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I. Introduction

A previous report (Fuller and Ward, 1969) discussed the solution for a concentrically layered moon in a plane electromagnetic field and presented frequency response curves for particular layered models derived from the work of England et al (1968) and Ward (1969). It is somewhat naive to expect planets to be composed of perfectly homogeneous concentric layers. A concentric model can be instructive, nevertheless. The utility of a layered model to simulate a planet is usually based upon the assumption that a continuous distribution of electrical properties may be approximated by an arbitrarily large number of discrete layers. Wait (1961) has remarked that the convergence of this process is usually very satisfactory provided that the refractive index is not near zero or infinitely large. The intent of this report is to investigate that assumption for a particular conductivity distribution and to illustrate the convergence of the layered model solution to the solution for a sphere with a conductivity which is functionally dependent upon the radial coordinate r. This work is intended to justify the utility of layered models in theoretical computations which are directed toward the goal of estimating the electrical parameters of the inner moon by observing the behavior of the natural time varying magnetic field in the lunar environment. With this purpose in mind, the report will consider the problem only as it pertains to spherical geometry and low frequencies, such that the wavelength in the lunar sphere is of the order of the dimension of the sphere. A later report will consider the problem at high frequencies where plane geometry is a sufficient approximation and where the intent is to investigate the near surface conductivity distribution in the moon.

The problem of an inhomogeneous sphere with a conductivity which obeys some functional dependence with radial coordinate r has been discussed by Bremmer (1949), Tai (1958), Wait (1962), Negi (1962, 1967) and others. For particular functional dependencies, analytic solutions are obtainable and have been presented in many of the aforementioned references. This report will adhere to the notation developed by Fuller and Ward (1969) in order to effect convenient comparisons between the layered solution and the solution for the continuous distribution of conductivity.

II. Theoretical Formulation

In the case of inhomogeneity, or dependence of the electrical parameters upon the radial coordinate r, it is both necessary and desirable to consider the method of solution and the relationship of fields and potential in some detail in order to ensure that the solutions and potentials chosen do indeed satisfy all requirements of the problem. Either of two possible approaches is sufficient to establish the solution; one approach is to derive potentials from Maxwell's equations and the other is to assume relations between fields and potentials and demonstrate that the fields satisfy Maxwell's equations under certain conditions on the potentials. The latter approach is used here in order to retain the notation developed in a previous report (Fuller and Ward, 1969).

The geometry is illustrated in Figure 1. A plane wave, which is x-polarized and z-traveling is incident upon a sphere which has electrical properties which vary as a function of the radial coordinate. The electrical properties of the external medium are assumed constant and, in computations, will assume free space values. The outer radius, R_1, of the sphere is chosen to be one lunar radius (1738 Km).

With $e^{-i\omega t}$ time dependence suppressed, we seek to solve, in each medium, the following Maxwell equations:

$$\nabla \times \bar{E}(\bar{x},\omega) = \hat{z}\, \bar{H}(\bar{x},\omega) \tag{1}$$

$$\nabla \times \bar{H}(\bar{x},\omega) = \hat{y}\, \bar{E}(\bar{x},\omega) \tag{2}$$

where the position vector $\bar{x}$ denotes spatial dependence and angular frequency ω denotes frequency dependence. The impedivity, $\hat{z}$, and admittivity, $\hat{y}$, are given by

$$\hat{z} = i\omega\mu \tag{3}$$

$$\hat{y} = \sigma - i\omega\epsilon \tag{4}$$

and are, for this discussion, allowed to be functions of the radial coordinate r as well as of frequency. In accordance with Fuller and Ward (1969) we define a transverse electric potential Ψ and transverse magnetic potential Φ such that the fields are given by

$$\bar{E}(\bar{x},\omega) = \nabla\times\Psi r\bar{e}_r + \frac{1}{\hat{y}}\nabla\times\nabla\times\Phi r\bar{e}_r \tag{5}$$

$$H(\bar{x},\omega) = \nabla\times\Phi r\bar{e}_r + \frac{1}{\hat{z}}\nabla\times\nabla\times\Psi r\bar{e}_r \tag{6}$$

Equations (5) and (6) may be considered, at the outset, as merely assumptions. It remains to justify these assumptions by substituting the field expressions (5) and (6) into the Maxwell equations (1) and (2) in order to arrive at the conditions which Ψ and Φ must satisfy to ensure that the fields given by (5) and (6) satisfy Maxwell's equations. Substitution of (5) and (6) into (1) yields

$$\nabla\times\frac{1}{\hat{y}}\nabla\times\nabla\times\Phi r\bar{e}_r = \hat{z}\,\nabla\times\Phi r\bar{e}_r \tag{7}$$

as a condition on the potential Φ. Making use of the vector identity (Davis, 1961, p. 119)

$$\nabla\times f\bar{A} = f\,\nabla\times\bar{A} + \overline{\nabla f}\times\bar{A} \tag{8}$$

we may write the right hand side of (7) as

$$\hat{z}\,\nabla\times\Phi r\bar{e}_r = \nabla\times\hat{z}\,\Phi r\bar{e}_r - \nabla\hat{z}\times\Phi r\bar{e}_r \tag{9}$$

If $\hat{z} = \hat{z}(r)$ is a function of r only, then $\nabla \hat{z}$ and $\Phi r\bar{e}_r$ are parallel vectors, the second term on the right hand side of equation (9) is zero, and

$$\hat{z}\,\nabla \times \Phi r\bar{e}_r = \nabla \times \hat{z}\,\Phi r\bar{e}_r \tag{10}$$

Substitution of (10) into (7) gives

$$\nabla \times \left[\frac{1}{\hat{y}}\,\nabla \times \nabla \times \Phi r\bar{e}_r - \hat{z}\,\Phi r\bar{e}_r\right] = 0 \tag{11}$$

A curl-free vector may be expressed as the gradient of a scalar function so that, making use of the relation

$$k^2 = \hat{y}\,\hat{z} = \omega^2\mu\epsilon + i\mu\sigma\omega \tag{12}$$

we may write

$$\nabla \times \nabla \times \Phi r\bar{e}_r - k^2\Phi r\bar{e}_r = -\hat{y}\,\nabla f \tag{13}$$

where f is an arbitrary scalar function. Expanding equation (13) and equating the vector components in the θ and ϕ directions yields, respectively,

$$\frac{\partial^2(\Phi r)}{\partial r\,\partial\theta} = -\hat{y}\,\frac{\partial f}{\partial\theta} \tag{14}$$

$$\frac{\partial^2(\Phi r)}{\partial r\,\partial\phi} = -\hat{y}\,\frac{\partial f}{\partial\phi} \tag{15}$$

Since $\hat{y}$ is a function of r only, equations (14) and (15) may be integrated with respect to θ and ϕ respectively and will be identically satisfied if we require that the heretofore arbitrary scalar function f satisfy

$$\frac{\partial(\Phi r)}{\partial r} = -\hat{y} f \tag{16}$$

Equating the r components of the vector equation (13) and making use of (16) to substitute for f yields the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} + k^2\Phi = 0 \tag{17}$$

Equation (17) is the scalar Helmholtz equation in spherical coordinates (Harrington, 1961, p. 449) and may be written as

$$\nabla^2\Phi + k^2\Phi = 0 \tag{18}$$

The same procedure may be carried out to obtain the condition under which the fields derived from the potential Ψ will satisfy Maxwell's equations. Substitution of (5) and (6) into (2) gives

$$\nabla\times\frac{1}{\hat{z}}\nabla\times\nabla\times\psi r\bar{e}_r = \hat{y}\nabla\times\psi r\bar{e}_r \tag{19}$$

Making use of the vector identity (8) and the fact that $\hat{y}$ is a function of r only, equation (19) leads to the relation

$$\nabla\times\nabla\times\psi r\bar{e}_r - k^2\psi r\bar{e}_r = -\hat{z}\nabla g \tag{20}$$

where g is an arbitrary scalar function. The equality of the θ and ϕ components of equation (20) is ensured if the scalar function g satisfies

$$\frac{\partial(\psi r)}{\partial r} = -\hat{z}\, g \qquad (21)$$

Equating the r components of equation (20) and substituting (21) gives the condition

$$\nabla^2\psi + K^2\psi = 0 \qquad (22)$$

Equations (18) and (22) allow that $K^2 = \hat{y}\,\hat{z}$ is a function of r only, or a constant. In summary, the fields derived from equations (5) and (6) satisfy Maxwell's equation under the conditions (18) and (22) on Φ and Ψ, respectively, for electrical parameters $\hat{y}$ and $\hat{z}$ which are either constant or functions of r only.

The case when $K^2 = K^2(r)$ is of interest in this report and has been discussed by Tai (1958), Wait (1961, 1962), Negi (1962, 1967), Bremmer (1949) and others. By separation of variables (Harrington, 1961, p. 264), the solution of equation (22) may be written in the form

$$\psi(r,\theta,\phi) = S(r)\,P(\theta)\,H(\phi) \qquad (23)$$

where $S(r), P(\theta), H(\phi)$ respectively satisfy the following differential equations:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial S(r)}{\partial r}\right) + \left[K^2r^2 - n(n+1)\right]S(r) = 0 \qquad (24)$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right)+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]P(\theta)=0 \tag{25}$$

$$\frac{\partial^2 H(\phi)}{\partial\phi^2}+m^2 H(\phi)=0 \tag{26}$$

The solutions to equations (25) and (26) are (Harrington, 1961, p. 265)

$$P(\theta) = P_n^m(\cos\theta),\ Q_n^m(\cos\theta) \tag{27}$$

$$H(\phi) = \cos m\phi,\ \sin m\phi \tag{28}$$

where $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$ are the associated Legendre functions of the first and second kind, respectively. In order that $H(\phi)$ be single valued in the range $0 \le \phi < 2\pi$, m must be an integer (Harrington, 1961, p. 266). In order that $P(\theta)$ is finite in the range $0 \le \theta < \pi$, the Legendre function of the first kind, with n an integer, must be chosen (Harrington, 1961, p. 266).

For K^2 = constant, equation (24) reduces to the spherical Bessel equation with solutions given by the spherical Bessel functions (Morse and Feshbach, 1953, p. 622)

$$S(r) = j_n(Kr),\ y_n(Kr),\ h_n^{(1)}(Kr),\ h_n^{(2)}(Kr) \tag{29}$$

The solutions of equation (29) are thus appropriate to the region surrounding the sphere, in which the electrical parameters are constant. For $e^{-i\omega t}$ time dependence, $j_n(Kr)$ and $h_n^{(1)}(Kr)$ are linearly independent solutions which represent incoming and outgoing waves, respectively.

The general solutions to equation (24), when $K^2 = \hat{y}\hat{z}$ is allowed to be a function of r, depend upon the particular functional dependence which is assumed. This report will investigate one such functional dependence, which has a convenient analytic solution, in order to test the convergence of the layered solution to the analytic solution.

Let K_1 denote the propagation constant within the sphere and let

$$K_1 = a^2/r^2 \tag{30}$$

where a^2 is constant in space, but a function of frequency. At the low frequencies (10^{-8} - 1 Hz) of interest for investigation of a sphere of lunar size, displacement currents are usually considered to be negligible so that we may set

$$a^2 = i\mu_0 \sigma \omega R_1^2 \tag{31}$$

where μ_0 is the permeability of free space and where σ is a constant. Equation (30) then corresponds to the case when the radial dependence of the conductivity is given by

$$\sigma_1(r) = \sigma/r^2 \tag{32}$$

and the conductivity decreases, according to an inverse square law, toward the outer radius of the sphere. It is not intended that equation (32) represent a particular lunar model, but only that it serve to test the applicability of a layered solution to a convenient continuously varying model.

Substitution of equation (30) into equation (24) gives the differential equation to be solved:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial S(r)}{\partial r}\right) + \left[a^2 - n(n+1)\right] S(r) = 0 \tag{33}$$

Taking advantage of the results of Negi (1967) for a different problem formulation, let us seek solutions of the type

$$S(r) = r^{\alpha} \tag{34}$$

Substituting (34) into (33), we have

$$\alpha(\alpha+1) r^{\alpha} + \left[a^2 - n(n+1)\right] r^{\alpha} = 0 \tag{35}$$

which is satisfied for all r only if

$$\alpha^2 + \alpha + \left[a^2 - n(n+1)\right] = 0 \tag{36}$$

The solution to the quadratic equation (36) is

$$\alpha = \frac{-1 \pm \sqrt{1 - 4(a^2 - n(n+1))}}{2} \tag{37}$$

This agrees with the solution obtained by Negi (1967). The two solutions represented by equations (34) and (37) may be written as

$$S_n(r) = A r^{-\frac{1}{2}+\beta} + B r^{-\frac{1}{2}-\beta} \tag{38}$$

where A and B are arbitrary constant and where β is given by

$$\beta = \frac{\sqrt{1-4(a^2-n(n+1))}}{2} \tag{39}$$

The fact that the two solutions of equation (38) are independent is illustrated by the non-zero value of the Wronskian W:

$$W(r^{-\frac{1}{2}+\beta}, r^{-\frac{1}{2}-\beta}) = -2\beta r^{-2} \tag{40}$$

Letting conduction currents predominate so that a^2 is purely imaginary, as given by equation (31), and taking the principal value of the square root ensures that the real part of β is always $> 1/2$. The solution of the form $r^{-\frac{1}{2}+\beta}$ is then finite at the origin and represents an incoming wave (analogous to $j_n(Kr)$ while the solution of the form $r^{-\frac{1}{2}-\beta}$ is infinite at the origin and represents an outgoing wave (analogous to $h_n^{(1)}(Kr)$).

The solution to the stated problem is then similar to the solution for a homogeneous sphere, which is a special case of the results of Fuller and Ward (1969), with the exception that, inside the sphere, the Bessel function solutions are replaced by the solutions given by equation (38).

Following Fuller and Ward (1969) the primary plane wave, assuming unit amplitude ($|\bar{E}_0| = 1$), may be expanded in terms of the potentials as

$$\Phi_p = j\left(\frac{\hat{y}_0}{\hat{z}_0}\right)^{1/2} \sum_{n=1}^{\infty} (-j)^n \frac{(2n+1)}{n(n+1)} j_n(K_0 r)\cos\phi \, P_n^1(\cos\theta) \tag{41}$$

$$\Psi_p = \sum_{n=1}^{\infty} (-j)^n \frac{(2n+1)}{n(n+1)} j_n(K_0 r)\sin\phi \, P_n^1(\cos\theta) \tag{42}$$

Utilizing the general form of the solution to the Helmholtz equation, and making use of the form of equations (41) and (42), the solutions for the potentials in the external medium may be written down as

$$\Phi^{\circ} = j\left(\frac{\hat{y}_0}{\hat{z}_0}\right)^{1/2}\cos\phi \sum_{n=1}^{\infty}(-j)^n \frac{(2n+1)}{n(n+1)} P_n^1(\cos\theta)\left[a_n^{\circ} j_n(K_0 r) + b_n^{\circ} h_n^{(1)}(K_0 r)\right] \tag{43}$$

$$\psi^{\circ} = \sin\phi \sum_{n=1}^{\infty}(-j)^n \frac{(2n+1)}{n(n+1)} P_n^1(\cos\theta)\left[c_n^{\circ} j_n(K_0 r) + d_n^{\circ} h_n^{(1)}(K_0 r)\right] \tag{44}$$

where a_n°, b_n°, c_n°, d_n° are as yet undetermined coefficients. The primary wave is the only incoming wave in the external medium so that $a_n^{\circ} = c_n^{\circ} = 1$ by comparison of (43) with (41) and (44) with (42). The solution inside the sphere may be written as

$$\Phi' = j\left(\frac{\hat{y}_0}{\hat{z}_0}\right)^{1/2}\cos\phi \sum_{n=1}^{\infty}(-j)^n \frac{(2n+1)}{n(n+1)} P_n^1(\cos\theta)\left[a_n' r^{-\frac{1}{2}+\beta} + b_n' r^{-\frac{1}{2}-\beta}\right] \tag{45}$$

$$\psi' = \sin\phi \sum_{n=1}^{\infty}(-j)^n \frac{(2n+1)}{n(n+1)} P_n^1(\cos\theta)\left[c_n' r^{-\frac{1}{2}+\beta} + d_n' r^{-\frac{1}{2}-\beta}\right] \tag{46}$$

Finiteness at the origin requires that $b_n' = d_n' = 0$. The remaining four coefficients a_n', c_n', b_n°, d_n° may be determined by application of the four boundary conditions which are sufficient to ensure the continuity of tangential $\bar{E}$ and $\bar{H}$:

$$\psi' = \psi^{\circ}\Big|_{r=R_1} \tag{47}$$

$$\Phi' = \Phi^{\circ}\Big|_{r=R_1} \tag{48}$$

$$\frac{1}{\hat{z}_1}\frac{\partial}{\partial r}\left(r\,\psi'\right) = \frac{1}{\hat{z}_0}\frac{\partial}{\partial r}\left(r\,\psi^{\circ}\right)\Big|_{r=R_1} \tag{49}$$

$$\frac{1}{\hat{y}_1}\frac{\partial}{\partial r}\left(r\,\Phi'\right) = \frac{1}{\hat{y}_0}\frac{\partial}{\partial r}\left(r\,\Phi^{\circ}\right)\Big|_{r=R_1} \tag{50}$$

Application of the boundary conditions given by equations (47) through (50) yields the following expressions for b_n° and d_n° in the external medium:

$$b_n^{\circ} = \frac{\hat{y}_1\,\alpha_n(y_1) - \hat{y}_0\,j_n(y_1)\left[\frac{1}{2} + \beta_n\right]}{\hat{y}_0\,h_n^{(1)}(y_1)\left[\frac{1}{2} + \beta_n\right] - \hat{y}_1\,\gamma_n(y_1)} \tag{51}$$

$$d_n^{\circ} = \frac{\hat{z}_1\,\alpha_n(y_1) - \hat{z}_0\,j_n(y_1)\left[\frac{1}{2} + \beta_n\right]}{\hat{z}_0\,h_n^{(1)}(y_1)\left[\frac{1}{2} + \beta_n\right] - \hat{z}_1\,\gamma_n(y_1)} \tag{52}$$

where $\hat{y}_1$ ($= \sigma_1$, neglecting displacement currents), is understood to be evaluated at the boundary $r = R_1$ and where

$$y_1 = K_0 R_1 \tag{53}$$

$$\alpha_n(y_1) = \frac{\partial}{\partial y}\left[y\,j_n(y)\right]_{y=y_1} \tag{54}$$

$$\gamma_n(y_1) = \frac{\partial}{\partial y}\left[y\,h_n^{(1)}(y)\right]_{y=y_1} \tag{55}$$

If we define model reflection coefficients for the TE and TM potentials as

$$R_{n\psi} = d_n^{\circ} \frac{h_n^{(1)}(K_o R_1)}{j_n(K_o R_1)} \tag{56}$$

$$R_{n\Phi} = b_n^{\circ} \frac{h_n^{(1)}(K_o R_1)}{j_n(K_o R_1)} \tag{57}$$

then the solutions for the potentials in the esterior medium, equations (43) and (44) may be written as

$$\Phi^{\circ} = j\left(\frac{\hat{y}_o}{\hat{z}_o}\right)^{1/2} \cos\phi \sum_{n=1}^{\infty} (-j)^n \frac{(2n+1)}{n(n+1)} P_n^1(\cos\theta)\left[j_n(K_o r) + R_{n\Phi} \frac{j_n(K_o R_1)}{h_n^{(1)}(K_o R_1)} h_n^{(1)}(K_o r)\right] \tag{58}$$

$$\psi^{\circ} = \sin\phi \sum_{n=1}^{\infty} (-j)^n \frac{(2n+1)}{n(n+1)} P_n^1(\cos\theta)\left[j_n(K_o r) + R_{n\psi} \frac{j_n(K_o R_1)}{h_n^{(1)}(K_o R_1)} h_n^{(1)}(K_o r)\right] \tag{59}$$

By equations (58) and (59) the solution is determined everywhere in the exterior medium once $R_{n\psi}$ and $R_{n\Phi}$ have been determined. The fields are obtained through the vector relationships of equations (5) and (6).

The solutions of Fuller and Ward (1969) for an arbitrarily layered sphere are in exactly the same form as equations (58) and (59). To show convergence of an appropriate layered solution to the particular case of conductivity distribution considered here, it suffices to show that $R_{n\Phi}$ and $R_{n\psi}$, for an appropriate layered model, respectively converge to $R_{n\Phi}$ and $R_{n\psi}$ for the continuously varying model.

III. Computations

Figure 2 illustrates the chosen continuous model of a sphere in which the conductivity varies according to the inverse square of radius, as given by eqn. (32). The conductivity at the surface of the sphere is 10^{-5} mhos/m and is within the range of anticipated lunar conductivities, though both this value and the functional dependence of conductivity upon lunar radius are used for illustrative purposes only and are not an expression of our belief of the conductivity structure of the moon. All model calculations presented in this study will assume free space as the external medium. In this instance, for frequencies and observation points such that $K_o r \ll 1$, the observed magnetic field is primarily dependent upon the TE potential $R_{n\psi}$, the contribution of $R_{n\Phi}$ being negligible. Quantitative justification of this result is available in Fuller and Ward (1969) and Fuller (1969). In addition, it is shown in those references that, where the sphere conductivity is sufficiently high that displacement currents may be neglected, any value of conductivity yields the same $R_{n\Phi}$ in the quasi-static range of frequencies. For these reasons, this report will be concerned only with the TE reflection coefficients.

The evaluation of eqns. (52) and (56) yields $R_{n\psi}$ for the continuously varying model of Figure 2 and is presented in Figure 3 for n = 1 through 4. The features of the frequency response of the continuously varying model are of particular interest when compared to the frequency response of a homogeneous model. For this reason, the response coefficient $R_{n\psi}$ for a homogeneous model of conductivity 10^{-5} mhos/m is presented in Figure 4 (calculated according to Fuller and Ward (1969). In comparison to the homogeneous model of Figure 4, we may note the following features of the continuously varying model of Figure 3.

(1) The response of Figure 3 begins at a lower frequency than that of Figure 4 due to the higher internal conductivity.

(2) The response curves of Figures 3 and 4 tend to coincide at the higher frequencies because of the close agreement of the conductivities in the outer regions of the two models.

(3) The response of the continuously varying model is spread over a wider frequency range than that of the homogeneous model, and quadrature (imaginary part) peak amplitudes are lower than those for the homogeneous model.

(4) Higher modes of the reflection coefficient appear to agree more closely than the first mode, indicating that the higher modes are more affected by near surface conductivity.

(5) Except for (1) through (4) above, there are no marked differences between the particular continuous model chosen and the homogeneous model. This would indicate that the response of the very high conductivity region at the center of the continuous model is sufficiently attenuated through the conductive outer portion that its effect is negligible.

Figure 5 is a two layer approximation of the continuous model, calculated according to Fuller and Ward (1969). The layer parameters are illustrated on the figure. Beyond about 4 Hz, the quasi-static approximation breaks down within (though not outside of) the outer layer so as to cause computational difficulties; for this reason, the response is not presented beyond that frequency. The increased conductivity of the internal part of the sphere results in fair agreement, at low frequencies, to the response of Figure 3. At the higher frequencies, however, the agreement is not as good as between Figures 3 and 4, due to the choice of a higher conductivity for the outer layer of the model of Figure 5.

The three layer model of Figure 6 yields a close approximation to the response of Figure 3 at the high and low frequencies, though the sharpness of the quadrature peaks, as opposed to those of Figure 3, indicate that a greater number of layers is necessary to simulate the response of the continuous model.

Figure 7 illustrates the calculated response of a ten layer approximation to the continuous model, with layer parameters illustrated on the figure. The responses, or reflection coefficients of Figures 3 and 7 are almost exactly coincident for all modes and for both real and imaginary parts. The ten layer model thus accomplishes, for all practical purposes, an exact electromagnetic simulation of the continuously varying model at low frequencies.

IV. Summary and Conclusions

It has been demonstrated that a concentrically layered spherical model may be utilized to simulate electromagnetically a sphere which has a continuous distribution of conductivity as a function of radius, for sufficiently low frequencies. The frequency limitation implied requires that the layers be electrically thin; that is, the wavelength in a given layer must be much greater than the layer thickness in order to avoid waveguide effects.

The intent of this report has been to demonstrate the utility of layered models as theoretical approximations to bodies in which the electrical parameters obey some functional distribution. This has been accomplished for a particular conductivity distribution which is smoothly varying as a function of radius. A less smooth conductivity distribution, one in which the conductivity changes sharply at particular values of radius, would be expected to be more conducive to approximation by a layered model.

The value of using layered models lies in the fact that the computational procedure remains general for any arrangement of layers. The exact solution for a particular conductivity distribution, on the other hand, depends upon the functional dependence assumed.

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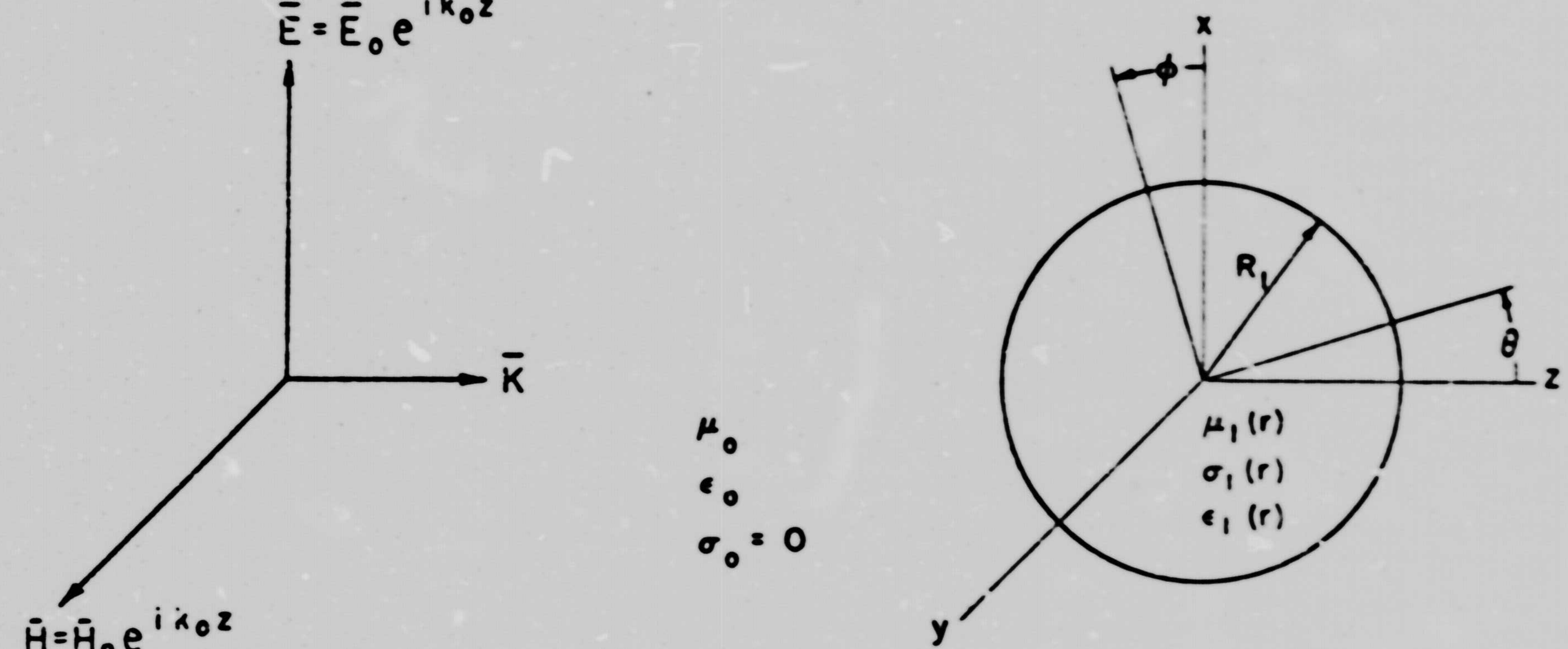
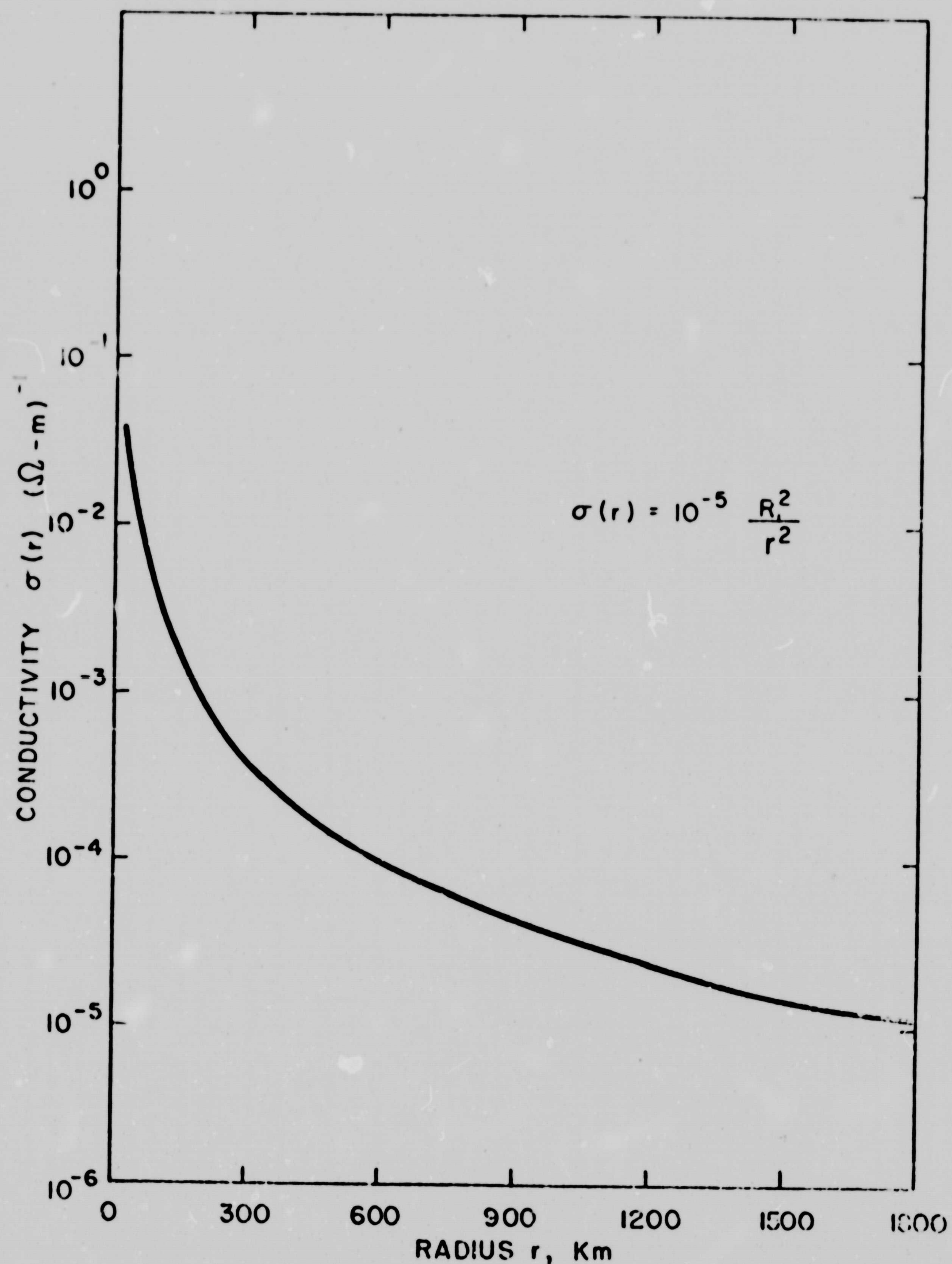
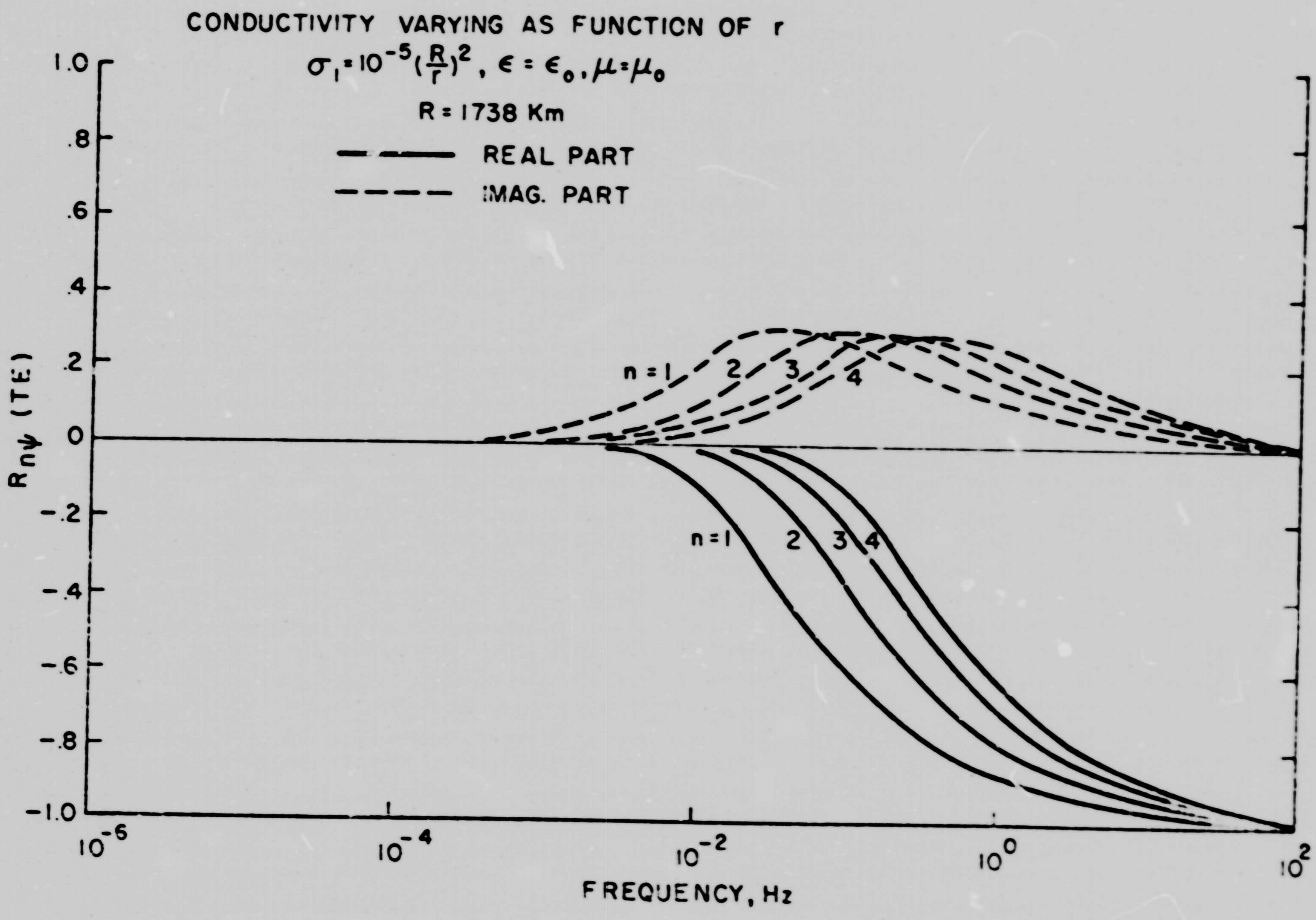

$\bar{E} = \bar{E}_0 e^{ik_0 z}$
$\bar{K}$
$\bar{H} = \bar{H}_0 e^{ik_0 z}$
μ_0
ϵ_0
$\sigma_0 = 0$
x
ϕ
R_1
θ
z
y
$\mu_1(r)$
$\sigma_1(r)$
$\epsilon_1(r)$

FIGURE 1



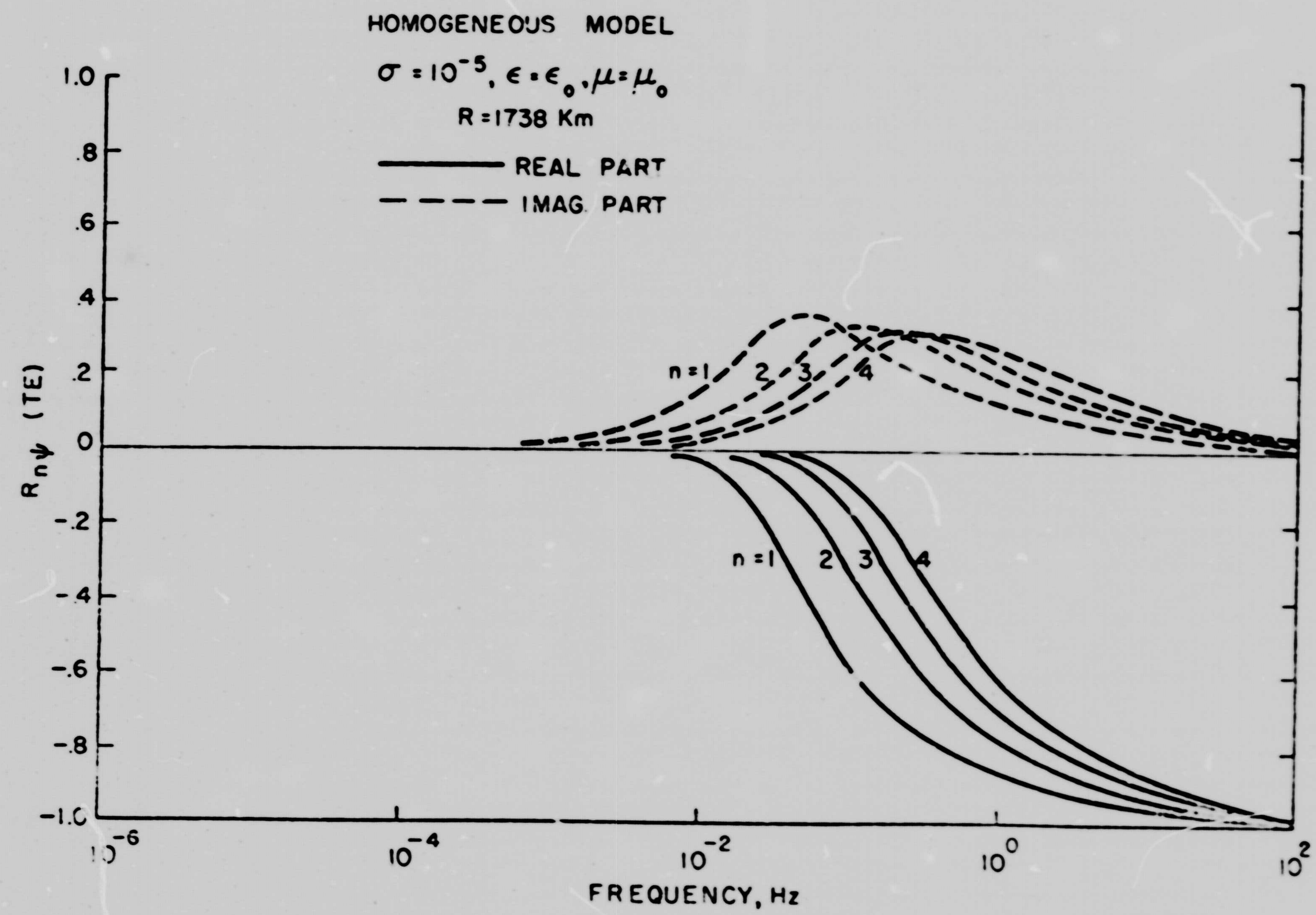
CONDUCTIVITY σ(r) (Ω-m)⁻¹
10⁰
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
10⁻⁶
σ(r) = 10⁻⁵ R₁²/r²
0
300
600
900
1200
1500
1800
RADIUS r, Km

FIGURE 2



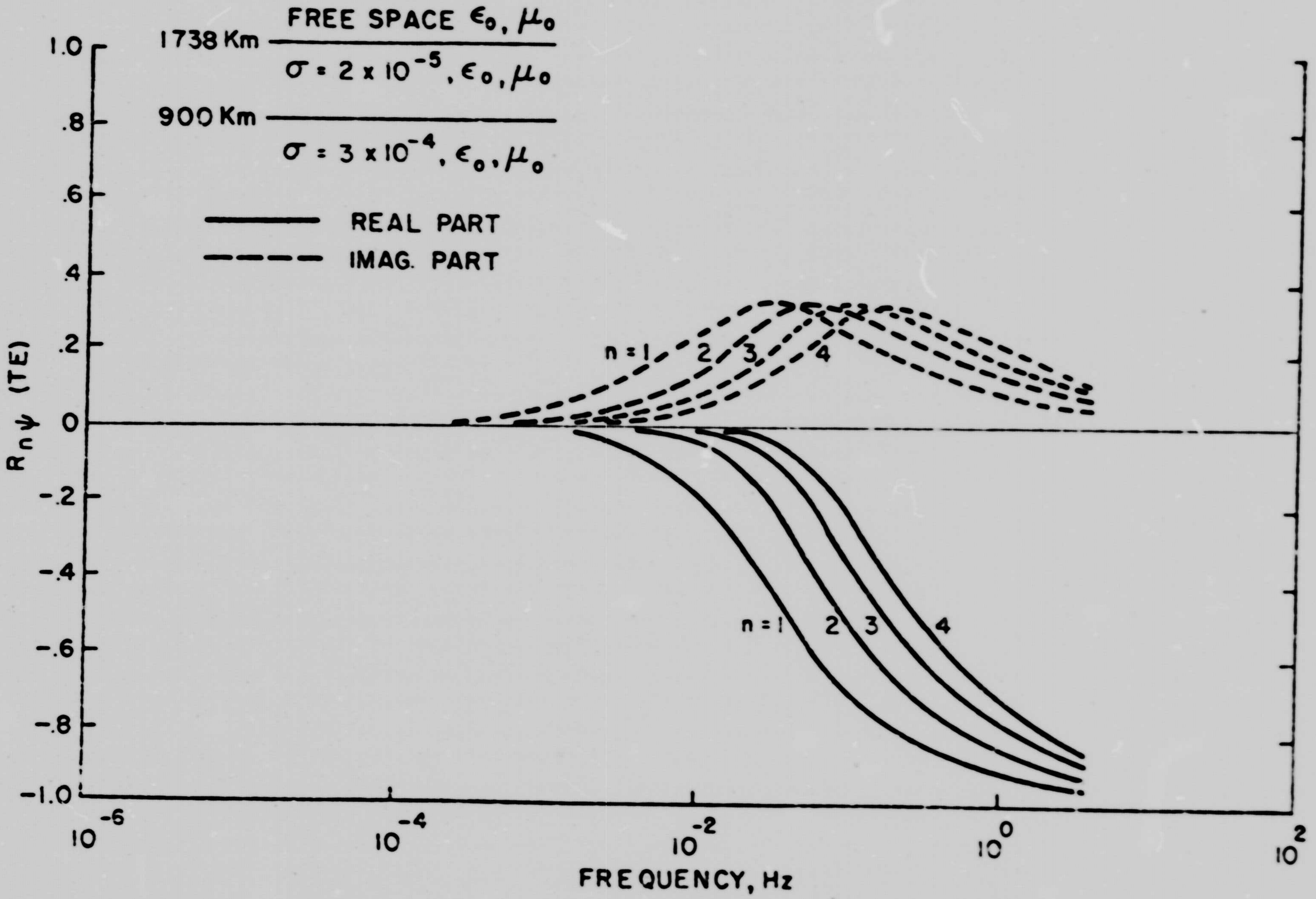
CONDUCTIVITY VARYING AS FUNCTION OF r
$\sigma_1 = 10^{-5}(\frac{R}{r})^2$, $\epsilon = \epsilon_0$, $\mu = \mu_0$
R = 1738 Km
REAL PART
IMAG. PART
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
$R_{n\psi}$ (TE)
n = 1
2
3
4
n = 1
2
3
4
10^{-6}
10^{-4}
10^{-2}
10^{0}
10^{2}
FREQUENCY, Hz

FIGURE 3



HOMOGENEOUS MODEL
σ = 10^-5, ε = ε0, μ = μ0
R = 1738 Km
REAL PART
IMAG. PART
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
n = 1
2
3
4
n = 1
2
3
4
R_nψ (TE)
10^-6
10^-4
10^-2
10^0
10^2
FREQUENCY, Hz

FIGURE 4



1738 Km
FREE SPACE ϵ_0, μ_0
$\sigma = 2 \times 10^{-5}, \epsilon_0, \mu_0$
900 Km
$\sigma = 3 \times 10^{-4}, \epsilon_0, \mu_0$
REAL PART
IMAG. PART
n = 1
2
3
4
$R_n\psi$ (TE)
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
10^{-6}
10^{-4}
10^{-2}
10^{0}
10^{2}
FREQUENCY, Hz

FIGURE 5

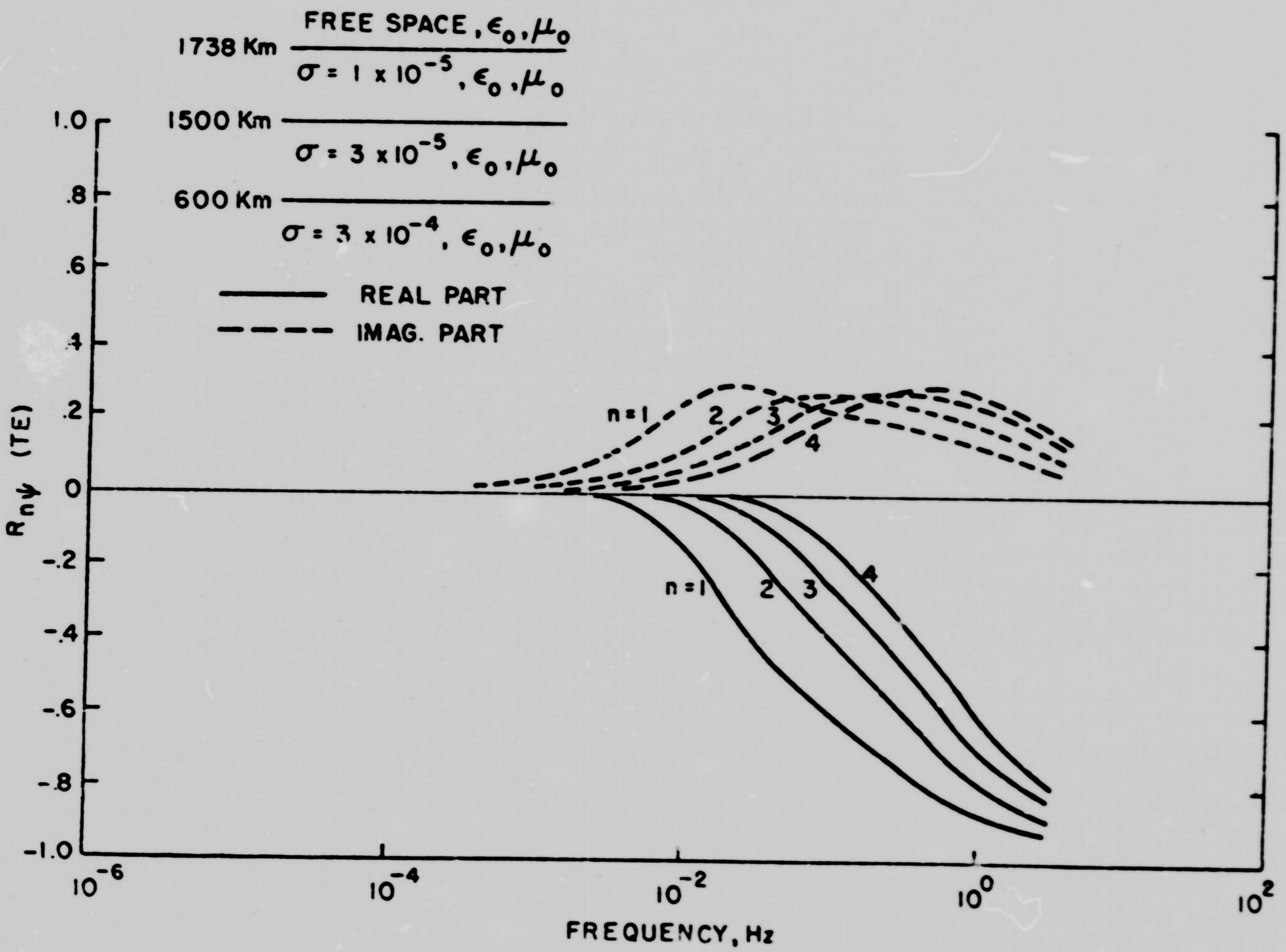
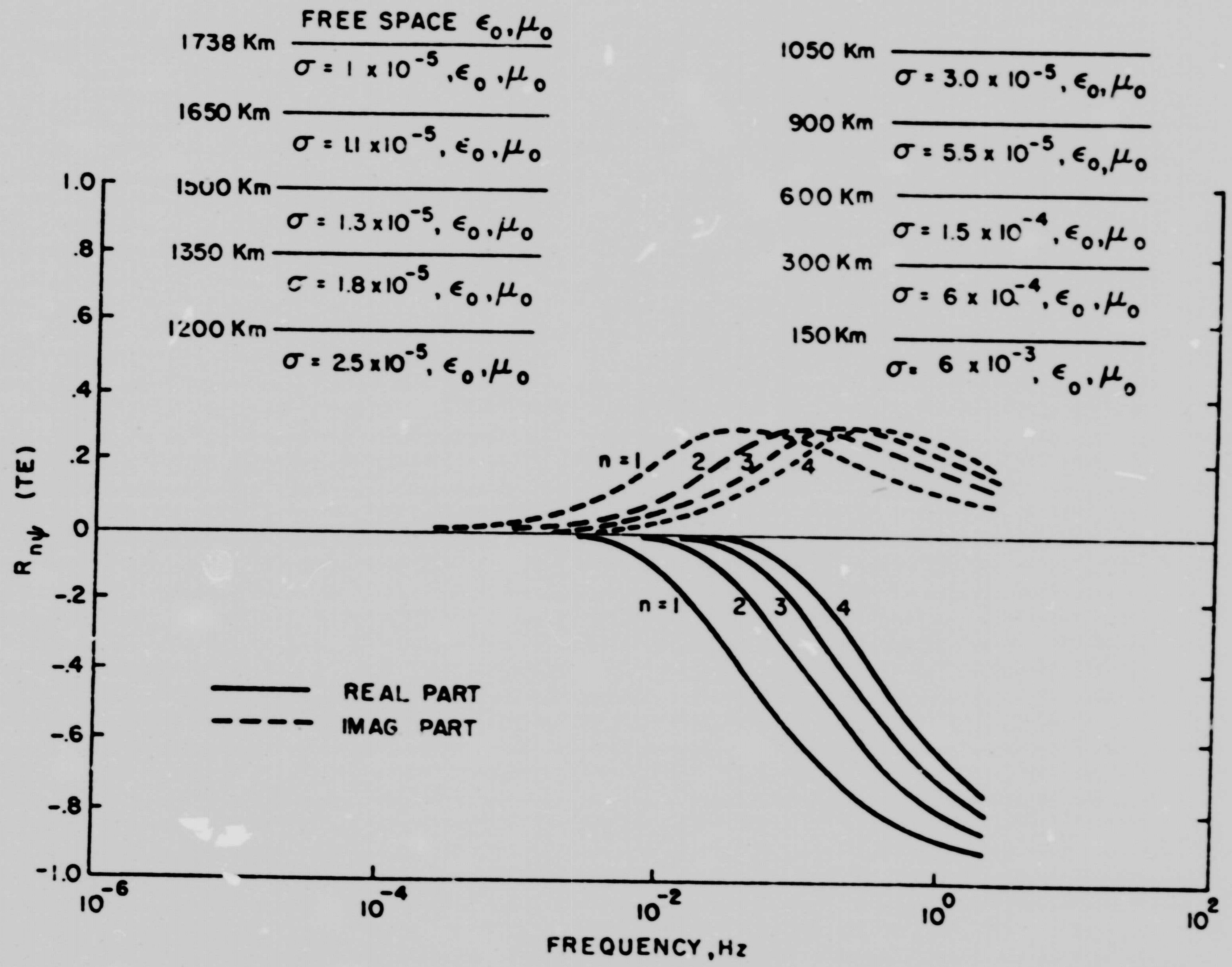

1738 Km
FREE SPACE, ϵ_0, μ_0
$\sigma = 1 \times 10^{-5}, \epsilon_0, \mu_0$
1500 Km
$\sigma = 3 \times 10^{-5}, \epsilon_0, \mu_0$
600 Km
$\sigma = 3 \times 10^{-4}, \epsilon_0, \mu_0$
REAL PART
IMAG. PART
$R_{n\psi}$ (TE)
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
n=1
2
3
4
10^{-6}
10^{-4}
10^{-2}
10^{0}
10^{2}
FREQUENCY, Hz

FIGURE 6



FREE SPACE ϵ_0, μ_0
1738 Km
$\sigma = 1 \times 10^{-5}, \epsilon_0, \mu_0$
1650 Km
$\sigma = 1.1 \times 10^{-5}, \epsilon_0, \mu_0$
1500 Km
$\sigma = 1.3 \times 10^{-5}, \epsilon_0, \mu_0$
1350 Km
$\sigma = 1.8 \times 10^{-5}, \epsilon_0, \mu_0$
1200 Km
$\sigma = 2.5 \times 10^{-5}, \epsilon_0, \mu_0$
1050 Km
$\sigma = 3.0 \times 10^{-5}, \epsilon_0, \mu_0$
900 Km
$\sigma = 5.5 \times 10^{-5}, \epsilon_0, \mu_0$
600 Km
$\sigma = 1.5 \times 10^{-4}, \epsilon_0, \mu_0$
300 Km
$\sigma = 6 \times 10^{-4}, \epsilon_0, \mu_0$
150 Km
$\sigma = 6 \times 10^{-3}, \epsilon_0, \mu_0$
$R_{n\psi}$ (TE)
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
n = 1
2
3
4
REAL PART
IMAG PART
10^{-6}
10^{-4}
10^{-2}
10^{0}
10^{2}
FREQUENCY, Hz

FIGURE 7