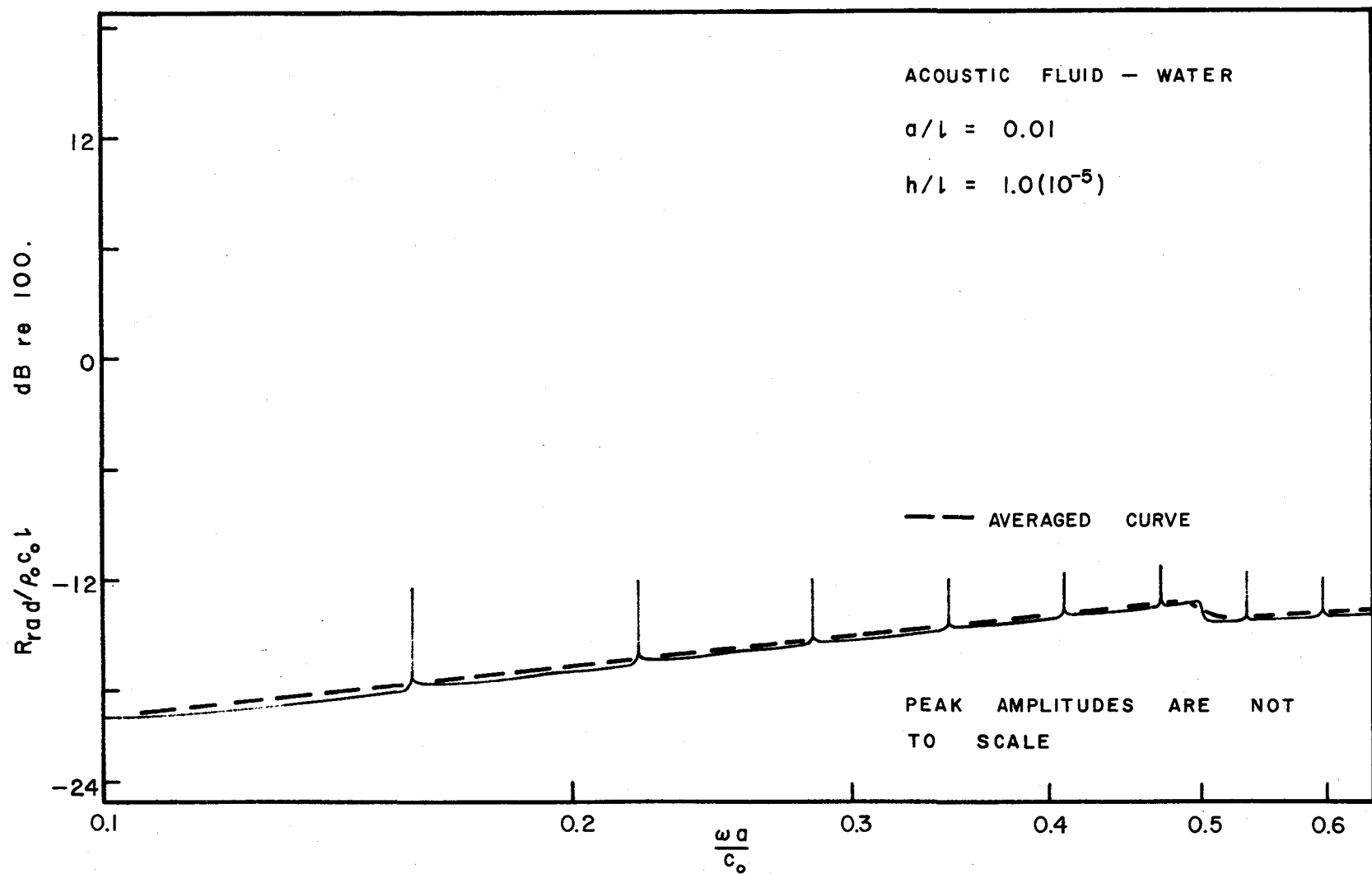


FIGURE 5.2

DIMENSIONLESS RADIATION RESISTANCE VS  $(\frac{\omega a}{c_0})$

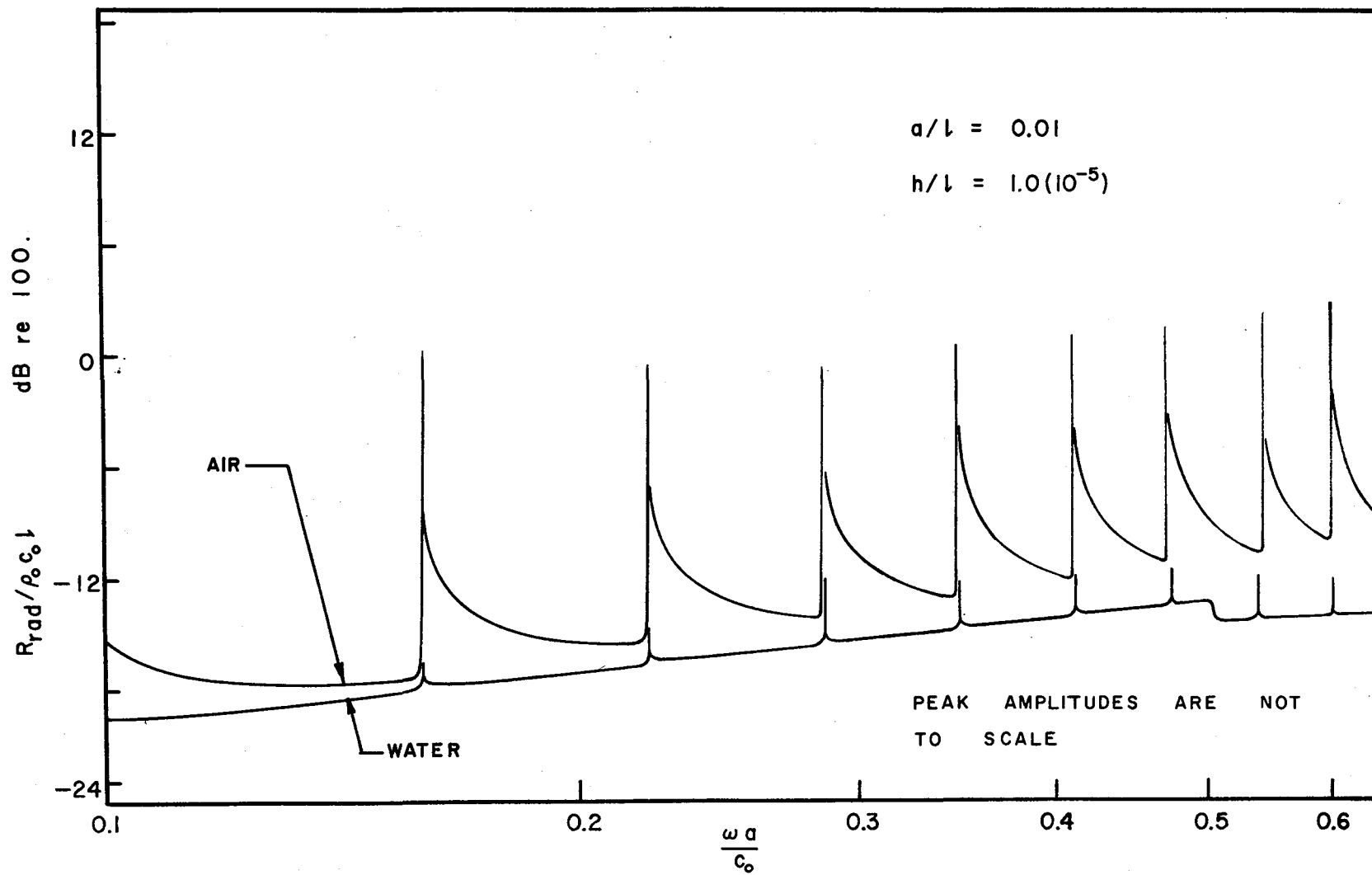


FIGURE 5.3 DIMENSIONLESS RADIATION RESISTANCE VS  $(\frac{\omega a}{c_0})$

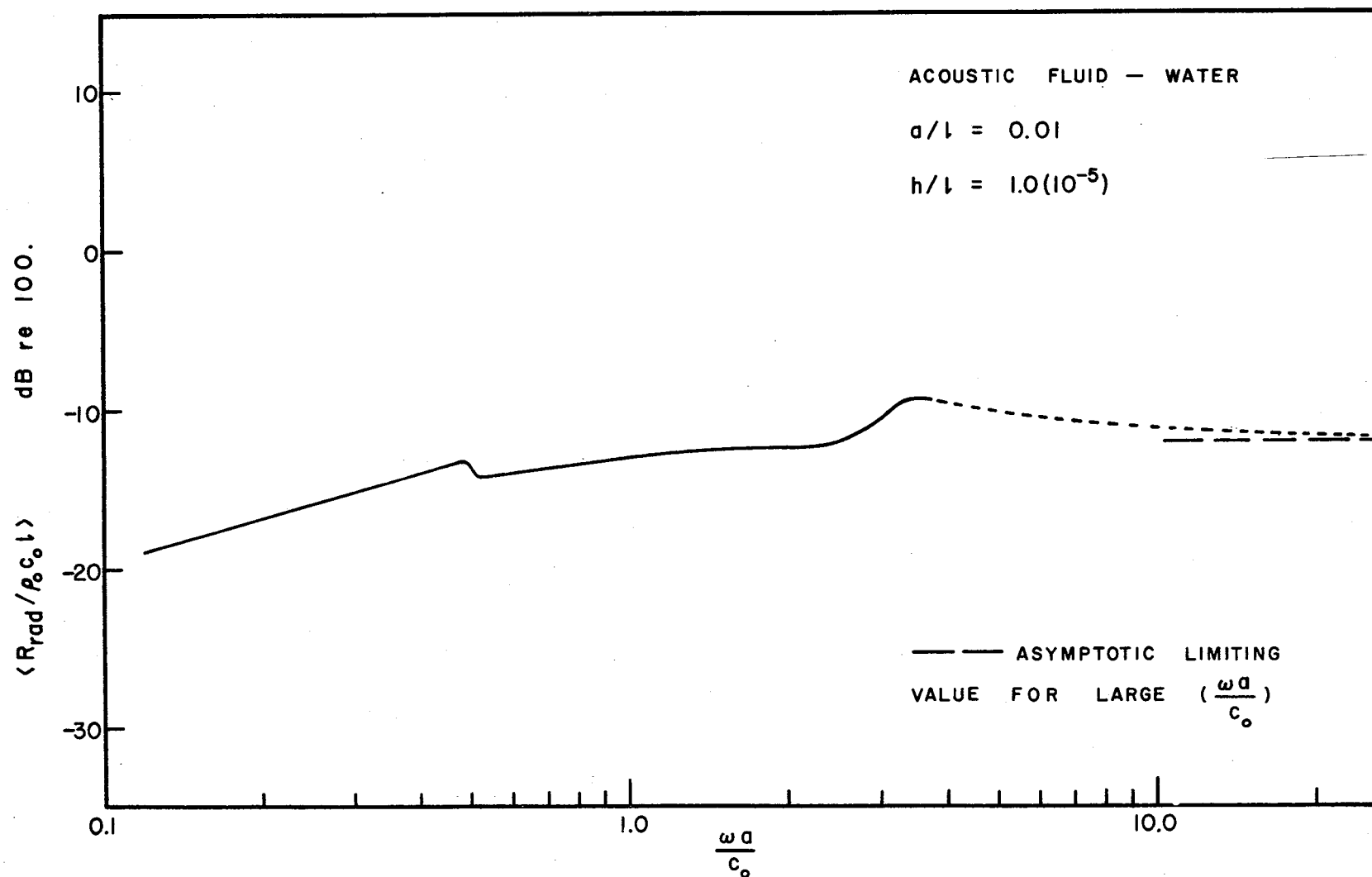


FIGURE 5.4 AVERAGED DIMENSIONLESS RADIATION RESISTANCE VS  $(\frac{\omega a}{c_0})$

Table 5.2 Dimensionless radiation resistance for various values of  $h/L$

| $\left(\frac{\omega a}{c_o}\right)$ | Dimensionless Radiation Resistance |                      |                      |
|-------------------------------------|------------------------------------|----------------------|----------------------|
|                                     | $a/L = 0.01$                       | Acoustic Fluid - Air |                      |
|                                     | $h/L = 1.0(10^{-5})$               | $h/L = 1.0(10^{-4})$ | $h/L = 1.0(10^{-3})$ |
| .3400                               | 5.202                              | 5.200                | 5.200                |
| .3600                               | 12.100                             | 12.090               | 12.090               |
| .3800                               | 7.972                              | 7.966                | 7.965                |
| .4000                               | 6.669                              | 6.665                | 6.665                |
| .4200                               | 17.170                             | 17.150               | 17.150               |
| .4400                               | 10.360                             | 10.350               | 10.350               |
| .4600                               | 8.334                              | 8.327                | 8.326                |
| .4800                               | 24.380                             | 24.350               | 24.340               |
| .5000                               | 13.20                              | 13.18                | 13.18                |
| .5200                               | 10.06                              | 10.05                | 10.05                |
| .5400                               | 35.66                              | 35.62                | 35.61                |
| .5600                               | 16.40                              | 16.36                | 16.35                |
| .5800                               | 11.90                              | 11.88                | 11.88                |
| .6000                               | 58.18                              | 58.12                | 58.12                |
| .6200                               | 20.36                              | 20.30                | 20.29                |
| .6400                               | 14.20                              | 14.16                | 14.15                |
| .6600                               | 222.70                             | 229.70               | 229.70               |
| .6800                               | 25.29                              | 25.19                | 25.18                |
| .7000                               | 16.65                              | 16.59                | 16.58                |
| .7200                               | 13.08                              | 13.04                | 13.03                |



Table 5.3 Dimensionless radiation resistance for various shell materials in contact with air

| $\left(\frac{\omega a}{c_o}\right)$ | Dimensionless Radiation Resistance |                      |           |
|-------------------------------------|------------------------------------|----------------------|-----------|
|                                     | $a/L = 0.01$                       | $h/L = 1.0(10^{-5})$ |           |
|                                     | 7075<br>Aluminum                   | Titanium             | Beryllium |
| .1000                               | 2.870                              | 2.870                | 2.870     |
| .1200                               | 1.732                              | 1.732                | 1.732     |
| .1400                               | 2.869                              | 2.869                | 2.869     |
| .1600                               | 8.213                              | 8.213                | 8.212     |
| .1800                               | 3.186                              | 3.186                | 3.186     |
| .2000                               | 5.450                              | 5.449                | 5.448     |
| .2200                               | 76.570                             | 76.570               | 76.57     |
| .2400                               | 5.155                              | 5.154                | 5.154     |
| .2600                               | 9.447                              | 9.445                | 9.444     |
| .2800                               | 5.264                              | 5.264                | 5.263     |
| .3000                               | 7.794                              | 7.792                | 7.791     |
| .3200                               | 16.060                             | 16.050               | 16.050    |
| .3400                               | 7.362                              | 7.361                | 7.360     |
| .3600                               | 11.360                             | 11.350               | 11.350    |
| .3800                               | 29.530                             | 29.53                | 29.520    |
| .4000                               | 9.980                              | 9.977                | 9.975     |
| .4200                               | 16.290                             | 16.280               | 16.280    |
| .4400                               | 153.200                            | 153.200              | 153.100   |
| .4600                               | 13.270                             | 13.270               | 13.260    |

continued

Table 5.3 (continued)

| $\left(\frac{\omega a}{c}\right)_0$ | Dimensionless Radiation Resistance |          |                      |
|-------------------------------------|------------------------------------|----------|----------------------|
|                                     | $a/L = 0.01$                       |          | $h/L = 1.0(10^{-5})$ |
|                                     | 7075<br>Aluminum                   | Titanium | Beryllium            |
| .4800                               | 23.440                             | 23.430   | 23.430               |
| .5000                               | 12.050                             | 12.050   | 12.040               |
| .5200                               | 17.380                             | 17.370   | 17.370               |
| .5400                               | 34.810                             | 34.800   | 34.790               |
| .5600                               | 14.850                             | 14.840   | 14.840               |
| .5800                               | 22.750                             | 22.740   | 22.730               |
| .6000                               | 57.660                             | 57.650   | 57.630               |
| .6200                               | 18.460                             | 18.450   | 18.450               |
| .6400                               | 30.110                             | 30.100   | 30.080               |
| .6600                               | 229.700                            | 229.700  | 229.700              |
| .6800                               | 22.940                             | 22.930   | 22.920               |
| .7000                               | 40.690                             | 40.670   | 40.650               |
| .7200                               | 19.440                             | 19.430   | 19.420               |

## 5.2 Numerical Evaluation for Lobar Mode Shapes

Expression (4.121) for the radiation resistance in the case of lobar mode shapes is evaluated numerically by noting that  $\xi$ , and  $v_m/\chi_m$  can be written in terms of more fundamental parameters which are related to the geometry of the shell and the physical properties of the shell material and the fluid.

Equation (4.121) is expressed in terms of the dimensionless parameters,  $\eta$ , and  $\xi$ . Since  $\eta = (\omega a)/c_o$  where  $\omega$  is the as yet unspecified and therefore arbitrary forcing frequency of the applied surface load,  $q(\theta, t)$ , it is convenient and desirable to hypothesize that  $\xi$  is a dependent variable in the problem and write it in terms of  $\eta$ , the independent variable. To accomplish this, consider equation (4.100) and write it as

$$\left(\frac{v_m}{\chi_m}\right)^2 = \frac{1}{1 - \frac{\rho_o a}{m_s} \left( \frac{J_m}{m J_m(\eta) - \eta J_{m+1}(\eta)} \right)} \quad (5.10)$$

Note that  $m_s$  is the mass per unit area of the shell surface and is, therefore, equal to the shell material density,  $\rho_s$ , multiplied by the shell thickness,  $h$ . Hence the above expression can be written as

$$\left(\frac{v_m}{\chi_m}\right)^2 = \frac{1}{1 - \left(\frac{\rho_o}{\rho_s}\right) \left(\frac{a}{h}\right) \left( \frac{J_m(\eta)}{m J_m(\eta) - \eta J_{m+1}(\eta)} \right)} \quad (5.11)$$

This result is expressed in terms of the desired quantity,  $\eta$ , as well as the density ratio,  $\rho_o/\rho_s$ , and  $a/h$ , the ratio of cylinder radius to cylinder wall thickness. Equation (5.10) can be employed to write an

expression for  $\xi$  as

$$\xi^2 = \left( \frac{v_m a}{c_o} \right)^2 = \left( \frac{\chi_m a}{c_o} \right)^2 \left( \frac{v_m}{\chi_m} \right)^2 . \quad (5.12)$$

But from equation (4.96)

$$\chi_m^2 = \frac{Dm^4}{m_s a^4} , \quad (5.13)$$

and equation (5.7),

$$\left( \frac{\chi_m a}{c_o} \right)^2 = \frac{1}{12} \left( \frac{C_L}{c_o} \right)^2 \left( \frac{h}{a} \right)^2 m^4 . \quad (5.14)$$

Consequently

$$\xi^2 = \left( \frac{v_m}{\chi_m} \right)^2 \left[ \frac{1}{12} \left( \frac{C_L}{c_o} \right)^2 \left( \frac{h}{a} \right)^2 m^4 \right] . \quad (5.15)$$

All quantities in equation (4.121) for the radiation resistance are now represented in terms of the forcing frequency parameter,  $\eta$ ; the shell geometry parameter,  $h/a$ ; and the material properties,  $\rho_o/\rho_s$ ,  $C_L/c_o$ , and  $\nu$  with  $Q_m/Q_o = 1$ .

The numerical analysis scheme is a parameter study of the problem in terms of the previously mentioned parameters: the ranges of values for  $\eta$  and  $h/a$  are

$$10^{-2} \leq \eta \leq 10^3 ,$$

and

$$\frac{1}{1000} \leq \frac{h}{a} \leq \frac{1}{20} .$$

The shell material and fluid property parameters to be employed in the numerical analysis are identical to those employed in the case of axisymmetric mode shapes and are given in Table 5.1. The details of the computer program employed in the parameter study are given in Appendix 10.3.3. In contrast to the case for axisymmetric mode shapes, the series in equation (4.121) are theoretically infinite series but for the purposes of numerical evaluation, summation is terminated as each series converges to a suitably stable value.

The results of machine computation are shown in Figures 5.5 and 5.6. Figure 5.5 shows the radiation resistance for a cylinder in contact with air for large values of  $\eta$  while Figure 5.6 presents the same information for a cylinder in contact with water.

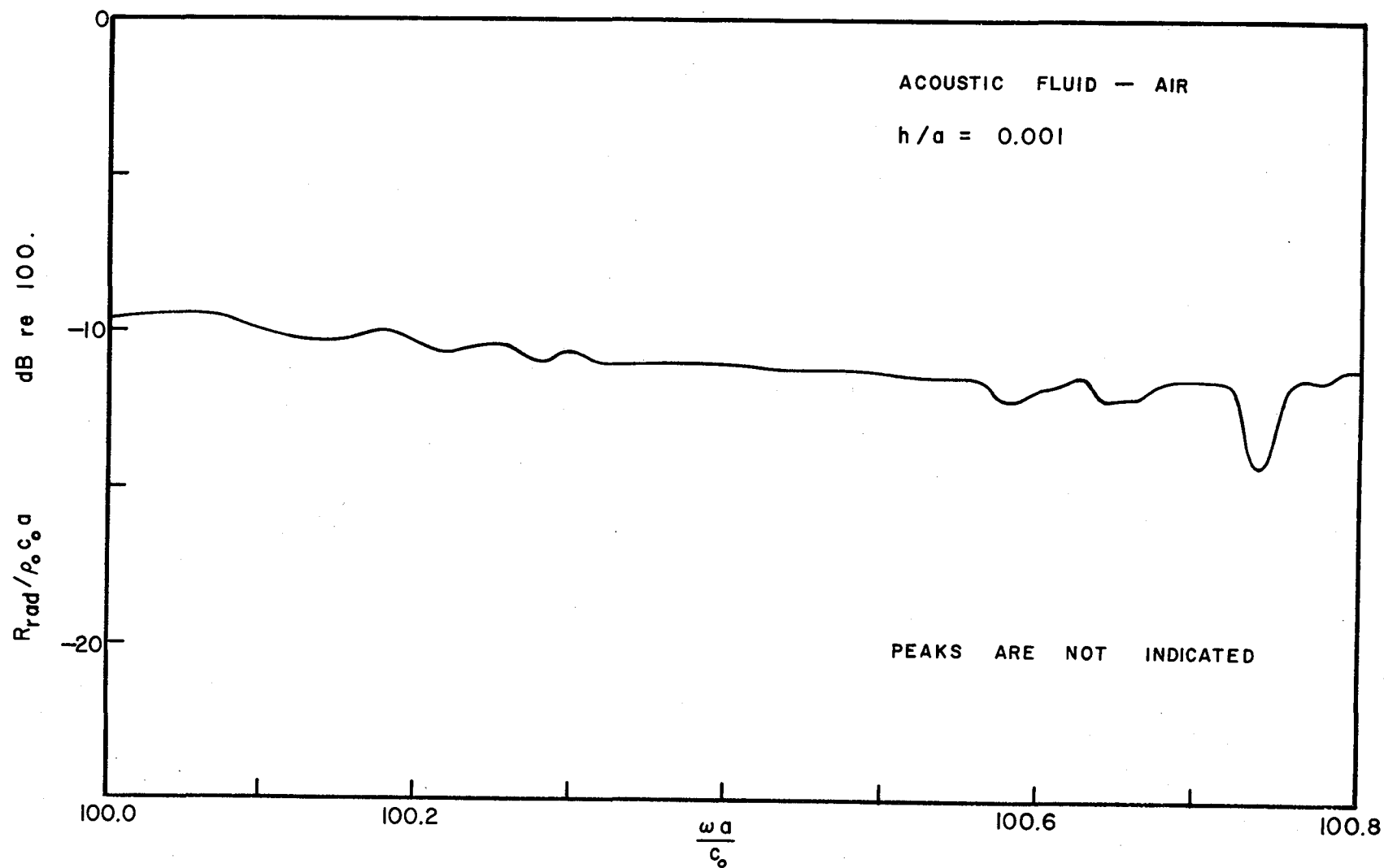


FIGURE 5.5 DIMENSIONLESS RADIATION RESISTANCE VS  $(\frac{\omega a}{c_0})$  FOR LOBAR MODE SHAPES

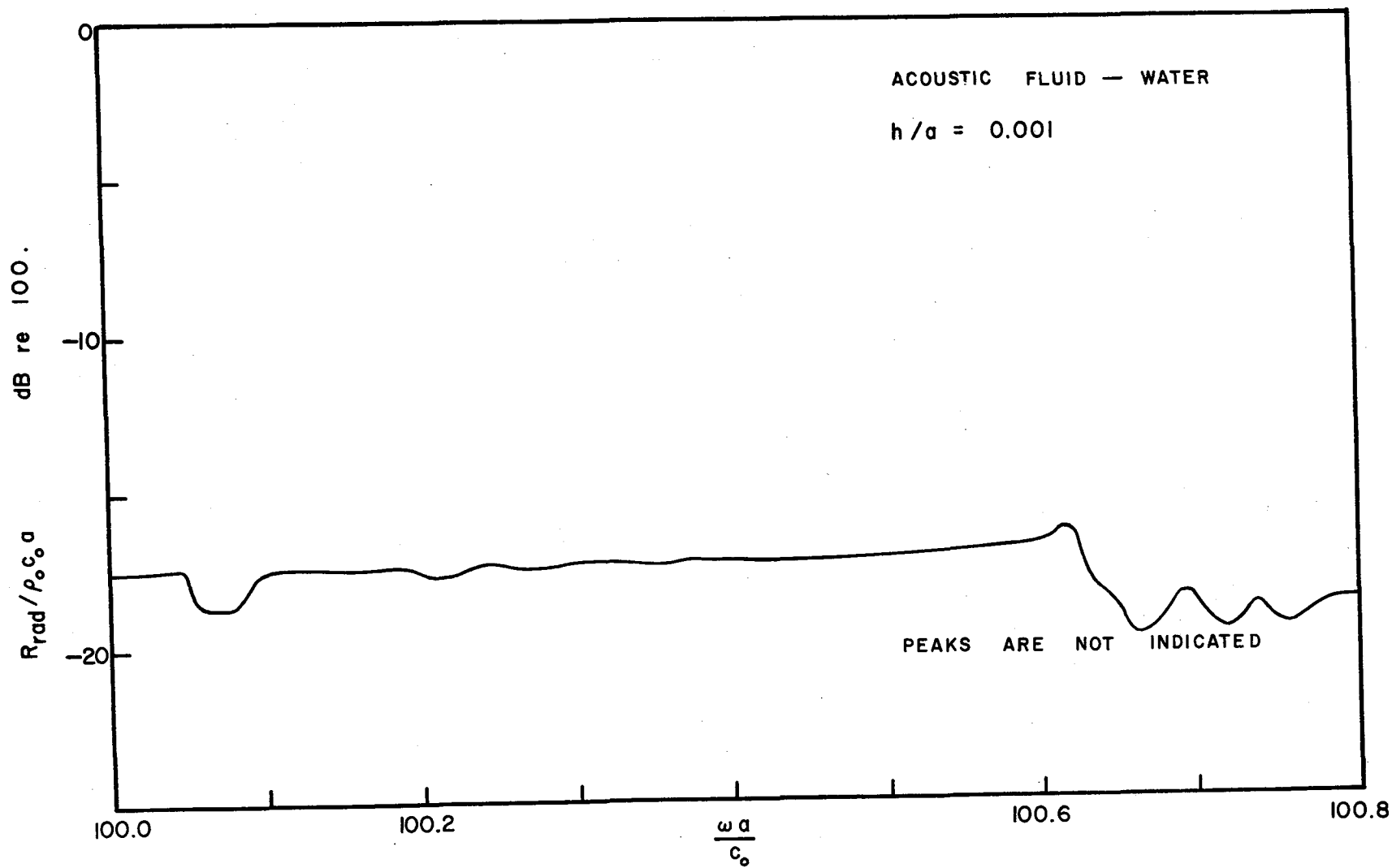


FIGURE 5.6 DIMENSIONLESS RADIATION RESISTANCE VS  $(\frac{\omega a}{c_0})$  FOR LOBAR MODE SHAPES

## 6. DISCUSSION OF RESULTS

The results presented in Chapter 5 exhibit excellent agreement with the basic characteristics of previous work on the radiation resistance of finite cylindrical shells (Manning and Maidanik, 1964). Although the work reviewed in this report is theoretically applicable to long or mathematically infinite cylindrical shells, Figure 5.4, which indicates the averaged behavior of  $R_{\text{rad}}$  over a wide range of dimensionless frequency values, shows quite clearly the characteristics of previously published experimental data with regard to the two peaks in the low- to middle-frequency values and the asymptotic approach to the radiation resistance of a flat plate of equal area for large values of the dimensionless frequency parameter,  $\eta$ . The two peaks are identified as the ring frequency,  $\omega_r$ , the frequency at which the longitudinal wave length in the cylinder material is equal to its circumference, and the critical frequency,  $\omega_g$ , the frequency at which the flexural-wave speed in a flat plate of equivalent thickness is equal to the speed of sound in the surrounding acoustic medium, respectively. For large values of  $\eta$ , the results as indicated in Figure 5.4 oscillate between the dotted curve which represents the upper bound and the dashed curve which represents the asymptotic limit for values of the radiation resistance of the cylinder vibrating in axisymmetric mode shapes.

Due to the substantial differences in the shell and acoustic environment in the two cases--the theoretical work reported herein applying to an infinite, thin cylindrical shell in contact with an



unbounded ideal fluid and the experimental work of Manning and Maidanik (1964) applying to a finite, flanged cylinder in contact with a reverberant airspace--it is felt that the results of this work show good agreement with previous experimental studies. The decaying oscillation of the radiation resistance as  $\eta$  becomes large is perhaps due to the fact that this study is formulated in terms of an anechoic acoustic environment instead of a reverberant one. A reverberant acoustic environment would have a more pronounced averaging influence on the motion of the shell than would an anechoic condition which essentially serves as an energy sink compared to a reverberant environment which is more aptly described in terms of energy storage.

The results for the case of lobar mode shapes are pointedly less useful than the results for the axisymmetric mode shapes because the results are valid in the case of the former only for large values of the dimensionless frequency,  $\eta$ .

## 7. SUMMARY AND CONCLUSIONS

A mathematical model consisting of simultaneous partial differential equations, one describing the motion of the shell; the other, the motion of the acoustic medium, is developed. The descriptive equations are solved subject to a velocity compatibility boundary condition at the shell-fluid interface and the classic radiation condition at large distances from the surface of the cylindrical shell. The solution is employed to obtain the total radiated power and the mean-square surface velocity. The radiation resistance is calculated from this information in the case of axisymmetric and lobar mode shapes. The results are obtained in the form of a ratio of dimensionless series which is evaluated numerically for realistic values of the dimensionless parameters characterizing the problem. The results are presented in the form of graphs and tables in Chapter 5.

In summary, it should be noted that the results obtained in this work are predicated by hypotheses pertaining to both the shell and its environment. Specifically, the shell is assumed to be a long cylindrical shell constructed from an isotropic elastic material and obeying thin shell equations of deformation. The equations of motion for the shell do not include internal mechanical damping effects such as would be caused by joints and fasteners or hysteresis damping within the material. Damping occurs only in terms of the effect of the acoustic medium upon the motion of the shell. The acoustic medium is hypothesized to be an ideal compressible fluid that satisfies the perfect gas law, that is constituted in such a way as to permit propagation of a sound wave in an adiabatic fashion, and that has a viscosity of zero.

Hence energy dissipation in the fluid by both thermal and viscous means is not considered.

Considering the differences between the mathematical model utilized in this work and the environment within which experimental work on the subject has been done, the results presented in Chapter 5 exhibit good agreement with experimental results in the literature both in terms of the peak at the critical frequency and the asymptotic approach to the radiation resistance of a flat plate of equal area at large values of the dimensionless frequency parameter. However a need exists for more extensive experimental study of this problem. A need also exists for more theoretical work based on general mode shapes for infinite cylindrical shells and for finite cylindrical shells. Although it will be much more difficult to handle in a mathematically rigorous manner, future theoretical work will obviously be of greater value if the acoustic environment is included in terms of a reverberant condition.

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## 9. LIST OF SYMBOLS

|                 |                                                      |
|-----------------|------------------------------------------------------|
| $a$             | mean-radius of cylinder                              |
| $C_L$           | $C_L^2 = E/[\rho_s(1-\nu^2)]$                        |
| $c_o$           | speed of sound in the acoustic medium                |
| $D$             | plate stiffness, $Eh^3/[12(1-\nu^2)]$                |
| $E$             | Young's Modulus                                      |
| $F_u, F_v, F_w$ | components of force applied to the cylindrical shell |
| $h$             | thickness of shell                                   |
| $i$             | $i^2 = -1$                                           |
| $k_r$           | radial wave number; defined by equation (4.26)       |
| $k_z$           | axial wave number; defined by equation (4.33)        |
| $m$             | integer                                              |
| $m_s$           | mass of shell material per unit of area              |
| $M$             | lumped mass of the system                            |
| $n$             | integer                                              |
| $n_s(\omega)$   | modal density of structure                           |
| $p$             | acoustic pressure                                    |
| $P_o$           | ambient pressure in the acoustic medium              |
| $q$             | applied load on cylinder surface per unit of area    |
| $r$             | radial coordinate in cylindrical coordinate system   |
| $R$             | total resistance                                     |
| $R_{rad}$       | radiation resistance                                 |
| $R_m$           | mechanical resistance                                |
| $s$             | spring constant                                      |
| $S_a(\omega)$   | acceleration spectral density of the structure       |

|               |                                                                            |
|---------------|----------------------------------------------------------------------------|
| $S_p(\omega)$ | acoustic pressure spectral density in the medium surrounding the structure |
| $t$           | time                                                                       |
| $T$           | $T = 2\pi m/\omega$                                                        |
| $T_s$         | reverberation time of structure                                            |
| $u$           | axial velocity of shell surface                                            |
| $\vec{u}$     | acoustic-medium particle velocity                                          |
| $u_r$         | radial fluid particle velocity                                             |
| $v$           | tangential velocity of shell                                               |
| $w$           | radial velocity of shell surface                                           |
| $W$           | total dissipated power                                                     |
| $W_i$         | input power                                                                |
| $W_r$         | radiated power                                                             |
| $x$           | coordinate in single-degree-of-freedom problem                             |
| $z$           | axial coordinate in cylindrical coordinate system                          |
| $Z_r$         | radiation impedance                                                        |
| $\alpha$      | dimensionless axial coordinate                                             |
| $\beta$       | $\beta = \theta$                                                           |
| $\gamma$      | ratio of specific heats ( $c_p/c_v$ )                                      |
| $\epsilon$    | $\epsilon^2 = D/a^4 m_s$                                                   |
| $\zeta$       | damping factor                                                             |
| $\theta$      | polar angle in cylindrical coordinate system                               |
| $\mu$         | resistance ratio                                                           |
| $\nu$         | Poisson's ratio                                                            |
| $\rho_0$      | ambient density of the acoustic medium                                     |
| $\rho_s$      | density of shell material                                                  |



|            |                                                      |
|------------|------------------------------------------------------|
| $\phi$     | acoustic velocity potential                          |
| $\omega$   | frequency of forcing function                        |
| $\omega_r$ | ring frequency                                       |
| $\omega_g$ | critical frequency                                   |
| $\omega_n$ | natural frequency of single-degree-of-freedom system |

## 10. APPENDICES

### 10.1 Response Analysis of a Randomly Excited Rigid Piston in an Infinite Baffle

#### 10.1.1 Introduction

The problem of a baffled, rigid piston radiating into a semi-infinite acoustic medium was originally investigated by Rayleigh (1945). Since Rayleigh's initial report, numerous methods have been employed in further study of the problem (McLachlan, 1932; King, 1934; Williams and Labaw, 1945; Pachner, 1951; Guptill, 1953; Quint, 1959; Mangulis, 1964; Williams, 1964; and Greenspan, 1966). The present work is concerned with analysis of the problem when the piston is excited by a random force.

The problem is examined in terms of a single-degree-of-freedom vibrational system with retarding forces due to mechanical stiffness, mechanical damping, inertia, and the effects of the acoustic medium. Solution of the differential equation of motion permits determination of the admittance of the system. The mean-square response velocity of the piston is calculated for a random exciting force of uniform spectral density. The result is an integral form solution that is evaluated numerically.

Numerical results are obtained for the general case, for  $2ka \ll 1$ , and for  $2ka$  large. When  $2ka$  is large, an analytical result can be obtained and a comparison of numerical and analytical results gives good agreement. Numerical results are presented in graphical form and discussed.

### 10.1.2 Analytical Development

Consider a circular piston mounted flush with the surface of an infinite plane baffle, Figure 10.1. Motion perpendicular to the plane of the piston is initiated by application of a random force. Resisting forces tending to retard motion are generated by mechanical damping, mechanical stiffness and the effects of the acoustic medium. A study of this motion will be accomplished in terms of the mathematically equivalent model shown in Figure 10.2, where  $F_F(t)$  is the forcing function and  $F_R(t)$  is the reaction force due to the surrounding acoustic medium.

Describing the system in terms of the coordinate  $X$  gives the differential equation of motion of the system as

$$m\ddot{X} + R_m\dot{X} + SX = F_F(t) - F_R(t) . \quad (10.1)$$

The reaction force due to the acoustic medium can be written in terms of the acoustic or radiation impedance,  $Z_r$ ; then for a simple harmonic forcing function of amplitude  $F$  and circular frequency  $\omega$ , equation (10.1) becomes

$$m\ddot{X} + R_m\dot{X} + SX = Fe^{i\omega t} - Z_r Ue^{i\omega t} \quad (10.2)$$

where  $Ue^{i\omega t}$  is the velocity of the piston. The solution of this equation is expressible as  $Ce^{i\omega t}$  where  $C$  is a complex constant, i.e.

$$X = \frac{[F - Z_r U]e^{i\omega t}}{[S - m\omega^2] + i R_m \omega} . \quad (10.3)$$

The velocity of the piston has the form

$$u = \dot{X} = Ue^{i\omega t} = \frac{i\omega[F - Z_r U]e^{i\omega t}}{[S - m\omega^2] + iR_m \omega} ; \quad (10.4)$$

so that the complex velocity amplitude can be written as

$$U = \frac{i\omega F}{[S - m\omega^2] + i\omega[R_m + Z_r]} . \quad (10.5)$$

By writing the radiation impedance,  $Z_r$ , in terms of the radiation resistance,  $R_r$ , and the radiation reactance,  $X_r$ , the velocity amplitude becomes

$$U = \frac{F}{[R_m + R_r] + i[X_r + m\omega(1 - (\frac{\omega}{\omega_n})^{-2})]} , \quad (10.6)$$

where  $\omega_n^2 = S/m$  is the natural frequency of an equivalent spring-mass system. Consequently the admittance function for the system can be written as

$$Y(\omega) = \frac{U}{F} = \frac{1}{[R_m + R_r] + i[X_r + m\omega(1 - (\frac{\omega}{\omega_n})^{-2})]} . \quad (10.7)$$

Expressions for the radiation resistance and radiation reactance can be obtained from a number of sources in the literature (Rayleigh, 1945; Kinsler and Frey, 1962; Stephens and Bate, 1966; and Morse, 1948). The approach employed by Kinsler and Frey (1962) is to solve the wave equation for a hemispherical point velocity source in an infinite baffle and obtain an expression for the acoustic pressure generated

in the surrounding acoustic medium. If the piston is composed of a large number of these point sources, then the total pressure at a point in the acoustic medium is made up of components of pressure with each component resulting from a point velocity source on the piston face. If the surface area of the piston which corresponds to an elemental point source is represented by  $dA$  then the desired result can be obtained by integrating over the surface of the piston. The resulting pressure at a point  $(r, \theta)$  of the acoustic medium is

$$p = \frac{i \rho_o c_o k a^2 U e^{i(\omega t - kr)}}{2r} \left[ \frac{2J_1(ka \sin \theta)}{ka \sin \theta} \right] \quad (10.8)$$

The coordinate  $r$  is the distance between the center of the piston face and the point of interest while  $\theta$  is the angle between a line segment joining the center of the piston and the point of interest and the axis of rotational symmetry of the piston, Figure 10.1. The ambient density of the acoustic medium is  $\rho_o$ , while  $c_o$  is the speed of sound in the medium, and  $k = \omega/c_o$ . Furthermore the radius of the piston is  $a$ , and  $J_1(\xi)$  is a Bessel function of the first kind of order one.

The reaction force on the piston will now be determined. The pressure,  $p'$ , acting on an element of piston area,  $dA'$ , due to the motion of other area elements,  $dA$ , is

$$p' = \iint \frac{i \rho_o c_o k U e^{i(\omega t - kr)}}{2\pi r} dA \quad (10.9)$$

Hence the total reaction force is obtained by integrating with respect to  $dA'$

$$F_R(t) = \iint p' dA' \quad (10.10)$$

Integration yields an expression for the reaction force which when divided by the piston velocity,  $Ue^{i\omega t}$ , gives the acoustic or radiation impedance

$$Z_r = \pi \rho_o c_o a^2 [R_1(2ka) + iX_1(2ka)] , \quad (10.11)$$

where

$$R_1(\xi) = 1 - 2 \frac{J_1(\xi)}{\xi} , \quad (10.11a)$$

$$X_1(\xi) = 2 \frac{K_1(\xi)}{\xi^2} , \quad (10.11b)$$

and

$$K_1(\xi) = \frac{4\xi}{\pi} \int_0^{\frac{\pi}{2}} \sin^2(\frac{1}{2}\xi \sin \theta) \sin \theta d\theta . \quad (10.11c)$$

Examine the case of the piston subjected to a random exciting force; this excitation is a stationary process consisting of ideal white noise with uniform spectral density,  $S_f$ , in the frequency domain. The spectral density of response velocity will be related to the spectral density of the forcing function by

$$S_u(\omega) = |Y(\omega)|^2 S_f(\omega) , \quad (10.12)$$

where  $Y(\omega)$  is the admittance of the piston-acoustic medium system as noted in equation (10.7). The mean-square response velocity is expressible in terms of the spectral density of the response velocity:

$$\langle u^2 \rangle = \int_0^{\infty} S_u(\omega) d\omega . \quad (10.13)$$

Hence, by incorporating equations (10.7) and (10.12) with (10.13),

$$\langle u^2 \rangle = S_f \int_0^\infty |Y(\omega)|^2 d\omega ; \quad (10.14)$$

where

$$|Y(\omega)|^2 = \frac{1}{[R_m + R_r]^2 + [X_r + m\omega(1 - (\frac{\omega}{\omega_n})^{-2})]^2} , \quad (10.15)$$

and  $R_r$  and  $X_r$  are determined from equation (10.11).

For generality and convenience in numerical evaluation of the integral in (10.14), the following dimensionless parameters are introduced:

$$\lambda_c = \frac{c_o}{\omega_n a} , \quad \lambda_m = \frac{m}{\rho_o a^3} , \quad \text{and} \quad q = \frac{\omega}{\omega_n} .$$

Utilizing these quantities in equation (10.14), an expression for the mean-square response velocity can be written as

$$\frac{\langle u \rangle}{S_f} = \int_0^\infty \frac{q^2 dq}{q^4 + C_1 q^2 + C_2 + C_3 K_1(\frac{2q}{\lambda_c}) + C_4 K_1^2(\frac{2q}{\lambda_c}) + C_5 J_1(\frac{2q}{\lambda_c}) + C_6 J_1^2(\frac{2q}{\lambda_c})} , \quad (10.16)$$

where

$$C_1 = [(\pi \frac{\lambda_c}{\lambda_m} + 2\zeta)^2 - 2] ,$$

$$C_2 = 1 ,$$

$$C_3 = [\pi \frac{\lambda_c^3}{\lambda_m} (q - \frac{1}{q})] ,$$

$$C_4 = \left[ \frac{\pi^2}{4} \left( \frac{\lambda_c^3}{\lambda_m} \right)^2 \left( \frac{1}{q} \right) \right] ,$$

$$C_5 = - \left[ 2\pi \frac{\lambda_c^2}{\lambda_m} \left( \pi \frac{\lambda_c}{\lambda_m} + 2\zeta q \right) \right] ,$$

and

$$C_6 = \left[ \pi^2 \frac{\lambda_c^4}{\lambda_m^2} \right] .$$

For the special case where  $ka = \frac{2q}{\lambda_c} \ll 1$ , the expressions for  $R_r$  and  $X_r$  simplify and equation (10.16) becomes:

$$\frac{\langle u^2 \rangle}{\frac{S_f}{m^2 \omega_n^2}} = \frac{4\lambda_m^2 \lambda_c^2}{\pi^2} \int_0^\infty \frac{q^2 dq}{q^6 + D_1 q^4 + D_2 q^2 + D_3} , \quad (10.17)$$

where

$$D_1 = \frac{8\zeta \lambda_m \lambda_c^2}{\pi} + \frac{256\lambda_c^2}{9\pi^2} + \frac{64\lambda_m \lambda_c^2}{3\pi^2} + \frac{4\lambda_m^2 \lambda_c^2}{\pi^2} ,$$

$$D_2 = \left[ \frac{16\zeta^2 \lambda_m^2 \lambda_c^2}{\pi^2} - \frac{64\lambda_m \lambda_c^2}{3\pi^2} - \frac{8\lambda_m^2 \lambda_c^2}{\pi^2} \right] ,$$

and

$$D_3 = \frac{4\lambda_m^2 \lambda_c^2}{\pi^2} .$$



Likewise when  $ka = \frac{2q}{\lambda_c}$  is large, equation (10.16) simplifies and

$$\frac{\langle u^2 \rangle}{\frac{S_f}{m \omega_n^2}} = \int_0^\infty \frac{q^2 dq}{q^4 + E_1 q^2 + E_2}, \quad (10.18)$$

where

$$E_1 = \left[ \left( \pi \frac{\lambda_c}{\lambda_m} + 2\zeta \right)^2 + 2 \left( 2 \frac{\lambda_c^2}{\lambda_m} - 1 \right) \right],$$

and

$$E_2 = \left[ 2 \left( \frac{\lambda_c}{\lambda_m} \right)^2 - 1 \right].$$

Equation (10.18) is evaluated analytically (Bierens de Haan, 1858) with the following result:

$$\frac{\langle u^2 \rangle}{\frac{S_f}{m \omega_n^2}} = \frac{\pi}{2 \sqrt{\left( \pi \frac{\lambda_c}{\lambda_m} + 2\zeta \right)^2 + 4 \left( 2 \frac{\lambda_c^2}{\lambda_m} - 1 \right)}}. \quad (10.19)$$

### 10.1.3 Discussion of Results

Equations (10.16), (10.17), and (10.18) for the dimensionless mean-square response velocity of the piston were evaluated numerically by application of Simpson's rule. The behavior of the integrands in these equations as a function of  $q$  is illustrated in Figures 10.3, 10.4, 10.5 and 10.6. The monotonic decreasing character of these quantities as  $q$  becomes large allows the use of a large finite upper

limit in the numerical evaluation process. The validity of this approach was verified in the case of equation (10.18) by use of an exact expression for the integral. A comparison of results by the two methods involves differences of no more than 1/2 of 1 percent.

In Figure 10.3 the admittance function of the system as described by equation (10.16) is shown for fixed  $\lambda_c$  and  $\zeta$  as  $\lambda_m$  is varied; the effects of the acoustic medium are aptly illustrated here since in the limit as  $\lambda_m \rightarrow \infty$ ,  $\lambda_c \rightarrow 0$ , the acoustic medium becomes a vacuum. Figure 10.4 shows the variation of the admittance for fixed  $\lambda_c$  and  $\lambda_m$  as  $\zeta$  is varied. Increased damping simply reduces and shifts the resonance peak as with a simple damped spring-mass system. Figure 10.5 shows the admittance function for equation (10.17) for fixed  $\lambda_c$  and  $\zeta$  as  $\lambda_m$  is varied; Figure 10.6 shows the admittance function for equation (10.18) subject to the same conditions on  $\lambda_c$ ,  $\zeta$ , and  $\lambda_m$ .

Figure 10.7 depicts the dimensionless mean-square response velocity (equation 10.16) as a function of  $\lambda_m$  for various fixed values of  $\lambda_c$  and  $\zeta$ . It should be noted that these curves approach a limiting value of velocity as  $\lambda_m$  increases. In fact, the asymptotic value of velocity approached is the velocity of the piston when in contact with a vacuum. For the case of the piston-vacuum system, the mean-square velocity attained is a function of  $\zeta$  alone or

$$\frac{\langle u^2 \rangle}{\frac{S_f}{m \omega_n^2}} = \frac{\pi}{4\zeta} \quad (10.20)$$

For  $\zeta = 0.01, 0.02, 0.04$ , and  $0.08$ , the limiting value of dimensionless

mean-square velocity becomes 78.54, 39.27, 19.63, and 9.817 respectively. Figures 10.7 - 10.13 accurately exhibit this behavior. Figures 10.7 - 10.9 are graphical presentations of equation (10.16). Figures 10.10 and 10.11 are graphical forms of equation (10.17), and Figures 10.12 and 10.13 are graphical forms of equation (10.18). Figure 10.12 should be noted, in particular, since only in the region below the dashed line do the numerical and analytical results agree. This fact rather firmly establishes the respective values of  $\lambda_c$  and  $\lambda_m$  which allow valid usage of equation (10.18); i.e.

$$\lambda_c > \sqrt{\frac{\lambda_m}{2}} . \quad (10.21)$$

#### 10.1.4 Conclusions

The problem of a baffled, rigid piston radiating into a semi-infinite acoustic medium is examined in terms of a single-degree-of-freedom vibrational system. Solution of the differential equation of motion permits determination of the admittance of the system. The mean-square response velocity of the piston is calculated for a random exciting force of uniform spectral density. The result is an integral form solution that is evaluated numerically.

The results indicate the general ranges of usefulness of the two asymptotic results, equations (10.17) and (10.18), compared with the general results, equation (10.16). Equation (10.19) is obviously the most useful result, but it is restricted to small values of  $\lambda_c$  and  $\lambda_m$  being related to  $\lambda_c$  by equation (10.21). Neither equation (10.16) nor (10.17) can be handled conveniently without the use of a digital

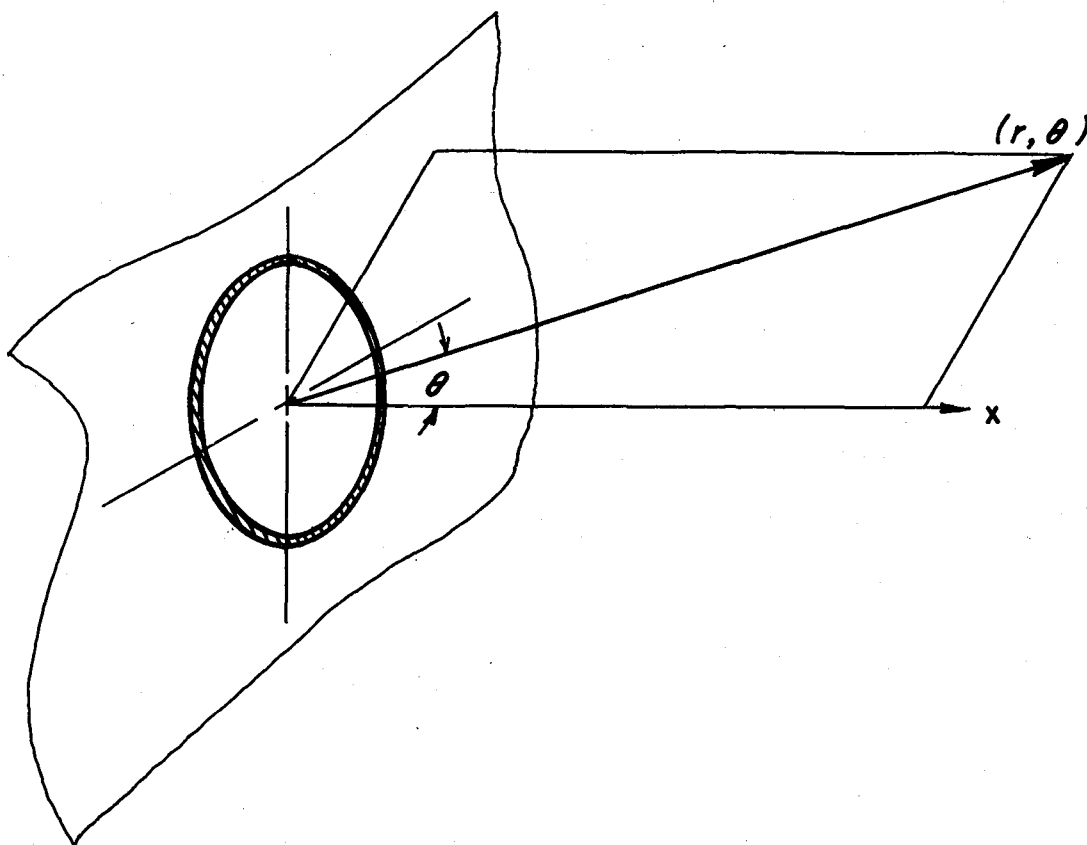


FIGURE 10.1

RIGID PISTON IN INFINITE BAFFLE

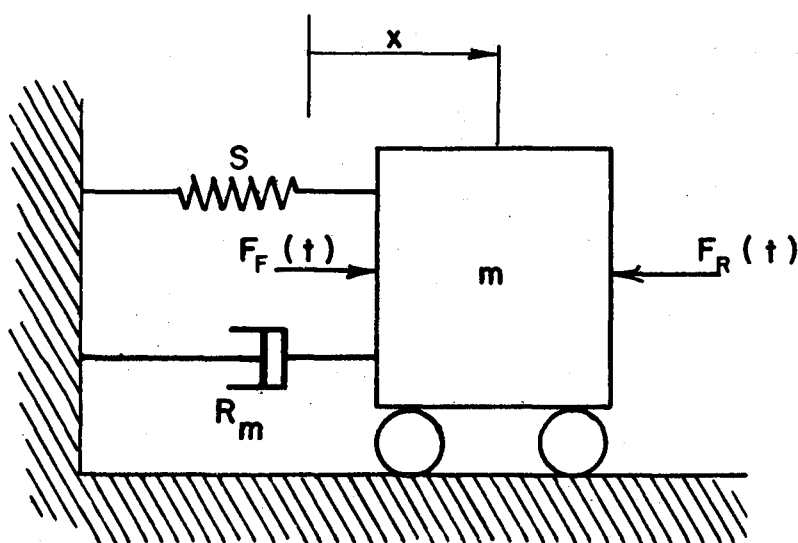


FIGURE 10.2

EQUIVALENT SYSTEM

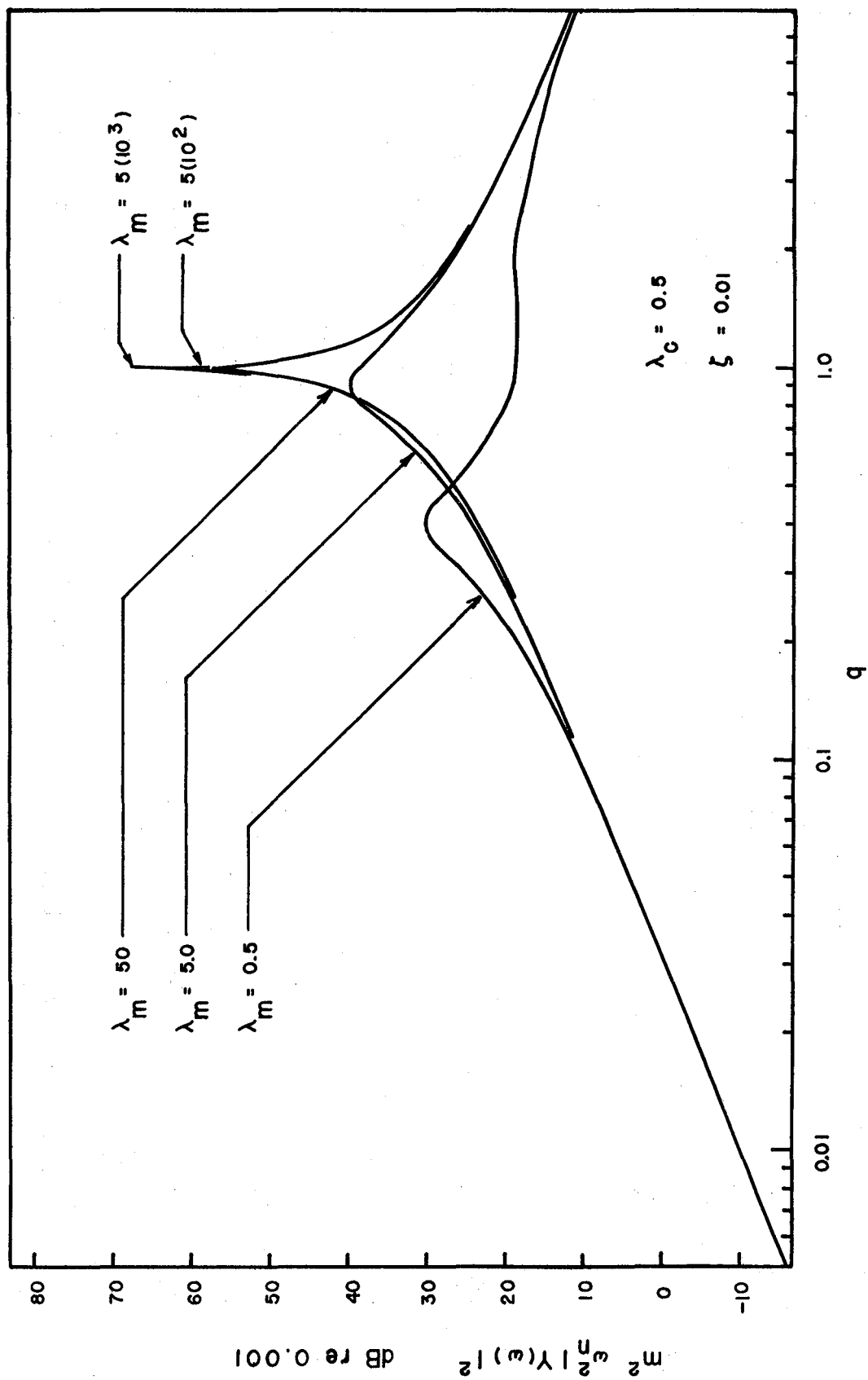


FIGURE 10.3 DIMENSIONLESS ADMITTANCE VS DIMENSIONLESS FREQUENCY

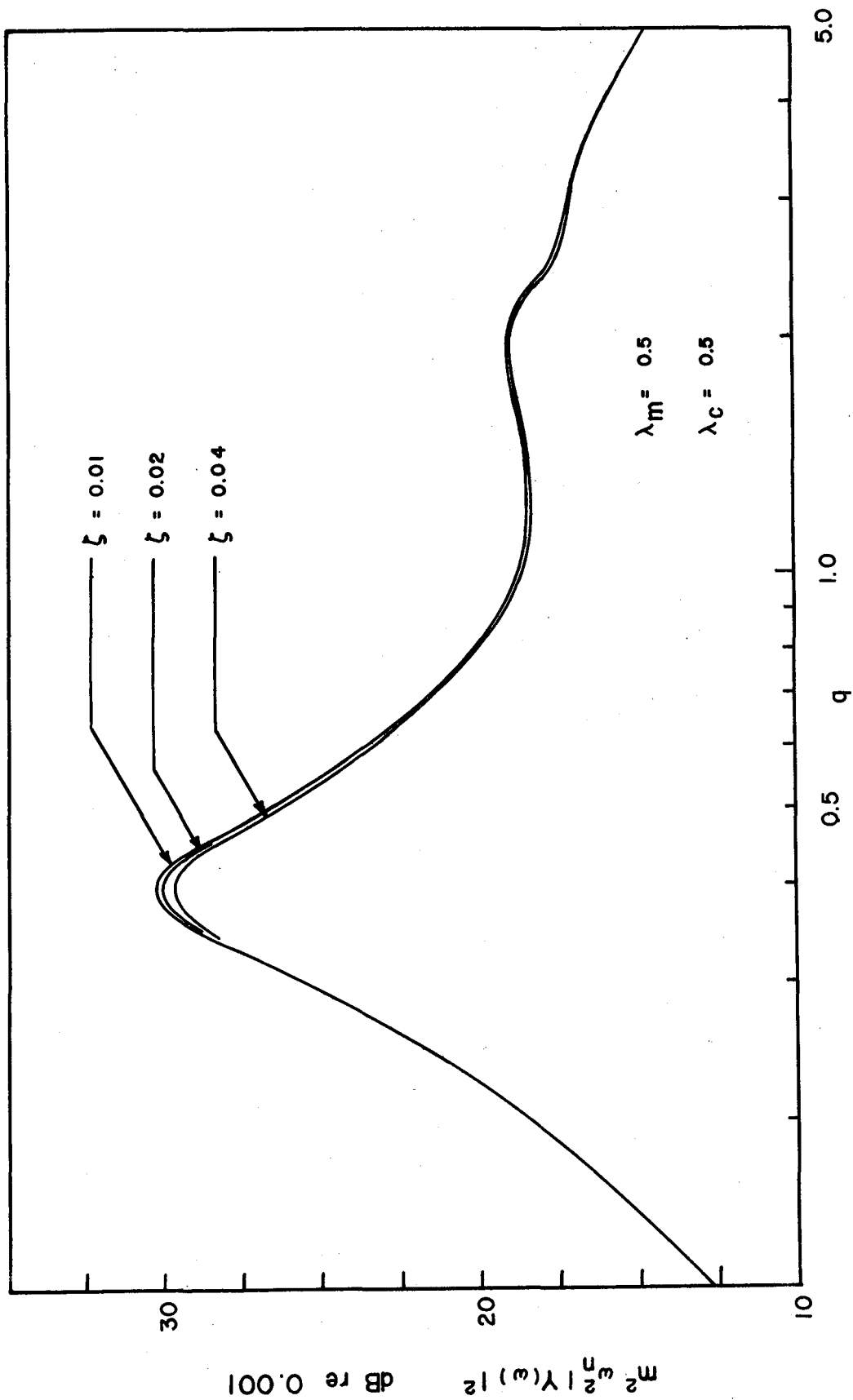


FIGURE 10.4 DIMENSIONLESS ADMITTANCE VS DIMENSIONLESS FREQUENCY

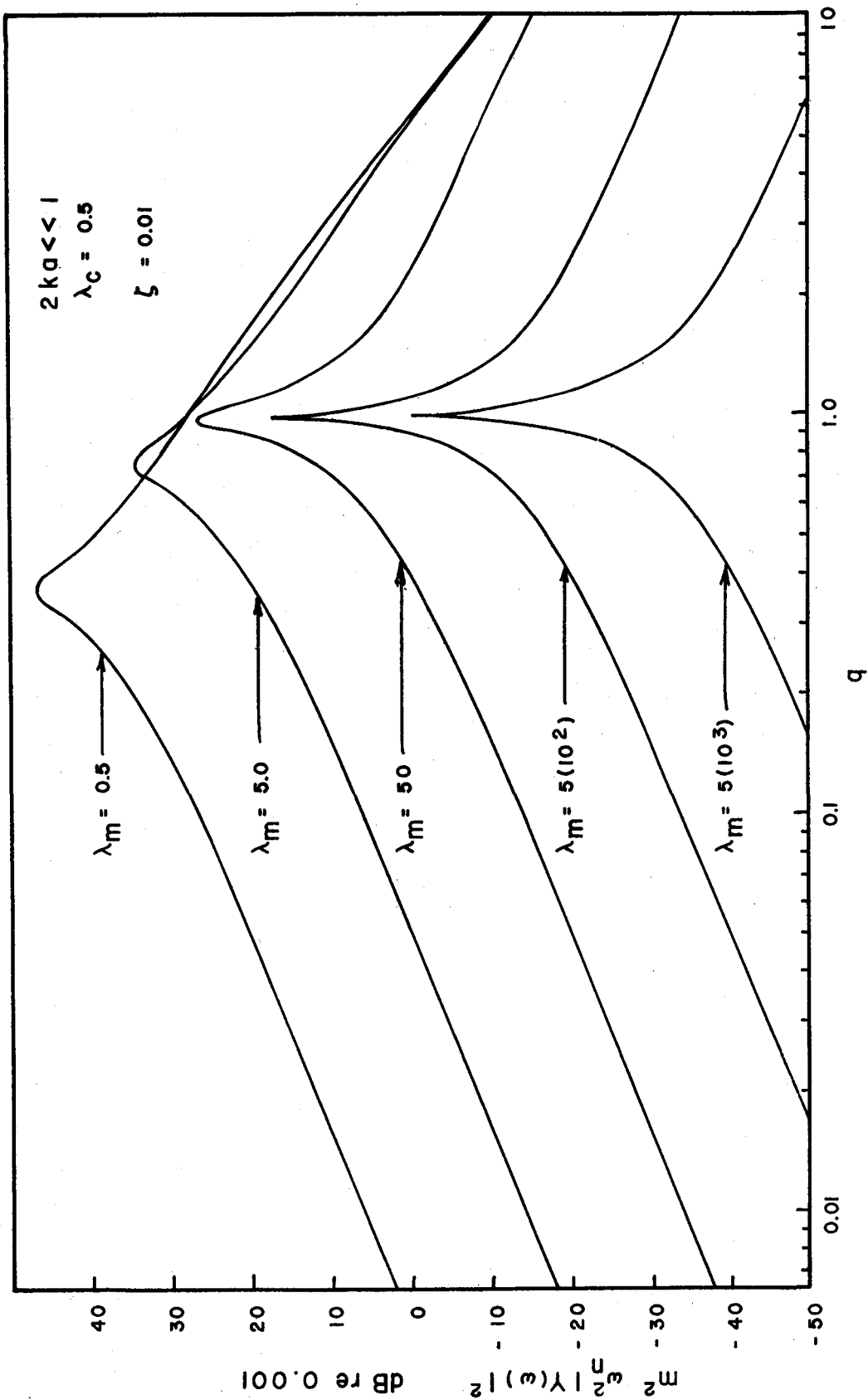


FIGURE 10.5 DIMENSIONLESS ADMITTANCE VS DIMENSIONLESS FREQUENCY

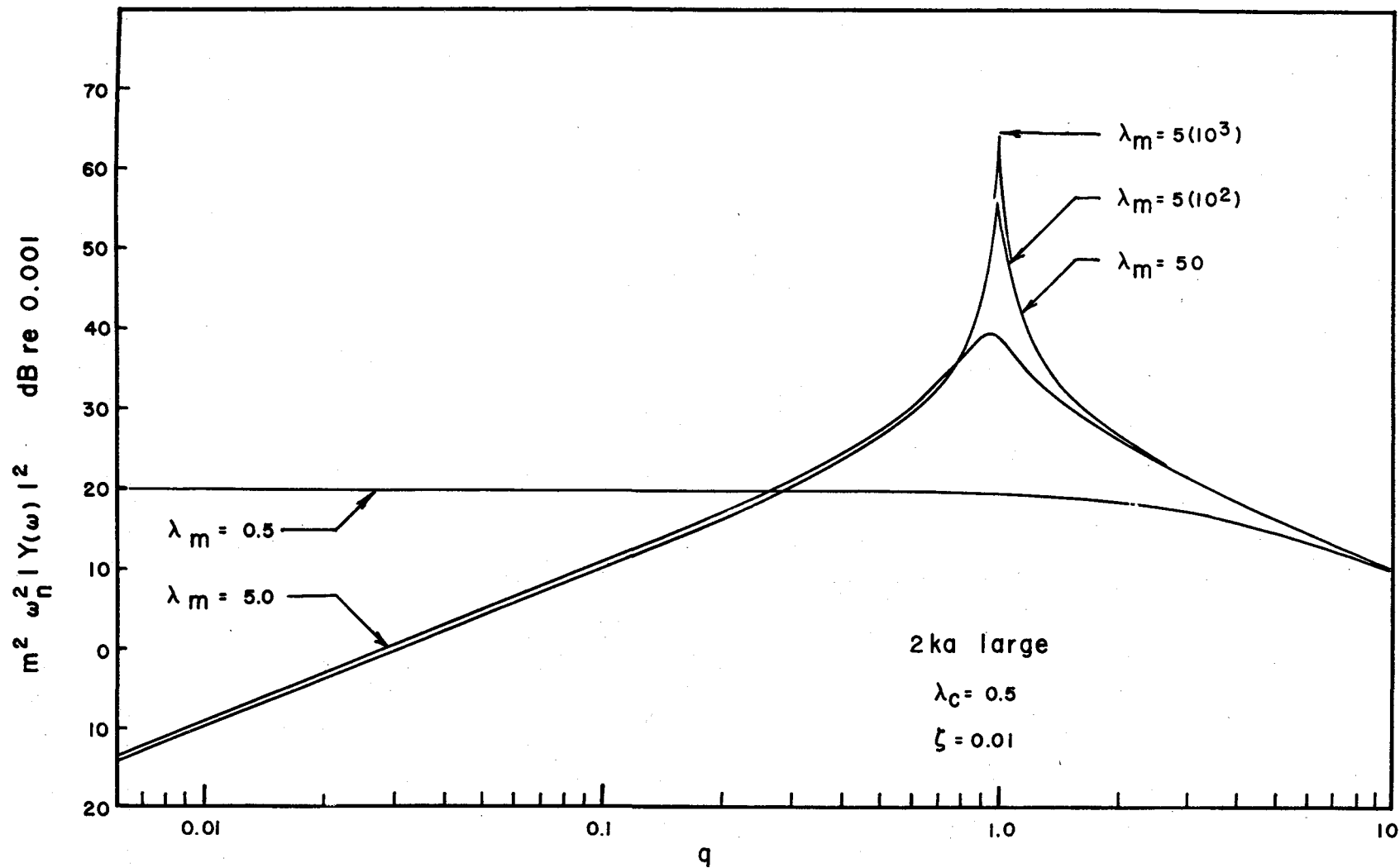


FIGURE 10.6 DIMENSIONLESS ADMITTANCE VS DIMENSIONLESS FREQUENCY



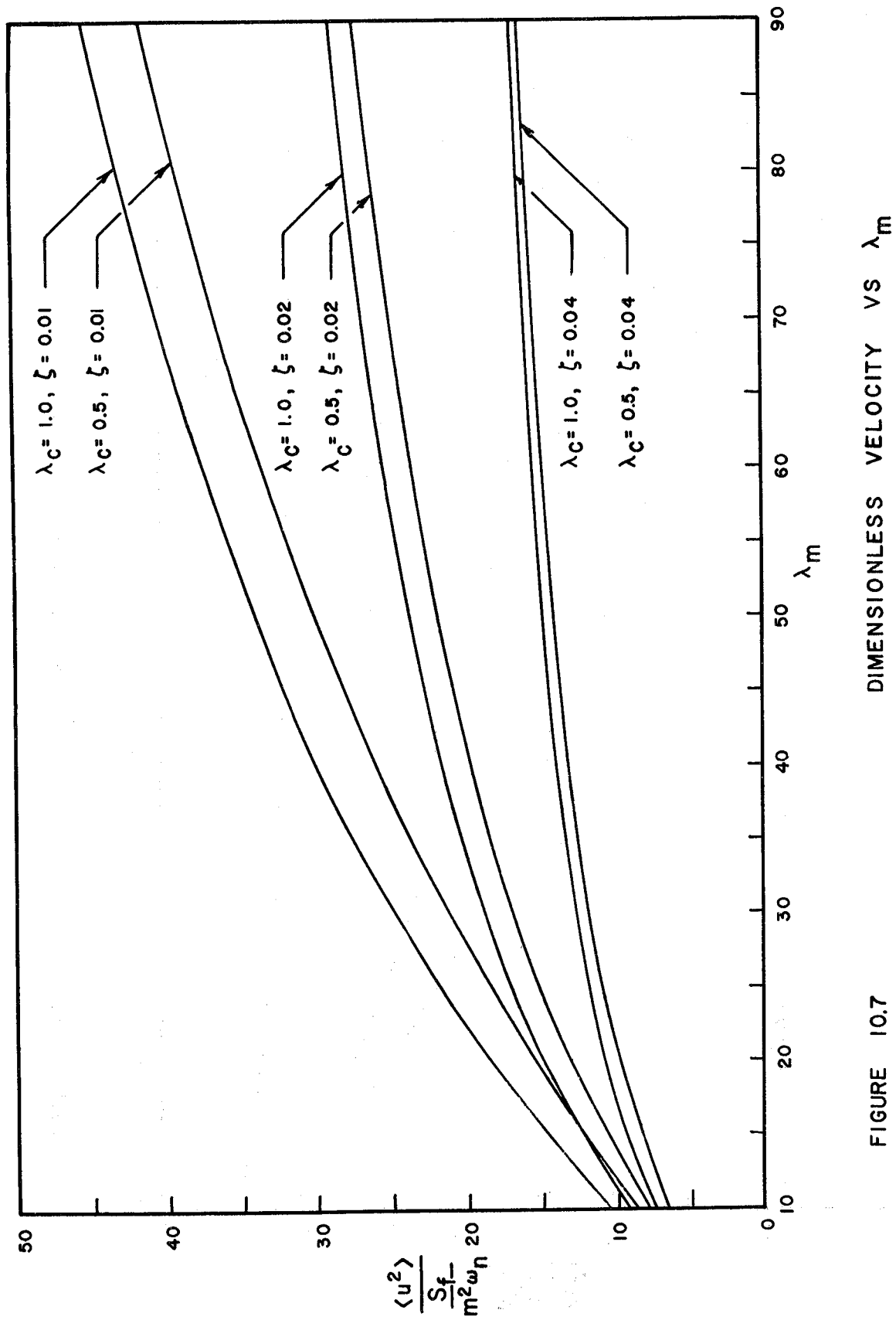


FIGURE 10.7

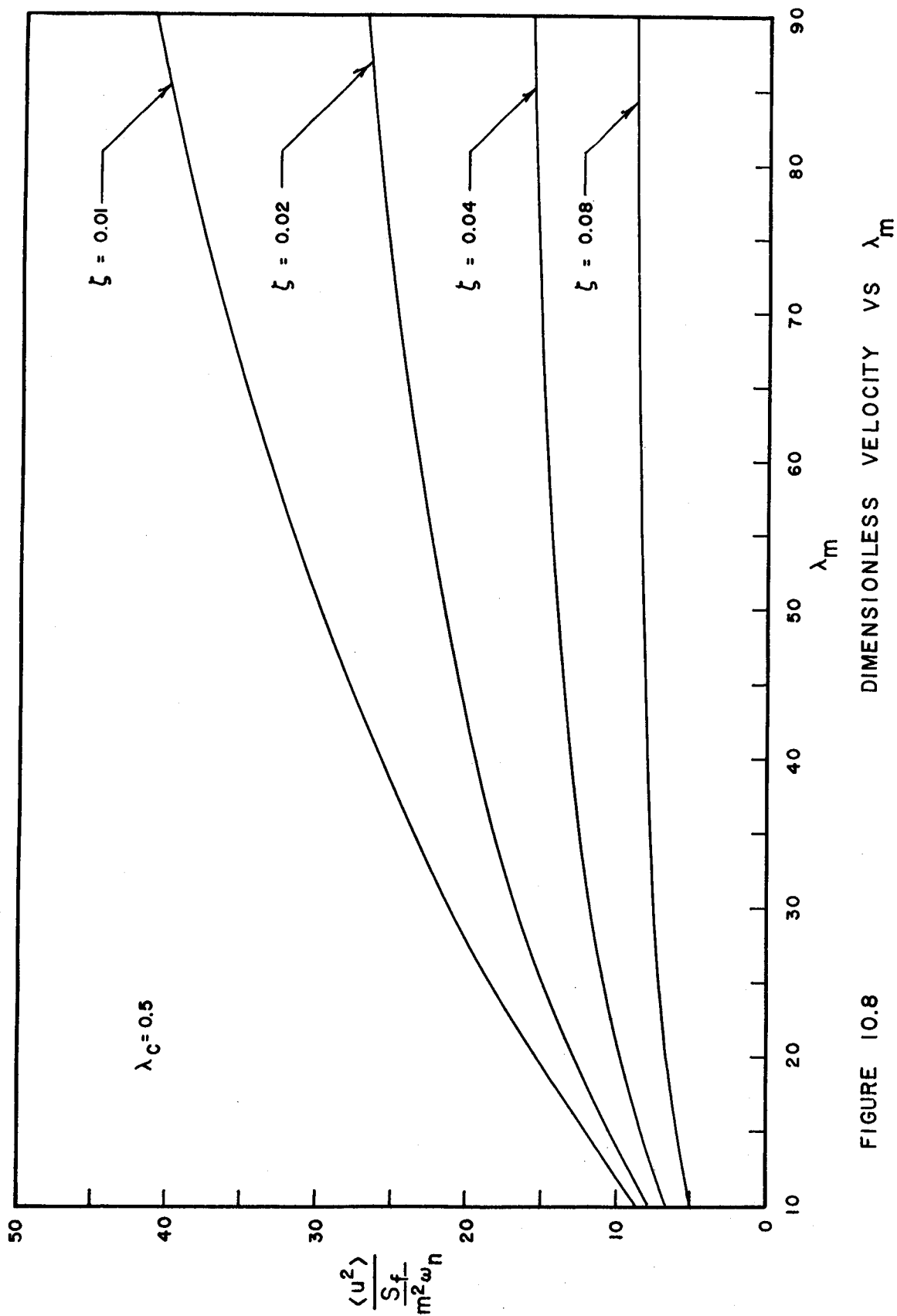


FIGURE 10.8

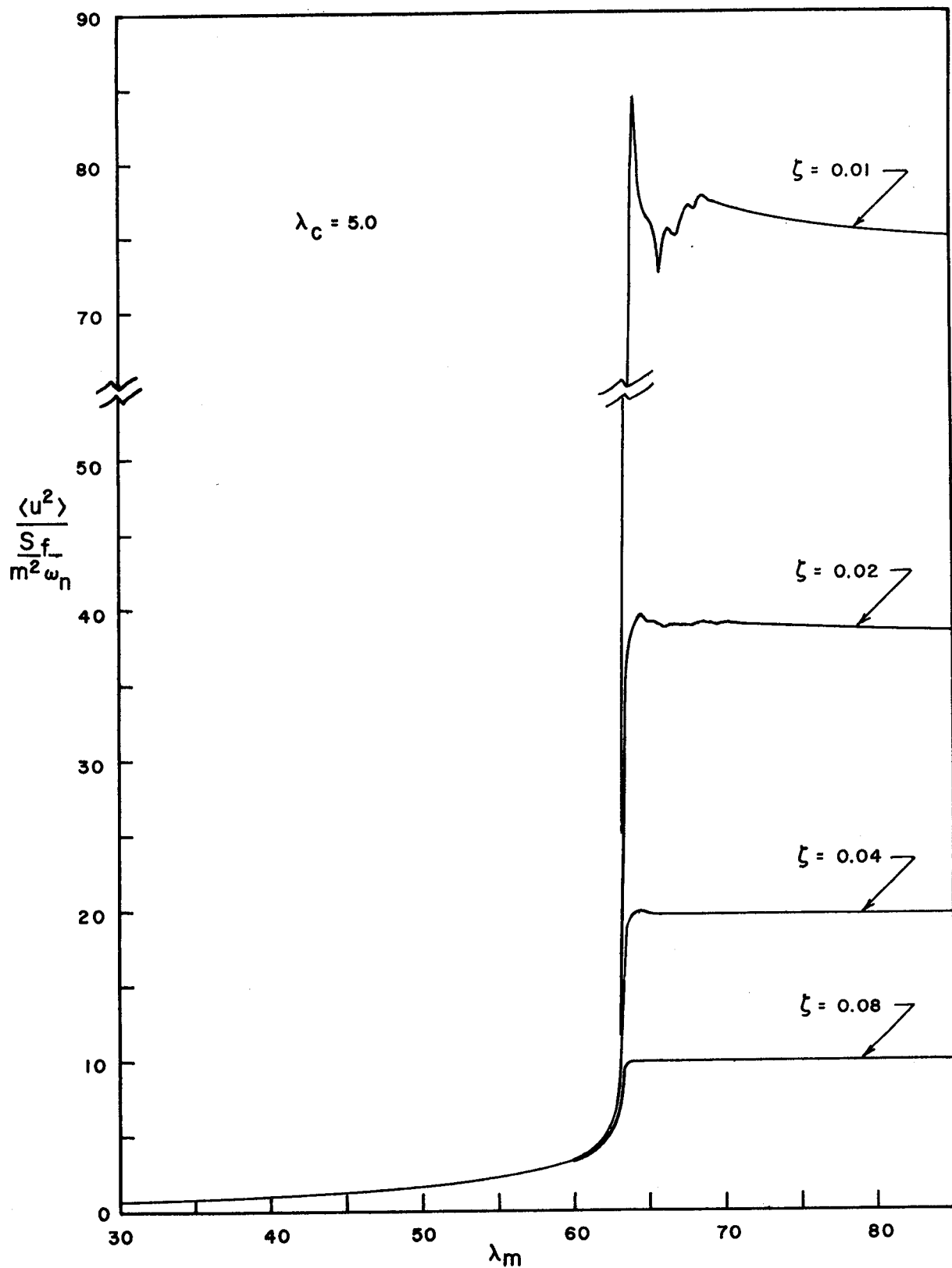


FIGURE 10.9

DIMENSIONLESS VELOCITY VS  $\lambda_m$

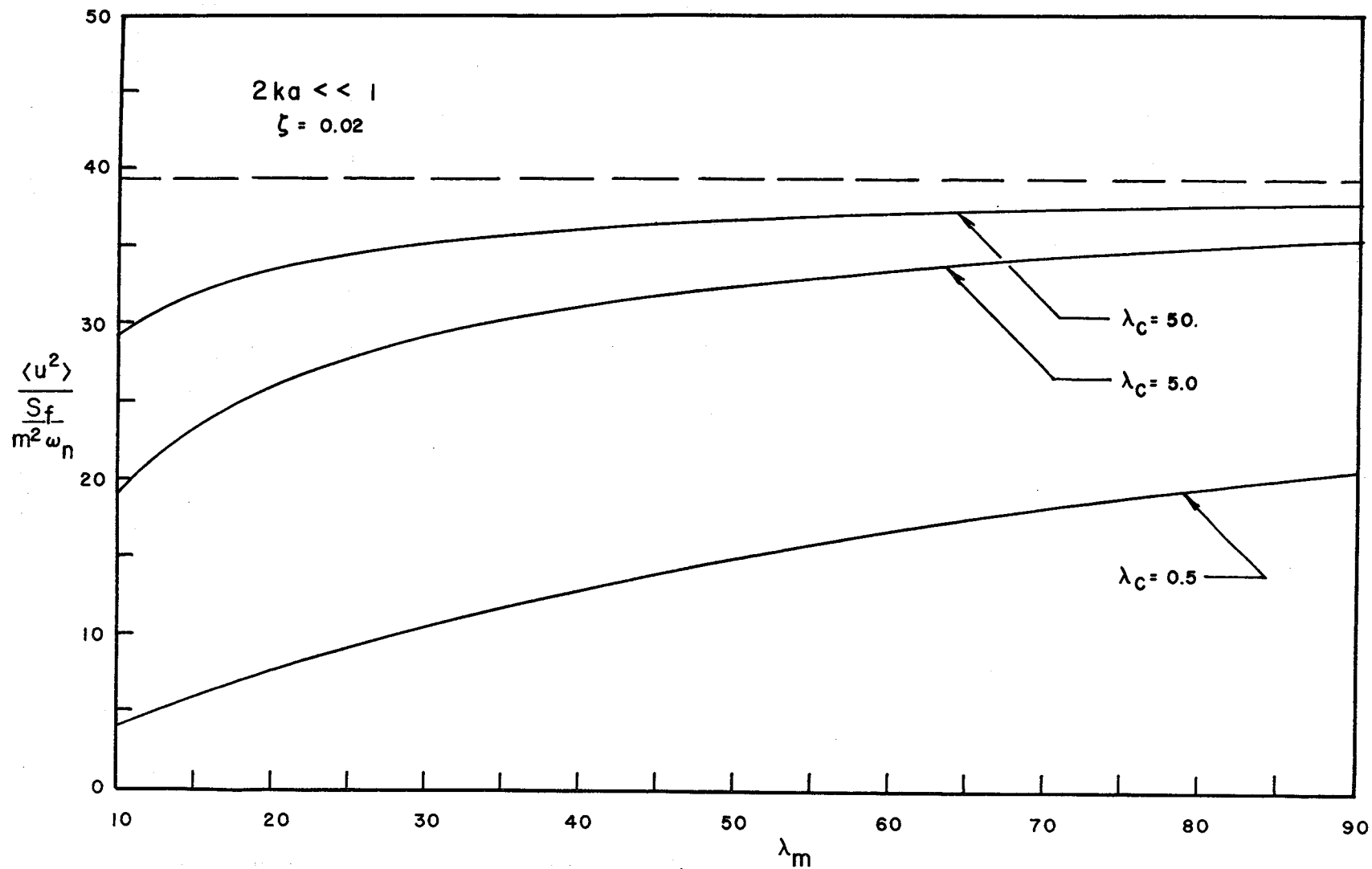


FIGURE 10.10

DIMENSIONLESS VELOCITY VS  $\lambda_m$

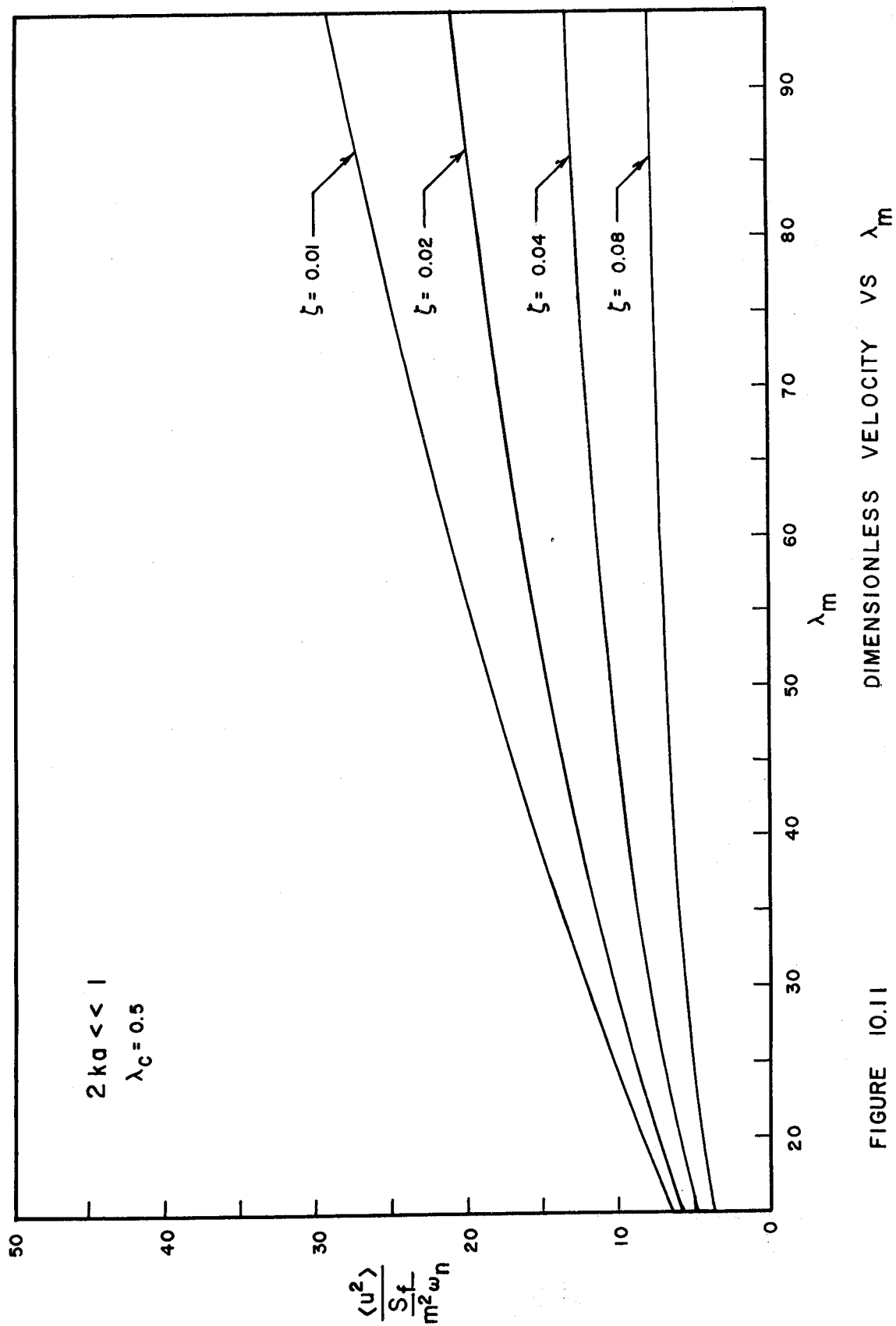
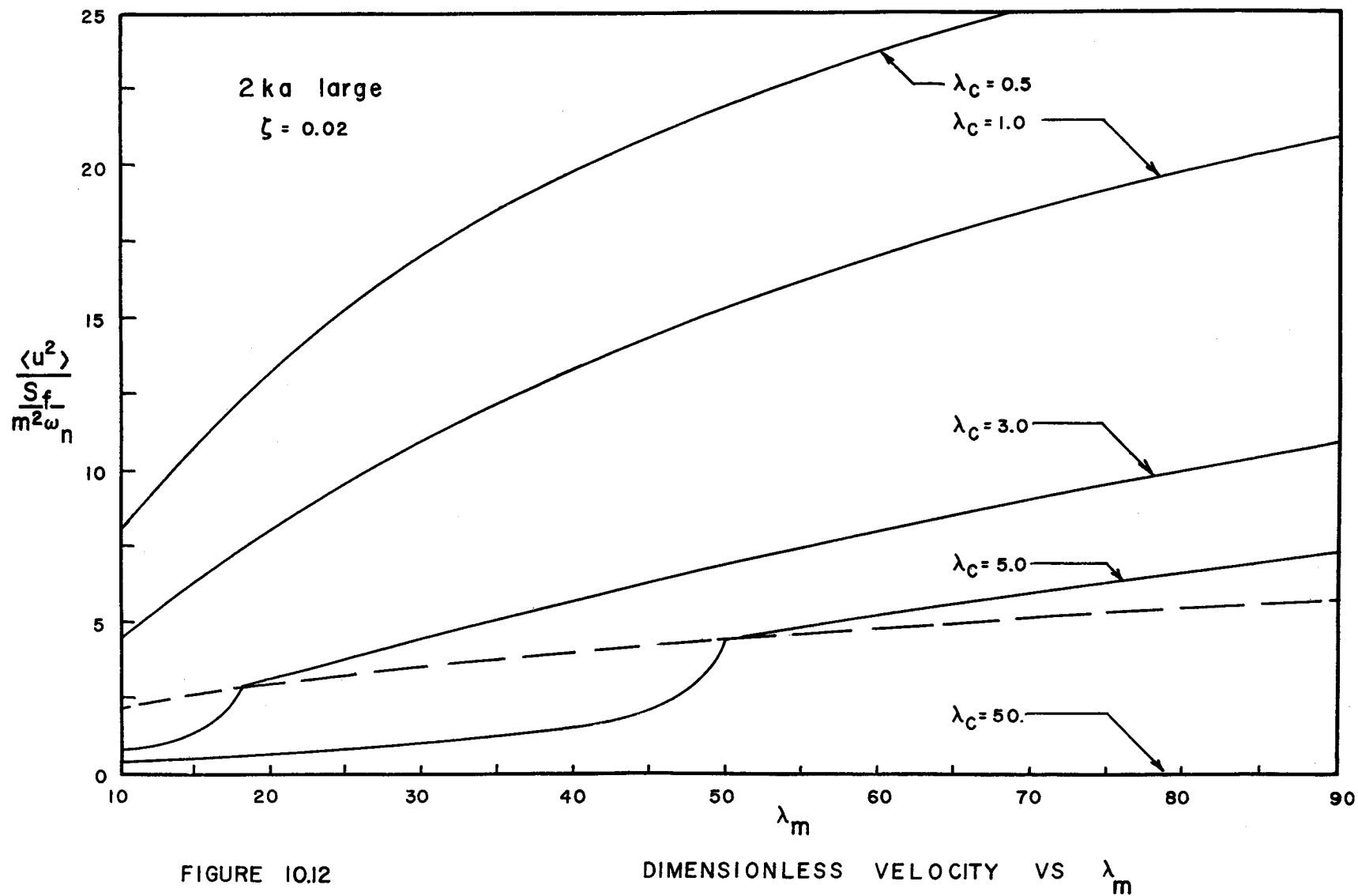


FIGURE 10.11



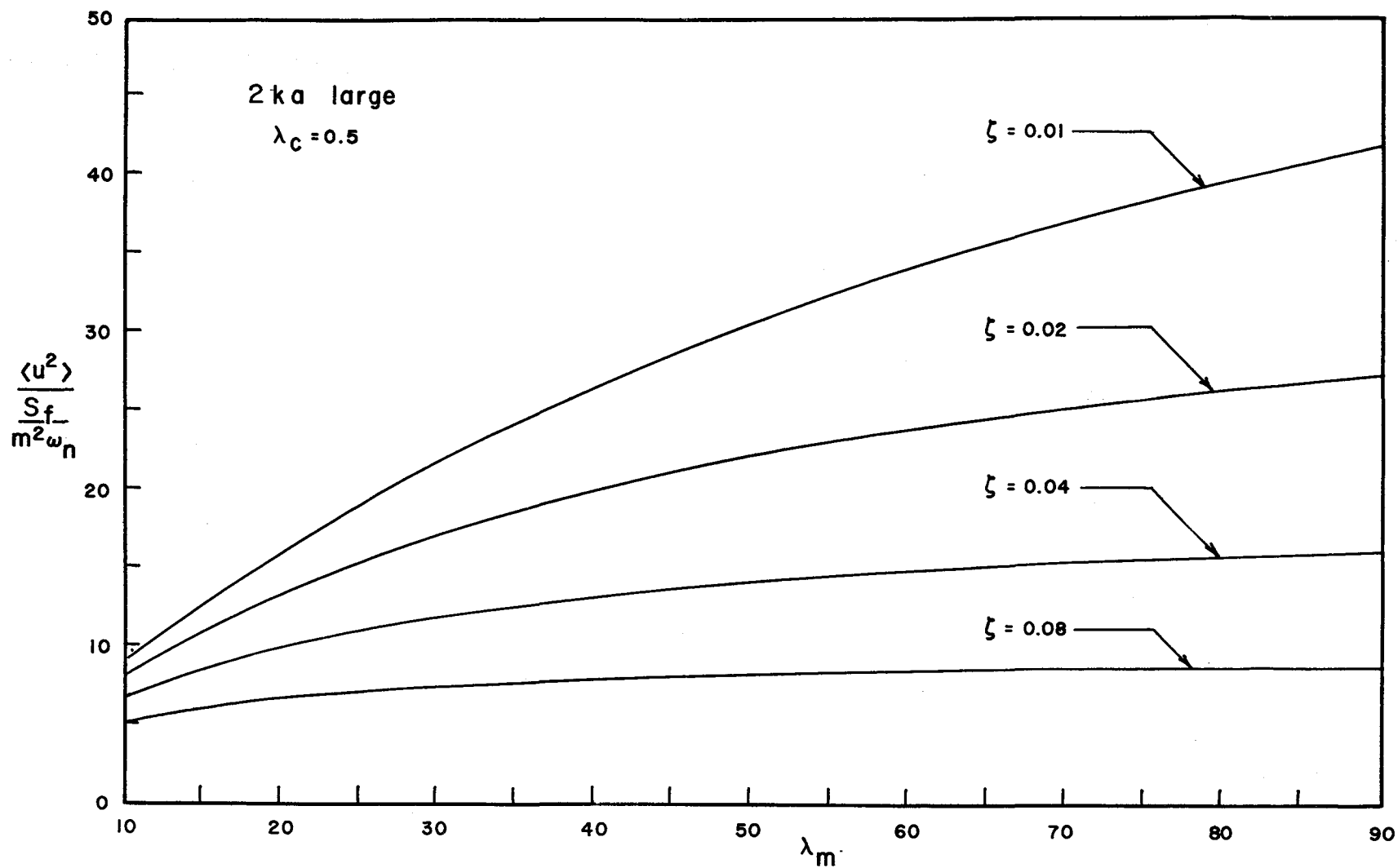


FIGURE 10.13

DIMENSIONLESS VELOCITY VS  $\lambda_m$

computer. Hence the results, equation (10.16) and (10.17), are not as useful as would be a closed-form solution.

## 10.2 Radiation Condition Verification

The general solution of the ordinary differential equation in the radial coordinate of the separated acoustic wave equation in cylindrical coordinates is

$$R(r) = C^{(1)} H_0^{(1)}(k_r r) + C^{(2)} H_0^{(2)}(k_r r), \quad (10.22)$$

where  $H_0^{(1)}$  and  $H_0^{(2)}$  are Hankel functions of the first and second kind of order one respectively for the case of a real separation constant,  $k_r > 0$ .

The radiation condition written in terms of  $R(r)$  is

$$\lim_{r \rightarrow \infty} \sqrt{r} [R'(r) - ik_r R(r)] = 0 \quad (10.23)$$

for  $k_r > 0$ . The importance of the radiation condition in an oscillation problem generated by sources in the finite domain is that the radiation condition is then sufficient to insure a unique solution. In other words, the radiation condition guarantees that sources are, in fact, sources--not sinks. As a result, the only admissible solution to the radial part of the separated wave equation in cylindrical coordinates is an outgoing wave front.

Equation (10.22) is substituted into equation (10.23) and asymptotic expressions for the Hankel functions and their derivatives are employed as follows:



$$H_o^{(1)}(k_r r) = \sqrt{\frac{2}{\pi k_r r}} e^{i(k_r r - \frac{\pi}{4})} = \frac{A^{(1)}}{\sqrt{r}} e^{i k_r r} \quad (10.24)$$

and

$$H_o^{(2)}(k_r r) = \sqrt{\frac{2}{\pi k_r r}} e^{-i(k_r r - \frac{\pi}{4})} = \frac{A^{(2)}}{\sqrt{r}} e^{-i k_r r} . \quad (10.25)$$

Further

$$\frac{d}{dr} H_o^{(1)}(k_r r) = A^{(1)} e^{i k_r r} \left( \frac{i k_r r - \frac{1}{2}}{(\sqrt{r})^3} \right) , \quad (10.26)$$

and

$$\frac{d}{dr} H_o^{(2)}(k_r r) = A^{(2)} e^{-i k_r r} \left( \frac{-i k_r r - \frac{1}{2}}{(\sqrt{r})^3} \right) . \quad (10.27)$$

Substitution gives that

$$\begin{aligned} \lim_{r \rightarrow \infty} \sqrt{r} [R'(r) - i k_r R(r)] &= \lim_{r \rightarrow \infty} [C^{(1)} A^{(1)} e^{i k_r r} \\ &\times \left( \frac{i k_r r - \frac{1}{2}}{r} - i k_r \right) + C^{(2)} A^{(2)} e^{-i k_r r} \left( \frac{-i k_r r - \frac{1}{2}}{r} - i k_r \right)] . \end{aligned}$$

Rewriting and simplifying produces

$$\begin{aligned} \lim_{r \rightarrow \infty} \sqrt{r} [R'(r) - i k_r R(r)] &= \lim_{r \rightarrow \infty} [C^{(1)} A^{(1)} e^{i k_r r} \\ &\times \left( \frac{-\frac{1}{2}}{r} + C^{(2)} A^{(2)} e^{-i k_r r} \left( \frac{-2i k_r r - \frac{1}{2}}{r} \right) \right] . \end{aligned} \quad (10.28)$$

The limit of a sum is equal to the sum of the limits provided that the

separate limits exist. The limit of the first term of equation (10.28) indeed exists and is equal to zero in the limit of large  $r$  but the second term does not have a limit. Hence in order for the radiation condition to be fulfilled,  $C^{(2)}$  must equal zero. This requirement eliminates the second kind of Hankel function from equation (10.22), and the result is

$$R(r) = C^{(1)} H_0^{(1)}(k_r r) \quad (10.29)$$

for  $k_r$  real and positive.

Proceeding in a similar manner for  $k_r = ik_r > 0$ , the analogue to equation (10.22) is

$$R(r) = B^{(1)} H_0^{(1)}(ik_r r) + B^{(2)} H_0^{(2)}(ik_r r), \quad (10.30)$$

and the radiation condition is

$$\lim_{r \rightarrow \infty} \sqrt{r} [R'(r) - \bar{k}_r R(r)] = 0. \quad (10.31)$$

Substituting equation (10.30) into equation (10.31) and utilizing asymptotic expressions for the Hankel functions yields

$$\begin{aligned} \lim_{r \rightarrow \infty} \sqrt{r} [R'(r) - \bar{k}_r R(r)] &= \lim_{r \rightarrow \infty} [B^{(1)} A^{(1)} e^{-\bar{k}_r r} \\ &\times (-2\bar{k}_r - \frac{1}{r}) + B^{(2)} A^{(2)} e^{\bar{k}_r r} (-\frac{1}{r})] . \end{aligned} \quad (10.32)$$

It can be shown in this case that  $B^{(2)}$  must equal zero in order for the limit of the sum to exist and be zero; hence equation (10.30) becomes

$$R(r) = B^{(1)} H_0^{(1)}(ik_r r) \quad \text{for} \quad \bar{k}_r > 0. \quad (10.33)$$

Both equations (10.29) and (10.33) are the solutions to the radial part of the separated wave equation employed in the previous analytical developments of Chapter 4.

### 10.3 Computer Programs

This section presents the computer programs employed to evaluate the radiation resistance as given in equations (4.73) and (4.121).

The program contained in section 10.3.1 is employed to evaluate the radiation resistance for axisymmetric mode shapes. Input data about shell material and geometry as well as fluid properties are employed to calculate the individual terms of the series. Summation is terminated at the point such that  $n$  is greater than  $\eta/(\pi a/L)$  or upon convergence of the series to the degree that the absolute value of the last term divided by the sum of all previously calculated terms is less than or equal to 0.1 of one percent.

Section 10.3.2 presents a program which is employed to compute an average radiation resistance. The program of section 10.3.1 is utilized as a function subprogram in a Simpson's Rule integration scheme which averages the radiation resistance for axisymmetric mode shapes.

The program of section 10.3.3 is utilized to evaluate the radiation resistance for lobar mode shapes. Input data concerning shell material and geometry as well as fluid properties are employed to calculate the individual terms of the series. Summation is terminated at the point such that the absolute value of the last term divided by the sum of all previously calculated terms is less than or equal to 0.1 of one percent.

### 10.3.1 Program to Compute Radiation Resistance for Axisymmetric Mode Shapes

```
1 FORMAT('1',19X,'DIMENSIONLESS RADIATION RESISTANCE')
2 FORMAT(' ',23X,'FOR AN AXISYMMETRIC SHELL')
3 FORMAT(' ',31X,'MODE SHAPE')
4 FORMAT('0',23X,'SHELL MAT L -7075-ALUMINUM')
40 FORMAT(' ',5X,'DENSITY RATIO = ',E11.4,10X,'SPEED RATIO = ',E11.4)
5 FORMAT('0',9X,'A/L = ',E11.4,20X,'H/L = ',E11.4)
6 FORMAT('0',14X,'ETA',05X,'RADIATION RESISTANCE',4X,'RDB',11X,'N')
7 FORMAT('0',10X,E11.4,5X,E11.4,5X,E11.4,5X,I4)
  X(A,N,B)=A**2-(3.1415927*N*B)**2
  CHIN(A,B,N,C,D,E)=A*B*(((3.1415927*N)**2*C*D)**2/12.+(1.-E**2))
  WRITE(3,1)
  WRITE(3,2)
  WRITE(3,3)
  PI=3.1415927

C
C  SHELL MATERIAL AND FLUID PROPERTY PARAMETERS
C  SHELL MATERIAL - 7075 ALUMINUM
C

  E=10.4E+06
  RHOS=0.101
  PRATO=0.333
  DRATO=4.4276E-05/RHOS
  SRATO=(32.1739*E)/(RHOS*(1.-PRATO**2)*(1117.))**2
  WRITE(3,4)
  WRITE(3,40) DRATO,SRATO

C
C  GEOMETRY PARAMETERS
C

  HOL=.1E-04
  AOL=.1E-01
  WRITE(3,5) AOL,HOL
  WRITE(3,6)
  DO 50 K=1,9
  DO 50 J=0,9
  DO 50 I=0,4
  ETA=.100*K+.01*J+.002*I
  TOPSUM=0.0
  BOTSUM=0.0
  N=1
10 CONTINUE
  IF((N.GT.ETA/(PI*AOL)).AND.(N.EQ.1)) GO TO 25
  IF(N.EQ.1) GO TO 12
  IF(ABS(TOPTER/TOPSUM).LE.0.0001) GO TO 15
12 CONTINUE
  XR=SQRT(X(ETA,N,AOL))
  IF(XR.GE.50.) GO TO 13
  CALL BESJ(XR,0,B0,0.0005,IE0)
  CALL BESJ(XR,1,B1,0.0005,IE1)
```

```

      GO TO 14
13  CONTINUE
      BO=SQRT(2./ (PI*XR)) *COS(XR-PI/4.)
      B1=SQRT(2./ (PI*XR)) *COS(XR-(3.*PI/4.))
14  CONTINUE
      QRATO=1./N
      FRATO=1./ (1. +(DRATO*AOL*BO)/ (HOL*XR*B1))
      FRATO=FRATO*QRATO
      ZET=CHIN(FRATO,SRATO,N,HOL,AOL,PRATO)
      TOPTER=(FRATO/(ETA**2-ZET))**2*(1./ (XR**3*B1**2))
      TOPSUM=TOPSUM+TOPTER
15  CONTINUE
      IF(N.EQ.1) GO TO 17
      IF(ABS(BOTTER/BOTSUM).LE.0.0001) GO TO 20
17  CONTINUE
      IF(XR.LT.0.5) GO TO 18
      CALL BESY(XR,1,BY1,IEY1)
      GO TO 19
18  CONTINUE
      BY1=-2./ (PI*XR)
19  CONTINUE
      BOTTER=(FRATO/(ETA**2-ZET))**2*(1.+BY1**2/B1**2)
      BOTSUM=BOTSUM+BOTTER
      N=N+2
      IF(N.GT.ETA/(PI*AOL)) GO TO 20
      GO TO 10
20  CONTINUE
      RRAD=4.*ETA**2*(TOPSUM/BOTSUM)
      RDB=10.*ALOG10(RRAD/100.)
      GO TO 30
25  CONTINUE
      RRAD=0
      RDB=-.1E30
      WRITE(3,7) ETA,RRAD,RDB,N
30  CONTINUE
      WRITE(3,7) ETA,RRAD,RDB,N
50  CONTINUE
      STOP
      END

```

### 10.3.2 Program to Average Radiation Resistance for Axisymmetric Mode Shapes

```

COMMON AOL,HOL,PI,DRATO,SRATO,PRATO
EXTERNAL F
DIMENSION DATA(7,8)
1  FORMAT('1',15X,'DIMENSIONLESS AVERAGE RADIATION RESISTANCE')
2  FORMAT(' ',23X,'FOR AN AXISYMMETRIC SHELL')
3  FORMAT(' ',31X,'MODE SHAPE')
4  FORMAT('0',23X,'SHELL MAT L -7075-ALUMINUM')

```

```

40 FORMAT(' ',5X,'DENSITY RATIO = ',E11.4,10X,'SPEED RATIO = ',E11.4)
5  FORMAT('0',9X,'A/L = ',E11.4,20X,'H/L = ',E11.4)
6  FORMAT('0',10X,'A',14X,'DEL',13X,'RPA',12X,'RPADB',9X,'N',6X,'IER'
*)
60 FORMAT(8F10.4)
7  FORMAT('0',05X,E11.4,05X,E11.4,05X,E11.4,05X,E11.4,04X,I4,04X,I4)
WRITE(3,1)
WRITE(3,2)
WRITE(3,3)
PI=3.1415927

```

C  
C  
C  
C

```

SHELL MATERIAL AND FLUID PROPERTY PARAMETERS
SHELL MATERIAL - 7075 ALUMINUM

```

```

E=10.4E+C6
RHOS=0.101
PRATO=0.333
DRATO=3.605E-02/RHOS
SRATO=(32.1739*E)/(RHOS*(1.-PRATO**2)*(4859.))**2)
WRITE(3,4)
WRITE(3,40) DRATO,SRATO

```

C  
C  
C

```

GEOMETRY PARAMETERS

```

```

HOL=.1E-04
AOL=.1E-01
WRITE(3,5) AOL,HOL
WRITE(3,6)
A=PI*AOL+.0000001
READ(1,60)((DATA(I,J),J=1,8),I=1,7)
DO 50 I=1,7
DO 50 J=1,8
B=DATA(I,J)
CALL INTRL(F,A,B,0.001,12,SII,S,N,IER)
DEL=B-A
RPA=S/DEL
RPADB=10.*ALOG10(RPA/100.)
WRITE(3,7) A,DEL,RPA,RPADB,N,IER
A=B

```

```

50 CONTINUE
STOP
END

```

```

SUBROUTINE INTRL(F,A,B,DEL,IMAX,SII,S,N,IER)
SII=0.0
S=0.0
N=0
BA=B-A
IF (BA) 20,19,20

```

```

19 IER=1
   RETURN
20 IF (DEL) 22,22,23
22 IFR=2
   RETURN
23 IF (IMAX-1) 24,24,25
24 IER=3
   RETURN
25 X=BA/2. +A
   NHALF=1
   SUMK=F(X)*BA*2./3.
   S=SUMK+(F(A)+F(B))*BA/6.
   DO 28 I=2, IMAX
     SII=S
     S=(S-SUMK/2.)/2.
     NHALF=NHALF*2
     ANHLF=NHALF
     FRSTX=A+(BA/ANHLF)/2.
     SUMK=F(FRSTX)
     XK=FRSTX
     KLAST=NHALF-1
     FINC=BA/ANHLF
     DO 26 K=1, KLAST
       XK=XK+FINC
26 SUMK=SUMK+F(XK)
   SUMK=SUMK*2. βBA/(3. ANHLF)
   S=S+SUMK
27 IF (ABS(S-SII)-ABS(DEL*S)) 29,28,28
28 CONTINUE
   IER=4
   GO TO 30
29 IER=0
30 N=2*NHALF
   RETURN
   END

```

```

FUNCTION F(XTA)
COMMON AOL, HOL, PI, DRATO, SRATO, PRATO
X(A,N,B)=A**2-(3.1415927*N*B)**2
CHIN(A,B,N,C,D,E)=A*B*((3.1415927*N)**2*C*D)**2/12.+(1.-E**2)
TOPSUM=0.0
BOTSUM=0.0
N=1
10 CONTINUE
   IF((N.GT.XTA/(PI*AOL)).AND.(N.EQ.1)) GO TO 25
   IF(N.EQ.1) GO TO 12
   IF(ABS(TOPTER/TOPSUM).LE.0.0001) GO TO 15
12 CONTINUE
   XR=SQRT(X(XTA,N,AOL))
   IF(XR.GE.50.) GO TO 13

```

```

      CALL BESJ(XR,0,B0,0.0005,IE0)
      CALL BESJ(XR,1,B1,0.0005,IE1)
      GO TO 14
13  CONTINUE
      B0=SQRT(2./(PI*XR))*COS(XR-PI/4.)
      B1=SQRT(2./(PI*XR))*COS(XR-(3.*PI/4.))
14  CONTINUE
      QRATO=1./N
      FRATO=1./(1.+(DRATO*AOL*B0)/(HOL*XR*B1))
      FRATO=FRATO*QRATO
      ZET=CHIN(FRATO,SRATO,N,HOL,AOL,PRATO)
      TOPTER=(FRATO/(XTA**2-ZET))**2*(1./(XR**3*B1**2))
      TOPSUM=TOPSUM+TOPTER
15  CONTINUE
      IF(N.EQ.1) GO TO 17
      IF(ABS(BOTTER/BOTSUM).LE.0.0001) GO TO 20
17  CONTINUE
      IF(XR.LT.0.5) GO TO 18
      CALL BESY(XR,1,BY1,IEY1)
      GO TO 19
18  CONTINUE
      BY1=-2./(PI*XR)
18  CONTINUE
      BOTTER=(FRATO/(XTA**2-ZET))**2*(1.+BY1**2/B1**2)
      BOTSUM=BOTSUM+BOTTER
      N=N+2
      IF(N.GT.XTA/(PI*AOL)) GO TO 20
      GO TO 10
20  CONTINUE
      RRAD=4.*XTA**2*(TOPSUM/BOTSUM)
      F=RRAD
      GO TO 30
25  CONTINUE
      RRAD=0.0
      F=RRAD
30  CONTINUE
      RETURN
      END

```

### 10.3.3 Program to Compute Radiation Resistance for Lobar Mode Shapes

```

1  FORMAT('1',19X,'DIMENSIONLESS RADIATION RESISTANCE')
2  FORMAT(' ',21X,'FOR A SHELL EXHIBITING A LOBAR')
3  FORMAT(' ',31X,'MODE SHAPE')
4  FORMAT('0',23X,'SHELL MAT L -7075-ALUMINUM')
40 FORMAT(' ',5X,'DENSITY RATIO = ',E11.4,10X,'SPEED RATIO + ',E11.4)
5  FORMAT('0',27X,'H/A = ',E11.4)
6  FORMAT('0',14X,'ETA',05X,'RADIATION RESISTANCE',4X,'RDB',11X,'M')
7  FORMAT('0',10X,E11.4,5X,E11.4,5X,E11.4,5X,I4)
      CHIM(A,B,M)=(1./12.)*A*(B**2)*M**4

```



```

WRITE(3,1)
WRITE(3,2)
WRITE(3,3)
PI=3.1415927

C
C SHELL MATERIAL AND FLUID PROPERTY PARAMETERS
C SHELL MATERIAL - 7075-ALUMINUM
C

E=10.4E+06
RHOS=0.101
PRATO=0.333
DRATO=3.605E-02/RHOS
SRATO=(32.1739*E)/(RHOS*(1.-PRATO**2)*(4859.))**2)
WRITE(3,4)
WRITE(3,40) DRATO,SRATO

C
C GEOMETRY PARAMETERS
C

DO 50 I=1,2
DO 50 J=1,5
HOA=(.001E-01)*J*10**I
WRITE(3,5) HOA
WRITE(3,6)
DO 50 K=0,100
DO 50 L=0,1
ETA=50.0+1.0*K+0.5*L
TOPSUM=0.0
BOTSUM=0.0
M=1
10 CONTINUE
IF(M.LE.2) GO TO 100
MFAC=MFAC*(M-1)
GO TO 11
100 CONTINUE
MFAC=1
11 CONTINUE
IF(M.EQ.1) GO TO 12
IF(ABS(TOPTER/TOPSUM).LE.0.0001) GO TO 15
12 CONTINUE
IF(ETA.GE.50.) GO TO 13
CALL BESJ(ETA,M,BO,0.0005,IEO)
MADD=M+1
CALL BESJ(ETA,MADD,B1,0.0005,IE1)
GO TO 14
13 CONTINUE
BO=SQRT(2./(PI*ETA))*COS(ETA-PI/4.-M*PI/2.)
MADD=M+1
B1=SQRT(2./(PI*ETA))*COS(ETA-PI/4.-MADD*PI/2.)
14 CONTINUE
FRATO=1./(1.-(DRATO*(1./HOA)*(BO/(M*BO-ETA*B1))))
ZET=CHIM(SRATO,HOA,M)*FRATO
TOPTER=(FRATO/(ZET-ETA**2))**2*(1./(M*BO-ETA*B1))**2

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```

      TOPSUM=TOPSUM+TOPTER
15  CONTINUE
      IF (M.EQ.1) GO TO 17
      IF (ABS(BOTTER/BOTSUM).LE.0.0001) GO TO 20
17  CONTINUE
      IF ((M.EQ.0).AND.(ETA.LT.0.5)) GO TO 200
      IF (ETA.LT.0.5) GO TO 18
      IF (ETA.GE.50.) GO TO 300
      CALL BESY(ETA,M,BYO,IEYO)
      CALL BESY(ETA,MADD,BY1,IEY1)
      GO TO 19
18  CONTINUE
      BYD=((-.2.**M*MFAC)/PI)*(1./ETA**M)
      BY1=((-.2.**MADD*MFAC*M)/PI)*(1./ETA**MADD)
      GO TO 19
200 CONTINUE
      BYO=(2./PI)*ALOG10(ETA)
      BY1=-.2./(PI*ETA)
      GO TO 19
300 CONTINUE
      BYO=SQRT(2./(PI*ETA))*SIN(ETA-PI/4.-M*PI/2.)
      BY1=SQRT(2./(PI*ETA))*SIN(ETA-PI/4.-MADD*PI/2.)
19  CONTINUE
      BOTTER=(FRATO/(ZET-ETA**2))**2*(1.+(M*BYO-ETA*BY1)**2/(M*BO-ETA*B1
*)**2)
      BOTSUM=BOTSUM+BOTTER
      M=M+2
      GO TO 10
20  CONTINUE
      RRAD=8.*ETA*(TOPSUM/BOTSUM)
      RDB=10.*ALOG10(RRAD/100.)
      WRITE(3,7) ETA,RRAD,RDB,M
50  CONTINUE
      STOP
      END

```