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INTEGRATION OF THE RELATIVISTIC EQUATIONS OF MOTION OF AN ARTIFICIAL EARTH SATELLITE

ABOLGHASSEM GHAFFARI

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Greenbelt, Maryland

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ABSTRACT

The Lindstedt perturbation method is applied to the motion of an artificial earth satellite which moves along a geodesic of the Schwarzschild metric of general relativity. The purpose of this analysis is to determine the extent to which general relativistic effects are detectable in range measurements of earth orbiting spacecraft.

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SUMMARY

The Lindstedt perturbation method is applied to the motion of an artificial earth satellite which moves along a geodesic of the Schwarzschild metric of general relativity theory. It is shown that both the amplitude and the frequency of the first approximate solution obtained is affected by the nonlinearity of the relativistic term appearing in the equations of motion. The approximate periodic solution is compared with the solution for the motion of an artificial earth 24 hours synchronous satellite orbit (Application Technology Satellite 3) in a Newtonian force field. Assuming that the coordinate time and the initial conditions were the same in both systems, it is deduced that the maximum of the magnitude of the radial deviation in both systems is of order 1.6 cm after half orbit period and then too small to be detected.

INTEGRATION OF THE RELATIVISTIC EQUATIONS OF MOTION OF AN ARTIFICIAL EARTH SATELLITE*

1. Introduction. In the theory of general relativity, the external gravitational field of a spherically symmetric massive body M, whose center lies at $r = 0$, is represented by the static Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{1}{c^2}\left[\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)\right]. \quad (1)$$

An exact solution, for such a metric, of a set of discrete bodies is possible only when one of the bodies is of finite mass whereas the rest are of infinitesimal small mass [1].†

If the massive body is taken to be the Earth, then an artificial earth satellite may be considered as an infinitesimal test particle whose gravitational field can be neglected.

The complete solution for the relativistic effects of the combined fields of the Earth and the Sun on an artificial earth satellite is complex. The metric must involve both the principal masses, and the Earth is orbiting the Sun, the field is no longer static. It is well known [2] that in general the motion of a

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*Some results of this paper have been presented at the Sixth Semi-Annual Astrodynamic Conference held at Goddard Space Flight Center, Greenbelt, Maryland on November 7 and 8, 1967.

†Figures in brackets indicate the literature references at the end of this paper.

satellite can be derived with a great deal of precision by considering the effects of the Earth and the Sun separately.

The relativistic effects of solar gravitation have been calculated by Corinaldesi and Papapetrou [3] and Papapetrou [4].

The use of artificial earth satellites seems more suitable than solar orbiting satellites for testing the effects of general relativity for the Sun has many planets and their mutual attractions are significant.

For convenience we suppose the earth gravitational field is spherically symmetric, and the Sun's gravitational effect on an artificial earth satellite is negligible. Bearing in mind the above mentioned assumptions, one knows that the motion of an artificial earth satellite is given by the equations of the ordinary geodesics of the Schwarzschild metric [1, 6] which in simplified form is

$$\left(\frac{d\,\frac{1}{r}}{d\phi}\right)^2 = 2\frac{m}{r^3} - \frac{1}{r^2} + 2m\frac{c^2}{h^2}\frac{1}{r} - \gamma$$

or

$$\left(\frac{du}{d\phi}\right)^2 = 2mu^3 - u^2 + \frac{2mc^2}{h^2}u - \gamma \tag{2}$$

where

$$u = \frac{1}{r}, \qquad \gamma = \frac{c^2}{h^2}(1 - \beta^2), \qquad \beta = \left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

$$m = \frac{GM}{c^2}, \qquad h = r^2\frac{d\phi}{ds},$$

r is the distance of the satellite from the center of the Earth, M is the mass of the Earth, v is the speed of satellite and c indicates the local speed of light, h and ϵ are constants of integration, the angle ϕ is the true anomaly and G is the gravitational constant.

2. Integration of the Equations of Motion

The classical equation (2) has been investigated and discussed completely in most standard texts on celestial mechanics and general relativity [5, 6, 7, 8 and 9]. The rigorous integration of (2) leads to the Weirstrass elliptic function σ (ϕ, g_2, g_3) which satisfies the differential equation

$$\left(\frac{d\sigma}{d\phi}\right)^2 = 4\sigma^3 - g_2\,\sigma - g_3 ,$$

where g_2 and g_3 are constants. However, in practice an approximate integration of (2) gives us the advance of the perigee.

The solution of equation (2) with reference to the Sun as a central body, and a planet as the test particle gave the perihilion advances in 100 years of the planets Mercury (43."03), Venus (8."64), Earth (3."84) and Mars (1."35), predicted by Einstein's law of gravitation.

A complete discussion of equation (2) has been given by J. Chazy [5], and McVittie [6] discussed and solved equation (2) by a different method.

Differentiating Eq. (2) with respect to the true anomaly ϕ and setting $m\,c^2/h^2 = 1/p$, we get

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{p} + 3mu^2, \tag{3}$$

provided $du/d\phi$ has only isolated zeros, i.e., eliminating the circular orbital solutions.

In the Newtonian law of gravitation the right-hand sides of (2) and (3) are quadratic and linear in u respectively. The presence of corrective terms $2\,mu^3$ in (2) and $3\,mu^2$ in (3) is due to the Einstein's law of gravitation.

Equation (3) was first derived by A. Eddington [7] and using a method of successive approximation he obtained, in ignoring the small term $3\,mu^2$, as a first approximation a Keplerian orbit, then substituting the first approximation in the small term $3\,mu^2$ he arrived at the second approximation with a secular term which presents the resonance case. P. Bergmann [9] also considered equation (3) and applying Fourier Series procedure he obtained the perihilion advances of the planets without giving the explicit expressions of planets orbits.

The purposes of this paper are:

1. Integration of equation (3) by the classical Lindstedt method, and obtain a first approximate periodic solution of (3) starting from the perigee.

2. Comparison of the approximate periodic solution with the solution for the motion of an artificial earth satellite in a Newtonian force field.

3. In order to have a numerical estimate of the comparison the central body is taken to be the Earth, and the test particle being identified with a long life earth synchronous satellite such as the Application Technology satellite 3.

The advantages of equation (3) over equation (2) are twofold:

a. Though equation (3) is of the second order, nevertheless it is nonlinear in u only, whereas equation (2) is nonlinear in both u and $du/d\phi$ and its integration leads to the Weierstrass elliptic functions.

b. We are primarily concerned with the periodic solution and its behavior of periodicity. The left-hand side of equation (3) suggests clearly the application of classical and modern approximation methods for derivation of approximations to periodic solutions.

3. Application of the Lindstedt Method. Let us assume that we wish to find an approximate periodic solution of Eq. (3) starting from the perigee i.e., satisfying the initial conditions:

$$u(0) = \frac{1 + e}{p}, \qquad \left(\frac{du}{d\phi}\right)_0 = 0 \; . \tag{4}$$

In order to find such a solution and investigate the perturbation in the basic frequency arising from the presence of the relativistic term on the right-hand side of equation (3), and also eliminate the secular term, we apply [10] the Lindstedt method to equation (3).

The Lindstedt method consists primarily of changing the independent variable ϕ to another independent variable α such that the determination of the available unknown coefficients enables us to eliminate gradually the secular terms in the subsequent approximations. As the frequency is altered it is of advantage [11] to replace the independent variable ϕ by a new independent variable α through the relation:

$$\phi = \alpha \left(1 + c_1 \epsilon + c_2 \epsilon^2 \ldots\right), \tag{5}$$

where $c_1, c_2 \ldots\ldots$ are unknown coefficients and ϵ is an arbitrary small positive parameter. The smallness of the gravitational radius m enables us to suppose $\epsilon = 3m$, and then the equation (3) becomes

$$\frac{d^2u}{d\alpha^2} + \left(1 + c_1 \epsilon + c_2 \epsilon^2 \ldots\right)^2 \left(u - \frac{1}{p} - \epsilon u^2\right) = 0$$

or

$$\frac{d^2u}{d\alpha^2} + u - \frac{1}{p} + \epsilon \left(-\frac{2c_1}{p} + 2c_1 u - u^2\right) + 0(\epsilon^2) = 0. \tag{6}$$

Now let us write the solution of (6) in a power series with respect to the small parameter ϵ:

$$u(\epsilon, \alpha) = \sum_{n=0}^{\infty} \epsilon^n u_n(\alpha), \tag{7}$$

and limit ourselves to the first order approximation such that

$$u = u_0 + \epsilon u_1 + O(\epsilon^2) . \tag{8}$$

Using the equations (6) and (7), one finds that the leading term u_0 is a solution of the unperturbed equation

$$\frac{d^2 u_0}{d\alpha^2} + u_0 = \frac{1}{p} , \tag{9}$$

and u_1 satisfies

$$\frac{d^2 u_1}{d\alpha^2} + u_1 = \frac{2 + e^2}{2p^2} - \frac{2c_1}{p} + \frac{2e}{p^2} (1 - c_1 p) \cos \alpha + \frac{e^2}{2p^2} \cos 2\alpha . \tag{10}$$

The unknown coefficient c_1 is to be determined such that no secular term appears in the solution of (10). Hence we chose c_1 so that

$$c_1 = \frac{1}{p} , \tag{11}$$

and therefore equation (10) assumes the form

$$\frac{d^2 u_1}{d\alpha^2} + u_1 = \frac{1}{2p^2} (e^2 - 2 + e^2 \cos 2\alpha) \tag{12}$$

which has the solution

$$u_1 = \frac{e^2 - 2}{2p^2} (1 - \cos 2\alpha) \tag{13}$$

satisfying the initial conditions

$$u_1(0) = 0, \quad \left(\frac{du_1}{d\alpha}\right)_0 = 0. \tag{14}$$

Therefore the first order approximate periodic solution of (3), satisfying the initial condition (4), is

$$u = u_0 + \epsilon u_1 + 0(\epsilon^2) = \frac{1 + e\cos\alpha}{p} + \frac{3m(e^2 - 2)}{2p^2}(1 - \cos 2\alpha) + 0(\epsilon^2), \tag{15}$$

where

$$\alpha = \frac{\phi}{1 + c_1\epsilon \ldots} = \frac{\phi}{1 + c_1\epsilon} + 0(\epsilon^2) = \left(1 - \frac{3m}{p}\right)\phi + 0(\epsilon^2). \tag{16}$$

We can deduce that the nonlinearity shown by the relativistic term in (3) affects not only the amplitude of the solution but also the frequency.

Formulas (15) and (16) show that the change in the frequency depends upon the amplitude a and the eccentricity e of the Keplerian orbit and also of the parameter $\epsilon = 3$ m, a property which belongs to the periodic solutions of all non-linear autonomous differential equations.

The period of the approximate periodic solution of (9) is equal to 2π, that is to say, the orbits are closed. The period of the exact solution of (3) differs from 2π by a small amount δ, which is the difference between the angles of two succeeding perigees, given by

$$\delta = \frac{2\pi}{1 - \frac{3m}{p}} - 2\pi = 2\pi \left(1 + \frac{3m}{p}\right) - 2\pi = \frac{6m\pi}{p} = \frac{6m\pi}{a(1 - e^2)} \text{ rad. per revol.} \tag{17}$$

Therefore the precession of the perigee of satellite orbit obtained by this method amounts to $6m\pi/p$ radians per revolution, which is in good agreement with the precession predicted by Einstein's theory of general relativity as well as with gravitational theories of Whitehead [12] and Birkhoff [13].

4. Comparison with Newtonian Force Field and Numerical Analysis

Now let a second artificial earth satellite with the same characteristics features as the first one, move in a planar elliptic orbit according to Newtonian law of motion. If r_N and ϕ are the classical polar coordinates of the second satellite in orbital plane, and $u_N = 1/r_N$, the classical Newtonian equation of motion is

$$u_N = \frac{1 + e \cos \phi}{p}, \tag{18}$$

where the true anomaly ϕ is measured from the perigee and

$$p = a\,(1 - e^2)$$

is the semi-latus rectum, $\underline{a}$ and $\underline{e}$ are the semi-major axis and the eccentricity of the Newtonian orbit respectively. We assume both satellites start from the perigee, i.e., they satisfy both the initial conditions:

$$\left.\begin{aligned}(r_N)_0 &= (r_R)_0 \\ (u_N)_0 &= (u_R)_0 \\ (u'_N)_0 &= (u'_R)_0\end{aligned}\right\} \tag{19}$$

where u_R and u_N stand for relativistic and Newtonian solutions respectively.

We are now in a position to compare the approximate periodic relativistic solution (15) with the classical Newtonian solution (18) by forming the difference

$$\begin{aligned}\Delta u = u_N - u_R = \frac{e}{p}&\left[\cos\phi - \cos\left(1 - \frac{3m}{p}\right)\phi\right] \\ &- \frac{3m(e^2 - 2)}{2p^2}\left[1 - \cos 2\left(1 - \frac{3m}{p}\right)\phi\right] .\end{aligned} \tag{20}$$

The difference Δu is a function of the Keplerian orbital elements and the gravitational radius m of the central body. As the primary body is taken to be the Earth, then m = 0.443 cm and the parameters a and e are constant for one revolution. Therefore, the difference $\Delta u\ (\phi)$ is a function of the true anomaly ϕ only for one revolution.

In order to have a numerical estimate of $\Delta u\ (\phi)$, formula (20) was applied to the case of the Earth as the central body and the Application Technology Satellite 3 with the elements:

$$a = 6.6109161 \times R$$

$$e = 0.0001703684$$

as a test particle, $R = 6.371 \times 10^8$ cm is the mean radius of the Earth. The results are plotted and shown in Table 1 and Figure 1 in the range $0 \leq \phi \leq 180°$ and the function $\Delta u(\phi)$, being a continuous function of ϕ, begins to increase again after 180°.

Now setting

$$r_R = r_N + \sigma = r_N \left(1 + \frac{\sigma}{r_N}\right), \tag{21}$$

where σ denotes the relativistic correction in radial deviation in both systems, one finds that

$$\Delta u = u_N - u_R = \frac{1}{r_N} - \frac{1}{r_R} = \frac{\sigma}{r_N^2}. \tag{22}$$

Table 1 and Figure 1 show that, in case of Application Technology Satellite 3, the relativistic correction to equation (3) is too small to be detected and the maximum value of $\Delta u \sim 1.5 \times 10^{-19}$ is obtained for $\phi = 89°59'59''.148792708$, correct to the nine decimal places given in the seconds. Therefore the maximum of the radial deviation in both systems is of the order

$$\sigma \sim r_N^2 \,(1.5 \times 10^{-19}) = (4 \times 10^9 \text{ cm})^2 \,(1.5 \times 10^{-19}) \sim 1.6 \text{ cm}, \tag{23}$$

and then too small to be detected. However, it is important to separate the relativistic effects from those due to the other causes, such as the oblateness of the Earth, magnetic fields, and the influence of high-altitude winds, etc.

$$\Delta u = \frac{e}{p}\left[\cos\phi - \cos\left(1 - \frac{3m}{p}\right)\phi\right] - \frac{3m(e^2 - 2)}{2p^2}\left[1 - \cos 2\left(1 - \frac{3m}{p}\right)\phi\right]$$

$(0° \leq \phi \leq 180°)$

[illegible]	[illegible]
[illegible]	[illegible]
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[illegible]	[illegible]
[illegible]	[illegible]
70	[illegible] $\times 10^{-18}$
[illegible]	[illegible]
[illegible]	[illegible] $\times 10^{-18}$
[illegible]	.1198160626 $\times 10^{-18}$
100	.1452961373 $\times 10^{-18}$
110	.1322857458 $\times 10^{-18}$
120	.1123541627 $\times 10^{-18}$
130	.8790553806 $\times 10^{-19}$
140	.6188892964 $\times 10^{-19}$
150	.3744260732 $\times 10^{-19}$
160	.1751553208 $\times 10^{-19}$
170	.4511663842 $\times 10^{-20}$
180	$-$.1050408293 $\times 10^{-27}$
190	.4525317707 $\times 10^{-20}$

Table 1. Variation of Δu in Terms of the True Anomaly ϕ

$$\Delta u = \frac{e}{p}\left[\cos\phi - \cos\left(1-\frac{3m}{p}\right)\phi\right] - \frac{3m(e^2-2)}{2p^2}\left[1-\cos 2\left(1-\frac{3m}{p}\right)\phi\right]$$

$$(0^\circ \le \phi \le 180^\circ)$$

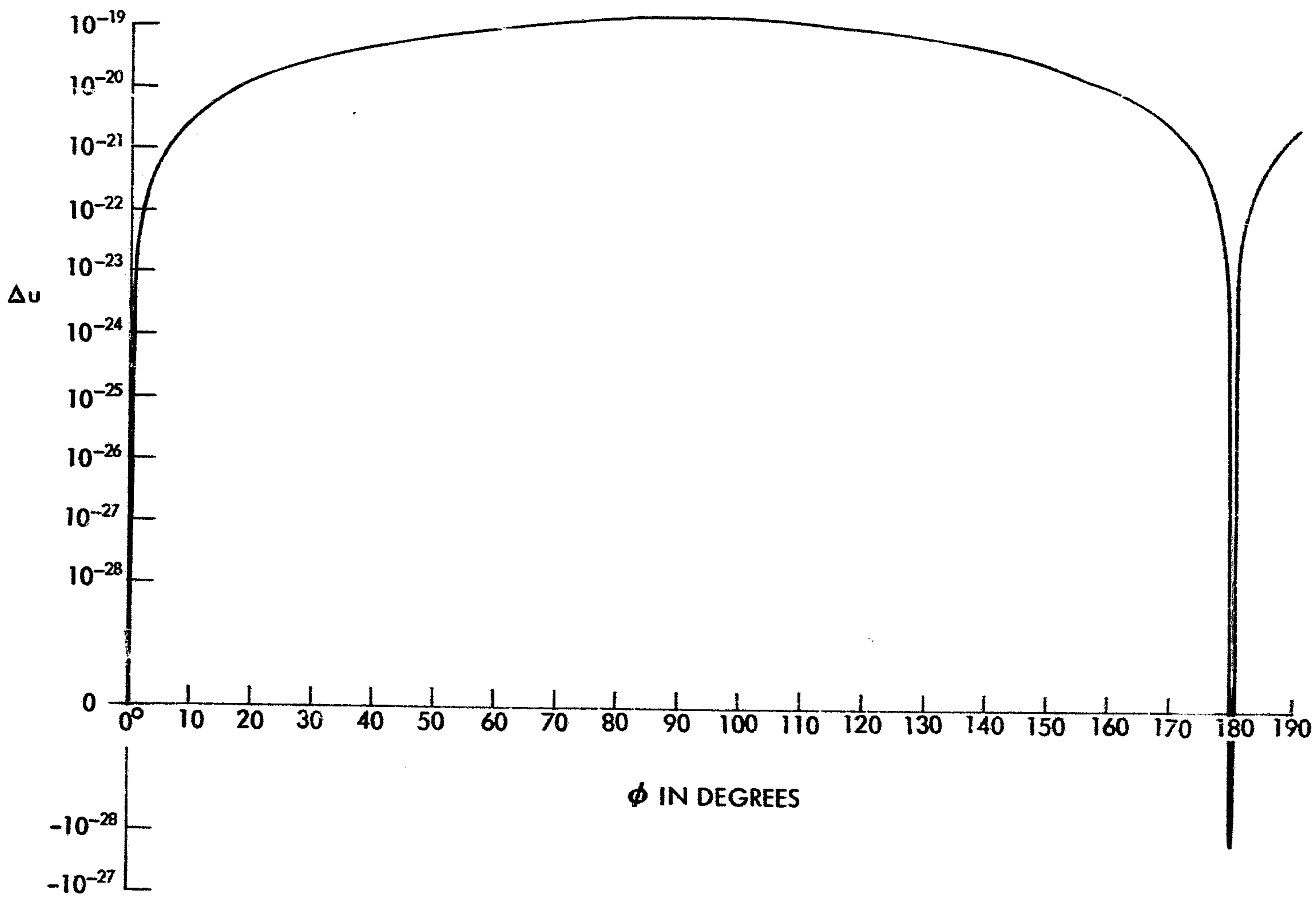

Figure 1–Radial Component of Range Perturbation

Application of the averaging method of Bogoliubov-Mitropolsky [14], which leads to a different approximate solution, and its comparison with the solution (15) should follow later.

5. Acknowledgment

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6. References

1. McVittie, G. C., Relativistic effects on Space Tracking Data. First Semi-Annual Status Report, NASA NGR 14-005-088 (1967), to appear in the Astronomical Journal.

2. Sitter, W. de., Monthly Notices, Roy. Astron. Soc. 77, 155 (1916).

3. Corinaldesi, E. and Papapetrou, A. Proc. Roy. Soc. A. 209 (1951) 259.

4. Papapetrou, A., Proc. Roy. Soc. A-209 (1951) 248.

5. Chazy, J. La théorie de la relativité et la mécanique Céleste, Tames I et II, Gauthier-Villars, Paris (1928) et (1930).

6. McVittie G., General relativity and Cosmology, 2nd Ed. Illinois University Press, Urbana (1965).

7. Eddington, A., The mathematical theory of relativity, Cambridge University Press (1924).

8. Synge, J. L., Relativity: The general theory, 2nd Printing (1964), North-Holland Publishing Company, Amsterdam.

9. Bergamn, P., Introduction to the Theory of Relativity, Prentice-Hall, Inc. (1942).

10. Ghaffari. A., On Some Applications of Approximate Methods in Relativistic Celestial Mechanics. To appear in the Proceedings of the Fifth International Conference on Nonlinear Oscillations, Kiev USSR, August 25-September 5, 1969.

11. Bellman, R. E., Perturbation Techniques in Mathematics, Physics, and Engineering, Holt, Rinehart and Winston, Inc. (1964) .

12. Whitehead, A. N., The principle of relativity, with applications to physical Science, Cambridge University Press (1922).

13. Birkhoff, G. D., Relativity and Modern Physics, with the Corporation of R. E. Langer, Harvard University Press, Cambridge (1923).

14. Bogoliubov-Mitropolsky, Asymptotic methods in the Theory of Nonlinear Oscillations. Translated from Russian (revised ed. 1958) by Gordon and Breach (1962).