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FINITE-LARMOR-RADIUS ANALYSIS OF LAMINAR COLLISIONLESS SHOCKS

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FINITE-LARMOR-RADIUS ANALYSIS
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by

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ABSTRACT

The structure of finite-β laminar collisionless fast and slow shock waves is investigated from the finite-Larmor-radius Chew-Goldberger-Low (FLR-CGL) hydromagnetic fluid equations. As a motivation for the shock study the linear dispersion relation for the three hydromagnetic waves is derived including ion FLR dispersion.

To calculate the shock structure two criteria must be satisfied: (1) a complete set of Rankine-Hugoniot conservation laws must exist; (2) the moment hierarchy of the Vlasov equation must be truncated. For perpendicular fast shocks these conditions are attained since the temperature parallel to the magnetic field is conserved across the shock and the parallel heat flow vanishes. For non-perpendicular shocks neither condition is satisfied, and some approximations are required to continue the fluid approach to collisionless shocks. Two approximations which close the Rankine-Hugoniot relations are considered: (1) for near-perpendicular fast shocks the parallel temperature is taken to be approximately constant; (2) for arbitrary propagation angle the parallel and perpendicular components of the pressure tensor are taken to be equal. The moment equations are closed by neglecting the parallel heat flow.

With these approximations a wave train differential equation for the magnetic field is derived in the weak shock approximation. A linearized analysis about the Rankine-Hugoniot stationary points is then performed to determine the wave train properties. The major results are: (1) near-perpendicular $\beta_+ > 1$ fast and slow shocks possess a trailing ion gyro-radius wave train; (2) oblique $\beta_+ < 1$ slow shocks have a trailing ion inertia wave train; (3) the structure of near-parallel $\beta_+ > 1$ fast shocks consists of a leading ion gyro-radius wave train; (4) near-parallel $\beta_+ > 1$ slow shocks have a trailing ion gyro-radius wave train.

Recent observations on the earth's bow shock obtained by the OGO-V satellite indicate an electron inertia trailing wave train to be the most persistent laminar feature. The solar wind never satisfies the low-β and almost perpendicular propagation criteria required for the classical electron inertia wave train to occur. To understand these observations a set of fluid equations is obtained to describe the short scale length layer. Both electron inertia and electron gyro-radius produce a trailing wave train in the magnitude of the magnetic field. This wave train is relatively insensitive to the shock propagation angle. It is then argued that for strong shocks the long scale length wave trains may possess insufficient group velocities to convect away the shock compression energy and thus limit the shock steepening. The shock continues to steepen until either a short dispersion scale length establishes a steady shock or the shock flow becomes turbulent. This effect is termed a dispersion discontinuity and is presented as an interpretation of the bow shock observations.

Finally, some non-self-consistent estimates are made of the turbulent ion sound dissipation observed at the electron inertia wave train in the bow shock. A crude argument on the amplitude of the saturation ion sound electric field and the quasi-linear linear theory permit a rough determination of anomalous electron-ion and ion-ion collision frequencies. With these estimates it is shown that the pressure should be approximately maintained at isotropy and that the electron and ion parallel heat flows should be suppressed by the turbulence. The length of the electron inertia wave train required to thermalize the solar wind ions is estimated and found consistent with observations.

Another possible turbulent dissipation mechanism, the parametric three-wave decay of the wave train, is considered. As an example, the trailing ion gyro-radius wave train of the perpendicular fast shock is found to be decay unstable to perturbations containing Alfvén waves.

1.0 Introduction

1.1 Collisionless Shocks

The development of a physical theory for collisionless plasmas has proceeded vigorously over the past decade and a half, spurred on primarily by the possibility of achieving a controlled thermonuclear reaction. Residing on the periphery and yet embodying many of the fundamental difficulties of such a theory is the problem of collisionless shocks. Here the basic non-linearity of shock phenomena combined with the requirement that the shock dissipation be accomplished by plasma turbulence forces consideration of the more advanced but least understood aspects of plasma physics.

Theoretical investigation of collisionless shocks has developed along two main schools of thought. The laminar shock theory employs fluid equations to resolve the shock structure into an oscillatory wave train; here the turbulent dissipation is of secondary importance. Proponents of this school include Sagdeev (1958; 1966, 1967), Adlam and Allen (1958), Davis _et al._ (1958), Gardner _et al._ (1958), Auer _et al._ (1961, 1962), Cavaliere and Englemann (1967), Kennel and Sagdeev (1967b) and Goldberg (1969). Alternatively Petschek (1958, 1965) and his co-workers, Fishman _et al._ (1960) and Camac _et al._ (1962), developed a fully turbulent shock theory in which standing whistler mode waves are amplified in the shock and produce dissipation by wave-wave scattering. In the same philosophy a high-β turbulent shock was constructed by Kennel and Sagdeev (1967a); here the turbulent dissipation is accomplished by firehose unstable Alfvén waves.

An attempt to unify the whistler and Alfvén turbulence theories is discussed in Kennel and Petschek (1968). A very high-β shock model based on ion sound turbulence driven by multi-streaming ions has been developed by Tidman (1967).

Although both of these general methods have received some confirmation from experiment (see Patrick and Pugh, 1969, for a discussion), the emphasis here will concentrate almost exclusively on the fluid approach. An unresolved question which persists, however, is to what extent can a collisionless plasma be described by fluid equations? Observations of collisionless fluid behavior, as differentiated from kinetic or Vlasov phenomena, are still scarce, but there exists considerable evidence that the space plasma, which is truly collision-free, exhibits some fluid properties. In the interaction of the super-magnetosonic solar wind with the earth's magnetic field a fast bow shock is formed whose average position and flow properties are well described by hydromagnetics (see the review by Spreiter and Alksne, 1969). In addition, many of the phenomena observed inside the magnetosphere are explicable in terms of an internal hydromagnetic convective flow (Axford and Hines, 1961; Levy _et al._, 1964; Axford _et al._, 1965; Axford, 1969).

Hence the fluid theory might provide a more accurate description of a collision-free plasma than had been previously anticipated. Collisionless turbulent dissipation arising from high frequency instabilities may often be sufficiently intense to permit low frequency-long wavelength phenomena to be described by fluid equations. Therefore the plasma turbulence occurring in the shock structure, although left qualitatively and quantitatively ill-defined, forms the philosophical foundation for the fluid theory of collisionless shocks. (For a more general discussion of these and related concepts the reader is referred to Kennel, 1969.)

1.2 Review of Laminar Shock Theory

The fluid approach to collisionless shocks commenced with the investigation of a hydromagnetic pulse or shock propagating perpendicular to the magnetic field into a cold plasma (Sagdeev, 1958; Adlam and Allen, 1958; Davis *et al.*, 1958; Gardner *et al.*, 1958). If dissipation is neglected, the time independent solution of the fluid equations consists of a non-linear pulse or solitary wave which maintains its shape as it propagates in the plasma and has a characteristic thickness given by the electron inertia scale length, C/ω_{p_-} (see section 2.0 for definition of symbols). Being dissipationless the fluid state ahead of and behind the soliton is identical. With the addition of a weak amount of dissipation, the solitary wave is converted into a shock. The upstream and downstream states, which satisfy the Rankine-Hugoniot conservation relations, are connected by an oscillatory wave train which trails the leading jump in the plasma parameters, and spatially damps to the uniform downstream state. The oscillation wavelength is also of the order of the electron inertial length. This particular case is reviewed in more detail in section 2.0.

The next evolution of the fluid theory was to relax the restriction of strict perpendicular propagation while still retaining the simplicity of the cold plasma approximation (Sagdeev, 1966; Cavaliere and Englemann, 1967). For angles sufficiently far from perpendicular, the contributions from ion inertia dominate those of electron inertia. The wave train is now found to lead the sharp jump in the magnetic field and consists of both rotational and compressional magnetic components; the oscillation wavelength is characterized by the ion inertia length, C/ω_{p_+} (see section 6.0). This work is restricted to fast shocks by the zero pressure assumption.

In several laboratory plasmas and most plasmas of space and astrophysical interest the thermal pressure is not negligible compared with the magnetic pressure. The scale length associated with the particle gyro-radius introduces new effects in the wave train structure, and therefore must be included in the fluid equations. Kennel and Sagdeev (1967b) and subsequently Goldberg (1969) attempted to construct a wave train for a finite-β perpendicular fast shock in which the ion gyro-radius was the dominant oscillation length. Unfortunately the fluid equations employed by these authors were incomplete, and the incorrect sign for the wave train dispersion was obtained (MacMahon, 1968; Fredricks and Kennel, 1968).

The above review is altogether too short to do justice to the above work. In addition, many interesting investigations such as Auer _et al._ (1961, 1962), Morowetz (1961, 1962), Moiseev and Sagdeev (1963), Kellogg (1964), and Bardotti _et al._ (1966) cannot be mentioned. The interested reader is referred to the collisionless shock literature.

1.3 Purpose and Content

The main direction of this work is motivated by recent observations on the structure of the earth's bow shock. At times the magnetic shock transition exhibits a laminar-like wave train which, at least on one occasion, possessed a multiple scale length structure (Fredricks and Coleman, 1968). Since in the solar wind both the electron and ion thermal pressures are comparable to the magnetic pressure, the wave train shock structure must be calculated from fluid equations which contain the effects of finite Larmor radius. The fluid set chosen here is the Chew-Goldberger-Low (CGL) (1956) hydromagnetics with finite Larmor radius (FLR) corrections as developed by MacMahan (1965). In addition to fast shocks, consideration of finite pressure permits an investigation of the structure of collisionless slow shocks, a hitherto virtually unexplored subject.

The structure of the FLR-CGL equations is quite complicated, and many assumptions and approximations, not all of which can be rigorously justified, will be made in order to pursue the fluid approach. The attitude taken here toward these equations is founded within the historical and philosophical development of the fluid theory. The primary purpose of the calculations is to determine the sign of dispersion and the scale lengths of the fast and slow shock wave trains. Unfortunately much important information about the shock structure will be sacrificed in the attainment of this objective. Little will be said about the details of the turbulent dissipation processes. However, it is explicitly assumed that the turbulence is of sufficient strength so that the plasma may be considered to behave as a fluid, but not so strong that the wave train structure is obliterated. Furthermore, only the leading and trailing edges of the shock are readily analysable; no details will be obtained about the structure in the middle of the shock.

The fundamental concepts of the fluid theory are most easily demonstrated by the example of what might be called "the classical hydromagnetic collisionless shock." This shock propagates perpendicular to the magnetic field into a plasma with $\beta_+ < M_-/M_+$; here the only important dispersion scale length is C/ω_{p_-}. The steepening of the shock is inhibited when the non-linearly excited harmonics attain wavelengths of the order of C/ω_{p_-} (section 2.2). A steady wave train is formed which is describable by a differential equation for the magnetic field (section 2.3); the weak shock approximation is used to simplify this equation and recover the results of Sagdeev (1966). To obtain the wave train structure, the differential equation is linearized about the Rankine-Hugoniot stationary points. Finally it is noted that many of the wave train properties could have been obtained by examining the linear fast wave dispersion relation. The methods developed and reviewed in this section form the basis for the remaining work.

The investigation of the FLR-CGL equations commences by studying the linear response of the plasma. The equations themselves are exhibited and discussed in section 3.2. Dispersion relations for the fast, intermediate, and slow waves are derived in section 3.3. The dispersion properties of the fast and slow waves select the appropriate scale lengths and the sign of dispersion for the wave train solutions, and hence form a check on later work.

In order to calculate the wave train structure, two conditions must be satisfied by the FLR-CGL fluid equations: (1) a closed set of Rankine-Hugoniot conservation relations must exist; (2) the moment equations must be truncated. Only perpendicular fast shocks satisfy these criteria. Here the temperature parallel to the magnetic field is isothermal so that the parallel heat flow vanishes. For non-perpendicular propagation of both fast and slow shocks there are insufficient conservation laws to determine the plasma variables uniquely; furthermore, the parallel heat flow is not specified by the moment equations (section 4.2). The implication of these results is that the downstream state is no longer independent of the details of the shock structure; determination of the energy distribution between the parallel and perpendicular degrees of freedom depends on the turbulent dissipation.

To proceed with the fluid approach within the FLR-CGL equations, several assumptions are required. For near-perpendicular fast shocks taking the parallel temperature to be an approximate constant of the motion closes the Rankine-Hugoniot relations (section 4.3). If the analysis is to include arbitrary propagation angles and slow shocks, a relation between the parallel and perpendicular pressures is needed to complete the conservation relations; here the calculations are performed assuming the two pressures to be equal (section 4.4). To truncate the moment equations it is assumed, without justification, that the turbulent dissipation suppresses the parallel heat flow (section 4.5).

With the above considerations separate wave train differential equations for the magnetic field are derived for each of the two approximations, constant parallel temperature (section 5.3) and isotropic pressure (section 5.4). The weak shock approximation is invoked to simplify the calculations. The dispersion properties of the wave train differential equation are analyzed by linearizing about the Rankine-Hugoniot stationary points (section 6.2). Both the near-perpendicular fast (section 6.3) and slow (section 6.5) high-β shocks possess trailing ion gyro-radius wave trains. For near-parallel propagation the fast high-β shock (section 6.3) has a leading ion gyro-radius wave train whereas the ion gyro-radius wave train trails for the analogous high-β slow shock (section 6.6). The low-β oblique slow shock structure consists of a trailing ion inertia wave train (section 6.5).

Recent results from the earth's bow shock have indicated that a C/ω_{p_-} trailing wave train is the most persistent laminar structure observed (Fredricks et al. 1968; Fredricks and Coleman, 1968; section 8.2). The solar wind never satisfies the low-β criteria required for the classical C/ω_{p_-} wave train to occur. In an attempt to understand the bow shock observations a wave train differential equation is derived and analyzed from a set of fluid equations based on the small parameter expansions $L_s/R_+ << 1$ and $R_-/L_s << 1$ (sections 7.2 and 7.3); L_s is a characteristic shock thickness. Both finite electron inertia and finite electron gyro-radius produce a trailing wave train (section 7.4).

To understand the occurrence of a short scale length wave train in a high-β shock it is noted that in the oblique fast wave (section 3.3) and in the near-perpendicular fast shock wave train differential equation (section 6.3) ion inertia and ion gyro-radius dispersion produce opposite contributions. Hence a mathematical cancellation between these two dispersion effects is possible, leaving only electron inertia and electron gyro-radius dispersion. However, the precision required for the cancellation effect to occur probably excludes it from being observed in physical shock flows.

A more reasonable explanation of the bow shock observations follows from a reconsideration of the dispersion limitation on the shock steepening (section 7.5). If the shock is weak, the long scale length wave trains are capable of convecting the compression energy out of the shock front and establishing a steady shock. For strong shocks, however, the long scale length wave trains may be inadequate to limit the shock steepening. Hence the shock continues to steepen until a short dispersion scale length establishes a steady wave train or the shock flow becomes turbulent. In analogy with the dissipation discontinuities in hydromagnetic shocks (Marshall, 1955; Coroniti, 1969) this effect is called a dispersion discontinuity. It is suggested that the C/ω_{p_-} wave train in the earth's bow shock can be interpreted as a dispersion discontinuity.

The weakest part of the fluid approach is the indefiniteness of the plasma turbulence providing the shock dissipation. In an attempt to at least partially rectify this situation, some quantitative, although non-self-consistent, estimates of the ion sound turbulence observed in the C/ω_{p_-} layer of the earth's bow shock are made in section 8.3. The currents supporting the sharp magnetic field gradients in the wave train are unstable to the emission of ion sound waves. A very crude argument which assumes that the turbulent wave spectrum saturates by ion absorption permits an estimate of the electric field amplitude.

Quasi-linear theory is then used to construct anomalous electron-ion and ion-ion collision frequencies. The resulting dissipation rates are sufficiently high to maintain the ion pressure isotropic and to suppress the electron and ion parallel heat flows. As a check on the ion sound turbulence calculations, the length of the C/ω_{p_-} wave train needed to provide ion thermalization is estimated, and found to be consistent with observations.

Another turbulence process which might contribute to the shock dissipation is the non-linear three-wave decay of the wave train. For the particular example of a high-β perpendicular fast shock the trailing ion gyro-radius wave train is found to be decay unstable to perturbations containing Alfvén and magneto-sonic waves (section 8.4). A decay length for the wave train is estimated using the instability growth rate.

Section 9.1 reviews the philosophy of the fluid approach to collisionless shocks and indicates how a self-consistent shock theory might be formulated. The problems associated with the dispersion discontinuity are further commented upon in section 9.2. Finally, section 9.3 mentions the importance of slow shocks in providing magnetic dissipation in neutral sheet flows, and speculates on a possible turbulent structure for low-β oblique slow shocks.

2.0 The Fluid Description of Collisionless Shocks

2.1 Introduction

The fluid approach to the structure of collisionless shocks commences with the equations of ideal hydromagnetics. Although the time independent, conservation or Rankine-Hugoniot form of these equations permit discontinuous or shock solutions, the equations contain no basic scale length; hence a more general set of equations is required to resolve the shock structure. The appropriate modification of hydromagnetics is to include the dispersive effects of finite electron and ion inertia characterized by the induction scale lengths $C/\omega p_{\pm}$, $\omega p_{\pm} = [(4\pi N_{\pm} e^{2}_{\pm})/M_{\pm}]^{\frac{1}{2}}$, and of finite Larmor radius (FLR), $R_{\pm} = C_{\pm}/\Omega_{\pm}$; $C_{\pm}$ is a typical thermal speed, $\Omega_{\pm} = (e_{\pm} B)/M_{\pm} C)$, the gyro-frequency, $e_{\pm} = \pm e$, where e is the magnitude of the electronic charge, B the magnetic field strength, $M_{\pm}$ the ion and electron mass, and C is the velocity of light. Gaussian units are used throughout. Only a single species of ions, protons, will be considered. The shock thickness will be proportional to one or more of these basic scale lengths.

The investigation of the collisionless shock structure begins with a discussion of the limitation of the shock steepening. Section 2.2 reviews the arguments presented by Sagdeev (1966), Petschek (1965), Kennel and Sagdeev (1967b), and many others. To introduce the methods of the fluid approach and to further elaborate the fundamental concepts, a particularly simple example of a fast shock which includes only the effects of finite electron inertia is discussed in section 2.3.

2.2 Limitation of Shock Steepening

Consider a finite amplitude, spatially limited compressional wave which propagates at an arbitrary angle to a homogeneous magnetic field and obeys the hydromagnetic dispersion relation $\omega = kC_{HM}$; C_{HM} is either the fast or slow propagation speed for linear hydromagnetic waves defined by (Kantrowitz and Petschek, 1966)

$$\left.\begin{array}{c} C_F^2 \\ C_{SL}^2 \end{array}\right\} = \frac{C_A^2 + C_S^2}{2} \pm \left[\left(\frac{C_A^2 + C_S^2}{2} \right)^2 - C_S^2 C_I^2 \right]^{1/2}$$

$C_A^2 = B^2/4\pi\rho$ is the Alfvén speed, $C_s^2 = \gamma\, p/\rho$ is the sound speed, and $C_I^2 = C_A^2 \cos^2\theta$ is the intermediate speed; θ is the angle between the wave vector $\underline{k}$ and $\underline{B}$; ρ is the mass density, P is the pressure, and γ the ratio of specific heats. Being compressional, the wave steepens into a shock. If the wave is resolved into its Fourier components, the steepening can be viewed as the non-linear excitation of higher harmonics of the fundamental frequency generated by the non-linear terms in the hydromagnetic equations. If the dispersion relation $\ell\omega = \ell kC_{HM}$, where ℓ is a harmonic number, is satisfied, the higher harmonics propagate with the phase velocity of the fundamental, and therefore remain in the wave front. Eventually, however, the wavelengths of the excited modes become comparable to one of the basic scale lengths thus resulting in dispersive propagation.

To understand dispersive propagation, consider the effect of finite ion inertia on a magnetically polarized hydromagnetic wave. The wave electric field accelerates the ions (and electrons) to a velocity of the order of the Alfvén speed. At long wavelengths all the ions experience the same average phase of the electric field since the spatial gradient of the electric field is much smaller than the "gyro-radius" based on the Alfvén speed, $kC_A/\Omega_+ = kC/\omega_{p_+} << 1$. When $kC_A/\Omega_+ \sim 1$, ions at different points in their gyro-orbits experience a different phase of the electric field; the result is a phase lag between the force and ion response which depends on the wavelength. Dispersive propagation results since different wavelengths propagate at different phase velocities. Similar arguments can be constructed to describe the dispersive effects of C/ω_{p_-} and $R_\pm$.

The non-linear excitation of short wavelength modes produces waves which no longer remain in the wave front but propagate either ahead of or behind depending on whether the dispersion increases the phase velocity above or decreases it below the hydromagnetic velocity. A steady state is possible in which the compression energy associated with the non-linear excitation is balanced by the energy loss due to dispersive propagation. Thus a non-linear pulse or solitary wave, investigated by Adlam and Allen (1958), Davis et al. (1958), Gardner et al. (1958) and Sagdeev (1958), is formed which, since it propagates non-dispersively, maintains its shape. Since no dissipation processes are involved, the plasma state behind the wave is the same as that ahead.

2.3 Example of C/ω_{p-}, Perpendicular Shock

To demonstrate the above concepts and to clarify the transition from a solitary wave to a shock solution, a differential equation is derived which describes the shock structure for the following parameters: shock propagation perpendicular to the magnetic field, $\beta_+ < M_-/M_+$ and $\beta_- << 1$; $\beta_\pm = (8\pi N_\pm T_\pm)/B^2$ where $N_\pm$ is the number density and $T_\pm$ the temperature. This case was studied by Sagdeev (1966) in the zero β limit; the derivation here is only slightly more general in that the energy equation is used to account for a small, but finite, pressure. For the above parameters, C/ω_{p_-} exceeds $R_\pm$; C/ω_{p_+} does not enter into the equations. The physical principles uncovered by this example as well as the methods used are those employed throughout the remainder of this work.

Consider a plane, time stationary shock propagating in the x-direction perpendicular to a magnetic field in the z-direction. In the shock frame, the equations which describe the shock structure are

$$\rho u = \rho_1 u_1 \tag{2.1}$$

$$\rho u (u - u_1) + P - P_1 - \frac{B^2 - B_1^2}{8\pi} = 0 \tag{2.2}$$

$$\rho u \left(\frac{u^2 - u_1^2}{2}\right) + \frac{\gamma}{\gamma - 1}\left(Pu - P_1 u_1\right) + \frac{u_1 B_1 (B - B_1)}{4\pi} = 0 \tag{2.3}$$

$$M_- u \frac{d}{dx}\left(V_y^+ - V_y^-\right) = \frac{e}{c}\left(u_1 B_1 - uB\right) - \nu_{ei} M_-\left(V_y^+ - V_y^-\right) \tag{2.4}$$

$$-\frac{dB}{dx} = \frac{4\pi N e}{c}\left(V_y^+ - V_y^-\right) \tag{2.5}$$

U is the flow velocity in the x-direction and $\underline{V}^{\pm}$ the ion and electron velocity. Subscript 1(2) denotes upstream (downstream) values. ν_{ei} is the electron-ion collision frequency, arising either from weak Coulomb collisions or from anomalous turbulent dissipation, and introduces irreversibility into the above equations. In deriving 2.1-2.5, quasi-neutrality was assumed thus restricting consideration to plasmas where $\Omega_-/\omega_{p_-} << 1$ or $C_-/C << 1$. Quasi-neutrality will be assumed throughout this work; thus electrostatic shocks, which require Debye scale lengths, will not be considered. The above equations differ from those of hydromagnetics only in the presence of the electron inertia term on the LHS of 2.4.

Differentiation of 2.5 with respect to x, and use of the differential form of 2.1, 2.5 yields

$$\frac{d}{dx}\left(v_y^+ - v_y^-\right) = \frac{c}{4\pi N e}\frac{d^2 B}{dx^2} + \frac{\left(v_y^+ - v_y^-\right)}{u}\frac{du}{dx} \tag{2.6}$$

Eliminating P from 2.3 by substitution of 2.2, the following equation for the flow velocity as a function of the magnetic field strength is obtained

$$u^2 - \frac{2\gamma}{\gamma+1}u\left[u_1 + \frac{P_1}{\rho_1 u_1} - \frac{B^2 - B_1^2}{8\pi\rho_1 u_1}\right] + \frac{\gamma-1}{\gamma+1}u_1^2$$
$$+\frac{\gamma}{\gamma+1}\frac{P_1}{\rho_1} - \frac{2(\gamma-1)}{\gamma+1}\frac{B_1(B-B_1)}{4\pi\rho_1} = 0 \tag{2.7}$$

Solution of this quadratic gives U = U(B). Differentiating both 2.2 and 2.3, and eliminating dP/dx between them gives

$$\frac{dU}{dx} = \frac{-U}{U^2 - C_s^2} \frac{B_1}{4\pi\rho_1} \frac{dB}{dx} \tag{2.8}$$

where C_s^2 depends on U(B). Finally substituting 2.8 into 2.6 and the result into 2.5, yields a differential equation for the magnetic field

$$\frac{c^2}{\omega_{p_-}^2} \frac{U^2}{U_1} \frac{d^2B}{dx^2} + \frac{c^2}{\omega_{p_-}^2} \frac{U\nu_{ei}}{U_1} \frac{dB}{dx} - \frac{c^2}{\omega_{p_-}^2} \frac{U^2}{U^2 - C_s^2} \frac{B_1}{4\pi\rho_1 U_1} \left(\frac{dB}{dx}\right)^2$$

$$= UB - U_1B_1$$

2.9)

where $\omega_{p_-}^2 = 4\pi N_1 e^2/M_-$ and U and P are to be eliminated by 2.2 and 2.7.

First note that on the RHS of 2.9 only hydromagnetic terms, i.e., no dispersion or dissipation derivatives, occur. With 2.7 the solution $UB - U_1B_1 = 0$ yields the values of U and B at either the upstream or downstream stationary points of the differential equation. (In general more than one solution exists at the downstream point; however, only one solution is consistent with the constancy of the tangential component of the electric field across the shock.) Therefore the RHS is a form of the Rankine-Hugoniot relations.

The differential operator on the LHS of 2.9 is extremely complicated since it is not only non-linear, but the coefficients of the derivatives are functions of x. Also the RHS is a non-linear function of B. Clearly an analytic solution, even in this simplest of all possible cases, is impossible, and numerical methods must be employed. However, it is not always necessary to solve the differential equation in order to obtain important physical information about the solutions. Simplifying assumptions and linearization techniques permit continuation of the fluid approach to collisionless shock waves.

One such simplifying assumption is the weak shock approxition. Here the shock strength, $M_F^2 - 1$, where $M_F = U_1/C_{F_1}$ is the Mach number, is assumed small compared to unity. Returning to 2.9, it is anticipated that the derivative terms are proportional to $M_F^2 - 1 << 1$. Hence a legitimate approximation is to "linearize" or evaluate the coefficients of the derivative terms about the upstream or downstream stationary points. Secondly the non-linear term in 2.9 is of order (dB/dx) dB/dx $\approx$ 0 since dB/dx vanishes at a stationary point. This assumption, however, requires some care since the coefficient of $(dB/dx)^2$ becomes unbounded at the sonic point $U = C_s$. Hence the neglect of the non-linear term in the weak shock approximation also requires that U not become comparable to C_s. If this is satisfied, 2.9 becomes

$$\frac{c^2}{\omega_{p_-}^2}\frac{d^2B}{dx^2} + \frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u_1}\frac{dB}{dx} = \frac{uB}{u_1} - B_1 \equiv \frac{\partial\phi(B)}{\partial B} \qquad (2.10)$$

The differential equation is now in the form obtained by Sagdeev (1966) and is readily interpretable. The RHS can be written as the derivative with respect to B of an effective potential, $\varphi(B)$, where $\varphi(B)$ is in principle known from an integration of the Rankine-Hugoniot relations with the normalizing condition $\varphi(B_1) = 0$. In practice, however, $[\partial \varphi(B)]/\partial B$ is a complicated non-linear function of B, and integration requires numerical techniques. The LHS is in the form of the harmonic oscillator equation, including resistive dissipation, if x is interpreted as time and B as position. Therefore 2.10 is interpretable in terms of an analogic particle B moving in a non-linear potential $\varphi(B)$ (Figure 1).

Being a non-linear differential equation, 2.10 is still too difficult to solve analytically. However, the basic features of the solution are obtainable by studying the behavior in the vicinity of the Rankine-Hugoniot stationary points by linearization techniques. It is convenient to linearize the set of equations 2.1-2.5 directly by taking, for example, $U = U_1 + \delta U$, $\delta U/U << 1$. Neglecting the non-linear terms, 2.1-2.5 reduce to

$$\frac{c^2}{\omega_{p-}^2}\frac{d^2\delta B}{dx^2} + \frac{c^2}{\omega_{p-}^2}\frac{\nu_{ei}}{u}\frac{d\delta B}{dx} = \frac{u^2 - c_F^2}{u^2 - c_s^2}\,\delta B$$

$$= -\frac{\partial^2 \phi(B)}{\partial B^2}\,\delta B \qquad (2.11)$$

where the subscripts 1 and 2 have been dropped.

The RHS of 2.11 can be interpreted as the second derivative of the potential. If $\partial^2\varphi/\partial B^2 < 0$ as it is for the upstream point, since $U_1 > C_{F_1}$ by the shock evolutionary conditions (Kantrowitz and Petschek, 1966), the analogic particle is at an unstable equilibrium in the potential well, and the perturbation grows. For the downstream point $C_{F_2} > U_2 > C_{s_2}$, $\partial^2\varphi/\partial B^2$ is positive, and the analogic particle undergoes damped oscillations about the downstream stationary point. If $\nu_{ei} = 0$, the particle would make one traversal of the potential well and return to its original position. This solution is the dissipationless solitary wave. With $\nu_{ei} \neq 0$ the particle loses energy as it moves in the potential well and eventually must come to rest at the potential minimum. Since at the final state U and B satisfy the downstream Rankine-Hugoniot relations, the shock transition has been accomplished.

Performing the ansatz $\delta B \sim \exp(\lambda x)$, the solution of 2.11 is

$$\lambda = -\frac{c^2}{\omega_p^2}\frac{\nu_{ei}}{2u} \pm \frac{1}{\frac{c}{\omega_p}}\left[\frac{u^2 - C_F^2}{u^2 - C_s^2}\right]^{1/2} \tag{2.12}$$

where terms of order ν_{ei}^2 have been neglected. The boundary condition on the solution is that the perturbation not grow at $x \to \pm\infty$. For the upstream stationary point, $U_1 > C_{F_1}$, the square root is real and the appropriate solution is the positive sign. The magnetic field undergoes an exponential rise with a scale length given by

$$L_s \sim \frac{M_{F_1}\frac{c}{\omega_p}}{(M_{F_1}^2 - 1)^{1/2}} \tag{2.13}$$

if $\beta_- << 1$. Therefore a typical scale length or shock thickness is of the order of C/ω_{p_-}. About the downstream point if $C_{s_2} < U_2 < C_{F_2}$, the radical is imaginery so that the magnetic field undergoes damped oscillations about its downstream Rankine-Hugoniot value with wavelength

$$L_s \sim \frac{M_{F_2} \frac{c}{\omega_{p_-}}}{|M_{F_2}^2 - 1|^{1/2}} \tag{2.14}$$

and damping length

$$L_D \sim \frac{2u}{\nu_{ei}} \frac{1}{c^2/\omega_{p_-}^2} \tag{2.15}$$

The laminar structure, sketched in Figure 2, is often called a wave train. For C/ω_{p_-} dispersion, the wave train trails the leading jump in the magnetic field.

If $U_2 < C_{s_2}$, 2.12 yields an unbounded solution which does not satisfy the boundary conditions. In this situation, resistivity alone is no longer sufficient to provide the shock dissipation, and a stronger dissipation mechanism, such as viscosity, is required (see Marshall, 1955; Kantrowitz and Petschek, 1966; Coroniti, 1969). Further discussion of this point is given in sections 5.0 and 6.0.

The detailed properties of the above solution depended on the sign of the potential term as compared with that of the dis-persive term in 2.11. In many problems such as the low β

oblique fast shock (Sagdeev, 1966; Cavaliere and Englemann, 1967), the relative sign between these two terms is opposite to that of 2.11. Hence the upstream solution is oscillatory so that the wave train leads the jump in the magnetic field. This type of solution is sketched in Figure 3.

Reconsider the above example by returning to 2.9, the full non-linear differential equation. If 2.9 had been linearized about the stationary points, 2.11 could have been obtained directly, as long as $U \neq C_s$, without utilization of the weak shock approximation since the non-linear term does not contribute to the linearized solution in lowest order. Also the coefficients of the derivatives would then be evaluated at the respective stationary points. Therefore the perturbation solution obtained in the weak shock approximation actually applies to strong shocks provided $U \neq C_s$.

Many of the properties of the C/ω_{p_-} wave train could have been derived from the dispersion characteristics of the linear perpendicular fast wave. The low β dispersion relation is (Formisano and Kennel, 1969)

$$\frac{\omega^2}{k^2} = \frac{C_A^2}{1 + \frac{k^2 C^2}{\omega_{p_-}^2}} + C_s^2 \tag{2.16}$$

When $kC/\omega_{p_-} >> 1$, finite electron inertia decouples the fast wave from the magnetic field and slows the phase velocity to the sound speed. To form a wave train a wave on the same branch of the dispersion relation as the shock wave must stand in the flow. Since for $kC/\omega_{p_-} \sim 1$, the fast wave speed is reduced below the hydromagnetic speed, the wave stands in the downstream flow and hence is trailing. The approximate oscillation wavelength can be obtained from 2.16 by setting $\omega/k = U_2$ and solving for k. In low β, the wavelength is $\sim C/\omega_{p_-} \; [M_{F_2}/|1-M^2_{F_2}|^{\frac{1}{2}}]$, in agreement with 2.14. The dispersion relation is sketched in Figure 4. Note that at the intersection of the dispersion curve with the downstream flow velocity, the group velocity, $\partial\omega/\partial k$, is negative so that energy is directed away from the shock front as required from the steepening arguments of section 2.2.

2.4 Summary

Although somewhat lengthy, the physical principles reviewed in this section are fundamental in the fluid approach to collisionless shocks. In addition the methods employed in discussing the C/ω_{p_-} wave train will be used repeatedly throughout this work to discuss the structure of both fast and slow shocks in finite β plasmas. Two points should be stressed again:

1) Although the differential operator governing the shock structure is non-linear with variable coefficients, the linearized solutions about the stationary points obtained from a simplified differential operator in the weak shock approximation are valid for strong shocks.

2) The character of the wave train solutions is obtainable from the dispersion properties of linear waves.

Therefore the next section investigates linear waves from the finite Larmor radius — Chew-Goldberger-Low (FLR-CGL) equations.

3.0 Linear Waves from the FLR-CGL Fluid Equations

3.1 Introduction

The simplicity of the C/ω_{p_-} shock was achieved only by severely restricting the available parameter space to very low β and motion perpendicular to the magnetic field. Since some laboratory plasmas and most space plasmas occupy a much larger volume of parameter space, the above restrictions must be relaxed. The appropriate equations to continue the fluid approach to collisionless shocks are the FLR-CGL equations which are presented in section 3.2. The discussion of section 2.0, having demonstrated the conceptual benefits of knowing the dispersive properties of linear waves in selecting the scale lengths and direction of wave trains, motivates the discussion of linear wave propagation from the FLR-CGL equations in section 3.3.

3.2 The FLR-CGL Equations

Various methods have been employed to derive fluid and kinetic equations which describe the consequences of a finite particle gyro-radius; all involve expansion in the small parameter $\epsilon \equiv R_{\pm}/L_{\perp}$ where $L_{\perp}$ is a basic scale length of the system perpendicular to the magnetic field. The FLR corrections to the CGL equations have been derived by Frieman et al. (1966) by systematically expanding the Vlasov equation to first order in ϵ. They obtained a set of fluid equations to describe the motion perpendicular to the magnetic field while retaining the kinetic equation for the dynamics along the magnetic field. MacMahon (1965), expanding the exact moments of the Vlasov equation to first order in ϵ, obtained a set of fluid equations for both parallel and perpendicular motion to the magnetic field. In considering a fluid approach to collisionless shocks, MacMahon's equations are more tractable than those of Frieman et al.; however, they are more restricted in applicability since phenomena which depend on details of the particle distribution functions, such as the parallel Landau resonance and parallel heat flow, cannot be treated.

The FLR-CGL equations can be written in the following form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0 \tag{3.1}$$

$$\rho \frac{d\underline{V}}{dt} + \nabla \cdot \underset{\approx}{P}^{(1)} = \frac{\underline{J} \times \underline{B}}{c} = - \frac{\underline{B} \times (\nabla \times \underline{B})}{4\pi} \tag{3.2}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\rho V^2}{2} + P_\perp^{(1)} + \frac{P_\parallel^{(1)}}{2} + \frac{B^2}{8\pi} \right) &+ \nabla \cdot \Big[\rho \underline{V} \left(\frac{V^2}{2} + \frac{P_\perp^{(1)}}{\rho} + \frac{P_\parallel^{(1)}}{2\rho} \right) \\ &+ \underline{V} \cdot \underset{\approx}{P}^{(1)} + \frac{\underline{B} \times (\underline{V} \times \underline{B})}{4\pi} + \underline{q}_\parallel^{\parallel(0)} + \underline{q}_\parallel^{\perp(1)} \\ &+ \underline{q}_\perp^{\parallel(1)} + \underline{q}_\perp^{\perp(1)} \Big] = 0 \end{aligned} \tag{3.3}$$

$$\nabla \times \underline{E} = - \frac{1}{c} \frac{\partial \underline{B}}{\partial t} \tag{3.4}$$

$$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J} \tag{3.5}$$

$$e\left[\underline{E} + \frac{\underline{V}\times\underline{B}}{c}\right] = \frac{M_-}{Ne}\frac{d\underline{J}}{dt} + M_+\frac{d\underline{V}}{dt} + \frac{1}{N}\nabla\cdot\underset{\approx}{P}^{(1)+} \tag{3.6}$$

$$\frac{d\underset{\approx}{P}^{\pm}}{dt} + \underset{\approx}{P}^{\pm}\,\nabla\cdot\underline{V}^{\pm} + \nabla\cdot\underset{\approx}{Q}^{\pm} + \underset{\approx}{P}^{\pm}\cdot\nabla\underline{V}^{\pm} + (\underset{\approx}{P}^{\pm}\cdot\nabla\underline{V}^{\pm})^{T}$$

$$= \Omega_{\pm}\left[\underset{\approx}{P}^{\pm}\times\hat{e}_3 - \hat{e}_3\times\underset{\approx}{P}^{\pm}\right] \tag{3.7}$$

Equations 3.1, 3.2, and 3.3 are the equations of continuity, momentum and energy, respectively; 3.4 and 3.5 are the two self-consistent Maxwell equations; 3.6 is the generalized collisionless Ohm's law. Equation 3.6 has already been linearized by replacing in the electron inertia term $d/dt(\underline{V}^{+} - \underline{V}^{-})$ by $1/N_e$ $d\underline{J}/dt$. An alternative form for the Ohm's law, which is useful in the linear wave calculation, is obtained by taking the curl of 3.6 and using 3.4 and 3.5 to find

$$\left(\frac{\partial}{\partial t} + \frac{c^2}{\omega_{p-}^2}\frac{d}{dt}\nabla^2\right)\underline{B} = \nabla\times(\underline{V}\times\underline{B})$$

$$-\frac{M_+c}{e}\frac{d}{dt}(\nabla\times\underline{V}) - \frac{c}{e}\nabla\times\left(\frac{\underset{\approx}{P}^{(1)+}}{N}\right) \tag{3.8}$$

Note that quasi-neutrality has been assumed in 3.6 and 3.8.

Equation 3.7 advances the pressure tensor for ions and electrons, $\underset{\approx}{P}^{\pm} \equiv M_{\pm}\int(\underline{V} - \underline{V}^{\pm})(\underline{V} - \underline{V}^{\pm})\, f^{\pm}\, d^3V$, in time. $\underset{\approx}{Q}^{\pm}$ is the heat flow tensor. Since the RHS of 4.7 is multiplied by $\Omega_{\pm}$, it is $\mathcal{O}(1/\epsilon)$ larger than the LHS; the lowest order pressure tensor is found by setting the RHS equal to zero.

Defining an orthogonal set of unit vectors $\hat{e}_i$ such that $\hat{e}_3 = \underline{B}/|\underline{B}|$, $\hat{e}_1$ in the direction of the principal normal, and $\hat{e}_2 = \hat{e}_3 \times \hat{e}_1$, the lowest order solution of $\underset{\approx}{P}^{\pm}$ is

$$\underset{\approx}{P}^{(0)\pm} = P_{\parallel}^{(0)\pm}\hat{e}_3\hat{e}_3 + P_{\perp}^{(0)\pm}(\underset{\approx}{I} - \hat{e}_3\hat{e}_3) \tag{3.9}$$

$_\parallel$ and $_\perp$ denote projections parallel and perpendicular to the magnetic field; $\underset{\approx}{I}$ is the unit tensor; the zero superscript denotes $\mathcal{O}(\epsilon^{o})$. Equation 3.9 is the familiar CGL pressure tensor.

The FLR corrections to pressure tensor were obtained by MacMahon (1965) by substituting 3.9 into the RHS of 3.7 and using appropriate tensor annihilators to determine the first order corrections (denoted by superscript (1)). The results are

$$P_{\delta}^{(1)\pm} = \frac{-1}{2\Omega_{\pm}}\underset{\approx}{I}_{\gamma} : \left[P_{\perp}^{\pm}\nabla\underline{V}^{\pm} + q_{\parallel}^{\perp\pm}\nabla\hat{e}_3\right]^{(0)} \tag{3.10}$$

$$P_{12}^{(1)\pm} = \frac{1}{2\Omega_{\pm}}\underset{\approx}{I}_{\delta} : \left[P_{\perp}^{\pm}\nabla\underline{V}^{\pm} + q_{\parallel}^{\perp\pm}\nabla\hat{e}_3\right]^{(0)} \tag{3.11}$$

$$P_{31}^{(1)\pm}\hat{e}_1 + P_{32}^{(1)\pm}\hat{e}_2 = \frac{\hat{e}_3\times}{\Omega_{\pm}}\Big[(2P_{\parallel}^{\pm} - P_{\perp}^{\pm})\hat{e}_3\cdot\nabla\underline{V}^{\pm}$$

$$+ P_{\perp}^{\pm}(\nabla\underline{V}^{\pm})\cdot\hat{e}_3 + 2(q_{\parallel}^{\parallel\pm} - q_{\parallel}^{\perp\pm})\hat{e}_3\cdot\nabla\hat{e}_3 \tag{3.12}$$

$$+ \nabla_{\perp}q_{\parallel}^{\perp\pm}\Big]^{(0)}$$

where $\underset{\approx}{I}_\gamma = \hat{e}_1\hat{e}_2 + \hat{e}_2\hat{e}_1$ and $\underset{\approx}{I}_\delta = \hat{e}_1\hat{e}_1 - \hat{e}_2\hat{e}_2$; $\nabla_\perp = \underset{\approx}{I}_\perp \cdot \nabla$, $\underset{\approx}{I}_\perp = \hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2$. Note that since $\Omega_+ = (M_-/M_+)|\Omega_-|$, the FLR corrections in 3.10-3.12 are usually only needed for the ions unless $\varepsilon < M_-/M_+$.

In the fluid approach it is often convenient to define the pressure tensor with respect to the fluid velocity $\underline{V} = M_+\underline{V}^+ + M_-\underline{V}^-/M_+ + M_-$, $\underset{\approx}{P}^\pm = M_\pm \int (\underline{V}' - \underline{V})(\underline{V}' - \underline{V})\, f^\pm(\underline{V}')\, d^3V'$. Summing the equivalent of 3.7 for electrons and ions, the following equation for $\underset{\approx}{P}^{(1)} = \underset{\approx}{P}^{(1)+} + \underset{\approx}{P}^{(1)-}$ is obtained (Goldberg, 1969)

$$\frac{d\underset{\approx}{P}^{(1)}}{dt} + \underset{\approx}{P}^{(1)}\nabla\cdot\underline{v} + \underset{\approx}{P}^{(1)}\cdot\nabla\underline{v} + (\underset{\approx}{P}^{(1)}\cdot\nabla\underline{v})^T + \nabla\cdot\underset{\approx}{Q}^{(1)}$$

$$- \underline{J}\left[\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right] - \left[\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right]\underline{J} = 0 \qquad (3.13)$$

where $\underset{\approx}{Q}^{(1)} = \underset{\approx}{Q}^{(1)+} + \underset{\approx}{Q}^{(1)-}$. $P_\parallel^{(1)}$ and $P_\perp^{(1)}$ are still needed since they are not determined by the annihilation procedure. Double dotting 3.13 with $\hat{e}_3\hat{e}_3$ and $\underset{\approx}{I}_\perp$ gives

$$\frac{1}{2}\frac{dP_\parallel^{(1)}}{dt} + P_\parallel^{(1)}\left(\frac{3}{2}\nabla\cdot\underline{v} - \nabla_\perp\cdot\underline{v}\right) + \nabla\cdot\underline{q}_\perp^{\parallel(1)}$$

$$+ P_\parallel^{(0)}\alpha_1 - \alpha_2 + \frac{1}{Ne}\underline{J}_\parallel\cdot\left[\nabla P_\parallel^- + \frac{\nabla B}{B}(P_\parallel^- - P_\perp^-)\right]^{(0)} \qquad (3.14)$$

$$= T - T_\perp$$

$$\frac{dP_\perp^{(1)}}{dt} + P_\perp^{(1)}(\nabla\cdot\underline{v} + \nabla_\perp\cdot\underline{v}) + \nabla\cdot\underline{q}_\perp^{\perp(1)} - P_\perp^{(0)}\alpha_1$$

$$+ \alpha_2 + \frac{1}{Ne}\underline{J}_\perp\cdot\left[\nabla\left(P_\perp^- + \frac{B^2}{8\pi}\right) + \hat{e}_3\cdot\nabla\hat{e}_3\left(P_\parallel^- - P_\perp^- - \frac{B^2}{4\pi}\right)\right]^{(0)}$$

$$= \Gamma_\perp \tag{3.15}$$

where

$$\alpha_1 = \sum_{\pm}\alpha_1^{\pm}\ ;\quad \alpha_1^{\pm} = \frac{\hat{e}_3}{2\Omega_\pm}\cdot\left[(\hat{e}_3\cdot\nabla\underline{v}^{\pm})\times(\nabla\underline{v}^{\pm})\cdot\hat{e}_3\right]$$

$$(3.16)$$

$$\alpha_2 = \sum_{+-}\alpha_2^{\pm}$$

$$\alpha_2^{\pm} = \frac{\hat{e}_3}{\Omega_\pm}\cdot\left[\left(\nabla R_1^{\pm} - \frac{P_\parallel^{(0)\pm}}{NM_\pm}\nabla P_\perp^{(0)\pm}\right)\times(\hat{e}_3\times\nabla\hat{e}_3)\right] \tag{3.17}$$

Γ and $\Gamma_\perp$ are proportional to the zero order heat flow, $q_\parallel^{\perp(0)}$ and $q_\parallel^{\parallel(0)}$, where

$$q_\parallel^{\perp(0)} = \sum_{\pm} q_\parallel^{\perp(0)\pm}\ ;\quad q_\parallel^{\perp(0)\pm} = M_\pm\int(v_\parallel - V_\parallel^{\pm})(v_\perp - V_\perp^{\pm})f^{\pm}d^3v$$

$$q_\parallel^{\parallel(0)} = \sum_{+-} q_\parallel^{\parallel(0)\pm}\ ;\quad q_\parallel^{\parallel(0)\pm} = M_\pm\int(v_\parallel - V_\parallel^{\pm})^3 f^{\pm}d^3v$$

and will not be written here (see MacMahon, 1965).

$q_{\parallel}^{\parallel(0)}$ is the flow of parallel heat along the magnetic field; $q_{\parallel}^{\perp(0)}$ is the analagous flow of perpendicular heat.

The first order FLR heat flows perpendicular to the magnetic field, $\underline{q}_{\perp}^{\perp(1)}$ and $\underline{q}_{\perp}^{\parallel(1)}$, are obtained by a similar annihilation procedure as in the pressure equation of the third moment equation for $\underset{\approx}{Q}^{\pm}$. The components needed are

$$\underline{q}_{\perp}^{\parallel(1)\pm} = \frac{\hat{e}_3 \times}{2\Omega_{\pm}}\Big[\nabla R_1^{\pm} - \frac{P_{\parallel}^{(0)\pm}}{NM_{\pm}}\nabla\cdot\underset{\approx}{P}^{(1)\pm} + (R_3^{\pm} - 3R_1^{\pm})\,\hat{e}_3\cdot\nabla\hat{e}_3$$

$$+2\left(q_{\parallel}^{\parallel(0)\pm} - q_{\parallel}^{\perp(0)\pm}\right)\hat{e}_3\cdot\nabla\underline{V}^{\pm} + 2\,q_{\parallel}^{\perp(0)\pm}(\nabla\underline{V}^{\pm})\cdot\hat{e}_3\Big] \tag{3.18}$$

$$\underline{q}_{\perp}^{\perp(1)\pm} = \frac{\hat{e}_3 \times}{2\Omega_{\pm}}\Big[\nabla R_2^{\pm} - \frac{4\,P_{\perp}^{(0)\pm}}{NM_{\pm}}\nabla\cdot\underset{\approx}{P}^{(1)\pm}$$

$$+4\left(R_1^{\pm} - \frac{R_3^{\pm}}{4}\right)\hat{e}_3\cdot\nabla\hat{e}_3 + 4\,q_{\parallel}^{\perp(0)\pm}\hat{e}_3\cdot\nabla\underline{V}^{\pm}\Big] \tag{3.19}$$

$R_i^{\pm}$ are the zero order 4th order moments defined as

$$R_1^{\pm} = M_{\pm}\int (v_{\parallel} - V_{\parallel}^{\pm})^2 (v_{\perp} - V_{\perp}^{\pm})^2 f^{\pm}\, d^3v \tag{3.20}$$

$$R_2^{\pm} = M_{\pm} \int (v_{\perp} - v_{\perp}^{\pm})^4 f^{\pm} d^3v \tag{3.21}$$

$$R_1^{\pm} = M_{\pm} \int (v_{\parallel} - v_{\parallel}^{\pm})^4 f^{\pm} d^3v \tag{3.22}$$

Both $\underset{\approx}{Q}^{\pm}$ and $\underset{\approx}{R}^{\pm}$ are advanced in time by equations that can be found in MacMahon (1965).

Several important features of the FLR-CGL equations should be noted:

1. The moment equations are not closed since lower order moments are determined in terms of higher order ones. Therefore, some truncation approximation is anticipated.
2. The moment equations do not determine the zero order heat flows $q_{\parallel}^{\parallel(0)}$ and $q_{\parallel}^{\perp(0)}$. Detailed knowledge of the distribution function and the collision operator is needed to specify these.
3. Since these equations are derived by expanding in powers of $\epsilon = R_{\pm}/L_{\perp}$, they can only describe phenomena on a space scale larger than the gyro-radius. For example, if $k_{\perp}$ is a typical wave number in the wave spectrum, the FLR-CGL equations are valid only for $k_{\perp} R_{+} \leqslant 1$.

4. The equations for $P_{\parallel}^{(1)}$ and $P_{\perp}^{(1)}$, 3.14 and 3.15 are coupled by α_1, and α_2. α_1 arises from two terms, $\underset{\approx}{P} : \hat{e}, d\hat{e}/dt$ and $\underset{\approx}{P} : \nabla \underline{V}$. α_1 is often called the collisionless or gyro-viscosity since, being quadratic in $\underline{V}$, it resembles the ordinary collisional viscosity. Unlike collisional viscosity, however, gyro-viscosity couples the three degrees of freedom but does not produce any dissipation. In the pressure equations the gyro-viscosity accomplishes a transfer between fluid and thermal energy. α_2 arises from the term $\hat{e}_3 \cdot \underset{\approx}{Q} : \nabla \hat{e}_3$ and couples the flow of heat between the parallel and perpendicular directions.

5. In the FLR heat flows, $\underline{q}_{\perp}^{\perp(1)}$ and $\underline{q}_{\perp}^{\parallel(1)}$, the term $\nabla \cdot \underset{\approx}{P}^{(1)}$ occurs rather than, as might be expected, $\nabla \cdot \underset{\approx}{P}^{(0)}$. This term is in the same order as those retained in the momentum equation since, by coupling the degrees of freedom, the lowest order FLR correction is the product of two FLR terms, and hence is of the same order as $1/\Omega_{\pm} \nabla \cdot \underset{\approx}{P}^{(1)}$. MacMahon (1968) demonstrated that for the perpendicular fast wave the above term provides the dominant dispersion.

The FLR-CGL equations are obviously too complicated to analyze in complete generality and throughout this work various simplifying assumptions are necessary to render the calculations tractable. However these assumptions often raise significant questions which are unresolvable within the context of the FLR-CGL equations and require knowledge of the distribution function, the turbulent wave spectrum, and the precise dymanics of the fluid.

3.3 Linear Wave Theory

To begin the investigation of the FLR-CGL equations and to establish an intuitive foundation for later work, the linear response of the above equations is determined in the form of a dispersion relation. As written the equations contain 4 scale lengths which produce dispersive propagation, $C/\omega_{p_\pm}$ and $R_\pm$. Since $R_- \sim \sqrt{M_-/M_+}\ R_+$, R_- effects can be neglected in the long wavelength limit, $k_\perp R_- << 1$, and only ion gyro-radius corrections retained. For $\beta_+ < 1$, C/ω_{p_+} exceeds R_+ so that ion inertia is the dominate dispersion effect except for near perpendicular propagation; likewise for $\beta_+ > 1$, R_+ dispersion should dominate.

Consider the equilibrium configuration $\underline{B}_o = B_o\hat{e}_z$, $\hat{e}_z = \hat{e}_3$, $\hat{e}_1 = \hat{e}_x$, $\hat{e}_2 = \hat{e}_y$, $\underline{V}_o = \underline{E}_o = q_{\parallel}^{\parallel(0)} = q_{\parallel}^{\perp(0)} = 0$. The calculation proceeds by expanding the plasma variables about their equilibrium value, e.g., $\rho = \rho_o + \delta\rho$, $\delta\rho/\rho_o << 1$, and neglecting non-linear terms in the perturbation. In calculating the dispersion relation an assumption which greatly simplifies the algebra and permits a clearer presentation of the dispersion properties, which are of primary interest here, is to take the pressures isotropic, i.e., $P_{\parallel}^{(0)\pm} = P_{\perp}^{(0)\pm}$, and $P_{\parallel}^{(1)\pm} = P_{\perp}^{(1)\pm}$. Alternatively it could be assumed that the pressure anisotropy is small, $P_\parallel - P_\perp/B^2 \sim k_\perp R_+ \approx \epsilon$, so that it need only be retained in the lowest order, or CGL terms.

Here, however, the effect of anisotropic pressure in producing the firehose or mirror instabilities is well understood.

A possible justification for the above assumption in collisionless plasmas is to note that there are many higher frequency modes such as ion and electron cyclotron waves, electrostatic loss cone instabilities, etc., which are destabilized by anisotropic velocity space distributions. The nonlinear effects of such instabilities is to reduce the available free energy by isotropizing the distribution function (Kennel and Englemann, 1966). Hence even collisionless plasmas often tend toward isotropy on a time scale more rapid than that of interest in the fluid equations.

Fourier analyzing the perturbed quantities in space and time as $\exp[i(k_\perp x + k_\parallel z) - i\omega t]$ the following dispersion relation is derived in Appendix A.1.

$$\left[\frac{\omega^2}{k^2} - c_T^2\right]\left[\frac{\omega^2}{k^2} - c_F^2\right]\left[\frac{\omega^2}{k^2} - c_{SL}^2\right] - \frac{k^2 c_A^2}{\Omega_+^2} c_T^2 \left\{\left[\frac{\omega^2}{k^2} - \frac{2P_0^{(1)}}{\rho_0}\sin^2\theta\right]\cdot\left[\frac{\omega^2}{k^2} - \frac{3P_0^{(1)}}{\rho_0}\cos^2\theta\right] - \sin^2\theta\cos^2\theta\left(\frac{P_0^{(1)}}{\rho_0}\right)^2\right\}$$
$$+ \frac{\omega^4}{k^4} M - \frac{\omega^2}{K^2} N + L = 0 \tag{3.23}$$

where

$$\left.\begin{matrix} c_F^2 \\ c_{SL}^2 \end{matrix}\right\} = \frac{c_A^2 + \frac{2P_0^{(1)}}{\rho_0}\sin^2\theta + \frac{3P_0^{(1)}}{\rho_0}\cos^2\theta}{2} \pm \left[\left(\frac{c_A^2 + \frac{2P_0^{(1)}}{\rho_0}\sin^2\theta - \frac{3P_0^{(1)}}{\rho_0}\cos^2\theta}{2}\right)^2 + \left(\frac{P_0^{(1)}}{\rho_0}\right)^2 \sin^2\theta\cos^2\theta\right]^{1/2} \tag{3.24}$$

and $P_o^{(1)} = P_{\perp o}^{(1)} = P_{\parallel o}^{(1)}$. Note that even though isotropy was assumed, use of 3.14 and 3.15 retains the collisionless CGL ratios of specific heats. The term multiplied by $k^2C_A^2/\Omega_+^2$ contains the ion inertial corrections to the hydromagnetic waves (Stringer, 1963; Formisano and Kennel, 1969); N, M, and L, defined by A1.22, A1.23 and A1.24 are the lowest order FLR corrections. Note that the electron inertial correction, $k^2C^2/\omega_{p_-}^2$, is here contained in the definition of the Alfvén speed, $C_A^2 = (B_o^2/4\pi\rho_o)[1/(1 + k^2C^2/\omega_{p_-}^2)]$.

3.3.1 Oblique Fast Wave

When $\cos\theta << 1$, $C_F >> C_I$, C_{SL}; dropping C_I and C_{SL} with respect to ω/k, 3.23 for the oblique fast wave becomes

$$\frac{\omega^2}{k^2} = C_F^2\left[1 + \frac{r\,C_I^2}{C_F^2}\left(1 - \frac{2P_o^{(1)}}{\rho_o C_F^2}\right) - \frac{M}{C_F^2}\left(1 - \frac{N}{MC_F^2}\right)\right] \tag{3.24}$$

where $M \sim (3/4)(P_o^{(0)+}/\rho_o)\, k_\perp^2 R_+^2$, $N/MC_F^2 \sim \cos^2\theta$, and $r = k^2C_A^2/\Omega_+^2$. First take $\cos\theta = 0$; the ion inertial term, proportional to r, drops out. If $\beta_+ < M_-/M_+$, C/ω_{p_-} exceeds R_+ and the leading dispersion comes from the electron inertial term contained in $C_F^2 = C_A^2 + (2P_o^{(1)}/\rho_o)$. As noted in section 2.3, electron inertia slows down the fast wave. C/ω_{p_-} continues to dominate over the range of angles $\theta < \pi/2 - \sqrt{M_-/M_+}$ if $\beta_+ < M_-/M_+$, a very small region of parameter space. If $\beta_+ > M_-/M_+$, and $\cos^2\theta < \beta_+$, R_+ dispersion dominates and also slows down the fast wave. If $\cos^2\theta > \beta_+$, $\beta_+ < 1$, and $\theta > \pi/2 - \sqrt{M_-/M_+}$, the ion inertial term exceeds both C/ω_{p_-} and R_+, and increases the phase velocity of the fast wave. The dispersive effects of ion and electron inertia have also been found by Stringer (1963), Formisano and Kennel (1969), and many others.

If $\beta_+ > 1$, ion FLR dispersion dominates, and decreases the fast wave phase velocity, in agreement with MacMahon (1968) and Fredricks and Kennel (1968). The transition between ion inertia and ion FLR dispersion occurs approximately for $\cot\theta \approx \sqrt{3}/4\ \beta_+/[1 - (\langle 2P_o^{(1)}/\rho_o\rangle/C_F^2)]^{\frac{1}{2}}$ $\approx \sqrt{3}/4\ \beta_+ C_F/C_A$ if $\beta_+ > M_-/M_+$. When this transition occurs, it is possible that ion inertia and ion FLR dispersion cancel. Care must be taken here, however, since in obtaining 3.24 only the lowest order FLR terms, $\mathcal{O}(k_\perp^2 R_+^2 < 1)$, were retained. On the other hand, even if all the FLR corrections were retained to arbitrary order in $k_\perp^2 R_+^2$, the FLR term in 3.23 would only be slightly modified from its present form; a small change in either θ or $\beta_+ \lesssim 1$ would still produce a cancellation. If this occurs, the only dispersion term remaining in 3.24 is electron inertia; of course if $\beta_- \sim 1$, electron FLR corrections are comparable to electron inertia and must also be included.

For linear wave propagation this cancellation effect is not significant since it will not be valid up to wavelengths $\sim C/\omega_{p_-}$. When $kR_+ >> 1$, the fast wave dispersion changes sign, and the wave speeds up to become the high frequency whistler if $\beta_+ < \sqrt{M_+/M_-}$ (see section 7.0; Formisano and Kennel, 1969). However in determining the wave train solution the important wave is the one that stands at a given point in the flow. If the wave train cannot be formed by C/ω_{p_+} or R_+ because of a cancellation, the only possibility that remains is a C/ω_{p_-} - R_- wave train. Hence it is possible, even for $\beta_+ \sim 1$ and $\theta >> \pi/2 - \sqrt{M_-/M_+}$, that the dispersion properties of the oblique fast wave train be dominated by electron inertia and possibly electron FLR. This effect is further considered at the end of this section and in sections 7.0 and 8.0 where the C/ω_{p_-} - R_- wave train is calculated, and is compared with some recent observations of the structure of the earth's bow shock wave.

3.3.2 Oblique Intermediate Wave

Although the intermediate wave does not steepen to form a shock (Kantrowitz and Petschek, 1966), its dispersion properties are included here for completeness. Performing the opposite expansion in 3.23, $\omega/k << C_F$, yields a bi-quadratic for the intermediate and slow waves. Solving this equation and taking the intermediate wave root yields

$$\frac{\omega^2}{k^2} = C_I^2\left[1 + r\,\frac{C_I^2}{C_F^2}\,\frac{\dfrac{2P_0^{(1)}}{\rho_0} - C_I^2}{C_I^2 - C_{SL}^2}\right.$$
$$+\, s r\,\frac{C_I^2}{C_F^2}\,\frac{\dfrac{P_0^{(1)}}{\rho_0}\cos^2\theta}{C_I^2}\,\frac{\dfrac{P_0^{(1)}}{\rho_0}}{C_I^2 - C_{SL}^2}$$
$$\left. +\, \frac{M}{C_F^2}\,\frac{C_I^2}{C_I^2 - C_{SL}^2}\left(1 - \frac{N}{C_I^2 M} - \frac{L}{C_I^4 M}\right)\right] \tag{3.25}$$

First consider $\beta_+ < 1$ so that ion inertia dominates. Since C_I always exceeds C_{SL}, ion inertia slows down the intermediate wave if $\cos^2\theta > \beta$ and speeds up the wave if $\cos^2\theta < \beta \leqslant 1$, in agreement with the results of Stringer (1963) and Coroniti and Kennel (1969). For $\beta_+ > 1$ from A1.22-A1.24, $N/C_I^2M \sim L/C_I^4M \sim \beta > 1$ so that ion FLR dispersion slows down the intermediate wave.

3.3.3 Oblique Slow Wave

Taking the slow wave root of the bi-quadratic when $\omega/k << C_F$ yields

$$\frac{\omega^2}{k^2} = C_{SL}^2\left[1 - r\,\frac{C_I^2}{C_F^2}\,\frac{\left(\dfrac{2P_0^{(1)}}{\rho_0} - C_{SL}^2\right)}{C_I^2 - C_{SL}^2}\right.$$
$$+\, s r\,\frac{C_I^2}{C_F^2}\,\frac{\dfrac{P_0^{(1)}}{\rho_0}\cos^2\theta}{C_{SL}^2}\,\frac{\dfrac{P_0^{(1)}}{\rho_0}}{C_I^2 - C_{SL}^2} \tag{3.26}$$
$$\left. -\, \frac{M}{C_F^2}\,\frac{C_{SL}^2}{C_I^2 - C_{SL}^2}\left(1 - \frac{N}{M C_{SL}^2} + \frac{L}{M C_{SL}^4}\right)\right]$$

For $\beta << 1$, $C_{SL}^2 \approx 3P_o^{(1)}/\rho_o \cos^2\theta$; neglecting the FLR terms, 3.26 becomes

$$\frac{\omega^2}{k^2} = C_{SL}^2\left[1 - \gamma\,\frac{C_I^2}{C_F^2}\,\frac{\frac{1}{3}\frac{P_o^{(1)}}{\rho_o} - C_{SL}^2}{C_I^2 - C_{SL}^2}\right] \tag{3.27}$$

Since $P_o^{(1)}/\rho_o >> C_{SL}^2$, finite ion inertia decreases the phase velocity of the slow wave.

If $\beta_+ > 1$, ion FLR dispersion dominates. Substituting the expressions for M, N, L to lowest order in $(k_\perp R_+)^2$ and neglecting terms $\mathcal{O}(1/\beta)$ 3.26 becomes

$$\frac{\omega^2}{k^2} = C_{SL}^2\left[1 - \frac{2\left(\frac{P_o^{(0)+}}{\rho_o}\right)^2 \cos^2\theta\, k_\perp^2 R_+^2}{C_F^2\left(C_I^2 - C_{SL}^2\right)}\left(1 - \frac{1}{8}\,\frac{P_o^{(1)}}{P_o^{(0)+}}\right)\right] \tag{3.28}$$

Ion FLR dispersion also decreases the slow wave phase velocity unless $P_o^{(1)}/P_o^{(0)+} > 8$, which can happen only if $T_-/T_+ > 8$.

3.3.4 Near-Parallel Fast Wave

Taking the opposite limit $\cos\theta \sim 1$ in 3.23, the dispersion relation can again be split into a quadratic and bi-quadratic form in the high and low β limits. For

$\beta << 1$, $C_{SL} << C_I$, C_F; dropping C_{SL}^2 compared to ω^2/k^2 in 3.23 yields a bi-quadratic for the fast and intermediate waves. The fast wave root is

$$\frac{\omega^2}{k^2} = C_F^2\left[1 + r\,\frac{C_I^2}{C_F^2}\right] \tag{3.29}$$

Finite ion inertia again speeds up the fast wave.

If $\beta >> 1$, $C_F >> C_I$, C_{SL}; retaining terms to lowest order in $k_{\parallel}^2 R_+^2$ and $k_\perp^2 R_+^2$ in the expressions for N, M, L, the dispersion relation becomes

$$\frac{\omega^2}{k^2} = C_F^2\left[1 + \frac{P_0^{(0)+}}{3\rho_0 C_F^2}\,k_\perp^2 R_+^2\right] \tag{3.30}$$

For $k_\perp = 0$ the parallel high β fast wave is unaffected by ion FLR dispersion, or, as it is easy to show, ion inertia dispersion. This wave is electrostatically polarized, being the ordinary sound wave, and is non-dispersive until wavelengths the order of the Debye length are reached. For $k_\perp \neq 0$, and $k_\perp R_+ > k_\parallel \lambda_D^\pm$, $\lambda_D^\pm = C_\pm/\omega_{p_\pm}$ the Debye length, ion FLR increases the fast wave speed.

3.3.5 Near-Parallel Intermediate Wave

In low β the intermediate wave root of the bi-quadratic gives

$$\frac{\omega^2}{k^2} = C_I^2\left[1 - r\,\frac{C_I^2}{C_F^2 - C_I^2}\right] \tag{3.31}$$

Since $C_F > C_I$, ion inertia slows down the wave. In high β, the intermediate and slow waves are coupled in a bi-quadratic. To lowest order in $k_\parallel^2 R_+^2$ the intermediate wave root gives

$$\frac{\omega^2}{k^2} = C_I^2\left[1 + \frac{\frac{P_0^{(0)+}}{\rho_0}\,k_\parallel^2 R_+^2}{C_I^2 - C_{SL}^2}\right] \tag{3.32}$$

Ion FLR dispersion increases the phase velocity, an effect opposite to that of ion inertia.

3.3.6 Near-Parallel Slow Wave

Taking the limit $\omega/k << C_F$, C_I and $\beta << 1$ in 3.23, the dispersion relation for the slow wave is

$$\frac{\omega^2}{k^2} = C_{SL}^2\left[1 - \frac{\frac{P_0^{(1)}}{\rho_0}\sin^2\theta}{3\,C_F^2}\right] \tag{3.33}$$

Note that for parallel propagation the slow wave is non-dispersive up to $k\lambda_D \sim 1$, and is the ordinary ion sound wave. It should be recalled from kinetic theory that the ion sound wave is heavily Landau damped unless $T_-/T_+ >> 1$. For slightly oblique propagation, ion inertia slows the wave speed.

The slow wave root of the bi-quadratic in high β yields

$$\frac{\omega^2}{k^2} = C_{SL}^2 \left[1 - \frac{\frac{P_a^{(0)+}}{P_0} k_\parallel^2 R_+^2}{C_S^2 - C_{SL}^2} \right] \qquad (3.34)$$

Ion FLR decreases the slow wave phase velocity.

3.4 Discussion

Section 2 demonstrated that by knowing the dispersive properties of linear waves, information about the laminar wave train structure could be determined. If dispersion increased (decreased) the wave phase velocity, the corresponding wave train stands in the upstream (downstream) flow. The oscillation wavelength is comparable to the dispersion scale length. The results of section 3.3 are summarized in terms of the wave train structure for fast and slow shocks.

A. Near-Perpendicular Fast Shocks

1. $\beta_+ < M_-/M_+$, $\theta < \pi/2 - \sqrt{M_-/M_+}$, trailing C/ω_{p_-} wave train.
2. $\beta_+ > M_-/M_+$, $\theta < \pi/2 - \sqrt{M_-/M_+}$, trailing R_+ wave train.
3. $1 \geqslant \beta_+$, $\theta > \pi/2 - \sqrt{M_-/M_+}$, leading C/ω_{p_+} wave train

4. $\beta_+ > 1$ trailing R_+ wave train.

5. $\beta_+ \sim 4/\sqrt{3}\ (1 + 2P_o^{(1)}/\rho_o C_F^2)\cot\theta$, ion inertia and ion FLR dispersion cancel. In this region the only dispersion left is C/ω_{p_-} and R_-, both much smaller than either C/ω_{p_+} or R_+. Before a wave train can be formed, two difficulties must be resolved. At scale lengths of order $C/\omega_{p_-} - R_-$, the FLR-CGL equations break down since the $R_+/L_S < 1$ expansion is no longer valid. To describe the $C/\omega_{p_-} - R_-$ structure a new expansion of the fluid equations in which $L_S/R_+ << 1$ and $R_-/L_S \leqslant 1$ is required. This is performed in section 7.0. Secondly, in order for the C/ω_{p_-}-R_- structure to occur, the shock must steepen to these short scale lengths; which implies that the long scale length wave trains be insufficient to limit shock steepening. Hence the C/ω_{p_-}-R_- structure might be important only for strong shocks. In this case the steepening arguments given in section 7.0 suggest that the C/ω_{p_-}-R_- wave train might occur independent of whether or not the ion inertia and ion FLR dispersion cancel; hence this short scale length wave train could be a more general feature of strong shocks. The steepening to shorter dispersion scale lengths is the collisionless analogue of the dissipation discontinuities discussed by

Coroniti (1969); therefore the C/ω_{p_-} -R_- structure will be referred to as a dispersion discontinuity.

B. Near-Perpendicular Slow Shocks

1. $1 \geqslant \beta_+$, trailing C/ω_{p_+} wave train
2. $\beta_+ > 1$, trailing R_+ wave train

C. Near-Parallel Fast Shocks

1. $\beta_+ \leqslant 1$ leading C/ω_{p_+} wave train
2. $\beta_+ > 1$, $k_\perp = 0$, non-dispersive until $k\lambda_D \sim 1$
3. $\beta_+ > 1$, $k_\perp \neq 0$, $k_\perp R_+ > k_\parallel \lambda_D$, leading R_+ wave train.

D. Near-Parallel Slow Shocks

1. $\beta_+ < 1$, $k_\perp = 0$, non-dispersive until $k_\parallel \lambda_D \sim 1$
2. $\beta_+ < 1$, $k_\perp \neq 0$, $k_\perp C/\omega_{p_+} > k_\parallel \lambda_D$, trailing C/ω_{p_+} wave train
3. $\beta_+ \geqslant 1$, trailing R_+ wave train.

These results are summarized in Figures 5-8.

4.0 Discussion of FLR-CGL Equations

4.1 Introduction

The theory of linear waves provided a deceptively simple method for investigating the laminar wave train structure, deceptively simple since the basic non-linear features of the equations were suppressed from the outset. Although virtually no progress will be made here toward solving the non-linear equations (recall that even the relatively simple C/ω_{p_-} low β fast perpendicular shock involved an almost hopeless differential equation), it is imperative to struggle with the non-linear aspects of these equations in order to completely understand the implications and limitations of the FLR-CGL equations. This section will be devoted to a discussion of the difficulties encountered and the approximations required to solve the FLR-CGL equations for the laminar shock structure. The full calculation, following the methods outlined in section 2.0, is performed in sections 5.0 and 6.0.

To carry out the discussion, the time independent form of FLR-CGL equations is needed. Again choose a coordinate system moving with the shock such that the shock propagates in the x direction, $x = -\infty$ being upstream, with the upstream (and downstream by the co-planarity theorem) magnetic field lying in the x-z plane. All quantities have spatial dependence only in the x-direction. The three conservation laws possess first integrals and can be written as

$$\rho u = \rho_1 u_1 \qquad (4.1)$$

$$\rho u(u-u_1) + P_{xx}^{(1)} - P_{xx_1}^{(1)} + \frac{B_y^2 + B_z^2 - B_{z_1}^2}{8\pi} = 0 \tag{4.2}$$

$$\rho u V_y + P_{xy}^{(1)} - \frac{B_x B_y}{4\pi} = 0 \tag{4.3}$$

$$\rho u V_z + P_{xz}^{(1)} - P_{xz_1}^{(1)} - \frac{B_x(B_z - B_{z_1})}{4\pi} = 0 \tag{4.4}$$

$$\begin{aligned} &\rho u\left[\frac{u^2 + V_y^2 + V_z^2 - u_1^2}{2}\right] + u\left(P_\perp^{(1)} + \frac{P_{\parallel}^{(1)}}{2}\right) \\ &- u_1\left(P_{\perp_1}^{(1)} + \frac{P_{\parallel_1}^{(1)}}{2}\right) + u P_{xx}^{(1)} - u_1 P_{xx_1}^{(1)} + V_y P_{xy}^{(1)} \\ &+ V_z P_{xz}^{(1)} + \frac{u_1 B_{z_1}(B_z - B_{z_1})}{4\pi} \\ &+ \left(\underline{q}_{\parallel}^{\parallel(0)} + \underline{q}_{\parallel}^{\perp(0)} + \underline{q}_{\perp}^{\perp(1)} + \underline{q}_{\perp}^{\parallel(1)}\right)\cdot\hat{x} = 0 \end{aligned} \tag{4.5}$$

Using the Maxwell equation 3.5 and linearizing the coefficients of the derivative terms as in section 2.0, the Ohm's law becomes

$$\frac{c^2}{\omega_p^2} u_1 \frac{d^2 B_z}{dx^2} + \frac{c^2}{\omega_p^2} \nu_{ei} \frac{dB_z}{dx} - \frac{c B_x}{4\pi P_1 e} \frac{dB_y}{dx} =$$

$$u B_z - u_1 B_{z_1} - V_z B_x \tag{4.6}$$

$$\frac{c^2}{\omega_p^2} u_1 \frac{d^2 B_y}{dx^2} + \frac{c^2}{\omega_p^2} \nu_{ei} \frac{dB_y}{dx} + \frac{cB_x}{4\pi N_1 e} \frac{dB_z}{dx} = uB_y - V_y B_x \tag{4.7}$$

Note that even though $B_{y_1} = B_{y_2} = 0$, ion inertia couples the B_z and B_y components together so that $B_y \neq 0$ inside the shock. An effective electron-ion collision frequency is included to introduce irreversibility into the equations. The pressure equations 3.14 and 3.15 become

$$\frac{u}{2}\frac{dP_\parallel^{(1)}}{dx} + \frac{P_\parallel^{(1)}}{2}\frac{du}{dx} + \frac{d}{dx}\left(\underline{q}_\perp^{\parallel(1)}\cdot\hat{x}\right) + P_\parallel^{(0)+}\alpha_1 - \alpha_2 + \frac{1}{Ne}\underline{J}_\parallel\cdot\left[\nabla P_\parallel^- + \frac{\nabla B}{B}\left(P_\parallel^- - P_\perp^-\right)\right]^{(0)} = \Gamma - \Gamma_\perp \tag{4.8}$$

$$u\frac{dP_\perp^{(1)}}{dx} + 2P_\perp^{(1)}\frac{du}{dx} + \frac{d}{dx}\left(\underline{q}_\perp^{\perp(1)}\cdot\hat{x}\right) - P_\perp^{(0)+}\alpha_1 + \alpha_2 + \frac{1}{Ne}\underline{J}_\perp\cdot\left[\nabla\left(P_\perp^- + \frac{B^2}{8\pi}\right) + \hat{e}_3\cdot\nabla\hat{e}_3\left(P_\parallel^- - P_\perp^- - \frac{B^2}{4\pi}\right)\right]^{(0)} = \Gamma_\perp \tag{4.9}$$

where $\nabla = \hat{e}_x\,(d/dx)$. The quantities $P_{xx}^{(1)}$, $P_{xy}^{(1)}$, $P_{xz}^{(1)}$, $\underline{q}_\perp^{\perp(1)}\cdot\hat{e}_x$, $\underline{q}_\perp^{\parallel(1)}\cdot\hat{e}_x$, α_1, and α_2 are derived in Appendix A.2.

Recall from section 2.0 that the wave train differential equation is derived by using the conservation or Rankine-Hugoniot relations to eliminate the velocity in the Ohm's law.

This method, of course, is usable only if there exist sufficient conservation relations to determine all the variables, i.e., the equations are closed. Section 4.2 discusses the Rankine-Hugoniot relations for the FLR-CGL equations and the physical reasons why they are not closed. To continue the fluid approach an approximation is required which closes the Rankine-Hugoniot relations. Two such approximations are considered in sections 4.3 and 4.4: (1) for perpendicular and near-perpendicular shocks the parallel pressure equation provides the required conservation law; (2) assuming the pressure to be isotropic throughout the flow also closes the Rankine-Hugoniot relations. Furthermore, the moment equations are not closed until the zero order parallel heat flow is specified. Since this determination requires knowledge of the particle distribution function and hence is beyond the scope of the fluid approach, a further assumption is necessary and is discussed in section 4.5.

4.2 The Rankine-Hugoniot Relations

Before proceeding to the FLR-CGL equations, the significance of the Rankine-Hugoniot relations is best appreciated by reviewing hydromagnetic shocks. The hydromagnetic equations are a closed set of non-linear hyperbolic partial differential equations; the Cauchy problem is, therefore, well posed, and the solution is obtainable by the method of characteristics (Kantrowitz and Petschek, 1966). The time independent hydromagnetic equations can be written in the form $\nabla \cdot \underset{\sim}{j} = 0$ where $\underset{\sim}{j}$ is the mass, momentum, or energy flux. Hence there exist three integrals of the motion which are the Rankine-Hugoniot conservation laws.

The time independent equations permit discontinuous changes in the characteristics with the final state determined from the initial one by the conservation laws. The significance for shock waves is that no information about the internal structure and dissipation processes is required to determine the downstream state.

Now consider the CGL system of equations assuming that the characteristics are real, i.e., that the conditions for the firehose or mirror instabilities are not satisfied (Goldberg, 1969; Morioka and Spreiter, 1968). There are nine variables, $\underline{V}$, $\underline{B}$, ρ, $P_{\parallel}$, $P_{\perp}$, but ten equations, continuity, 3 momentum, energy, 3 Ohm's and 2 pressure equations, only eight of which are in the form of conservation laws. Therefore one further relation in the form $\nabla \cdot \underline{j} = 0$ is needed for a closed set of Rankine-Hugoniot relations.

It might be thought that the above difficulty could be resolved by writing separate equations for the parallel and perpendicular energies. Within the CGL system the following two energy equations are obtained

$$\frac{\partial}{\partial t}\left(\frac{\rho u_{\perp}^2}{2} + P_{\perp} + \frac{B^2}{8\pi}\right) + \nabla \cdot \left[\rho \underline{v}\left(\frac{u_{\perp}^2}{2} + \frac{P_{\perp}}{\rho}\right) + \underline{u}_{\perp} \cdot \underset{\approx}{P} + \frac{\underline{B} \times \underline{v} \times \underline{B}}{4\pi}\right] + \rho u_{\parallel} \underline{u}_{\perp} \cdot \frac{d\hat{e}_3}{dt} = 0 \tag{4.10}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho u_{\parallel}^2}{2} + \frac{P_{\parallel}}{2}\right) + \nabla \cdot \left[\rho \underline{v}\left(\frac{u_{\parallel}^2}{2} + \frac{P_{\parallel}}{2\rho}\right) + \underline{u}_{\parallel} \cdot \underset{\approx}{P}\right] + \rho u_{\parallel} \underline{u}_{\perp} \cdot \frac{d\hat{e}_3}{dt} = 0 \tag{4.11}$$

If the last terms in 4.10 and 4.11 were ignored, these equations would be in the form of conservation laws and the Rankine-Hugoniot relations would be closed. Abraham Shrauner (1967) has discussed the conservation laws in this form.

However, except for perpendicular shocks where $d\hat{e}_3/dt = 0$, it is incorrect to neglect the last term in 4.10 and 4.11. This term couples the parallel and perpendicular energies, and represents the change in energy arising from the centripedal acceleration of the plasma along the magnetic field due to variations in the direction of the field. Since the Vlasov equation always conserves total energy, changes in parallel energy must come, in part, at the expense of the perpendicular energy and vice-versa. In hydromagnetics this coupling never occurs since the assumption of pressure isotropy implies that any gain in parallel energy is transferred back to perpendicular energy by collisions.

Another possibility for obtaining a closed set of Rankine-Hugoniot relations is to consider the separate equations for $P_{\parallel}$ and $P_{\perp}$. In the CGL system use of the continuity equation and Ohm's law reduces the pressure equations to the well-known double adiabatic laws

$$\frac{d}{dt}\left(\frac{P_{\perp}}{\rho B}\right) = 0 \tag{4.12}$$

$$\frac{d}{dt}\left(\frac{P_{\parallel} B^2}{\rho^3}\right) = 0 \tag{4.13}$$

Equation 4.12 expresses conservation of the first adiabatic invariant, 4.13 the conservation of the longitudinal invariant. With 4.12 and 4.13, there are ten conservation equations in nine unknowns so that the shock problem in the CGL system is over-determined.

To resolve this difficulty Goldberg (1969) suggested that if the FLR terms were retained in the pressure equations, only one of the double adiabatic invariants (4.13) is conserved across the shock. As discussed in section 4.3 this argument is valid only for perpendicular shocks. For oblique propagation there are additional coupling terms in the equations for $P_{\perp}^{(1)}$ and $P_{\parallel}^{(1)}$ arising from the FLR corrections to $\underset{\approx}{P} : \hat{e}_3 \, d\hat{e}_3/dt$ and $\underset{\approx}{P} : \nabla \underline{V}$ as well as coupling through the heat flow tensor of the form $\hat{e}_3 \cdot Q : \nabla \hat{e}_3$. ($\alpha_1$ and α_2 in 4.14 and 4.15 represent the sum of these couplings.) None of these terms is in the form $\nabla \cdot \underline{j} = 0$.

In neither the CGL or the FLR-CGL system of equations does there exist a well-defined set of Rankine-Hugoniot relations. Hence these equations are not of hyperbolic form, and the solution of the initial value problem does not permit discontinuous changes in the characteristics. To solve for the downstream state, it is necessary to know the dissipation processes and the detailed changes of the fluid parameters within the shock: in general these must be determined by numerical integration of the equations of motion through the shock, including at least the kinetic equation for parallel motion. Therefore, unless some simplifying assumptions are made, the fluid approach to collisionless shocks stops here.

4.3 Perpendicular and Near Perpendicular Shocks

The quest for appropriate assumptions starts by considering perpendicular shocks. Here $\hat{e}_3 \cdot \nabla = 0$ implies that $\alpha_1 = \alpha_2 = 0$; since no gradients exist along the lines of force, $q_{\parallel}^{\parallel(0)} = q_{\parallel}^{\perp(0)} = 0$. With $\underline{J}_{\parallel} \cdot \nabla = 0$, 4.8 upon rewriting becomes

$$\frac{d}{dx}\left(u P_{\parallel}^{(1)} + 2\, \underline{q}_{\perp}^{\parallel(1)} \cdot \hat{x} \right) = 0 \tag{4.14}$$

and yields the required additional conservation law to close the Rankine-Hugoniot relations. Since $\underline{q}_{\perp}^{\parallel(1)}$ depends only on the gradients in the flow, it vanishes upstream and downstream so that $P_{\parallel 1}^{(1)} U_1 = P_{\parallel 2}^{(1)} U_2$. Alternatively, $P_{\parallel}^{(1)}/\rho = T_{\parallel}/M_+ =$ constant, and the longitudinal invariant is conserved through the shock, a result found by Goldberg (1969) but without considering the FLR heat flow.

Now consider 4.9 for the perpendicular pressure. From the x component of Ampere's equation $J_x = 0$ so that 4.9 becomes

$$u \frac{d P_{\perp}^{(1)}}{dx} + 2 P_{\perp}^{(1)} \frac{du}{dx} + \frac{d}{dx}\left(\underline{q}_{\perp}^{\perp(1)} \cdot \hat{x} \right) = 0 \tag{4.15}$$

or using the continuity equation

$$\rho_1 u_1 \rho \frac{d}{dx}\left(\frac{P_{\perp}^{(1)}}{\rho^2} \right) + \frac{d}{dx}\left(\underline{q}_{\perp}^{\perp(1)} \cdot \hat{x} \right) = 0 \tag{4.16}$$

Since 4.16 is not in the form $\nabla \cdot \underline{j} = 0$, in the FLR-CGL system the magnetic moment is never conserved, even for weak shocks as assumed by Kennel and Sagdeev (1967a) and "substantiated" by Goldberg (1969).

For near perpendicular shocks, 4.14 should constitute an approximate conservation law since the parallel temperature change across the shock should be small compared to that of the perpendicular temperature. Comparing terms in 4.14, first note that since $J_x = 0$, $\underline{J}_\parallel \cdot \nabla$ always vanishes. From A.2.29 an order of magnitude estimate for $d/dx(\underline{q}_\perp^{\parallel(1)} \cdot \hat{e}_x)$, if $q_\parallel^{\parallel(0)}$ and $q_\parallel^{\perp(0)}$ are neglected, is

$$\frac{d}{dx}\left(\underline{q}_\perp^{\parallel(1)} \cdot \hat{x}\right) \sim \frac{\left(P_\perp^{(0)+}\right)^2}{\rho \Omega_+^2} \frac{d^3 u}{dx^3} \tag{4.17}$$

Estimating α_1 from A.2.27, using $dV_y/dx \sim (P_\perp^{(0)+}/4\pi\rho\Omega_+)\, d^2U/dx^2$ from the y momentum equation and $P_{xy}^{(1)}$ from A.2.19, α_1 is of order

$$\frac{P_\parallel^{(0)+} \alpha_1}{\frac{d}{dx}\left(\underline{q}_\perp^{\parallel(1)} \cdot \hat{x}\right)} \sim \frac{\frac{P_\parallel^{(0)+}}{\Omega_+} \frac{B_x^2}{B^2} \frac{du}{dx} \left(\frac{P_\perp^{(0)+}}{4\pi\rho\Omega_+} \frac{d^2u}{dx^2}\right)}{\left(\frac{P_\parallel^{(0)+}}{\rho\Omega_+^2}\right) \frac{d^3u}{dx^3}} \sim \frac{B_x^2}{B^2} \ll 1 \tag{4.18}$$

where $B_z/B \sim 1$. Similarly estimating $R_1 \sim P_\parallel^{(1)} P_\perp^{(0)}/\rho$, from A.2.28 α_2 is of order

$$\alpha_2 \sim \frac{P_\perp^{(0)+}}{\Omega_+} \frac{d}{dx}\left(\frac{P_\parallel^{(0)+}}{\rho}\right) \frac{B_x}{B} \frac{dB_y}{dx} \tag{4.19}$$

Since from 4.14 $d/dx(q_\perp^{\parallel(1)} \cdot \hat{e}_x) \sim \rho U (d/dx)(P_\parallel/\rho)$, the importance of α_2 can be estimated as

$$\frac{\alpha_2}{\frac{d}{dx}\left(q_\perp^{\parallel(1)} \cdot \hat{x}\right)} \sim \frac{P_\perp^{(0)+}}{\Omega_+ \rho u} \frac{B_x B_y}{B^2 L_s} \sim \frac{R_+}{L_s} \frac{B_x^2}{B^2} \tag{4.20}$$

Hence for the range of angles near perpendicular such that $B_x/B << 1$, 4.14 will be an approximate conservation law to $\mathcal{O}(B_x^2/B^2)$. Note, however, that this approximation is valid only for fast shocks, where the magnetic field increases across the shock, and not for oblique slow shocks, where the magnetic field decreases in magnitude. Furthermore the above estimates were made neglecting $q_\parallel^{\parallel(0)}$ and $q_\parallel^{\perp(0)}$, a still unjustified assumption.

4.4 Pressure Isotropy Assumption

A simpler approximation which closes the Rankine-Hugoniot conditions without restricting the angle of propagation or discriminating against slow shocks is to take the pressure isotropic, $P_\parallel = P_\perp$, in the energy equation. As mentioned in section 3.3, a collisionless plasma may often be maintained at approximate pressure isotropy on fluid time scales by high frequency wave turbulence. Care must be taken, however, since anisotropy driven turbulence could be an important part of the shock structure, as it is for Alfvén shocks (Kennel and Sagdeev, 1967a) and might be for finite-β whistler shocks (Kennel and Petschek, 1968). Here both the non-resonant and resonant anisotropy instabilities must be investigated from the Vlasov equation (see Kennel and Scarf, 1968), and such shocks are, therefore, not amenable to the fluid approach. If the pressure is sufficiently anisotropic so that the firehose or mirror instability conditions are satisfied, the basic assumption of time independence is also violated; alternatively stated, the fluid characteristics in such unstable flows are imaginary.

Most of the calculations to be done in sections 5.0 and 6.0 are based on the isotropy assumption since both arbitrary propagation angle and slow shocks can be treated. However, for any particular case, the validity of this assumption must be checked against either a calculation of the turbulent collision operator or observations. The effects of pressure anisotropy are discussed further in sections 5.3, 6.3 and 8.3.

4.5 Problem of the Parallel Heat Flow

By far the most difficult and uncertain aspect of the fluid approach to collisionless shocks is the indeterminancy within the moment equations of the zero order parallel heat flow. Since non-perpendicular shocks have temperature gradients along the magnetic field, there could exist a large heat flow upstream from the shock layer which could either greatly broaden the shock transition or might, in certain circumstances, render the whole concept of a shock transition meaningless.

The usual argument used in constructing the CGL equations is that only a few collisions are needed to suppress the heat flow along the lines of force. Since collisionless shocks must involve turbulent dissipation which replaces ordinary Coulomb collisions, the parallel heat flow will probably be suppressed within the shock layer. Every line of force must pass through the turbulent region in the shock layer so that if the heat flow is negligible there, it can also be neglected in the upstream and downstream flow.

Furthermore, Coroniti (1969) showed that heat flow alone can provide the required dissipation only for very weak shock waves; therefore, for most shocks other types of dissipation are necessary. With the above considerations the zero order parallel heat flow will be neglected in the calculations that follow; it is mandatory, however, that before these calculations can be applied to the interpretation of collisionless shock data, this approximation must be justified. In section 8.0 the parallel heat flow problem for the particular example of the earth's bow shock is considered where some knowledge of the turbulent dissipation permits an estimate of its effect. Section 9.0 discusses the likelihood that this approximation be valid for other shock flows.

5.0 The Wave Train Differential Equation

5.1 Introduction

With the approximations of section 4.0 the Rankine-Hugoniot relations can be closed thus permitting the reduction of 4.1-4.7 to a coupled set of differential equations for the magnetic field. Symbolically, these equations can be written as

$$\left(\hat{L}_{\frac{c}{\omega_{p_-}}} + \hat{L}_{R_+}\right)B_z + \hat{L}_{\nu_{ei}}B_z - \hat{L}_{\frac{c}{\omega_{p_+}}}B_y = -\frac{\partial \varphi(B_z, B_y)}{\partial B_z} \tag{5.1}$$

$$\left(\hat{L}_{\frac{c}{\omega_{p_-}}} + \hat{L}'_{R_+}\right)B_y + \hat{L}_{\nu_{ei}}B_y + \hat{L}'_{\frac{c}{\omega_{p_+}}}B_z = -\frac{\partial \varphi(B_z, B_y)}{\partial B_y} \tag{5.2}$$

where $\hat{L}_{C/\omega_{p_-}}$, $\hat{L}^{(')}_{R_+}$, $\hat{L}^{(')}_{C/\omega_{p+}}$ are differential operators representing the dispersive effects of C/ω_{p_-}, R_+, and C/ω_{p_+} respectively; $\hat{L}_{\nu ei}$ represents the dissipation, either collisional or anomalous. The C/ω_{p_-} operator is retained to make contact with the $\beta_+ < M_-/M_+$ perpendicular shock; for $\beta_+ > M_-/M_+$ this term will contribute negligibly except for fast oblique shocks when ion inertia and FLR dispersion may cancel. $\varphi(B_z, B_y)$ is an effective potential obtained by integration of the Rankine-Hugoniot conditions with $\varphi(B_{z_1}, 0) = 0$.

In general the differential operators in 5.1 and 5.2 are non-linear (recall the discussion of 2.9); $\varphi(B_z, B_y)$ is also a non-linear function of its arguments. In order to obtain an algebraically tractable differential equation, two simplifying assumptions will be made:

1. Weak Shock Approximation. Here the non-linear terms in the $\hat{L}$ operators are assumed small compared to the linear terms; the coefficients of the derivatives are linearized about the upstream stationary point. Equations 5.1 and 5.2 then reduce to coupled non-linear harmonic oscillator differential equations.

2. The operators $\hat{L}_{R_+}$ and $\hat{L}_{C/\omega_{p+}}$ are calculated only to the first non-vanishing order in ϵ; for example, $\hat{L}_{R_+} \sim (R_+/L_s)^2 + \mathcal{O}(\epsilon^3)$. Therefore only the lowest order FLR effects are considered thereby restricting possible shock scale lengths to $R_+/L_s \leqslant 1$.

The calculation proceeds in section 5.2 by eliminating V_y and V_z from the energy equation 4.5. This equation is then reduced in the two limits corresponding to the two approximations of sections 4.3 and 4.4. Then the differential equation for near-perpendicular fast shocks is dervied in section 5.3. Shocks with isotropic pressure are considered in section 5.4.

In this section the only dissipation process which is included in the analysis is resistivity. Viscosity could, in principle, be included but the equations become much more difficult algebraically; in addition any approximation scheme involves an internal ordering between the coefficients of viscosity and the FLR terms, a difficult consideration since the magnitude of the viscosity is unknown. However, it is anticipated from Coroniti (1969) and section 2.3 that difficulty will be encountered at the sonic point and for sub-sonic flow, which must occur behind $\beta \gg 1$ fast shock waves. This problem will be considered further in section 6.7.

5.2 Reduction of the Energy Equation

In Appendices A.2, A.3 and A.4 the required components of $\underset{\approx}{P}^{(1)}$ are calculated to $\mathcal{O}(R_+^2/L_s^2)$ for both near perpendicular propagation (A.3) and arbitrary propagation for pressure isotropy (A.4). Following the general method presented in section 2.0, 4.1-4.5 are to be reduced to expressions involving only U, B_y and B_z in preparation for substitution into 4.6 and 4.7.

Substituting A.2.3 into 4.2 and solving for $P_\perp^{(1)}$ gives

$$\frac{P_\perp^{(1)}}{\tilde{\gamma}-1} = -P_\parallel^{(1)}\frac{B_x^2}{B^2} + P_{\perp 1}^{(1)}\frac{B_{z1}^2}{B_1^2} + P_{\parallel 1}^{(1)}\frac{B_x^2}{B_1^2} - \rho_1 u_1 (u-u_1) - T_{xx}^{(1)} - \frac{B_y^2 + B_z^2 - B_{z1}^2}{8\pi} \qquad (5.3)$$

where the "effective" ratio of specific heats, $\tilde{\gamma}$, is defined by

$$\frac{\tilde{\gamma}}{\tilde{\gamma}-1} = 1 + \frac{B_x^2 B_z^2}{B_\perp^2 B^2} + \frac{B_y^2}{B_\perp^2}$$

Note that for perpendicular propagation $\tilde{\gamma} = 2$, the usual CGL value. Similarly substituting A.2.17 into 4.3 gives

$$V_y = \frac{1}{\rho_1 u_1}\left[\frac{B_x B_y}{4\pi} - \frac{(P_{\parallel}^{(1)} - P_{\perp}^{(1)}) B_x B_y}{B^2} - T_{xy}^{(1)}\right] \tag{5.4}$$

and A.2.22 into 4.4 gives

$$V_z = \frac{1}{\rho_1 u_1}\left[\frac{B_x(B_z - B_{z_1})}{4\pi} - \frac{(P_{\parallel}^{(1)} - P_{\perp}^{(1)}) B_x B_z}{B^2} \right.$$

$$\left. + \frac{(P_{\parallel_1}^{(1)} - P_{\perp_1}^{(1)}) B_x B_{z_1}}{B_1^2} - T_{xz}^{(1)}\right] \tag{5.5}$$

Substitution of 5.3, 5.4 and 5.5 into the energy equation yields, after a little manipulation,

$$\rho_1 u_1 \left(\frac{1 - 2\tilde{\gamma}}{2}\right) u^2 + u\left[\tilde{\gamma}\rho_1 u_1^2 - \tilde{\gamma}\,\frac{B_y^2 + B_z^2 - B_{z_1}^2}{8\pi} + P_{\parallel}^{(1)}\left(\frac{1}{2}\right.\right.$$

$$\left.\left. - (\tilde{\gamma}-1)\frac{B_x^2}{B^2}\right) + \tilde{\gamma}\left(P_{\perp_1}^{(1)}\frac{B_{z_1}^2}{B^2} + P_{\parallel_1}^{(1)}\frac{B_x^2}{B^2}\right)\right] - u_1 P_{\perp_1}^{(1)}\left(1 + \frac{B_{z_1}^2}{B_1^2}\right)$$

$$- u_1 P_{\parallel_1}^{(1)}\left(\frac{1}{2} + \frac{B_x^2}{B_1^2}\right) - u(\tilde{\gamma}-1)T_{xx}^{(1)} + \frac{B_x^2\left[B_y^2 + (B_z - B_{z_1})^2\right]}{2(4\pi)^2 \rho_1 u_1}$$

$$- \rho_1 u_1 \frac{u_1^2}{2} - \frac{(P_{\parallel}^{(1)} - P_{\perp}^{(1)})^2 B_x^2 (B_y^2 + B_z^2)}{2 B^4 \rho_1 u_1}$$

$$+ \frac{(P_{\parallel_1}^{(1)} - P_{\perp_1}^{(1)}) B_x^2 B_z (B_z - B_{z_1})}{B_1^2\, 4\pi \rho_1 u_1} + \frac{(P_{\parallel_1}^{(1)} - P_{\perp_1}^{(1)})^2 B_x^2 B_{z_1}^2}{2\rho_1 u_1 B_1^4}$$

$$- \frac{(P_{\parallel}^{(1)} - P_{\perp}^{(1)}) B_x (B_y T_{xy}^{(1)} + B_z T_{xz}^{(1)})}{2\rho_1 u_1 \; B^2}$$

$$- \frac{(T_{xy}^{(1)})^2 + (T_{xz}^{(1)})^2}{2\rho_1 u_1} + \left(\underline{q}_{\parallel}^{\parallel(0)} + \underline{q}_{\parallel}^{\perp(0)} + \underline{q}_{\perp}^{\parallel(1)} + \underline{q}_{\perp}^{\perp(1)} \right) \cdot \hat{x}$$

$$+ \frac{u_1 B_{z_1} (B_z - B_{z_1})}{4\pi} = 0$$

(5.6)

Note that $P_{\parallel}^{(1)}$ and $P_{\perp}^{(1)}$ have not been entirely eliminated. The term involving $(T_{xy}^{(1)})^2$ and $(T_{xz}^{(1)})^2$ is non-linear in the derivatives, and will henceforth be dropped, as will the zero order heat flows.

For nearly perpendicular fast shocks the parallel pressure can be eliminated by the approximate conservation law 4.14. Then to order (B_x/B), 5.6 becomes

$$\rho_1 u_1 \left(\frac{1 - 2\tilde{\gamma}}{2} \right) u^2 + u \left[\tilde{\gamma} \rho_1 u_1^2 - \tilde{\gamma} \frac{B_y^2 + B_z^2 - B_{z_1}^2}{8\pi} + \tilde{\gamma} P_{\perp_1}^{(1)} \frac{B_{z_1}^2}{B_1^2} \right]$$

$$- u_1 P_{\perp_1}^{(1)} \left(1 + \frac{B_{z_1}^2}{B_1^2} \right) - u (\tilde{\gamma} - 1) T_{xx}^{(1)} + \frac{B_x^2 \left[B_y^2 + (B_z - B_{z_1})^2 \right]}{2 (4\pi)^2 \rho_1 u_1}$$

$$- \rho_1 u_1 \frac{u_1^2}{2} + \underline{q}_{\perp}^{\perp(1)} \cdot \hat{x} + \frac{u_1 B_{z_1} (B_z - B_{z_1})}{4\pi} = 0 \qquad (5.7)$$

Note that the term proportional to $[(P_{\|}^{(1)}-P_{\perp}^{(1)})/B^2]\, B_x$ $(B_y\, T_{xy}^{(1)} - B_z T_{xz}^{(1)})$ was dropped since it is order B_x^2/B^2 times derivative terms. Also note that $\underline{q}_{\perp}^{\|(1)} \cdot \hat{x}$ drops out; in this approximation the lines of force are isothermal.

For isotropic shocks 5.6 is readily reduced to the following equation

$$u^2 - \frac{5}{4} u \left[u_1 - \frac{B_y^2 + B_z^2 - B_{z_1}^2}{8\pi \rho_1 u_1} + \frac{P_1^{(1)}}{\rho_1 u_1} \right] + \frac{3}{4} \frac{u T_{xx}^{(1)}}{\rho_1 u_1}$$

$$+ \frac{5}{4} \frac{P_1^{(1)}}{\rho_1} + \frac{u_1^2}{4} - \frac{B_x^2 \left[B_y^2 + (B_z - B_{z_1})^2 \right]}{4 (4\pi)^2 (\rho_1 u_1)^2}$$

$$+ \frac{(\underline{q}_{\perp}^{\|(1)} + \underline{q}_{\perp}^{\perp(1)}) \cdot \hat{x}}{2 \rho_1 u_1} - \frac{B_{z_1} (B_z - B_{z_1})}{8\pi \rho_1} = 0 \qquad (5.8)$$

Note that here the sum of the FLR heat flows enters. On comparing 5.8 with 2.7 the ratio of specific heats has become 5/3, the isotropic value.

5.3 Near-Perpendicular Fast Shocks

The calculation proceeds by eliminating the flow velocity in terms of the magnetic field components B_y and B_z. Equation 5.7 is a quadratic equation for U and hence is easily solvable. Note first, however, that under the radical sign of this solution there will be two separate types of terms, hydromagnetic and FLR.

Since the FLR terms involve spatial derivatives and are assumed smaller than hydromagnetic terms by the weak shock approximation, the radical can be expanded in the usual manner and only the lowest order FLR terms retained. Solving for U, performing the above expansion, and linearizing all coefficients of FLR terms about the upstream state yields

$$U = U(\underline{B}) + \frac{u_1}{(\tilde{\gamma}_1 - 1)u_1^2 - \frac{\tilde{\gamma}_1 P_{\perp_1}^{(1)}}{\rho_1}} \left\{ -\frac{(\tilde{\gamma}_1 - 1)}{\rho_1} T_{xt}^{(1)} + \underline{q}_{\perp}^{\perp(1)} \cdot \hat{x} \right\} \tag{5.9}$$

where $\tilde{\gamma}_1 = 1 + B_{z_1}^2 / B_1^2$ and

$$\begin{aligned} U(\underline{B}) = {} & \frac{\tilde{\gamma}}{2\tilde{\gamma}-1}\left[u_1 + \frac{P_{\perp_1}^{(1)}}{\rho_1 u_1}\frac{B_{z_1}^2}{B_1^2} - \frac{B_y^2 + B_z^2 - B_{z_1}^2}{8\pi\rho_1 u_1}\right] \\ & - \left\{\frac{\tilde{\gamma}^2}{(2\tilde{\gamma}-1)^2}\left[-u_1 - \frac{P_{\perp_1}^{(1)}}{\rho_1 u_1}\frac{B_{z_1}^2}{B_1^2} + \frac{B_y^2 + B_z^2 - B_{z_1}^2}{4\pi\rho_1 u_1}\right]^2 \right. \\ & + \frac{1}{2\tilde{\gamma}-1}\frac{B_x^2\left[B_y^2 + (B_z - B_{z_1})^2\right]}{(4\pi\rho_1 u_1)^2} \\ & + \frac{2}{2\tilde{\gamma}-1}\frac{B_{z_1}(B_z - B_{z_1})}{4\pi\rho_1} - \frac{1}{2\tilde{\gamma}-1}u_1^2 \\ & \left. - \frac{2}{2\tilde{\gamma}-1}\frac{P_{\perp_1}^{(1)}}{\rho_1}\left(1 + \frac{B_{z_1}^2}{B_1^2}\right)\right\}^{1/2} \end{aligned} \tag{5.10}$$

The square root is to be taken such that $U(\underline{B}_1) = U_1$. Equation 5.9 is now in the form to be substituted into the Ohm's law, 4.6 and 4.7.

Eliminating V_z by 5.5 and using 5.9, 4.6 becomes

$$\frac{c^2}{\omega_{p_-}^2} u_1 \frac{d^2 B_z}{dx^2} + \frac{c^2}{\omega_{p_-}^2} \nu_{ei} \frac{dB_z}{dx} - \frac{cB_x}{4\pi N_1 e} \frac{dB_y}{dx} =$$

$$u(\underline{B}) B_z - u_1 B_{z_1} + \frac{u_1 B_{z_1}}{(\tilde{\gamma}_1 - 1)u_1^2 - \frac{\tilde{\gamma}_1 P_{\perp_1}^{(1)}}{\rho_1}} \left\{ - \frac{(\tilde{\gamma}_1 - 1)}{\rho_1} T_{xx}^{(1)} \right.$$

$$\left. + \underline{q}_\perp^{\perp(1)} \cdot \hat{x} \right\} - \frac{B_x}{\rho_1 u_1} \left[\frac{B_x (B_z - B_{z_1})}{4\pi} \right. \tag{5.11}$$

$$\left. + \frac{(P_{\parallel_1}^{(1)} - P_{\perp_1}^{(1)}) B_x B_{z_1}}{B_1^2} - \frac{(P_{\parallel}^{(1)} - P_{\perp}^{(1)}) B_x B_z}{B^2} - T_{xz}^{(1)} \right]$$

Substituting A.2.17 for $P_{xy}^{(1)}$, using A.36 for $T_{xy}^{(1)}$, and noting that the $P_{xz}^{(1)}$ is of order (B_x^2/B^2), A.2.30 for $\underline{q}_\perp^{\perp(1)} \cdot \hat{x}$ becomes

$$\underline{q}_\perp^{\perp(1)} \cdot \hat{x} = \frac{2}{\Omega_+} \left[\frac{(P_{\parallel_1}^{(1)} - P_{\perp_1}^{(1)}) B_x B_{z_1}}{B_1^3} \frac{P_{\perp_1}^{(0)+}}{\rho_1} \frac{dB_y}{dx} \right.$$

$$+ \frac{1}{\tilde{\gamma}_1 - 1} \left(\frac{P_{\perp_1}^{(0)+}}{\Omega_+} \right)^2 \frac{B_{z_1}^2}{2B_1^2} \frac{d^2 u}{dx^2} \tag{5.12}$$

$$\left. + \left(R_1^{(0)+} - \frac{R_2^{(0)+}}{4} \right) \frac{B_x}{B_1^3} \left(B_y \frac{dB_z}{dx} - B_z \frac{dB_y}{dx} \right) \right]$$

Finally substituting 5.12, A.3.5 for $T^{(1)}_{xx}$, and A.3.7 for $T^{(1)}_{xz}$ into 5.11, the y component of the Ohm's law becomes

$$
\begin{aligned}
&\frac{c^2}{\omega_p^2} u_1 \frac{d^2 B_z}{dx^2} - \frac{3}{4}\frac{1}{\tilde{\gamma}_1 - 1} \frac{P_{\perp 1}^{(1)}/\rho_1}{\left[(\tilde{\gamma}_1-1)u_1^2 - \frac{\tilde{\gamma}_1 P_{\perp 1}^{(1)}}{\rho_1}\right]} \frac{B_{z_1}^2}{B_1^2} B_{z_1} R_+^2 \frac{d^2 u}{dx^2} \\
&+ \frac{c^2}{\omega_p^2} \nu_{ei} \frac{dB_z}{dx} - \frac{cB_x}{4\pi N_1 e}\frac{dB_y}{dx}\Bigg\{1 - \frac{1}{\left[(\tilde{\gamma}_1-1)u_1^2 - \frac{\tilde{\gamma}_1 P_{\perp 1}^{(1)}}{\rho_1}\right]}\Bigg[\\
&\frac{3}{2}\frac{P_{\perp 1}^{(0)+}}{\rho_1} \frac{4\pi(P_{\parallel 1}^{(1)} - P_{\perp 1}^{(1)})}{B^2}\frac{B_{z_1}^2}{B_1^2} + \frac{1}{2}\frac{P_{\perp 1}^{(0)+}}{\rho_1}\frac{B_{z_1}^2}{B_1^2} \\
&- 8\pi\left(R_1^{(0)+} - \frac{R_2^{(0)+}}{4}\right)\frac{B_{z_1}^2}{B_1^4}\Bigg]\Bigg\} + \frac{1}{\tilde{\gamma}-1}\frac{2\pi P_{\perp}^{(0)+}}{B_1^2}\frac{B_y}{u}\frac{du}{dx} \\
&= U(B) B_z - u_1 B_{z_1} - \frac{B_x^2 (B_z - B_{z_1})}{4\pi \rho_1 u_1} \qquad (5.13) \\
&\quad + \frac{B_x^2 B_z (P_{\parallel}^{(1)} - P_{\perp}^{(1)})}{\rho_1 u_1 B^2} - \frac{B_x^2 B_{z_1}(P_{\parallel 1}^{(1)} - P_{\perp 1}^{(1)})}{\rho_1 u_1 B_1^2}
\end{aligned}
$$

where here $R_+^2 \equiv P_{\perp 1}^{(0)+}/\Omega_+^2 \rho_1$. Equation 5.13 is not yet completely reduced since $P_\parallel^{(1)}$ and $P_\perp^{(1)}$ occur on the RHS and the LHS contains the d^2U/dx^2 term. $P_\parallel^{(1)}$ could be eliminated by 4.14 and $P_\perp^{(1)}$ by 5.2. Since these terms are small, multiplied by B_x^2/B^2, their effect on the fast shock B_z structure is probably negligible. This conclusion is substantiated by noting that the oblique fast wave is relatively insensitive to pressure anisotropy since its driving force is the total pressure and not the tension along the magnetic field; d^2U/dx^2 could be calculated from 5.9. However, it is more convenient to note that for near perpendicular shocks $UB_z \approx U_1 B_{z_1}$; in the same spirit of approximation that went into deriving 5.13, $B_z(d^2U/dx^2)$ can be replaced by $-U_1(d^2B_z/dx^2)$. Furthermore the derivative term multiplied by B_y has been retained even though $B_{y_1} = 0$.

Without actually performing the above substitutions, note that the C/ω_{p_-} and R_+ derivative term have the same sign. From section 2.3 the C/ω_{p_-} wave train trails; hence when R_+ is the dominant derivative term, the R_+ wave train will also trail. The ion inertia derivative term, that multiplied by $cB_x/4\pi$, contains FLR corrections which depend on the pressure anisotropy. ($R_1^{(0)+} - R_2^{(0)+}/4$ is also roughly proportional to $P_\parallel^{(0)+} - P_\perp^{(0)+}$.) It might be thought possible that if the pressures were sufficiently anisotropic, the sign of the ion inertia term could be changed. However this would require $\beta_+ >> 1$ in which case the R_+ derivative term dominates any way.

A similar series of substitutions into 4.7 yields the other component of the Ohm's law

$$\frac{c^2}{\omega_{p_-}^2} U_1 \frac{d^2 B_y}{dx^2} - \frac{3}{4}\frac{1}{\tilde{\gamma}_1 - 1} \frac{P_{\perp 1}^{(0)}/\rho_1}{\left[(\tilde{\gamma}_1 - 1)U_1^2 - \frac{\tilde{\gamma}_1 P_{\perp 1}^{(0)}}{\rho_1}\right]} \frac{B_{z_1}^2}{B_1^2} B_y R_+^2 \frac{d^2 U}{dx^2}$$

$$+ \frac{cB_x}{4\pi N_1 e}\frac{dB_z}{dx} + \frac{c^2}{\omega_{p_-}^2}\nu_{ei}\frac{dB_y}{dx} - \frac{cB_x}{4\pi N_1 e}\frac{2\pi P_{\perp 1}^{(0)+}}{(\tilde{\gamma}_1 - 1)B_1^2}\frac{B_z}{U}\frac{dU}{dx}$$

$$= U(\underline{B})B_y - \frac{B_x^2 B_y}{4\pi\rho_1 U_1} + \frac{B_x^2 B_y (P_\parallel^{(0)} - P_\perp^{(0)})}{\rho_1 U_1 B^2} \qquad (5.14)$$

5.4 Isotropic Shocks

If isotropic pressure is assumed in order to close the Rankine-Hugoniot relations, the starting point in obtaining the wave train differential equation is the reduced energy equation 5.8. Proceeding exactly as in 5.3, the quadratic is solved for U, the FLR terms under the radical are expanded to first order, and the coefficients of the FLR terms are linearized about the upstream point. The result is

$$U = U'(\underline{B}) + \frac{U_1}{U_1^2 - \frac{5}{3}\frac{P_1^{(1)}}{\rho_1}}\left[-\frac{3}{4}\frac{T_{xx}^{(1)}}{\rho_1} + \frac{\left(\underline{q}_\perp^{\perp(1)} + \underline{q}_\perp^{\parallel(1)}\right)\cdot\hat{x}}{2\rho_1 U_1}\right] \tag{5.15}$$

where

$$\begin{aligned} U'(\underline{B}) = {} & \frac{5}{8}\left[U_1 + \frac{P_1^{(1)}}{\rho_1 U_1} - \frac{B_y^2 + B_z^2 - B_{z1}^2}{8\pi\rho_1 U_1}\right] - \left\{\left(\frac{5}{8}\right)^2\left[-U_1\right.\right. \\ & \left. - \frac{P_1^{(1)}}{\rho_1 U_1} + \frac{B_y^2 + B_z^2 - B_{z1}^2}{8\pi\rho_1 U_1}\right]^2 + \frac{B_x^2\left[B_y^2 + (B_z - B_{z1})^2\right]}{4(4\pi)^2\rho_1^2 U_1^2} \\ & \left. + \frac{B_{z1}(B_z - B_{z1})}{8\pi\rho_1} - \frac{5}{4}\frac{P_1^{(1)}}{\rho_1} - \frac{U_1^2}{4}\right\}^{1/2} \end{aligned} \tag{5.16}$$

The square root is again to be taken such that $U(\underline{B}_1) = U_1$.

The appropriate components of $\underset{\approx}{P}^{(1)}$ are derived to order $(R_+/L_s)^2$ in the Appendix A.4. Substituting for $P_{xy}^{(1)}$ and $P_{xz}^{(1)}$ in A.2.29 and A.2.30, dropping the fourth order moment terms $R_3 - 4R_1$ and $R_1 - R_2/4$ since both are $\sim (P_\parallel - P_\perp)$, and summing, the total FLR heat flow becomes

$$
\begin{aligned}
(\underline{q}_\perp^{\perp(1)} + \underline{q}_\perp^{\parallel(1)})\cdot\hat{x} = \frac{5}{2\Omega_+}\Bigg[& \frac{(P_1^{(0)+})^2}{2\Omega_+\rho_1}\frac{B_{z_1}^2}{B_1^2}\left(1+3\frac{B_x^2}{B_1^2}\right)\frac{d^2u}{dx^2} \\
& + \frac{5}{2}\frac{B_y^2}{B_1^2}\frac{(P_1^{(0)+})^2}{\rho_1\Omega_+}\frac{d^2u}{dx^2} + \frac{(P_1^{(0)+})^2}{2\rho_1\Omega_+}\frac{B_xB_y}{B_1^2}\left(\frac{B_{z_1}^2}{B_1^2}-\frac{2B_x^2}{B_1^2}\right)\frac{B_x}{4\pi\rho_1u_1}\frac{d^2B_y}{dx^2} \\
& + \frac{(P_1^{(0)+})^2}{2\Omega_+\rho_1}\frac{B_xB_{z_1}}{B_1^2}\left(1-3\frac{B_x^2}{B_1^2}\right)\frac{B_x}{4\pi\rho_1u_1}\frac{d^2B_z}{dx^2}\Bigg] \qquad (5.17)
\end{aligned}
$$

Note that 5.17 vanishes for $B_z = 0$, that is for parallel shocks.

Substituting for $T_{xx}^{(1)}$ from A.4.1, $T_{xz}^{(1)}$ from A.4.3, and 5.7 into 4.6, the y component of Ohm's law becomes

$$
\begin{aligned}
& \frac{c^2}{\omega_{p_-}^2}u_1\frac{d^2B_z}{dx^2} - \frac{7}{12}\frac{P_1^{(0)+}/\rho_1}{u_1^2-\frac{5P_1^{(0)}}{3\rho_1}}\left(1+3\frac{B_x^2}{B_1^2}\right)\left(1-\frac{9}{7}\frac{B_{z_1}^2}{B_1^2}\right)\frac{B_{z_1}^2}{B_1^2}R_+^2B_z\frac{d^2u}{dx^2} \\
& - \frac{7}{12}\frac{P_1^{(0)+}/\rho_1}{u_1^2-\frac{5P_1^{(0)}}{3\rho_1}}\left(1-3\frac{B_x^2}{B_1^2}\right)\left(1-\frac{9}{7}\frac{B_x^2}{B_1^2}\right)\frac{B_x^2}{B_1^2}\frac{B_{z_1}^2}{4\pi\rho_1u_1}R_+^2\frac{d^2B_z}{dx^2} \\
& - \frac{5}{12}\frac{P_1^{(0)+}/\rho_1}{u_1^2-\frac{5P_1^{(0)}}{3\rho_1}}\frac{B_y^2}{B_1^2}\left(7-9\frac{B_x^2}{B_1^2}\right)B_zR_+^2\frac{d^2u}{dx^2} - \frac{P_1^{(0)+}}{4\rho_1u_1^2}\left(1+3\frac{B_x^2}{B_1^2}\right)\left(\frac{B_{z_1}^2}{B_1^2}-\frac{2B_x^2}{B_1^2}\right)\frac{B_x}{B_1}B_zR_+^2\frac{d^2u}{dx^2} \\
& - \frac{P_1^{(0)+}}{4\rho_1u_1^2}\left(1-3\frac{B_x^2}{B_1^2}\right)\left(\frac{B_{z_1}^2}{B_1^2}-\frac{2B_x^2}{B_1^2}\right)\frac{B_x^2}{B_1^2}\frac{B_x}{4\pi\rho_1u_1}R_+^2\frac{d^2B_z}{dx^2} + \frac{P_1^{(0)+}/\rho_1}{u_1^2-\frac{5P_1^{(0)}}{3\rho_1}}\frac{B_yB_{z_1}}{2B_1^2}\left(1+3\frac{B_x^2}{B_1^2}\right)\frac{cB_x}{4\pi N_1e}\frac{dB_z}{dx} \\
& + \frac{c^2}{\omega_{p_-}^2}\nu_{ei}\frac{dB_z}{dx} - \frac{cB_x}{4\pi N_1e}\frac{dB_y}{dx}\Bigg[1+\frac{1}{2}\frac{P_1^{(0)+}/\rho_1}{u_1^2-\frac{5P_1^{(0)}}{\rho_1}}\left(1+3\frac{B_x^2}{B_1^2}\right)\frac{B_{z_1}^2}{B_1^2} \\
& - \frac{1}{2}\frac{P_1^{(0)+}}{\rho_1u_1^2}\left(\frac{B_{z_1}^2}{B_1^2}-\frac{2B_x^2}{B_1^2}\right)\frac{B_x^2}{B_1^2}\Bigg] + \frac{10\pi P_1^{(0)+}}{B_1^2}\frac{cB_x}{4\pi N_1e}\frac{B_y}{u}\frac{du}{dx} \qquad (5.18) \\
& - \frac{\frac{5}{3}\frac{P_1^{(0)+}}{\rho_1}}{u_1^2-\frac{5P_1^{(0)}}{3\rho_1}}\frac{B_yB_{z_1}}{B_1^2}\left(\frac{B_{z_1}^2}{B_1^2}-\frac{2B_x^2}{B_1^2}\right)\frac{B_x^2}{4\pi\rho_1u_1}R_+^2\frac{d^2B_y}{dx^2} = u'(\underline{B})B_z \\
& \qquad - u_1B_{z_1} - \frac{B_x^2(B_z-B_{z_1})}{4\pi\rho_1u_1}
\end{aligned}
$$

After a similar series of substitutions, the z-component of Ohm's law 4.7 becomes

$$
\frac{c^2}{\omega_p^2} u_1 \frac{d^2 B_z}{dx^2} - \frac{7}{12} \frac{P_1^{(0)+}/\rho_1}{u_1^2 - \frac{5P_1^{(0)}}{3\rho_1}} \left(1 + 3\frac{B_x^2}{B_1^2}\right)\left(1 - \frac{9}{7}\frac{B_x^2}{B_1^2}\right) \frac{B_{z1}^2}{B_1^2} R_+^2 B_y \frac{d^2 u}{dx^2}
$$

$$
- \frac{5 P_1^{(0)+}}{4\rho_1 u_1^2}\left(1 - 3\frac{B_x^2}{B_1^2}\right)\frac{B_x^2}{B_1^2} R_+^2 B_y \frac{d^2 u}{dx^2} - \frac{5}{12}\frac{P_1^{(0)+}/\rho_1}{u_1^2 - \frac{5P_1^{(0)}}{3\rho_1}} \frac{B_y^2}{B_1^2}\left(7 - 9\frac{B_x^2}{B_1^2}\right) B_y R_+^2 \frac{d^2 u}{dx^2}
$$

$$
- \frac{P_1^{(0)+}}{4\rho_1 u_1^2}\left(1 - 3\frac{B_x^2}{B_1^2}\right)\left(\frac{B_{z1}^2}{B_1^2} - \frac{2B_x^2}{B_1^2}\right)\frac{B_x^2}{B_1^2}\frac{B_x^2}{4\pi\rho_1 u_1} R_+^2 \frac{d^2 B_y}{dx^2}
$$

$$
- \frac{5}{3}\frac{P_1^{(0)+}/\rho_1}{u_1^2 - \frac{5P_1^{(0)}}{3\rho_1}} \frac{B_y^2}{B_1^2}\left(\frac{B_{z1}^2}{B_1^2} - \frac{2B_x^2}{B_1^2}\right)\frac{B_x^2}{4\pi\rho_1 u_1} R_+^2 \frac{d^2 B_y}{dx^2} + \frac{c^2}{\omega_p^2}\nu_{ei}\frac{dB_y}{dx}
$$

$$
+ \frac{cB_x}{4\pi N_1 e}\frac{dB_z}{dx}\left[1 - \frac{P_1^{(0)+}}{2\rho_1 u_1^2}\left(1 - 3\frac{B_x^2}{B_1^2}\right)\frac{B_x^2}{B_1^2}\right] + \frac{P_1^{(0)+}/\rho_1}{u_1^2 - \frac{5P_1^{(0)}}{3\rho_1}}\frac{B_y^2}{2B_1^2}\left(1 + 3\frac{B_x^2}{B_1^2}\right)\frac{cB_x}{4\pi N_1 e}\frac{dB_z}{dx} \tag{5.19}
$$

$$
- \frac{2\pi P_1^{(0)+}}{B_1^2}\left(1 + 3\frac{B_x^2}{B_1^2}\right)\frac{cB_x}{4\pi N_1 e}\frac{B_z}{u}\frac{du}{dx} = U'(\underline{B})B_y - \frac{B_x^2 B_y}{4\pi\rho_1 u_1}
$$

Again, 5.18 and 5.19 are not yet completely reduced since dU/dx and d^2U/dx^2 occur; these could be eliminated by differentiating 5.16 or by the approximation mentioned in section 5.3. Further note that in 5.13, 5.14, 5.18 and 5.19, the coefficient of some of the FLR terms is proportional to $(U^2 - C_s^2)^{-1}$, so that the familiar difficulty with the sonic point remains. Recall also that the above differential equations are valid only if $B_z \neq 0$. For parallel propagation the low β slow shock and at least the leading edge of the high β fast shock have no magnetic structure since the shock steepens from an electrostatic wave. For these shocks Debye scale lengths become important (Tidman, 1967) and must be included in any attempt at a laminar differential equation.

Even though these wave train differential equations have been drastically simplified by invoking the weak shock approximation to linearize the coefficients of the derivatives, and by expanding the derivatives to lowest order in $(R_+/L_s)^2$, they are still far too difficult to solve analytically. Most of the physical information about the solutions can be obtained by studying their behavior near the stationary points, a familiar method in the study of non-linear differential equations which permits determination of the phase plane structure.

6.0 Dispersion Properties of the Wave Train Differential Equation

6.1 Introduction

Although perhaps suitable for numerical integration in the weak shock limit, the wave train differential equations derived in section 5.0 do not reveal the maximum information obtainable about the shock structure. As may be recalled from the discussion of section 2.0, the investigation of a differential equation linearized about the Rankine-Hugoniot stationary points determined the general physical features, although not the precise details, of the shock structure. Furthermore the discussion of section 2.3 demonstrated that in the analysis by means of the linearized equations the weak shock approximation is overly restrictive and may even be unnecessary. In this section the linearization approach will be pursued to determine the laminar wave train structure of fast and slow shocks.

On comparing 5.13 and 5.14 with 5.18 and 5.19 in the oblique fast shock limit, i.e., drop all dispersion terms $\mathcal{O}(B_x^2/B^2)$ note that the only significant difference between them, aside from small differences in the numerical factors multiplying the derivative terms, is the presence of pressure anisotropy corrections in the ion inertial term of 5.13. Therefore, to limit the discussion below to reasonable length, only the linearized analogues of 5.18 and 5.19 are discussed; the effects of pressure anisotropy on the oblique fast wave are then commented on separately. In section 6.2 the coupled linearized differential equations are derived starting from 4.1-4.7. Sections 6.3-6.6 analyze these equations for fast and slow shocks in the near-perpendicular and near-parallel limits. In order to resolve the difficulty with the sonic transition, the fast high-β perpendicular shock with viscous dissipation is discussed in section 6.7. Section 6.8 summarizes this aspect of the work.

6.2 Linearized Wave Train Differential Equation

At the upstream or downstream Rankine-Hugoniot points U, B_z and V_z are finite but $B_y = V_y = 0$; all derivatives on plasma quantities also vanish. Assuming that $P_{\perp}^{(1)} = P_{\parallel}^{(1)}$, the linearized form of 4.2-4.7 becomes

$$\rho u\,\delta u + \delta P_{xx}^{(1)} + \frac{B_z\,\delta B_z}{4\pi} = 0 \tag{6.1}$$

$$\rho u\,\delta V_y + \delta P_{xy}^{(1)} - \frac{B_x\,\delta B_y}{4\pi} = 0 \tag{6.2}$$

$$\rho u\,\delta V_z + \delta P_{xz}^{(1)} - \frac{B_x\,\delta B_z}{4\pi} = 0 \tag{6.3}$$

$$\begin{aligned} &\rho u\left[u\,\delta u + V_z\,\delta V_z\right] + \tfrac{3}{2}\,u\,\delta P^{(1)} + \tfrac{3}{2}\,P^{(1)}\,\delta u \\ &+ u\,\delta P_{xx}^{(1)} + P_{xx}^{(1)}\,\delta u + V_z\,\delta P_{xz}^{(1)} + \frac{u_1 B_{z_1}\,\delta B_z}{4\pi} \\ &+ \left(\delta \underline{q}_{\perp}^{\perp(1)} + \delta \underline{q}_{\perp}^{\parallel(1)}\right)\cdot\hat{x} = 0 \end{aligned} \tag{6.4}$$

$$\frac{c^2}{\omega_p^2} u \frac{d^2 \delta B_z}{dx^2} + \frac{c^2}{\omega_p^2} V_{ei} \frac{d\delta B_z}{dx} - \frac{cB_x}{4\pi Ne} \frac{d\delta B_y}{dx} = u\,\delta B_z + B_z\,\delta u - B_x\,\delta V_z \tag{6.5}$$

$$\frac{c^2}{\omega_p^2} u \frac{d^2 \delta B_y}{dx^2} + \frac{c^2}{\omega_p^2} V_{ei} \frac{d\delta B_y}{dx} + \frac{cB_x}{4\pi Ne} \frac{d\delta B_z}{dx} = u\,\delta B_y - B_x\,\delta V_y \tag{6.6}$$

The linearized forms of $P^{(1)}_{xx} = P^{(1)} + T^{(1)}_{xx}$, $P^{(1)}_{xy}$, and $P^{(1)}_{xz}$ are obtained from A.4.1-A.4.3. Note that the coefficients of all derivative terms are to be evaluated at either the upstream or downstram stationary points.

Using 6.1 to eliminate $\delta P^{(1)}$, 6.3 and 4.4 to eliminate δV_z and V_z, 6.4 becomes

$$\left(\rho u^2 - \frac{5}{3} P^{(1)}\right)\delta u + u\,\delta T^{(1)}_{xx} + \frac{5}{3}\frac{u B_z \delta B_z}{4\pi} - \frac{2}{3}\frac{B_x^2 \delta B_z}{(4\pi)^2 \rho u} - \frac{2}{3}\frac{u_1 B_{z_1} \delta B_z}{4\pi} - \frac{2}{3}\left(\delta \underline{q}_{\perp}^{\perp(1)} + \delta \underline{q}_{\perp}^{\parallel(1)}\right)\cdot\hat{x} = 0 \tag{6.7}$$

4.6 and 4.4 evaluated at a stationary point yield

$$u_1 B_{z_1} = u B_z - \frac{B_x^2 (B_z - B_{z_1})}{4\pi \rho u} \tag{6.8}$$

Substituting 6.8 into 6.7, the following expression for δU is obtained

$$\delta U = \frac{1}{\rho(U^2 - C_s^2)}\Big[-U\,\delta T_{xx}^{(1)} - \frac{U B_z \delta B_z}{4\pi} + \frac{2}{3}\left(\delta \underline{q}_\perp^{\perp(1)} + \delta \underline{q}_\perp^{\parallel(1)}\right)\cdot\hat{x}\Big] \tag{6.9}$$

where $C_s^2 = (5/3)(P^{(1)}/\rho)$.

Substituting 6.9 for δU and 6.3 for δV_z, 6.5 becomes

$$\frac{c^2}{\omega_{p-}^2} U \frac{d^2 \delta B_z}{dx^2} + \frac{c^2}{\omega_{p-}^2} \nu_{ei} \frac{d\delta B_z}{dx} - \frac{c B_x}{4\pi N e}\frac{d\delta B_y}{dx} - \frac{B_x}{\rho U}\delta T_{xz}^{(1)} + \frac{U B_z}{\rho(U^2 - C_s^2)}\delta T_{xx}^{(1)} - \frac{2}{3}\frac{B_z}{\rho(U^2 - C_s^2)}\left(\delta \underline{q}_\perp^{\perp(1)} + \delta \underline{q}_\perp^{\parallel(1)}\right)\cdot\hat{x} = \frac{(U^2 - C_F^2)(U^2 - C_{SL}^2)}{(U^2 - C_s^2)U}\,\delta B_z \tag{6.10}$$

The FLR terms contain derivatives with respect to U which must be eliminated to obtain a homogeneous equation in δB_z. Noting from 6.5 and 6.8 that in hydromagnetics, neglecting ν_{ei}, δU is

$$\delta U = -\left(1 - \frac{C_I^2}{U^2}\right)\frac{U\,\delta B_z}{B_z} \tag{6.11}$$

where $C_I^2 = B_x^{\,2}/4\pi\rho$; 6.11 is valid only if $B_z \neq 0$. Therefore the zero order δU of 6.11 can be substituted into the $\mathcal{O}(\epsilon^2)$ FLR terms. Eliminating $\delta T_{xx}^{(1)}$ and $\delta T_{xz}^{(1)}$ by A.4.1 and A.4.3, $(\delta \underline{q}_{\perp}^{\perp(1)} + \delta \underline{q}_{\perp}^{\parallel(1)}) \cdot \hat{x}$ by 5.17, 6.10 can be written as

$$\frac{c^2}{\omega_p^2}\frac{d^2\delta B_z}{dx^2} + D R_+^2 \frac{d^2 \delta B_z}{dx^2} + \frac{c^2}{\omega_p^2}\frac{\nu_i}{u}\frac{d\delta B_z}{dx} - \frac{c_I}{u}\frac{c}{\omega_{p+}} F \frac{d\delta B_y}{dx} = C_z \delta B_z \tag{6.12}$$

where

$$C_z = \frac{(u^2 - c_F^2)(u^2 - c_{sL}^2)}{u^2(u^2 - c_s^2)} \tag{6.13}$$

$$\begin{aligned} D = {} & \frac{7}{12}\frac{p^{(0)+}/\rho}{u^2 - c_s^2}\frac{B_z^2}{B^2}\left(1 + 3\frac{B_x^2}{B^2}\right)\left(1 - \frac{5}{7}\frac{B_x^2}{B^2}\right)\left(1 - \frac{c_I^2}{u^2}\right) \\ & - \frac{7}{12}\frac{p^{(0)+}/\rho}{u^2 - c_s^2}\left(1 - 3\frac{B_x^2}{B^2}\right)\left(1 - \frac{5}{7}\frac{B_x^2}{B^2}\right)\frac{B_z^2}{B^2}\frac{c_I^2}{u^2} \\ & + \frac{p^{(0)+}}{4\rho u^2}\left(1 + 3\frac{B_x^2}{B^2}\right)\left(\frac{B_z^2}{B^2} - \frac{2B_x^2}{B^2}\right)\frac{B_x^2}{B^2}\left(1 - \frac{c_I^2}{u^2}\right) \\ & - \frac{p^{(0)+}}{4\rho u^2}\left(1 - 3\frac{B_x^2}{B^2}\right)\left(\frac{B_z^2}{B^2} - \frac{2B_x^2}{B^2}\right)\frac{B_x^2}{B^2}\frac{c_I^2}{u^2} \end{aligned} \tag{6.14}$$

$$F = 1 + \frac{1}{2}\,\frac{P^{(0)+}/\rho}{u^2 - c_s^2}\left(1 + 3\,\frac{B_x^2}{B^2}\right)\frac{B_z^2}{B^2}$$

$$- \frac{1}{2}\,\frac{P^{(0)+}}{\rho u^2}\left(\frac{B_z^2}{B^2} - \frac{2B_x^2}{B^2}\right)\frac{B_x^2}{B^2} \tag{6.15}$$

Note that the term in 5.18 proportional to $(B_y/U)\,(dU/dx)$ does not appear in 6.12 since it is of order $\delta B_y\,\delta U$.

A similar series of substitutions reduces 6.6 to

$$\frac{c^2}{\omega_{p-}^2}\,\frac{d^2\delta B_y}{dx^2} + G R_+^2\,\frac{d^2\delta B_y}{dx^2} + \frac{c^2}{\omega_{p-}^2}\,\frac{\nu_{ei}}{u}\,\frac{d\,\delta B_y}{dx}$$

$$+ \frac{c_I}{u}\,\frac{c}{\omega_{p+}}\,H\,\frac{d\,\delta B_z}{dx} = C_y\,\delta B_y \tag{6.16}$$

where

$$C_y = 1 - \frac{c_I^2}{u^2} \tag{6.17}$$

$$G = -\frac{P^{(0)+}}{4\rho u^2}\left(1 - 3\,\frac{B_x^2}{B^2}\right)\left(\frac{B_z^2}{B^2} - \frac{2B_x^2}{B^2}\right)\frac{B_x^2}{B^2}\,\frac{c_I^2}{u^2} \tag{6.18}$$

$$H = 1 - \frac{1}{2}\,\frac{P^{(0)+}}{\rho u^2}\left(1 - 3\,\frac{B_x^2}{B^2}\right)\frac{B_x^2}{B^2}$$

$$+ \frac{2\pi P^{(0)+}}{B^2}\left(1 + 3\,\frac{B_x^2}{B^2}\right)\left(1 - \frac{c_I^2}{u^2}\right) \tag{6.19}$$

Again note that the terms in 5.19 ~ $B_y(d^2U/dx^2)$ do not appear in 6.16.

If $B_z = 0$, the substitution 6.11 is invalid; the correct equation follows immediately, however, from 6.10 after substituting for $\delta T_{xz}^{(1)}$

$$\frac{c^2}{\omega_p^2}\frac{d^2\delta B_z}{dx^2} - \frac{p^{(1)+}}{\rho u^2}\frac{c_I^2}{u^2}R_+^2\frac{d^2\delta B_z}{dx^2} + \frac{c^2}{\omega_p^2}\frac{\nu_{ei}}{u}\frac{d\delta B_z}{dx} - \frac{c_I}{u}\frac{c}{\omega_{p+}}\frac{d\delta B_y}{dx} = \frac{u^2 - c_A^2}{u^2}\delta B_z \tag{6.20}$$

Similarly the equation for δB_y is

$$\frac{c^2}{\omega_p^2}\frac{d^2\delta B_y}{dx^2} - \frac{p^{(1)+}}{\rho u^2}\frac{c_I^2}{u^2}R_+^2\frac{d^2\delta B_y}{dx^2} + \frac{c^2}{\omega_p^2}\frac{\nu_{ei}}{u}\frac{d\delta B_y}{dx} + \frac{c_I}{u}\frac{c}{\omega_{p+}}\frac{d\delta B_y}{dx} = \frac{u^2 - c_A^2}{u^2}\delta B_y \tag{6.21}$$

Note that except for an opposite sign in the ion inertia term, which is just a phase difference, 6.20 and 6.21 are identical as required by symmetry. Recall, however, that for $B_z = 0$ these equations are not valid for low β slow shocks since they do not include Debye length dispersion; also the $\beta > 1$ parallel slow shock is of zero strength.

In analyzing 6.12 and 6.16 it is convenient to have asymptotic forms for 6.14, 6.15, 6.18, and 6.19 in the near perpendicular and near parallel propagation limits; the limits for fast and slow shocks differ.

1. Fast Shocks

a. Near-perpendicular - $B_x/B << 1$; $C_I/U << 1$

$$D \sim \frac{7}{12} \frac{P^{(0)+}/\rho}{U^2 - C_s^2} \tag{6.22}$$

$$F \sim 1 + \frac{1}{2} \frac{P^{(0)+}/\rho}{U^2 - C_s^2} \tag{6.23}$$

$$G \sim -\frac{P^{(0)+}}{4\rho U^2} \frac{B_x^2}{B^2} \frac{C_I^2}{U^2} \tag{6.24}$$

$$H \sim 1 + \frac{2\pi P^{(0)+}}{B^2} \tag{6.25}$$

b. Near-parallel - $B_z/B << 1$; $B_z^2/B^2 \cdot$

$$[(P^{(0)+}/\rho)/(U^2 - C_s^2)] << 1.$$

$$D \sim -\frac{P^{(0)+}}{\rho U^2}(2 - C_I^2/U^2) \tag{6.26}$$

$$F \sim 1 + \frac{P^{(0)+}}{\rho U^2} \tag{6.27}$$

$$G \sim -\frac{2 P^{(0)+}}{\rho U^2} \frac{C_I^2}{U^2} \tag{6.28}$$

$$H \sim 1 + \frac{P^{(0)+}}{\rho u^2} + \frac{8\pi P^{(0)+}}{B^2}\left(1 - \frac{C_T^2}{u^2}\right) \tag{6.29}$$

Equations 6.26-6.29 are not valid about the downstream point for very strong switch-on fast shocks since B_z/B need no longer be small. If $B_z/B \sim 1$, 6.22-6.25 would be appropriate for the strong switch-on limit. Furthermore, note that for $B_x/B << 1$, $G \sim (B_x/B)^4$ is much smaller than the other terms. Hence for the high β oblique fast wave the FLR coupling to B_y is very weak and only ion inertia couples the two polarizations.

2. Slow Shocks

a. Near-perpendicular - $B_x/B << 1$, $U << C_s$

$$D \sim \left(\frac{7}{12}\frac{P^{(0)+}}{\rho C_s^2} - \frac{1}{4}\frac{P^{(0)+}}{\rho u^2}\frac{B_x^2}{B^2}\right)\left(2\frac{C_T^2}{u^2} - 1\right) \tag{6.30}$$

$$F \sim 1 - \frac{1}{2}\frac{P^{(0)+}}{\rho C_s^2} - \frac{1}{2}\frac{P^{(0)+}}{\rho u^2}\frac{B_x^2}{B^2} \tag{6.31}$$

$$G \sim -\frac{P^{(0)+}}{4\rho u^2}\frac{C_T^2}{u^2}\frac{B_x^2}{B^2} \tag{6.32}$$

$$H \sim 1 - \frac{1}{2}\frac{P^{(0)+}}{\rho u^2}\frac{B_x^2}{B^2} - \frac{2\pi P^{(0)+}}{B^2}\left(\frac{C_T^2}{u^2} - 1\right) \tag{6.33}$$

Note that the terms proportional to B_x^2/B^2 are important for slow oblique shocks since $U^2 \sim B_x^2/B^2\ C_s^2$ for $\beta << 1$ or $U^2 \sim B_x^2/B^2\ C_A^2$ for $\beta >> 1$. Furthermore 6.30-6.33 are invalid about the downstream point of nearly complete switch-off shocks since $B_z/B << 1$; here the appropriate expressions are the near parallel ones. Assuming for $\beta << 1$ that $U << C_I$, the signs of 6.30-6.33 are $D > 0$, $F > 0$, $G < 0$, $H > 0$; similarly for $\beta >> 1$, $D < 0$, $F < 0$, $G < 0$, and $H < 0$. Actually the above signs hold even if $U_1 = C_{I_1}$, the maximum strength switch-off shock.

b. Near-parallel $B_z/B << 1$, $(B_z^2/B^2[(P^{(0)+}/\rho)/U^2 - C_s^2)]$ $<< 1$;

$$D \sim \frac{P^{(0)+}}{\rho u^2}\left(\frac{C_I^2}{u^2} - 2\right) \tag{6.34}$$

$$F \sim 1 + \frac{P^{(0)+}}{\rho u^2} \tag{6.35}$$

$$G \sim -\frac{P^{(0)+}}{\rho u^2}\frac{C_I^2}{u^2} \tag{6.36}$$

$$H \sim 1 + \frac{P^{(0)+}}{\rho u^2} - \frac{8\pi P^{(0)+}}{B^2}\left(\frac{C_I^2}{u^2} - 1\right) \tag{6.37}$$

Note that in going from $\beta << 1$ to $\beta >> 1$, D changes sign. Recall that for $B_z = 0$, $\beta > 1$, $U_1 = C_I$ and the slow shock is of zero strength.

Equations 6.12 and 6.16 form a set of coupled linear differential equations for δB_z and δB_y which are easily solved by assuming $\delta B_z = A_z \exp(\lambda x)$ and $\delta B_y = A_y \exp(\lambda x)$. Substitution yields the following fourth order algebraic equation for λ

$$\begin{aligned}
&\left(\frac{c^2}{\omega_{p_-}^2} + DR_+^2\right)\left(\frac{c^2}{\omega_{p_-}^2} + GR_+^2\right)\lambda^4 + \frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}\left[2\frac{c^2}{\omega_{p_-}^2}\right. \\
&\left. + (G+D)R_+^2\right]\lambda^3 + \left[-\left(\frac{c^2}{\omega_{p_-}^2} + DR_+^2\right)c_y\right. \\
&\left. -\left(\frac{c^2}{\omega_{p_-}^2} + GR_+^2\right)c_z + HF\frac{c_I^2}{u^2}\frac{c^2}{\omega_{p_+}^2} + \frac{c^4}{\omega_{p_-}^4}\frac{\nu_{ei}^2}{u^2}\right]\lambda^2 \\
&- \left[(c_y + c_z)\frac{\nu_{ei}}{u}\frac{c^2}{\omega_{p_-}^2}\right]\lambda + c_y c_z = 0
\end{aligned} \tag{6.38}$$

Note that dissipation occurs in odd powers of λ and dispersion in even powers. Setting $\nu_{ei} = 0$, 6.38 has solutions

$$\begin{aligned}
\lambda^2 = &\frac{c_y\left(\frac{c^2}{\omega_{p_-}^2} + DR_+^2\right) + c_z\left(\frac{c^2}{\omega_{p_-}^2} + GR_+^2\right) - HF\frac{c_I^2}{u^2}\frac{c^2}{\omega_{p_+}^2}}{2\left(\frac{c^2}{\omega_{p_-}^2} + DR_+^2\right)\left(\frac{c^2}{\omega_{p_-}^2} + GR_+^2\right)} \\
&\pm\left[\left(\frac{-c_y\left(\frac{c^2}{\omega_{p_-}^2} + DR_+^2\right) - c_z\left(\frac{c^2}{\omega_{p_-}^2} + GR_+^2\right) + HF\frac{c_I^2}{u^2}\frac{c^2}{\omega_{p_+}^2}}{2\left(\frac{c^2}{\omega_{p_-}^2} + DR_+^2\right)\left(\frac{c^2}{\omega_{p_-}^2} + GR_+^2\right)}\right)^2\right. \\
&\left. - \frac{c_y c_z}{\left(\frac{c^2}{\omega_{p_-}^2} + DR_+^2\right)\left(\frac{c^2}{\omega_{p_-}^2} + GR_+^2\right)}\right]^{1/2}
\end{aligned} \tag{6.39}$$

The relative magnitudes of A_z and A_y are

$$\frac{A_z}{A_y} = \frac{\frac{C_I}{u}\frac{c}{\omega_{p_+}} F \lambda}{\left(\frac{c^2}{\omega_{p_-}^2} + DR_+^2\right)\lambda^2 + \frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}\lambda - C_z} \tag{6.40}$$

$$= \frac{\left(\frac{c^2}{\omega_{p_-}^2} + GR_+^2\right)\lambda^2 + \frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}\lambda - C_y}{H \frac{C_I}{u}\frac{c}{\omega_{p_+}}\lambda} \tag{6.41}$$

The solutions 6.38-6.41, although formally correct, are often too cumbersome to manipulate. In addition, 6.37 gives only the dispersion properties of the solution; to satisfy the boundary conditions that no solutions grow at $x = \pm\infty$, it is necessary to include the dissipation, ν_{ei}. In what follows it is often more convenient to make approximations on 6.12 and 6.16 directly rather than attempt to solve 6.38.

6.3 Near-Perpendicular Fast Shocks

First consider the perpendicular fast shock. Since $C_I = 0$, the ion inertia term drops out and 6.12 and 6.16 decouple. The perpendicular fast wave has only a δB_z polarization so that 6.16 can be neglected. The solution of 6.12 is

$$\lambda_\pm = \frac{-\frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}}{2\left(\frac{c^2}{\omega_{p_-}^2} + DR_+^2\right)} \pm \frac{\left[\frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}^2}{u^2} + C_z\right]^{1/2}}{\left[\frac{c^2}{\omega_{p_-}^2} + DR_+^2\right]^{1/2}} \tag{6.42}$$

If $\beta_+ < M_-/M_+$ and $1 >> \beta_-$, but possibly exceeding M_-/M_+, upon neglect of ν_{ei}^2 compared to C_z the results of section 2.3 are immediately recovered

$$\lambda_{\pm} = -\frac{\nu_{ei}}{2U} \pm \frac{\sqrt{C_z}}{\frac{c}{\omega_{p-}}} \tag{6.43}$$

At the upstream point $C_z > 0$, and the magnetic field undergoes an exponential rise with a scale length given by 2.13. If $U_2 > C_{s_2}$, $C_z < 0$ and the solutions are damped oscillations with the wavelength given by 2.14 and the damping length by 2.15.

When $\beta_+ > M_-/M_+$, ion FLR dispersion dominates electron inertia. The solution at the upstream point is

$$\delta B_z = A_{z_1} \exp\left[\left(\frac{12}{7}\,\frac{C_{F_1}^2}{P_1^{(1)+}/\rho_1}\,(M_{F_1}^2-1)\right)^{1/2}\frac{x}{R_+}\right] \tag{6.44}$$

where 6.22 was substituted for D and ν_{ei} was neglected. About the downstream point the appropriate solution is

$$\delta B_z = A_{z_2}\, \exp\left[-\frac{6}{7}\,\frac{\frac{c^2}{\omega_{p-}^2}\,\frac{\nu_{ei}}{U_2}\,(U_2^2-C_{s_2}^2)}{\frac{P_2^{(2)+}}{\rho_-}\,R_+^2}\right] \cdot$$

$$\sin\left[\left(\frac{12}{7}\,\frac{C_{F_2}^2}{P_2^{(1)+}/\rho_2}\,|M_{F_2}^2-1|\right)^{1/2}\frac{x}{R_+^2}+\psi\right] \tag{6.45}$$

where ψ is a phase factor, which, along with A_{z_2}, can be determined when the origin $x = 0$ is selected. Equation 6.45 describes a trailing wave train with the oscillation wavelength characterized by R_+. If $U_2 > C_{s_2}$, the wave train is damped; however, if $U_2 < C_{s_2}$ as is likely if $\beta_2 > 1$, the solutions grow exponentially unless $\nu_{ei} = 0$. Here resistivity alone fails to provide the required dissipation, and viscosity is necessary to complete the shock transition. Viscous dissipation is considered in section 6.7 where it is found that the R_+ wave train trails and damps out.

Now consider the near perpendicular shock limit when ion inertia dispersion exceeds that of electron inertia, i.e., $C_I/U \; C/\omega_{p_+} > C/\omega_{p_-}$ or $B_x/B > U/C_A \sqrt{M_-/M_+}$. When $\beta_+ < M_-/M_+$, the ion FLR terms can be neglected in 6.39; then since the ion inertia term is larger than the term proportional to C_yC_z, the square root can be expanded. Keeping only the lowest order terms the solutions are

$$\lambda_+^2 \doteq - \frac{C_y C_z}{HF \frac{C_I^2}{U^2} \frac{C^2}{\omega_{p_+}^2}} \tag{6.46}$$

$$\lambda_-^2 \doteq - \frac{HF \frac{C_I^2}{U^2} \frac{C^2}{\omega_{p_+}^2}}{\frac{C^4}{\omega_{p_-}^4}} \tag{6.47}$$

Now note that 6.46 can be obtained from considering only the last three terms of 6.38 whereas 6.47 is obtainable from the first three terms of 6.38. Splitting 6.38 in this manner yields

$$\lambda_+ = \frac{(C_y + C_z)\frac{\nu_{ei}}{u}\frac{c^2}{\omega_{p_-}^2}}{2HF\frac{C_I^2}{u^2}\frac{c^2}{\omega_{p_+}^2}} \pm i\,\frac{\sqrt{C_y C_z}}{\sqrt{HF}\,\frac{C_I}{u}\frac{c}{\omega_{p_+}}} \tag{6.48}$$

$$\lambda_- = -\frac{\nu_{ei}}{u} \pm i\,\frac{\sqrt{HF}\,\frac{C_I}{u}\frac{c}{\omega_{p_+}}}{\frac{c^2}{\omega_{p_-}^2}} \tag{6.49}$$

These solutions have been obtained and exhaustively discussed by Cavaliere and Englemann (1967); therefore, the results will only be briefly mentioned here. Since $C_y\, C_{z_1} > 0$, 6.48 yields a leading C/ω_{p_+} wave train (see Figure 3) which grows exponentially if $\nu_{ei} \neq 0$; the oscillation wavelength is

$$L_s \sim \sqrt{H_1F_1}\left(\frac{u_1^2 - C_{s_1}^2}{C_{F_1}^2(M_{F_1}^2 - 1)}\right)^{1/2}\frac{C_{I_1}}{u_1}\frac{c}{\omega_{p_+}} \tag{6.50}$$

About the downstream point $C_{y_2} C_{z_2} < 0$, so that the solution damps out to a uniform state. For the upstream point 6.49 violates the boundary conditions and must be discarded. The downstream solution consists of very short wavelength damped oscillations, $L_s \sim \sqrt{M_-/M_+}\; C/\omega_{p_-}$, and is probably unphysical.

The short-scale solutions given by the first three terms of 6.38 are a general feature of most of the solutions developed here. Their origin can be understood by considering the dispersion relation for oblique whistlers (Formisano and Kennel, 1969; and equation 7.5) $\omega/k = C_A/[1 + (k^2C^2/\omega_{p_-}^2)]\, kC_A/\Omega_+ \cos\theta$. Setting $\omega/k = U$ and solving for k yields the two solutions $kC_A/\Omega_+ \sim U/C_{A}\cos\theta$ corresponding to 6.48 and $kC/\omega_{p_-} \sim C_A\cos\theta/U\,\sqrt{M_+/M_-}$ corresponding to the short wavelength solutions. From the kinetic theory for linear waves (Stix, 1962) the whistler is critically cyclotron damped by thermal electrons when $kC/\omega_{p_-} >> 1$. Hence the short wavelength solutions probably cannot be excited during the non-linear steepening of the shock, and therefore cannot establish a wave train. A similar situation occurs if the waves are heavily Landau damped.

Although when $1 > \beta_+ > M_-/M_+$ a similar splitting of 6.46 occurs, it is easier to consider 6.12 and 6.16 directly. From 6.18, $G \sim (B_x/B)^4 << 1$ so that in 6.16 both ion FLR and electron inertia are small compared to ion inertia. If ν_{ei} is also small, 6.16 can be solved for δB_y and the result substituted in 6.12 to obtain

$$\frac{c^2}{\omega_{p_-}^2}\frac{d^2\delta B_z}{dx^2} + DR_+^2\frac{d^2\delta B_z}{dx^2} - \frac{HF}{C_y}\frac{G_I^2}{u^2}\frac{c^2}{\omega_{p_+}^2}\frac{d^2\delta B_z}{dx^2} + \frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}\frac{d\delta B_z}{dx} = C_z\,\delta B_z \qquad (6.51)$$

For the same parameters the full wave train differential equations of section 6.0 could be combined to obtain the analogue of 6.51. The ion inertia and ion FLR derivative terms in 6.51 are of opposite sign, a result which was anticipated from the linear theory of section 3.0. When ion inertia dispersion exceeds that of ion FLR, the dispersion term is opposite that of the potential term, and the wave train leads. Alternatively if ion FLR dominates, the wave train trails.

It is possible, however, that the effects of ion inertia and ion FLR dispersion cancel. The cancellation occurs for approximately

$$D R_+^2 \approx \frac{HF}{c_Y} \frac{c_I^2}{u^2} \frac{c^2}{\omega_{p_+}^2} \tag{6.52}$$

or

$$\cos^2\theta \approx \frac{u^2}{u^2 - c_s^2} \frac{c_Y}{HF} \frac{7}{48} \beta_+^2 \tag{6.53}$$

Neither 6.53 nor the assumption of small G severely restrict the critical angle or β_+ for cancellation. Hence the cancellation of ion inertia and ion FLR should occur over a wide range of propagation angles and plasma β's. Recall that F, H, and D were calculated only to $\mathcal{O}(\epsilon^2)$ so that 6.53 is only approximate; however since the corrections to 6.53 are small, $\mathcal{O}(\epsilon^3)$, the above cancellation will still occur.

At the cancellation point the only remaining dispersion term is electron inertia. If $\beta_- \sim 1$, electron FLR dispersion must also be considered; this calculation is performed in section 7.0 where it is found that electron FLR contributes a dispersion term of the same sign as electron inertia. Since 6.51 determines the wave that stands in the flow, when ion inertia and ion FLR cancel, the only such wave has $C/\omega_{p_-} - R_-$ scale lengths. Care must be taken here, however. It would take only a slight deviation away from precise cancellation to also cancel the $C/\omega_{p_-} - R_-$ terms; on the other hand, the same slight deviation in the opposite sense would permit many oscillations of the $C/\omega_{p_-} - R_-$ wave train within one C/ω_{p_+} scale length. If the shock has $C/\omega_{p_-} - R_-$ lengths, the small ion FLR ordering, upon which 6.51 is based, breaks down, and a new calculation based on the ordering $L_s/R_+ << 1$, and $R_-/L_s \leqslant 1$ is required. This calculation is performed in section 7.0. Furthermore even if ion inertia and ion FLR cancel, the shock may not steepen to $C/\omega_{p_-} - R_-$ scale lengths. In section 7.0 it is argued that only sufficiently strong shocks might possess the short $C/\omega_{p_-} - R_-$ scale length structure; the short scale length wave train is the collisionless dispersion analogue of the dissipation discontinuity.

The solutions of 6.51 are

$$\lambda_\pm = \frac{-\frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}}{2\left[\frac{c^2}{\omega_{p_-}^2} + DR_+^2 - \frac{HF}{c_Y}\frac{c_I^2}{u^2}\frac{c^2}{\omega_{p_+}^2}\right]} \pm i\,\frac{\sqrt{c_t}}{\left[\frac{HF}{c_Y}\frac{c_I^2}{u^2}\frac{c^2}{\omega_{p_+}^2} - DR_+^2 - \frac{c^2}{\omega_{p_-}^2}\right]^{1/2}} \tag{6.54}$$

If ion inertia dominates, there is a leading wave train with oscillation wavelength

$$L_s \sim \left[\frac{u_1^2 - c_{s_1}^2}{c_{F_1}^2}\,\frac{1}{M_{F_1}^2 - 1}\right]^{1/2} \cdot \left[\frac{H_1F_1}{c_{Y_1}}\,\frac{c_{I_1}^2}{u_1^2}\,\frac{c^2}{\omega_{p_+}^2} - D_1R_+^2 - \frac{c^2}{\omega_{p_-}^2}\right]^{1/2} \tag{6.55}$$

The downstream solutions damp to a uniform state. If ion FLR dominates, there is an exponential rise in the magnetic field upstream with a scale length given by 6.55 with the signs of the ion inertia and ion FLR terms reversed. About the downstream point there is a trailing wave train given by

$$\delta B_z = A_{z_2} \exp\left[\frac{-\frac{c^2}{\omega_{p-}^2}\frac{\nu_{ei}}{2U_2}\, x}{\frac{c^2}{\omega_{p-}^2} + D_2 R_+^2 - \frac{H_1 F_2}{C_{Y_2}} \frac{C_{I_2}^2}{U_2^2} \frac{c^2}{\omega_{p+}^2}} \right] \cdot \tag{6.56}$$

$$\sin\left[\left(\frac{C_{F_2}^2}{U_2^2 - C_{s_2}^2} \left| M_{F_2}^2 - 1 \right| \right)^{1/2} \left[\frac{c^2}{\omega_{p-}^2} + D_2 R_+^2 - \frac{H_1 F_2}{C_Y} \frac{C_I^2}{U_2^2} \frac{c^2}{\omega_{p+}^2} \right]^{-1/2} x + \psi \right]$$

Note again that if $U_2 < C_{s_2}$, D is negative so that 6.54 is unbounded at $x = \infty$. Solutions for δB_y follow immediately from those of δB_z and 6.40, and won't be written here. Since the only coupling between δB_z and δB_y is ion inertia, if ion FLR dispersion dominates, δB_y is smaller than δB_z by roughly $C_I/U \; 1/\sqrt{\beta_+}$.

Consider the R_+ wave train when $\beta_+ >> 1$; from 6.55 the shock thickness is of order $L_s \sim [(P^{(0)+}/\rho)/C_F^2(M_F^2 - 1)]^{\frac{1}{2}} R_+$. For strong shocks, $M_F^2 - 1 >> 1$, $L_s < R_+$; hence the ion FLR ordering scheme breaks down in the strong shock limit, and 6.55 and 6.56 no longer describe the shock structure. The difficulty arises since FLR dispersion is basically weak, being effective only when $kR_+ \leqslant 1$; if $kR_+ >> 1$, section 7.0 demonstrates that ion FLR no longer contributes to wave dispersion.

Hence ion FLR alone limits the steepening only if the shock is weak; for strong shocks either another dispersion scale length enters or the wave train breaks resulting in a fully turbulent shock (see section 7.0).

Now return to 5.13 and 5.14 and consider the effect of anisotropic pressure on the near perpendicular fast shock. The anisotropy terms on the RHS of the differential equations have little effect since they are multiplied by $B_x^2/B^2 << 1$; the near perpendicular fast wave, driven primarily by total pressure, is relatively insensitive to changes in tension along the magnetic field. Since $R_1 - R_2/4$ is roughly equal to $P_\perp^{(0)+}/\rho\ (P_\parallel - P_\perp)$, the coefficient of the ion inertia term is approximately

$$1 + \frac{P_{\perp 1}^{(0)+}/\rho_1}{u_1^2 - \frac{\gamma P_{\perp 1}^{(0)}}{\rho_1}}\left[1 - \frac{4\pi\left(P_{\parallel 1}^{(0)} - P_{\perp 1}^{(0)}\right)}{B_1^2}\right]$$

Note that $1 - [4\pi(P_\parallel - P_\perp)/B^2]$ is the usual factor multiplying the Alfvén speed in CGL linear wave theory. For $1 - [4\pi(P_\parallel - P_\perp)/B^2]$ to be negative, and hence have the possibility of changing the sign of the ion inertia term, β must exceed unity (Kennel and Sagdeev, 1967a). However when $\beta > 1$, the ion FLR terms dominate the dispersion so that slight changes in the ion inertia term cannot be too important. Also note that if $1 - [4\pi(P_\parallel - P_\perp)/B^2]$ < 0, the flow is unstable to firehose waves, and the whole fluid approach breaks down.

6.4 Near-Parallel Fast Shocks

As discussed above for propagation at small angles to the magnetic field, 6.12 and 6.16 for $B_z \neq 0$, or 6.20 and 6.21 for $B_z = 0$ are of restricted validity. The leading edge of the $\beta < 1$ parallel fast shock can be investigated with 6.20 and 6.21; since this is the switch-on shock, however, 6.12 and 6.16 are needed about the downstream point. The $\beta > 1$ parallel fast shock and the $\beta < 1$ parallel slow shock cannot be investigated from the above equations since Debye length dispersion is required (recall from section 3.0 that the corresponding linear waves were non-dispersive when $k_\perp = 0$). These shocks are perhaps better investigated from the Vlasov equation directly (Bernstein et al., 1957; Montgomery and Joyce, 1969). For the $\beta > 1$ fast shock propagating at angles $\theta >> \lambda_D/R_+ \sim \Omega_+/\omega_{p_+}$, 6.12 and 6.16 are valid; similarly for $\theta >> \lambda_D/(C/\omega_{p_+}) \sim C_+/C$ the $\beta < 1$ slow shock can be treated by 6.12 and 6.16. Note that for typical solar wind conditions $\Omega_+/\omega_{p_+} << 1$, $C_+/C << 1$.

Turning now to the full expression for λ, (6.39), if $\beta_+ \leqslant 1$, ion inertia dominates ion FLR and electron inertia, and the radical can be expanded in an analogous fashion as for the fast oblique shock. The last three terms of 6.3 yield the solutions

$$\lambda_\pm = \frac{(C_y + C_z)\frac{\nu_{ei}}{u}\frac{c^2}{\omega_{p_-}^2}}{\left[\frac{HF\,C_x^2}{u^2}\frac{c^2}{\omega_{p_+}^2} - DR_+^2 C_y - GR_+^2 C_z\right]} \tag{6.57}$$

$$\pm\, i\,\frac{\sqrt{C_y C_z}}{\left[\frac{HF\,C_x^2}{u^2}\frac{c^2}{\omega_{p_+}^2} - DR_+^2 C_y - GR_+^2 C_z\right]^{1/2}}$$

Since D and G are negative, no cancellation between dispersion terms is possible; C/ω_{p_-} dispersion has been dropped. About the upstream point C_{y_1}, $C_z > 0$, and 6.57 describes a leading wave train with an oscillation wavelength of the order of

$$L_s \sim \sqrt{H_1F_1}\,\frac{C_{I_1}}{(U_1^2 - C_{I_1}^2)^{1/2}}\left[\frac{U_1^2 - C_{S_1}^2}{C_{F_1}^2(M_{F_1}^2 - 1)}\right]^{1/2}\frac{C}{\omega_{p_+}} \tag{6.58}$$

if $\beta_+ < 1$. The solutions of 6.57 about the downstream point damp exponentially to a uniform state if $U_2 > C_{s_2}$ and are exponentially growing oscillations if $U_2 < C_{s_2}$.

The solutions given by the first three terms of 6.38 are

$$\lambda_\pm = \frac{-\frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}\left[2\frac{c^2}{\omega_{p_-}^2} + (G+D)R_+^2\right]}{\left(\frac{c^2}{\omega_{p_-}^2} + DR_+^2\right)\left(\frac{c^2}{\omega_{p_-}^2} + GR_+^2\right)}$$

$$\pm i\,\frac{\left[HF\frac{C_I^2}{u^2}\frac{c^2}{\omega_{p_+}^2} - C_yDR_+^2 - C_zGR_+^2\right]^{1/2}}{\left[\left(c^2/\omega_{p_-}^2 + DR_+^2\right)\left(c^2/\omega_{p_-}^2 + GR_+^2\right)\right]^{1/2}} \tag{6.59}$$

If $\beta_+ < M_-/M_+$, the same result as 6.49 is obtained giving a very short scale length trailing wave train, $L_s \sim \sqrt{M_-/M_+}\,(C/\omega_{p_-})$; the upstream solution violates the $x = -\infty$ boundary condition. If $1 > \beta_+ > M_-/M_+$, 6.59 describes a leading wave train with an oscillation length

$$L_s \sim \frac{P^{(0)+}}{\rho u^2}\,\frac{R_+^2}{c/\omega_{p_+}} \tag{6.60}$$

which is much less than the C/ω_{p_+} wave train given by 6.58. The downstream solutions are exponentially growing. These short scale length solutions are probably unphysical since the corresponding linear waves, which form the wave train, should be critically Landau or cyclotron damped.

When $\beta_+ >> 1$, ion inertia is unimportant so that 6.12 and 6.16 are almost decoupled. Here 6.38 splits into two quadratic forms with solutions

$$\lambda_+ = \frac{\frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}}{|G|R_+^2} \pm \frac{i\sqrt{c_y}}{\sqrt{|G|}\,R_+} \tag{6.61}$$

$$\lambda_- = \frac{\frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}}{|D|R_+^2} \pm \frac{i\sqrt{c_z}}{\sqrt{|D|}\,R_+} \tag{6.62}$$

Equation 6.61 (6.62) is the solution of 6.16 (6.12). The ion inertia corrections to 6.61 and 6.62 are obtained by expanding 6.39 to lowest order in $C^2/\omega_{p_+}^2$. The results are

$$\lambda_+ = \frac{\frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}}{|G|R_+^2} \pm i\frac{\sqrt{c_y}}{\sqrt{|G|}R_+}\left(1 + \frac{HF\frac{c_I^2}{u^2}\frac{c^2}{\omega_{p_+}^2}}{(|D|c_y - |G|c_z)R_+^2}\right)^{1/2} \tag{6.63}$$

$$\lambda_- = \frac{\frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}}{|D|R_+^2} \pm i\frac{\sqrt{c_z}}{\sqrt{|D|}R_+}\left(1 - \frac{HF\frac{c_I^2}{u^2}\frac{c^2}{\omega_{p_+}^2}}{(|D|c_y - |G|c_z)R_+^2}\right)^{1/2} \tag{6.64}$$

It is easily shown that $|D|C_y - |G|C_z > 0$.

Equation 6.64 describes a leading wave train in B_z with oscillation scale length

$$L_s \sim \left[\frac{\frac{p_1^{(6)+}}{\rho_1 U_1^2}\left(2 - \frac{C_{I_1}^2}{U_1^2}\right)}{\frac{C_{F_1}^2 (M_{F_1}^2 - 1)}{U_1^2 - C_{S_1}^2}} \right]^{1/2} \left[1 - \frac{H_{F_1} \frac{C_{I_1}^2}{U_1^2} \frac{c^2}{\omega_{p_+}^2}}{(|D_1| C_y - |G_1| C_z) R_+^2} \right]^{-1/2} \tag{6.65}$$

The second bracket in 6.65 is roughly equal to $(1 - 1/\beta_+)^{-\frac{1}{2}}$. If $\beta > 1$, $U_2 < C_{s_2}$ must hold so that downstream 6.64 exhibits exponentially growing oscillations which must be excluded by the boundary conditions. The leading oscillations in B_y have a different wavelength than those in B_z; if $\beta_+ >> 1$, the two components of the magnetic field are decoupled. About the downstream point 6.63 also gives exponentially growing oscillations.

6.5 Near-Perpendicular Slow Shocks

A curious anomaly in the literature on collisionless shocks is that virtually nothing is known about the structure of slow shock waves. Even the question of whether or not slow shocks exist has, apparently, not been fully resolved. According to the criteria discussed by Akhiezer *et al.* (1959), the hydromagnetic equations admit stable slow shock solutions; except for the complete switch-off shock, there are always six waves emanating from the shock front as required to satisfy the six boundary conditions at the front. Germain (1960), by studying the integral curves between stationary points, concluded that most slow shocks were stable but did construct an unstable counter example. Leonard (1966), however, demonstrated that this example violated the evolutionary conditions of Kantrowitz and Petschek (1966).

Anderson (1963) studied the shock stability problem by examining the topology of the shock integral curves and the stability to hydromagnetic perturbations. Although mathematical difficulties sometimes prevented an absolute determination, he concluded that most slow shocks were stable. The complete switch-off shock, however, he found to be unstable to intermediate wave perturbations. (A similar conclusion was reached with respect to the complete switch-on shock.) Here the confluence of the flow velocity with the intermediate speed appears to allow intermediate waves to be driven resonantly unstable and collect in the shock front; since this situation cannot continue, the shock must break up into other shocks or discontinuities. An alternate resolution of this difficulty (Petschek and Thorne, 1967) might be for the intermediate wave and switch-off shock to simply propagate together in the fluid.

Recently Lessen and Deshpande (1967) investigated the stability of fast and slow shocks to two dimensional hydromagnetic perturbations which excluded the intermediate wave. By utilizing a computer to determine the temporal eigenvalues of the perturbed shock system, they found that the slow shock was generally unstable over a wide range of propagation angles. There are several troublesome aspects about their results, however. First they found that the $\beta < 1$ parallel slow shock, the ordinary gas dynamic shock, was unstable. Second many of their slow shock parameters violated the shock evolutionary conditions. It is, therefore, not clear what confidence should be placed in their results.

The discussion here will attempt to avoid the existence problem of slow shocks; however, certain aspects of the shock structure solutions indicate that it may be difficult to maintain a steady slow shock structure. These problems will be discussed in section 9.0.

Consider now obliquely propagating slow shocks so that $B_x/B << 1$ and $U << C_s$. (The slow shock, of course, does not propagate perpendicular to the magnetic field but reduces to a non-propagating tangential discontinuity across which hydrostatic pressure balance must hold. The perpendicular slow shock limit changes the magnitude of the magnetic field across the discontinuity; the perpendicular intermediate wave limit accomplishes the rotation of the magnetic field direction.) If $\beta_+ < 1$, ion inertia is the dominant dispersion term so that 6.39 again splits into solutions representing the first three and last three terms of 6.38. The latter solution is given by 6.57 provided that the shock strength is far from the switch-off shock limit where $C_{y_1} = 0$.

About the upstream point $C_{z_1} > 0$ and $C_{y_1} < 0$; for $\beta_1 < 1$, $HF > 0$ so that the magnetic field undergoes an exponential decrease with the characteristic scale length

$$L_s \sim \frac{\sqrt{H_1 F_1}\;\dfrac{C_{I_1}}{u_1}\,\dfrac{c}{\omega_{p_+}}}{\left[\left|1 - \dfrac{C_{I_1}^2}{u_1^2}\right| \dfrac{C_{F_1}^2}{C_{S_1}^2}\,\dfrac{|M_{SL_1}^2 - 1|}{M_{SL_1}^2}\right]^{1/2}} \tag{6.66}$$

which for $\beta << 1$ and $U_1 << C_{I_1}$ is approximately

$$L_s \sim \frac{M_{SL_1}}{|M_{SL_1}^2 - 1|^{1/2}} \sqrt{\beta_1}\, \frac{c}{\omega_{p_+}} \tag{6.67}$$

where $M_{SL} = U/C_{SL}$. At the downstream point $C_{z_2} < 0$, so that 6.57 describes a trailing wave train given by

$$\delta B_z = A_{z_2} \exp\left[- \frac{|C_y + C_z| \frac{\nu_{ci}}{u_2} \frac{c^2}{\omega_{p_-}^2}}{H_2 F_2 \frac{C_{I_1}^2}{u_1^2} \frac{c^2}{\omega_{p_+}^2}} \right] \cdot \tag{6.68}$$

$$\sin\left[\left(\left|1 - \frac{C_{I_2}^2}{u_2^2}\right| \frac{C_{F_2}^2}{C_{S_2}^2} \frac{|M_{SL_2}^2 - 1|}{M_{SL_2}^2} \right)^{1/2} \left(H_2 F_2 \frac{C_{I_2}^2}{u_2^2} \frac{c^2}{\omega_{p_+}^2} \right)^{-\frac{1}{2}} x + \varphi \right]$$

The solution is sketched in Figure 9. The solutions arising from the first three terms of 6.38 either do not satisfy the boundary conditions at $\pm\infty$ or yield oscillation lengths which are so short as to be critically Landau damped.

When $\beta > 1$, the linear slow wave propagation speed differs little from the intermediate speed so that the wave is almost transverse polarized and incompressible (Kantrowitz and Petschek, 1966). Considering the evolutionary conditions the high β slow shocks are very weak; hence $1 - C_I^2/U^2 << 1$ and $M_{SL}^2 - 1 << 1$, which hold about both stationary points since, from the continuity equation, the flow velocity can change only slightly across the shock. Therefore 6.39 can again be expanded since $C_y C_z << 1$, the solutions of 6.38 split, and 6.57 is again valid.

First note that for $\beta >> 1$, and $B_x/B << 1$, $HF(C_I^2/U^2)(C^2/\omega_{p_+}^2) \approx \frac{1}{4}(C_I^4/U^4)(P^{(0)+}/\rho U^2)(B_x^2/B^2)R_+^2$, and is the same order as DR_+^2 and GR_+^2. Since $C_{y_1} < 0$, 6.57 describes an exponential decrease in the magnetic field about the upstream point with a scale length

$$L_s \sim \frac{\frac{1}{2}\frac{C_{I_1}^2}{u_1^2}\left(\frac{P_1^{(0)+}}{\rho_1 u_1^2}\right)^{1/2}\frac{B_x}{B_1}R_+}{\left[\left(\frac{C_{I_1}^2}{u_1^2}-1\right)\frac{M_{SL_1}^2-1}{M_{SL_1}^2}\right]^{1/2}} \tag{6.69}$$

Note that 6.69 is not valid for switch-off shocks. About the downstream point there is a trailing wave train given by

$$\delta B_z = A_{z_2}\exp\left[\frac{-|C_y+C_z|\frac{\nu_{ci}}{u_2}\frac{c^2}{\omega_{p_-}^2}}{\frac{1}{4}\frac{C_{I_2}^4}{u_2^4}\frac{P_2^{(0)+}}{\rho_2 u_2^2}\frac{B_x^2}{B_2^2}R_+^2}\right]\cdot$$

$$\sin\left[\frac{\left(\left|\frac{C_{I_2}^2}{u_2^2}-1\right|\frac{|M_{SL_2}^2-1|}{M_{SL_2}^2}\right)^{1/2}}{\frac{1}{2}\frac{C_{I_2}^2}{u_2^2}\left(\frac{P_2^{(0)+}}{\rho_2 u_2^2}\right)^{1/2}}\frac{x}{R_+}+\phi\right] \tag{6.70}$$

Note that since $C^2/\omega_{p_-}^2 << R_+^2$, the damping of the wave train is very slow so that the oscillations should persist far downstream (this holds for 6.68 also).

Equation 6.59 yields the solutions from the first three terms of 6.38. Since $D < 0$ and $G < 0$, there is a leading oscillatory solution with a wavelength of the order of

$$L_s \sim \frac{1}{2}\left(\frac{p^{(0)+}}{\rho u^2}\right)^{1/2} \frac{B_x}{B} R_+ \tag{6.71}$$

which is much shorter than 6.69. These waves are probably heavily ion Landau damped. The downstream solutions grow exponentially and are disallowed by the boundary conditions.

6.6 Near-Parallel Slow Shocks

For $\beta < 1$ slow shocks propagating along the magnetic field, the equations developed here are invalid since Debye scale lengths are required to resolve the shock structure. Here an electrostatic treatment starting from the Vlasov equation (Bernstein et al., 1957; Montgomery and Joyce, 1969) is more appropriate than the fluid approach. From 3.33 if $\theta > C_+/C$ $1/\sqrt{\beta_+} \approx C_A/C$, ion inertia produces a larger dispersive effect than does the Debye length. Hence at least the magnetic structure is describable by 6.38 and 6.39. Coroniti (1969) showed that for near parallel slow shocks resistively provides the shock dissipation only for weak shocks with $U_1 < C_{s_1}$. When $\beta > 1$, the parallel slow shock is of zero strength since the flow velocity equals the intermediate speed; slightly oblique shocks are, therefore, very weak.

First consider the low β near parallel slow shocks for which 6.57 and 6.59 are again valid. If $U_1 > C_{s_1}$, both $C_{z_1} < 0$ and $C_{y_1} < 0$ so that 6.57 yields leading oscillations which violate the upstream boundary condition. Here the slow shock is almost the gas dynamic shock so that the magnetic field changes across the shock are negligible. If $C_{s_1} > U_1 > C_{SL}$, $C_{z_1} > 0$ and $C_y + C_{z_1} > 0$; 6.57 describes an exponential decrease in the magnetic field with a scale length

$$L_s \sim \frac{\sqrt{H_1 F_1}\ \frac{C_{I_1}}{U_1}\ \frac{c}{\omega_{p+}}}{\left|\left(\frac{C_{I_1}^2}{U_1^2}-1\right)\ \frac{C_{F_1}^2}{C_{S_1}^2}\right|^{1/2}} \tag{6.72}$$

Equation 6.72 is invalid in the switch-off shock limit. The downstream state is characterized by a trailing wave train given by

$$\delta B_t = A_{t_2}\ \exp\left[\frac{-\left|C_y + C_z\right|\ \frac{\nu_{ci}}{U_2}\ \frac{c^2}{\omega_{p+}^2}}{H_2 F_2 \left(C_{I_2}^2/U_2^2\right)\ c/\omega_{p+}}\right]$$

$$\sin\left[\frac{\left|\left(\frac{C_{I_2}^2}{U_2^2}-1\right)\frac{C_{F_2}^2}{U_2^2-C_{S_2}^2}\ \frac{\left(M_{SL_2}^2-1\right)}{M_{SL_2}^2}\right|^{1/2}}{\sqrt{H_2 F_2}\ C_{I_2}/U_2}\ \frac{x}{c/\omega_{p+}} + \varphi\right] \tag{6.73}$$

Again the solutions from 6.59 are either of very short wavelengths, and hence damping, or violate the boundary conditions.

Now consider $\beta > 1$ slow shocks; since $C_y C_z << 1$, 6.57 and 6.59 can again be used. For the upstream flow $C_{z_1} > 0$ and $C_{y_1} < 0$, and the magnetic field decreases exponentially with a scale length

$$L_s \sim \frac{\frac{C_{I_1} C_{A_1}}{U_1^2}\left(\frac{P_1^{(1)+}}{\rho_1 U_1^2}\right)^{1/2} R_+}{\left[\left(\frac{C_{I_1}^2}{U_1^2}-1\right)\ \frac{C_{F_1}^2}{C_{S_1}^2}\ \frac{M_{SL_1}^2-1}{M_{SL_1}^2}\right]^{1/2}} \tag{6.74}$$

Note that 6.74 is much larger than R_+. The downstream solution yields a trailing wave train

$$\delta B_z = A_{z_2} \exp\left[\frac{-\left|C_y + C_z\right| \frac{\nu_{ei}}{u_2} \frac{c^2}{\omega_p^2}}{\frac{C_{I_2}^2 C_{A_2}^2}{u_2^4}\left(\frac{p_2^{(0)+}}{\rho_2 u_2^2}\right) R_+^2}\right] \cdot \tag{6.75}$$

$$\sin\left[\frac{\left|\left(\frac{C_{I_2}^2}{u_2^2}-1\right)\frac{C_{F_2}^2}{C_{S_2}^2}\frac{M_{SL_2}^2-1}{M_{SU_2}^2}\right|^{1/2}}{\frac{C_{I_2} C_{A_2}}{u_2^2}\left(\frac{p_2^{(0)+}}{\rho_2 u_2^2}\right)^{1/2}} \frac{x}{R_+} + \psi\right]$$

The downstream oscillations damp very slowly.

Equation 6.59 describes a leading wave train with a characteristic oscillation length given by

$$L_s \sim \left(\frac{p^{(0)+}}{\rho\, c_A^2}\right)^{1/2} R_+ \tag{6.76}$$

Since 6.76 is much less than 6.74, it probably is ion Landau damped. The downstream solutions from 6.59 violate the boundary conditions.

6.7 Perpendicular Fast Shock with Viscosity

The dissipation for fast shocks in which the downstream sound speed exceeds the flow velocity cannot be provided by resistivity alone but requires a stronger dissipation mechanism such as viscosity. Hence the trailing R_+ wave train (6.56) for the $\beta_2 > 1$ fast shock exhibited unstable growth if only resistivity is included. To resolve this difficulty the particularly simple case of a perpendicular fast shock is considered for which it is assumed that viscosity provides all the dissipation.

The desired demonstration is obtained by passing immediately to the linearized equations. For simplicity, resistivity is neglected. The linearized equations are

$$\rho u \, \delta u + \delta P_\perp^{(1)} + \delta T_{xx}^{(1)} + \frac{B_z \delta B_z}{4\pi} = \mu \frac{d\,\delta u}{dx} \qquad (6.77)$$

$$\rho u \, \delta v_y + \delta T_{xy}^{(1)} = \eta \frac{d\,\delta v_y}{dx} \qquad (6.78)$$

$$\rho u^2 \delta u + u\,\delta P_\perp^{(1)} + P_\perp^{(1)} \delta u + u\,\delta T_{xx}^{(1)} + \delta \underline{q}_\perp^{\perp(1)} \cdot \hat{x} + \frac{u_1 B_{z1} \delta B_z}{4\pi} = \mu u \frac{d\,\delta u}{dx} \qquad (6.79)$$

$$\delta u \, B_z + B_z \, \delta u = 0 \qquad (6.80)$$

where $\mu = \frac{4}{3}\eta + \zeta$; ζ and η are the two coefficients of viscosity. $\delta P_\parallel^{(1)}$ and $\delta \underline{q}_\parallel^{\parallel\,(1)} \cdot \hat{x}$ have been eliminated from the energy equation by 4.14. Neglecting products of ion FLR terms and viscosity terms, $\delta T_{xx}^{(1)}$ and $\delta \underline{q}_\perp^{\perp(1)} \cdot \hat{x}$ become

$$\delta T_{xx}^{(1)} = \frac{1}{4} \frac{R_+^2}{u} \frac{d^2 \delta u}{dx^2} \qquad (6.81)$$

$$\delta \underline{q}_\perp^{\perp(1)} \cdot \hat{x} = R_+^2 \frac{d^2 \delta u}{dx^2} \qquad (6.82)$$

$\delta T_{xy}^{(1)}$ is simply

$$\delta T_{xy}^{(1)} = \frac{P_{\perp}^{(0)+}}{2\Omega_+} \frac{d\delta u}{dx} \tag{6.83}$$

After eliminating $\delta P_{\perp}^{(1)}$ and δB_z, 6.77-6.80 become

$$\frac{3}{4} R_+^2 \frac{d^2 \delta u}{dx^2} + \frac{\mu u^2}{P_{\perp}^{(0)+}} \frac{d\delta u}{dx} - \frac{u^2 - c_F^2}{P_{\perp}^{(0)+}/\rho} \delta u = 0 \tag{6.84}$$

Neglecting terms proportional to μ^2, the solutions of 6.84 are

$$\lambda_{\pm} = \frac{-\mu u^2}{\frac{3}{2} P_{\perp}^{(0)+} R_+^2} \pm \left[\frac{u^2 - c_F^2}{\frac{3}{2} P_{\perp}^{(0)+}/\rho} \right]^{1/2} \frac{1}{R_+} \tag{6.85}$$

The upstream flow velocity undergoes an exponential decrease with a characteristic scale length

$$L_s \sim \left[\frac{3}{4} \frac{P_{\perp_1}^{(0)+}}{\rho_1 c_{F_1}^2} \frac{1}{M_{F_1}^2 - 1} \right]^{1/2} R_+ \tag{6.86}$$

Downstream there is a trailing wave train given by

$$\delta u = \delta u_2 \exp\left[\frac{-\mu u_2^2}{\frac{3}{2} P_{\perp_2}^{(0)+} R_+^2} \right] \cdot$$

$$\sin\left[\left(\frac{4}{3} \frac{c_{F_2}^2 |M_{F_2}^2 - 1|}{P_{\perp_2}^{(0)+}/\rho_2} \right)^{1/2} \frac{x}{R_+} + \varphi \right] \tag{6.87}$$

As anticipated, viscosity damps the trailing R_+ wave train to a downstream uniform state.

6.8 Discussion

The calculations of this section have considerably extended the range of parameter space investigated by the fluid theory. Whereas previous efforts have been concerned primarily with very low β fast shocks (Cavaliere and Englemann, 1967) and some early work on finite β perpendicular fast shocks (Kennel and Sagdeev, 1967b; Goldberg, 1969), the fluid approach taken here has covered for arbitrary β and propagation angle not only fast shocks, but the hitherto virtually unexplored problem of slow shock waves. Unfortunately it has not been possible here to develop a completely general fluid shock theory since several restricting assumptions were necessary in order to pursue the fluid approach. For all but perpendicular fast shocks the FLR-CGL Rankine-Hugoniot conditions do not close; to even initiate the fluid calculations a closure assumption was required. The approximate conservation law relating $P_{\|}^{(1)}$ and $\underline{q}_{\perp}^{\|(1)} \cdot \hat{x}$, 4.14, restricted consideration to near perpendicular fast shocks. The more useful, but far less general, assumption of pressure isotropy permitted arbitrary propagation of both fast and slow shocks to be investigated. Furthermore, the effect of zero order parallel heat flow, which is not determined by the moment equations, was neglected.

In deriving the wave train differential equation only the lowest order ion FLR corrections to the fluid equations were considered. In addition, the weak shock approximation was invoked, an assumption which greatly simplified the algebra. In performing the linearized analysis about the Rankine-Hugoniot stationary points, however, the weak shock approximation was relaxed, and the results are probably valid for at least moderate strength shocks. For strong shocks the wave train probably breaks, and a fully turbulent theory is required (Sagdeev, 1966).

The results of the linearized analysis confirm the predictions of section 3.0 based on the linear wave theory; however, better quantitative estimates of the shock thickness and wave train oscillation scale lengths were obtained. The new results of section 6.0 are summarized as follows:

1. The oblique fast shock structure consists of a trailing R_+ wave train when either $\theta < \pi/2 - \sqrt{M_-/M_+}$, $\beta_+ > M_-/M_+$ (6.45), or $\beta_+ \geqslant 1$ (6.56 and 6.87). A possible cancellation between ion inertia and ion FLR dispersion exists if, at any point in the flow, 6.53 is satisfied. The only dispersive terms remaining (6.51) are electron inertia and electron FLR; hence a short scale length sub-layer or dispersion discontinuity may be part of the shock structure. This sub-layer is further investigated in section 7.0.
2. For slightly oblique propagation when ion FLR dominates Debye length dispersion, the $\beta_+ > 1$ fast shock possess a leading R_+ wave train (6.65).
3. The structure of near-perpendicular propagating slow shocks consists of a trailing C/ω_{p_+} (6.68) wave train if $\beta_+ < 1$ and a trailing R_+ (6.70) wave train if $\beta_+ > 1$.

4. The slightly oblique $\beta < 1$ slow shock possess a trailing C/ω_{p_+} (6.75) wave train if $\theta > C_A/C$ and $U_1 < C_{s_1}$. If $\beta > 1$, and $\theta \neq 0$, the shock structure consists of a trailing R_+ wave train.

The above results are summarized in Figure 10 for fast shocks and Figure 11 for slow shocks. where the angle θ, to be interpreted as either the upstream or downstream angle as appropriate, is plotted against the ion β.

The solutions obtained in this section are not to be considered to represent the structure of actual collisionless shock waves. First only the structure in the vicinity of the Rankine-Hugoniot stationary points has been investigated; full details of the shock structure required solution of the complete non-linear differential equations. However the analysis does determine the qualitative nature of the shock structure. Second, the fundamental assumptions of pressure isotropy and neglect of the parallel heat flow are not valid apriori and must be verified by a determination of the turbulent dissipation. Some qualitative speculations on the validity and possible breakdown of these assumptions are given in section 9.0. Also recall that the theory of turbulent Alfvén shocks (Kennel and Sagdeev, 1967a) requires a firehose unstable pressure anisotropy; here the fluid approach breaks down entirely.

Finally to obtain a complete collisionless shock theory the phenomenological dissipation coefficients employed in this work must be replaced with ones determined by plasma turbulence theory. Here a self-consistent solution of the fluid equations incorporating a turbulent collision operator as well as the energy and momentum flux of the turbulent wave fields is required.

This problem is further considered in section 8.0 where non-self-consistent estimates of the turbulent dissipation possibly appropriate to the earth's bow shock wave are discussed. Here the usefulness of the fluid theory in arriving at such estimates is demonstrated.

7.0 The C/ω_{p_-} - R_- Dispersion Discontinuity

7.1 Introduction

The analysis of the previous section indicated that for the $\beta \leqslant 1$ oblique fast shock the competing effects of ion inertia and ion FLR dispersion may cancel in the wave train differential equation. At this point the long scale length waves no longer stand in the flow to form the wave train leaving only the possibility that short scale length waves on the fast branch might stand. Whether or not the above mathematical cancellation would occur in a physical shock is somewhat dubious, however, since the disparity between the long and short scale length terms in the wave train differential equation renders the precision required for the cancellation effect to occur unlikely. On the other hand, recent observations on the earth's bow shock, to be reviewed in section 8.0, have demonstrated the occasional existence of a short scale C/ω_{p_-} structure. The well-established turbulent nature of the solar wind flow argues against the possibility that the cancellation effect might account for the C/ω_{p_-} layer. Yet this structure occurs in a shock flow which almost certainly violates the restrictions on propagation angle and β required by the classical theory. Hence an alternative explanation is sought.

First the short scale length structure must be described in terms of the fluid equations. The only other dispersion lengths that affect the fast wave are electron inertia and electron FLR. The ordering of the FLR-CGL fluid equations, however, breaks down at the short scale length sub-layer since $R_+/L_s >> 1$.

Clearly then a reordering of the fluid equations which has L_s/R_+ and R_-/L_s as small parameters is required to describe a C/ω_{p_-} - R_- wave train. Since for $L_s \geqslant C/\omega_{p_-} \sim R_-$, $L_s/R_+ \sim \sqrt{M_-/M_+} << 1$, only the zero order ion terms and first order electron FLR terms need be retained. The appropriate expansion of the fluid equations is performed in section 7.2. Proceeding as before, the wave train differential equation is calculated in section 7.3 with the same basic assumptions as discussed in sections 4.0 and 5.0. A linearized analysis about the Rankine-Hugoniot stationary points, section 7.4, shows that both electron inertia and electron FLR dispersion produce trailing wave trains.

Section 7.5 reconsiders the dispersion limitation of the shock steepening, reviewed in section 2.2, and a qualitative argument is presented which suggests that only sufficiently strong fast shocks might possess a short scale length wave train. In analogy with the viscous dissipation discontinuity, this sub-layer is termed the C/ω_{p_-} - R_- dispersion discontinuity.

7.2 Re-expansion of the Fluid Equations

The equations of continuity, momentum, and energy (3.1-3.3), having been obtained by summing electron and ion contributions, remain valid under the new expansion and yield the familiar time independent conservation laws (4.1-4.5). The Ohm's law, 3.6, is also valid to terms $\mathcal{O}$ (M_-/M_+). It remains to order 3.7 to obtain equations for $\underset{\approx}{P}^{\pm}$ when $L_s/R_+ << 1$ and $R_-/L_s \leqslant 1$.

Considering 3.7 for the ion pressure, terms on the RHS are $\mathcal{O}(\epsilon)$ smaller than those on the LHS, opposite to the ion FLR ordering, so that to lowest order $\underset{\approx}{P}^{(0)+}$ satisfies

$$\frac{d\underset{\approx}{P}^{(0)+}}{dt} + \underset{\approx}{P}^{(0)+}\nabla\cdot\underline{V}^{+} + \underset{\approx}{P}^{(0)+}\cdot\nabla\underline{V}^{+}$$

$$+ \left(\underset{\approx}{P}^{(0)+}\cdot\nabla\underline{V}^{+}\right)^{T} + \nabla\cdot\underset{\approx}{Q}^{(0)+} = 0 \qquad (7.1)$$

Note that the only term in 7.1 which is influenced by the presence of the magnetic field is $\underset{\approx}{Q}^{(0)+}$. If the zero order ion heat flow is neglected, it is only consistent to take $\underset{\approx}{P}^{(0)+}$ isotropic, $\underset{\approx}{P}^{(0)+} = P^{(0)+}\underset{\approx}{I}$. Higher order terms in L_s/R_+ will be neglected.

The ordering of 3.7 for $\underset{\approx}{P}^{-}$ is exactly analogous to that used to obtain the ion FLR-CGL equations. Hence the solution of 3.7 is given by 3.9 to lowest order in R_-/L_s, and 3.10-3.12, and 3.14-3.19 to first order where all terms are evaluated only for electrons. Recall that $\Omega_- = -eB/M_-C$. Also note that it is $\underline{V}^{-}$ which occurs in the first order terms.

Attention now centers on formulating the appropriate Ohm's law. First consider the components of Newton's law for ions parallel and perpendicular to the magnetic field

$$\underline{V}_{\perp}^{+} - \frac{c\,\underline{E}\times\hat{e}_3}{|\underline{B}|} = \frac{1}{\Omega_+}\hat{e}_3\times\left[\frac{d\underline{V}^{+}}{dt}\right.$$

$$\left. + \frac{1}{NM_+}\nabla\cdot\underset{\approx}{P}^{+}\right] \qquad (7.2)$$

$$\hat{e}_3 \cdot \left[\frac{d\underline{v}^+}{dt} + \frac{1}{NM_+} \nabla \cdot \underset{\approx}{P}^+ \right] - \frac{e}{NM_+} E_{\parallel} = 0 \tag{7.3}$$

Ordering 7.2 for $L_s/R_+ << 1$, the RHS is $\mathcal{O}(1/\epsilon)$ compared to the LHS; hence to lowest order the RHS can be set equal to zero. In 7.3 the term $E_{\parallel}$ is multiplied by the small parameter e/M_+, and can be neglected to lowest order. Therefore 7.2 and 7.3 become

$$\frac{d\underline{v}^+}{dt} + \frac{1}{NM_+} \nabla \cdot \underset{\approx}{P}^+ = 0 + \Theta\left(\frac{L_s}{R_+}\right) \tag{7.4}$$

Since $\underset{\approx}{P}^+$ is isotropic, 7.4 describes the ion motion as independent of the magnetic field to lowest order; the ion trajectories are now straight lines. By way of comparison and interpretation, the high frequency electrostatic dispersion relation for the loss cone instability obtained by Rosenbluth and Post (1965) was based on the assumption that the ion orbits were straight lines. The justifying argument was that since the waves of interest had frequencies much larger than the ion cyclotron frequency and wavelengths much shorter than the ion Larmor radius, the ions would move essentially rectilinearly on the time and space scales of the wave. Similarly here on space scales the order of $C/\omega_{p_-} - R_-$, the ions are decoupled from the magnetic field and therefore have straight trajectories.

Now consider the Ohm's law 3.6 or 3.7. Since $\underset{\approx}{P}^{+}$ is isotropic, the only dispersion terms are electron and ion inertia. Suppose the linear waves described by the set 3.1-3.6 were determined with the wavelengths now restricted to satisfy $kR_+ >> 1$ and $kR_- << 1$. If kC/ω_{p_-} is also small, ion inertia is the only dispersion term. Therefore the linear wave propagation is described by the high frequency, $\Omega_+ << \omega << |\Omega_-|$, dispersion relation obtained by Formisano and Kennel (1969) for $\cos\theta \neq 0$ and $P_\parallel = P_\perp$

$$\left[\left(\frac{\omega^2}{k^2 C_A^2}\right)^2 - \frac{\omega^2\cos^2\theta}{\Omega_+^2}\right]\left[1 - \frac{\omega^2}{k^2 C_s^2}\right] = 0 \tag{7.5}$$

where $C_A^2 = (B_o^2/4\pi\rho_o)(1/1 + \langle k^2C^2/\omega_{p_-}^2\rangle)$. The first bracket is the high frequency whistler mode; the second is the isotropic sound wave or electroacoustic mode. Ion FLR dispersion has no effect on the wave propagation. Therefore ion FLR dispersion is basically weak since it is important only when $kR_+ \sim 1$ (see Figure 5).

Passing now to consideration of the C/ω_{p_-} - R_- sub-layer, ion inertia effects must also be eliminated since $(C/\omega_{p_+})/L_s >> 1$. Noting that $\underline{V} \approx \underline{V}^+$, 7.4 can be substituted into 3.6 to obtain to lowest order in L_s/R_+, $L_s/(C/\omega_{p_+})$

$$e\left[\underline{E} + \frac{\underline{V}\times\underline{B}}{c}\right] = \frac{M_-}{Ne}\frac{d\underline{J}}{dt} + O\left(\frac{M_-}{M_+}, \frac{L_s}{R_+}\right) \tag{7.6}$$

Equation 7.6 is the desired Ohm's law since only electron inertia dispersion remains.

7.3 C/ω_{p_-} -R_- Wave Train Differential Equation

The time independent set of equations are 4.1-4.5; the Ohm's law 4.6 and 4.7 is replaced by

$$\frac{c^2}{\omega_{p_-}^2} u_1 \frac{d^2 B_z}{dx^2} + \frac{c^2}{\omega_{p_-}^2} \nu_{ei} \frac{dB_z}{dx} = u B_z - u_1 B_{z_1} - V_z B_x \tag{7.7a}$$

$$\frac{c^2}{\omega_{p_-}^2} u_1 \frac{d^2 B_y}{dx^2} + \frac{c^2}{\omega_{p_-}^2} \nu_{ei} \frac{dB_y}{dx} = u B_y - V_y B_x \tag{7.7b}$$

Note that the ion inertia coupling of B_y and B_z is absent. It remains to determine the elements of $\underset{\approx}{P}^-$ with electron FLR corrections. Again assuming the electron pressure to be isotropic, A.2.13, A.2.17 and A.2.22 become

$$T_{xx}^{(1)} = -\frac{P^{(0)-}}{2\Omega_-} \frac{B_z}{B}\left(1+3\frac{B_x^2}{B^2}\right)\frac{dV_y^-}{dx} + \frac{P^{(0)-}}{2\Omega_-} \frac{B_y}{B}\left(1+3\frac{B_x^2}{B^2}\right)\frac{dV_z^-}{dx} \tag{7.8}$$

$$T_{xy}^{(1)} = \frac{P^{(0)-}}{2\Omega_-} \frac{B_z}{B}\left(1+3\frac{B_x^2}{B^2}\right)\frac{du^-}{dx} + \frac{P^{(0)-}}{2\Omega_-} \frac{B_x}{B}\left(1-3\frac{B_x^2}{B^2}\right)\frac{dV_z^-}{dx} \tag{7.9}$$

$$T_{xz}^{(1)} = -\frac{S}{2}\,\frac{p^{(0)-}}{\Omega_-}\,\frac{B_y}{B}\,\frac{dU^-}{dx} - \frac{p^{(0)-}}{2\Omega_-}\,\frac{B_x}{B}\left(\frac{B_z^2}{B^2} - \frac{2B_x^2}{B^2}\right)\frac{dV_y^-}{dx} \tag{7.10}$$

To eliminate U^-, V_y^- and V_z^- in terms of the fluid velocity and magnetic field, consider Ampere's law. When the magnetic field gradients are of order C/ω_{p_-}, the ion contribution to the current is $\mathcal{O}(\epsilon)$ smaller than that of the electrons. Hence to $\mathcal{O}(\epsilon)$ the y and z components of 3.5 are

$$V_y^- = \frac{c}{4\pi N e}\,\frac{dB_z}{dx} \tag{7.11}$$

$$V_z^- = -\frac{c}{4\pi N e}\,\frac{dB_y}{dx} \tag{7.12}$$

Since the x-component of curl $\underline{B}$ vanishes, $U^- = U^+$. In the weak shock or linearized approximation, 7.8-7.10 become

$$T_{xx}^{(1)} = -\frac{p^{(0)-}}{2\Omega_-}\,\frac{B_z}{B}\left(1+3\frac{B_x^2}{B^2}\right)\frac{c}{4\pi N e}\,\frac{d^2B_z}{dx^2} - \frac{p^{(0)-}}{2\Omega_-}\,\frac{B_y}{B}\left(1+3\frac{B_x^2}{B^2}\right)\frac{c}{4\pi N e}\,\frac{d^2B_y}{dx^2} \tag{7.13}$$

$$T_{xy}^{(1)} = \frac{p^{(0)-}}{2\Omega_-}\frac{B_z}{B}\left(1+3\frac{B_x^2}{B^2}\right)\frac{du}{dx} - \frac{p^{(0)-}}{2\Omega_-}\frac{B_x}{B}\left(1-3\frac{B_x^2}{B^2}\right)\frac{c}{4\pi n e}\frac{d^2B_z}{dx^2} \tag{7.14}$$

$$T_{xz}^{(1)} = -\frac{5}{2}\frac{p^{(0)-}}{\Omega_-}\frac{B_y}{B}\frac{du}{dx} - \frac{p^{(0)-}}{2\Omega_-}\frac{B_x}{B}\left(\frac{B_z^2}{B^2}-\frac{2B_x^2}{B^2}\right)\frac{c}{4\pi n e}\frac{d^2B_z}{dx^2} \tag{7.15}$$

The sum of the two electron FLR heat flows is

$$\left(\underline{q}_\perp^{I(1)} + \underline{q}_\perp^{II(1)}\right)\cdot\hat{x} = \frac{5}{2}\frac{p^{(0)-}}{\Omega_- n m_-}\left[\frac{B_z^2}{B^2}\frac{p^{(0)-}}{2\Omega_-}\left(1+3\frac{B_x^2}{B^2}\right)\frac{d^2u}{dx^2} + \frac{5}{2}\frac{B_y^2}{B^2}\frac{p^{(0)-}}{\Omega_-}\frac{d^2u}{dx^2}\right] \tag{7.16}$$

The reduction of the energy equation and the solution for U in terms of B_y and B_z proceeds exactly as in section 5.0. After iterating the Ohm's law with 6.11 the results are

$$\frac{c^2}{\omega_{p_-}^2} u_1 \frac{d^2 B_z}{dx^2} + \frac{\frac{5}{3}\frac{P_1^{(0)-}}{\rho_1}}{u_1^2 - c_{s_1}^2}\left(1+3\frac{B_x^2}{B_1^2}\right)\left(1-\frac{c_{I_1}^2}{u_1^2}\right)\frac{B_{z_1}^2}{B_1^2} u_1 R_-^2 \frac{d^2 B_z}{dx^2}$$

$$+\frac{1}{2}\frac{B_{z_1}^2/4\pi\rho_1}{u_1^2 - c_{s_1}^2}\left(1+3\frac{B_x^2}{B_1^2}\right)u_1 R_-^2 \frac{d^2 B_z}{dx^2} - \left(\frac{B_{z_1}^2}{B_1^2} - \frac{2B_x^2}{B_1^2}\right)\frac{c_{I_1}^2}{2u_1} R_-^2 \frac{d^2 B_z}{dx^2}$$

$$-\frac{5}{2}\frac{u_1 c_{s_1}^2}{u_1^2 - c_{s_1}^2}\frac{B_y^2}{B_1^2}\left(1-\frac{c_{I_1}^2}{u_1^2}\right)R_-^2\frac{d^2 B_z}{dx^2} + \frac{1}{2}\frac{B_{z_1}B_y/4\pi\rho_1}{u_1^2 - c_{s_1}^2}\left(1+3\frac{B_x^2}{B_1^2}\right)u_1 R_-^2 \frac{d^2 B_y}{dx^2}$$

$$+\frac{c^2}{\omega_{p_-}^2}\nu_{ei}\frac{dB_z}{dx} + \frac{5}{2}\sqrt{\frac{m_-}{m_+}}\frac{B_y B_x}{B_{z_1} B_1}\left(1-\frac{c_{I_1}^2}{u_1^2}\right)\sqrt{\frac{P_1^{(0)-}}{\rho_1}} R_- \frac{dB_z}{dx}$$

$$= u(B)B_z - u_1 B_{z_1} - \frac{B_x^2(B_z - B_{z_1})}{4\pi\rho_1 u_1}$$

(7.17)

$$\frac{c^2}{\omega_{p_-}^2} u_1 \frac{d^2 B_y}{dx^2} + \frac{c_{s_1}^2}{u_1^2 - c_{s_1}^2}\left(1+3\frac{B_x^2}{B_1^2}\right)\frac{B_z B_y}{B_1^2} u_1 R_-^2 \frac{d^2 B_z}{dx^2}$$

$$+\frac{1}{2}\frac{B_y B_z/4\pi\rho_1}{u_1^2 - c_{s_1}^2}\left(1+3\frac{B_x^2}{B_1^2}\right)u_1 R_-^2\frac{d^2 B_z}{dx^2} - \frac{1}{2}\left(1-3\frac{B_x^2}{B_1^2}\right)\frac{c_{I_1}^2}{u_1} R_-^2 \frac{d^2 B_y}{dx^2}$$

$$+\frac{1}{2}\frac{B_y^2/4\pi\rho_1}{u_1^2 - c_{s_1}^2}\left(1+3\frac{B_x^2}{B_1^2}\right)u_1 R_-^2 \frac{d^2 B_y}{dx^2}$$

$$-\frac{5}{2}\frac{c_{s_1}^2}{u_1^2 - c_{s_1}^2}\frac{B_y^3}{B_1^2 B_{z_1}}\left(1-\frac{c_{I_1}^2}{u_1^2}\right)u_1 R_-^2\frac{d^2 B_z}{dx^2} + \frac{c^2}{\omega_{p_-}^2}\nu_{ei}\frac{dB_y}{dx}$$

$$-\frac{1}{2}\sqrt{\frac{m_-}{m_+}}\left(1+3\frac{B_x}{B_1}\right)\frac{B_x}{B_1}\left(1-\frac{c_{I_1}^2}{u_1^2}\right)\sqrt{\frac{P_1^{(0)-}}{\rho_1}} R_- \frac{dB_z}{dx}$$

$$= u(B)B_y - \frac{B_x^2 B_y}{4\pi\rho_1 u_1} \qquad (7.18)$$

Note that the dominant electron FLR derivatives have the same sign as the electron inertia term so that R_- dispersion also slows down the fast wave.

7.4 Linearized Analysis

The differential equation describing the linearized response about the Rankine-Hugoniot stationary points is obtained as in section 6.2. Considering only oblique fast shocks for simplicity, terms of order $(B_x/B)^2$ and C_I^2/U^2 can be dropped. The resulting equations are

$$\frac{c^2}{\omega_{p_-}^2}\frac{d^2\delta B_z}{dx^2} + \frac{C_s^2 + B_z^2/8\pi\rho}{u^2 - C_s^2} R_-^2 \frac{d^2\delta B_z}{dx^2} + \frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}\frac{d\delta B_z}{dx} = \frac{(u^2 - C_F^2)(u^2 - C_{SL}^2)}{u^2(u^2 - C_s^2)}\delta B_z \quad (7.19)$$

$$\frac{c^2}{\omega_{p_-}^2}\frac{d^2\delta B_y}{dx^2} + \frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{u}\frac{d\delta B_y}{dx} = \frac{u^2 - C_I^2}{u^2}\delta B_y \quad (7.20$$

where terms $\mathcal{O}(\sqrt{M_-/M_+})$ have been dropped in 7.20. Note that B_y and B_z are completely decoupled. If a completely arbitrary propagation angle were considered, from 7.17 the sign, but not the magnitude, of the R_-^2 coefficient in the linearized equation would be the same as in 7.19. Hence the wave train results obtained below are qualitatively correct for a wide range of propagation angles; i.e., the C/ω_{p_-} - R_- structure is relatively insensitive to shock angle.

Performing the familiar ansatz, the solution of 7.19 is

$$\lambda_{\pm} = \frac{-\frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{2U}}{\left[\frac{c^2}{\omega_{p_-}^2} + \frac{c_s^2 + B_z^2/8\pi\rho}{U^2 - c_s^2}R_-^2\right]} \pm \left[\frac{U^2 - c_F^2}{U^2 - c_s^2}\right]^{1/2}\left[\frac{c^2}{\omega_{p_-}^2} + \frac{c_s^2 + B_z^2/8\pi\rho}{U^2 - c_s^2}R_-^2\right]^{-1/2} \quad (7.21)$$

At the upstream point the z component of the magnetic field undergoes an exponential rise whose scale length is

$$L_s \sim \frac{\left[\frac{c^2}{\omega_{p_-}^2} + \frac{c_{s_1}^2 + B_{z_1}^2/8\pi\rho_1}{U_1^2 - c_{s_1}^2}R_-^2\right]^{1/2}}{\left[\frac{c_{F_1}^2(M_{F_1}^2 - 1)}{U_1^2 - c_{s_1}^2}\right]^{1/2}} \quad (7.22)$$

a combination of C/ω_{p_-} and R_- if $\beta_- \sim 1$. The downstream solution is a trailing wave train given by

$$\delta B_z = A_{z_2} \exp\left[-\frac{\frac{c^2}{\omega_{p_-}^2}\frac{\nu_{ei}}{2U_2}}{\left(\frac{c^2}{\omega_{p_-}^2} + \frac{c_{s_2}^2 + B_{z_2}^2/8\pi\rho_2}{U_2^2 - c_{s_2}^2}R_-^2\right)}\right] \cdot \quad (7.23)$$

$$\sin\left[\left(\frac{c_{F_2}^2|M_{F_2}^2 - 1|}{U_2^2 - c_{s_2}^2}\right)^{1/2}\frac{x}{\left[\frac{c^2}{\omega_{p_-}^2} + \frac{c_{s_2}^2 + B_{z_2}^2/8\pi\rho_2}{U_2^2 - c_{s_2}^2}R_-^2\right]^{1/2}} + \phi\right]$$

Equation 7.23 exhibits the usual difficulty if $U_2 < C_{s_2}$ and viscous dissipation is required.

The solutions of 7.20 are exponentially growing at the upstream point and damping about the downstream point; hence the only complete solution is the trivial one, $\delta B_y = 0$. In the C/ω_{p_-} - R_-

sublayer, only the magnitude of the magnetic field changes since ion inertia, which drives the circular polarized component, is absent.

7.5 Dispersion Discontinuities

The physical process by which dispersion limits the steepening of a shock wave was reviewed in section 2.2. If dissipation is present, the resulting shock structure consists of a standing wave train whose group velocity is directed away from the shock. The wave train establishes a non-linear steady state in which the compression energy of the shock is convected away by the wave train group velocity.

Consider a plasma with very weak dissipation in which a pulse is undergoing steepening from long wavelengths as it propagates. By the above arguments the first wave that stands, which will have the longest wavelength, will form a wave train which attempts to prevent further shock steepening. If the group velocity is sufficient to convect away the compression energy, a steady shock is formed. For a strong pulse, however, the group velocity of the initial long scale length wave train may be incapable of preventing further steepening. Here either one of two possibilities exist. The pulse continues to steepen until another wave, on the same branch of the dispersion relation, stands whose group velocity is capable of maintaining a non-linear energy balance. The short scale length wave, having different dispersion characteristics from the long wavelength modes, will in general stand in another part of the flow. Hence the strong shock structure may consist of several different flow regions, each with a separate scale length standing wave train. There may, however, not exist a shorter dispersion length wave which stands, or even if there are such waves, their group velocities may be insufficient to limit the shock steepening. Here the wave train structure must break, and a fully turbulent shock results.

The above argument suggests that a C/ω_{p_-} - R_- wave train may be a general property of reasonably strong fast shock flows. The wave train calculation of section 7.4 indicated that this short scale length layer was relatively insensitive to the shock propagation angle and the plasma β, and therefore might occur for a wide variety of upstream flow conditions.

To develop the above argument in mathematical terms is quite difficult since it would involve solving the kinetic equation for the non-linearly excited wave modes in time and wave number. Here it is possible to give only a very crude argument which at least outlines a probable approach to the problem. Assuming the pulse to have already reached steady state, the time independent kinetic equation for waves in a flow U is (Camac et al., 1962; Galeev and Karpman, 1963)

$$\left(U + \frac{\partial \omega}{\partial k}\right)\frac{\partial N_k}{\partial x} - k\frac{dU}{dx}\frac{\partial N_k}{\partial k} = \left(\frac{\partial N_k}{\partial t}\right)_{NL} \approx \gamma_{NL} N_k \tag{7.24}$$

N_k is the wave action in the k^{th} mode; the RHS is the non-linear excitation rate which has been expressed in the form $\gamma_{NL}\, N_k$. If the explicit dependence of N_k on k is ignored, a very poor assumption, 7.24 balances the non-linear generation of wave action with its convection loss.

Since the non-linear excitation rate is very difficult to calculate, γ_{NL} is estimated on physical grounds. First γ_{NL} should be proportional to the amplitude of the pulse, $\rho_2/\rho_1 - 1$, so that $\gamma_{NL} \to 0$ when $M_1^2 - 1 \to 0$. Second the minimum excitation time, maximum γ_{NL}, that can exist in steady state must be of the order of the time it takes the wave to cross the pulse, τ_w. Clearly if $\gamma_{NL}\tau_w >> 1$, wave energy accumulates inside the pulse, and the wave train cannot be established. τ_w is estimated as $\tau_w \sim L_s/(\omega/k)$. Hence 7.24 becomes

$$\left(u + \frac{\partial \omega}{\partial k}\right) \frac{\partial N_k}{\partial x} \simeq \left(\frac{\rho_2}{\rho_1} - 1\right) N_k \frac{\omega}{k L_s} \tag{7.25}$$

Since all characteristic gradients are the order of the shock thickness, $\partial \ln N_k/\partial x$ is estimated as L_s^{-1} so that 7.25 becomes

$$\frac{ku}{\omega} + \frac{k}{\omega}\frac{\partial \omega}{\partial k} \simeq \left(\frac{\rho_2}{\rho_1} - 1\right) \simeq \frac{2(M_1^2 - 1)}{M_1^2 + 2} \equiv \beta_w \tag{7.26}$$

The Rankine-Hugoniot relations for a perpendicular fast shock were used to estimate $\rho_2/\rho_1 - 1$. Note that for $M_1^2 \to \infty$, $\beta_w = 2$. Since the wave stands in the shock frame, $\omega/k = U$ for either the upstream or downstream flow.

As examples, consider separate wave trains formed by ion FLR, electron inertia, and ion inertia. From 3.24 for perpendicular shocks the group velocity of the fast wave with ion FLR dispersion is

$$\frac{\partial \omega}{\partial k} = \frac{\omega}{k} \; \frac{1 - 3/2\, k_\perp^2 R_+^2}{1 - 3/4\, k_\perp^2 R_+^2} \tag{7.27}$$

Similarly for electron inertia, 2.16 yields

$$\frac{\partial \omega}{\partial k} = \frac{\omega}{k} - \frac{k c_A^2}{\omega} \; \frac{k^2 c^2/\omega_{p-}^2}{\left(1 + k^2 c^2/\omega_{p-}^2\right)^2} \tag{7.28}$$

Note that when $kC/\omega_{p_-} >> 1$, corresponding to strong shocks, the group velocity approaches the phase velocity. For oblique whistlers, 7.5 gives the well-known result

$$\frac{\partial \omega}{\partial k} = 2\, \frac{\omega}{k} \tag{7.29}$$

Upon substitution of 7.27, the relation 7.26 for the ion FLR wave train becomes

$$\beta_\omega \lesssim 1 + \frac{1 - \frac{3}{2} k_\perp^2 R_+^2}{1 - \frac{3}{4} k_\perp^2 R_+^2} \tag{7.30}$$

Setting $\omega^2/k^2 = C_F^2(1 - \frac{3}{4} k_\perp^2 R_+) = U^2$, 7.30 becomes

$$\beta_\omega \lesssim 3 - \frac{1}{M_2^2} \tag{7.31}$$

For weak shocks, $M_2 \sim 1$, $\beta_w < 2$, and ion FLR dispersion limits the shock steepening. However for strong shocks $M_2^2 \to 0$ and 7.31 is no longer satisfied. Hence, as mentioned above, ion FLR, being an essentially weak dispersion effect, cannot prevent further steepening if the shock is strong, and another dispersion scale length is required.

Now consider the electron inertia wave train. Using 7.28 for strong shocks, $kC/\omega_{p_-} >> 1$, 7.26 becomes

$$\beta_\omega \lesssim 1 + \frac{k}{\omega}\frac{\partial\omega}{\partial k} = 2 \tag{7.32}$$

Equation 7.32 is always satisfied for arbitrarily strong shocks, $M_1^2 \to \infty$; electron inertia is always capable of limiting the shock steepening. Also note from 2.16 that the C/ω_{p_-} wave train stands ahead of or near the sonic point.

Since the oblique whistler wave train stands in the upstream flow, the flow velocity inhibits the convection of wave energy in 7.25. After substituting 7.29, 7.27 becomes

$$\beta_\omega \lesssim -1 + \frac{k}{\omega}\left(\frac{2\omega}{k}\right) = 1 \tag{7.33}$$

Equation 7.33 is satisfied only if $M_1^2 \leqslant 4$; otherwise the whistler wave train alone does not prevent further steepening.

The above semi-quantitative arguments should probably not be taken too seriously but only as an indication that a dispersion discontinuity is possible. The actual structure of strong shocks will undoubtedly turn out to be extremely complicated, and no more than qualitative "hints" at various possibilities can be given here. In the next section some recent observations of the earth's bow shock are discussed for which a possible interpretation can be made in terms of a laminar wave train structure and a $C/\omega_{p_-} - R_-$ dispersion discontinuity.

8.0 The Earth's Bow Shock Wave

8.1 Introduction

The preceding calculations were performed in the spirit of investigating the content of the FLR-CGL equations with respect to laminar shock wave trains. Certain assumptions about the pressure isotropy and the parallel heat flow were required to close the Rankine-Hugoniot relations; although the mathematical validity of the equations was thereby severely restricted, the dispersion properties of the wave train solutions are probably approximately valid for a wider class of shock flows. Attention must now be concentrated on understanding the plasma turbulence responsible for the shock dissipation since not only the shock structure but also the final downstream state is dependent on it.

Plasma turbulence theory contains a bewildering variety of unstable modes and several different non-linear relaxation processes. In such a situation abstract theorizing upon the turbulent shock dissipation may not be fruitful, and the problem is placed on far surer ground if experimental information is available for guidance. Laboratory experiments have verified some aspects of the low-β, C/ω_{p_-} and C/ω_{p_+} laminar solutions (Paul et al., 1965; Paul et al., 1967; Kurtmullaev et al., 1966; Robeson et al., 1968), but problems of steady flow conditions and a fully collisionless plasma still persist. Observations performed on the earth's bow show afford, perhaps, the best opportunity to investigate finite-β collisionless shocks over a wide range of upstream propagation conditions. Observations from OGO-V have revealed that the shock transition often, but not always, has a laminar form; these are reviewed in section 8.2 and compared with the above calculations.

To legitimately make a comparison with experiment the assumptions of this work must be justified by determining some of the basic features of the turbulent dissipation. Guided by experiment a crude estimate of current driven ion sound turbulence, which is associated with sharp gradients in the magnetic field (Fredricks et al., 1968; Fredricks and Coleman, 1968), is performed in section 8.3. The electron and ion parallel heat flows are suppressed by this turbulence. Section 8.4 discusses the stability of the wave train to three wave non-linear decay. The trailing R_+ wave train for perpendicular fast shocks is unstable to perturbations containing Alfvén waves.

8.2 Comparison with the Earth's Bow Shock

Although there have been many observations of the bow shock, most suffered from either poor time resolution, inadequate diagnostic capability, or both. The most definitive results on the shock structure have been obtained by OGO-V, and hence the discussion here will be limited to data reported by Fredricks et al. (1968), Fredricks and Coleman (1968), and Fredricks et al. (1969). Caution should be exercised in interpreting these results as typical of the earth's bow shock since only a limited data sample has been analyzed to date. In addition, only those shocks which appear to be of laminar form are discussed. The bow shock at times also exhibits a turbulent magnetic structure which has been interpreted as large amplitude whistlers standing in the shock, which supports the shock theory of Fishman _et al._ (1960) and Camac _et al._ (1962).

Data for the shock crossing of March 12, 1968 from the UCLA fluxgate magnetometer and the TRW electric field experiment is reproduced in Figure 12. The shock parameters were $M_A \sim 9.5$, $B_1 \sim 6 \times 10^{-5}$ gauss, $N_1 \sim 10\ cm^{-3}$, $\beta_1 \sim 2$, $T_-/T_+ \sim 10$ where $M_A \equiv U_1/C_{A_1}$. The magnitude of the magnetic field at the leading edge rises sharply to roughly seven times its upstream value, and then oscillates in a quasi-regular manner, reminiscent of a trailing laminar wave train. The data, of course, is observed as a temporal signal in the satellite frame. If, however, it is assumed that the relative speed of the shock and satellite was small, ≤ 10 km/sec, and that the wave train-like structure was time stationary, the temporal signal is interpretable as a length. Viewed in this manner the scale length of the leading edge and the trailing oscillatory wavelength is of the order of several times C/ω_{p_-}. The C/ω_{p_-} layer is the most frequently observed laminar structure in the bow shock (Fredricks, private communication).

The electric field amplitude in the 1.3 khz channel of the TRW experiment was generally enhanced throughout this layer with peak amplitudes, about 27 mV/meter, occurring in association with the maximum magnetic field gradients, i.e., the maximum current density. Comparison with the high frequency magnetic field detector reveals that these waves are almost purely electrostatic. Since the wave frequencies are near the ion plasma frequency, Fredricks and Coleman (1968) suggested that the wave turbulence is driven by an ion sound current instability. Furthermore, the Lockheed ion spectrometer observed that the ions were heated to roughly their downstream temperature at the C/ω_{p_-} layer (Fredricks, private communication). Hence it appears that the electrostatic ion sound turbulence is responsible for the shock dissipation in the C/ω_{p_-} shocks.

A more complicated shock crossing was observed on March 10, 1968, a sketch of which is shown in Figure 13. The shock parameters were $M_A \sim 7$, $B_1 \sim 8 \times 10^{-5}$ gauss, $N_1 \sim 10$ cm^{-3}, and $\beta \sim .5$. The upstream flow consisted of a standing oscillatory wave which Fredricks and Coleman (1968) interpreted as being a C/ω_{p_+} wave train. In the middle of the shock there again occurred a short C/ω_{p_-} structure with its attendant electrostatic turbulence. A third oscillatory wave train stood in the downstream flow which Fredricks and Coleman (1968) did not identify. If it is assumed that the relative velocity between the shock and satellite was about 10 km/sec, this trailing structure has a scale length of the order of 10^2 km. The thermal ion gyroradius downstream is of the same order.

In attempting to interpret these shock observations in terms of the above calculations several points should be borne in mind: (1) the above analysis is valid only for the leading and trailing edge of a time stationary shock flow; (2) no restriction on the shock strength was made; (3) the assumptions of pressure isotropy and zero parallel heat flow have not been justified; (4) the dissipation coefficients were completely ad hoc so that no estimate of the wave train growth or damping length can be made. For the March 10 shock the interpretation of the upstream structure as a leading C/ω_{p_+} wave train appears secure. From the calculations the downstream oscillations are attributable only to a trailing R_+ wave train, a conclusion consistent with the estimated 100 km scale length. Hence at this time the bow shock was either propagating obliquely to the upstream interplanetary magnetic field or was sufficiently strong so that the downstream magnetic field was near-perpendicular.

The occurrence of the C/ω_{p_-} structure as the most dominant and persistent feature of the laminar-like bow shock observations is inexplicable in terms of the wave train analysis of section 6.0 since the solar wind ion β far exceeds M_-/M_+. A possible, but perhaps not exclusive, explanation of the C/ω_{p_-} layer is in terms of a dispersion discontinuity which occurs since the bow shock is of moderate strength. The analysis of section 7.0 then indicates that the scale length should be a combination of C/ω_{p_-} and R_- if in the solar wind $\beta_- \sim 1$. Several difficulties or questions with this interpretation remain unresolved, however: (1) aside from the crude arguments of section 7.5, which are at best illustrative, there is as yet no determination of the Mach number at which the C/ω_{p_-} - R_- dispersion discontinuity becomes important; (2) since the C/ω_{p_-} - R_- discontinuity did not arise from a self-consistent, general wave train differential equation, its location in the shock flow cannot be calculated; (3) why does the C/ω_{p_-} - R_- layer occur alone in most bow shock crossings without the long scale length wave trains? Perhaps the strong turbulence associated with the C/ω_{p_-} - R_- wave train provides all the required shock dissipation thus obviating the necessity of further wave trains. The resolution of the above points awaits a more detailed analysis of strong shock flows, possibly with the assistance of a computer.

8.3 Ion Sound Turbulence in the C/ω_{p_-} - R_- Layer

8.3.1 Linear Theory of Ion Sound Instability

From the bow shock observations the currents required to support the sharp magnetic field gradients in the C/ω_{p_-} - R_- wave train appear to become unstable to the emission of ion sound waves. An estimate of the effective electron current velocity (recall that the ion trajectories are straight lines, and therefore do not contribute to the current) is obtained from Ampere's law

$$V_D^- \sim \frac{c}{4\pi N e}\frac{dB}{dx} \sim \frac{c}{4\pi N e}\frac{B}{c/\omega_{p_-}} \qquad (8.1)$$

$$\sim \sqrt{\frac{M_+}{M_-}}\, c_A$$

Hence V_D^- is close to the electron thermal speed if, as in the solar wind, $\beta_- \sim 1$. Since T_-/T_+ is estimated to be of order 10 in the upstream flow, the conditions for the ion sound instability are well satisfied.

As a zeroth order approximation the complicated C/ω_{p_-} - R_- structure is replaced by a uniform plasma with an electron current drift velocity V_D^-. Since the unstable waves have frequencies around ω_{p_+} which generally exceeds the electron cyclotron frequency, the effects of the magnetic field can, in this crude approximation, be neglected. For this simplified model the solution of the linear stability problem follows from the electrostatic dispersion relation (Sagdeev and Galeev, 1966)

$$k^2 = -\sum_{+-} \omega_{p\pm}^2 \int \frac{\underline{k}\cdot\frac{\partial f_0^\pm}{\partial \underline{v}}\, d^3v}{\nu_k - \underline{k}\cdot\underline{v}} \tag{8.2}$$

where $\nu_k = \omega_k + i\gamma_k$ and γ_k/ω_k is assumed small. Choosing the phase velocity intermediate between the ion and electron thermal speeds, the denominator can be expanded, and a dispersion relation is obtained which for ω_k, yields

$$\omega_k^2 = \frac{k^2 C_s^2}{1 + k^2 \lambda_D^2} \tag{8.3}$$

where $C_s^2 = T_-/M_+$ and $\lambda_D = C_s/\omega_{p_+}$. Assuming a drifting Maxwellian for the electrons, the imaginary part of the dispersion relation gives an estimate of the growth rate

$$\gamma_k \sim \omega_{p_+} \frac{V_D^-}{c_-} \tag{8.4}$$

which is sufficiently accurate for the purpose here.

8.3.2 Non-Linear Estimates

Since the electron current velocity greatly exceeds the ion sound speed, the resulting turbulence will be strong.

The electron current must maintain the sharp magnetic field gradient so that non-linear saturation of the wave spectrum will probably not proceed by diffusion of electrons but rather by absorption of the wave energy by the ions. Non-linearly the wave spectrum will broaden until the thermal ions are in resonance. The wave spectrum should saturate if the turbulent electric fields can accelerate an ion to C_s in one resonance time, τ_{res}. Since $C_+ < C_s$, the resonance time is less than the wave period by C_+/C_s, or $\omega_{p_+} \tau_{res} \sim \sqrt{T_-/T_+}$. Hence the turbulent wave amplitude can be estimated as

$$\sum_k e^2 E_k^2 \sim \left(\frac{M_+ C_s}{\tau_{res}} \right)^2$$
$$\sim \left[M_+ \omega_{p_+} C_s \left(\frac{T_-}{T_+} \right)^{1/2} \right]^2 \qquad (8.5)$$

where E_k is the wave electric field. Equation 8.5, of course, is an upper limit to E_k.

The non-linear evolution of the electron distribution is crudely given by the quasi-linear theory (Drummond and Pines, 1962; Vedenov et al., 1961; Kennel and Englemann, 1966). An effective or anomalous electron-ion collision frequency, which is a measure of the electron momentum loss to ion sound radiation, is obtained by taking the velocity moment of the quasi-linear diffusion equation to find (Sagdeev and Galeev, 1966)

$$\frac{\partial}{\partial t}\int f^- v\, d^3v \sim c_- \nu_{ei}$$

$$\sim \sum_k \frac{e^2 E_k^2}{M_-^2}\int \frac{i}{\omega_k - kv}\frac{\partial f^-}{\partial v}\, dv \qquad (8.6)$$

Retention of the resonance contribution to 8.6 and substitution for E_k from 8.5 yields

$$\nu_{ei} \sim \omega_{p_+}\frac{V_D^-}{c_-}\frac{T_-}{T_+} \qquad (8.7)$$

Equation 8.7 is consistent with the result obtained by Sagdeev and Galeev (1966) from a much more elegant treatment employing the kinetic equation for waves and the quasi-linear theory. An effective ion-ion collision frequency, ν_{ii}, is smaller than ν_{ei} by roughly $\sqrt{M_-/M_+}$.

Since from 8.1 $V_D^-/C_- \sim 1$ and $\omega_{p_+} \sim |\Omega_-|$ for the solar wind, $\nu_{ei} \sim |\Omega_-|$; hence in the C/ω_{p_-} - R_- wave train the effective collision frequency is large enough to satisfy the usual criteria for the hydromagnetic assumption of pressure isotropy. Similarly $\nu_{ii} >> \Omega_+$ so that the ion pressure will also be isotropic in this layer.

8.3.3 Estimate of the Parallel Heat Flow

In section 4.0 the moments of the Vlasov equation were closed by neglecting the flow of heat along the lines of force. Since the heat flow depends on the detailed distribution function, justification of the above assumptions requires information about the collision operator in the kinetic equation, i.e., about the dissipation coefficients. For the $C/\omega_{p_-} - R_-$ layer in the bow shock, the anomalous collision frequencies arising from ion sound turbulence, albeit extremely crude, permit an estimate of the thermal conductivity if it is assumed that

$$q_{\parallel}^{\pm} \simeq \kappa_{\parallel}^{\pm} \nabla T_{\pm} \tag{8.8}$$

The thermal conductivity can be related to the effective electrical conductivity by dimensional considerations as

$$\kappa_{\parallel}^{\pm} \sim \frac{T_{\pm}}{e^2} \sigma_{\pm} \tag{8.9}$$

where

$$\sigma_{\pm} \simeq \frac{N e^2}{M_{\pm} \nu_{\substack{ii \\ ei}}} \tag{8.10}$$

For the bow shock the ions were observed to approximately thermalize across the $C/\omega_{p_-} - R_-$ wave train so that the maximum temperature gradient occurs there. Since every line of force passes through this wave train, the heat flow upstream will depend primarily on the value of $\kappa_{\parallel}^{\pm}$ in this layer. The relative importance of electron thermal conduction to resistive dissipation follows by comparing terms in the energy equation

$$\frac{c^2 B}{(4\pi)^2 \sigma_-} \frac{dB}{dx} : \kappa_{\parallel}^{-} \frac{dT_-}{dx} \tag{8.11}$$

Substitution of 8.9, 8.10 and 8.7 into 8.11 yields

$$\frac{c^2}{c_-^2} \left(\frac{T_-}{T_+}\right)^2 \frac{d \ln B}{dx} : \beta_-^2 \frac{d \ln T_-}{dx} \tag{8.12}$$

Hence if the magnetic field and the temperature gradients have approximately the same scale length, electron thermal conduction contributes negligibly to the dissipation unless $\beta_- >> 1$. Ion thermal conductivity is smaller than that for electrons by roughly $\sqrt{M_-/M_+}$.

The above estimates are probably only valid for the main part of the electron or ion distribution where anomalous collisions are frequent enough to suppress the heat flow. If a high energy tail develops in the shock, it may not be so constrained and could contribute a flow of heat upstream.

8.3.4 Estimate of the C/ω_{p_-}-R_- Wave Train Length

A possible check on the reasonableness of the above turbulence estimates is to determine the number of C/ω_{p_-} - R_- oscillations necessary to accomplish the shock dissipation and compare with observations. A phenomenological quasi-linear velocity space ion diffusion coefficient can be constructed as

$$D^{+} \sim \frac{\sum_k e^2 E_k^2}{2 M_+^2 \nu_{ii}} \tag{8.13}$$

For a reasonably strong fast shock the upstream flow kinetic energy is roughly converted to thermal energy downstream so that $\Delta T^{+} \sim \frac{1}{2} M_+ U_1^2$. On the other hand total temperature change is approximately the number of steps $n \sim L/(C/\omega_{p_-})$, where L is the length of the wave train, times the energy diffusion per step

$$\Delta T_+ \sim n M_+ D^{+} \Delta\tau \tag{8.14}$$

where $\Delta\tau$ is the step time and should be approximately $\Delta\tau \sim (C/\omega_{p_-})/U$. Substitution for $\sum_k E_k^2$ from 8.5 and ν_{ii} from 8.7 into D^{+}, the length L can be estimated as

$$L \sim \left(\frac{U}{c_s}\right)^2 \sqrt{\frac{M_+}{M_-}}\, \frac{U}{c} \sqrt{\frac{2}{\beta_-}}\, \frac{c}{\omega_{p_-}} \tag{8.15}$$

For typical solar wind parameters 8.15 yields $L \sim 5$ to 10 C/ω_{p_-} in reasonable agreement with Figures 12 and 13. Furthermore, since this length is still less than an ion gyro-radius, the approximation of straight line ion orbits used in analyzing this layer remains valid.

8.4 Non-Linear Decay Instability of the Trailing Perpendicular R_+ Wave Train

In the previous section some estimates were made of the shock dissipation resulting from ion sound turbulence in the $C/\omega_{p_-} - R_-$ wave train. Another dissipation process that has received some discussion in the literature is the three wave parametric decay of the wave train structure (Galeev and Karpman, 1963; Sagdeev, 1966; Sagdeev and Galeev, 1966; Kennel and Sagdeev, 1967b). The generation of long wavelength turbulence, which does not stand in the flow, produces an irreversible shock transition; the particle distribution functions adjust by damping, or possibly amplifying, the turbulent waves, eventually producing a uniform downstream state. After briefly reviewing the fundamental concepts of the three wave decay instability, the stability of a trailing R_+ wave train propagating perpendicular to the magnetic field is considered as an example.

A non-linear wave is capable of interacting with two other waves such that a resonant exchange of energy occurs between them. If the wave amplitudes are sufficiently small so that the waves propagate, to lowest order, as if in a homogeneous system, perturbation theory permits a calculation of the matrix elements for the interaction. The matrix element couples the two perturbing waves with the non-linear wave, and vanishes unless the so-called decay conditions are satisfied

$$\underline{k}_1 = \underline{k}_0 + \underline{k}_2 \tag{8.16}$$

$$\omega_1 = \omega_0 + \omega_2 \tag{8.17}$$

ω_o and k_o refer to the non-linear wave; ω_1, $\underline{k}_1$ and ω_2, $\underline{k}_2$ refer to the perturbing waves. In analogy with quantum mechanics, 8.16 expresses the conservation of momentum and 8.17 the conservation of energy. In addition to 8.16 and 8.17 the matrix element must be non-zero and have the appropriate sign for the three wave decay to occur. Three waves on the same branch of the dispersion relation may mode couple only if ω increases with increasing k, a requirement imposed by 8.16 and 8.17. Interactions between waves on different branches are possible, but may be prohibited by polarization restrictions.

As an example, consider a trailing R_+ wave train propagating perpendicular to the magnetic field. Assume that the wave amplitude has been reduced sufficiently by either collisional or anomalous damping, so that the wave train may be approximated by a sinusoidal oscillation of frequency ω_o and wave vector k_o, and obeys the linear polarization relations; alternatively, consideration could be restricted to weak shocks so that the wave amplitudes are always small. The wave train is to be perturbed by a slightly oblique fast wave, ω_1 and $\underline{k}_1$, and an intermediate wave, ω_2 and $\underline{k}_2$. For simplicity the ion β is taken large enough so that ion and electron inertia are negligible and only ion FLR dispersion is important. Pressure isotropy is again assumed.

The calculation follows the procedure outlined by Sagdeev and Galeev (1966) and hence will be only briefly sketched here. The polarization of the R_+ wave train is given by δB_z, δV_x, and δV_y; the perturbing fast wave is characterized by $\underline{V}_1$ and $\underline{b}_1$, and it is anticipated that $k_{\perp_1} >> k_{\parallel_1}$; the intermediate wave polarization is b_{y_2} and V_{y_2}, and $k_{\perp_2}$ is taken to be zero. Reduction of Ohm's law in the form of 3.8 yields the following relations

$$\delta B_z = \frac{k_{\perp_0} B_0}{\omega_0} \delta V_x \tag{8.18}$$

$$b_{z_1} = \frac{k_{\perp_1} B_0}{\omega_1} V_{x_1} \tag{8.19}$$

$$b_{y_1} = -\frac{k_{\parallel_1} V_{y_1} B_0}{\omega_1} + \frac{k_{\perp_1}}{\omega_1} \delta V_x b_{y_2} \tag{8.20}$$

$$b_{x_1} = -\frac{k_{\parallel_1} B_0 V_{x_1}}{\omega_1} \tag{8.21}$$

$$b_{y_2} = -\frac{k_{\parallel_2} B_0 V_{y_2}}{\omega_2} - \frac{k_{\parallel_2}}{\omega_2}\left(\delta V_y^* b_{z_1} + V_{y_1} \delta B_z^* \right) \tag{8.22}$$

where * denotes the complex conjugate. Averaging over phases has eliminated the non-resonant terms so that only those satisfying the decay relations remain. Note that $k_{\parallel_1} = k_{\parallel_2}$. Similarly, the perturbed momentum equations are

$$\omega_1 V_{x_1} = \frac{k_{\perp_1} \Delta P_{\perp_1}^{(1)}}{\rho_0} - i k_{\perp_1}^2 \left(\frac{P^{(0)}}{2\Omega_+ \rho_0} \right) V_{y_1} + \frac{B_0}{4\pi\rho_0} b_{z_1} - \frac{B_0}{4\pi\rho_0} b_{x_1} \tag{8.23}$$

$$\omega_1 v_{y_1} = i k_{\perp_1}^2 \left(\frac{p^{(0)+}}{2\Omega_+ \rho_0} \right) v_{x_1} - \frac{k_{\parallel_2} \delta B_z b_{y_2}}{4\pi \rho_0} - \frac{k_{\parallel_1} B_0 b_{y_1}}{4\pi \rho_0} \tag{8.24}$$

$$\omega_1 v_{z_1} = \frac{k_{\parallel_1} \Delta P_{\parallel_1}^{(1)}}{\rho_0} \tag{8.25}$$

$$\omega_2 v_{y_2} = - \frac{k_{\parallel_2} B_0}{4\pi \rho_0} b_{y_2} - \frac{k_{\parallel_1} b_{y_1} \delta B_z^*}{4\pi \rho_0} + k_{\perp_1} \delta v_x^* v_{y_1} - k_{\perp_0} v_{x_1} \delta v_y^* - k_{\parallel_0} v_{z_1} \delta v_y^* \tag{8.26}$$

$$\delta v_y = i \frac{k_{\perp_0}^2 \left(\frac{p^{(0)+}}{2\Omega_+ \rho_0} \right)}{\omega_0} \delta v_x \tag{8.27}$$

where

$$\Delta P_{\perp_1}^{(1)} = \frac{P_\perp^{(1)} \left[2 k_{\perp_1} v_{x_1} + k_{\parallel_1} v_{z_1} \right]}{\omega_1} - \frac{p^{(0)+} k_{\perp_1}^2 R_+^2}{\omega_1} k_{\perp_1} v_{x_1} \tag{8.28}$$

$$\Delta P_{\parallel_1}^{(1)} = \frac{P_\parallel^{(1)} \left[k_{\perp_1} v_{x_1} + 3 k_{\parallel_1} v_{z_1} \right]}{\omega_1} - \frac{p^{(0)+} k_{\perp_1}^2 R_+^2}{\omega_1} k_{\parallel_1} v_{z_1} \tag{8.29}$$

Ion FLR terms proportional to $k_{\parallel_1}$ have been neglected compared to those containing $k_{\perp_1}$.

The above equations could be reduced to obtain a growth rate for the non-linear three-wave decay. The result is more transparent, however, if the equations are reformulated in a Hamiltonian representation. The perturbations are now taken to be of the form $V_{x_1} = V_{x_1}(t) \exp[-i\omega_1 t + i\underline{k}_1 \cdot r]$ where $V_{x_1}(t)$ is considered to be slowly varying.

Introduction of the action wave amplitudes

$$C_0 = \sqrt{\frac{\rho_0}{|\omega_0|}}\, \delta V_x \;; \quad C_1 = \sqrt{\frac{\rho_0}{|\omega_1|}}\, V_{x_1} \;;$$

$$C_2 = \sqrt{\frac{\rho_0}{|\omega_2|}}\, V_{x_2} \tag{8.30}$$

and use of the above linear polarizations yields the following equations for C_1 and C_2

$$\frac{\partial C_1}{\partial t} = - \frac{k_\parallel^2 C_A^2\, k_{\perp 1}^3 \left(\frac{p^{(0)+}}{\rho_0}\right)}{2\Omega + \sqrt{\rho_0}\, \omega_1 \omega_2} \left[\frac{1}{\omega_0} + \frac{1}{\omega_1}\right] \left|\frac{\omega_0 \omega_1}{\omega_1}\right|^{1/2} C_0 C_2$$

(8.31)

$$\frac{\partial C_2}{\partial t} = \frac{k_{\perp 0}^3 \left(\frac{p^{(0)+}}{\rho_0}\right)}{2\Omega + \sqrt{\rho_0}} \left[\frac{1}{\omega_0} + \frac{1}{\omega_1}\right] \left|\frac{\omega_0 \omega_1}{\omega_2}\right|^{1/2} C_1 C_0^*$$

(8.32)

where $k_\parallel C_A/\omega_1 << 1$ was used. A similar equation could be written for $C_0(t)$.

If the amplitude C_o is considered to be much larger than the perturbation amplitudes C_1 and C_2, the time dependence of C_o can be neglected as a first approximation; 8.31 and 8.32 then combine to give

$$\frac{\partial^2 C_1}{\partial t^2} = - \frac{k_{\parallel}^2 k_{\perp}^2 C_A^2 \left[k_{\perp}^2 \left(\frac{p^{(0)+}}{\rho_0} \right) \right]^2}{4 \Omega_+^2 \rho_0} \frac{\omega_0}{\omega_1 \omega_2} \left[\frac{1}{\omega_2} + \frac{1}{\omega_1} \right]^2 |C_0|^2 C_1 \tag{8.33}$$

If $C_1(t) = C_1 e^{\gamma t}$, the square of the growth rate, γ^2, for the decay is given by the RHS of 8.33. There will be instability only if sign $(\omega_1 \omega_2) < 0$; otherwise the perturbations produce a non-linear shift in the frequency of C_o. From 8.17, implies that $|\omega_o| > |\omega_1|, |\omega_2|$ so that decay only occurs to waves of lower frequency.

With the above determination of the non-linear decay rate, a crude estimate is possible for the length of the R_+ trailing wave train. The wave train should persist downstream for a length of the order of $L \sim M_2 C_F / \gamma$. If the shock is assumed to be weak, $|C_o|^2$ is estimated as

$$|C_0|^2 \sim \frac{\rho_0 \omega_0}{k_{\perp_0}^2} \left(M_1^2 - 1 \right)^2 \tag{8.34}$$

The wavelength of the R_+ wave train is approximately $k_{\perp_o}^{-1} \sim R_+ (M_1 / \left|M_1^2 - 1\right|^{\frac{1}{3}})$. Substitution into 8.33 for γ then yields

$$L_s \sim \frac{M_1^2}{(M_1^2 - 1)^2} R_+ \tag{8.35}$$

as an estimate of the damping length.

The fast waves generated by the three-wave decay are probably ion transit-time damped in the downstream flow since their propagation is oblique to the magnetic field and $\beta_+ > 1$ (Barnes, 1967). Actually for the decay to occur, the non-linear decay rate must exceed the transit-time damping decrement within the wave train. Since the intermediate wave is undamped at any propagation angle, Alfvén turbulence should persist in the downstream flow for large distances. Note, however, that the shock theories of Camac _et al._ (1962) and Kennel and Sagdeev (1967a) also predict downstream Alfvén turbulence.

The above calculation is illustrative of only one possible three-wave mode coupling; in principle, decay into other wave modes should also be considered. Hence 8.35 is an upper limit to the damping length. However the calculation was restricted to consideration of only slightly non-linear wave trains. If the oscillation amplitude is large, which is certainly the case in Figure 13, there is no assurance that the above decay would occur and limit the wave train length; the application of this calculation to the earth's bow shock is somewhat in question.

8.5 Discussion

This section has attempted to illuminate several diverse points which may be summarized as follows:

1. Observations of the earth's bow shock from OGO-V indicate that the magnetic field transition often assumes a laminar form. The leading oscillations are probably correctly interpreted as a standing whistler with

scale length C/ω_{p_+}. The trailing wave train is here interpreted as having an ion gyroradius scale length.

2. The most persistent laminar feature of the bow shock is a short scale length C/ω_{p_-} wave train. In association with the maximum magnetic field gradients the electrostatic wave amplitude with frequencies near ω_{p_+} attain their largest value. This turbulence is thought to result from an electron current driven ion sound instability.

3. The presence of the C/ω_{p_-} wave train is here interpreted as a dispersion discontinuity required to limit the strong shock steepening. Since in the solar wind $\beta_- \sim 1$, the observed short scale length in the bow shock should also be associated with the electron gyroradius.

4. The non-linear effects of the ion-sound instability were very crudely estimated to yield an anomalous dissipation rate for electron-ion and ion-ion collisions. The electron and ion parallel heat flows were suppressed by these collisions so that thermal conduction makes a negligible contribution to the total shock dissipation. The length of the C/ω_{p_-} - R_- wave train required to thermalize the ion distribution was estimated as 5 to 10 C/ω_{p_-}.

5. An additional dissipation mechanism may occur due to the parametric three-wave decay of the wave train. A particular example of a trailing perpendicular R_+ wave train was shown to be unstable to perturbations containing oblique fast waves and intermediate waves;

an estimate of the R_+ wave train damping length was made. Since the oblique fast waves should be damped when $\beta_+ > 1$, the downstream flow should be characterized by Alfvén turbulence.

The discussion of this section should not be considered in any way to constitute a theory of the earth's bow shock. The assumptions required to perform the wave train analysis and the imprecise formulation of the dispersion discontinuity permit only the denotation of resemblances between theory and experiment. In addition, the estimates of the turbulent dissipation, aside from being very approximate, are not self-consistent.

9.0 Discussion

Since adequate summaries of the calculational results are contained at the end of their respective sections and in the figures, this section will be devoted to a few comments, generally non-rigorous, about collisionless shocks. After reviewing the fluid approach philosophy in section 9.1, some new directions for collisionless shock studies are outlined.

9.1 The Philosophy of the Fluid Approach

The fluid approach describes the collisionless shock structure as an ordered or laminar wave train in which a standing wave pattern connects the uniform upstream and downstream states. The wavelength is characterized by one or more of the dispersion scale lengths which are present in the theory of linear wave propagation derived from either the Vlasov theory or generalized hydromagnetic equations. Provided that a complete set of conservation laws exist and the moment hierarchy of the Vlasov equation can be truncated, the fluid equations reduce to a system of differential equations from which the shock structure can be deduced. In this theory the shock dissipation is provided by very weak Coulomb collisions, micro-turbulence associated with the wave train structure, or the non-linear three-wave decay of the wave train itself.

The main direction here has been to apply the fluid approach to finite-β collisionless shocks by investigating the FLR-CGL equations. An immediate difficulty was encountered, however: the FLR-CGL equations do not form a complete set of Rankine-Hugoniot conservation relations. Since the FLR dynamics couples the degrees of freedom, the division of energy between the parallel and perpendicular directions cannot be followed independently. Stated in the language of classical mechanics, the number of particles is conserved, translational invariance implies momentum is conserved, and time reversal invariance yields conservation of total energy. Except for perpendicular shocks where the Hamiltonian would be independent of the angle between the flow velocity and magnetic field direction, there are no rotational symmetries for oblique plane shock flows which might yield further conservation relations. Separation of the diagonal pressure tensor elements into parallel and perpendicular components introduces an additional degree of freedom into the system which is not accounted for in the fundamental conservation laws.

In low-β plasmas where pressure effects are relatively unimportant, the moment equations can be closed by neglecting the third order heat flow moments along the magnetic field. Also since perpendicular shock flows are two dimensional, the parallel heat flow does not enter the fluid equations. For finite-β non-perpendicular shocks, however, the moment equations are not closed since the parallel heat flow is not determined by the fluid equations.

To continue pursuit of the fluid approach, two assumptions were made: (1) the pressure was taken to be isotropic thus reducing by one the number of degrees of freedom and closing the Rankine-Hugoniot relations; (2) the parallel heat flow was neglected. Clearly neither of these assumptions is wholly satisfactory since important details of the shock flow are not determined.

To improve upon the above calculations, however, requires a self-consistent formulation for the kinetic theory of plasma turbulence within the structure of the fluid equations. Such a turbulence theory would provide a relation between the parallel and perpendicular pressure thus specifying the rate of isotropization by either macro- or micro-turbulence or describing the approach to instability. Furthermore, knowledge of the turbulent "collision" operator in the kinetic equation would permit a determination of the parallel heat flow.

Given the above assumptions, what are the restrictions placed upon the validity of the wave train solutions obtained here? The assumption of pressure isotropy could probably be considerably relaxed without seriously modifying the solutions provided that the anisotropy never attains a sufficiently large value to satisfy the mirror or firehose instability conditions. To close the Rankine-Hugoniot relations all that is really required is some relation between $P_{\|}$ and $P_{\perp}$; for example, assuming that the anisotropy was a constant through the shock constitutes an alternative choice to the isotropy assumption. Actually the linearized analysis about the stationary points to determine the wave train structure depended only on taking $\delta P_{\|} = \delta P_{\perp}$; as long as the fluid instabilities are avoided, the anisotropy $P_{\|} \neq P_{\perp}$ could have been retained without substantially modifying the wave train solutions. Of course, taking $P_{\|} = P_{\perp}$ greatly simplified the algebra.

The restrictions imposed by neglecting the parallel heat flow are more difficult to evaluate. For both fast and slow hydromagnetic shocks thermal conduction alone can provide the required shock dissipation only if the shock is weak (Coroniti, 1969);

resistivity and/or viscosity are needed for moderate strength shocks. Thus it might appear that the above solutions would not be seriously altered by neglecting the parallel heat flow. In a collisionless shock, however, it may not always be possible to separate the various modes of plasma turbulence into categories analogous to the collisional dissipation coefficients. For example, the propagation of unstable waves or waves generated by three-wave decay out of the shock might contribute a large radiation transport or effective heat flow (see Camac et al., 1962 and Kennel and Sagdeev, 1967a). If the waves propagate a large distance before they are damped, this effective heat flow would greatly broaden the shock structure and contribute non-negligibly to the total dissipation. Therefore the importance of heat flow cannot be assessed without first determining the turbulent dissipation.

To proceed further with the fluid theory of collisionless shocks will require a reformulation of the fluid equations which includes a prescription for calculating the dissipation coefficients in a self-consistent manner. Starting from an appropriate kinetic theory of plasma turbulence, the moments of the kinetic equation will provide a closed set of fluid equations which can then be solved to determine the shock structure. Weak plasma turbulence or quasi-linear theory will probably prove inadequate to describe the shock dissipation since only small wave amplitudes and often special resonant subsets of the particle distributions are considered. Perhaps a kinetic theory based on the methods of Dupree (1966) might be useful for some shock applications since the diffusive broadening of the velocity space resonances implies that a large fraction of the distribution function is affected by the wave turbulence. In any case no complete and self-consistent fluid shock theory can be developed until an adequate theory of strong plasma turbulence has been formulated.

9.2 Dispersion Discontinuity

Turning now to future applications of the fluid approach, the concept of the dispersion discontinuity and its possible importance in interpreting the observed structure of the earth's bow shock may provide an initiating point for a self-consistent shock theory. Before such a theory can be formulated, however, the dispersion discontinuity must be placed on firmer theoretical ground.

First the dispersion discontinuity should be deducible from a self-consistent set of wave train differential equations. An indication of this is obtainable by considering the $M \rightarrow \infty$ limit of 6.12 and 6.16, the linearized differential equations but without the weak shock approximation. The coefficients of ion inertia and ion FLR vanish in this limit leaving only C/ω_{p_-} dispersion; the potential terms remain finite. Since 6.12 and 6.16 decouple, the equations now describe a C/ω_{p_-} fast shock trailing wave train in the magnitude of the magnetic field which is completely insensitive to the shock propagation angle. Hence infinite strength fast shocks exhibit a C/ω_{p_-} dispersion discontinuity.

The earth's bow shock data indicates that the dispersion discontinuity becomes important for much smaller Mach numbers, however, and determination of the critical Mach number for the onset of this effect is an immediate theoretical problem. The self-consistent wave train differential equation may resolve this question. Perhaps, however, a more accurate formulation of the energy balance arguments of section 7.5 may be necessary to determine the critical Mach number.

9.3 Slow Shocks

Whereas fast collisionless shocks have been verified experimentally and have received considerable theoretical attention, the existence and structure of collisionless slow shocks remain in doubt. Yet the slow shock presents an intriguing problem for plasma turbulence theory since in the oblique, near switch-off limit where magnetic energy is converted into kinetic and thermal energy, an efficient turbulent resistive dissipation process must be found. In the astrophysics and space physics literature the large magnetic annihilation associated with neutral sheet flows bounded by slow shocks was originally investigated by Petschek (1964) in connection with solar flares and was also applied to the magnetospheric convective flow by Levy *et al.* (1964) and Axford *et al.* (1965). Considering its possible importance a few comments on the structure of slow shocks seems appropriate; discussion will be limited to oblique, low-β near switch-off shocks since the low-β parallel electrostatic shock has already received some attention (Bernstein *et al.*, 1957; Montgomery and Joyce, 1969).

The primary difficulty in constructing an oblique collisionless slow shock is that the shock propagation speed is much less than the ion thermal speed downstream, provided ions and electrons share the annihilated magnetic energy. The problem is, can the hot ions be contained behind the shock or will there exist a large ion flow upstream? (Electrons, because of their small inertia, can always be dragged through the shock by an electric field.) Penetration of the hot shocked ions upstream constitutes a large heat flow so that the slow shock may simply degenerate into a broad thermal diffusion layer; the concept of a shock in this instance may be meaningless.

A possible resolution of this difficulty is to postulate that only the electrons are heated in the shock with the ions remaining cold. Thermalization between the two species might occur far downstream. Since resistivity must be the primary dissipation process, a current instability is likely to be part of the shock structure and may preferentially heat electrons over ions. The wave train solutions of section 6.0 suggested that the magnetic gradients in the shock are of the order of C/ω_{p_+}. Again estimating the self-consistent electron current from Ampere's law yields a current drift velocity $V_D^- \sim C_A$ which, if $\beta << 1$ and $T_-/T_+ >> 1$, should destabilize the ion sound wave.

If the ions are heated by the resistive current instability, as they are in the earth's bow shock, then the ion heat flow upstream must be considered as part of the shock structure. For the near switch-off shock the downstream ions are flowing into a magnetic mirror at the leading edge of the shock; hence many of the ions will be reflected by the mirror, but some ions, and electrons, at small pitch angles will penetrate the upstream flow. The distribution functions for both species upstream will be of a Mott-Smith type. Since the contribution from the downstream particles is aligned along the magnetic field, $T_{\parallel\pm} >> T_{\perp\pm}$ for this component. If there are a sufficient number of hot particles, the combined distribution functions may be unstable to anisotropy-type instabilities; for example, the low frequency whistler mode is driven unstable if $T_{\parallel +} >> T_{\perp +}$, and the instability is greatly enhanced if the electrons also have $T_{\parallel -} >> T_{\perp -}$ (Kennel and Scarf, 1968). The non-linear effects of the instability will be to isotropize the ion distributions by pitch angle diffusion and to cool off the hot upstream ions. The unstable whistlers may be eventually damped by the cold upstream plasma, and hence contribute to the overall shock heating. In any case the anisotropy instability should suppress the upstream ion heat flow, and thus limit the shock to a finite and distinct thickness.

The above crude model suggests that the turbulent dissipation for the low-β oblique slow shock may be of two forms: a downstream current driven ion sound instability which dissipates magnetic energy and an upstream anisotropy instability which limits the ion heat flow. Clearly the model does not exhaust the possible modes of turbulence which might contribute to the dissipation, and the detailed shock structure remains to be calculated. In this regard the method employed by Tidman (1967) based on the Mott-Smith distribution might be useful in calculating the structure of the leading edge.

Appendices

A.1 Linear Wave FLR-CGL Dispersion Relation

The linearized form of equations 4.1, 4.2, 4.8, 4.9, 4.10, 4.11, 4.12, 4.14, 4.15, 4.18, 4.19 after Fourier analysis become

$$\omega\,\delta\rho - \rho_0\,\underline{k}\cdot\delta\underline{v} = 0 \tag{A.1.1}$$

$$\omega\rho_0\,\delta\underline{v} = \underline{k}\cdot\delta\underset{\approx}{P}^{(1)} + \frac{\underline{B}_0\times\underline{k}\times\delta\underline{B}}{4\pi} \tag{A.1.2}$$

$$\omega\left(1+\frac{k^2c^2}{\omega_{p_+}^2}\right)\delta\underline{B} = -\underline{k}\times\delta\underline{v}\times\underline{B}_0 - i\frac{B_0}{\Omega_+}\omega\,\underline{k}\times\delta\underline{v}$$

$$+\frac{c}{Ne}\underline{k}\times\left(\underline{k}\cdot\delta\underset{\approx}{P}^{(1)}\right) \tag{A.1.3}$$

$$\delta\underset{\approx}{P}^{(1)} = \delta P_{\parallel}^{(1)}\hat{e}_z\hat{e}_z + \delta P_{\perp}^{(1)}\left(\underset{\approx}{I} - \hat{e}_z\hat{e}_z\right) + \left(\delta P_{ij}^{(1)}\right)_{i\neq j}\hat{e}_i\hat{e}_j$$

$$+\left(P_{\parallel}^{(0)} - P_{\perp}^{(0)}\right)\hat{e}_z\,\delta\left(\frac{\underline{B}}{|\underline{B}|}\right) \tag{A.1.4}$$

$$\omega\,\delta P_{\perp}^{(1)} - P_{\perp}^{(0)}\left(\underline{k}\cdot\delta\underline{v} + \underline{k}_{\perp}\cdot\delta\underline{v}\right) - \underline{k}\cdot\delta\underline{q}_{\perp}^{\perp(1)} = 0 \tag{A.1.5}$$

$$\omega\,\delta P_{\parallel}^{(1)} - P_{\parallel}^{(0)}\left(3\,\underline{k}\cdot\delta\underline{v} - 2\,\underline{k}_{\perp}\cdot\delta\underline{v}\right) - 2\,\underline{k}\cdot\delta\underline{q}_{\perp}^{\parallel(1)} = 0 \tag{A.1.6}$$

$$\delta P_{xx}^{(1)} = \delta P_{\perp}^{(1)} - i\,\frac{k_\perp P_\perp^{(0)+}}{2\Omega_+}\,\delta v_y \tag{A.1.7}$$

$$\delta P_{yy}^{(1)} = \delta P_{\perp}^{(1)} + i\,\frac{k_\perp P_\perp^{(0)+}}{2\Omega_+}\,\delta v_y \tag{A.1.8}$$

$$\delta P_{xy}^{(1)} = \delta P_{yx}^{(1)} = i\,\frac{k_\perp P_\perp^{(0)+}}{2\Omega_+}\,\delta v_x \tag{A.1.9}$$

$$\delta P_{xz}^{(1)} = \delta P_{zx}^{(1)} = -i\,\frac{(2P_\parallel^{(0)+} - P_\perp^{(0)+})}{\Omega_+}\,k_\parallel \delta v_y \tag{A.1.10}$$

$$\delta P_{yz}^{(1)} = \delta P_{zy}^{(1)} = \frac{i}{\Omega_+}\Big[(2P_\parallel^{(0)+} - P_\perp^{(0)+})\,k_\parallel \delta v_x + P_\perp^{(0)+}\,k_\perp \delta v_z\Big] \tag{A.1.11}$$

$$\delta P_{zz}^{(1)} = \delta P_\parallel^{(1)} \tag{A.1.12}$$

$$\delta \underline{q}_\perp^{(1)} = i\,\frac{\hat{e}_z \times}{2\Omega_+}\Big[\underline{k}\,\delta R_2^+ - \frac{4P_\perp^{(0)+}}{\rho_0}\,\underline{k}\cdot\delta \underset{\approx}{P}^{(1)} + 4\Big(R_1^+ - \frac{R_2^+}{4}\Big)k_\parallel\,\delta\Big(\frac{\underline{B}}{|\underline{B}|}\Big)\Big] \tag{A.1.13}$$

$$\delta \underline{q}_\perp^{(1)(1)} = i \, \frac{\hat{e}_z \times}{2\Omega_+} \Big[\underline{k} \, \delta R_1^+ - \frac{P_\parallel^{(1)+}}{\rho_0} \underline{k} \cdot \delta \underset{\approx}{P}^{(1)}$$

$$+ (R_3^+ - 3R_1^+) \, k_\parallel \, \delta \left(\frac{\underline{B}}{|\underline{B}|} \right) \Big] \tag{A.1.14}$$

$$\delta \left(\frac{\underline{B}}{|\underline{B}|} \right) = - \frac{k_\parallel}{\omega} \left(\delta V_x \hat{e}_x + \delta V_y \hat{e}_y \right) \tag{A.1.15}$$

The anisotropy will be retained until the final determination of the dispersion relation. Note that only the ion FLR corrections have been included.

The calculation proceeds by eliminating all the perturbed quantities in terms of $\underline{\delta V}$. Since this amounts to straightforward algebra, only the salient results are given. Expanding A.1.3 yields the components of $\underline{\delta B}$

$$\delta B_x = \frac{B_0}{1 + \frac{k^2 c^2}{\omega_{pi}^2}} \Big[- \frac{k_\parallel \delta V_x}{\omega} + i \, \frac{k_\parallel \delta V_y}{\Omega_+}$$

$$+ \frac{k_\parallel k_\perp^2 r_+^2}{\omega} \delta V_x + \frac{k_\parallel^2 k_\perp r_+^2 \Delta}{\omega} \delta V_x + \frac{2 k_\perp k_\parallel^2 r_+^2}{\omega} \delta V_z \tag{A.1.16}$$

$$+ i \, \frac{k_\parallel^3 \left(P_{\parallel 0}^{(1)} - P_{\perp 0}^{(1)} \right)}{\omega^2 \Omega_+ \rho_0} \delta V_y \Big]$$

$$\delta B_y = \frac{B_0}{1 + \frac{k^2c^2}{\omega_{p_0}^2}} \left[- \frac{k_\parallel \delta v_y}{\omega} + i \, \frac{(k_\parallel \delta v_x - k_\perp \delta v_z)}{\Omega_+} \right.$$

$$+ i \, \frac{k_\parallel k_\perp (\delta P_\perp^{(1)} - \delta P_\parallel^{(1)})}{\omega \rho_0 \Omega_+} + \frac{k_\parallel k_\perp^2 r_+^2}{\omega} \delta v_y + \frac{k_\parallel^3 r_+^2 \Delta}{\omega} \delta v_y$$

$$- i \, \frac{k_\parallel^3 (P_{\parallel 0}^{(1)} - P_{\perp 0}^{(1)})}{\omega^2 \Omega_+ \rho_0} \delta v_x - \frac{k_\parallel k_\perp^2 r_+^2 \Delta}{\omega} \delta v_y$$

$$\left. + i \, \frac{k_\parallel k_\perp^2 (P_{\parallel 0}^{(1)} - P_{\perp 0}^{(1)})}{\omega^2 \Omega_+ \rho_0} \delta v_x \right]$$

(A.1.17)

$$\delta B_z = \frac{B_0}{1 + \frac{k^2c^2}{\omega_{p_0}^2}} \left[\frac{k_\perp \delta v_x}{\omega} - i \, \frac{k_\perp \delta v_y}{\Omega_+} - \frac{k_\perp^3 r_+^2}{\omega} \delta v_x \right.$$

$$- \frac{k_\perp k_\parallel^2 r_+^2 \Delta}{\omega} \delta v_x - \frac{2 k_\parallel k_\perp^2 r_+^2}{\omega} \delta v_z$$

$$\left. - i \, \frac{k_\perp k_\parallel^2 (P_{\parallel 0}^{(1)} - P_{\perp 0}^{(1)})}{\omega^2 \Omega_+ \rho_0} \right] \qquad \text{(A.1.18)}$$

where $r_+^2 = P_\perp^{(0)+} / 2\rho_o \Omega_+^{\ 2}$ and $\Delta = 2\langle (2P_\parallel^{(0)+} / P_\perp^{(0)+}) - 1 \rangle$. The pressure perturbations are

$$\delta P_\perp^{(1)} = P_{\perp_0}^{(1)} \left[\frac{2k_\perp \delta V_x + k_\parallel \delta V_z}{\omega} \right] - \frac{2P_\perp^{(0)+} k_\perp^3 r_+^2}{\omega} \delta V_x$$

$$- \frac{4 P_\perp^{(0)+} k_\parallel^2 k_\perp r_+^2 \Delta}{\omega} \delta V_x - \frac{4 P_\perp^{(0)+} k_\perp^2 k_\parallel r_+^2}{\omega} \delta V_z$$

$$- i \frac{2 P_\perp^{(0)+}}{\rho_0 \Omega_+} \frac{k_\parallel^2 k_\perp (P_{\parallel_0}^{(1)} - P_{\perp_0}^{(1)})}{\omega^2} \delta V_y + i \frac{2 k_\parallel^2 k_\perp (R_1^+ - \frac{R_2^+}{4})}{\Omega_+ \omega^2} \delta V_y$$

(A.1.19)

$$\delta P_\parallel^{(1)} = P_{\parallel_0}^{(1)} \left[\frac{k_\perp \delta V_x + 3 k_\parallel \delta V_z}{\omega} \right] - \frac{P_\parallel^{(0)+} k_\perp^3 r_+^2}{\omega} \delta V_x$$

$$- \frac{2 P_\parallel^{(0)+} k_\parallel^2 k_\perp r_+^2 \Delta}{\omega} \delta V_x - \frac{2 P_\parallel^{(0)+} k_\perp^2 k_\parallel r_+^2}{\omega} \delta V_z$$

$$- i \frac{k_\parallel^2 k_\perp^2 P_\parallel^{(0)+} (P_{\parallel_0}^{(1)} - P_{\perp_0}^{(1)})}{\omega^2 \Omega_+ \rho_0} \delta V_y$$

$$+ i \frac{k_\parallel^2 k_\perp (R_3^+ - 3 R_1^+)}{\Omega_+ \omega^2} \delta V_y \qquad \text{(A.1.20)}$$

$\delta \underset{\approx}{P}^{(1)}$ and $\underline{\delta B}$ are now determined in terms of $\underline{\delta V}$. Substitution of A.1.16-A.1.20 into A.1.2 yields a 3 x 3 matrix, the determinant of which gives the dispersion relation. Assuming pressure isotropy the determinant can be written in the reasonably simple form

$$
\begin{vmatrix}
\omega^2 - k^2 C_A^2 \psi - k_\perp^2\left(\dfrac{2P_o^{(1)}}{\rho_o}\right)\psi' & i\dfrac{\omega}{\Omega_+}\left(k^2C_A^2 + \epsilon_{12}\right) & -\left[k_\parallel k_\perp\left(\dfrac{P_o^{(1)}}{\rho_o}\right)\chi - 2k^2C_A^2\, k_\parallel k_\perp r_+^2\right] \\[2ex]
-i\dfrac{\omega}{\Omega_+}\left[k_\parallel^2 C_A^2\left(1 - \dfrac{k_\perp^2\left(\dfrac{P_o^{(1)}}{\rho_o}\right)}{\omega^2}\psi'\right) + \epsilon_{12}\right] & \omega^2 - k_\parallel^2 C_A^2\left(\psi + 2k_\perp^2 r_+^2\right) & -i\dfrac{\omega}{\Omega_+}\left[k_\parallel k_\perp\left(\dfrac{P_o^{(0)+}}{\rho_o}\right) - k_\parallel k_\perp C_A^2\left(1 - \dfrac{k_\parallel^2\left(\dfrac{2P_o^{(1)}}{\rho_o}\right)}{\omega^2}\chi'\right)\right] \\[2ex]
-k_\parallel k_\perp\left(\dfrac{P_o^{(1)}}{\rho_o}\right)\psi' & i\dfrac{\omega}{\Omega_+}k_\parallel k_\perp\left(\dfrac{P^{(0)+}}{\rho_o}\right) & \omega^2 - k_\parallel^2\left(\dfrac{3P_o^{(1)}}{\rho_o}\right) + k_\parallel^2\left(\dfrac{2P^{(0)+}}{\rho_o}\right)k_\perp^2 r_+^2
\end{vmatrix}
= 0
$$

where $k^2 = k_\perp^2 + k_\parallel^2$, $C_A^2 = (B_o^2/4\pi\rho)(1/\langle 1+k^2C^2/\omega_{p_-}^2\rangle)$, $P_o^{(1)} = P_{\perp o}^{(1)} = P_{\parallel o}^{(1)}$, $\psi = 1 - k_\perp^2 C_+^2 - 2k_\parallel^2 r_+^2$, $\psi' = 1 - \alpha k_\perp^2 r_+^2 - 2\alpha k_\parallel^2 r_+^2$, $\chi = 1 - 4\alpha k_\perp^2 r_+^2$, $\chi' = 1 + \alpha k_\perp^2 r_+^2$, $\alpha = P_o^{(0)+}/P_o^{(1)}$, $\epsilon_{12} = k_\perp^2(P^{(0)+}/2\rho_o) + k_\parallel^2(P^{(0)+}/\rho_o)$.

In expanding the determinant to obtain the first order corrections to CGL hydromagnetic terms of order $(k^2 r_+{}^2)^2$ and $(k^2 r_+{}^2)\, k^2 C_A^2/\Omega_+{}^2$ can be neglected. The dispersion relation is

$$\left[\frac{\omega^2}{k^2} - C_I^2\right]\left[\frac{\omega^2}{k^2} - C_F^2\right]\left[\frac{\omega^2}{k^2} - C_{SL}^2\right] - \frac{k^2 C_A^2}{\Omega_+^2} C_I^2 \left[\left(\frac{\omega^2}{k^2} - \frac{2P_0^{(1)}}{\rho_0}\sin^2\theta\right)\left(\frac{\omega^2}{k^2} - \frac{3P_0^{(1)}}{\rho_0}\cos^2\theta\right) - \sin^2\theta\cos^2\theta\left(\frac{P_0^{(1)}}{\rho_0}\right)^2\right]$$

$$+ \frac{\omega^4}{k^4} M - \frac{\omega^2}{k^2} N + L = 0$$

(A.1.21)

where

$$M = \frac{P^{(0)+}}{\rho_0}\left[\frac{3}{4}\sin^2\theta\, K_\perp^2 R_+^2 + \cos^2\theta\, K_\perp^2 R_+^2 - \cos^2\theta\, K_\parallel^2 R_+^2\right] \tag{A.1.22}$$

$$N = C_I^2\left[C_A^2\left(K_\perp^2 R_+^2 + 2K_\parallel^2 R_+^2\right) + \sin^2\theta \frac{P^{(0)+}}{\rho_0} K_\perp^2 R_+^2 + \frac{3P^{(0)+}}{\rho_0}\cos^2\theta\, K_\parallel^2 R_+^2\right] + \frac{P_0^{(1)}\cos^2\theta}{\rho_0}\left[\frac{11}{4}\frac{P^{(0)+}}{\rho_0}\sin^2\theta\, K_\perp^2 R_+^2 + 4\frac{P^{(0)+}}{\rho_0}\cos^2\theta\, K_\perp^2 R_+^2 - \frac{3P^{(0)+}}{\rho_0}\cos^2\theta\, K_\parallel^2 R_+^2\right]$$

$$- 2\cos^2\theta\sin^2\theta\left(\frac{P^{(0)+}}{\rho_0}\right)^2 K_\perp^2 R_+^2 \tag{A.1.23}$$

$$L = C_I^2 \Bigg\{ C_A^2 \Bigg[\frac{P_o^{(1)}}{\rho_o} \cos^2\theta \, K_\perp^2 R_+^2 + \frac{P^{(0)+}}{\rho_o} \cos^2\theta \, K_\perp^2 R_+^2$$

$$+ \frac{6 P_o^{(1)}}{\rho_o} \cos^2\theta \, K_\parallel^2 R_+^2 \Bigg]$$

$$+ 5 \left(\frac{P_o^{(1)}}{\rho_o} \right) \left(\frac{P^{(0)+}}{\rho_o} \right) \sin^2\theta \cos^2\theta \left[\frac{K_\perp^2 R_+^2}{2} + K_\parallel^2 R_+^2 \right] \Bigg\}$$

(A.1.24)

where $R_+^2 = P_o^{(0)+} / 2\Omega_+^2 \rho_o$, and θ is the angle between $\underline{k}$ and $\underline{B}_o$.

A.2 Components of $\underset{\approx}{P}^{(1)}$, $\underset{\sim}{q}_{\perp}^{\perp(1)}$, $\underset{\sim}{q}_{\parallel}^{\parallel(1)}$, α_1 and α_2.

In formulating the FLR-CGL equations it is convenient to employ a coordinate system oriented about the magnetic field direction, the unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$. Plane shocks, however, are basically cartesian in that the plasma quantities are taken to vary only in the x-direction. It is, therefore, necessary to transform the various tensors and heat flow vectors into a cartesian system.

The magnetic field basis vectors become in cartesian coordinates

$$\hat{e}_1 = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta \tag{A.2.1}$$

$$\hat{e}_2 = \hat{y}\cos\phi - \hat{x}\sin\phi \tag{A.2.2}$$

$$\hat{e}_3 = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta \tag{A.2.3}$$

where

$$\cos\theta = \frac{B_z}{B}\ ; \quad \sin\theta = \frac{B_\perp}{B}\ ; \quad B_\perp = (B_x^2 + B_y^2)^{1/2}$$

$$\cos\phi = \frac{B_x}{B_\perp}\ ; \quad \sin\phi = \frac{B_y}{B_\perp} \tag{A.2.4}$$

The various tensor elements required in the calculation of $\underset{\approx}{P}^{(1)}$ are

$$\hat{e}_3\hat{e}_3 = \hat{x}\hat{x}\,\sin^2\theta\cos^2\phi + (\hat{x}\hat{y}+\hat{y}\hat{x})\sin^2\theta\cos\phi\sin\phi$$
$$+ (\hat{x}\hat{z}+\hat{z}\hat{y})\sin\theta\cos\theta\cos\phi + \hat{y}\hat{y}\,\sin^2\theta\sin^2\phi$$
$$+ (\hat{y}\hat{z}+\hat{z}\hat{y})\sin\theta\cos\theta\sin\phi$$
$$+ \hat{z}\hat{z}\cos^2\theta \tag{A.2.5}$$

$$\hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2 = \hat{x}\hat{x}(\cos^2\theta\cos^2\phi + \sin^2\phi)$$
$$- (\hat{x}\hat{y}+\hat{y}\hat{x})\sin^2\theta\sin\phi\cos\phi$$
$$- (\hat{x}\hat{z}+\hat{z}\hat{x})\cos\theta\sin\theta\cos\phi$$
$$+ \hat{y}\hat{y}(\cos^2\theta\sin^2\phi + \cos^2\phi)$$
$$- (\hat{y}\hat{z}+\hat{z}\hat{y})\cos\theta\sin\theta\sin\phi$$
$$+ \hat{z}\hat{z}\sin^2\theta \tag{A.2.6}$$

$$\underset{\approx}{I}_{\delta} = \hat{e}_1\hat{e}_1 - \hat{e}_2\hat{e}_2 = \hat{x}\hat{x}(\cos^2\theta\cos^2\phi - \sin^2\phi)$$
$$+ (\hat{x}\hat{y}+\hat{y}\hat{x})\cos\phi\sin\phi(1+\cos^2\theta)$$
$$- (\hat{x}\hat{z}+\hat{z}\hat{x})\cos\theta\sin\theta\cos\phi$$
$$+ \hat{y}\hat{y}(\cos^2\theta\sin^2\phi - \cos^2\phi)$$
$$- (\hat{y}\hat{z}+\hat{z}\hat{y})\cos\theta\sin\theta\sin\phi$$
$$+ \hat{z}\hat{z}\sin^2\theta \tag{A.2.7}$$

$$\underset{\approx}{I}_\gamma = \hat{e}_1\hat{e}_2 + \hat{e}_2\hat{e}_1 = -2\hat{x}\hat{x}\cos\theta\cos\phi\sin\phi$$
$$+ (\hat{x}\hat{y}+\hat{y}\hat{x})\cos\theta(\cos^2\phi - \sin^2\phi)$$
$$+ 2\hat{y}\hat{y}\cos\theta\cos\phi\sin\phi$$
$$+ (\hat{x}\hat{z}+\hat{z}\hat{x})\sin\theta\sin\phi$$
$$- (\hat{y}\hat{z}+\hat{z}\hat{y})\sin\theta\cos\phi \qquad \text{(A. 2. 8)}$$

Using A. 2. 8, 4. 10 becomes

$$\underset{\approx}{P}_\delta^{(1)} = \underset{\approx}{I}_\delta\Big\{-\frac{P_\perp^{(0)}}{2\Omega}\Big[-2\cos\theta\cos\phi\sin\phi\frac{du}{dx}$$
$$+\cos\theta(\cos^2\phi-\sin^2\phi)\frac{dV_y}{dx}+\sin\theta\sin\phi\frac{dV_z}{dx}\Big]$$
$$-\frac{q_\parallel^{\perp(0)}}{2\Omega}\Big[-2\cos\theta\cos\phi\sin\phi\frac{d}{dx}(\sin\theta\cos\phi) \qquad \text{(A. 2. 9)}$$
$$+\cos\theta(\cos^2\phi-\sin^2\phi)\frac{d}{dx}(\sin\theta\sin\phi)+\sin\theta\sin\phi\frac{d}{dx}(\cos\theta)\Big]\Big\}$$

where all quantities are to be evaluated for both ions and electrons. Similarly, 4. 11 is

$$\underset{\approx}{P}_\gamma^{(1)} = \underset{\approx}{I}_\gamma\Big\{\frac{P_\perp^{(0)}}{2\Omega}\Big[(\cos^2\theta\cos^2\phi-\sin^2\phi)\frac{du}{dx}$$
$$+\cos\phi\sin\phi(1+\cos^2\theta)\frac{dV_y}{dx}-\cos\theta\sin\theta\cos\phi\frac{dV_z}{dx}\Big]$$
$$+\frac{q_\parallel^{\perp(0)}}{2\Omega}\Big[(\cos^2\theta\cos^2\phi-\sin^2\phi)\frac{d}{dx}(\sin\theta\cos\phi)$$
$$+\cos\phi\sin\phi(1+\cos^2\theta)\frac{d}{dx}(\sin\theta\sin\phi)$$
$$-\sin\theta\cos\theta\cos\phi\frac{d}{dx}(\cos\theta)\Big]\Big\} \qquad \text{(A. 2. 10)}$$

Equation 4.12 becomes

$$\underset{\approx}{P}_{31}^{(1)} + \underset{\approx}{P}_{13}^{(1)} = \frac{1}{\Omega}\Big\{ 2P_{11}^{(0)} \sin\theta \sin\phi \cos\phi \frac{du}{dx} + \Big[P_{\perp}^{(0)} \sin\theta$$

$$- 2P_{11}^{(0)} \sin\theta \cos^2\phi \Big] \frac{dV_y}{dx} + P_{\perp}^{(0)} \cos\theta \sin\phi \frac{dV_z}{dx}$$

$$+ 2\left(q_{11}^{\parallel(0)} - q_{11}^{\perp(0)} \right) \sin\theta \cos^2\phi \frac{1}{B} \frac{dB_y}{dx} + \sin\phi \frac{dq_{11}^{\perp(0)}}{dx} \Big\} \cdot$$

$$\Big\{ 2\hat{x}\hat{x} \sin\theta \cos\theta \cos^2\phi + 2(\hat{x}\hat{y} + \hat{y}\hat{x}) \sin\theta \cos\theta \cdot$$

$$\sin\phi \cos\phi + (\hat{x}\hat{z} + \hat{z}\hat{x}) \cos\phi (\cos^2\theta - \sin^2\theta)$$

$$+ 2\hat{y}\hat{y} \sin\theta \cos\theta \sin^2\phi$$

$$+ (\hat{y}\hat{z} + \hat{z}\hat{y}) \sin\phi (\cos^2\theta - \sin^2\theta)$$

$$- 2\hat{z}\hat{z} \sin\theta \cos\theta \Big\}$$

(A.2.11)

$$\underset{\approx}{P}_{23}^{(1)} + \underset{\approx}{P}_{32}^{(1)} = \frac{1}{\Omega}\Big\{ 2P_{11}^{(0)} \sin\theta \cos\theta \cos^2\phi \frac{du}{dx}$$

$$+ 2P_{11}^{(0)} \sin\theta \cos\theta \cos\phi \sin\phi \frac{dV_y}{dx} + \Big(P_{\perp}^{(0)} \cos\phi$$

$$- 2P_{11}^{(0)} \sin^2\theta \cos\phi \Big) \frac{dV_z}{dx} + 2\left(q_{11}^{\parallel(0)} - q_{11}^{\perp(0)} \right) \sin\theta \cos\phi \cdot$$

$$\left(- \frac{\sin\theta}{B} \frac{dB_z}{dx} + \frac{\sin\phi \cos\theta}{B} \frac{dB_y}{dx} \right) + \cos\phi \cos\theta \frac{dq_{11}^{\perp(0)}}{dx} \Big\} \cdot$$

$$\Big\{ -2\hat{x}\hat{x} \sin\theta \sin\phi \cos\phi + (\hat{x}\hat{y} + \hat{y}\hat{x}) \sin\theta (\cos^2\phi - \sin^2\phi)$$

$$- (\hat{x}\hat{z} + \hat{z}\hat{x}) \cos\theta \sin\phi$$

$$+ (\hat{y}\hat{z} + \hat{z}\hat{y}) \cos\phi \cos\theta$$

$$+ 2\hat{y}\hat{y} \sin\theta \sin\phi \cos\phi \Big\}$$

(A.2.12)

For $\underset{\approx}{P}^{(1)}$ only the components $P^{(1)}_{xx}$, $P^{(1)}_{xy}$, and $P^{(1)}_{xz}$ are needed. It is convenient for approximations to use A.2.4 in place of the trigonometric funtions. Taking the $\hat{x}\hat{x}$, $\hat{x}\hat{y}$, and $\hat{x}\hat{z}$ components of A.2.9-A.2.12 as well as those of $P_\perp^{(1)} \underset{\approx}{I}_\perp + P_\parallel^{(1)} \hat{e}_3\hat{e}_3$, the needed components become

$$P_{xx}^{(1)} = P_\perp^{(1)} \left(\frac{B_x^2 B_z^2}{B_\perp^2 B^2} + \frac{B_y^2}{B^2} \right) + P_\parallel^{(1)} \frac{B_x^2}{B^2} + T_{xx}^{(1)} \tag{A.2.13}$$

where

$$T_{xx}^{(1)} = T_{xx}^{y} \frac{dV_y}{dx} + T_{xx}^{z} \frac{dV_z}{dx} \tag{A.2.14}$$

$$+ \frac{1}{\Omega} \left\{ \frac{q_\parallel^{\perp(0)} B_z}{B_\perp^2 B^4} \left[4 B_x^2 (B_y^2 - B_x^2) - \frac{(B_y^2 B^2 + B_x^2 B_z^2)}{2} \right] + \frac{4 q_\parallel^{\parallel(0)} B_z B_x^2}{B^4 B_\perp^2} (B_x^2 - B_y^2) \right\} \frac{dB_y}{dx}$$

$$+ \frac{1}{\Omega} \left\{ \frac{q_\parallel^{\perp(0)} B_y}{B_\perp^2 B^4} \left[\frac{B_y^2 B^2 + B_x^2 B_z^2}{2} - 4 B_x^2 B_\perp^2 \right] + \frac{4 q_\parallel^{\parallel(0)} B_y B_x^2}{B^4} \right\} \frac{dB_z}{dx}$$

and

$$T_{xx}^{y} = -\frac{1}{\Omega}\left\{ P_{\perp}^{(0)}\left[\frac{B_z}{2B}\left(\frac{B_x^2 B_z^2}{B_\perp^2 B^2} + \frac{B_y^2}{B^2}\right) - \frac{2B_z B_x^2}{B^3}\right] + 4P_{\parallel}^{(0)}\frac{B_z B_x^2}{B^3}\right\} \qquad \text{(A.2.15)}$$

$$T_{xx}^{z} = \frac{1}{\Omega}\left\{ P_{\perp}^{(0)}\left[\frac{B_y}{2B}\left(\frac{B_x^2 B_z^2}{B_\perp^2 B^2} + \frac{B_y^2}{B^2}\right) - \frac{2B_y B_x^2}{B^3}\right] + 4P_{\parallel}^{(0)}\frac{B_y B_x^2}{B^3}\right\} \qquad \text{(A.2.16)}$$

Similarly

$$P_{xy}^{(1)} = \left(P_{\parallel}^{(1)} - P_{\perp}^{(1)}\right)\frac{B_x B_y}{B^2} + T_{xy}^{(1)} \qquad \text{(A.2.17)}$$

$$T_{xy}^{(1)} = T_{xy}^{x}\frac{du}{dx} + T_{xy}^{y}\frac{dv_y}{dx} + T_{xy}^{z}\frac{dv_z}{dx}$$

$$+ \frac{2}{\Omega}\frac{\left(q_{\parallel}^{\parallel(0)} - q_{\parallel}^{\perp(0)}\right)B_x B_y B_z}{B_\perp^2 B^4}\left(3B_x^2 - B_y^2\right)\frac{dB_y}{dx} \qquad \text{(A.2.18)}$$

$$+ \frac{1}{\Omega}\left\{\frac{q_{\parallel}^{\perp(0)} B_x}{B^4}\left[\frac{B_y^2 B_\perp^2 + B_x^2 B_z^2}{2B_\perp^2} + 2\left(B_x^2 - B_y^2\right)\right.\right.$$

$$\left.\left. - \frac{2q_{\parallel}^{\parallel(0)} B_x\left(B_x^2 - B_y^2\right)}{B^4}\right\}\frac{dB_z}{dx} + \frac{1}{\Omega}\frac{B_x B_z}{B^2}\frac{dq_{\parallel}^{\perp(0)}}{dx}$$

where

$$T_{xy}^{x} = \frac{1}{\Omega}\left\{ P_\perp^{(0)} \frac{B_z}{2B}\left(\frac{B_x^2 B_z^2}{B_\perp^2 B^2} + \frac{B_y^2}{B_\perp^2}\right) + 2P_\parallel^{(0)} \frac{B_z B_x^2}{B^3}\right\} \tag{A.2.19}$$

$$T_{xy}^{y} = \frac{2\left(P_\perp^{(0)} - P_\parallel^{(0)}\right)}{\Omega}\,\frac{B_x B_y B_z}{B^3} \tag{A.2.20}$$

$$T_{xy}^{z} = \frac{1}{\Omega}\left\{ \frac{P_\perp^{(0)}}{2}\,\frac{B_x}{B}\left(1 + \frac{B_x^2}{B^2}\right) - \frac{2P_\parallel^{(0)} B_x^3}{B^3} - 2\left(P_\perp^{(0)} - P_\parallel^{(0)}\right)\frac{B_x B_y^2}{B^3}\right\} \tag{A.2.21}$$

Finally

$$P_{xz}^{(1)} = \left(P_\parallel^{(1)} - P_\perp^{(1)}\right)\frac{B_x B_z}{B^2} + T_{xz}^{(1)} \tag{A.2.22}$$

$$\begin{aligned} T_{xz}^{(1)} = {} & T_{xz}^{x}\frac{du}{dx} + T_{xz}^{y}\frac{dV_y}{dx} + T_{xz}^{z}\frac{dV_z}{dx} \\ & + \frac{1}{\Omega}\left\{ \frac{q_\parallel^{\perp(0)} B_x}{B_\perp^2 B^4}\left[\frac{B_y^2 B^2 + B_z^2 B_x^2}{2} + 2\left(B_x^2 B_\perp^2 \right.\right.\right. \\ & \left.\left. + B_y^2 B_z^2 - B_x^2 B_z^2\right)\right] - \frac{2 q_\parallel^{\parallel(0)} B_x}{B^4 B_\perp^2}\left(B_x^2 B_\perp^2 + B_y^2 B_z^2 \right. \\ & \left.\left. - B_x^2 B_z^2\right)\right\}\frac{dB_y}{dx} \\ & + \frac{2\left(q_\parallel^{\parallel(0)} - q_\parallel^{\perp(0)}\right) B_x B_y B_z}{\Omega\, B^4}\,\frac{dB_z}{dx} - \frac{1}{\Omega}\frac{B_x B_y}{B^2}\frac{d q_\parallel^{\perp(0)}}{dx} \end{aligned} \tag{A.2.23}$$

$$T_{xz}^{x} = \frac{-1}{\Omega}\left\{ \frac{P_\perp^{(0)}}{2}\frac{B_y}{B}\left(\frac{B_x^2 B_z^2}{B_\perp^2 B^2} + \frac{B_y^2}{B_\perp^2}\right) + 2P_\parallel^{(0)}\frac{B_y B_x^2 B_z^2}{B^3 B_\perp^2}\right\}$$

(A. 2. 24)

$$T_{xz}^{y} = \frac{1}{\Omega}\left\{ P_\perp^{(0)}\left[\frac{B_x}{2B}\left(\frac{B_x^2 B_z^2}{B_\perp^2 B^2} + \frac{B_y^2}{B_\perp^2}\right)\right.\right.$$

(A. 2. 25)

$$\left.\left. + \frac{B_x}{B}\frac{B_z^2 - B_\perp^2}{B^2}\right] + 2P_\parallel^{(0)}\frac{B_x}{B}\frac{B_x^2 - B_z^2}{B^2}\right\}$$

$$T_{xz}^{z} = \frac{2\left(P_\parallel^{(0)} - P_\perp^{(0)}\right)}{\Omega}\frac{B_x B_y B_z}{B^3}$$

(A. 2. 26)

Performing the indicated operations in 4. 16, α_1 becomes

$$\alpha_1 = \frac{2}{\Omega}\left[\frac{B_x}{B}\frac{du}{dx} + \frac{B_y}{B}\frac{dV_y}{dx} + \frac{B_z}{B}\frac{dV_z}{dx}\right] \cdot$$

$$\left[\frac{B_y}{B}\frac{dV_z}{dx} - \frac{B_z}{B}\frac{dV_y}{dx}\right]\frac{B_x}{B}$$

(A. 2. 27)

Similarly, 4. 17 becomes

$$\alpha_2 = \left[\frac{dR_1^{(1)}}{dx} - \frac{P_\parallel^{(0)}}{NM}\frac{dP_\perp^{(1)}}{dx}\right]\frac{B_x}{B^2} \cdot$$

(A. 2. 28)

$$\left[\frac{B_z}{B}\frac{dB_y}{dx} - \frac{B_y}{B}\frac{dB_z}{dx}\right]$$

Finally, noting that only the x-components of the FLR heat flow are needed, 4.18 and 4.19 become

$$\begin{aligned} \underline{q}_\perp^{\parallel(1)} \cdot \hat{x} = \frac{1}{2\Omega} \Bigg\{ & \frac{B_z}{B} \frac{P_\parallel^{(0)}}{NM} \frac{dP_{xy}^{(1)}}{dx} - \frac{B_y}{B} \frac{P_\parallel^{(0)}}{NM} \frac{dP_{xz}^{(1)}}{dx} \\ & + \left(R_3^{(0)} - 4R_1^{(0)}\right) \frac{B_x}{B^3} \left(B_y \frac{dB_z}{dx} - B_z \frac{dB_y}{dx} \right) \\ & + 2\left(2q_\parallel^{\parallel(0)} - q_\parallel^{\perp(0)}\right) \frac{B_x}{B} \left(\frac{B_y}{B} \frac{dV_z}{dx} - \frac{B_z}{B} \frac{dV_y}{dx} \right) \Bigg\} \end{aligned} \tag{A.2.29}$$

$$\begin{aligned} \underline{q}_\perp^{\perp(1)} \cdot \hat{x} = \frac{2}{\Omega} \Bigg\{ & \frac{B_z}{B} \frac{P_\perp^{(0)}}{NM} \frac{dP_{xy}^{(1)}}{dx} - \frac{B_y}{B} \frac{P_\perp^{(0)}}{NM} \frac{dP_{xz}^{(1)}}{dx} \\ & + \left(R_1^{(0)} - \frac{R_2^{(0)}}{4}\right) \frac{B_x}{B^3} \left(B_y \frac{dB_z}{dx} - B_z \frac{dB_y}{dx} \right) \\ & + q_\parallel^{\perp(0)} \frac{B_x}{B} \left(\frac{B_y}{B} \frac{dV_z}{dx} - \frac{B_z}{B} \frac{dV_y}{dx} \right) \Bigg\} \end{aligned} \tag{A.2.30}$$

Recall that all the above equations are to be evaluated for both ions and electrons.

A.3 Calculation of $T_{ij}^{(1)}$ for Near-Perpendicular Shocks

In the wave train of section 6.0 calculation the components for the ions of $\underset{\approx}{P}^{(1)}$, $T_{ij}^{(1)+}$ are needed to order $(R_+/L_s)^2$ and as a function of dU/dx, i.e., the derivatives dV_y/dx and dV_z/dx should be eliminated. For the ions, to $\mathcal{O}(M_-/M_+)$ this can be accomplished by using the y and z components of the momentum equations

$$\frac{dV_y}{dx} = \frac{1}{\rho u}\left[\frac{B_x}{4\pi}\frac{dB_y}{dx} - \frac{(P_\parallel^{(1)} - P_\perp^{(1)})B_x}{B^2}\frac{dB_y}{dx} - \frac{B_x B_y}{B^2}\frac{d}{dx}(P_\parallel^{(1)} - P_\perp^{(1)}) - \frac{dT_{xy}^{(1)}}{dx}\right] \tag{A.3.1}$$

$$\frac{dV_z}{dx} = \frac{1}{\rho u}\left[\frac{B_x}{4\pi}\frac{dB_z}{dx} - \frac{(P_\parallel^{(1)} - P_\perp^{(1)})B_x}{B^2}\frac{dB_z}{dx} - \frac{B_x B_z}{B^2}\frac{d}{dx}(P_\parallel^{(1)} - P_\perp^{(1)}) - \frac{dT_{xz}^{(1)}}{dx}\right] \tag{A.3.2}$$

where $T_{xy}^{(1)}$ and $T_{xz}^{(1)}$ are to be evaluated for the ions but $P_\parallel^{(1)}$ and $P_\perp^{(1)}$ are the total pressures.

For near perpendicular shocks, $T_{ij}^{(1)}$ are needed only to $\mathcal{O}(B_x/B)$ so that all terms $\mathcal{O}(B_x^2/B^2,\ B_xB_y/B^2,\ B_y^2/B^2)$ can be dropped. Since $T_{ij}^{(1)}$ is a functional of the velocity derivatives and they in turn are functionals of $T_{ij}^{(1)}$, the equations must be expanded up to order $(R_+/L_s)^2$. The calculation will be performed for $T_{xx}^{(1)}$ and the results simply quoted for the other components. In what follows the zero order heat flows will be neglected.

Substituting A.3.1 and A.3.2 into A.2.1 and using 2.18 and 2.23 gives

$$T_{xx}^{(1)} = T_{xx}^{y}\left\{\frac{B_x}{4\pi\rho u}\frac{dB_y}{dx} - \frac{(P_{\parallel}^{(1)} - P_{\perp}^{(1)})B_x}{B^2}\frac{dB_y}{dx}\right.$$

$$- \frac{1}{\rho u}\frac{d}{dx}\left[T_{xy}^{x}\frac{du}{dx} + T_{xy}^{y}\frac{dv_y}{dx} + T_{xy}^{z}\frac{dv_z}{dx}\right]\Bigg\} \qquad \text{(A.3.3)}$$

$$+ T_{xx}^{z}\left\{\frac{B_x}{4\pi\rho u}\frac{dB_z}{dx} - \frac{(P_{\parallel}^{(1)} - P_{\perp}^{(1)})B_x}{B^2\rho u}\frac{dB_z}{dx}\right.$$

$$- \frac{B_x B_z}{B^2\rho u}\frac{d}{dx}(P_{\parallel}^{(1)} - P_{\perp}^{(1)}) - \frac{1}{\rho u}\frac{d}{dx}\left[T_{xz}^{x}\frac{du}{dx}\right.$$

$$\left.\left. + T_{xz}^{y}\frac{dv_y}{dx} + T_{xz}^{z}\frac{dv_z}{dx}\right]\right\}$$

Noting that T_{xx}^{z}, T_{xz}^{x}, T_{xz}^{y}, T_{xz}^{z} are $\mathcal{O}(B_x/B)$, the last term of A.33, proportional to T_{xx}^{z}, can be dropped. Also T_{xy}^{y} is $\mathcal{O}(B_x^2/B^2)$ and can be neglected. Substitution of A.3.1 and A.3.2 into A.3.3 gives

$$T_{xx}^{(1)} = T_{xx}^{y}\left\{\frac{B_x}{4\pi\rho u}\frac{dB_y}{dx} - \frac{(P_{\parallel}^{(1)} - P_{\perp}^{(1)})B_x}{B^2}\frac{dB_y}{dx}\right.$$

$$- \frac{1}{\rho u}\frac{d}{dx}\left(T_{xy}^{x}\frac{du}{dx}\right) - \frac{1}{\rho u}\frac{d}{dx}\left[\frac{T_{xy}^{z}}{\rho u}\left(\frac{B_x}{4\pi}\frac{dB_z}{dx}\right.\right.$$

$$- \frac{(P_{\parallel}^{(1)} - P_{\perp}^{(1)})B_x}{B^2}\frac{dB_z}{dx} - \frac{B_x B_z}{B^2}\frac{d}{dx}(P_{\parallel}^{(1)} - P_{\perp}^{(1)})$$

$$\left.\left.\left. - \frac{d}{dx}T_{xz}^{(1)}\right)\right]\right\} \qquad \text{(A.3.4)}$$

Since T^z_{xy} is $\mathcal{O}(B_x/B)$, the first three terms on the LHS of A.3.4 proportional to T^z_{xy} can be dropped. Now consider the order of the last term $d/dx(T^z_{xy}\langle dT^{(1)}_{xz}/dx\rangle)$. From A.2.23, this becomes

$$\frac{d}{dx}\left(T^z_{xy}\frac{dT^{(1)}_{xz}}{dx}\right) \sim T^z_{xy}\,T^x_{xz}\,\frac{d^3u}{dx^3}$$

$$\sim \left(\frac{R_+}{L}\right) T^y_{xx}\,T^x_{xy}\,\frac{d^2u}{dx^2}$$

or $\mathcal{O}(\epsilon)$ smaller the lowest order terms required. Actually here since $T^z_{xy}T^x_{xz} \sim \mathcal{O}(B_x^2/B^2)$, this term could have been dropped by the small angle expansion; however this will not always be true for the other terms below.

In evaluating A.3.4 the weak shock approximation is used to "linearize" the coefficients of dU/dx so that

$$\frac{d}{dx}\left(T^x_{xy}\frac{du}{dx}\right) \approx T^x_{xy}\frac{d^2u}{dx^2}$$

Expanding T^y_{xx} and T^x_{xy} to lowest order in B_x/B, A.3.4 becomes

$$T^{(1)}_{xx} = \frac{-1}{\tilde{\gamma}-1}\,\frac{P_\perp^{(0)+}}{2\Omega_+}\,\frac{B_z}{B}\left[\frac{B_x}{4\pi\rho u}\frac{dB_y}{dx} - \frac{(P_\parallel^{(1)}-P_\perp^{(1)})B_x}{B^2\rho u}\frac{dB_y}{dx}\right.$$

(A.3.5)

$$- \frac{P_\perp^{(0)+}}{2\Omega_+ \rho U} \frac{B_z}{B} \frac{1}{\tilde{\gamma}-1} \frac{d^2U}{dx^2} \Big] + O\left(\frac{B_x^2}{B^2}, \epsilon^3\right)$$

Similar calculations for $T_{xy}^{(1)}$ and $T_{xz}^{(1)}$ give

$$T_{xy}^{(1)} = \frac{1}{\tilde{\gamma}-1} \frac{P_\perp^{(0)+}}{2\Omega_+} \frac{B_z}{B} \frac{dU}{dx} + O\left(\frac{B_x^2}{B^2}, \epsilon^3\right) \tag{A.3.6}$$

$$\begin{aligned} T_{xz}^{(1)} = & \frac{-1}{\tilde{\gamma}-1} \frac{P_\perp^{(0)+}}{2\Omega_+} \frac{B_y}{B} \frac{dU}{dx} \\ & - \frac{1}{\tilde{\gamma}-1} \frac{P_\perp^{(0)+}}{2\Omega_+^2 \rho U} \frac{B_z}{B} \Big[P_\perp^{(0)+} \Big(\frac{1}{\tilde{\gamma}-1} \frac{B_x}{2B} \\ & + \frac{B_x B_z^2}{B^3} \Big) - \frac{2 P_\parallel^{(0)+} B_x B_z^2}{B^3} \Big] \frac{d^2U}{dx^2} \\ & + O\left(B_x^2/B^2, \epsilon^3 \right) \end{aligned} \tag{A.3.7}$$

Note that the term $B_y \, dU/dx$ has been retained even though about the upstream point $B_y = 0$.

A.4 Calculation of $T_{ij}^{(1)}$ for Isotropic Pressure

Now consider the components of $T_{ij}^{(1)}$ when $P_{\parallel}^{(0)+} = P_{\perp}^{(0)+}$ and $P_{\parallel}^{(1)} = P_{\perp}^{(1)}$ but making no approximation on the angle B_x/B. By calculations analagous to those of A.3, to $\mathcal{O}\,(R_+/L)^2$ the desired terms are

$$
\begin{aligned}
T_{xx}^{(1)} = &- \frac{p^{(0)+}}{2\Omega_+}\frac{B_z}{B}\left(1+3\frac{B_x^2}{B^2}\right)\frac{B_x}{4\pi\rho u}\frac{dB_y}{dx} \\
&+ \frac{\left(p^{(0)+}\right)^2}{4\Omega_+^2\rho u}\left(1+3\frac{B_x^2}{B^2}\right)^2\frac{B_z^2}{B^2}\frac{d^2u}{dx^2} + \left(\frac{p^{(0)+}}{2\Omega_+}\right)^2\frac{B_xB_z}{B^2}\frac{B_x}{4\pi\rho^2u^2}\left(1-9\frac{B_x^2}{B^2}\right)\frac{d^2B_z}{dx^2} \\
&+ \frac{p^{(0)+}}{2\Omega_+}\frac{B_y}{B}\left(1+3\frac{B_x^2}{B^2}\right)\frac{B_x}{4\pi\rho u}\frac{dB_z}{dx} + \frac{5}{4}\left(\frac{p^{(0)+}}{\Omega_+}\right)^2\frac{B_y^2}{B^2}\left(1+3\frac{B_x^2}{B^2}\right)\frac{1}{\rho u}\frac{d^2u}{dx^2} \\
&+ \mathcal{O}(\epsilon^3)
\end{aligned}
\tag{A.4.1}
$$

$$
\begin{aligned}
T_{xy}^{(1)} = &\frac{p^{(0)+}}{2\Omega_+}\frac{B_z}{B}\left(1+3\frac{B_x^2}{B^2}\right)\frac{du}{dx} + \frac{p^{(0)+}}{2\Omega_+}\frac{B_x}{B}\left(1-3\frac{B_x^2}{B^2}\right)\frac{B_x}{4\pi\rho u}\frac{dB_z}{dx} \\
&+ \frac{5}{4}\left(\frac{p^{(0)+}}{\Omega_+}\right)^2\frac{B_xB_y}{B^2\rho u}\left(1-3\frac{B_x^2}{B^2}\right)\frac{d^2u}{dx^2} \\
&+ \left(\frac{p^{(0)+}}{2\Omega_+}\right)^2\frac{B_x^2}{B^2}\left(1-3\frac{B_x^2}{B^2}\right)\left(\frac{B_z^2}{B^2} - \frac{2B_x^2}{B^2}\right)\frac{B_x}{4\pi\rho^2u^2}\frac{d^2B_y}{dx^2} \\
&+ \mathcal{O}(\epsilon^3)
\end{aligned}
\tag{A.4.2}
$$

$$
\begin{aligned}
T_{xz}^{(1)} = &- \frac{5}{2} \frac{p^{(0)+}}{\Omega_+} \frac{B_y}{B} \frac{du}{dx} \\
&- \frac{p^{(0)+}}{2\Omega_+} \frac{B_x}{B} \left(\frac{B_z^2}{B^2} - \frac{2B_x^2}{B^2} \right) \frac{B_x}{4\pi\rho u} \frac{dB_y}{dx} \\
&+ \left(\frac{p^{(0)+}}{2\Omega_+} \right)^2 \frac{B_x B_z}{B^2 \rho u} \left(1 + 3 \frac{B_x^2}{B^2} \right) \left(\frac{B_z^2}{B^2} - \frac{2B_x^2}{B^2} \right) \frac{d^2u}{dx^2} \\
&+ \left(\frac{p^{(0)+}}{2\Omega_+} \right)^2 \frac{B_x^2}{B^2} \left(1 - 3 \frac{B_x^2}{B^2} \right) \left(\frac{B_z^2}{B^2} - \frac{2B_x^2}{B^2} \right) \frac{B_x}{4\pi\rho u^2} \frac{d^2 B_z}{dx^2}
\end{aligned}
\tag{A.4.3}
$$

In calculating the above, the weak shock approximation was again invoked to linearize the coefficients.

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Figure Captions

Figure 1: A sketch of the non-linear potential $\varphi(B)$ against B; $B_1(B_2)$ is the upstream (downstream) value of the magnetic field. Interpreting B as a particle moving in a potential well, B leaves the upstream unstable maximum and commences oscillating. If there is no dissipation, B makes one oscillation and returns to B_1; this motion corresponds to the solitary wave. With dissipation B fails to return to B_1, having lost energy, and eventually must come to rest at the potential minimum, B_2. Since B_2 satisfies the downstream Rankine-Hugoniot relations, a shock transition has been effected.

Figure 2: A sketch of the perpendicular fast shock C/ω_{p_-} wave train for $\beta_+ < M_-/M_+$, $\beta_- << 1$. The potential well motion of Figure 1 is here translated into the magnetic field and flow velocity as a function of the spatial distance through the shock. At the leading edge of the shock B undergoes an exponential rise from its upstream value with a scale length proportional to C/ω_{p_-}. Behind this rise B oscillates about its downstream value; the oscillation length is also $\sim C/\omega_{p_-}$. The wave train oscillations eventually damp due to the weak dissipation.

Figure 3: A sketch of a leading C/ω_{p_+} wave train for a low-β_+ oblique fast shock. Here the dispersion term in the wave train differential equation has the opposite sign to the potential term; oscillations develop about the leading edge of the shock and grow exponentially in space until the downstream conditions are reached.

Figure 4: A plot of the phase velocity, ω/k, vs. k for the linear perpendicular fast wave in a $\beta_+ < M_-/M_+$ plasma. When $kC/\omega_{p_-} \sim 1$, the fast wave is decoupled from the magnetic field, and is slowed to the sound speed, C_s. Possible upstream, U_1, and downstream, U_2, shock flow velocities are also sketched. If the phase velocity equals either U_1 or U_2, it is possible for a wave to stand in the flow; dispersive propagation is required since $U_1 > C_F > U_2$ by the shock evolutionary conditions. If the group velocity, $\partial\omega/\partial k$, is directed away from the shock, the standing wave forms a wave train. Here a wave with a wavelength $\sim C/\omega_{p_-}$ stands in the downstream flow and forms the wave train described in Figure 2.

Figure 5: A sketch of the linear fast wave dispersion relation for propagation oblique to the magnetic field with $\beta_+ < 1$ and $\beta_+ > 1$. If $\beta_+ < 1$, ion inertia speeds up the fast wave until electron inertia becomes dominant and slows the wave to C_s. Since the phase velocity can intersect U_1, a leading C/ω_{p_+} wave train is possible. If $\beta_+ > 1$, ion FLR dispersion dominates that of ion inertia and slows down the fast wave; a trailing R_+

wave train is possible. Ion FLR dispersion is effective only when $kR_+ \sim 1$. For $kR_+ >> 1$, the ion motion, with respect to the wave, becomes decoupled from the magnetic field; the ion trajectories can be taken to be straight lines. Since only ion inertia remains when $kC/\omega_{p_-} << 1$, the wave speeds up to become the high frequency whistler.

Figure 6: A sketch of the oblique slow wave dispersion relation for $\beta_+ < 1$ and $\beta_+ > 1$. Trailing C/ω_{p_+} ($\beta_+ < 1$) and R_+ ($\beta_+ > 1$) wave trains are possible.

Figure 7: A sketch of the fast wave dispersion relation for near-parallel propagation. When $\beta_+ > 1$, the propagation angle θ must exceed Ω_+/ω_{p_+} so that Debye length dispersion can be neglected. A leading C/ω_{p_+} ($\beta_+ < 1$) and R_+($\beta_+ > 1$) wave trains are possible.

Figure 8: A sketch of the near-parallel slow wave dispersion relation. When $\beta_+ < 1$, neglect of Debye length dispersion requires $\theta > C_+/C$. Since both C/ω_{p_+} and R_+ slow down the wave, trailing slow shock wave trains are possible.

Figure 9: A sketch of the magnetic field profile for a low-β oblique slow shock. Note that the magnetic field decreases across the shock. The trailing wave train has an oscillation length $\sim C/\omega_{p_+}$.

Figure 10: A parameter space summary of the fast shock wave train calculations. The shock propagation angle θ and the ion-β_+ are to be interpreted as upstream (downstream) quantities depending on whether the wave train is leading (trailing). Note that when $\theta < \Omega_+/\omega_{p_+}$ and $\beta_+ > 1$ Debye length effects become important. The dispersion lengths simply represent the scaling length for the shock thickness. Boundaries between various regions are only approximate, and near the edges, shocks are probably characterized by a combination of scale lengths. The region representing typical solar wind and bow shock β_+'s is shaded.

Figure 11: Similar to Figure 10 except for slow shocks. Again note electrostatic region where Debye length dispersion is important.

Figure 12: (After Fredricks _et al._, 1968 and Fredricks and Coleman, 1968). The first two graphs are a dynamic spectrum and the electric field amplitude obtained by the TRW electric field experiment on board OGO-V for the March 12, 1968 crossing of the earth's bow shock. Upstream is to the left. The third graph is the magnitude of the magnetic field as measured by the UCLA fluxgate magnetometer. The magnetic field undergoes a sharp rise to roughly seven times its upstream value and then oscillates about a value roughly four times that upstream. Assuming the relative velocity between the satellite and shock to be

~ 10 km/sec, the temporal oscillations can be interpreted as a scale length which is of the order of several times C/ω_{p_-}. The electric field amplitude attains its maximum value at the maximum magnetic field gradients, i.e., the maximum current density.

Figure 13: (After Fredricks and Coleman, 1968) A sketch of a bow shock crossing obtained on March 12, 1968. Same format for the data display as in Figure 12. Upstream the magnetic field oscillates with an interpreted scale length of C/ω_{p_+}. In the center of the shock a short C/ω_{p_-} scale length layer is observed; here the electric field attains maximum amplitude. Downstream the magnetic field possesses an oscillatory structure labeled C/ω'_{p_+}. These oscillations are interpreted on the basis of the calculations performed here as a trailing ion gyro-radius scale length wave train.

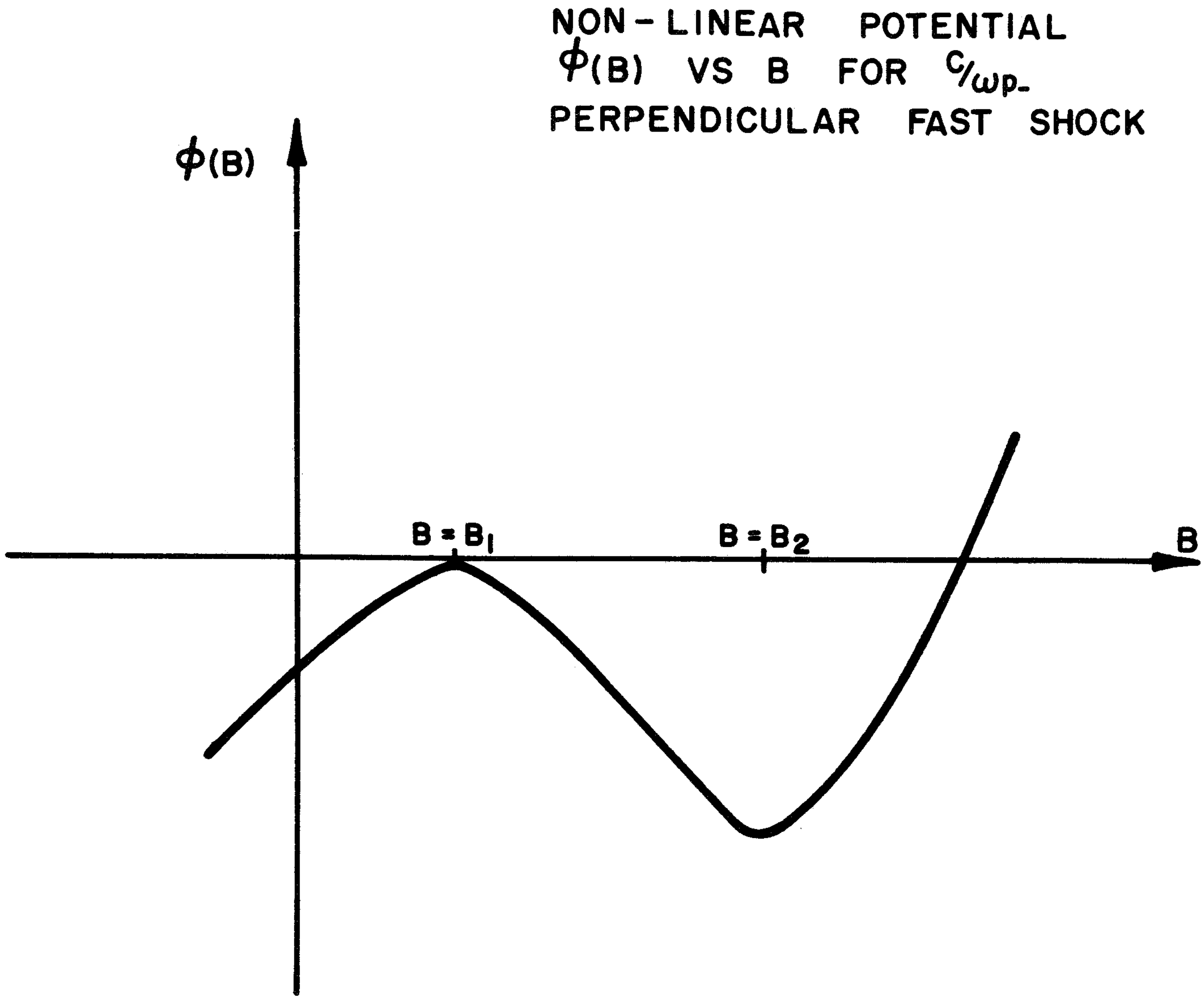

NON-LINEAR POTENTIAL
$\phi(B)$ VS B FOR c/ω_{p-}
PERPENDICULAR FAST SHOCK
$\phi(B)$
B
B = B_1
B = B_2

Figure 1

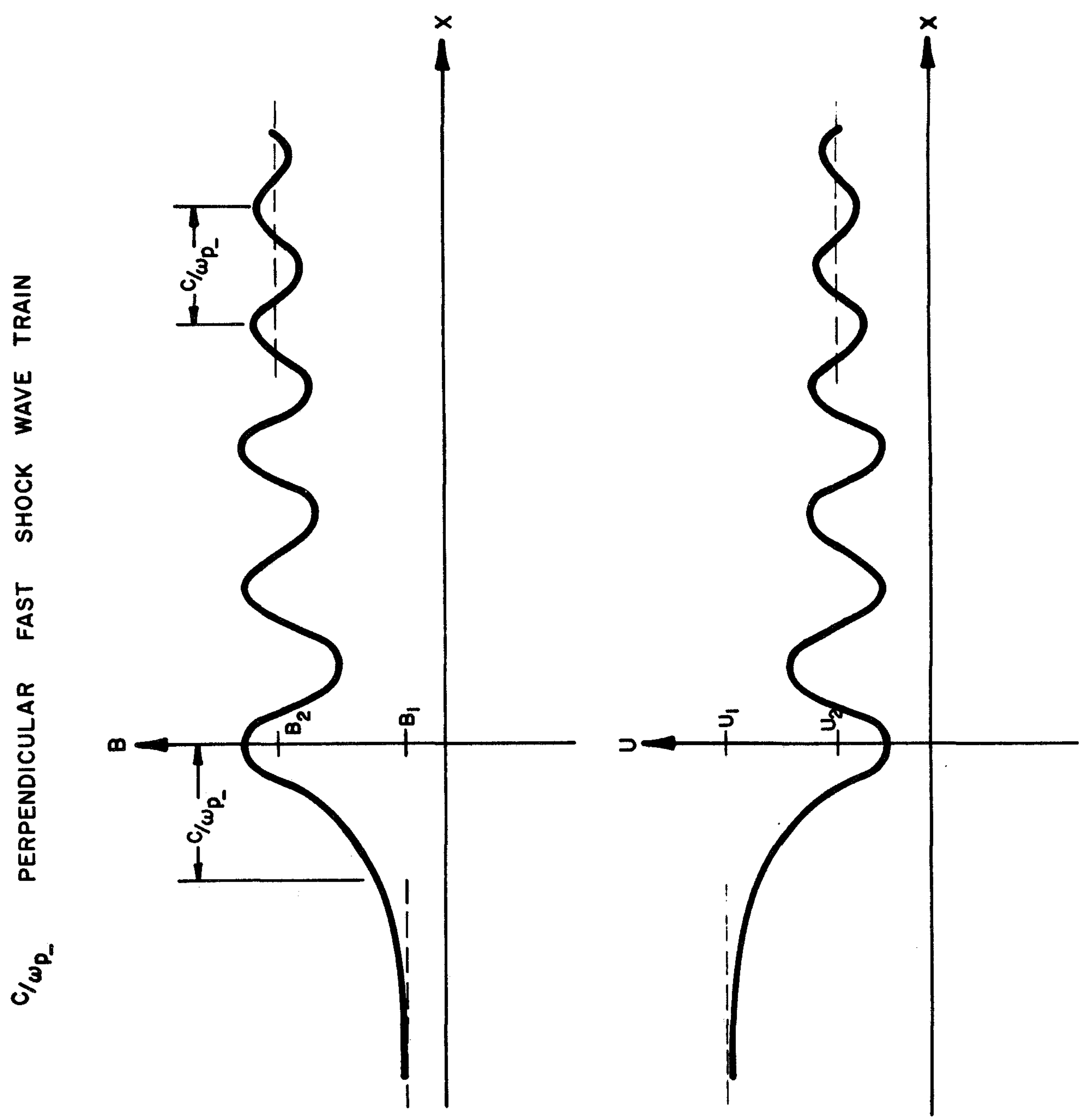

c/ω_{p-}
PERPENDICULAR FAST SHOCK WAVE TRAIN
B
X
B_1
B_2
c/ω_{p-}
c/ω_{p-}
U
X
U_1
U_2

Figure 2

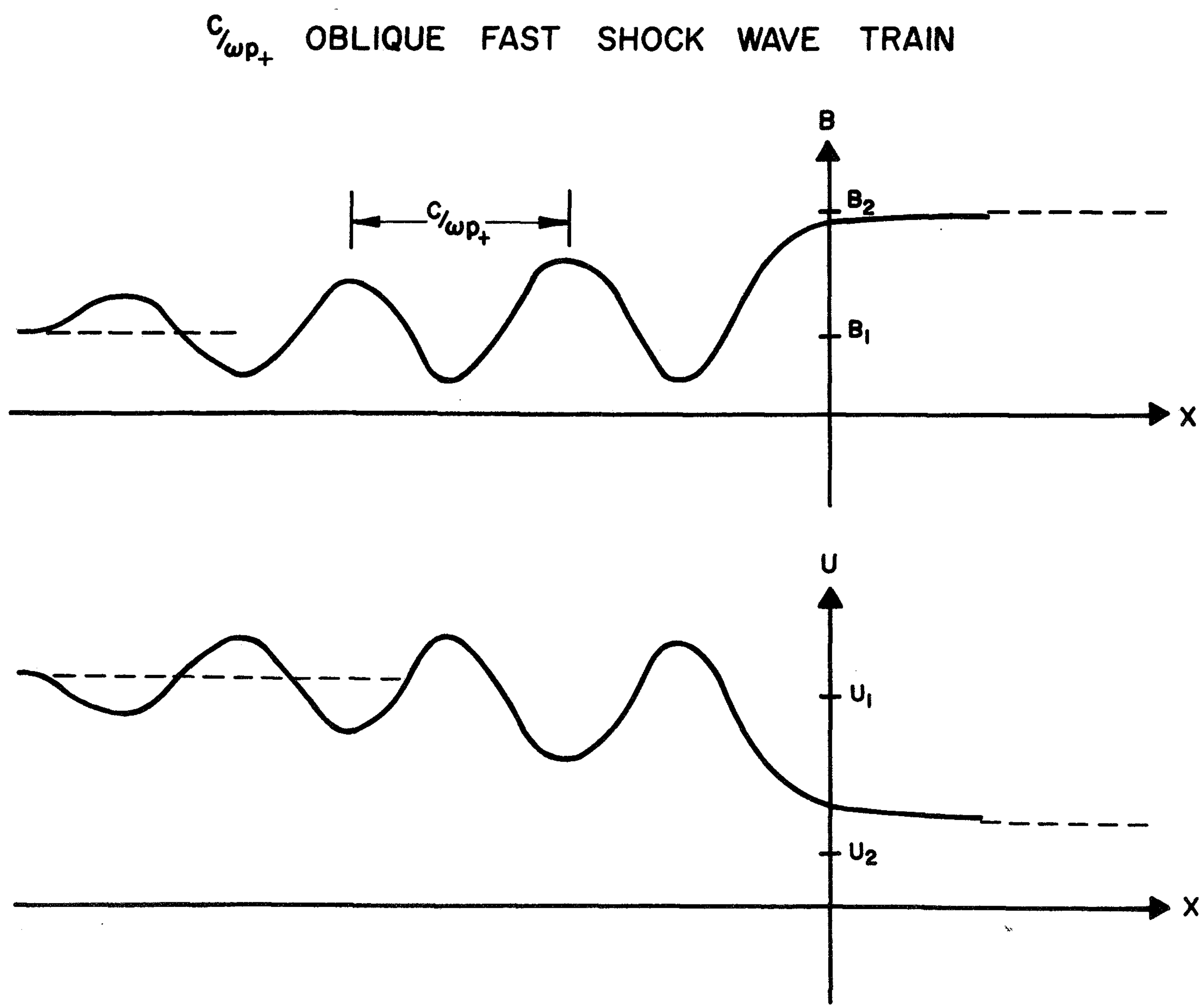

$c/\omega p_+$ OBLIQUE FAST SHOCK WAVE TRAIN
B
B_2
B_1
$c/\omega p_+$
X
U
U_1
U_2
X

Figure 3

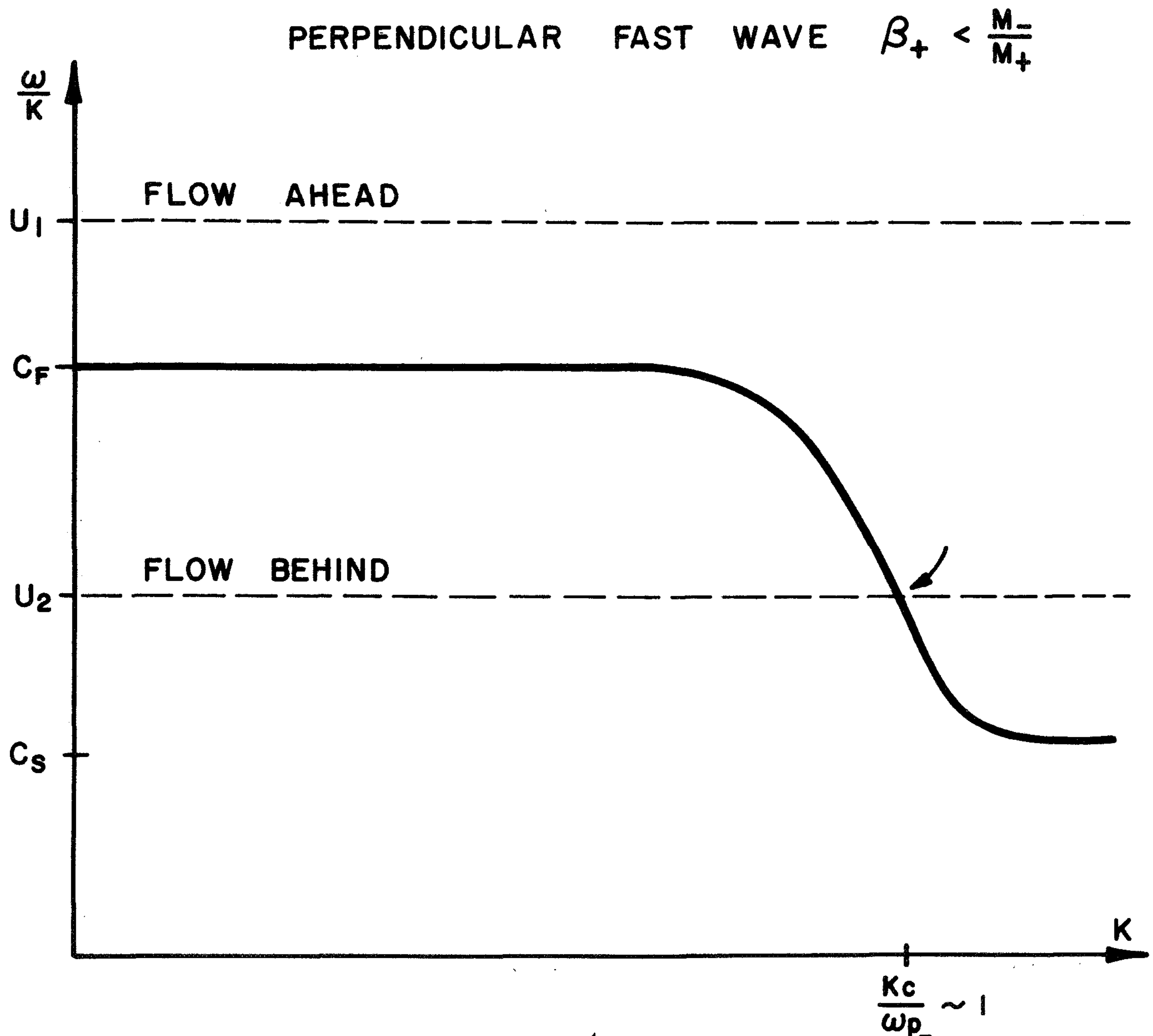

PERPENDICULAR FAST WAVE $\beta_+ < \frac{M_-}{M_+}$
$\frac{\omega}{K}$
FLOW AHEAD
U_1
C_F
FLOW BEHIND
U_2
C_S
K
$\frac{Kc}{\omega_{p_-}} \sim 1$

Figure 4

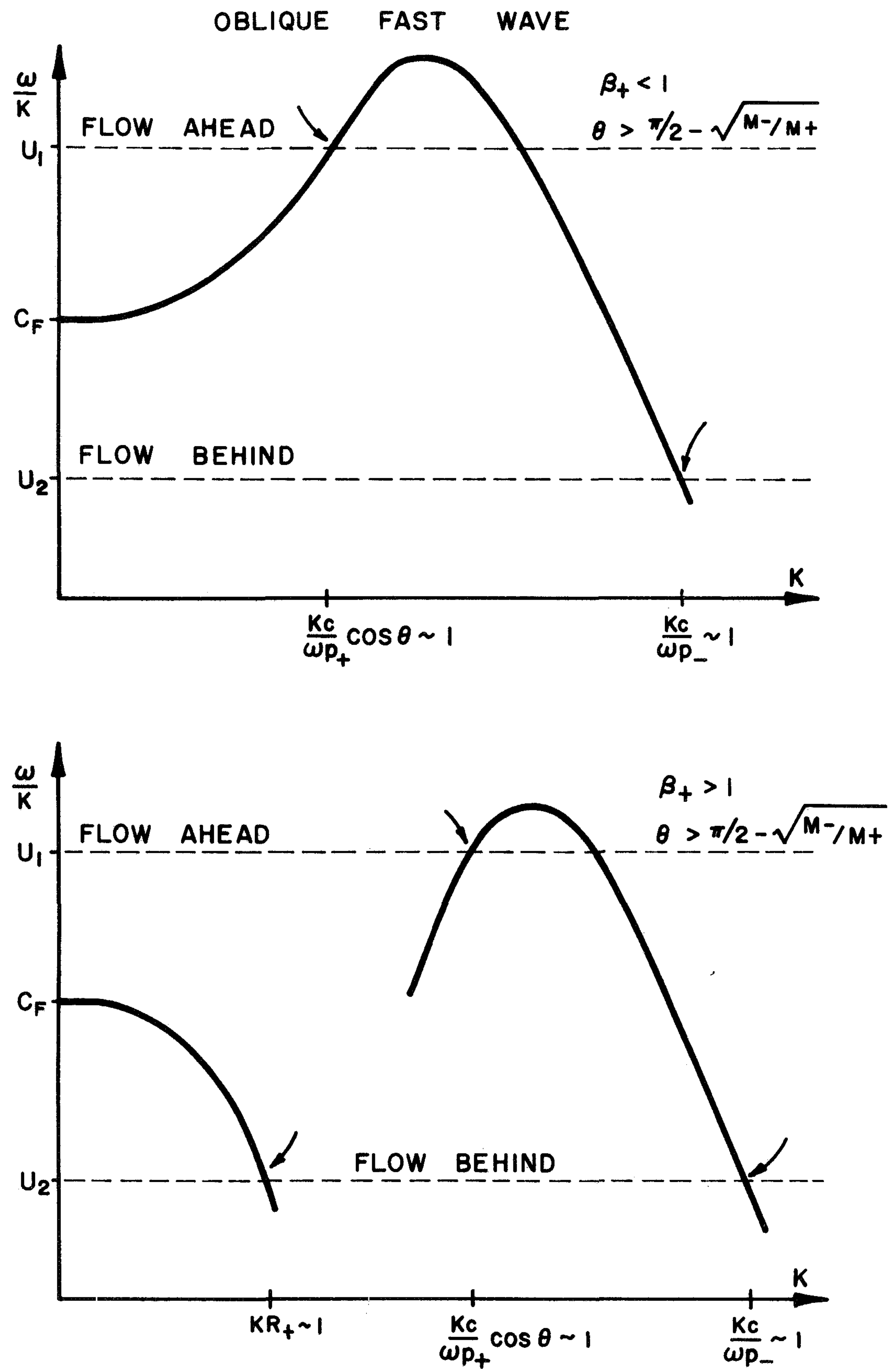

OBLIQUE FAST WAVE
ω/K
FLOW AHEAD
U1
CF
FLOW BEHIND
U2
β+ < 1
θ > π/2 - √M-/M+
K
Kc/ωp+ cos θ ~ 1
Kc/ωp- ~ 1
ω/K
FLOW AHEAD
U1
CF
FLOW BEHIND
U2
β+ > 1
θ > π/2 - √M-/M+
K
KR+ ~ 1
Kc/ωp+ cos θ ~ 1
Kc/ωp- ~ 1

Figure 5

OBLIQUE SLOW WAVE

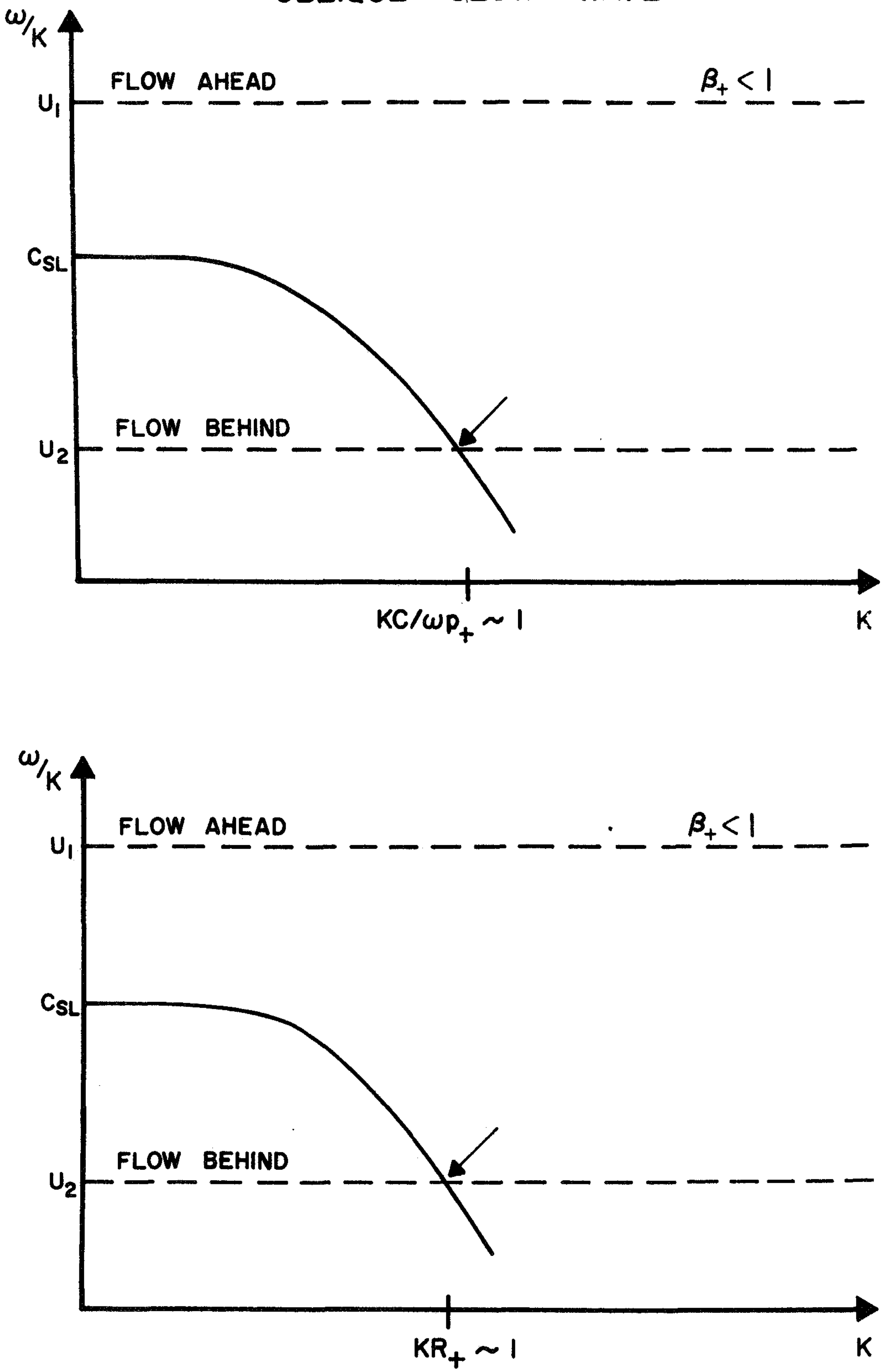


Figure 6

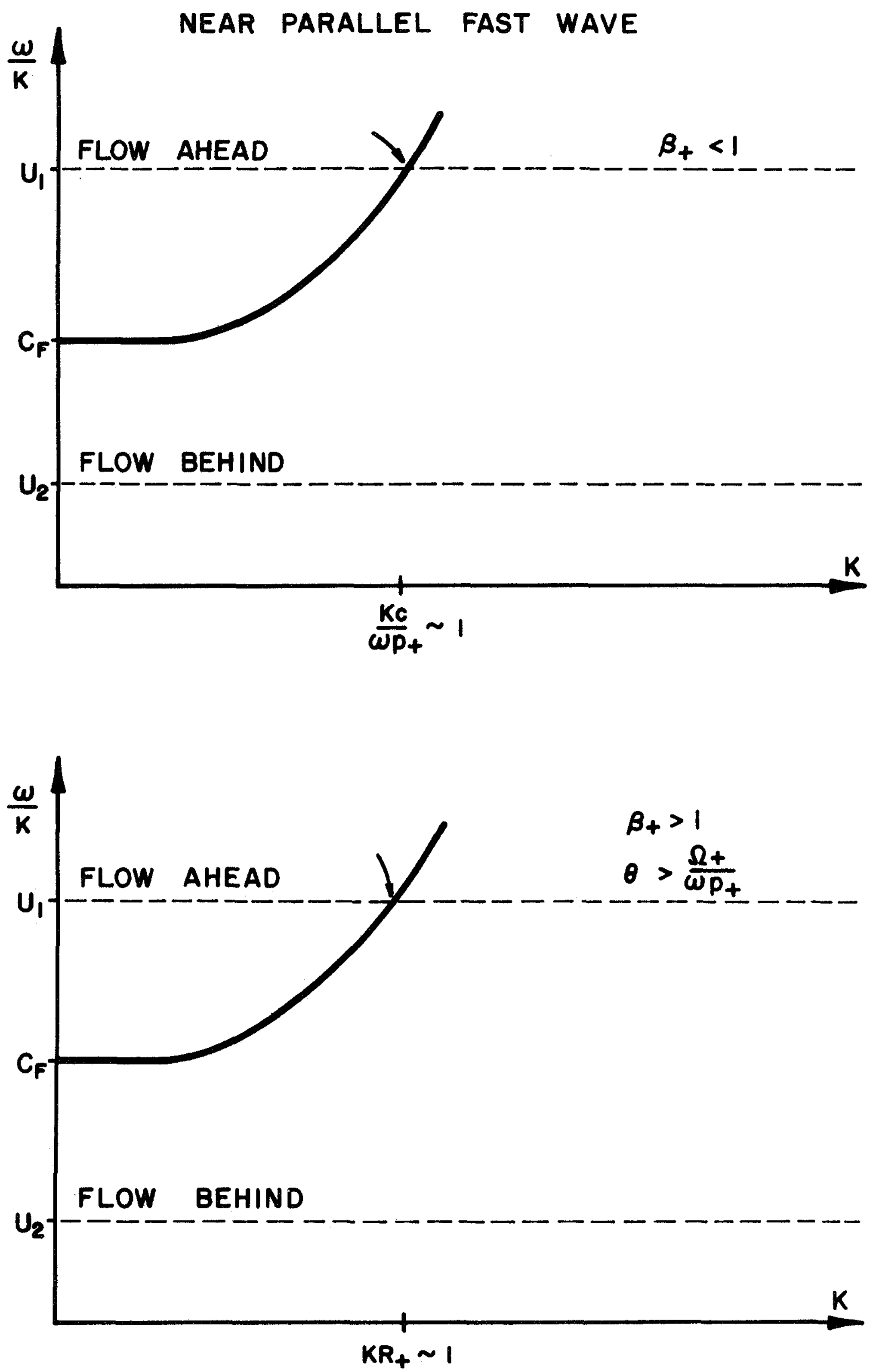
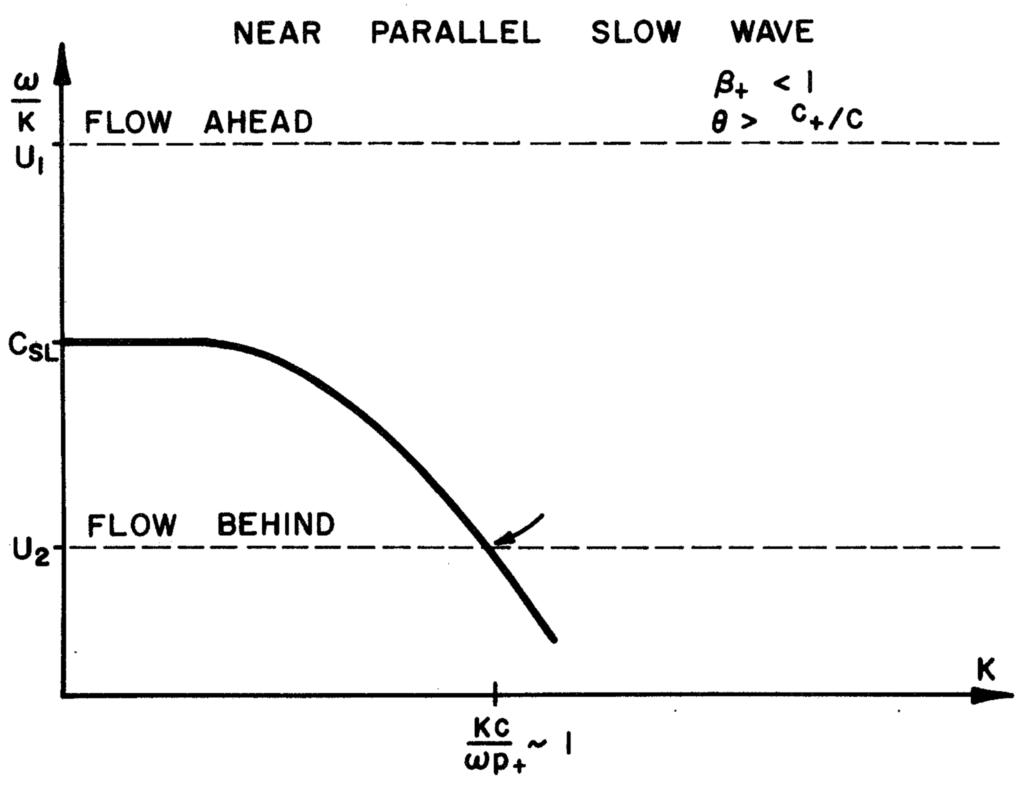

NEAR PARALLEL FAST WAVE
ω/K
FLOW AHEAD
U1
β+ < 1
CF
FLOW BEHIND
U2
K
Kc/ωp+ ~ 1
ω/K
β+ > 1
θ > Ω+/ωp+
FLOW AHEAD
U1
CF
FLOW BEHIND
U2
K
KR+ ~ 1

Figure 7


NEAR PARALLEL SLOW WAVE
ω/K
FLOW AHEAD
β+ < 1
θ > C+/C
U1
CSL
FLOW BEHIND
U2
K
Kc/ωp+ ~ 1

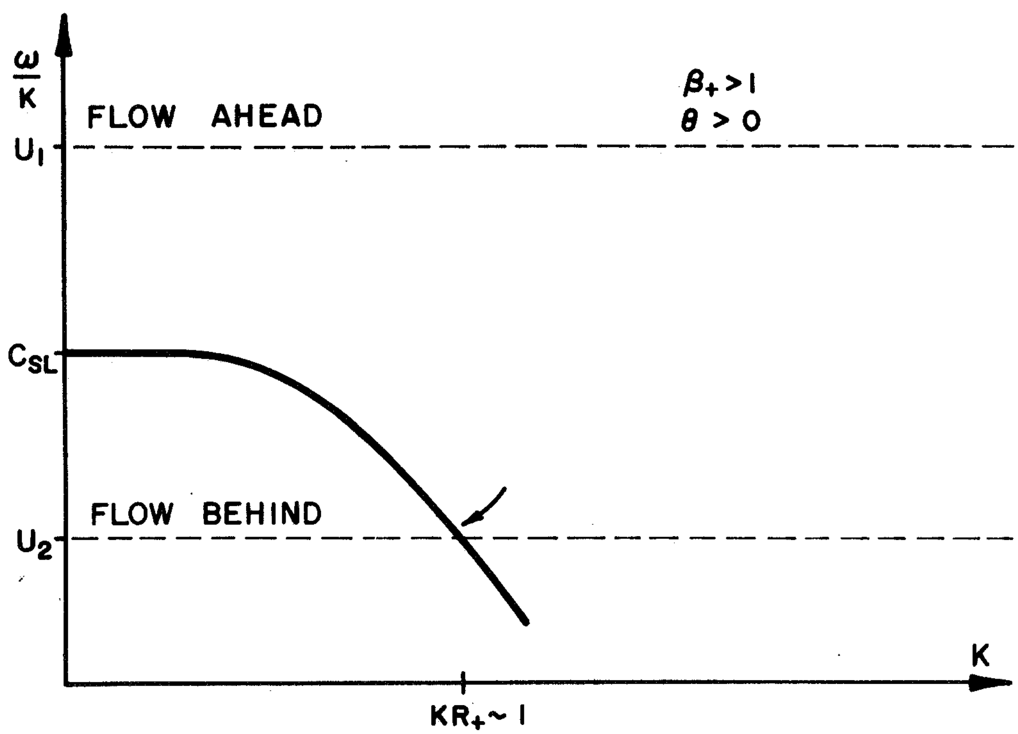

ω/K
FLOW AHEAD
β+ > 1
θ > 0
U1
CSL
FLOW BEHIND
U2
K
KR+ ~ 1

Figure 8

SLOW SHOCK TRAILING c/ω_{p_+} WAVE TRAIN

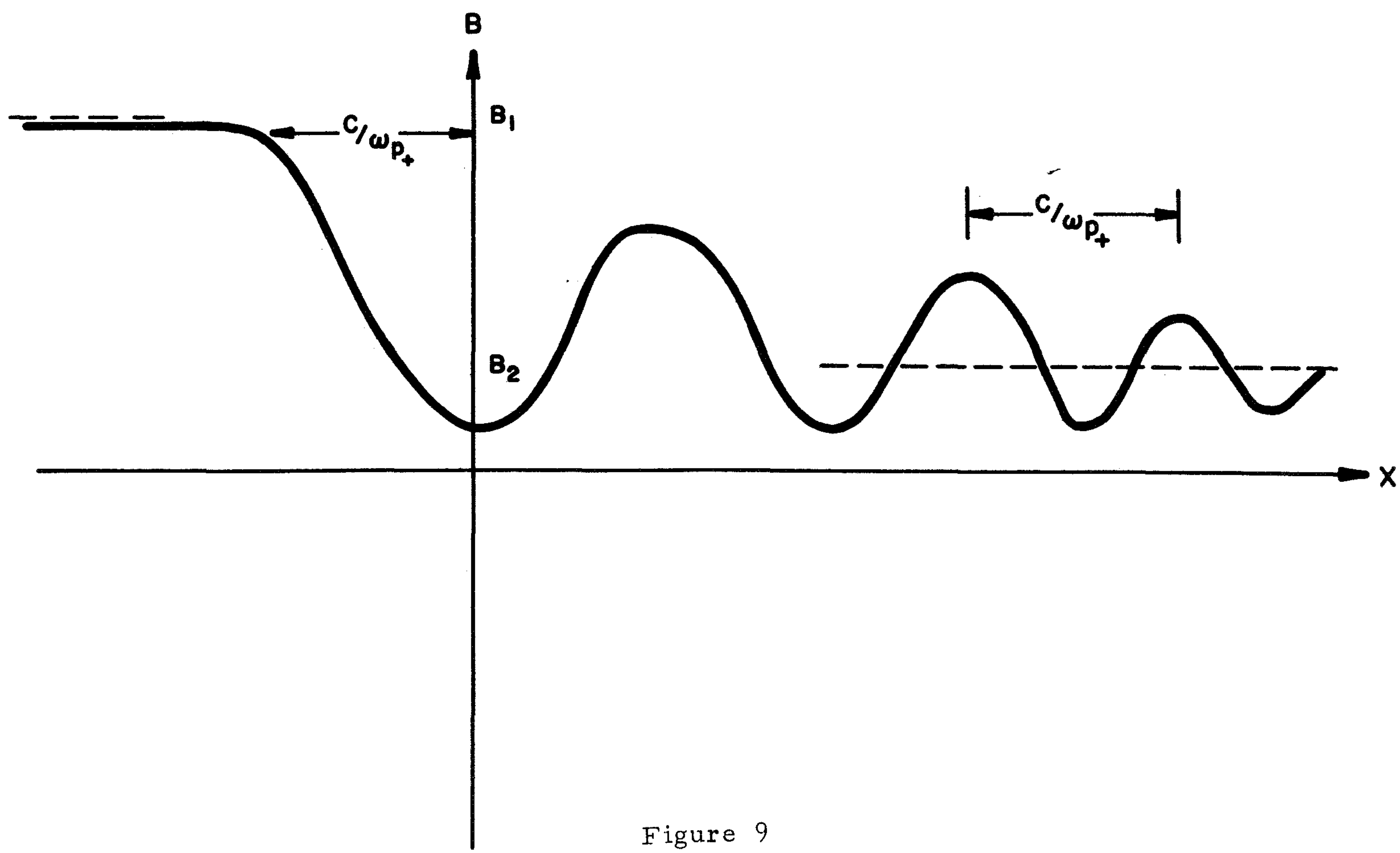


Figure 9

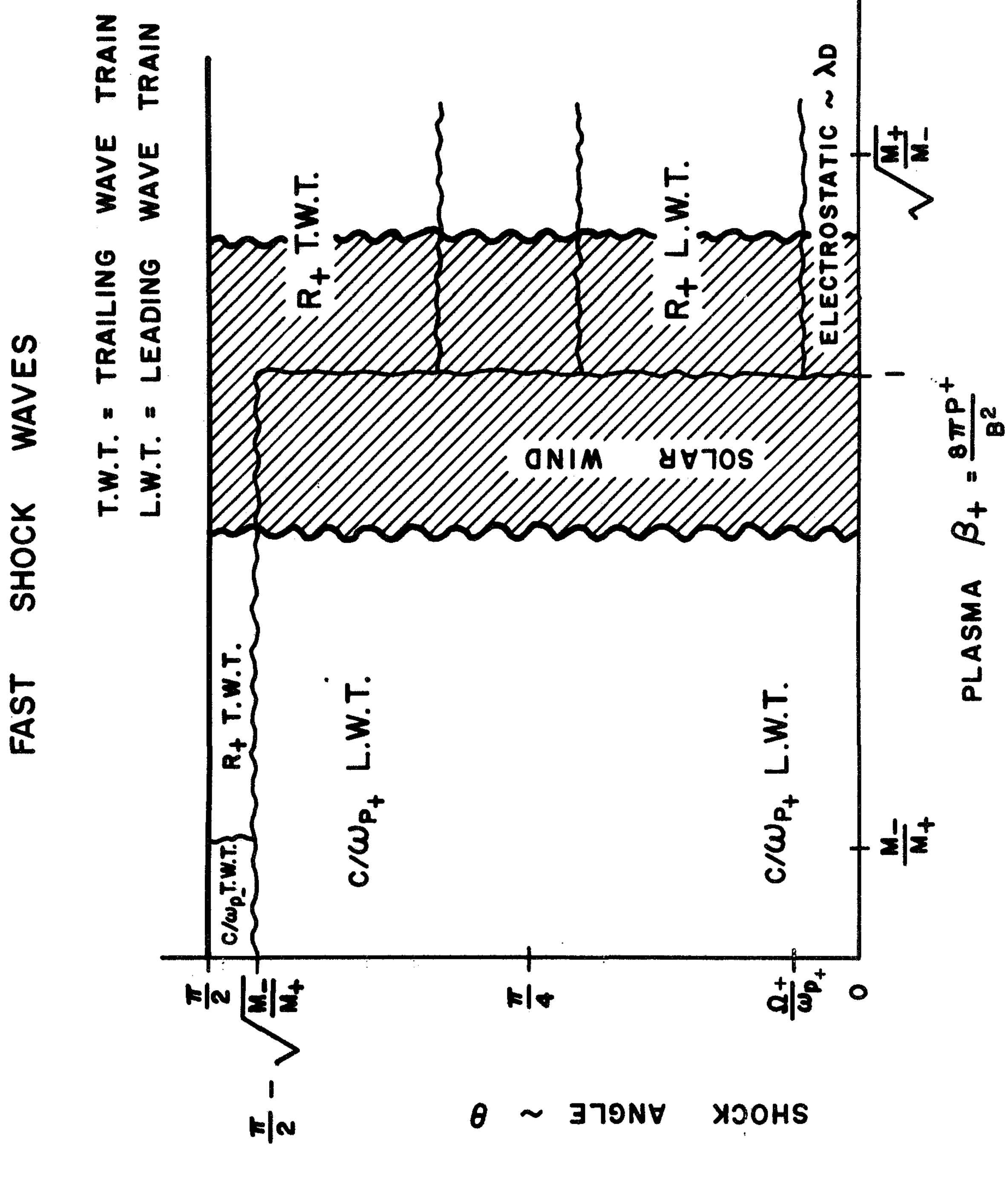

FAST SHOCK WAVES
T.W.T. = TRAILING WAVE TRAIN
L.W.T. = LEADING WAVE TRAIN
R_+ T.W.T.
R_+ L.W.T.
ELECTROSTATIC ~ λD
SOLAR WIND
C/ω_{p_+} L.W.T.
C/ω_{p_+} L.W.T.
R_+ T.W.T.
C/ω_{p_-} T.W.T.
$\frac{M_-}{M_+}$
$\sqrt{\frac{M_+}{M_-}}$
PLASMA $\beta_+ = \frac{8\pi P^+}{B^2}$
SHOCK ANGLE ~ θ
$\frac{\pi}{2}$
$\frac{\pi}{2} - \sqrt{\frac{M_-}{M_+}}$
$\frac{\pi}{4}$
$\frac{\Omega^+}{\omega_{p_+}}$
0

Figure 10

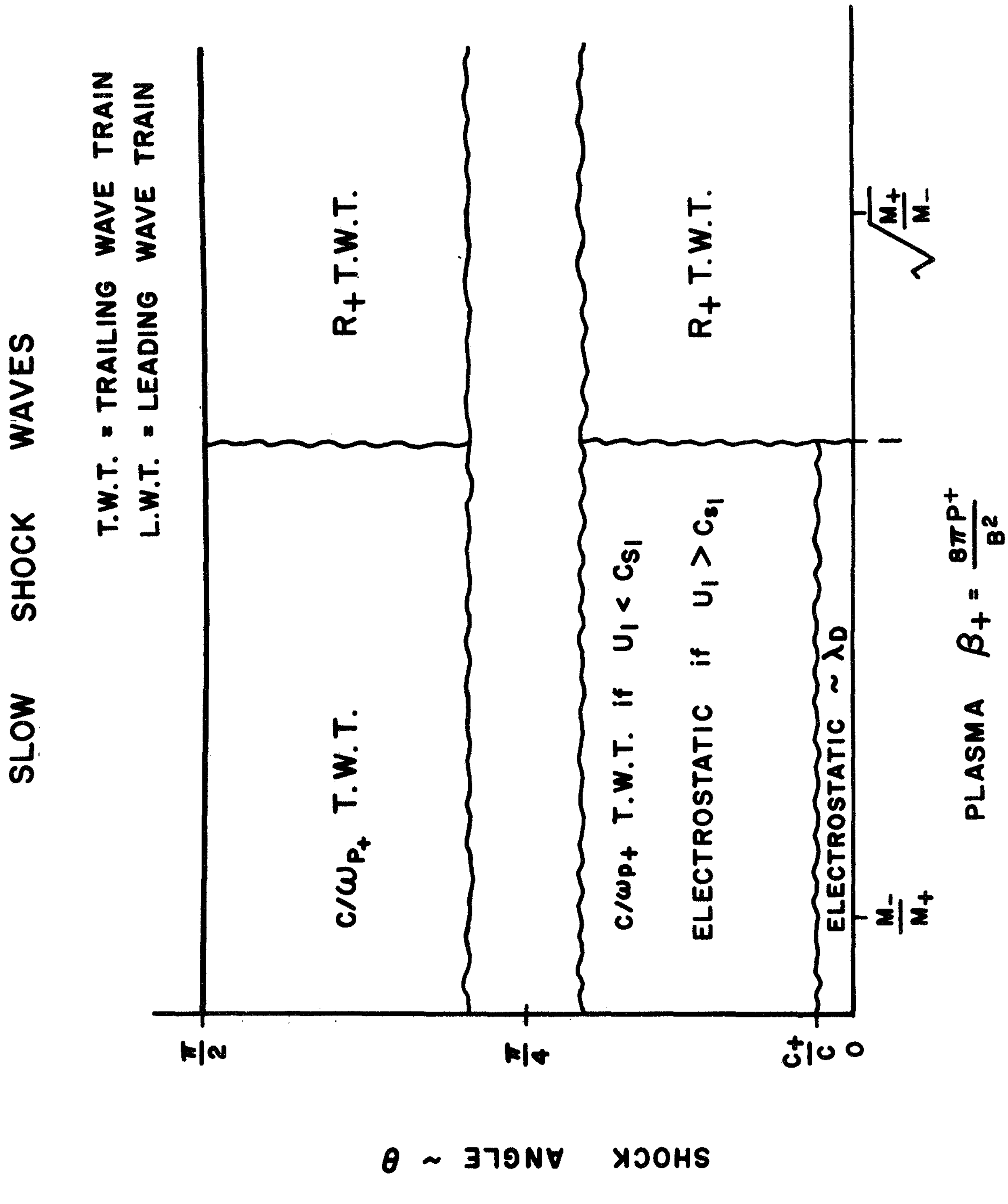
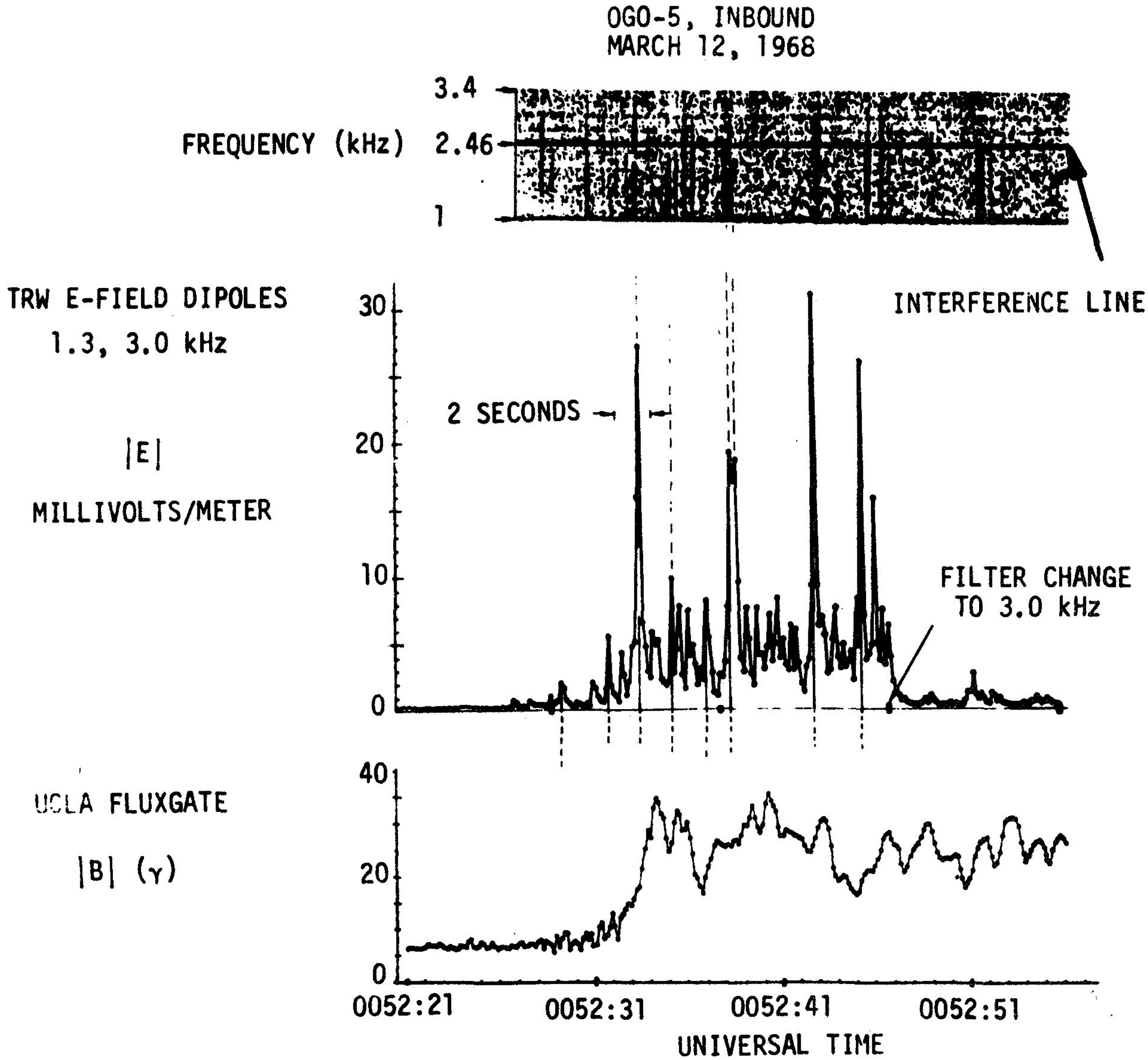
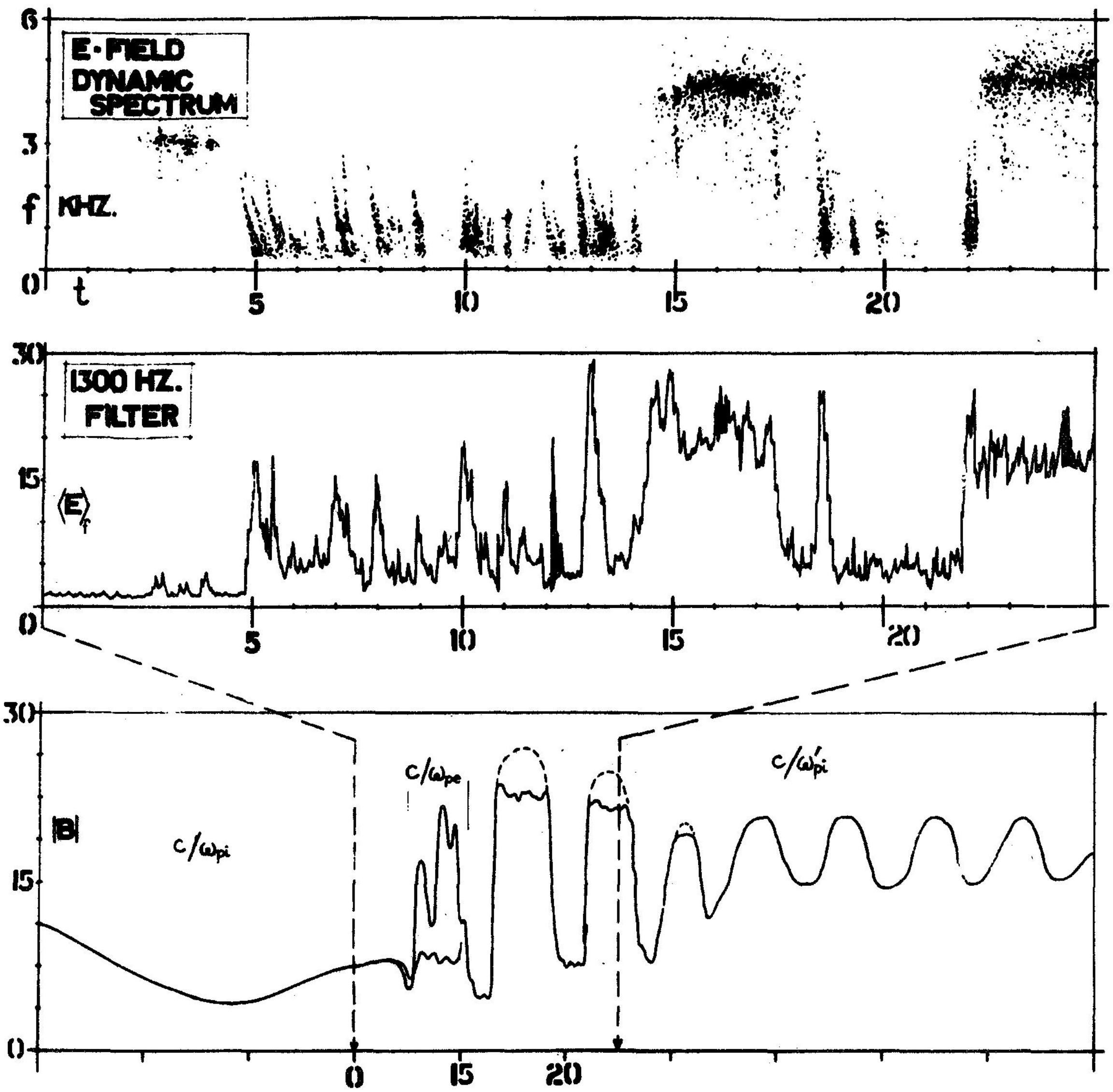

SLOW SHOCK WAVES
T.W.T. = TRAILING WAVE TRAIN
L.W.T. = LEADING WAVE TRAIN
R_+ T.W.T.
C/ω_{P_+} T.W.T.
R_+ T.W.T.
C/ω_{P_+} T.W.T. if $U_1 < C_{S1}$
ELECTROSTATIC if $U_1 > C_{S1}$
ELECTROSTATIC $\sim \lambda_D$
$\frac{M_-}{M_+}$
$\sqrt{\frac{M_+}{M_-}}$
PLASMA $\beta_+ = \frac{8\pi P^+}{B^2}$
$\frac{\pi}{2}$
$\frac{\pi}{4}$
$\frac{C_+}{C}$
0
SHOCK ANGLE $\sim \theta$

Figure 11



OGO-5, INBOUND
MARCH 12, 1968
FREQUENCY (kHz)
3.4
2.46
1
INTERFERENCE LINE
TRW E-FIELD DIPOLES
1.3, 3.0 kHz
|E|
MILLIVOLTS/METER
30
20
10
0
2 SECONDS
FILTER CHANGE
TO 3.0 kHz
UCLA FLUXGATE
|B| (γ)
40
20
0
0052:21
0052:31
0052:41
0052:51
UNIVERSAL TIME

Figure 12



E·FIELD
DYNAMIC
SPECTRUM
f KHZ.
t
1300 HZ.
FILTER
⟨E⟩f
|B|
c/ωpi
c/ωpe
c/ω'pi

Figure 13