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70N 11608

NASA TECHNICAL NOTE

NASA TN D-5564

NASA TN D-5564

CASCADE IMPEDANCE SYNTHESIS USING AN EXTENSION OF THE FIALKOW-GERST THEOREM

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • NOVEMBER 1969

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TN D-5564		

4. Title and Subtitle	5. Report Date
CASCADE IMPEDANCE SYNTHESIS USING AN EXTENSION OF THE FIALKOW-GERST THEOREM	November 1969
	6. Performing Organization Code
7. Author(s) John P. Barranger	8. Performing Organization Report No. E-4591
9. Performing Organization Name and Address Lewis Research Center National Aeronautics and Space Administration Cleveland, Ohio 44135	10. Work Unit No. 120-27
	11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546	13. Type of Report and Period Covered Technical Note
	14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

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17. Key Words (Suggested by Author(s))	18. Distribution Statement
Network synthesis	Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	66	$3.00

*For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 32151

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SUMMARY

A new driving point impedance synthesis is found based on an extension of the Fialkow-Gerst Theorem. The resultant network is a cascade of lossless network sections where each section corresponds to zeros of the even part of the driving point impedance. The final termination is a resistance.

If the even part zeros are on the imaginary axis, the network consists of a reactance and a transformer. The section is terminated by a combination of a resonant circuit and an impedance of reduced degree.

The network section corresponding to complex even part zeros consists of reactances, a transformer, and a gyrator. A reduced degree impedance terminates the section. The proof of the validity of this network is based on a new technique of impedance operator even part identification.

An advantage of the new realization is the reduction in the number of gyrators used compared with the Fialkow-Gerst cascaded network. The synthesis is illustrated by an example.

INTRODUCTION

The Problem

Power conditioning systems and automatic control systems rely on network synthesis techniques to realize electric filters and compensating networks. In many systems applications, power must be transferred to a resistive load through a lossless network. One of the most important network theory problems associated with these systems is that of driving point impedance synthesis. In 1954, Fialkow and Gerst (ref. 1) found a synthesis procedure based on a new theorem of positive real functions. The cascade form of the synthesis yields the desired lossless network terminated in a resistance.

This report provides a new driving point impedance synthesis based on an extension of the Fialkow-Gerst theorem. The network realization always consists of one or more lossless cascaded sections terminated in a resistance.

Definitions

Positive real function. - A rational polynomial function $Z(s)$ of the complex variable s is a positive real function if it satisfies the following conditions:

(1) $Z(s)$ is real when s is real

(2) $\mathrm{Re}\{Z(s)\} \geq 0$ when $\mathrm{Re}\{s\} \geq 0$

where $\mathrm{Re}\{Z(s)\}$ means the real part of $Z(s)$. Alternatively, $Z(s)$ is a positive real function if

(1) $Z(s)$ is real when s is real

(2) $\mathrm{Re}\{Z(s)\} \geq 0$ when $\mathrm{Re}\{s\} = 0$

(3) $Z(s)$ is analytic in the right half s-plane

(4) The poles of $Z(s)$ on the imaginary axis are simple with positive residues

(All symbols are defined in appendix A.)

A positive real function with no poles or zeros on the imaginary axis is defined as a minimum reactance function. A positive real function whose real part vanishes at some $s = j\omega$ is defined as a minimum resistive function. A positive real function that is simultaneously a minimum reactance function and a minimum resistive function is defined as a minimum function.

If $Z(s)$ is written as a quotient of polynomials in s, the degree of $Z(s)$ is defined as the degree of the numerator or the denominator whichever is greater. The even part of $Z(s)$ is defined as

$$\mathrm{Ev}\{Z(s)\} = \frac{Z(s) + Z(-s)}{2}$$

while the odd part of $Z(s)$ is defined as

$$\mathrm{Od}\{Z(s)\} = \frac{Z(s) - Z(-s)}{2}$$

Driving point impedance. - Let the terminated two-port network of figure 1 be described by the following set of equations:

$$E_1(s) = z_{11}I_1(s) + z_{12}I_2(s) \tag{1}$$

$$E_2(s) = z_{21}I_1(s) + z_{22}I_2(s) \tag{2}$$

$$E_2(s) = -\zeta I_2(s) \tag{3}$$

where z_{11}, z_{12}, z_{21}, and z_{22} are the open circuit impedance parameters. Further, let the driving point impedance $Z(s)$ be defined as

$$Z(s) = \frac{E_1(s)}{I_1(s)} \tag{4}$$

Substituting equations (1) to (3) into equation (4) results in

$$Z(s) = \frac{z_{11}\zeta + (z_{11}z_{22} - z_{12}z_{21})}{z_{22} + \zeta} \tag{5}$$

If $z_{12} = z_{21}$, the network is said to be reciprocal. Brune (ref. 2) showed the very important fact that driving point impedances are positive real functions.

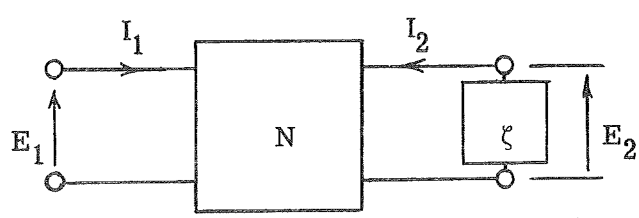


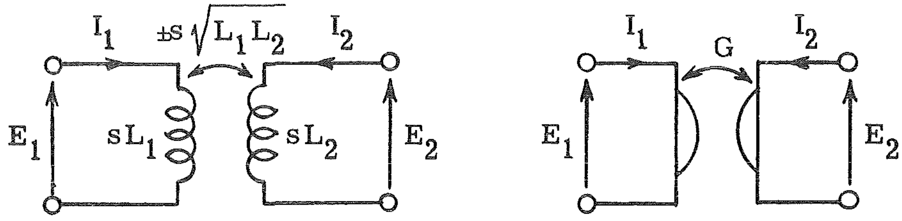
Figure 1. - Terminated two-port network.

Unity coupled transformer and gyrator. - The unity coupled transformer, shown in figure 2(a), is a circuit element that is described by the following equations:

$$E_1(s) = sL_1I_1(s) \pm s\sqrt{L_1L_2}\ I_2(s) \tag{6}$$

$$E_2(s) = \pm s\sqrt{L_1L_2}\ I_1(s) + sL_2I_2(s) \tag{7}$$

Since $z_{12} = z_{21}$, the unity coupled transformer is a reciprocal network.



(a) Unity coupled transformer. (b) Gyrator.

Figure 2. - Circuit elements.

The gyrator, shown in figure 2(b), is a circuit element that is described by the following equations

$$E_1(s) = GI_2(s) \tag{8}$$

$$E_2(s) = -GI_1(s) \tag{9}$$

where G is a real number. Since $z_{12} \neq z_{21}$, the gyrator is a nonreciprocal network.

Together with the capacitor and the inductor, the unity coupled transformer and the gyrator are considered lossless networks since they do not dissipate power.

History

Brune algorithm. - Brune (ref. 2) was the first to find a procedure that would synthesize all driving point impedances. The Brune algorithm is as follows:

(1) The positive real function is made minimum reactive by successive removal of all imaginary axis poles and zeros.

(2) The minimum resistance is found and removed.

(3) The remaining minimum function is realized by using the unity coupled transformer.

The corresponding network N of figure 1 consists of inductors, capacitors, unity coupled transformers, and resistances. Thus N is not a lossless network.

Darlington synthesis. - A lossless network terminated in a resistance was described by Darlington (ref. 3). By setting equation (5) equal to

$$\frac{m_1 + n_1}{m_2 + n_2}$$

where m and n are the even and odd functions of s, he obtained the open circuit impedance parameters. In some cases, z_{12} turns out to be irrational. Darlington solved this problem by multiplying the numerator and denominator of Z(s) by a polynomial in s called a surplus factor. The corresponding network N contains inductors, capacitors, and unity coupled transformers.

Richards theorem. - In 1947, Richards (ref. 4) showed that if Z(s) is a positive real function, the function R(s) defined by

$$R(s) = \frac{sZ(s) - kZ(k)}{sZ(k) - kZ(s)} \tag{10}$$

is also a positive real function for positive k. Although he only proved it for k = 1, the preceding statement is usually called Richards theorem.

Bott-Duffin synthesis. - Bott and Duffin (ref. 5) were the first to find a procedure that would synthesize all positive real functions without unity coupled transformers. By using Richards theorem, they were able to realize a minimum resistive function in the balanced bridge network of figure 3. Their procedure replaces step (3) of the Brune algorithm. A cascade representation of the Bott-Duffin synthesis was found by Hazony and Schott (ref. 6).

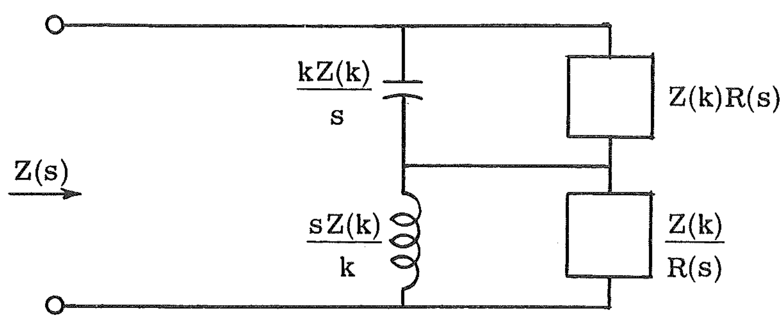


Figure 3. - Network associated with Bott-Duffin synthesis.

Fialkow-Gerst theorem. - Fialkow and Gerst (ref. 1) found a synthesis procedure based on a new theorem on positive real functions. Their original network realization was a set of balanced bridge networks without transformers. An equivalent cascaded form can also be found by using the procedure of Hazony and Schott (ref. 6). The resultant synthesis is a cascade of lossless network sections terminated in a resistive load. An advantage of their procedure is that it does not require finding the minimum resistance. Thus, the often difficult computation associated with the resistive minimization procedure is completely eliminated.

Hazony synthesis. - Hazony (refs. 7 to 9) was the first to take advantage of iteration of theorems on positive real functions. By applying Richards theorem twice, with k now complex, he found a new theorem and a new network. His theorem states that if Z(s) is a positive real function, the function $\zeta_1(s)$ defined by

$$\zeta_1(s) = \frac{(1 + s^2 D)Z(s) - sA}{1 + s^2 B - Z(s)sC} \tag{11}$$

is also a positive real function. The numbers A, B, C, and D are all positive and are given by

$$A = \frac{Z(\alpha)Z(\beta)(\alpha^2 - \beta^2)}{\alpha\beta[\alpha Z(\alpha) - \beta Z(\beta)]}$$

$$B = \frac{\alpha Z(\alpha) - \beta Z(\beta)}{\alpha\beta[\alpha Z(\beta) - \beta Z(\alpha)]}$$

$$C = \frac{\alpha^2 - \beta^2}{\alpha\beta[\alpha Z(\beta) - \beta Z(\alpha)]}$$

$$D = \frac{\alpha Z(\beta) - \beta Z(\alpha)}{\alpha\beta[\alpha Z(\alpha) - \beta Z(\beta)]}$$

where α and β are positive or conjugate complex with a nonnegative real part.

The Hazony synthesis replaces steps (2) and (3) of the Brune algorithm. The resultant realization is a cascade of lossless network sections terminated in a resistance. The network section associated with the theorem is shown in figure 4, where E^2 is positive and is given by

$$E^2 = AC - \left(\sqrt{B} - \sqrt{D}\right)^2$$

The network contains inductors, capacitors, unity coupled transformers, and gyrators.

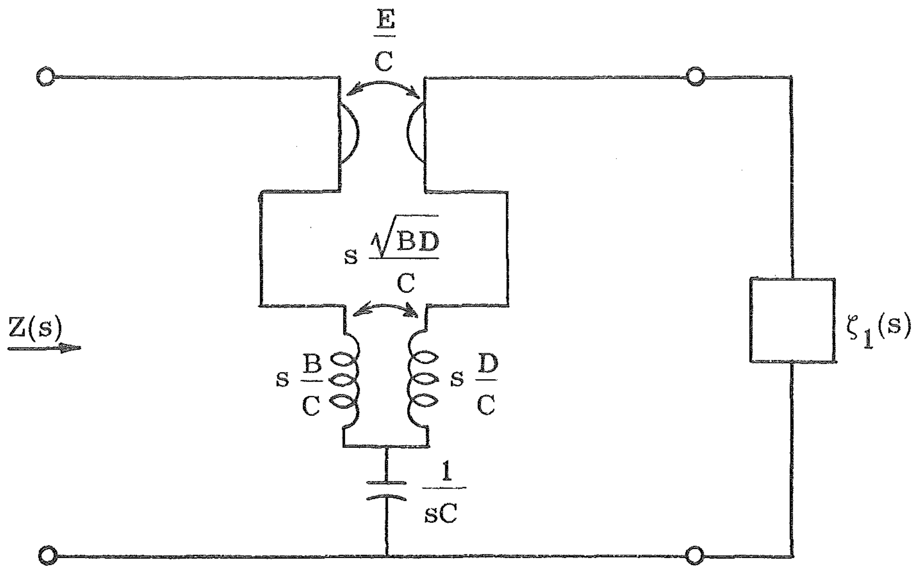


Figure 4. - Network associated with Hazony synthesis.

Present Work

In this report, a new driving point impedance synthesis is found based on an extension of the Fialkow-Gerst theorem. The resultant realization is a cascade of lossless network sections terminated in a resistance.

In the section EXTENSION OF FIALKOW-GERST THEOREM, a theorem is written based on applying the Fialkow-Gerst theorem twice. It is shown that the new function is a positive real function and is of degree not exceeding the degree of $Z(s)$. The conditions under which the degree can be reduced below that of $Z(s)$ are also discussed. The networks corresponding to the theorem are realized using three impedance operators. The open circuit impedance parameters are found for each operator and all are determined to be realizable. In the concluding section, the cascading of operators is discussed and new cascaded operator networks are proposed.

In the section DRIVING POINT IMPEDANCES OF DEGREE TWO, the second degree driving point impedance is studied for the case corresponding to complex even part zeros. The existence of certain constants on which the validity of the synthesis depends is proved by using an extension of even part synthesis. The network consists of a cascade of two sections terminated in a resistance of 1 ohm.

The second degree solution of the section DRIVING POINT IMPEDANCES OF DEGREE TWO is extended to driving point impedances of any degree in the section COMPLEX EVEN PART ZEROS AND OPERATOR EVEN PART IDENTIFICATION. It is found that all $Z(s)$ containing complex even part zeros may be realized by the new cas-

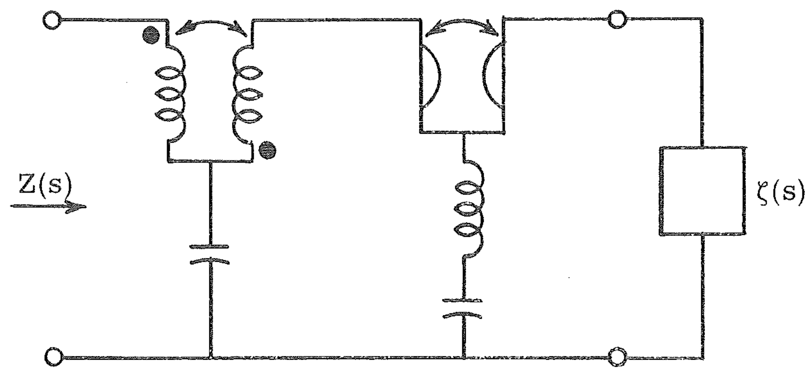


Figure 5. - New network for realizing complex even part zeros.

caded impedance operator network illustrated in figure 5. The driving point impedance is designated as $Z(s)$, while $\zeta(s)$ denotes the terminating impedance. Figure 5 differs from the cascade form of the Fialkow-Gerst network in that one gyrator has been replaced by the transformer.

EXTENSION OF FIALKOW-GERST THEOREM

As already stated, the objective of this report is to synthesize a driving point impedance as a cascade of one or more network sections terminated in a resistance. In this section, the power transfer problem and the concept of cascaded networks are reviewed.

Let the power conditioning network be described by the doubly terminated two-port network of figure 6. The generator is represented as a voltage source E_g in series with a resistance R_g, while the load voltage is denoted by E_d and the load resistance by R_d. The network is lossless in order to minimize system power losses. Suppose the network is to act as an electric filter to attenuate unwanted generator frequencies.

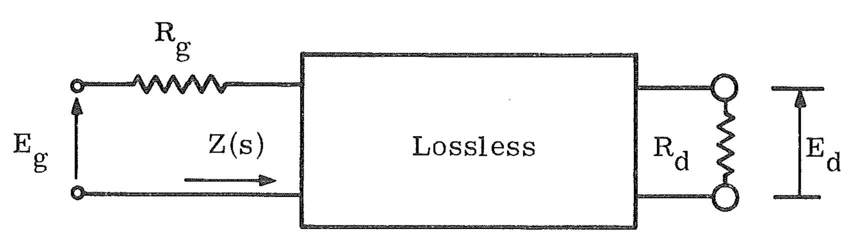


Figure 6. - Doubly terminated two-port network.

The network can be described by comparing the power delivered to the load with the maximum power available from the source. Let the ratio of these two quantities be designated as $G(j\omega)$; that is,

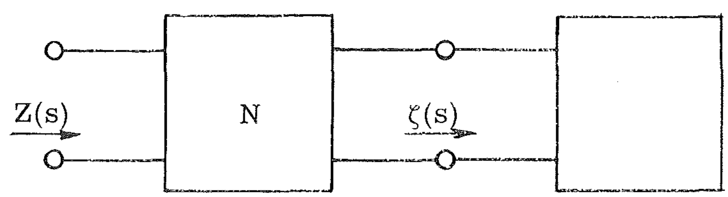
$$G(j\omega) = \frac{P_d(j\omega)}{P_g(j\omega)}$$

where the quantities are functions of $j\omega$. Examination of figure 6 yields the result

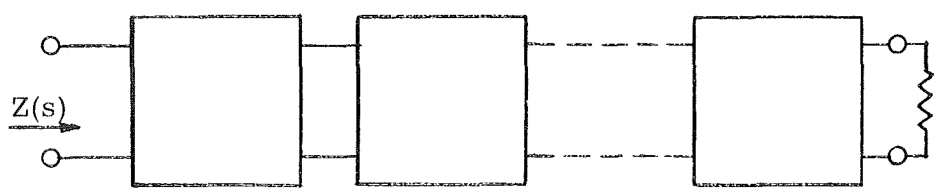
$$G(j\omega) = \frac{|E_d(j\omega)|^2}{R_d} \frac{4R_g}{|E_g(j\omega)|^2} = \frac{4R_g}{R_d} \left| \frac{E_d(j\omega)}{E_g(j\omega)} \right|^2$$

Darlington (ref. 3) showed that given $G(j\omega)$, the driving point impedance of the network input, designated $Z(s)$ in figure 6, can always be found. Thus, the power transfer problem can be effectively reformulated in terms of the driving point impedance. A synthesis of $Z(s)$ therefore also yields a synthesis of $G(j\omega)$.

Let each section of a cascaded network be described by the terminated two-port network of figure 7(a). The driving point impedance is designated $Z(s)$, while the terminating impedance is denoted by $\zeta(s)$. A cascaded network is a chain of two-port sections



(a) Terminated two-port network.



(b) Cascade of network sections terminated in a resistance.

Figure 7. - Network termination.

connected so that the terminating impedance of one section is the driving point impedance of the following section. Figure 7(b) illustrates a cascade of network sections where the final terminating impedance is a resistance. It therefore represents the ultimate network configuration.

Connection of two-port sections in cascade places certain restrictions on the terminating impedance of figure 7(a). First, being a terminating impedance and the driving point impedance of the following section, $\zeta(s)$ must of course be a positive real function. Second, the degree of $\zeta(s)$ is required to be no greater than the degree of $Z(s)$. Thus, no section of the cascaded network of figure 7(b) is allowed to raise the degree of its respective driving point impedance. This condition, however, is insufficient to guarantee a final terminating impedance equal to a resistance, since no section can reduce the degree. As a final condition on figure 7(a), then, there must be some mechanism by which the degree of $\zeta(s)$ can be made lower than the degree of $Z(s)$. Thus, under certain circumstances, $\zeta(s)$ must possess the capability of being reduced in degree.

For the rest of this section, a theorem is discussed which meets the needs of $\zeta(s)$ and which also has a physically realizable network representation. First, the Fialkow-Gerst theorem is extended and found to conform to the requirements of $\zeta(s)$. Then the corresponding networks are synthesized. And finally, the cascading of operators is discussed and new cascaded operator networks are proposed.

The Theorem

Fialkow and Gerst (ref. 1) wrote their theorem as an extension of Richards theorem (ref. 4). The positive real function associated with Richards theorem was called $R(s)$ (see section History, Richards theorem). Rewriting it in the form

$$R(s) = \frac{\left(\frac{s}{k}\right)Z(s) - Z(k)}{\left(\frac{s}{k}\right)Z(k) - Z(s)}$$

indicates that the function s/k is a reactance. Fialkow and Gerst found that, subject to certain conditions, $R(s)$ would still be a positive real function if the reactance of a series or parallel resonant circuit were used instead of s/k. They extended these simple reactances to a more generalized form for use in their theorem.

A new theorem on positive real functions is now written based on applying the Fialkow-Gerst theorem twice. The original theorem corresponds to the reactance of equation (12) and the positive real function of equation (14).

Theorem I:

(1) Let $a_1, a_2, \ldots, a_p$ be a set of real or complex numbers having $\mathrm{Re}\{a_i\} > 0$ (where $i = 1, 2, \ldots, p$) and such that the complex a_i occur in conjugate complex pairs. Let the function

$$Z_1(s) = \frac{\prod_{i=1}^{p} (s + a_i) + \prod_{i=1}^{p} (s - a_i)}{\prod_{i=1}^{p} (s + a_i) - \prod_{i=1}^{p} (s - a_i)} \tag{12}$$

be defined. Further, let $b_1, b_2, \ldots, b_q$ be a set of real or complex numbers having $\mathrm{Re}\{b_j\} > 0$ (where $j = 1, 2, \ldots, q$) and such that the complex b_j occur in conjugate complex pairs. Let the function

$$Z_2(s) = \frac{\prod_{j=1}^{q} (s + b_j) + \prod_{j=1}^{q} (s - b_j)}{\prod_{j=1}^{q} (s + b_j) - \prod_{j=1}^{q} (s - b_j)} \tag{13}$$

also be defined. Then $Z_1(s)$ and $Z_2(s)$ are reactances.

(2) Let $Z(s)$ be a positive real function of degree n. If $Z(a_i) = C_1 > 0$ (where $i = 1, 2, \ldots, p$) and $Z_1'(b_j) = C_2 > 0$ (where $j = 1, 2, \ldots, q$), where C_1 and C_2 are constants,

$$Z_1'(s) = \frac{Z_1(s)Z(s) - C_1}{C_1 Z_1(s) - Z(s)} \tag{14}$$

and

$$Z_2'(s) = \frac{Z_2(s)Z_1'(s) - C_2}{C_2 Z_2(s) - Z_1'(s)} \tag{15}$$

are positive real functions.

Proof:

Fialkow and Gerst proved that $Z_1(s)$ of equation (12) is a reactance and $Z_1'(s)$ of equation (14) is a positive real function. Their results are used herein to prove the extensions represented by equations (13) and (15). In equation (13), $Z_2(s)$ is of the same form as $Z_1(s)$ except $a_1, a_2, \ldots, a_p$ has been replaced by $b_1, b_2, \ldots, b_q$. Thus $Z_2(s)$ is also a reactance. This proves Part (1) of the theorem.

For Part (2), if equations (15) and (14) are compared, $Z_2'(s)$ is of the same form as $Z_1'(s)$ except that $Z_1(s)$, $Z(s)$, and C_1 have been replaced by $Z_2(s)$, $Z_1'(s)$, and C_2, respectively. Since $Z_1'(s)$ is a positive real function, $Z_2'(s)$ is also a positive real function. This completes the proof of the theorem.

Stated in a less formal way, the preceding theorem states that under certain conditions, if $Z_1(s)$ and $Z_2(s)$ are reactances, there exist functions $Z_1'(s)$ and $Z_2'(s)$ that are positive real functions. Suppose now that the terminal impedance $\zeta(s)$ of figure 7(a) is associated with the new functions. Suppose also that $\zeta(s)$ is made equal to a constant times $Z_1'(s)$ or $Z_2'(s)$. Since the latter functions are positive real functions, it follows that $\zeta(s)$ is also a positive real function. Similarly, if $\zeta(s)$ is made equal to a constant times the reciprocal of $Z_1'(s)$ or $Z_2'(s)$, $\zeta(s)$ is also a positive real function. Thus, the functions corresponding to Theorem I meet the first requirement of $\zeta(s)$.

For the second condition for $\zeta(s)$, it must be shown that the degree of $Z_1'(s)$ and $Z_2'(s)$ is at most equal to n, the degree of $Z(s)$. Finally, it must also be shown under what circumstances the degree of $Z_1'(s)$ and $Z_2'(s)$ can be reduced below n.

According to equations (12) and (14), the degree of $Z_1'(s)$ is $n + p$ since the degree of $Z(s)$ is n and the degree of $Z_1(s)$ is p. Fialkow and Gerst proved that the product

$$\prod_{i=1}^{p} (s - a_i)$$

is always a common factor of the numerator and denominator of $Z_1'(s)$. Consequently, when these factors are canceled, the degree of $Z_1'(s)$ reduces to $n + p - p$ or n. Similarly, since the degree of the uncanceled $Z_1'(s)$ is $n + p$, by equations (13) and (15), the degree of $Z_2'(s)$ is $n + p + q$. Extending the results of Fialkow and Gerst, the product

$$\prod_{i=1}^{p} (s - a_i) \prod_{j=1}^{q} (s - b_j)$$

is a common factor of the numerator and denominator of $Z_2'(s)$, and after cancellation the degree of $Z_2'(s)$ reduces to $n + p + q - (p + q)$ or n. Thus, since the degree of both $Z_1'(s)$ and $Z_2'(s)$ is at most n, the degree of the associated terminal impedance $\zeta(s)$ is at most n.

Now the final task is to determine the conditions under which the degree of $Z_1'(s)$ or $Z_2'(s)$ can be reduced below the degree of $Z(s)$. Fialkow and Gerst proved that the degree of $Z_1'(s)$ can be reduced below n if and only if the number a_i for some $i = 1, 2, \ldots, p$ is a zero of the even part of $Z(s)$. If there are r such numbers, the degree is reduced to $n - r$; that is, if all other conditions are met, the degree of $\zeta(s)$ in figure 7(a) is less than the degree of $Z(s)$ when a_i is chosen so as to be equal to a zero of the even part of $Z(s)$. Thus, for the cascaded network of figure 7(b), each section that reduces the degree corresponds to one or more zeros of the even part of $Z(s)$.

A similar condition can be obtained for $Z_2'(s)$. By extending the preceding results, the degree of $Z_2'(s)$ can be reduced below n if and only if the number a_i for some $i = 1, 2, \ldots, p$ or the number b_j for some $j = 1, 2, \ldots, q$ is a zero of the even part of $Z(s)$. If there are t such numbers, the degree is reduced to $n - t$. For the cascaded network of figure 7(b), each section that reduces the degree corresponds to one or more zeros of the even part of $Z(s)$.

Fialkow and Gerst also showed that if a_i does not correspond to a zero of the even part of $Z(s)$, the even part zeros of $Z(s)$ and $Z_1'(s)$ are identical.

Network Synthesis

In the previous section, Theorem I was shown to provide a powerful tool for the determination of $\zeta(s)$, the terminating impedance of figure 7(a). It was also shown that under certain conditions, the degree of $\zeta(s)$ can be made lower than the degree of $Z(s)$. This is really only half the story since $\zeta(s)$ is useless without an associated network N which is also physically realizable. It is the purpose of this section to synthesize realizable networks that correspond to $Z_1'(s)$ and $Z_2'(s)$ of Theorem I. At this point, all the conditions of Theorem I are assumed to be satisfied; that is, there exist numbers $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q, C_1$, and C_2 that meet the requirements of the theorem. Proofs of the existence of these numbers and procedures to find them are discussed later.

For the sake of completeness, Theorem I was written in its most general form. It is most convenient, however, to use more restricted versions of the theorem. All the synthesis procedures developed in this report fall under one or more of the following three special cases:

Special Case 1:

If $p = 1$ and $a_1 = k > 0$, then by equation (12)

$$Z_1(s) = \frac{s}{k} \tag{16}$$

and $Z_1'(s)$ reduces to

$$Z_1'(s) = \frac{sZ(s) - kZ(k)}{sZ(k) - kZ(s)} \tag{17}$$

Equation (17) is recognized as the function associated with Richards theorem.

Special Case 2:

If $p = 2$, the reactance $Z_1(s)$ becomes

$$Z_1(s) = \frac{s^2 + a_1 a_2}{s(a_1 + a_2)} \tag{18}$$

where a_1 and a_2 are a pair of conjugate complex numbers having positive real parts. The impedance $Z_1'(s)$ can be written

$$Z_1'(s) = \frac{\left(s^2 + a_1 a_2\right)Z(s) - sC_1(a_1 + a_2)}{C_1\left(s^2 + a_1 a_2\right) - Z(s)s(a_1 + a_2)} \tag{19}$$

Special Case 3:

If $p = q = 1$ and $a_1 = b_1 = a > 0$, $Z_2'(s)$ becomes

$$Z_2'(s) = \frac{\left(s^2 + a^2 C_2\right)Z(s) - saC_1(1 + C_2)}{\left(s^2 C_2 + a^2\right)C_1 - saZ(s)(1 + C_2)} \tag{20}$$

Since $C_1 = Z(a)$, C_1 may be easily evaluated from $Z(s)$ for any a. For the evaluation of C_2, the following equation is written using equation (14):

$$C_2 = Z_1'(a) = \lim_{s \to a} Z_1'(s) = \lim_{s \to a} \frac{sZ(s) - aZ(a)}{sZ(a) - aZ(s)}$$

If the limit is taken, C_2 is indeterminate of the form O/O. Using L'Hospital's rule results in

$$C_2 = \lim_{s \to a} \frac{s \frac{dZ(s)}{ds} + Z(s)}{Z(a) - a \frac{dZ(s)}{ds}} \tag{21}$$

Equation (21) can be used for the evaluation of C_2.

Now the network realizations are found for Special Cases 1, 2, and 3, just discussed. Impedance operator methods (refs. 8 to 11) are utilized to find the networks corresponding to N of figure 7(a).

First, let $Z(s)$ be minimum reactive. This can always be achieved by successive removal of series and shunt reactances. It was shown earlier (see section Definitions, Driving point impedance) that the driving point impedance of the two-port network of figure 1 or figure 7(a) can be written in terms of the open circuit impedance parameters in the form

$$Z(s) = \frac{z_{11}\zeta + (z_{11}z_{22} - z_{12}z_{21})}{z_{22} + \zeta} \tag{22}$$

where ζ is the network terminating impedance. Let an impedance operator $V(s)$ be defined as

$$V(s) = Z(s)\big|_{\zeta=1} \tag{23}$$

that is, the impedance operator is equal to the driving point impedance with the network terminating impedance equal to 1. Thus, the synthesis of $Z(s)$ consists of, first, realizing a network corresponding to $V(s)$ and, second, setting the load impedance equal to ζ.

The balance of this section is divided into two parts: two operators for $Z_1'(s)$ are discussed in Part (1) and one operator for $Z_2'(s)$ is discussed in Part (2).

(1) Operators for $Z_1'(s)$. These operators correspond in general to equations (12) and (14), and in particular to Special Cases 1 and 2. Rewriting $Z(s)$ from equation (14) gives

$$Z(s) = \frac{[C_1Z_1(s)][C_1Z_1'(s)] + (C_1)^2}{C_1Z_1(s) + C_1Z_1'(s)} \tag{24}$$

where $Z_1(s)$ is obtained from equation (12), (16), or (18). Now the two equations for $Z(s)$ (eqs. (22) and (24)) are compared. The impedance ζ can be identified with $C_1Z_1'(s)$, since $Z_1'(s)$ has been shown to be a positive real function. Now let the first impedance operator be defined as

$$V_1(s) = Z(s)\Big|_{C_1 Z_1'(s)=1} \tag{25}$$

where ζ has been replaced by $C_1 Z_1'(s)$. Identifying the remaining terms in equations (22) and (24) yields

$$z_{11} = C_1 Z_1(s)$$

$$z_{22} = z_{11}$$

$$z_{11} z_{22} - z_{12} z_{21} = (C_1)^2$$

The transfer impedances z_{12} and z_{21} can be chosen as

$$z_{12} = C_1 Z_1(s) + C_1$$

$$z_{21} = C_1 Z_1(s) - C_1$$

These identifications are valid if and only if the resultant network is realizable. Figure 8 represents the lossless network corresponding to $V_1(s)$. Since $C_1 > 0$ by hypothesis in Theorem I, and $Z_1(s)$ is a reactance, the network is realizable and the identification is valid.

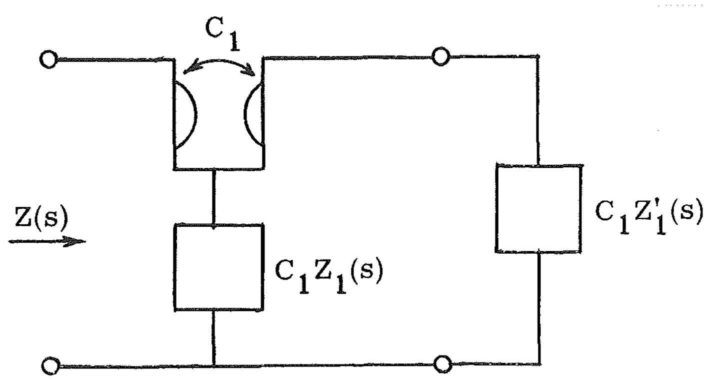


Figure 8. - Cascade network realization of Fialkow-Gerst theorem using operator $V_1(s)$.

Equation (24) can be rearranged to write $Z(s)$ as

$$Z(s) = \frac{\left[\frac{C_1}{Z_1(s)}\right]\left[\frac{C_1}{Z_1'(s)}\right] + (C_1)^2}{\frac{C_1}{Z_1(s)} + \frac{C_1}{Z_1'(s)}} \tag{26}$$

Now let the second impedance operator be defined as

$$V_2(s) = Z(s)\Big|_{C_1/Z_1'(s)=1} \tag{27}$$

where ζ has been replaced by $C_1/Z_1'(s)$. Identifying corresponding terms in equations (22) and (26) results in

$$z_{11} = \frac{C_1}{Z_1(s)}$$

$$z_{22} = z_{11}$$

$$z_{11}z_{22} - z_{12}z_{21} = (C_1)^2$$

The transfer impedances z_{12} and z_{21} can be chosen as

$$z_{12} = \frac{C_1}{Z_1(s)} + C_1$$

$$z_{21} = \frac{C_1}{Z_1(s)} - C_1$$

Figure 9 represents the lossless network corresponding to $V_2(s)$. Since $C_1 > 0$ by hypothesis in Theorem I, and since $1/Z_1(s)$ is a reactance, the network is realizable and the identification is valid.

Since figures 8 and 9 are realizations corresponding to $Z_1'(s)$, they are called the cascade circuit representations of the Fialkow-Gerst theorem.

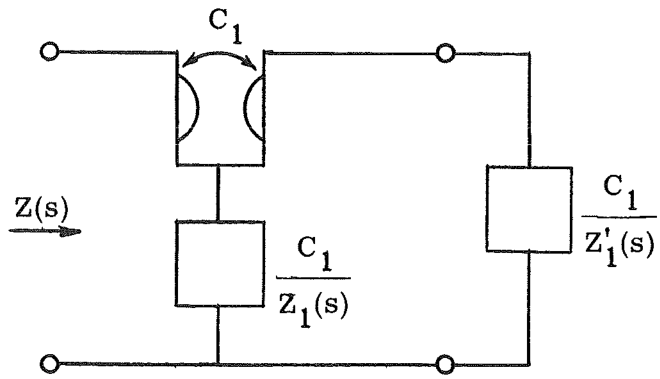


Figure 9. - Cascade network realization of Fialkow-Gerst theorem using operator $V_2(s)$.

(2) Operator for $Z_2'(s)$. This operator corresponds to Special Case 3. Rewriting Z(s) from equation (20) results in

$$Z(s) = \frac{\dfrac{C_1\left(a^2 + s^2C_2\right)}{as(C_2 + 1)}\left[\dfrac{C_1}{C_2} Z_2'(s)\right] + \dfrac{(C_1)^2}{C_2}}{\dfrac{C_1\left(C_2a^2 + s^2\right)}{asC_2(C_2 + 1)} + \left[\dfrac{C_1}{C_2} Z_2'(s)\right]} \tag{28}$$

Now let the impedance operator for this case be defined as

$$V_3 = Z(s)\Big|_{(C_1/C_2)Z_2'(s)=1} \tag{29}$$

where ζ has been replaced by $(C_1/C_2)Z_2'(s)$. Identifying corresponding terms in equations (22) and (28) gives

$$z_{11} = \frac{C_1a}{s(1 + C_2)} + \frac{sC_1C_2}{a(1 + C_2)}$$

$$z_{22} = \frac{C_1a}{s(1 + C_2)} + \frac{sC_1}{C_2a(1 + C_2)}$$

$$z_{11}z_{22} - z_{12}z_{21} = \frac{(C_1)^2}{C_2}$$

The transfer impedances z_{12} and z_{21} can be chosen as

$$z_{12} = z_{21} = \frac{C_1 a}{s(1 + C_2)} - \frac{sC_1}{a(1 + C_2)}$$

The network corresponding to $V_3(s)$ is shown in figure 10. This realization was first described by Murdoch in reference 10, section 3.4 By definition of the unity coupled transformer, the transformer has unity coupling since its z_{12} is the square root of the product of the primary and secondary inductances. Since a, C_1, and C_2 are positive by hypothesis in Theorem I, the network is realizable and the identifications are valid.

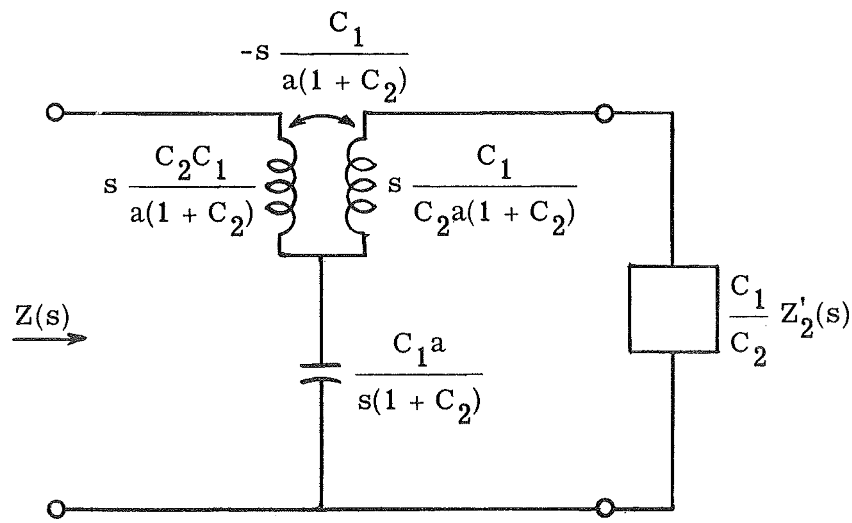
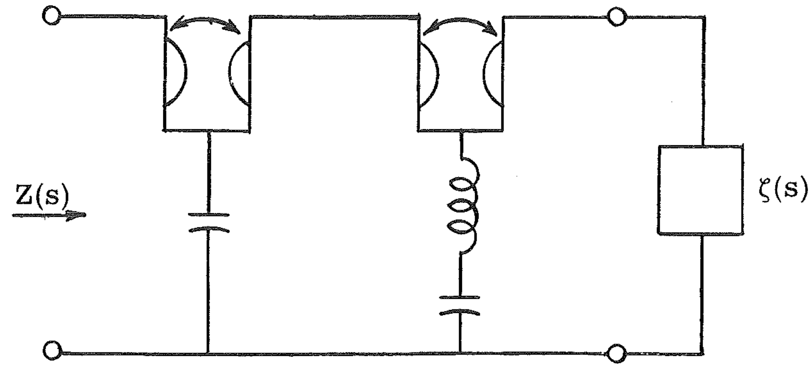


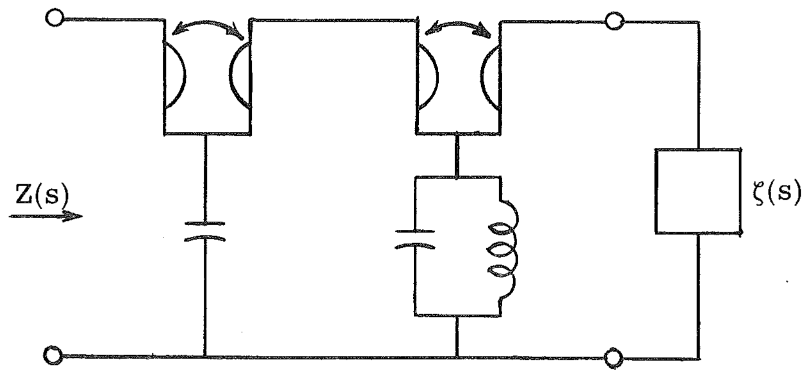
Figure 10. - Network realization of $V_3(s)$.

A New Cascaded Operator Network

In the section Network Synthesis, a number of networks were synthesized corresponding to $Z_1'(s)$ and $Z_2'(s)$. The impedance operators were treated separately and independently. The question arises as to whether there exists a combination of two operators that might be more advantageous together than individually. Fialkow and Gerst found some combinations that proved to be useful for their purposes. Figures 11(a) and (b) represent two such cascaded operator networks. They are important networks due to their versatility as sections for cascaded network synthesis. The networks can be used



(a) Series resonant.



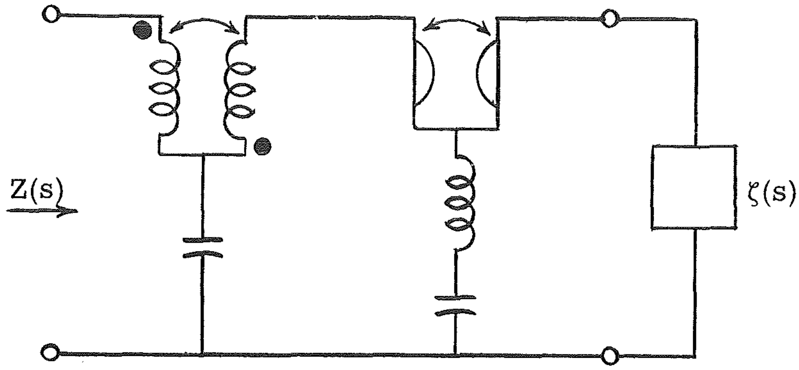
(b) Parallel resonant.

Figure 11. - Fialkow-Gerst cascaded operator networks.

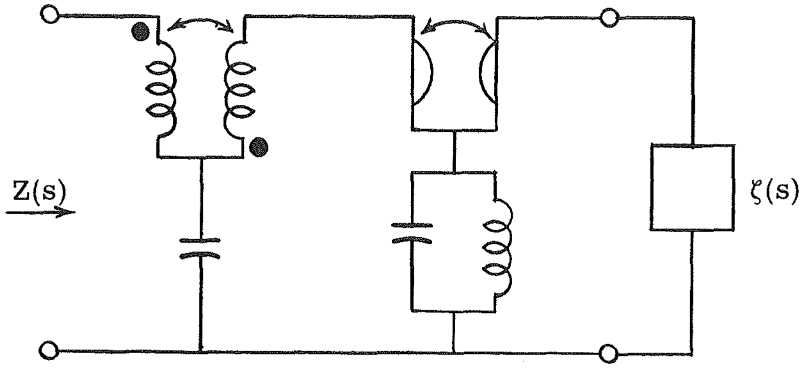
as is or certain network elements can be deleted to synthesize all driving point impedances. Murdoch (ref. 10) gives a detailed account of a number of different cascaded operators.

New cascaded operator networks are now proposed, as illustrated in figures 12(a) and (b). Part of the networks are recognized as figure 10, the network realization of operator $V_3(s)$. Comparison with figure 11 shows that one of the gyrators has been replaced by a transformer.

It is the purpose of the rest of this report to prove that all driving point impedances can be realized by using figure 12 or modifications thereof. Specifically, it must be shown that element values can always be found that are physically realizable. Since these values are related to the constants of Theorem I, conditions under which the theorem can be applied are in fact being determined.



(a) Series resonant.



(b) Parallel resonant.

Figure 12. - New cascaded operator networks.

DRIVING POINT IMPEDANCES OF DEGREE TWO

A number of cascade network representations for the extension of the Fialkow-Gerst theorem were found by assuming that all the conditions of Theorem I had been satisfied; that is, there existed numbers $a_1, a_2, \ldots, a_p$; $b_1, b_2, \ldots, b_q$; C_1; and C_2 that met the requirements of the theorem. Since these numbers are to be used for the cascade synthesis of figure 7(b), they must also be chosen so as to reduce the degree of the associated terminal impedance. Thus, the numbers must perform the dual role of meeting the explicit requirements of the theorem as well as degree reduction. The purpose of the remainder of the text is to prove the existence of these numbers and to find procedures for their evaluation.

Suppose that $Z(s)$ has been made minimum reactive by removal of all the imaginary axis poles and zeros. The zeros of the even part of $Z(s)$ can be written as the values of s that satisfy

$$\mathrm{Ev}\{Z(s)\} = \frac{Z(s) + Z(-s)}{2} = 0$$

If $\mathrm{Ev}\{Z(s)\}$ is expressed as a ratio of polynomials of s and since $Z(s)$ is minimum reactive, the zeros of the even part are also the zeros of the numerator of the even part. It is known that the numerator of the even part of $Z(s)$ is a polynomial of degree $2n$, whose zeros lie symmetrically about both the real and imaginary axes. Further, all the imaginary axis zeros are of even multiplicity. For these reasons, all even part zeros must fall under one of the following categories:

(1) Real even part zeros. These zeros occur in pairs and have symmetry about the imaginary axis. The positive real even part zero will be designated as u, where $u > 0$.

(2) Imaginary even part zeros. Since these zeros possess symmetry about the real axis, the imaginary even part zero can be designated as jv, where $v \neq 0$.

(3) Complex even part zeros. These zeros lie symmetrically about both the real and imaginary axes. Therefore, the complex even part zeros with a positive real part can be designated as $u + jv$, where $u > 0$ and $v \neq 0$.

Networks for the case corresponding to complex even part zeros are developed in this section and in the section COMPLEX EVEN PART ZEROS AND OPERATOR EVEN PART IDENTIFICATION. The results are extended to include the imaginary even part zeros (category (2)). Real even part zeros can be synthesized by direct application of any of the operators of the section EXTENSION OF FIALKOW-GERST THEOREM. The driving point impedances of degree two are discussed in this section, and the results are extended to impedances of any degree in the section COMPLEX EVEN PART ZEROS AND OPERATOR EVEN PART IDENTIFICATION.

One reason the driving point impedances of degree two is studied is the fact that all impedances can be reduced to a form similar to the second degree case. Thus, treatment of the simpler second degree driving point impedance points the way to the solution of the more complex situation. (See section COMPLEX EVEN PART ZEROS AND OPERATOR EVEN PART IDENTIFICATION.)

The complex even part zero is designated by $u + jv$, where $u > 0$ and $v \neq 0$. Because of the positive real function character of $Z(s)$, $Z(u + jv)$ can be positive or can be complex with a positive real part. When $Z(u + jv)$ is positive, $Z_1'(s)$ is shown to be found directly from equation (19) and the network corresponds to figure 8 or 9 with the terminating impedance equal to a resistance. If $Z(u + jv)$ is complex, it is shown that the realizations are the cascaded operator networks of figures 11(a) and 12(a), where $\zeta(s)$ has been set to 1 ohm.

In the next section, important coefficient conditions are found corresponding to $Z(s)$

of degree two. The cases for positive and complex $Z(u+jv)$ are discussed in subsequent sections.

Coefficient Conditions

Analysis of all realizable functions of second degree requires the complete description of the functions in their most general form. In appendix B it is shown that all second degree driving point impedances can be written in the form

$$Z(s) = \frac{1 + sa + s^2 b}{1 + sc + s^2 d} \tag{30}$$

where the coefficients a, b, c, and d are positive numbers.[1] The even part of $Z(s)$ can be written

$$\mathrm{Ev}\{Z(s)\} = \frac{Z(s) + Z(-s)}{2} = \frac{1 + s^2(b + d - ac) + s^4 bd}{(1 + s^2 d)^2 - (sc)^2} \tag{31}$$

Completing the square in the numerator gives equation (31) in the form

$$\mathrm{Ev}\{Z(s)\} = \frac{\left(1 + s^2\sqrt{bd}\right)^2 - s^2 e^2}{(1 + s^2 d)^2 - s^2 c^2} \tag{32}$$

where

$$e^2 = ac - \left(\sqrt{b} - \sqrt{d}\right)^2 \tag{33}$$

Thus equation (32) is the even part of $Z(s)$ in terms of the coefficients of equation (30).

In order to find the first coefficient condition, the fact that $Z(s)$ is a positive real function is used. It was stated in the section Definitions that $Z(s)$ is a positive real function only if

$$\mathrm{Re}\{Z(s)\} \geq 0 \qquad \text{when} \quad \mathrm{Re}\{s\} = 0 \tag{34}$$

If $s = j\omega$, equation (34) can be written as

$$\mathrm{Re}\{Z(j\omega)\} \geq 0 \tag{35}$$

---

[1] The letter a, in this section only, is used to represent one of the coefficients of $Z(s)$. The constant a, associated with equation (20) is designated a_0.

and since

$$\mathrm{Re}\{Z(j\omega)\} = \mathrm{Ev}\{Z(s)\}\Big|_{s=j\omega} \tag{36}$$

then by equation (32), equation (35) becomes

$$\frac{\left(1 - \omega^2\sqrt{bd}\right)^2 + \omega^2 e^2}{(1 - \omega^2 d)^2 + \omega^2 c^2} \geq 0 \tag{37}$$

Now since b, c, and d are positive numbers, equation (37) can be true for all ω if and only if the coefficient condition

$$e^2 = ac - \left(\sqrt{b} - \sqrt{d}\right)^2 \geq 0 \tag{38}$$

is satisfied. In equation (38), the equality sign corresponds to zeros of the even part on the imaginary axis.

The second coefficient condition is found by first noting that the zeros of the numerator of the even part satisfy

$$\left(1 + s^2\sqrt{bd}\right)^2 - s^2 e^2 = 0 \tag{39}$$

The four zeros of the even part of $Z(s)$ are therefore

$$s = \frac{\pm e \pm \sqrt{ac - \left(\sqrt{b} + \sqrt{d}\right)^2}}{2\sqrt{bd}} \qquad e \geq 0 \tag{40}$$

Since zeros of the even part that have an imaginary part not equal to zero ($v \neq 0$) are being dealt with, the coefficient condition

$$ac - \left(\sqrt{b} + \sqrt{d}\right)^2 < 0 \tag{41}$$

must be satisfied. Combining conditions (38) and (41) gives

$$\left(\sqrt{b} + \sqrt{d}\right)^2 > ac \geq \left(\sqrt{b} - \sqrt{d}\right)^2 \tag{42}$$

for all $Z(s)$ having an even part zero with a nonzero imaginary part. Note that the right inequality of equation (42) was obtained from consideration of the positive real function character of $Z(s)$ and therefore is true for all $Z(s)$ of degree two. The left inequality, however, does not apply to all $Z(s)$ since it was derived only for even part zeros that are not located on the real axis. For complex even part zeros with a positive real part, equation (42) becomes

$$\left(\sqrt{b} + \sqrt{d}\right)^2 > ac > \left(\sqrt{b} - \sqrt{d}\right)^2 \tag{43}$$

The inequalities of equation (43) play an important role in the proofs of the following sections.

Complex Even Part Zeros with $Z(u + jv)$ Positive

Since $Z(u + jv) = C_1$ is positive, Theorem I can be directly applied. The identifications in Special Case 2 of the Network Synthesis section, $a_1 = u + jv$ and $a_2 = u - jv$ are used in equations (18) and (19) to give

$$Z_1(s) = \frac{s^2 + u^2 + v^2}{2su} \tag{44}$$

and

$$Z_1'(s) = \frac{(s^2 + u^2 + v^2)Z(s) - 2suC_1}{C_1(s^2 + u^2 + v^2) - Z(s)2su} \tag{45}$$

respectively. The network synthesis is the same as the one given in the section Operators for $Z_1'(s)$, where the network corresponds to figure 8 or 9. Since a_1 and a_2 correspond to two zeros of the even part, the degree of $Z_1'(s)$ is two less than $Z(s)$, and the termination is now resistive. The rest of this section is devoted to finding specific values for the components of the networks in terms of the coefficients of $Z(s)$.

First, $Z_1(s)$ and $Z_1'(s)$ must be determined in terms of the coefficients of $Z(s)$. Since by equation (40)

$$u + jv = \frac{e + j\sqrt{\left(\sqrt{b} + \sqrt{d}\right)^2 - ac}}{2\sqrt{bd}} \tag{46}$$

the definitions for u and v can be written as

$$\left.\begin{aligned} u &= \frac{e}{2\sqrt{bd}} \\ v &= \frac{\sqrt{\left(\sqrt{b} + \sqrt{d}\right)^2 - ac}}{2\sqrt{bd}} \end{aligned}\right\} \tag{47}$$

Equations (44) and (45) now become

$$Z_1(s) = \frac{1 + s^2\sqrt{bd}}{se} \tag{48}$$

and

$$Z_1'(s) = \frac{\left(1 + s^2\sqrt{bd}\right)Z(s) - seC_1}{C_1\left(1 + s^2\sqrt{bd}\right) - Z(s)se} \tag{49}$$

respectively. Thus, $Z_1(s)$ and $Z_1'(s)$ are expressed in terms of the coefficients of $Z(s)$.

Synthesis of the networks of figures 8 and 9 from equations (48) and (49) could proceed, but when $Z(u + jv) > 0$, there is another constraint on the coefficients of $Z(s)$ in addition to the one associated with equation (43). Letting $s = u + jv$ in equation (30) results in

$$C_1 = Z(u + jv) = \frac{1 + (u + jv)a + (u + jv)^2 b}{1 + (u + jv)c + (u + jv)^2 d} \tag{50}$$

Taking the imaginary part of $Z(u + jv)$ shows that

$$\operatorname{Im}\{Z(u + jv)\} = \frac{v\left[(bc - ad)(u^2 + v^2) + 2u(b - d) + a - c\right]}{\left[1 + uc + (u^2 - v^2)d\right]^2 + v^2(c + 2ud)^2} \tag{51}$$

Since $Z(u + jv) > 0$, equation (51) must be equal to zero. The denominator of $\operatorname{Im}\{Z(u + jv)\}$ cannot be zero by virtue of the fact that $c + 2ud > 0$. Thus, the numerator of equation (51) must satisfy the condition

$$v\left[(bc - ad)(u^2 + v^2) + 2u(b - d) + a - c\right] = 0 \tag{52}$$

Replacing $u^2 + v^2$ and u by their values from equation (47) gives equation (52) in the form

$$v\left[\frac{bc - ad}{\sqrt{bd}} + \frac{e(b - d)}{\sqrt{bd}} + a - c\right] = 0 \tag{53}$$

or after factoring

$$v\left[\sqrt{\frac{d}{b}}\,(a + e) + e + c\right]\left(\sqrt{\frac{b}{d}} - 1\right) = 0 \tag{54}$$

Since $v \neq 0$ by hypothesis, and since the terms in the first bracket are all positive, equation (54) can be true if and only if $b = d$. Thus, the condition $Z(u + jv) > 0$ has been shown to imply that $b = d$.

The task of finding network component values remains. If $b = d$, $e = \sqrt{ac}$ and $C_1 = Z(u + jv) = \sqrt{a/c}$. Equations (48) and (49) can be used to write $Z_1(s)$ and $Z_1'(s)$ as

$$Z_1(s) = \frac{1 + s^2 b}{s\sqrt{ac}} \tag{55}$$

and

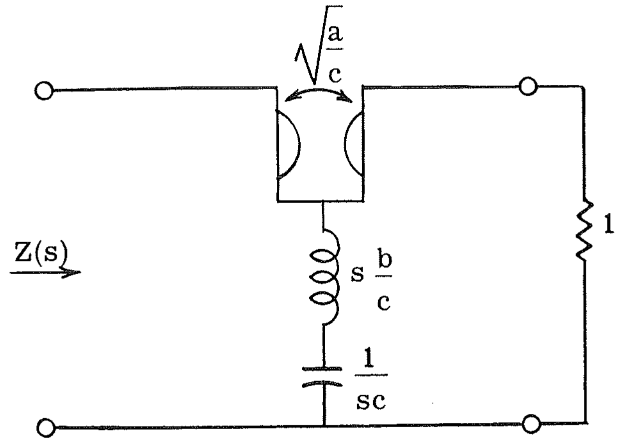
$$Z_1'(s) = \frac{(1 + s^2 b)Z(s) - sa}{\sqrt{\frac{a}{c}}\,(1 + s^2 b) - Z(s)s\sqrt{ac}} \tag{56}$$

Replacing $Z(s)$ by equation (30) gives equation (56) in the form

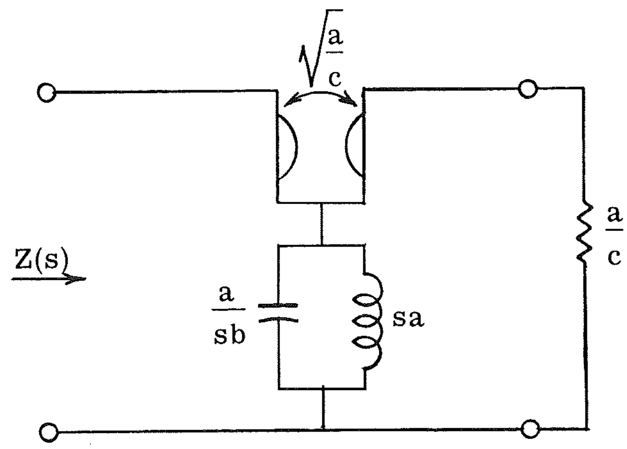
$$Z_1'(s) = \sqrt{\frac{c}{a}} \tag{57}$$

Thus, the degree of the terminating impedance has been reduced to zero, two less than $Z(s)$, the driving point impedance.

The networks corresponding to figures 8 and 9 are given in figures 13(a) and (b). All the element values are in terms of the coefficients of $Z(s)$, and the terminations are resistances.



(a) Series resonant.



(b) Parallel resonant.

Figure 13. - Fialkow-Gerst network realization for Z(s) of degree two when Z(u + jv) is positive.

Complex Even Part Zeros with Z(u + jv) Complex - Even Part Identification

Since $Z(u + jv) = C_1$ is not positive, Theorem I cannot be applied directly. Instead, procedures are used in which the network realization consists of the cascaded operator network discussed in the section A New Cascaded Operator Network.

This section is divided into two parts. Part (1) shows that Z(s) can be realized as the cascaded operator network of figure 11(a) with the terminating impedance equal to a resistance of 1 ohm. The synthesis is arrived at by using Special Case 1 followed by Special Case 2, discussed in the section Network Synthesis. Although this was also shown by Fialkow and Gerst, a new proof is developed based on even part identification. The new proof is also used in Part (2) where it is shown that $Z(s)$ can be realized as the new cascaded operator network of figure 12(a) where the terminal impedance is again a resistance of 1 ohm. That synthesis can be achieved by using Special Case 3 followed by Special Case 2.

(1) The second degree driving point impedance is written in the form

$$Z(s) = \frac{1 + sa + s^2 b}{1 + sc + s^2 d} \qquad b \neq d \tag{58}$$

The inequality $b \neq d$ is an added constraint since it was shown previously that $b = d$ implies $Z(u + jv) > 0$, which contradicts the hypothesis that $Z(u + jv)$ is complex. Suppose the synthesis of equation (58) is the cascade of the two network sections shown in figure 14. Section 1 is obtained from figure 9 by applying Special Case 1 of the section Network Synthesis and equations (16) and (17). The value of C_1 is $Z(k)$, where k is arbitrary and yet to be determined. The impedance terminating network Section 1 is given by

$$Z_3(s) = \frac{C_1}{Z_1'(s)}$$

In general, $Z_3(u + jv)$ is complex if $Z(u + jv)$ is complex. The arbitrariness of k can be removed by assuming that a value of k can be found that forces the imaginary part

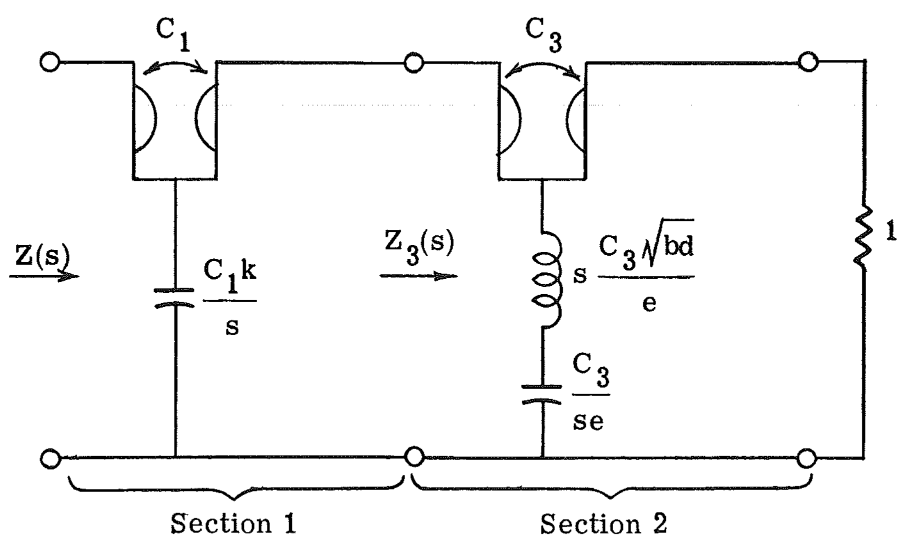


Figure 14. - Fialkow-Gerst network realization for $Z(s)$ of degree two when $Z(u + jv)$ is complex.

of $Z_3(u + jv)$ to be zero. Under this condition $Z_3(u + jv)$ is now positive and Section 2 can be constructed by using Special Case 2 and figure 8 as detailed in the preceding section. The identifications $a_1 = u + jv$ and $a_2 = u - jv$ are used to obtain $Z_1(s)$ from equation (48) since the even part zeros of $Z_3(s)$ and $Z(s)$ are identical. The quantity C_3 is defined by

$$C_3 = Z_3(u + jv)$$

The terminating impedance is a resistance since a_1 and a_2 were chosen to correspond to zeros of the even part of $Z(s)$.

As indicated previously, the validity of figure 14 depends on the existence of a value of k such that C_3 is positive. The balance of this part is devoted to proving this can always be done. First, $Z(s)$ is augmented by a surplus factor and then the even part of the augmented $Z(s)$ is identified with the even part of $Z(s)$ obtained from figure 14.

In equation (58), augment $Z(s)$ by multiplying the numerator and denominator by the factor $s + k$, where $k > 0$. The driving point impedance now becomes

$$Z(s) = \frac{m_1 + n_1}{m_2 + n_2} \tag{59}$$

where

$$m_1 = (1 + s^2 b)k + s^2 a$$

$$n_1 = s(1 + s^2 b + ak)$$

$$m_2 = (1 + s^2 d)k + s^2 c$$

$$n_2 = s(1 + s^2 d + ck)$$

Using the definition of the even part of $Z(s)$ results in

$$\mathrm{Ev}\,\{Z(s)\} = \frac{\left[\left(1 + s^2\sqrt{bd}\right)^2 - s^2 e^2\right](k^2 - s^2)}{m_2^2 - n_2^2} \tag{60}$$

A comparison of equations (60) and (32) shows that additional even part zeros now exist at $s = \pm k$.

It is known that any driving point impedance can be found from its even part (refs.

12 and 13). Further, if the impedance is a minimum reactance positive real function, Z(s) found in this way is also unique. Since Z(s) of equation (59) is being identified with the Z(s) of figure 14, the even part of Z(s), equation (60), can also be identified with the even part of Z(s) obtained from an analysis of the network.

Network Section 1 of figure 14 can be described by the equation

$$Z(s) = \frac{C_1 k Z_3(s) + sC_1^2}{C_1 k + sZ_3(s)} \tag{61}$$

while the analysis of network Section 2 yields the relation

$$Z_3(s) = \frac{C_3\left(1 + s^2\sqrt{bd}\right) + seC_3^2}{C_3\left(1 + s^2\sqrt{bd}\right) + se} \tag{62}$$

Combining equations (61) and (62) results in

$$Z(s) = \frac{m_3 + n_3}{m_4 + n_4} \tag{63}$$

where

$$m_3 = \left(1 + s^2\sqrt{bd}\right)k + s^2 e \frac{C_1}{C_3}$$

$$n_3 = s\left[keC_3 + \left(1 + s^2\sqrt{bd}\right)C_1\right]$$

$$m_4 = \left(1 + s^2\sqrt{bd}\right)k + s^2 e \frac{C_3}{C_1}$$

$$n_4 = s\left[ke\frac{1}{C_3} + \left(1 + s^2\sqrt{bd}\right)\frac{1}{C_1}\right]$$

and the even part of Z(s) becomes

$$\mathrm{Ev}\,\{Z(s)\} = \frac{\left[\left(1 + s^2\sqrt{bd}\right)^2 - s^2 e^2\right](k^2 - s^2)}{m_4^2 - n_4^2} \tag{64}$$

To identify equation (64), the even part of Z(s) obtained from the network, with equation (60), the even part of Z(s) found from augmenting Z(s), note first, that the numerators are identical and, second, that both denominators are of the form

$$m^2 - n^2$$

where m and n are even and odd polynomials of s, respectively. Identifying the even polynomials m_2 and m_4 of equations (60) and (64), respectively, gives

$$(1 + s^2 d)k + s^2 c = \left(1 + s^2 \sqrt{bd}\right)k + s^2 e \frac{C_3}{C_1} \tag{65}$$

and equating corresponding coefficients of like powers results in

$$s^0 : k = k$$

$$s^2 : dk + c = \sqrt{bd}\; k + e \frac{C_3}{C_1} \tag{66}$$

Solving for C_3 in equation (66) results in

$$C_3 = \frac{C_1\left(dk + c - \sqrt{bd}\; k\right)}{e} \tag{67}$$

Similarly, identifying the odd polynomials n_2 and n_4 gives

$$s(1 + s^2 d + ck) = s\left(\frac{ke}{C_3} + \frac{1 + s^2\sqrt{bd}}{C_1}\right)$$

and equating corresponding coefficients results in

$$s^1 : 1 + ck = \frac{ke}{C_3} + \frac{1}{C_1} \tag{68}$$

$$s^3 : d = \frac{\sqrt{bd}}{C_1} \qquad \text{or} \qquad C_1 = \sqrt{\frac{b}{d}} \tag{69}$$

Again solving for C_3 in equation (68) results in

$$C_3 = \frac{kC_1 e}{C_1(1 + ck) - 1} \tag{70}$$

Replacing C_1 with its value from equation (69) results in equations (67) and (70) in the form

$$C_3 = \frac{k\sqrt{bd}\left(1 - \sqrt{\frac{b}{d}}\right) + \sqrt{\frac{b}{d}}\, c}{e} \tag{71}$$

and

$$C_3 = \frac{ke\sqrt{\frac{b}{d}}}{kc\sqrt{\frac{b}{d}} + \left(\sqrt{\frac{b}{d}} - 1\right)} \tag{72}$$

respectively.

In order to prove the existence of a positive k, it would be convenient to have an equation in terms of k and the coefficients of $Z(s)$. Equating (71) and (72) gives

$$0 = k^2\left(\sqrt{\frac{b}{d}} - 1\right) + \frac{k}{\sqrt{bd}}\left(a - c\sqrt{\frac{b}{d}}\right) - \frac{\sqrt{\frac{b}{d}} - 1}{\sqrt{bd}} \tag{73}$$

where equation (33) has been used for e^2. Since $b \neq d$ is a constraint on $Z(s)$, divide by $\sqrt{b/d} - 1$. Thus, equation (73) becomes

$$0 = k^2 + k\,\frac{a - c\sqrt{\frac{b}{d}}}{b - \sqrt{bd}} - \frac{1}{\sqrt{bd}} \tag{74}$$

Now it can be proved that a positive k exists which satisfies equation (74). There are a number of ways of showing this fact, for instance, solving for k reveals that one positive root exists:

$$k = \frac{-\left(\frac{a - c\sqrt{\frac{b}{d}}}{b - \sqrt{bd}}\right) \pm \sqrt{\left(\frac{a - c\sqrt{\frac{b}{d}}}{b - \sqrt{bd}}\right)^2 + \frac{4}{\sqrt{bd}}}}{2} \tag{75}$$

Clearly, k possesses one positive and one negative root. Descarte's rule of signs yields the same result. Another method is the change of sign technique. This method is applied by letting the right side of equation (74) be designated g(k); that is,

$$g(k) = k^2 + k\frac{a - c\sqrt{\frac{b}{d}}}{b - \sqrt{bd}} - \frac{1}{\sqrt{bd}} \tag{76}$$

where k is now considered a parameter. At $k = 0$,

$$g(0) = -\frac{1}{\sqrt{bd}}$$

At $k = \infty$,

$$g(\infty) = \lim_{k \to \infty} g(k) = +\infty$$

Thus, at $k = 0$, g(0) is a negative number, while as k approaches ∞, g(k) approaches ∞. Hence, since g(k) is a continuous function of k, there exists a value of $k = k_0$, where $0 < k_0 < \infty$, such that $g(k_0) = 0$. To put it another way, g(k) crosses the axis at least once as k varies from 0 to ∞.

It was stated before that the realization of Z(s) by figure 14 is valid if a positive k and a positive C_3 can be found. Knowing now that a positive k exists it can easily be proved that C_3 is also positive. Since $b \neq d$ in equation (58), either $b < d$ or $b > d$. Examining equation (71) shows that if $b < d$, since $k > 0$, C_3 is also positive. Similarly, in equation (72), if $b > d$, C_3 is again positive. Thus, C_3 is always positive regardless of the relative magnitudes of b and d.

Part (1) has proved that figure 14 is indeed a valid representation of Z(s) of equation (30) when Z(u + jv) is complex. Moreover, it has been shown that there exists a k such that $Z_3(u + jv)$ is positive.

(2) In this part it is shown that $Z(s)$ can be realized as a cascade of two networks, one containing a transformer and one containing a gyrator. Let the synthesis of equation (58) be the one shown in figure 15. Network Section 1 is obtained from figure 10 by applying Special Case 3 of the Network Synthesis section and equation (20). The value of C_1 is $Z(a_0)$ and the value of C_2 is $Z_1'(a_0)$ which can be found by using equation (21). The impedance terminating network Section 1 is

$$Z_3(s) = \frac{C_1}{C_2} Z_2'(s)$$

Using the same reasoning as in Part (1), it is assumed that a value of a_0 can be found such that $Z_3(u + jv)$ is positive. Network Section 2 is the same as Section 2 of figure 14 Again the existence of a_0 is proved by augmenting $Z(s)$ by a surplus factor and then identifying the even part of the augmented $Z(s)$ with the even part of $Z(s)$ obtained from figure 15.

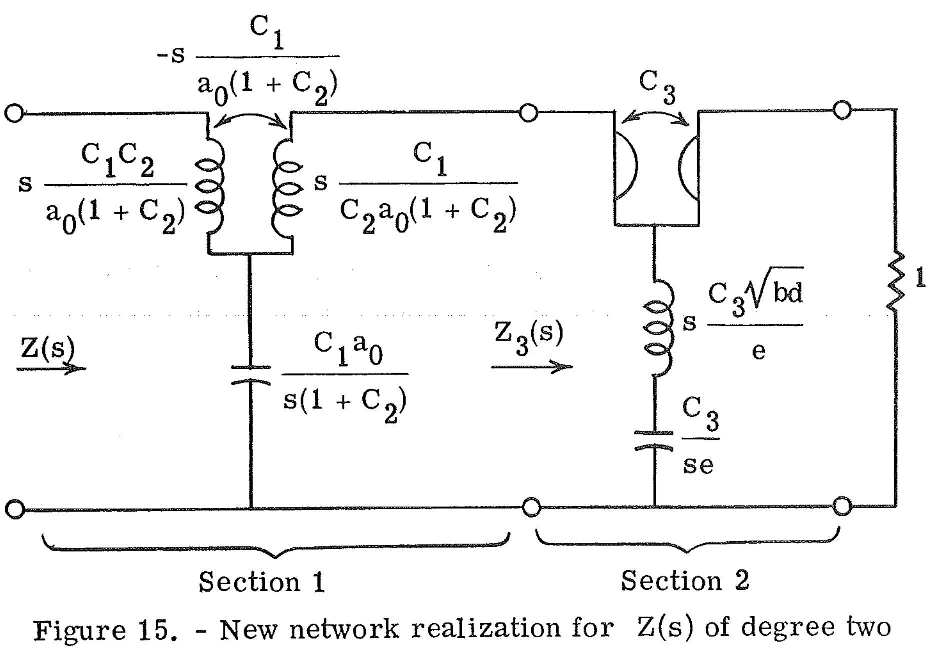


Figure 15. - New network realization for $Z(s)$ of degree two when $Z(u + jv)$ is complex.

In equation (58), augment $Z(s)$ by the factor $(s + a_0)^2$, where $a_0 > 0$. Multiplying numerator and denominator by the factor gives $Z(s)$ as

$$Z(s) = \frac{m_7 + n_7}{m_8 + n_8} \tag{77}$$

where

$$m_7 = \left(a_0^2 + s^2\right)(1 + s^2 b) + s^2 2a_0 a$$

$$n_7 = s\left[\left(a_0^2 + s^2\right)a + (1 + s^2 b)2a_0\right]$$

$$m_8 = \left(a_0^2 + s^2\right)(1 + s^2 d) + s^2 2a_0 c$$

$$n_8 = s\left[\left(a_0^2 + s^2\right)c + (1 + s^2 d)2a_0\right]$$

and the even part of $Z(s)$ is given by

$$\mathrm{Ev}\{Z(s)\} = \frac{\left[\left(1 + s^2\sqrt{bd}\right)^2 - s^2 e^2\right]\left(a_0^2 - s^2\right)^2}{m_8^2 - n_8^2} \tag{78}$$

Comparison of equations (78) and (32) shows that now there are additional even part zeros at $s = \pm a_0$ of order two. The same argument as used in Part (1) is used to identify the even part of $Z(s)$ in equation (78) with the even part of $Z(s)$ obtained by analyzing the network.

The procedures of Part (1) are used in appendix C to show that there exist two positive values of a_0 and, further, that C_3 is always positive. Summarizing the results of this part and appendix C, then, it has been demonstrated that figure 15 is a valid representation of $Z(s)$ when $Z(u + jv)$ is complex. The values of a_0, C_1, C_2, and C_3 are positive and can be found from the following equations:

$$0 = a_0^2 + a_0\left\{\frac{a - c\sqrt{\frac{b}{d}}}{b - \sqrt{bd}} \pm \left[\left(\frac{a - c\sqrt{\frac{b}{d}}}{b - \sqrt{bd}}\right)^2 + \frac{\left(\sqrt{b} + \sqrt{d}\right)^2 - ac}{bd}\right]^{1/2}\right\} - \frac{1}{\sqrt{bd}} \tag{C13}$$

$$C_1 = Z(a_0)$$

$$C_2 = \sqrt{\frac{b}{d}}$$

$$C_3 = \frac{C_1\left[\left(1 - a_0^2\sqrt{bd}\right)\left(1 - \sqrt{\frac{d}{b}}\right) + 2a_0c\right]}{ea_0\left(1 + \sqrt{\frac{b}{d}}\right)} \tag{C7}$$

or

$$C_3 = \frac{C_1e\left[\left(\frac{a_0^2\sqrt{bd} - 1}{1 - \sqrt{\frac{d}{b}}}\right)2\sqrt{\frac{d}{b}} + a_0c\right]}{\left(1 + \sqrt{\frac{b}{d}}\right)\left[\left(\frac{a_0^2\sqrt{bd} - 1}{1 - \sqrt{\frac{d}{b}}}\right)c + 2a_0\sqrt{bd}\right]} \tag{C11}$$

where the $\pm$ of equation (C13) is used to represent the two equations needed for the determination of a_0. As indicated in appendix C, the quantity within the brackets (raised to the 1/2 power) in equation (C13) is positive by virtue of the inequality equation (41); that is,

$$\left(\sqrt{b} + \sqrt{d}\right)^2 > ac$$

This fact assures that the coefficient of a_0 is real and, hence, that the required solutions of equation (C13) exist.

This investigation has been limited to complex even part zeros with a positive real part. The question is now could the results be somehow extended to even part zeros on the imaginary axis? In the section Coefficient Conditions, it was found that

$$e^2 = ac - \left(\sqrt{b} - \sqrt{d}\right)^2 = 0$$

corresponds to imaginary axis even part zeros. The coefficient of a_0 in equation (C13) can be rewritten as

$$\frac{a - c\sqrt{\dfrac{b}{d}}}{b - \sqrt{bd}} \pm \left[\left(\frac{a - c\sqrt{\dfrac{b}{d}}}{b - \sqrt{bd}}\right)^2 + \frac{4\sqrt{bd} - e^2}{bd}\right]^{1/2}$$

Letting $e^2 = 0$ shows that the expression inside the brackets is still positive and therefore there again exist two values of a_0 that satisfy equation (C13). What about C_3? In equation (C11), if $e = 0$, $C_3 = Z_3(jv) = 0$. This corresponds to a zero on the imaginary axis for $Z_3(s)$. The network of figure 15 is modified by deleting the gyrator, as shown in figure 16. Moreover, equation (C13) can also be used to find the values of a_0 for

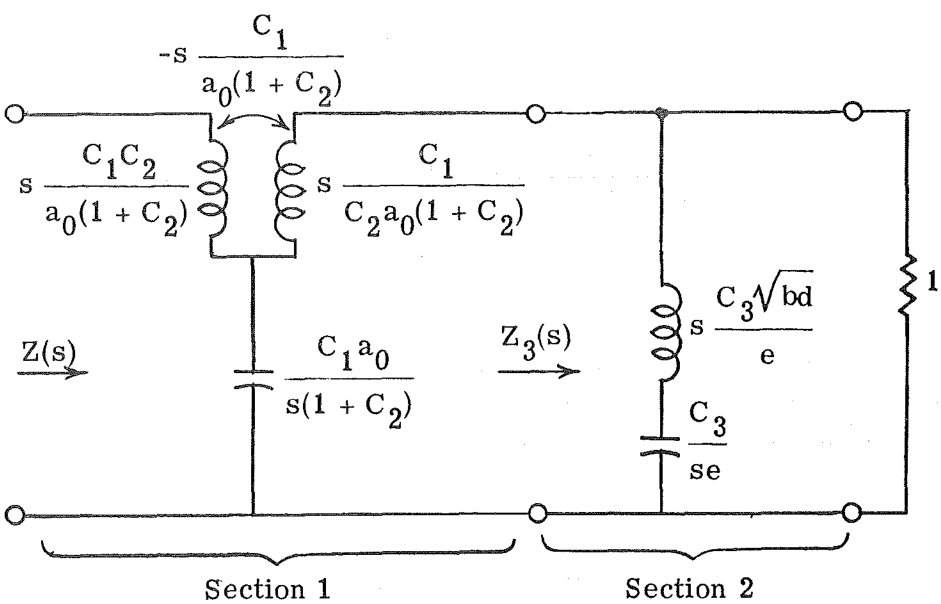


Figure 16. - New network realization for $Z(s)$ of degree two for imaginary even part zeros.

$Z(s)$ of degree two with even part zeros on the imaginary axis.

Part (2) has shown that $Z(s)$ of degree two with $Z(u + jv)$ complex can be realized by the network of figure 15. In fact, since there are two values of a_0 that satisfy the realizability conditions, two networks are possible for $Z(s)$, each with its own a_0.

COMPLEX EVEN PART ZEROS AND OPERATOR EVEN PART IDENTIFICATION

Driving point impedances of degree two were already discussed and now those results are extended to driving point impedances of any degree.

The complex even part zero is again designated by $u + jv$, where $u > 0$ and $v \neq 0$. As in the second degree case, $Z(u + jv)$ can be positive or complex with a positive real part. If $Z(u + jv)$ is positive, $Z_1'(s)$ is shown to be found directly from equation (19) and the network corresponds to figure 8 or 9. When $Z(u + jv)$ is complex, the resultant networks are the same as figures 11(a) and 12(a).

In the following section, extended coefficient conditions are shown for $Z(s)$ corresponding to those found in the section Coefficient Conditions. The results for the cases for positive and complex $Z(u + jv)$ are discussed in the sections Complex Even Part Zeros with $Z(u + jv)$ Positive and Complex Even Part Zeros with $Z(u + jv)$ Complex - Operator Even Part Identification, respectively.

Only the results are shown herein; the proofs are given in reference 14.

Extended Coefficient Conditions

A set of coefficient conditions similar to equations (42) and (43) must be found in order to extend the results of the section DRIVING POINT IMPEDANCES OF DEGREE TWO. As previously mentioned, Hazony discovered that if $Z(s)$ is a positive real function, $\zeta_1(s)$ defined by

$$\zeta_1(s) = \frac{(1 + s^2D)Z(s) - sA}{(1 + s^2B) - Z(s)sC} \tag{79}$$

is also a positive real function. Thus, $Z(s)$ can be written in the form

$$Z(s) = \frac{(1 + s^2B)\zeta_1(s) + sA}{(1 + s^2D) + \zeta_1(s)sC} \tag{80}$$

The numbers A, B, C, and D are all positive and are given by

$$A = \frac{Z(\alpha)Z(\beta)(\alpha^2 - \beta^2)}{\alpha\beta[\alpha Z(\alpha) - \beta Z(\beta)]} \tag{81}$$

$$B = \frac{\alpha Z(\alpha) - \beta Z(\beta)}{\alpha\beta[\alpha Z(\beta) - \beta Z(\alpha)]} \tag{82}$$

$$C = \frac{\alpha^2 - \beta^2}{\alpha\beta[\alpha Z(\beta) - \beta Z(\alpha)]} \tag{83}$$

$$D = \frac{\alpha Z(\beta) - \beta Z(\alpha)}{\alpha\beta[\alpha Z(\alpha) - \beta Z(\beta)]} \tag{84}$$

where α and β are positive or conjugate complex with a nonnegative real part. The numbers A, B, C, and D are called the extended coefficients of Z(s). As can be easily verified, the corresponding second degree Z(s) of equation (30) is obtained from equation (80) by letting $\zeta_1(s) = 1$.

Let α be identified with $u + jv$ and β with $u - jv$. Under these circumstances, Hazony showed that the degree of $\zeta_1(s)$ is two less than the degree of Z(s). It is shown in reference 14 that the coefficient conditions

$$\left(\sqrt{B} + \sqrt{D}\right)^2 > AC \geq \left(\sqrt{B} - \sqrt{D}\right)^2 \tag{85}$$

and

$$\left(\sqrt{B} + \sqrt{D}\right)^2 > AC > \left(\sqrt{B} - \sqrt{D}\right)^2 \tag{86}$$

correspond to equations (42) and (43), respectively. The equal sign in equation (85) corresponds to zeros of the even part on the imaginary axis.

The following relations, derived in reference 14, are useful for future work and are used in the remainder of the text

$$u^2 + v^2 = \frac{1}{\sqrt{BD}} \tag{87}$$

$$u = \frac{E}{2\sqrt{BD}} \tag{88}$$

where

$$E^2 = AC - \left(\sqrt{B} - \sqrt{D}\right)^2$$

Equations (87), (88), as well as the inequalities of (86) are used in the proofs of the following sections.

Complex Even Part Zeros with $Z(u + jv)$ Positive

Since $Z(u + jv) = C_1$ is positive, Theorem I can be applied immediately. Using the identifications $a_1 = u + jv$ and $a_2 = u - jv$ gives equations (18) and (19) in the form

$$Z_1(s) = \frac{s^2 + u^2 + v^2}{2su} \tag{89}$$

and

$$Z_1'(s) = \frac{(s^2 + u^2 + v^2)Z(s) - C_1 2su}{C_1(s^2 + u^2 + v^2) - Z(s)2su} \tag{90}$$

respectively. Using equation (87) for $u^2 + v^2$ and equation (88) for u gives equations (89) and (90) as

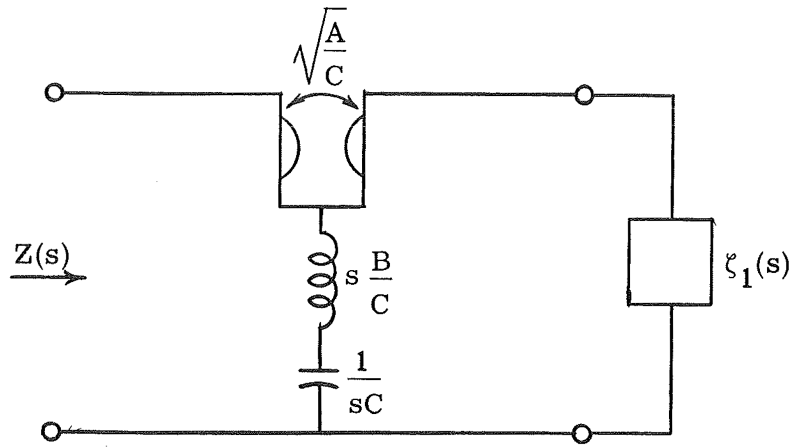
$$Z_1(s) = \frac{1 + s^2\sqrt{BD}}{sE} \tag{91}$$

and

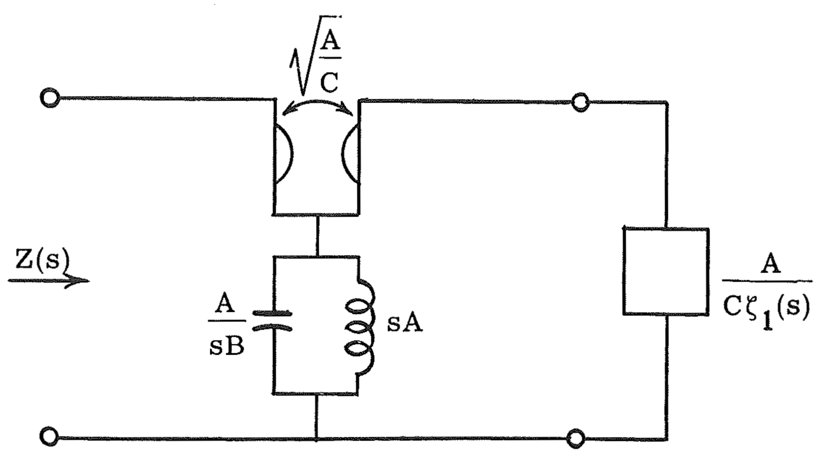
$$Z_1'(s) = \frac{\left(1 + s^2\sqrt{BD}\right)Z(s) - C_1 sE}{C_1\left(1 + s^2\sqrt{BD}\right) - Z(s)sE} \tag{92}$$

Thus, $Z_1(s)$ and $Z_1'(s)$ are now expressed in terms of the extended coefficients of $Z(s)$.

The operators $V_1(s)$ or $V_2(s)$ (see section Operators for $Z_1'(s)$) could be used to synthesize the networks of figure 8 or 9. When $Z(u + jv) > 0$, however, there exists another constraint on the extended coefficients of $Z(s)$ in addition to the inequalities of equation (86). The condition $Z(u + jv) > 0$ can be true if and only if $B = D$. The networks corresponding to figures 8 and 9 are shown in figures 17(a) and (b). The element values are in terms of the extended coefficients of $Z(s)$ and the degree of the terminating



(a) Series resonant.



(b) Parallel resonant.

Figure 17. - Fialkow-Gerst network realization when $Z(u + jv)$ is positive.

impedances is two less than the degree of $Z(s)$. The network of figure 17(a) could also have been obtained by letting $B = D$ in figure 4, the network associated with the Hazony synthesis.

Complex Even Part Zeros with $Z(u + jv)$ Complex - Operator Even Part Identification

Since $Z(u + jv) = C_1$ is not positive, Theorem I cannot be applied directly. Instead, procedures are found for which the network realization consists of the cascaded operator networks discussed in the section A New Cascaded Operator Network.

This section is divided into two parts. In Part (1), it is shown that $Z(s)$ can be realized as the cascaded operator network of figure 11(a). Although this was also shown by Fialkow and Gerst, a new proof is used here based on an extension of the even part identification technique. (See section Complex Even Part Zeros with $Z(u + jv)$ Complex - Even Part Identification.) The new proof is also used in Part (2), where it is shown that $Z(s)$ can be realized as the new cascaded operator network of figure 12(a).

(1) The driving point impedance is written as

$$Z(s) = \frac{(1 + s^2 B)\zeta_1(s) + sA}{1 + s^2 D + \zeta_1(s) sC} \qquad B \neq D \tag{93}$$

where A, B, C, and D are found by using equations (81) to (84). The inequality $B \neq D$ is a constraint on $Z(s)$ since, as already shown, $B = D$ implies that $Z(u + jv)$ is positive, which contradicts the present condition that $Z(u + jv)$ is complex. Suppose the synthesis of equation (93) is the cascade of two network sections as illustrated in figure 18. Section 1 is obtained from figure 9 by applying Special Case 1 of the section Network Synthesis and equations (16) and (17). The value of C_1 is $Z(k)$, and the terminating impedance of the section is given by

$$Z_3(s) = \frac{C_1}{Z_1'(s)}$$

The same reasoning as was used in DRIVING POINT IMPEDANCES OF DEGREE TWO is used here where a value of k is assumed which forces the imaginary part of $Z_3(u + jv)$ to be zero. Under this condition, $Z_3(u + jv)$ is now positive and Section 2 can be constructed by using Special Case 2 and figure 8. Using the identification $a_1 = u + jv$ and $a_2 = u - jv$, $Z_1(s)$ can be obtained from equation (91), since the even part zeros of $Z_3(s)$ and $Z(s)$ are identical. The quantity C_3 is defined by

$$C_3 = Z_3(u + jv)$$

The terminating impedance of Section 2 is two degrees less than $Z_3(s)$, since a_1 and a_2 were chosen to correspond to zeros of the even part.

The validity of figure 18 depends on the existence of a value of k such that C_3 is positive. This can be done by first augmenting $Z(s)$ by the function associated with Richards theorem. Then, identification is made of the even part of the augmented $Z(s)$

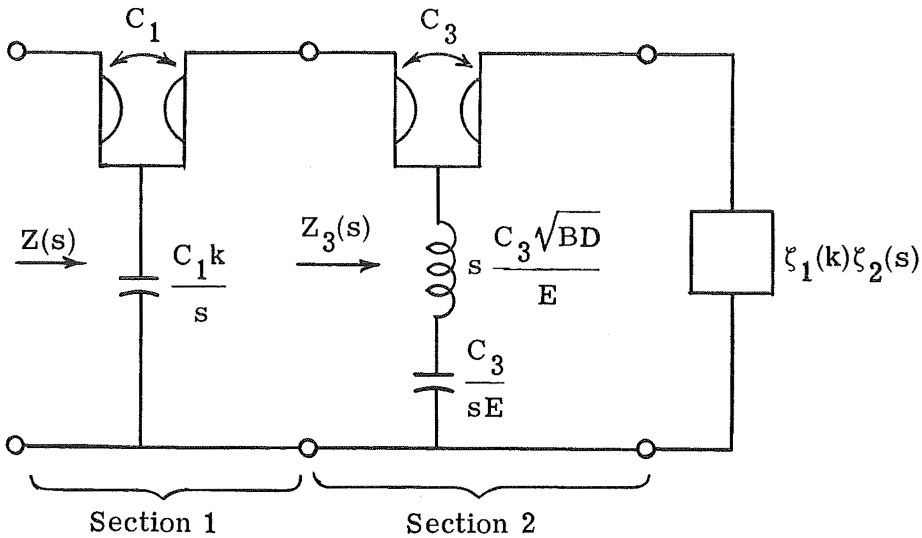


Figure 18. - Fialkow-Gerst network realization when $Z(u + jv)$ is complex.

with the even part of $Z(s)$ obtained from figure 18.

Let the function $\zeta_2(s)$ be defined as

$$\zeta_2(s) = \frac{s\zeta_1(s) - k\zeta_1(k)}{s\zeta_1(k) - k\zeta_1(s)} \tag{94}$$

where $k > 0$. Since $\zeta_1(s)$ is a positive real function, then by Richards theorem, $\zeta_2(s)$ is also a positive real function. Eliminating $\zeta_1(s)$ from equations (93) and (94) results in

$$Z(s) = \frac{M_1 + N_1\zeta_2(s)}{M_2\zeta_2(s) + N_2} \tag{95}$$

where

$$M_1 = (1 + s^2B)k\zeta_1(k) + s^2A$$

$$N_1 = s\left[(1 + s^2B)\zeta_1(k) + Ak\right]$$

$$M_2 = (1 + s^2D)k + s^2C\zeta_1(k)$$

$$N_2 = s\left[(1 + s^2D) + Ck\zeta_1(k)\right]$$

It is interesting to note the difference in the preceding augmenting procedure and the one used in Part (1) of the section Complex Even Part Zeros with $Z(u + jv)$ Complex - Even Part Identification. In equation (58), $Z(s)$ is augmented by multiplying the numerator and denominator by the factor $s + k$. The construction of $Z(s)$ in equation (93), however, suggests the introduction and substitution of the function $\zeta_2(s)$ of equation (94). The use of Richards theorem to augment a positive real function is also used by Hazony in reference 9, section 9.7.

A proof is developed that extends the even part identification method by combining it with the impedance operator procedure. The operator concept is useful here because it takes the problem of the driving point impedance of any degree and essentially reduces it to the second degree case. The combined impedance operator and even part synthesis is also used by Hazony in reference 9, section 9.7, where the procedure is called impedance operator even part synthesis. Therefore, the new proof is considered as an extension of impedance operator even part synthesis.

Finding a driving point impedance from its even part was discussed previously. Since an impedance operator is a driving point impedance, it can also be found from its even part. Further, if the impedance operator is a minimum reactance positive real function, the impedance operator found in this way is also unique. Suppose the operator is defined as

$$V(s) = Z(s)\Big|_{\zeta_2(s)=1} \tag{96}$$

where $Z(s)$ is both equation (95) and the driving point impedance of figure 18. Since $V(s)$ of equation (95) and the $V(s)$ of figure 18 are being identified, their respective even parts can also be identified.

It is shown in reference 14 that the values of k, C_1, and C_3 are positive and can be found from the following expressions:

$$0 = k^2 + k\left[\frac{\dfrac{A}{\zeta_1(k)} - \zeta_1(k)C\sqrt{\dfrac{B}{D}}}{B - \sqrt{BD}}\right] - \frac{1}{\sqrt{BD}} \tag{97}$$

$$C_1 = \zeta_1(k)\sqrt{\frac{B}{D}} \tag{98}$$

$$C_3 = \frac{\zeta_1(k)\left[\sqrt{BD}\; k\left(1 - \sqrt{\frac{B}{D}}\right) + \sqrt{\frac{B}{D}}\; C\zeta_1(k)\right]}{E} \tag{99}$$

or

$$C_3 = \frac{\zeta_1(k)kE\sqrt{\frac{B}{D}}}{\zeta_1(k)kC\sqrt{\frac{B}{D}} + \left(\sqrt{\frac{B}{D}} - 1\right)} \tag{100}$$

The constants A, B, C, and D are calculated from equations (81) to (84), and equation (79) is used to find $\zeta_1(s)$.

Part (1) is concluded by comparing the results here with the results of the second degree case. In order to reduce the Z(s) in equation (93) to the corresponding Z(s) in equation (58), set $\zeta_1(s) = 1$. The latter condition also implies that $\zeta_1(k) = 1$. Further, if $\zeta_1(s) = \zeta_1(k) = 1$ in the definition of $\zeta_2(s)$, equation (94), $\zeta_2(s) = 1$. It can be easily shown that setting $\zeta_1(s) = \zeta_1(k) = \zeta_2(s) = 1$ in the equations here reduces them to their corresponding equations of the second degree case. The network of figure 18 also corresponds to figure 14 when $\zeta_1(k) = \zeta_2(s) = 1$. Thus, the results of the second degree case can be considered as a special case of the results here.

(2) In this part, it is shown that Z(s) can be realized as the new cascaded operator network of figure 12(a). The driving point impedance Z(s) is written as

$$Z(s) = \frac{(1 + s^2B)\zeta_1(s) + sA}{1 + s^2D + \zeta_1(s)sC} \qquad B \neq D \tag{101}$$

Suppose the synthesis of equation (101) is the one shown in figure 19. Network Section 1 is obtained from figure 10 by applying Special Case 3 of the section Network Synthesis and equation (20). The value of C_1 is Z(a) and the value of C_2 is $Z_1'(a)$, which can be evaluated by using equation (21). The impedance terminating network Section 1 is

$$Z_3(s) = \frac{C_1}{C_2} Z_2'(s)$$

If the same argument is used here as was used in Part (2) of Complex Even Part Zeros with Z(u + jv) Complex - Operator Even Part Identification, it is assumed that a value

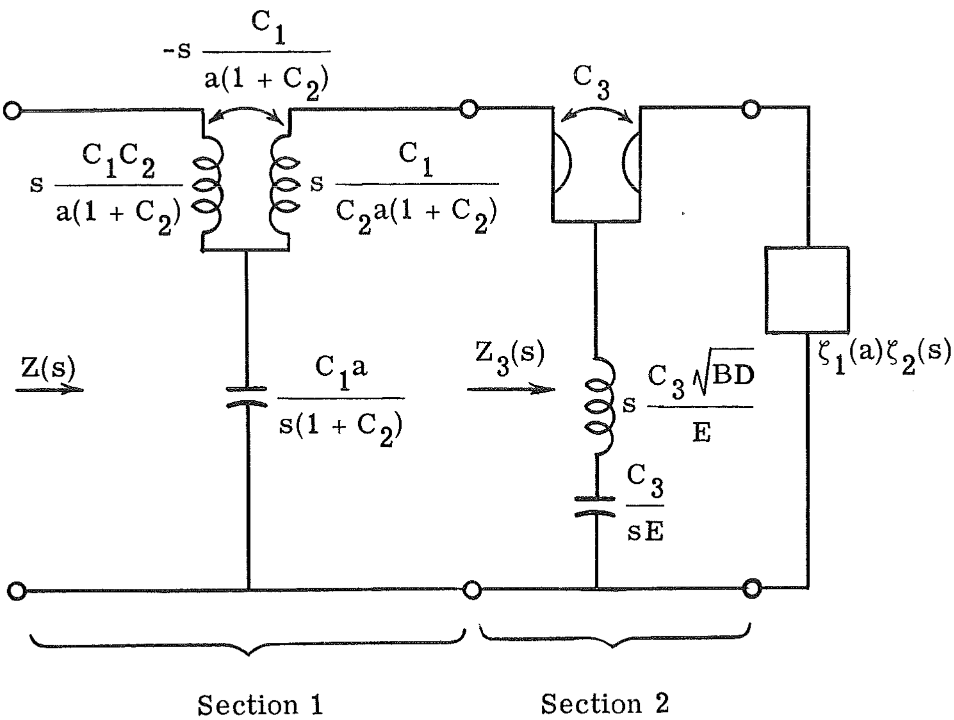


Figure 19. - New network realization when $Z(u + jv)$ is complex.

of a can be found such that $Z_3(u + jv)$ is positive. Network Section 2 is the same as Section 2 of figure 18. Again, the existence of a is proved by twice augmenting $Z(s)$ with the function associated with Richards theorem, and then identifying the even part of the augmented $Z(s)$ with the even part of $Z(s)$ obtained from figure 19.

Let the functions $\zeta_2(s)$ and $\zeta_3(s)$ be defined as

$$\zeta_2(s) = \frac{s\zeta_1(s) - a\zeta_1(a)}{s\zeta_1(a) - a\zeta_1(s)} \tag{102}$$

and

$$\zeta_3(s) = \frac{s\zeta_2(s) - a\zeta_2(a)}{s\zeta_2(a) - a\zeta_2(s)} \tag{103}$$

where $a > 0$. By Richards theorem, both $\zeta_2(s)$ and $\zeta_3(s)$ are positive real functions. Eliminating $\zeta_1(s)$ and $\zeta_2(s)$ from equations (101) to (103) results in

$$Z(s) = \frac{M_5\zeta_3(s) + N_5}{M_6 + N_6\zeta_3(s)} \tag{104}$$

where

$$M_5 = \left[a^2 + s^2\zeta_2(a)\right](1 + s^2B)\zeta_1(a) + s^2[1 + \zeta_2(a)]aA$$

$$N_5 = s\left\{\left[a^2\zeta_2(a) + s^2\right]A + (1 + s^2B)[1 + \zeta_2(a)]a\zeta_1(a)\right\}$$

$$M_6 = \left[a^2\zeta_2(a) + s^2\right](1 + s^2D) + s^2[1 + \zeta_2(a)]aC\zeta_1(a)$$

$$N_6 = s\left\{\left[a^2 + s^2\zeta_2(a)\right]C\zeta_1(a) + (1 + s^2D)[1 + \zeta_2(a)]a\right\}$$

The procedure of Part (1) is used in reference 14 to show that there exist two positive values of a and further that C_3 is always positive.

It has been proved that figure 19 is a network representation of Z(s) when Z(u + jv) is complex. The values of a, C_1, C_2, and C_3 are positive and can be found from the following expressions:

$$0 = a^4 + 2a^3\left[\frac{\dfrac{A}{\zeta_1(a)} - \zeta_1(a)C\sqrt{\dfrac{B}{D}}}{B - \sqrt{BD}}\right] + a^2\left(\frac{AC - B - D - 4\sqrt{BD}}{BD}\right) + 2a\left[\frac{\dfrac{A}{\zeta_1(a)} - \zeta_1(a)C\sqrt{\dfrac{B}{D}}}{\sqrt{BD}\left(\sqrt{BD} - B\right)}\right] + \frac{1}{BD} \tag{105}$$

or

$$0 = a^2 + a\left(\frac{\dfrac{A}{\zeta_1(a)} - \zeta_1(a)C\sqrt{\dfrac{B}{D}}}{B - \sqrt{BD}} \pm \left\{\left[\frac{\dfrac{A}{\zeta_1(a)} - \zeta_1(a)C\sqrt{\dfrac{B}{D}}}{B - \sqrt{BD}}\right]^2 + \frac{\left(\sqrt{B} + \sqrt{D}\right)^2 - AC}{BD}\right\}^{1/2}\right) - \frac{1}{\sqrt{BD}} \tag{106}$$

$$C_1 = Z(a)$$

$$C_2 = \sqrt{\frac{B}{D}} \tag{107}$$

$$C_3 = \frac{C_1\left[\left(1 - a^2\sqrt{BD}\right)\left(1 - \sqrt{\frac{D}{B}}\right) + \zeta_1(a)2aC\right]}{Ea\left(1 + \sqrt{\frac{B}{D}}\right)} \tag{108}$$

or

$$C_3 = \frac{C_1E\left[\left(\frac{a^2\sqrt{BD} - 1}{1 - \sqrt{\frac{D}{B}}}\right) 2\sqrt{\frac{D}{B}} + aC\zeta_1(a)\right]}{\left(1 + \sqrt{\frac{B}{D}}\right)\left[\left(\frac{a^2\sqrt{BD} - 1}{1 - \sqrt{\frac{D}{B}}}\right) C\zeta_1(a) + 2a\sqrt{BD}\right]} \tag{109}$$

where the $\pm$ of equation (106) is utilized to represent the two factors of equation (105). The quantity within the braces (raised to the 1/2 power) in equation (106) is positive as a consequence of the inequality of (86); that is,

$$\left(\sqrt{B} + \sqrt{D}\right)^2 > AC$$

This condition assures that the coefficient of a is real and hence that the required solutions of equation (105) exist.

Here the investigation is limited to complex even part zeros with a positive real part. Now the results will be extended to even part zeros on the imaginary axis. From equation (85) it was found that

$$E^2 = AC - \left(\sqrt{B} - \sqrt{D}\right)^2 = 0$$

corresponds to imaginary axis even part zeros. The coefficient of a in equation (106) can be rewritten as

$$\frac{\frac{A}{\zeta_1(a)} - \zeta_1(a) C \sqrt{\frac{B}{D}}}{B - \sqrt{BD}} \pm \left\{ \left[\frac{\frac{A}{\zeta_1(a)} - \zeta_1(a) C \sqrt{\frac{B}{D}}}{B - \sqrt{BD}} \right]^2 + \frac{4\sqrt{BD} - E^2}{BD} \right\}^{1/2}$$

Letting $E^2 = 0$, the expression inside the braces is still positive and therefore there again exist two values of a that satisfy equation (105). Also, in equation (109) if $E = 0$, $C_3 = Z_3(jv) = 0$. This corresponds to a zero on the imaginary axis for $Z_3(s)$, and the network of figure 19 is modified by deleting the gyrator, as shown in figure 20. Moreover, equation (105) can be used to find values of a for Z(s) with even part zeros on the imaginary axis.

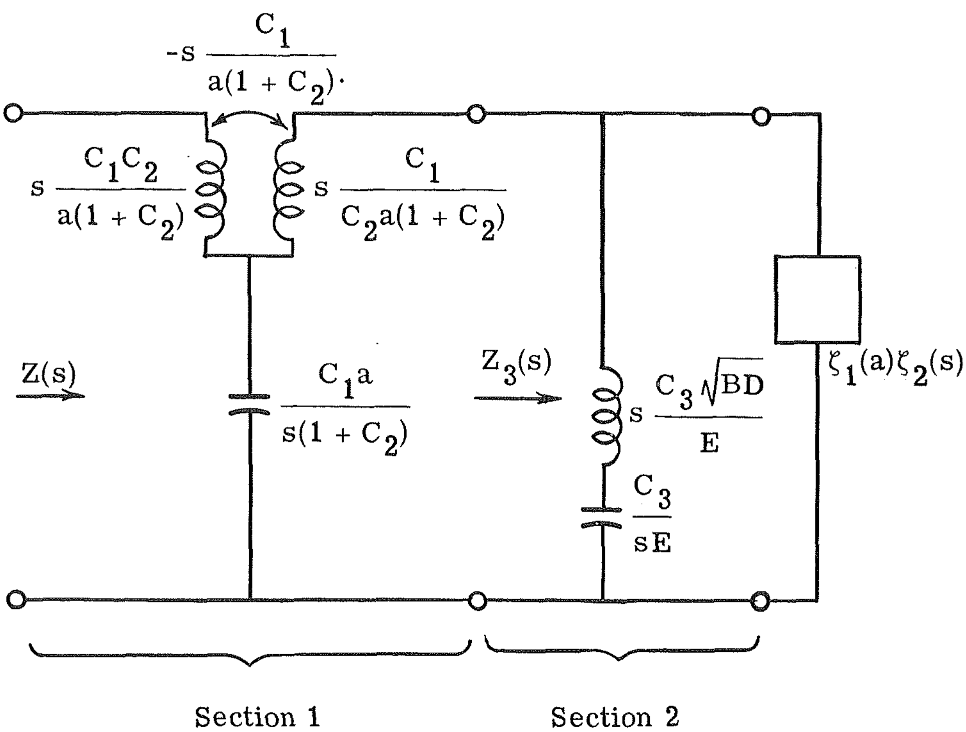


Figure 20. - New network realization for imaginary even part zeros.

The results of Part (2) can be summed up in the following theorem.

Theorem II:

Let $Z(s)$ be a minimum reactance positive real function such that $Z(u + jv)$ is complex, where $u + jv$, for $u > 0$ and $v \neq 0$, is a zero of the even part of $Z(s)$. Then $Z(s)$ can always be realized by the network of figure 19 in which the degree of the terminating impedance is at least two less than the degree of $Z(s)$.

A comparison of the results of Part (2) with the results of the second degree case is now made. At the end of Part (1), it was shown that this was equivalent to setting $\zeta_1(s) = \zeta_1(a) = \zeta_2(s) = 1$. In addition, if $\zeta_2(s) = \zeta_2(a) = 1$ in the definition of $\zeta_3(s)$ (eq. (103)), $\zeta_3(s) = 1$. By setting $\zeta_1(s) = \zeta_1(a) = \zeta_2(s) = \zeta_2(a) = \zeta_3(s) = 1$, the equations of this section reduce to their corresponding equations in Part (2) of the section Complex Even Part Zeros with $Z(u + jv)$ Complex - Even Part Identification. The network of figure 19 also corresponds to figure 15 when $\zeta_1(a) = \zeta_2(s) = 1$.

Example:

Let it be required to realize

$$Z(s) = \frac{9 + 34s + 3s^2 + 2s^3}{24 + 12s + 42s^2 + 3s^3}$$

as the network of figure 19. The zeros of the even part of $Z(s)$ are the numbers $+2$, -2, $1/\sqrt{2} + j\sqrt{5/2}$, $1/\sqrt{2} - j\sqrt{5/2}$, $-1/\sqrt{2} + j\sqrt{5/2}$, $-1/\sqrt{2} - j\sqrt{5/2}$. Choosing

$$u + jv = \alpha = \frac{1}{\sqrt{2}} + j\sqrt{\frac{5}{2}}$$

and

$$u - jv = \beta = \frac{1}{\sqrt{2}} - j\sqrt{\frac{5}{2}}$$

gives $Z(u + jv)$ and $Z(u - jv)$ as the complex quantities

$$Z\left(\frac{1}{\sqrt{2}} + j\sqrt{\frac{5}{2}}\right) = \frac{2}{9}\left(\sqrt{2} - j\sqrt{\frac{5}{2}}\right)$$

and

$$Z\left(\frac{1}{\sqrt{2}} - j\sqrt{\frac{5}{2}}\right) = \frac{2}{9}\left(\sqrt{2} + j\sqrt{\frac{5}{2}}\right)$$

The values of A, B, C, D, and E from equations (81) to (84) are as follows: $A = 2/3$, $B = 1/9$, $C = 1$, $D = 1$, and $E = \sqrt{2}/3$ Using equation (79) results in

$$\zeta_1(s) = \frac{3 + 6s}{8 + s}$$

Solving for the values of a in equation (105) gives $a = 1.00$ and $a = 5.54$. For $a = 1$, the numbers C_1 and C_2 can be obtained from

$$C_1 = Z(a) = Z(1) = \frac{16}{27}$$

$$C_2 = \sqrt{\frac{B}{D}} = \frac{1}{3}$$

and C_3 is the number

$$C_3 = \frac{4\sqrt{2}}{9}$$

where either equation (108) or (109) is used to find C_3. For the terminating impedance, $\zeta_1(a)\zeta_2(s)$,

$$\zeta_1(a) = \zeta_1(1) = 1$$

and

$$\zeta_2(s) = \frac{8 + 6s}{3 + s}$$

where $\zeta_2(s)$ is found from equation (102). Now since the numbers B, D, E, a, C_1, C_2, C_3, $\zeta_1(a)$, and the function $\zeta_2(s)$ are all available, the element values of the network corresponding to figure 19 can be obtained. Figure 21 illustrates the synthesis of $Z(s)$. Note that the terminal impedance is of degree two less than $Z(s)$.

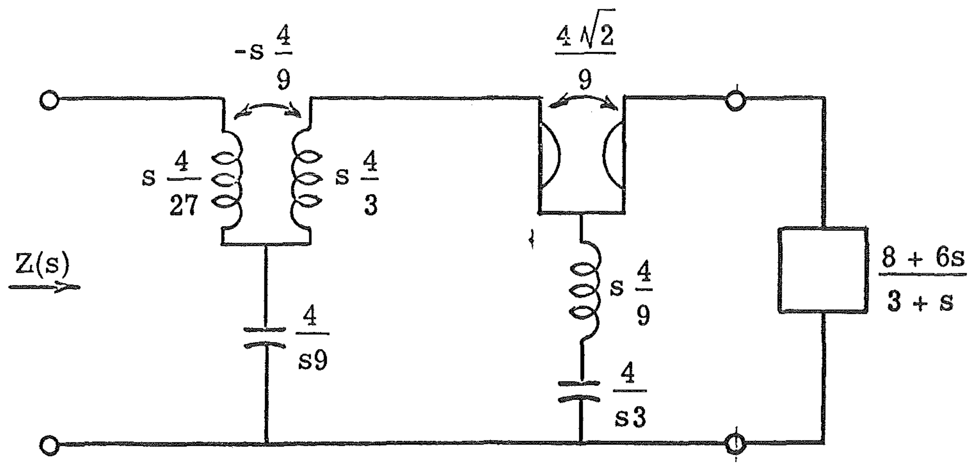


Figure 21. - Example synthesis.

CONCLUSION

A new cascaded driving point impedance synthesis has been found based on an extension of the Fialkow-Gerst theorem. In order to apply the new theorem it was necessary to determine a method of selection of constants such that ultimate degree reduction is assured. A new proof was developed based on an extension of impedance operator even part synthesis. This proof is sufficiently general to include always the case of the imaginary axis even part zeros and sometimes to include the case of even part zeros on the real axis. The resultant networks contain reactive elements, unity coupled transformers, and gyrators. The new realization reduces the number of gyrators used compared with the Fialkow-Gerst cascaded network. Moreover, any positive real function can be realized as a cascade of these lossless sections terminated in a resistance.

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Cleveland, Ohio, August 21, 1969,
120-27.

APPENDIX A

SYMBOLS

A, B, C, D	extended coefficients of any degree $Z(s)$
A_0, A_1, A_2	coefficients of most general second degree $Z(s)$
a, b, c, d	coefficients of second degree $Z(s)$
a_0	a value of a_1
$a_1, a_2, \ldots, a_p$	a set of real or complex numbers
B_0, B_1, B_2	coefficients of most general second degree $Z(s)$
$b_1, b_2, \ldots, b_q$	a set of real or complex numbers
C_1, C_2, C_3	positive numbers
E	number derived from coefficients of any degree $Z(s)$
$E_d, E_d(j\omega)$	load voltage
$E_g, E_g(j\omega)$	source voltage
$E_1, E_2, E_1(s), E_2(s)$	voltages
Ev	even part
e	number derived from coefficients of second degree $Z(s)$
G	real gyrator coefficient
$G(j\omega)$	power ratio
$g(k)$	a function of k
$I_1, I_2, I_1(s), I_2(s)$	currents
Im	imaginary part
$i=1, 2, \ldots, p$	set of numbers
$j=1, 2, \ldots, q$	set of numbers
k	a value of a_1
k_0	a value of k
L_1, L_2	inductances
M_1, M_2, M_5, M_6	even functions of s
$m_1, \ldots, m_{10}$	even functions of s

N	network
N_1, N_2, N_5, N_6	odd functions of s
n	degree of $Z(s)$
$n_1, \ldots, n_{10}$	odd functions of s
Od	odd part
$P_d(j\omega)$	power delivered to load
$P_g(j\omega)$	maximum power available from source
R_d	load resistance
R_g	source resistance
Re	real part
r	a set of numbers among i
s	complex variable
t	a set of numbers among j
$u + jv$	a value of a_1
$V(s), V_1(s), V_2(s), V_3(s)$	impedance operators
$(x - y)(x + y)$	factors
$dZ(s)/ds$	derivative of $Z(s)$ with respect to s
$\left.\begin{array}{l} Z(s), Z_1(s), Z_2(s), \\ Z_3(s), Z_1'(s), Z_2'(s) \end{array}\right\}$	positive real functions of s
$z_{11}, z_{12}, z_{21}, z_{22}$	open circuit impedance parameters
$\zeta, \zeta(s), \zeta_1(s), \zeta_2(s), \zeta_3(s)$	positive real functions of s
ω	imaginary part of s

APPENDIX B

FORM OF SECOND DEGREE DRIVING POINT IMPEDANCE

This appendix shows that all second degree driving point impedances can be written in the form

$$\frac{1 + sa + s^2 b}{1 + sc + s^2 d} \tag{B1}$$

The most general second degree driving point impedance can be written as

$$Z(s) = \frac{A_0 + sA_1 + s^2 A_2}{B_0 + sB_1 + s^2 B_2} \tag{B2}$$

where the coefficients of the polynomials are positive numbers. Equation (B2) can be factored into

$$Z(s) = \frac{A_0\left(1 + s\,\dfrac{A_1}{A_0} + s^2\,\dfrac{A_2}{A_0}\right)}{B_0\left(1 + s\,\dfrac{B_1}{B_0} + s^2\,\dfrac{B_2}{B_0}\right)} \tag{B3}$$

and multiplying both sides by B_0/A_0, gives equation (B3) in the form

$$\frac{B_0}{A_0}\,Z(s) = \frac{1 + s\,\dfrac{A_1}{A_0} + s^2\,\dfrac{A_2}{A_0}}{1 + s\,\dfrac{B_1}{B_0} + s^2\,\dfrac{B_2}{B_0}} \tag{B4}$$

Finally, the coefficients of equations (B1) and (B4) are identified to give

$$a = \frac{A_1}{A_0} \qquad b = \frac{A_2}{A_0}$$

$$c = \frac{B_1}{B_0} \qquad d = \frac{B_2}{B_0}$$

Thus, when it is required to synthesize the $Z(s)$ of equation (B2), equation (B4) can be realized instead, and then all the impedances in the network are multiplied by A_0/B_0.

APPENDIX C

PROOF OF EXISTENCE OF POSITIVE a_0 AND C_3

This appendix proves that two positive values of a_0 exist and further that C_3 is always positive. Examining figure 15 shows that the driving point impedance of Section 1 is given by

$$Z(s) = \frac{C_1\left(a_0^2 + s^2C_2\right)Z_3(s) + sa_0(1 + C_2)\dfrac{C_1^2}{C_2}}{C_1\left(a_0^2 + \dfrac{s^2}{C_2}\right) + Z_3(s)sa_0(1 + C_2)} \tag{C1}$$

and equation (62) can be used to describe Section 2. Combining equations (C1) and (62) results in

$$Z(s) = \frac{m_9 + n_9}{m_{10} + n_{10}} \tag{C2}$$

where

$$m_9 = \left(a_0^2 + s^2C_2\right)\left(1 + s^2\sqrt{bd}\right) + s^2e\,\frac{C_1}{C_2C_3}\,a_0(1 + C_2)$$

$$n_9 = s\left[\left(a_0^2 + s^2C_2\right)eC_3 + \left(1 + s^2\sqrt{bd}\right)\frac{C_1}{C_2}\,a_0(1 + C_2)\right]$$

$$m_{10} = \left(a_0^2 + \frac{s^2}{C_2}\right)\left(1 + s^2\sqrt{bd}\right) + s^2e\,\frac{C_3}{C_1}\,a_0(1 + C_2)$$

$$n_{10} = s\left[\left(a_0^2 + \frac{s^2}{C_2}\right)\frac{e}{C_3} + \left(1 + s^2\sqrt{bd}\right)\frac{a_0}{C_1}\,(1 + C_2)\right]$$

and the even part of $Z(s)$ is now

$$\mathrm{Ev}\{Z(s)\} = \frac{\left[\left(1 + s^2\sqrt{bd}\right)^2 - s^2e^2\right]\left(a_0^2 - s^2\right)^2}{m_{10}^2 - n_{10}^2} \tag{C3}$$

Comparison of equations (C3) and (78) shows that the numerators are identical. The procedure used in Part (1) is repeated to identify even and odd polynomials of s in the denominators. Identifying the even polynomials m_8 and m_{10} results in

$$\left(a_0^2 + s^2\right)(1 + s^2 d) + s^2 2a_0 c = \left(a_0^2 + \frac{s^2}{C_2}\right)\left(1 + s^2\sqrt{bd}\right) + s^2 e\,\frac{C_3}{C_1}\,a_0(1 + C_2) \tag{C4}$$

and equating coefficients of like powers gives

$$s^0 : a_0^2 = a_0^2$$

$$s^2 : 1 + 2a_0 c + a_0^2 d = \frac{1}{C_2} + e\,\frac{C_3}{C_1}\,a_0(1 + C_2) + a_0^2\sqrt{bd} \tag{C5}$$

$$s^4 : d = \frac{\sqrt{bd}}{C_2} \qquad \text{or} \qquad C_2 = \sqrt{\frac{b}{d}} \tag{C6}$$

Solving for C_3 in equation (C5) gives

$$C_3 = \frac{C_1\left[\left(1 - a_0^2\sqrt{bd}\right)\left(1 - \sqrt{\frac{d}{b}}\right) + 2a_0 c\right]}{ea_0\left(1 + \sqrt{\frac{b}{d}}\right)} \tag{C7}$$

where C_2 has been replaced by its value from equation (C6). Similarly, identifying the odd polynomials n_8 and n_{10} results in

$$s\left[\left(a_0^2 + s^2\right)c + (1 + s^2 d)2a_0\right] = s\left[\left(a_0^2 + \frac{s^2}{C_2}\right)\frac{e}{C_3} + \left(1 + s^2\sqrt{bd}\right)\frac{a_0}{C_1}(1 + C_2)\right] \tag{C8}$$

and equating corresponding coefficients yields

$$s^1 : 2a_0 + a_0^2 c = \frac{a_0}{C_1}(1 + C_2) + a_0^2 \frac{e}{C_3} \tag{C9}$$

$$s^3 : c + 2a_0 d = \frac{e}{C_2 C_3} + \frac{a_0}{C_1}\sqrt{bd}\,(1 + C_2) \tag{C10}$$

Taking the ratio of equations (C9) and (C10) results in

$$C_3 = \frac{C_1 e\left[\left(\dfrac{a_0^2\sqrt{bd} - 1}{1 - \sqrt{\dfrac{d}{b}}}\right)2\sqrt{\dfrac{d}{b}} + a_0 c\right]}{\left(1 + \sqrt{\dfrac{b}{d}}\right)\left[\left(\dfrac{a_0^2\sqrt{bd} - 1}{1 - \sqrt{\dfrac{d}{b}}}\right)c + 2a_0\sqrt{bd}\right]} \tag{C11}$$

The existence of a_0 can be proved from an equation in terms of a_0 and the coefficients of $Z(s)$. Equating (C7) and (C11) gives

$$0 = \frac{1}{bd} + 2a_0\left[\frac{a - c\sqrt{\dfrac{b}{d}}}{\sqrt{bd}\left(\sqrt{bd} - b\right)}\right] + a_0^2\left(\frac{ac - b - d - 4\sqrt{bd}}{bd}\right) + 2a_0^3\left(\frac{a - c\sqrt{\dfrac{b}{d}}}{b - \sqrt{bd}}\right) + a_0^4$$

or rearranging yields

$$0 = \left[a_0^2 + a_0\left(\frac{a - c\sqrt{\dfrac{b}{d}}}{b - \sqrt{bd}}\right) - \frac{1}{\sqrt{bd}}\right]^2 - a_0^2\left[\left(\frac{a - c\sqrt{\dfrac{b}{d}}}{b - \sqrt{bd}}\right)^2 + \frac{\left(\sqrt{b} + \sqrt{d}\right)^2 - ac}{bd}\right] \tag{C12}$$

The fourth degree equation (eq. (C12)) can be factored into two second degree equations with real coefficients. Consider (C12) as having the form

$$0 = x^2 - y^2 = (x - y)(x + y)$$

Factoring equation (C12) as indicated results in

$$0 = a_0^2 + a_0 \left\{ \frac{a - c\sqrt{\frac{b}{d}}}{b - \sqrt{bd}} \pm \left[\left(\frac{a - c\sqrt{\frac{b}{d}}}{b - \sqrt{bd}} \right)^2 + \frac{\left(\sqrt{b} + \sqrt{d}\right)^2 - ac}{bd} \right]^{1/2} \right\} - \frac{1}{\sqrt{bd}} \qquad \text{(C13)}$$

where the $\pm$ is used to represent the two equations. For each factor to have real coefficients, the quantity within the brackets (raised to the 1/2 power) is required to be positive. Already this is known to be true since by equation (41)

$$\left(\sqrt{b} + \sqrt{d}\right)^2 - ac > 0$$

This simplification turns out to be a most important key to the proof of the existence of a_0, for now each factor of equation (C13) is of a form similar to the equation in k (eq. (74)). Thus, the methods of Part (1) are applied to show that there exists one positive value of a_0 for each factor. Moreover, the equations represented by equation (C13) can be used to find these values.

The network of figure 15 is valid if it can be proved that a_0 and C_3 are positive. Having shown that there exist two values of a_0 that are positive, it can now be proved that C_3 is also positive. Consider the following expressions:

$$\left(1 - a_0^2\sqrt{bd}\right)\left(1 - \sqrt{\frac{d}{b}}\right) \qquad b \neq d \qquad \text{(C14)}$$

$$\frac{a_0^2\sqrt{bd} - 1}{1 - \sqrt{\frac{d}{b}}} \qquad b \neq d \qquad \text{(C15)}$$

A little thought reveals that regardless of the relative magnitudes of b and d, equations (C14) and (C15) are of opposite sign or both zero. If equation (C14) is positive or zero, then, since C_1 and a_0 are positive, C_3 of equation (C7) is also positive. Similarly, if equation (C15) is positive or zero, C_3 of equation (C11) is positive. Thus, C_3 is always positive.

In summary, then, it has been shown that there exist two positive values of a_0 as the solutions of the equations represented by equation (C13). Further, it was also proved that C_3 is always positive and can be found from equation (C7) or (C11).

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