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ESTIMATION OF THE LENGTH AND ORIENTATION OF THE LINE BETWEEN TWO CLOSELY CO-ORBITING SATELLITES

JOHN HRASTAR

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John Hrastar*
Goddard Space Flight Center
Greenbelt, Maryland

Abstract

The problem of estimating the baseline length and orientation using a combination of on-board sensor data and processed ground tracking data is considered. A minimum number (3) of simple on-board sensors are assumed. Some of the variables are assumed to be estimated from ground tracking information only. These estimates become part of the observations, along with the on-board sensor outputs, for estimation of the remaining variables. A linear estimate of the remaining variables is derived via a Kalman filter. This sequential processing obviates the need for the simultaneous processing of ground track and on-board data. The importance of deciding which variables are to be estimated from the ground and which are to be estimated using the combined data is discussed. An example demonstrates that baseline length and orientation can be estimated within a few feet and fractions of a degree.

INTRODUCTION

Recently interest has been expressed in using pairs and even clusters of satellites in close proximity to perform a variety of space missions.[1,2] One particular area in which dual satellite systems should prove very useful is in low frequency radio astronomy. A pair of satellites may be used as an interferometer to synthesize a large aperture radio telescope.[2,3] As the pair moves in orbit the location and frequency characteristics of various radio sources may be determined.

The satellites in these systems may be tethered,[4] thereby forming a very large "dumbbell" satellite, or they may be co-orbiting** but physically separate. In either case, however, when the pair of satellites is used as an interferometer, the length and orientation of the line between them (the baseline) must be determined to a relatively high degree of accuracy.

The problem considered here is the estimation of the baseline length and orientation of two separate but co-orbiting satellites. The basic assumption is that the baseline is small compared to the orbit dimension. For example, the radio astronomy mission may call for an altitude of 8000 miles and baseline of only one or two miles. Because of the closeness of the two satellites the baseline length and orientation cannot be adequately estimated from ground based measurements alone. Small errors in estimates of the position vectors of each satellite reflect as large errors in the baseline orientation estimate. Thus there is a requirement for some on-board sensors. The baseline length and orientation could be determined with an intersatellite ranging sensor and an optical sensor on one satellite viewing the second one against the star field background (the latter sensor determining baseline orientation). In this case on-board sensor data alone would solve the problem. An optical sensor of this type, however does not presently exist. The approach here was to use simpler, state-of-the-art sensors. These sensors however do not provide enough information to make the system observable. The solution of the problem therefore lies in the combination of the on-board sensor data and ground tracking data.

The problem is formulated in terms of the small perturbations from the nominal circular orbit common to both satellites. The orientation of the baseline may be described by two angles, one in the nominal orbit plane and one between the baseline and this plane. The linearized equations for the in-plane and out-of-plane motions are not coupled. The problem is therefore reduced to two simpler ones. The state variables are defined as the sums and differences of the satellite positions and velocities. Some of the variables are estimated using processed ground based measurements. These estimates, along with the on-board sensor information, are used to estimate the remaining variables. The estimates of the two angles and baseline length are derived from the state variable estimates.

PROBLEM FORMULATION

Assuming a circular earth the inertial frame to be used as a reference is the XYZ frame where X and Y define the nominal orbit plane and the origin is at the center of the earth (Figure 1).

The orientation of the baseline relative to this frame may be described by the angles β and γ. Beta is the

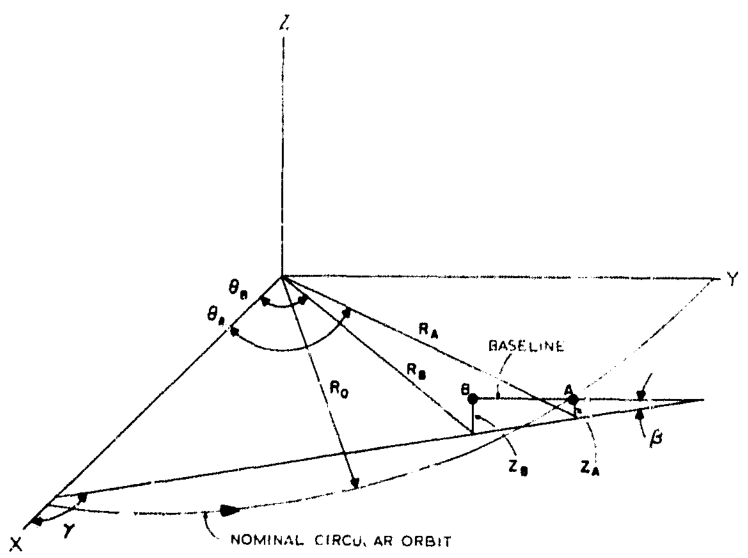

Figure 1. Orbital Configuration of the Dual Satellite System

*Aerospace Engineer

**Co-orbiting is taken to mean the two satellites are in the same nominal orbit and the difference in the times from perigee is a constant.

angle between the baseline and the X-Y plane and is the angle between the baseline planar projection and the X axis (Figure 1). These angles and the baseline length b may be determined from the variables shown in the figure. Thus the estimates of , and b are functions of the vectors locating the satellites A and B in the XYZ frame.

The equations of motion of each satellite with respect to this frame are:

$$\ddot{R} \quad R^{\cdot} \quad \frac{R}{(R^2 \quad z^2)^{3/2}} \quad 0 \qquad (1a)$$

$$R^{\cdot\cdot} \cdot 2\dot{R}^{\cdot} \quad 0 \qquad (1b)$$

$$\ddot{z} \cdot \frac{z}{(R^2 + z^2)^{3/2}} \quad 0 \qquad (1c)$$

In equations 1a, b, c cylindrical coordinates are used where R and are in the X-Y plane and z is the distance out of the plane. The gravitational constant is . The state variables for the A satellite are defined as components of the vector q.

$$q_1 \quad R_A \qquad q_4 \quad R_0 \dot{}_A$$

$$q_2 \quad R_0 \; _A \qquad q_5 \quad z_A$$

$$q_3 \quad \dot{R}_A \qquad q_6 \quad \dot{z}_A$$

The state variables for satellite B are similarly defined as components of the vector p. Components of the vectors q and p are indicated by numerical subscripts. The nominal orbit radius is R_0. When the equations are linearized with respect to the nominal circular orbit the system equations are:

$$\delta q(t) \quad \Phi(t, t_0)\, \delta q(t_0) + u(t) \qquad (2a)$$

$$\delta p(t) \quad \Phi(t, t_0)\, \delta p(t_0) + u(t) \qquad (2b)$$

$$\delta y(t) \quad H(t) \begin{bmatrix} \delta q(t) \\ ---- \\ \delta p(t) \end{bmatrix}^* + v(t) \qquad (3)$$

In equations (2) and (3) $\delta q(t)$ and $\delta p(t)$ are the first variations of $q(t)$ and $p(t)$. The dynamic noise is $u(t)$ (the same for both satellites because of their proximity) and the sensor noise is $v(t)$. The state transition matrix $\Phi(t, t_0)$ is listed in the appendix. The lack of coupling between the planar variables, δq_1, δp_1 through δq_4, p_4 and the non-planar variables δq_5, δp_5, δq_6, δp_6 is evident.

To determine the relationship between the output vector $\delta y(t)$ and the state vectors the matrix $H(t)$ must be defined. This requires definition of the sensors. The following sensors are assumed to be located on A.

1. An intersatellite range sensor measuring baseline length b.
2. A yaw sensor measuring the A satellite yaw angle (the attitude angle around the orbit radius vector).
3. An optical "B-tracker" locating the B satellite with respect to the A satellite attitude reference frame. This measures the angle .

The latter two angles are shown in Figure 2. Both , are nominally zero. The sensor is assumed to be independent of the small variation in .

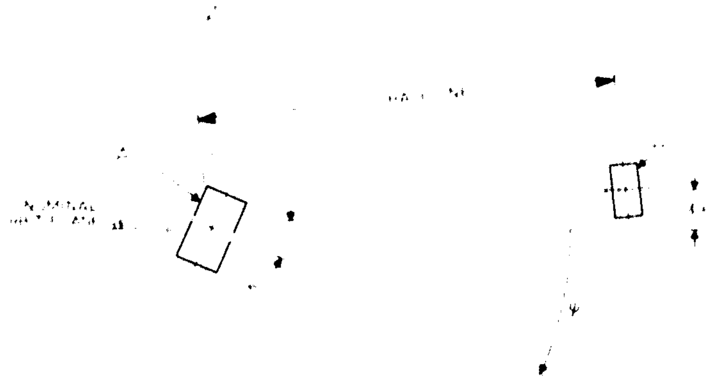

Figure 2. Definition of , ,

The relation between the state variables and the observables ($\delta y(t)$) may be determined from the geometry of the problem. As a function of the state variables, b is

$$b \quad \left(q_1^2 + p_1^2 - 2q_1 p_1 \cos\left(\frac{q_2 \quad p_2}{R_0}\right) + (q_5 - p_5)^2\right)^{1/2} \qquad (4)$$

The first variation in b is therefore

$$\delta b \quad (\delta q_1 + \delta p_1)\sin\left(\frac{\Delta\alpha}{2}\right) + (\delta q_2 \quad \delta p_2)\cos\left(\frac{\Delta\alpha}{2}\right) \qquad (5)$$

where $\Delta\alpha$ is the angle subtended by the nominal baseline B_0 when A and B are in the nominal orbit, i.e., $\sin(\quad 2) = B_0/2R_0$. Equation (5) is independent of the z variables δq_5, δp_5. The intersatellite range sensor is the only on-board sensor required for the estimation of b and . That the estimate of b may be derived from this information is obvious. It will be shown that the estimate may also be derived using only the intersatellite range information (b measurement) and processed ground tracking information.

The angle β is expressed as a function of the state variables as follows

$$\beta = \sin^{-1}\left(\frac{(p_5 \quad q_5)}{\left(q_1^2 + p_1^2 - 2q_1 p_1 \cos\left(\frac{q_2 \cdot p_2}{R_0}\right) + (p_5 \quad q_5)^2\right)^{1/2}}\right) \qquad (6)$$

This angle is not directly observable but is related to the angles ψ, η (Figure 2):

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*Brackets [ ] indicate a matrix

$$\eta - \rho \quad (7)$$

Using equations (6) and (7) the first variations in the observables ψ and η are $\delta\psi$ and

$$\delta\eta = \delta\psi + \frac{1}{B_0}(\delta p_5 - \delta q_5) \quad (8)$$

The introduction of $\delta\psi$ requires the introduction of the linearized yaw attitude dynamics

$$\delta\Psi(t) = \Phi_3(t,t_0)\,\delta\Psi(t_0) + w(t) \quad (9)$$

where $\delta\Psi(t)$ is the rx1 state vector with $\delta\psi$ as its first component and $w(t)$ is the dynamic noise. The corresponding transition matrix is $\Phi_3(t, t_0)$.

Examination of equations (5) and (8) shows that planar and non-planar observations are uncoupled just as the dynamics. The problem may therefore be considered as two uncoupled problems. Equations (2) and (3) hold for the planar problem (but only the first four components of δq and δp are used and $\Phi_1(t,t_0)$ replaces $\Phi(t,t_0)$). In the planar case $\delta y(t) = \delta\tilde{b}(t)$, $H = [\sin(\Delta\theta/2), \cos(\Delta\theta/2), 0, 0, \sin(\Delta\theta/2), -\cos(\Delta\theta/2), 0, 0]$ and $v_1(t)$ is the range sensor noise. The tilde ($\sim$) refers to the observable with measurement noise included.

The corresponding equations for the non-planar problem are equations (10), (11), and (12).

$$\begin{bmatrix} \delta\Psi(t) \\ \delta q_5(t) \\ \delta q_6(t) \\ \delta p_5(t) \\ \delta p_6(t) \end{bmatrix} = \begin{bmatrix} \Phi_3(t,t_0) & 0 & 0 \\ 0 & \Phi_2(t,t_0) & 0 \\ 0 & 0 & \Phi_2(t,t_0) \end{bmatrix} \begin{bmatrix} \delta\Psi(t_0) \\ \delta q_5(t_0) \\ \delta q_6(t_0) \\ \delta p_5(t_0) \\ \delta p_6(t_0) \end{bmatrix} + \begin{bmatrix} w(t) \\ u_5(t) \\ u_6(t) \\ u_5(t) \\ u_6(t) \end{bmatrix} \quad (10)$$

$$\delta y(t) = \begin{bmatrix} \delta\tilde{\psi}(t) \\ \delta\tilde{\eta}(t) \end{bmatrix} = H_1 \begin{bmatrix} \delta\psi(t) \\ \delta q_5(t) \\ \delta q_6(t) \\ \delta p_5(t) \\ \delta p_6(t) \end{bmatrix} + \begin{bmatrix} v_2(t) \\ v_3(t) \end{bmatrix} \quad (11)$$

$$H_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \underbrace{1 \quad 0 \quad \cdots \quad 0}_{r} & & & & -1/B_0 & 0 & 1/B_0 & 0 \end{bmatrix} \quad (12)$$

The ψ and η sensors have noise components $v_2(t)$ and $v_3(t)$ respectively. The components of $v(t)$ are all assumed to have zero mean.

Planar Solution

Estimates of the variables b and ρ may be determined as functions of the estimates of the planar state variables. These variables must therefore be related to the state variables. For b the relationship has already been shown to be equation (5). The first variation in ρ is

$$\delta\rho = \frac{1}{B_0}(\delta q_1 - \delta p_1) + \frac{1}{2R_0}(\delta q_2 + \delta p_2) \quad (13)$$

As the problem stands there are eight variables δq_i, δp_i, $i = 1,4$ and one observable $\delta\tilde{b}$. With the single onboard sensor the system is not observable. Not enough information is available to allow an estimate of δq and δp. This problem may be overcome by redefining the state variables as sums and differences of the δq_i's and δp_i's and then estimating some of the new variables from ground observations.

Let

$$x_i = \begin{cases} \delta q_i - \delta p_i & i = 1, 6 \quad (14a) \\ \delta q_{i-6} + \delta p_{i-6} & i = 7, 12 \quad (14b) \end{cases}$$

With this change of variables the planar system equations become

$$x^1(t) = \Phi_1(t,t_0)\, x^1(t_0) \quad (15)$$

$$x^2(t) = \Phi_1(t,t_0)\, x^2(t_0) \quad (16)$$

$$\delta\tilde{b} = x_2\cos\left(\frac{\Delta\theta}{2}\right) + x_7\sin\left(\frac{\Delta\theta}{2}\right) + v_1(t) \quad (17)$$

where

$$x^1 = [x_1, x_2, x_3, x_4]^T$$

$$x^2 = [x_7, x_8, x_9, x_{10}]^{T*}$$

In order to provide the necessary information to make the system observable it is now assumed x^2 is estimated from ground observations (Figure 3). Thus equation (16) is no longer needed. Note also that $\delta\tilde{b}$ contains components from both x^1 and x^2.

Denote the estimate of x_i by $\hat{x}_i$ and the error in the estimate by e_i.

$$x_i = \hat{x}_i - e_i \qquad i = 1, 12 \quad (18)$$

Substituting for x_7 in equation (17) results in

$$\delta\tilde{b} = x_2\cos\left(\frac{\Delta\theta}{2}\right) + \hat{x}_7\sin\left(\frac{\Delta\theta}{2}\right) - e_7\sin\left(\frac{\Delta\theta}{2}\right) + v_1(t) \quad (19)$$

At any time the estimate $\hat{x}_7$ is a constant entered from an external source (the output of the x^2 estimation process). Thus it may be incorporated in the measurement $\delta\tilde{b}$. The statistical properties of e_7 and $v_1(t)$

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*The T indicates transpose.

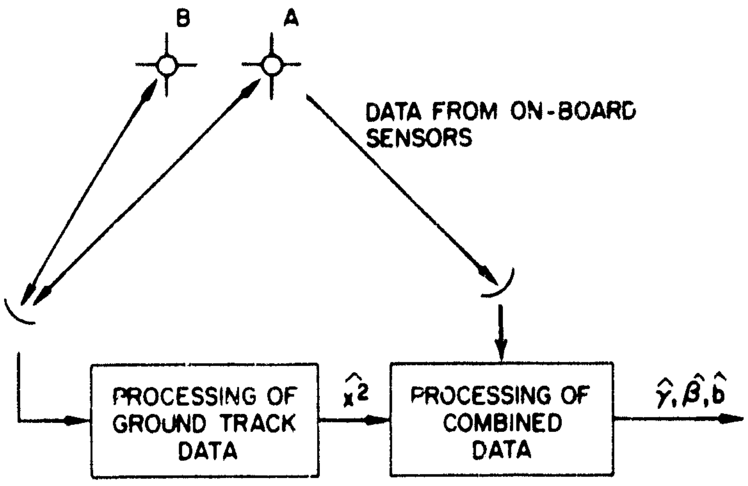

Figure 3. Information Flow Diagram

are assumed to be known and e_7 and $v_1(t)$ are uncorrelated. Taken together these terms constitute the total measurement noise. The output equation therefore reduces to

$$\tilde{\delta b} = \left[0,\ \cos\left(\frac{\Delta\theta}{2}\right),\ 0,\ 0\right] x^1 - e_7 \sin\left(\frac{\Delta\theta}{2}\right) + v_1(t) \qquad (20)$$

Equations (15) and (20) are in the correct form to allow x^1 to be estimated.

The vector x^1 may be estimated by means of the Kalman filter equations[5].

$$\hat{x}(t+1) = \Phi_1(t+1, t)\,\hat{x}(t) + K(t+1)\,(\delta y(t+1) - H(t+1)\,\Phi_1(t+1, t)\,\hat{x}(t)) \qquad (21)$$

$$K(t+1) = P_t(t+1)\,H^T(t+1)\cdot (H(t+1)\,P_t(t+1)H^T(t+1) + R(t+1))^{-1} \qquad (22)$$

The covariance matrix of the estimate error $P(t+1)$ is determined by equations (23) and (24).

$$P_t(t+1) = \Phi_1(t+1, t)\,P(t)\,\Phi_1^T(t+1) + Q(t) \qquad (23)$$

$$P(t+1) = P_t(t+1) - K(t+1)\,H^T(t+1)\,P_t(t+1) \qquad (24)$$

In equations (21) through (24) $\hat{x}(t+1)$ is the estimate given measurements through time $t+1$, $\hat{x}(t)$ is the estimate given measurements through time t. The measurement interval has been normalized to one in these equations. The covariance matrices of the dynamic noise and measurement noise are $Q(t)$ and $R(t)$ respectively. For the planar problem

$$R(t) = E\left\{\left(v_1(t) - e_7 \sin\left(\frac{\Delta\theta}{2}\right)\right)^2\right\} = \sigma_{v1}^2 + \sigma_7^2 \sin^2\left(\frac{\Delta\theta}{2}\right) \qquad (25)$$

In equation (25) $E\{\ \}$ is the expected value operator and σ_{v1}^2, σ_7^2 are the variances of the range sensor noise and x_7 estimate error respectively. Since $v_1(t)$ and e_7 are uncorrelated $E\{v_1(t)\,e_7\} = 0$.

By using equations (13) and (14) and $\hat{x}^1$ and $\hat{x}^2$ $\delta\gamma$ may be estimated

$$\hat{\delta\gamma} = -\frac{1}{B_0}\hat{x}_1 + \frac{1}{2R_0}\hat{x}_8. \qquad (26)$$

In equation (26) $\hat{x}_1$ is from $\hat{x}^1$, $\hat{x}_8$ is from $\hat{x}^2$ and $\hat{\delta\gamma}$ is the estimate of the variation $\delta\gamma$. The error of the estimate is $e_\gamma = \hat{\delta\gamma} - \delta\gamma$. The variance of the estimate error is

$$\sigma_\gamma^2 = \frac{\sigma_1^2}{B_0^2} + \frac{\sigma_8^2}{4R_0^2} - \frac{\rho\,\sigma_1\,\sigma_8}{R_0\,B_0} \qquad (27)$$

where σ_1^2, σ_8^2 are the variances of the $\hat{x}_1$ and $\hat{x}_8$ estimate errors and ρ is the correlation coefficient between these estimate errors. Because of the separate processing of $\hat{x}^1$ and $\hat{x}^2$, ρ is not directly available. However σ_γ^2 is bounded by

$$\sigma_{\gamma*}^2 = \frac{\sigma_1^2}{B_0^2} + \frac{\sigma_8^2}{4R_0^2} + \frac{\sigma_1\,\sigma_8}{R_0\,B_0} \qquad (28)$$

Because of the basic assumption that $B_0 \ll R_0$ it will turn out that $\sigma_{\gamma*} \cong \sigma_\gamma$. This will be demonstrated in the example.

The estimate of δb and variance of the estimate error are

$$\hat{\delta b} = H\,\hat{x}^1 = \cos\left(\frac{\Delta\theta}{2}\right)\hat{x}_2 \qquad (29a)$$

$$\sigma_b^2 = \cos^2\left(\frac{\Delta\theta}{2}\right)\sigma_2^2 \qquad (29b)$$

Non-Planar Solution

The angle β is defined as a function of the state variables in equation (6). With the variables defined in equation (15) the first variation is

$$\delta\beta = -\frac{1}{B_0}x_5 \qquad (30)$$

Thus the estimate of β is a function of the non-planar variable x_5 only.

The system equations for this problem (using equations (10), (11) and (12)) are

$$\begin{bmatrix} \delta\Psi(t) \\ \hline x_5(t) \\ x_6(t) \end{bmatrix} = \left[\begin{array}{c|c} \Phi_3(t,t_0) & 0 \\ \hline 0 & \Phi_2(t,t_0) \end{array}\right] \begin{bmatrix} \delta\Psi(t_0) \\ \hline x_5(t_0) \\ x_6(t_0) \end{bmatrix} + \begin{bmatrix} w(t) \\ \hline 0 \\ 0 \end{bmatrix} \qquad (31)$$

$$\begin{bmatrix} \delta\tilde{\psi} \\ \delta\tilde{\eta} \end{bmatrix} \quad \left[\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & 0 & & 0 \\ 1 & 0 & \cdots & 0 & 1 & B_0 & 0 \end{array}\right] \begin{bmatrix} \delta\Psi(t) \\ \hline x_5(t) \\ x_6(t) \end{bmatrix} + \begin{bmatrix} v_2(t) \\ v_3(t) \end{bmatrix} \quad (32)$$

Once an estimate of x_5 is available (via the filter equations) the estimate of $\delta\beta$ and variance of the estimate error are respectively

$$\hat{\delta\beta} = -\frac{1}{B_0}\hat{x}_5 \quad (33a)$$

$$\sigma_\beta^2 = \frac{1}{B_0^2}\sigma_5^2 \quad (33b)$$

EXAMPLE

Consider a dual satellite system with the mission of low frequency radio astronomy. The co-orbiting satellites are the receivers of an orbiting interferometer. In order to determine the location and frequency characteristics of the emitting sources the baseline length (b) and orientation (γ, β) must be known.

A digital simulation was carried out using the following parameters

$R_0 = 70 \cdot 10^6$ feet

$\omega_0 = 2.03 \cdot 10^{-4}$ sec^{-1}

$B_0 = 10^4$ feet

$\tau = 100$ sec (sampling interval)

The dynamic noise was assumed to be zero. The measurement noise was assumed to be white Gaussian with zero mean.

$\sigma_{v1} = 5$ feet

$\sigma_{v2} = \sigma_{v3} = 8.7$ milliradians (0.5°)

The estimate errors of $\hat{x}_7$ and $\hat{x}_8$ after processing of ground based measurements were assumed to have standard deviations

$\sigma_7 = 1500$ feet

$\sigma_8 = 300$ feet

Sample runs for the planar case are shown in Figures 4 and 5. These figures show the estimate errors of $\hat{x}_1$ and $\hat{x}_2$ (i.e. e_1 and e_2) and their respective standard deviations. Good estimates of x_1 and x_2 are available after 50 samples (5000 seconds). Using σ_1 and σ_2 after 50 samples, 35 feet and 2.5 feet respectively, in equations (28) and (29b) results in $\sigma_{\gamma*} \cong 35 \cdot 10^{-4}$ rad. (0.2°) and $\sigma_b = 2.5$ feet. Because of the assumption $B_0 << R_0$ the second and third terms of equation (28) are negligible. Thus $\sigma_{\gamma*} \cong \sigma_\gamma \cong \sigma_1/B_0$. For the non-planar case the sample run is shown in Figure 6. A simple second order system was assumed for the yaw attitude dynamics. The resulting σ_β after 50 samples is $25 \cdot 10^{-4}$ radians (0.14°).

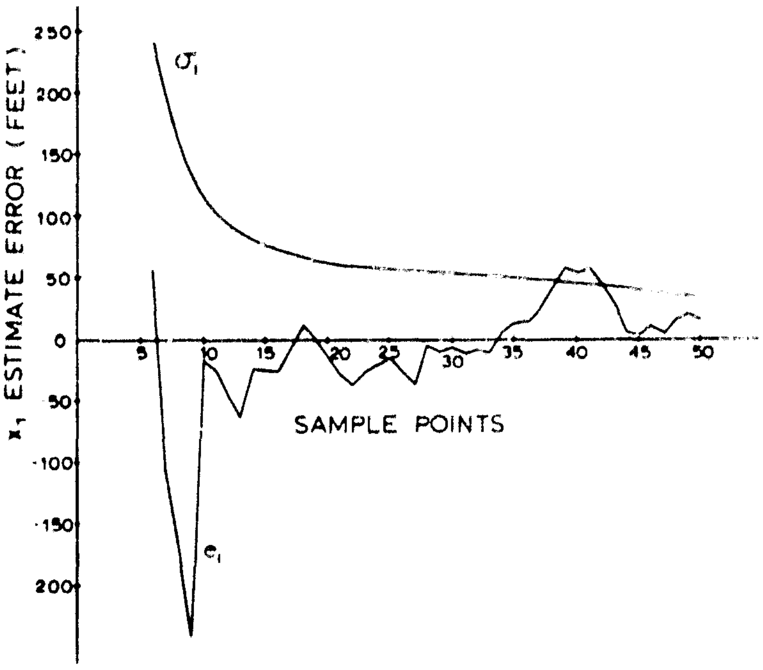

Figure 4. Estimate Error and Standard Deviation of $\hat{x}_1$

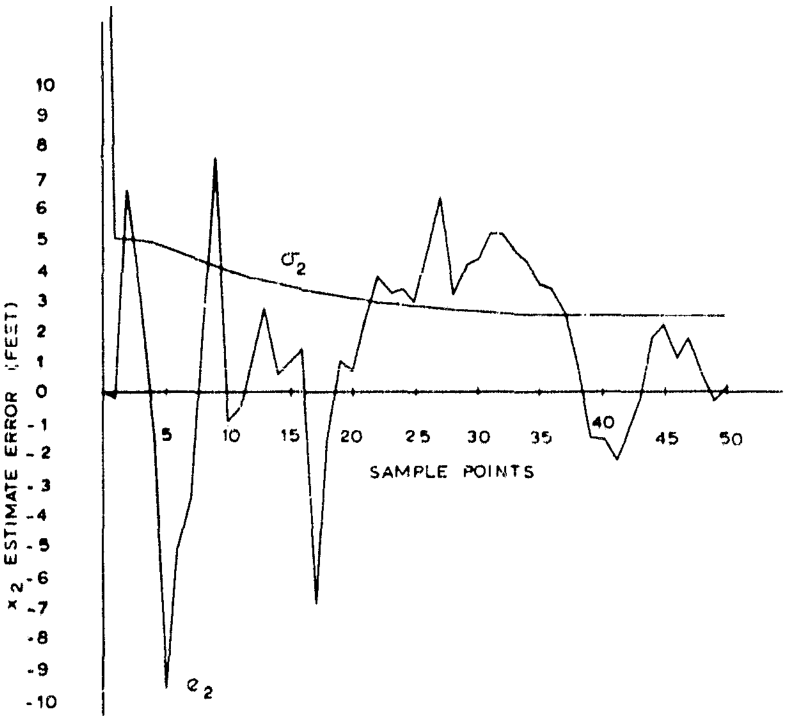

Figure 5. Estimate Error and Standard Deviation of $\hat{x}_2$

DISCUSSION

Two points should be emphasized with respect to the preceding results.

There may be cases in which the β estimate is not required either because the out-of-plane motion is not significant or because the sources of interest are in or near the orbit plane. In this case the quantities of

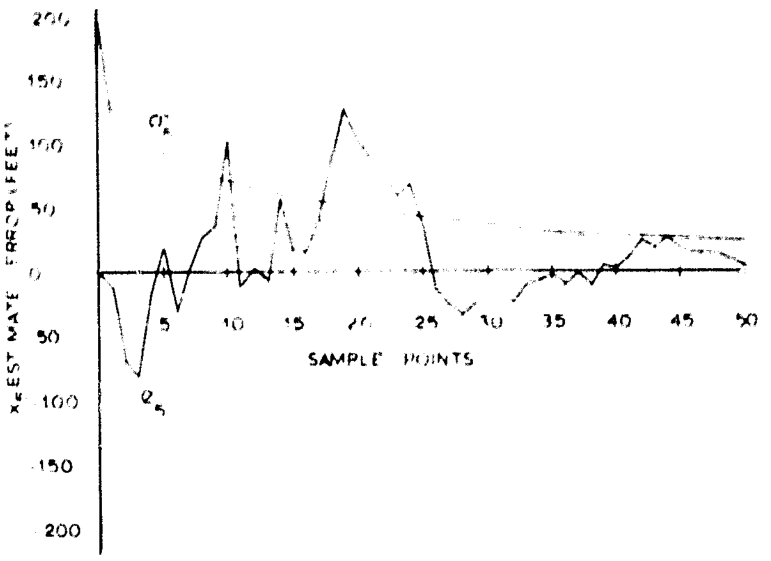

Figure 6. Estimate Error and standard Deviation of $\hat{x}_5$

interest γ and b may be estimated with a single on-board intersatellite range sensor; no optical angle sensors (B-trackers) are required on board the satellites.

Because the on-board sensors provide only part of the information required to make the system observable it was necessary to assume other information was available. This was done by redefining the variables (equation 15) and assuming x^2 was estimated from ground based measurements. Instead of doing this it could have been assumed that one of the vectors q or p is estimated from the ground. This would provide enough information to allow an estimate of the other vector. However, the large uncertainty in the radius variation δq_1 or δp_1 reflects directly into $\delta\hat{\gamma}$ as evidenced by equation (13). The resulting σ_γ would be much larger than σ_γ as determined by the described method. Thus the choice of the variables to be estimated is important. Since only the sums and differences of the variables are required to estimate $\delta\gamma$, δb and δ[illegible] only these are estimated and estimation of the absolute variations δq_1, δp_1 is avoided.

CONCLUSION

The three parameters describing the length and orientation of the line between two closely co-orbiting satellites cannot be adequately estimated from ground measurements alone because small uncertainties in the satellite location vectors contribute to large uncertainties in the baseline orientation. A minimum number of simple on-board sensors alone cannot do the job completely because they do not provide enough information in themselves. However, by combining the information provided by three on-board sensors (two angle sensors and an intersatellite ranging system) with the processed ground tracking information, a good linear estimate of these three parameters can be obtained. Care must be taken in defining the variables to be estimated to insure a low error in the final output.

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APPENDIX

The state transition matrix $\Phi(t, t_0)$ is

$$\Phi(t, t_0) = \left[\begin{array}{c|c} \Phi_1(t, t_0) & 0 \\ \hline 0 & \Phi_2(t, t_0) \end{array}\right] \quad \text{(A-1)}$$

$$\Phi_2(t, t_0) = \begin{bmatrix} \cos\omega_0\tau & \dfrac{\sin\omega_0\tau}{\omega_0} \\ -\omega_0\sin\omega_0\tau & \cos\omega_0\tau \end{bmatrix} \quad \text{(A-3)}$$

$$\tau = t - t_0$$

$$\Phi_1(t, t_0) = \begin{bmatrix} (4 - 3\cos\omega_0\tau) & 0 & \dfrac{\sin\omega_0\tau}{\omega_0} & \dfrac{2}{\omega_0}(1 - \cos\omega_0\tau) \\ 6(\sin\omega_0\tau - \omega_0\tau) & 1 & \dfrac{2}{\omega_0}(\cos\omega_0\tau - 1) & \dfrac{(4\sin\omega_0\tau - 3\omega_0\tau)}{\omega_0} \\ 3\omega_0\sin\omega_0\tau & 0 & \cos\omega_0\tau & 2\sin\omega_0\tau \\ 6\omega_0(\cos\omega_0\tau - 1) & 0 & -2\sin\omega_0\tau & (4\cos\omega_0\tau - 3) \end{bmatrix} \quad \text{(A-2)}$$