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ABSTRACT

Necessary conditions of optimality for state-variable inequality constrained problems are derived by examining the limiting behavior of the Kelley penalty function technique. The conditions so obtained differ from those presently known, with regard to the behavior of the adjoint variables at junctions of interior and boundary arcs. A second, rigorous, derivation is given; this confirms the necessary conditions obtained by the limiting argument. These conditions are related to those known earlier; in particular, it is shown that the earlier conditions over-specify the behavior of the adjoint variables at the junctions. An example is used to demonstrate that the earlier conditions may yield non-stationary trajectories.

For the regular case, it is shown that, under certain conditions, only boundary points, as opposed to boundary arcs, are possible. An analytic example illustrates this behavior.

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1. Introduction

Necessary conditions for the optimality of state-variable inequality constrained problems have been the subject of much research in the past ten years. Gamkrelidze [1], in 1960, approached the problem via the Pontryagin maximum principle. He adjoined the first time-derivative of the constraint -- which explicitly contained the control by his 'regularity' assumption -- to the cost functional and treated the resulting problem as one with a control constraint. Berkovitz [2], in 1962, derived the same conditions as in [1], by way of the classical calculus of variations. He too used the first time-derivative of the constraint, which, by his constraint qualifications, contained the control.

Bryson, Denham and Dreyfus [3], extended the above procedure to cases where the state-variable constraint was of order $p \geq 1$.† To ensure feasibility of the resulting trajectory, they adjoined a point equality constraint, consisting of the state constraint and its $(p - 1)$ time-derivatives, at the time of entry of the trajectory onto the constraint boundary. Their results reduce to those of Gamkrelidze and Berkovitz for the case of a first order constraint.

Chang [4], in 1962, used an entirely different approach. He adjoined the constraint violation to the cost functional by a penalty parameter and used a limiting procedure to obtain the necessary conditions directly. His proofs were limited to the first order case ($p = 1$). Dreyfus, in his book [5], clarified this direct procedure of adjoining the state-variable constraint per se, and obtained the same necessary conditions. Speyer [6] pointed out that Dreyfus' arguments failed for constraints of order $p > 1$, as then the adjoining multiplier may exhibit impulsive behavior. Dreyfus suggested the resolution of this matter as a research problem.

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† The constraint is assumed to be of p-th order, i.e. the p-th time-derivative of the constraint is the first to contain the control variable explicitly.

Speyer [7] extended the direct approach to constraints of higher order, by adjoining directly the state-variable constraint to the cost functional, together with point equality constraints at junctions of boundary and interior arcs. He obtained a set of necessary conditions which differed from, but were related to, those obtained in [3]. McIntyre and Paiewonsky [8] used a similar approach.

A third set of necessary conditions, differing considerably in form from those in [3] and [7] were obtained by Dreyfus in his Ph. D. thesis [9]. He used the constraint and its p - 1 time-derivatives to reduce, by p, the dimension of the state space along the constraint boundary. These results were related to those of [2] by Berkovitz and Dreyfus [10], for the case p = 1. Speyer has shown that these are related to the necessary conditions derived in [7].

Concurrently with these theoretical investigations, research was progressing on numerical methods for the solution of state variable inequality constrained problems. Denham and Bryson [11] used the results of [3] for a steepest ascent algorithm. Speyer [7] proposed a second order sweep algorithm. In 1962, Kelley [12] contributed by extending a device of Courant [13] to obtain a penalty function technique for the numerical solution of such problems. Other penalty procedures have been investigated by Lasdon, Waren and Rice [14], and Thrasher [26]. Kelley's procedure adjoins the square of the constraint violation to the cost by means of a penalty parameter; the resulting unconstrained problem is solved repeatedly for successively increasing penalty parameter values. The convergence of this type of procedure has been discussed by Butler and Martin [15], Russell [16], Lele and Jacobson [17], Cullum [18], and Beltrami [19]. Beltrami derived the generalized Kuhn-Tucker optimality conditions in a Hilbert space by investigating further the limiting behavior of the penalty method.

In this paper, following Chang and Beltrami we derive necessary conditions of optimality from the limiting form of the Kelley penalty function technique. This yields necessary conditions of optimality which are similar to those of Speyer [7], except at the junction points of boundary and interior arcs. At these points the influence functions exhibit fewer discontinuities than predicted in [7]. (The same result can be obtained from the Lasdon, Waren and Rice procedure [14].) This is confirmed, under weaker assumptions, with the aid of the generalized Kuhn-Tucker conditions. We show the relationship of our results to those of [3] and [7]. In particular, we demonstrate that Speyer's necessary conditions are identical to ours provided that all, except possibly one, of the multipliers adjoining the point constraint at the junction are zero. This leads us to the conclusion that, in addition to Speyer's stated conditions, <u>it is necessary that all his entry and exit point adjoining multipliers be zero except, possibly, the first</u>.

The necessary conditions of [3] can be derived directly, by integration by parts, from ours; this derivation indicates that <u>it is necessary that certain relationships hold between the entry (or exit) point multipliers</u>.

Note that for the case where $p = 1$, and for $p = 2$ if the Hamiltonian is regular, our results are equivalent to the above [7] known necessary conditions, since there is then only one adjoining multiplier at entry and exit points. We use a fourth order constrained problem to illustrate that the existing necessary conditions of Bryson, Denham, and Dreyfus and Speyer, can be satisfied by a <u>non-extremal</u> (i.e. non-stationary trajectory) and thus can yield an incorrect answer. In the particular example considered, a trajectory consisting of boundary and interior arcs is pieced together; this satisfies the existing necessary conditions [3] and [7]. However, it turns out that the unconstrained optimal trajectory is at all times feasible and yields a lower value of the cost.

The problem was deliberately chosen to be convex so that no stationary but non-optimal solutions exist. This confirms that the necessary conditions of Bryson, Denham and Dreyfus and Speyer can be satisfied by a non-extremal.

In our necessary conditions the influence functions may exhibit discontinuities at junction points of boundary and interior arcs only along the direction S_x.† For problems where the Hamiltonian is regular, yielding a continuous optimal control function of time [8], we are able to derive a particularly simple expression for the magnitude of this discontinuity. The form of this expression leads us to conclude that, in certain cases, problems with state constraints of odd order (where $p > 1$) will not exhibit any boundary arcs over non-zero intervals of time; i. e. the trajectory will, at most, only touch the constraint boundary but will not lie along it. This behavior is illustrated by a third order example which is solved analytically. As predicted by the theory, the constrained trajectories do not remain on the boundary for non-zero intervals of time.

In summary, we have, by directly adjoining the state variable constraint to the cost functional, obtained necessary conditions of optimality that are considerably simpler and 'sharper' (in that non-extremals cannot satisfy them) than those of [3]-[11]. Furthermore, we are able to prove that under certain conditions, problems with constraints of odd order ($p > 1$) cannot contain boundary arcs of non-zero length.

2. Preliminaries

We first state the basic problem to be considered and establish some notation.

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† $S(x(t)) \leq 0$, $t \in [0, T]$ is the state-variable inequality constraint. $S_x = \frac{\partial S}{\partial x}(x(t))$.

The following problem will be referred to as 'the basic problem' or 'problem (I)'.

Problem (I)

$$\underset{u}{\text{Minimize}}\ \Phi(x(T)) \tag{1}$$

subject to

$$\dot{x}(t) = f(x(t), u(t)) \quad ; \quad x(0) = x_o \tag{2}$$

and the scalar state-variable inequality constraint

$$S(x(t)) \leq 0 \quad ; \quad t \in [0, T] \quad . \tag{3}$$

Here,

$u(t)$ -- scalar control variable

$x(t)$ -- n-dimensional vector of state-variables

f -- n-dimensional vector function

S -- p-th order state-variable constraint

Φ -- scalar function of terminal value of state-variable

x_o -- initial value of state vector, assumed known

$\dot{}$ -- $\frac{d}{dt}(\ )$

$(\ )^{p}$ -- $\frac{d^p}{dt^p}(\ )$

$\|\cdot\|$ -- norm over the space under consideration

$\ni$ -- such that

$\in$ -- belonging to

$\forall$ -- for all

$\exists$ -- there exists

Assumptions

1. $u \in L_1[0, T] \equiv U$
2. $x \in C_1^n[0, T] \equiv X$

3. f is a continuous function of x and u and has bounded partial derivatives up to p-th order in both x and u on the interval $[0, T]$.
4. Φ is a continuous and differentiable function of $x(T)$.
5. $S \in C_p[0, T]$.
6. The basic problem has a feasible solution with finite cost v_o.

Notes

1. The basic problem is of the form of Mayer. However there is no loss of generality, for a problem in the form of Lagrange or Bolza can always be cast into the form of Mayer by defining an additional state-variable.
2. u and S are assumed to be scalars for simplicity, without loss of generality.
3. f, S, and Φ are assumed to be implicit functions of time, without any loss of generality.

3. Summary of previous results

We shall summarize only the results of [3] and [7], as these are the closest in form to ours.

3.1. Necessary conditions of Bryson, Denham and Dreyfus

In [3] Bryson, Denham and Dreyfus extended the approach of [1] and [2] to problems with state constraints of orders higher than the first.† They differentiated

$$S(x(t))$$

p times with respect to time to obtain the mixed control-state inequality constraint

$$\overset{p}{(S)}(x(t), u(t)) \leq 0 \quad . \tag{4}$$

They then transformed problem (I) into a control-constrained problem by applying the constraint (4) along boundary arcs. To ensure feasibility of the resulting trajectories they also imposed the following equality constraint

† See also [27].

at points of entry† of the trajectory onto the constraint boundary

$$\Psi(t_{entry}) \equiv \begin{pmatrix} S(x(t)) \\ \dot{S}(x(t)) \\ \vdots \\ \overset{p-1}{(S)}(x(t)) \end{pmatrix}_{t=t_{entry}} = 0 \quad . \tag{5}$$

Thus the single state constraint (3) was replaced by the point constraint (5) at entry points and the control inequality constraint (4) along the boundary (S = 0). The equivalent problem was

$$\underset{u}{\text{Minimize}}\ \Phi(x(T)) \tag{6}$$

subject to

$$\dot{x}(t) = f(x(t), u(t)) \quad ; \qquad x(0) = x_o$$

and

$$\Psi(t_{entry}) = 0$$

$$\overset{p}{(S)}(x(t), u(t)) \leq 0 \quad , \qquad t \in [t_{entry}, t_{exit}] \quad .$$

Adjoining (4) and (5) to (6) by a scalar multiplier function $\gamma(\cdot)$ (≥ 0) and vectors of multipliers, $\nu_b(t_i)$ (≥ 0) $i = 1, \ldots, N$ -- corresponding to the N entry points -- they obtained the following necessary conditions

$$\frac{\partial H}{\partial u} = 0 = \gamma \overset{p}{(S)}_u + f_u^T \lambda \tag{7}$$

and the adjoint equations

$$\left.\begin{aligned} -\dot{\lambda} &= \frac{\partial H}{\partial x} \\ &= f_x^T \lambda + \gamma \overset{p}{(S)}_x \end{aligned}\right\} \quad \lambda(T) = \left.\frac{\partial \Phi}{\partial x}\right|_{t=T} \tag{8}$$

where the Hamiltonian H was defined as

$$H = \gamma \overset{p}{(S)} + \lambda^T f \quad . \tag{9}$$

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† This equality constraint could equally well be imposed at the points of exit from the boundary.

Due to the point constraints (5), the influence functions $\lambda(\cdot)$ suffer discontinuities of the form:

$$\lambda(t_i^+) = \lambda(t_i^-) - \nu_b^T(t_i)\left(\frac{\partial\Psi}{\partial x}\right)_{t_i} \tag{10}$$

at $t = t_i$ $(i = 1, \ldots, N)$, the entry points.

3.2. Speyer's necessary conditions

Speyer [7] adjoins the state-variable constraint directly to the cost functional with a multiplier function $\mu(\cdot)$ (≥ 0). To ensure feasibility, he also adjoins the point constraint (5) both at entry and exit points of boundary arcs with multipliers $\nu_S(t_i)$ (≥ 0). He obtains the following set of necessary conditions:

$$\frac{\partial H}{\partial u} = 0 = f_u^T\lambda \tag{11}$$

and

$$\left.\begin{aligned} -\dot{\lambda} &= \frac{\partial H}{\partial x} \\ &= f_x^T\lambda + \mu S_x \end{aligned}\right\} \quad \lambda(T) = \left.\frac{\partial\Phi}{\partial x}\right|_T \tag{12}$$

where the Hamiltonian H is

$$H = \mu S + \lambda^T f \quad . \tag{13}$$

At junction points of interior and boundary arcs, the adjoint variables suffer discontinuities; the boundary conditions are

$$\lambda(t_i^+) = \lambda(t_i^-) - \nu_S^T(t_i)\left(\frac{\partial\Psi}{\partial x}\right)_{t_i} \tag{14}$$

$i = 1, \ldots, M$.

Speyer notes that, in going from an interior arc to a boundary arc, the jumps in $\lambda(\cdot)$ can be obtained as functions of $\lambda(\cdot)$ and $x(\cdot)$ immediately prior to the junction. Thus there are no more unknowns in his procedure than in the previous scheme [3].

4. Derivation of necessary conditions via the Kelley penalty function technique

The Kelley penalty function technique [12] converts the basic constrained problem (I) into the following unconstrained problem.

$$\underset{u}{\text{Minimize}}\ P(r_k) \equiv \Phi(x(T)) + \frac{1}{2} r_k^{-1} \int_0^T h(S)S^2\, dt \qquad (15)$$

subject to

$$\dot{x}(t) = f(x(t), u(t)) \quad ; \quad x(0) = x_o \qquad (16)$$

where

$$h(a) = \begin{cases} 1 & a > 0 \\ 0 & a \leqslant 0 \end{cases} \qquad (17)$$

and $r_k > 0$.

This problem is then solved repeatedly for an infinite sequence $\{r_k\}$ such that $r_k > r_{k+1} > 0$ and $\text{limit}_{k\to\infty} r_k = 0$. Lele and Jacobson [16] have shown that, under certain conditions, the above procedure converges. We state their main theorem below, with a slight modification.

Theorem 1. Let $\{r_k\}$ be an infinite sequence of positive numbers $\ni r_k > r_{k+1} > 0$ and $\text{limit}_{k\to\infty} r_k = 0$. Under certain conditions†, the P-function is minimized by a bounded control function u_k (not necessarily unique); further, every limit point of the sequence of control functions $\{u_k\}$ solves the problem (I).

Here u_k denotes the optimal control for the k-th problem, i.e. the unconstrained problem corresponding to the k-th member of the sequence $\{r_k\}$. Let us now examine the necessary conditions of optimality for this, unconstrained, problem. Defining the Hamiltonian H_k as

$$H_k \equiv \frac{1}{2} r_k^{-1} h(S)S^2 + \lambda^T f \qquad (18)$$

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† See [17] p. 165.

the necessary conditions of optimality are

$$\frac{\partial H_k}{\partial u} = 0 = f_u^T \lambda \tag{19}$$

and

$$\left.\begin{aligned} -\dot{\lambda} &= \frac{\partial H_k}{\partial x} \\ &= f_x^T \lambda + r_k^{-1} h(S)S.\, S_x \end{aligned}\right\} \lambda(T) = \left.\frac{\partial \Phi}{\partial x}\right|_T \quad . \tag{20}$$

If we denote the term $r_k^{-1} h(S)S$ by η_k, (20) may be re-written as

$$-\dot{\lambda} = f_x^T \lambda + \eta_k S_x \quad . \tag{21}$$

We can now derive the necessary conditions of optimality for problem (I) by considering the equations (18), (19) and (21) in the limit as $r_k \rightarrow 0^+$. For this purpose consider only the limit

$$\text{limit}_{k\rightarrow\infty}\, \eta_k(\cdot) \equiv \eta(\cdot) \quad . \tag{22}$$

Beltrami [19] has shown that, under certain additional restrictions on Φ, S and U, the above limit exists and is bounded. His proof is stated for the case when S maps X to E^1. It can be readily extended to the present case, with the essential change that

$$\text{limit}_{k\rightarrow\infty}\, \eta_k(t) \equiv \eta(t) < \text{constant a.e.} \tag{23}$$

This gives the following necessary conditions of optimality

$$\frac{\partial H}{\partial u} = 0 = f_u^T \lambda \tag{24}$$

and

$$\left.\begin{aligned} -\dot{\lambda} &= \frac{\partial H}{\partial x} \\ &= f_x^T \lambda + \eta S_x \end{aligned}\right\} \lambda(T) = \left.\frac{\partial \Phi}{\partial x}\right|_T \tag{25}$$

+ Chang [4] uses a similar approach. An alternative, rigorous derivation is given in Section 5.

where $\eta(\cdot) \geq 0$ and we have defined H as

$$H \equiv \text{limit}_{k\to\infty} H_k = \text{limit}_{k\to\infty} (\eta_k S + \lambda^T f) \quad . \tag{26}$$

From (23), it follows that the function $\eta(\cdot)$ may contain impulses of positive weight. This would lead to discontinuities in the influence functions of the form

$$\lambda(t_1^+) = \lambda(t_1^-) - \nu \left(\frac{\partial S}{\partial x}\right)_{t_1} \tag{27}$$

where $t_1 \in [0, T]$ is the time at which the impulse occurs and $\nu \geq 0$ is a scalar.

Equations (24)-(27) form the necessary conditions for problem (I). As can be seen, (27) is considerably different from either (10) or (14), which give the existing form of the discontinuities. In the next section, we derive the above necessary conditions directly and rigorously, and we show that the discontinuities in the influence functions occur at the junctions between boundary and interior arcs.

5. A direct derivation of the necessary conditions

We present below an alternative derivation of the necessary conditions of optimality, (24)-(27). Basically we rederive the generalized Kuhn-Tucker conditions [20] in a Banach space. Russell [21] has previously derived these conditions in a general topological space and applied them to a state constrained problem. Luenberger [22] has obtained similar results. However, these researchers have not related their work to that of [3], [7].

Our proof, in Section 5.1, follows closely that in [22], with the essential difference of no qualification on the constraint. This admits the possibility of an unbounded multiplier. After translating the necessary conditions to state space in Section 5.2, in Section 5.3 we prove the boundedness of the multiplier under a different qualification, more natural from the control-theoretic point of view.

5.1. A generalized Kuhn-Tucker Theorem

For the purposes of this section we write the basic problem in the form

$$\underset{u}{\text{Minimize}}\ \Phi(u) \tag{28}$$

subject to

$$\left.\begin{array}{l} S(u) \leqslant \theta \\ u \in U \end{array}\right\} \text{(II)}$$

Here θ denotes the null vector in $C_p[0,T]$. We will now consider necessary conditions of optimality for this problem.

All differentials and derivatives will be in the sense of Fréchet. The symbol $\langle x, T\rangle$ will denote the value of the linear functional $T(x)$ at a point $x \in X$. (cf [23] p. 21.)

Theorem 2. Let Φ be a real valued Fréchet differentiable function on U, and $S: U \longrightarrow C_p[0,T]$ a Fréchet differentiable mapping. Suppose $u^* \in U$ minimizes Φ subject to $S(u^*) \leqslant \theta$. Then there exists $r_o \geqslant 0$, $\eta^* \in L_\infty[0,T]$, $\eta^* \geqslant \theta$, such that the Lagrangian

$$r_o\Phi(u) + \langle S(u), \eta^*\rangle \tag{29}$$

is stationary at u^*. Further

$$\langle S(u^*), \eta^*\rangle = 0 \quad . \tag{30}$$

Proof. Define the following sets, A and B, on $W = R \times C_p[0,T]$.

$$A \equiv \{r, z \mid r \geqslant \delta\Phi(u^*; \delta u),\ z \geqslant S(u^*) + \delta S(u^*; \delta u) \text{ for some } \delta u \in U\} \tag{31}$$

and

$$B \equiv \{r, z \mid r \leqslant 0, z \leqslant \theta\} \quad . \tag{32}$$

The sets A and B are convex; B contains interior points as $C_p[0,T]$ has positive cone.

Denote by Int(B) the interior of B. Then

$$A \cap \mathrm{Int}(B) = \emptyset \tag{33}$$

the empty set. For, if $(r, z) \in A \ni r < 0,\ z < \theta$, then $\exists\, \delta u \in U \ni$

$$\delta\Phi(u^*; \delta u) < 0 \quad , \quad S(u^*) + \delta S(u^*; \delta u) < \theta \quad . \tag{34}$$

Then $\exists$ a sphere of radius ρ centered on $S(u^*) + \delta S(u^*; \delta u) \subseteq N$ (the negative cone in $C_p[0, T]$). For $0 < \alpha < 1$ the point $\alpha[S(u^*) + \delta S(u^*; \delta u)]$ is the center of an open sphere of radius $\alpha \cdot \rho$ contained in N; hence so is the point $(1 - \alpha)S(u^*) + \alpha[S(u^*) + \delta S(u^*; \delta u)] = S(u^*) + \alpha \cdot \delta S(u^*; \delta u)$. As for fixed δu

$$\| S(u^* + \alpha\delta u) - S(u^*) - \alpha \cdot \delta S(u^*; \delta u) \| = o(\alpha) \tag{35}$$

it follows that for sufficiently small α, $S(u^* + \alpha\delta u) < \theta$. A similar argument shows that $\Phi(u^* + \alpha\delta u) < \Phi(u^*)$ for sufficiently small α. This contradicts the optimality of u^*. Therefore

$$A \cap \mathrm{Int}(B) = \emptyset \quad .$$

So A and B are two convex sets in the normed space $R \times C_p[0, T]$ such that $A \cap \mathrm{Int}(B) = \emptyset$ and $\mathrm{Int}(B) \neq \emptyset$. Therefore $\exists$ a closed hyperplane seperating A and B, [24]. Hence $\exists\, r_o, \eta^*, \delta$ such that

$$r_o \cdot r + \langle z, \eta^* \rangle \geq \delta \qquad \forall\ (r, z) \in A \tag{36}$$

and

$$r_o \cdot r + \langle z, \eta^* \rangle \leq \delta \qquad \forall\ (r, z) \in B \quad . \tag{37}$$

As $(0, \theta) \in A \cap B$, $\delta = 0$; and from the nature of B it follows that $r_o \geq 0$, $\eta^* \geq \theta$. From the seperation property

$$r_o \cdot \delta\Phi(u^*; \delta u) + \langle S(u^*) + \delta S(u^*; \delta u), \eta^* \rangle \geq 0 \tag{38}$$

$\forall\ \delta u \in U$.

From the above inequality, $\delta u = \theta \Longrightarrow \langle S(u^*), \eta^* \rangle \geq 0$; but $S(u^*) \leq \theta$ and $\eta^* \geq \theta \Longrightarrow \langle S(u^*), \eta^* \rangle \leq 0$, hence

$$\langle S(u^*), \eta^* \rangle = 0 \quad . \tag{39}$$

It then follows that $\forall\ \delta u \in U$

$$r_o \delta\Phi(u^*; \delta u) + \langle \delta S(u^*; \delta u); \eta^* \rangle = 0 \tag{40}$$

i.e.

$$r_o \Phi_{u^*} + \eta^* S_{u^*} = \theta \tag{41}$$

where Φ_{u^*}, S_{u^*} denote the Fréchet derivatives of Φ and S evaluated at u*.

We will now translate these results into the more familiar state space form.

5.2. <u>The stationarity conditions in state space</u>

In state space, the equivalent of the Lagrangian of (29) is the adjoined cost functional

$$\overline{J} \equiv r_o \Phi(x(T)) + \int_0^T dt[\eta^*(t)S(x(t)) + \lambda(t)(f(x(t), u(t)) - \dot{x}(t))] \quad . \tag{42}$$

Then following the usual procedure [27] we consider variations in $\overline{J}$ for arbitrary variations δx, δu. This gives the following conditions for stationarity of $\overline{J}$:

$$\left.\begin{aligned} -\dot{\lambda} &= \frac{\partial H}{\partial x} \\ &= f_x^T \lambda + \eta^* S_x \end{aligned}\right\} \quad \lambda(T) = r_o \frac{\partial \Phi}{\partial x}\bigg|_T \tag{43}$$

and

$$\frac{\partial H}{\partial u} = 0 = f_u^T \lambda \tag{44}$$

where the Hamiltonian H is

$$H = \eta^* S + \lambda^T f \tag{45}$$

and from (30)

$$\eta^*(t) = \begin{cases} \geqslant 0 & S(x(t)) = 0 \\ 0 & S(x(t)) < 0 \end{cases} \tag{46}$$

From Section 5.1, these are the necessary conditions that a control u* and its associated trajectory x* satisfy if they solve (I).

5.3. Boundedness of the adjoining multiplier

Theorem 3. If $\frac{\partial}{\partial u(t)}[\overset{p}{(S)}(x^*(t), u^*(t))] \neq 0 \;\; \forall\, t \in [0, T]$ then $r_o \neq 0$.

Proof. For clarity, and without any loss of generality we will assume that the optimal trajectory x^* has only one boundary arc and two interior arcs.

Suppose $r_o = 0$. Then $\lambda(T) = 0$ (from (43)); hence

$$\lambda(t) = 0 \qquad t \in (t_{ex}, T] \tag{47}$$

where t_{ex} = time of exit from boundary arc. From (44), $\forall\, t \in [0, T]$

$$H_u = 0 \tag{48}$$

hence

$$\begin{cases} \dot{(H_u)} = 0 \\ \quad \vdots \\ \overset{p-1}{(H_u)} = 0 \end{cases} \tag{49}$$

and similarly for higher derivatives of H_u. In particular, at $t = t_{ex}$

$$\begin{cases} (H_u)^- = (H_u)^+ \\ (\dot{H}_u)^- = (\dot{H}_u)^+ \\ \overset{p-1}{(H_u)}{}^- = \overset{p-1}{(H_u)}{}^+ \end{cases} \tag{50}$$

where the - and + superscripts denote instants just prior to, and just after t_{ex}, respectively. From (43), we have that

$$\lambda(t_{ex}^-) = \lambda(t_{ex}^+) + \int_{t_{ex}^+}^{t_{ex}^-} \eta^*(t) S_x(x(t))dt \tag{51}$$

which substituted into

$$\overset{p-1}{(H_u)}{}^- = \overset{p-1}{(H_u)}{}^+$$

yields, after some manipulation (since $\lambda(t) = 0$; $t \in (t_{ex}, T]$)

$$\int_{t_{ex}^+}^{t_{ex}^-} \eta^* \overset{p}{(S)}_u \, dt = 0 \qquad . \tag{52}$$

As, by assumption $\overset{p}{(S)}_u \neq 0$, $\eta^*(t_{ex}^-) < \infty$. This gives $\lambda(t_{ex}^-) = 0$. Now, from

$$\overset{p}{(H_u)} = 0 \tag{53}$$

along S = 0 we have

$$\eta^*(t) = \frac{[\text{terms involving } \overset{p-1}{(f_u)} \text{ etc}]\lambda(t)}{\overset{p}{(S)}_u} \tag{54}$$

This gives a linear homogeneous differential equation for $\lambda(\cdot)$ along S = 0, with the initial condition $\lambda(t_{ex}^-) = 0$. Hence $\lambda(\cdot) = 0$ along the boundary. Thus,

$$\eta^*(\cdot) = 0 \quad \text{a.e.} \tag{55}$$

as well as $r_o = 0$. But $r_o = 0, \eta^* = \theta$ contradicts the fact that $\exists$ a closed seperating hyperplane (Section 5.1). Thus $r_o \neq 0$.

For convenience we set $r_o = 1$. We now show that η^* is bounded everywhere except possibly at junction points of boundary and interior arcs.

Theorem 4. The multiplier η^* is bounded everywhere along the boundary, except possibly at junction points of boundary and interior arcs. At such a junction point, it may exhibit a positive impulse of finite strength.

Proof. As in Theorem 3, we will assume, without loss of generality, that x* has only one boundary arc and two interior arcs.

From (54), $\eta^*(\cdot)$ along the boundary is given by

$$\eta^*(t) = \frac{[\text{terms involving } \overset{p-1}{(f_u)} \text{ etc}]\lambda(t)}{\overset{p}{(S)}_u} \quad . \tag{56}$$

Denote by t_1, the time of a junction between the boundary arc and an interior arc; superscripts - and + denote times prior to and after the junction, respectively. Then from (43)

$$\lambda(t_1^+) = \lambda(t_1^-) - \int_{t_1^-}^{t_1^+} \eta^* S_x \, dt \tag{57}$$

($\lambda(t_1^-)$ is assumed bounded†).

This coupled with (55), (43) and

$$(H_u^{p-1})^- = (H_u^{p-1})^+ \tag{58}$$

leads to the conclusion that

$$\int_{t_1^-}^{t_1^+} \eta^*(S)_u^p \, dt \qquad \text{unbounded} \Longrightarrow \eta^*(\cdot) \text{ unbounded.} \tag{59}$$

everywhere, which contradicts the fact that $\exists$ a closed hyperplane in Theorem 2. Hence

$$\int_{t_1^-}^{t_1^+} \eta^*(S)_u^p \, dt < \infty$$

and thus $\lambda(t_1^-)$ is bounded and hence $\lambda(t)$ is bounded along $S = 0$, so that

$$\eta^*(\cdot) < \infty \tag{60}$$

with the possible exception of junction points, where it may exhibit positive impulses ($\eta^* \geq \theta$) of finite strength.

This last possibility is conveniently summarized by adding the following junction condition to the necessary conditions (43), (44):

$$\lambda(t_1^+) = \lambda(t_1^-) - \nu(t_1)\left(\frac{\partial S}{\partial x}\right)_{t_1} \tag{61}$$

where t_1 is the time of the junction.

5.4. The necessary conditions summarized

We summarize below the necessary conditions derived in this section.

Theorem 5. The necessary conditions for optimality of problem (I) are

$$\frac{\partial H}{\partial u} = 0 = f_u^T \lambda \tag{62}$$

† A reasonable assumption.

$$\left.\begin{aligned} -\dot{\lambda} &= \frac{\partial H}{\partial x} \\ &= f_x^T\lambda + \hat{\eta} S_x \end{aligned}\right\} \quad \lambda(T) = \left.\frac{\partial \Phi}{\partial x}\right|_T \tag{63}$$

where

$$\hat{\eta}(t) = \begin{cases} \geqslant 0 & S(x(t)) = 0 \\ 0 & S(x(t)) < 0 \end{cases} \tag{64}$$

is a bounded function for $t \in [0, T]$.

At junction points t_i of boundary and interior arcs, the influence functions $\lambda(\cdot)$ may be discontinuous. The boundary conditions are

$$\lambda(t_i^+) = \lambda(t_i^-) - \nu(t_i)\left.\left(\frac{\partial S}{\partial x}\right)\right|_{t_i} \tag{65}$$

$\nu(t_i) \geqslant 0$. The Hamiltonian H, used above, is defined by

$$H \equiv \hat{\eta} S + \lambda^T f \tag{66}$$

5.5. Generalization

For the case where $S \in C_p^r[0, T]$, the necessary conditions are the same as those above, but $\hat{\eta}(\cdot)$ and ν are r-vectors. If terminal equality constraints $(\psi(x(t_f), t_f) = 0)$ are present, then the necessary conditions take the following form:

Theorem 6. (Equality Terminal Constraints) If in addition to the assumptions of Section 5, the following are true

i) $\delta\dot{x} = f_x(x^*(t), u^*(t))\delta x + f_u(x^*(t), u^*(t))\delta u$

is completely controllable

ii) $\psi_x(x^*(t_f), t_f)$ has rank q (ψ is a q-vector function)

then necessary conditions of optimality are:

$$\frac{\partial H}{\partial u} = 0 = f_u^T\lambda \tag{67}$$

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = f_x^T\lambda + \hat{\eta} S_x \quad ; \quad \lambda(T) = \left.\frac{\partial \Phi}{\partial x} + \sigma^T\psi_x\right|_T \tag{68}$$

σ is a q-vector of constant Lagrange multipliers and

$$\hat{\eta}(t) = \begin{cases} \geq 0 & S(x(t)) = 0 \\ 0 & S(x(t)) < 0 \end{cases} \quad . \tag{69}$$

At junction points of boundary and interior arcs:

$$\begin{cases} \lambda(t_i^+) = \lambda(t_i^-) - \nu(t_i)\left(\frac{\partial S}{\partial x}\right)_{t_i} & i = 1, \ldots, N \\ \nu(t_i) \geq 0 \end{cases} \tag{70}$$

<u>Proof</u>. We only sketch the proof here. The above problem may be considered in the following nonlinear programming formulation

$$\operatorname*{Min}_{u} \Phi(u)$$

subject to

$$S(u) \leq 0$$

$$\psi(u) = 0$$

Assumptions i), ii) above imply that ψ is regular at u^*.

Define the set

$$U_1 \equiv \{u : \psi_u(u^*)\delta u = 0, \ u^* + \delta u \in U\} \quad . \tag{71}$$

The set A (Theorem 2) is now defined as

$$A \equiv \{(r, z) : r \geq \delta f(u^*; \delta u), z \geq S(u^*) + \delta S(u^*; \delta u); \text{ for some } \delta u \in U_1\} \tag{72}$$

Using similar though more involved arguments than those of Theorem 2 it follows, since ψ is regular at u^*, that

$$\langle r_o \Phi_u + \eta^* S_u, \delta u \rangle = 0 \qquad \delta u \in U_1 \quad . \tag{73}$$

This leads to the conclusion

$$r_o \Phi_u + \eta^* S_u + \sigma^T \psi_u = 0 \quad . \tag{74}$$

Translating back into state space and using the assumption $\frac{\partial}{\partial u(t)} \overset{p}{(S)} \neq 0$; $t \in [0, T]$ yields the necessary conditions (67)-(70).

6. Relation to previous results

6.1. Bryson, Denham and Dreyfus

Let us rewrite (42) as the following (noting that $r_o = 1$)

$$\underset{u}{\text{Minimize}}\ J \equiv \Phi(x(T)) + \int_0^T d\tau\, \eta(\tau) S(x(\tau)) \tag{75}$$

subject to

$$\dot{x}(t) = f(x(t), u(t)) \quad ; \; x(0) = x_o \tag{76}$$

where we have written η for η^*. For simplicity, and with no loss of generality, we will assume that the optimal trajectory has only one constrained arc. Then integrating the cost functional by parts, equation (75) becomes (dropping the $\underset{u}{\text{Min}}$ operator for simplicity),

$$\Phi(x(\tau), T) + [\eta_1(\tau) S(x(\tau))] \Big|_{t=t_{en}}^{t=t_{ex}} - \int_0^T d\tau\, \eta_1(\tau) \frac{d}{d\tau} \Big[S(x(\tau))\Big] \tag{77}$$

where

$$\eta_1(t) = \int_0^t \eta(\tau) d\tau \quad . \tag{78}$$

Adding and subtracting $\eta_1(t_{ex}) S(t_{en})$, (77) becomes†

$$\Phi(x(\tau), T) + \nu_1 S(x) \Big|_{t=t_{en}} + \int_0^T \bar{\eta}_1(\tau) \frac{d}{d\tau} \Big[S(x(\tau))\Big] \tag{79}$$

where

$$\nu_1 \equiv \eta_1(t_{ex}) - \eta_1(t_{en}) \geqslant 0 \tag{80}$$

and

$$\bar{\eta}_1(t) = \eta_1(t_{ex}) - \eta_1(t) \geqslant 0 \qquad \forall\, t \in [t_{en}, t_{ex}] \quad . \tag{81}$$

---

† Note that $\int_0^T \eta(\tau) S(x(\tau)) d\tau \equiv \int_{t_{en}}^{t_{ex}} d\tau\, \eta S$

We can carry on this process of integration by parts until finally we obtain

$$J = \Phi(x(\tau), T) + \nu^T \Psi + \int_0^T \overline{\eta}_p(\tau) \frac{d^p}{d\tau^p} [S(x(\tau))] d\tau \tag{82}$$

where

$$\Psi^T = (S, \dot{S}, \ddot{S}, \ldots, \overset{p-1}{(S)}) \tag{83}$$

and

$$\nu^T = [\nu_1, \nu_2, \ldots, \nu_p] \tag{84}$$

and

$$\nu_i = \eta_i(t_{ex}) - \eta_i(t_{en}) \geqslant 0 \tag{85}$$

and

$$\overline{\eta}_p(t) = \eta_p(t_{ex}) - \eta_p(t) \tag{86}$$

and where

$$\eta_i(t) = \int_0^t \overline{\eta}_{i-1}(\tau) d\tau \tag{86}$$

and

$$\overline{\eta}_i(t) = \eta_i(t_{ex}) - \eta_i(t) \geqslant 0 \quad . \tag{88}$$

Identifying $\eta_p(\cdot)$ with $\gamma(\cdot)$ we see that (82) is equivalent to the adjoined cost functional of [3]. If we now use (82) as the functional to be minimized subject to equation (76) we obtain the same stationarity conditions as those given by equations (8)-(10). The noteworthy point is the set of equations (85)-(88). These indicate that the ν's and $\gamma(t)$ of [3] are <u>related</u> along the optimal trajectory.

6.2. Relation to Speyer's necessary conditions

Speyer's [7] necessary conditions reduce to those given by us, if his multipliers ν_{S_2} through ν_{S_p} are zero. For, if that is the case, (14) and (65) are the same, and setting $\mu(\cdot)$ equal to $\hat{\eta}(\cdot)$ completes the connection.

The fact that, along an extremal Speyer's multipliers ν_{S_2} through ν_{S_p} are zero leads to an interesting result, provided the Hamiltonian (66) has a unique minimum in $u(\cdot)$ for all $t \in [0, T]$, i.e. if the Hamiltonian is regular. In this case, Speyer [7] and McIntyre and Paiewonsky [8] have shown that $u(\cdot)$ must be continuous across the junction, and that $\nu_{S_p} = 0$. But if $\nu_{S_{p-1}}$, $\nu_{S_{p-2}}$, ν_{S_2} are all zero, then u and all its derivatives up to $\overset{p-2}{(u)}$ must be continuous across the junction. This result is easily obtained from [7] or equations (14) and (50).

7. <u>A further consequence of the new necessary conditions</u>

For the regular case, where, from the preceding discussion, u and its (p - 2) time derivatives are continuous we have:

<u>Theorem 7</u>. If the Hamiltonian H is regular, $S \in B_{2p-1}[0, T]^{\dagger}$ and the extremal path has a boundary arc of non-zero length, then

$$\nu(t_1) = (-1)^p \frac{H^-_{uu}[\overset{p-1}{(u)}{}^- - \overset{p-1}{(u)}{}^+]^2}{\overset{2p-1}{(S)}{}^-} \geqslant 0 \tag{89}$$

where $()^-$ denotes () on the interior arc at the junction time t_1.

<u>Proof</u>. We use (65) and

$$\overset{p-1}{(H_u)}{}^- = \overset{p-1}{(H_u)}{}^+ \tag{90}$$

which holds across a junction; noting that

$$\begin{aligned}\overset{p-1}{(H_u)} &= \frac{d^{p-1}}{dt^{p-1}}(f_u^T\lambda) \\ &= f_{uu}^T\lambda\overset{p-1}{(u)} + \text{terms of lower order time derivatives of u} \\ &\qquad + \text{terms in } f_u f_x \text{ etc.}\end{aligned}$$

$\dagger$ B_{2p-1} ≡ space of all functions whose (2p-1)th derivatives exist and are bounded.

we have the following expression for $\nu(t_1)$ (after simplifying it with the aid of (65) and the above expression for $(\overset{p-1}{H_u})$):

$$\nu(t_1) = \frac{(-1)^p (f_{uu}^T)^+ \lambda^- [\overset{p-1}{(u)}{}^- - \overset{p-1}{(u)}{}^+]}{\overset{p}{(S)}{}_u^+} \qquad (91)$$

As $S \in B_{2p-1}[0, T]$ by assumption, we have, from the general expression

$$\overset{2p-1}{(S)} = \overset{p}{(S)}_u \overset{p-1}{(u)} + \text{lower order time derivatives of } u + \text{terms in } f, (\dot{f}) \text{ etc.}$$

the relation

$$\overset{2p-1}{(S)}{}^- - \overset{2p-1}{(S)}{}^+ = \overset{p}{(S)}{}_u^+ [\overset{p-1}{(u)}{}^- - \overset{p-1}{(u)}{}^+]$$

whence,†

$$\nu(t_1) = \frac{(-1)^p (H_{uu})^- [\overset{p-1}{(u)}{}^- - \overset{p-1}{(u)}{}^+]^2}{\overset{2p-1}{(S)}{}^-} \qquad (92)$$

This expression for $\nu(t_1)$ is very significant. Note that $H_{uu} > 0$ (strengthened necessary condition for a minimum), $[\overset{p-1}{(u)}{}^- - \overset{p-1}{(u)}{}^+]^2 \geq 0$ and as S and its time derivatives up to $\overset{2p-2}{(S)}$ are continuous (therefore zero)

$$\overset{2p-1}{(S)}{}^- > 0$$

for the trajectory to reach the boundary. This implies that

$$\nu(t_1) \leq 0 \qquad (93)$$

for p odd. But $\nu(t_1) \geq 0$ and hence (93) implies that for odd order constraints the trajectory will, at most, only touch the boundary, if $\overset{p-1}{(u)}{}^- \neq \overset{p-1}{(u)}{}^+$. Note that for $p = 1$, $u^- = u^+$, so that, from (91) $\nu(t_1) = 0$; thus for the first order case boundary arcs <u>are</u> permitted.

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† $\overset{2p-1}{(S)}{}^+ = 0$ as S and all its time derivatives are zero along boundary.

This behavior and (92) are reminiscent of junction conditions in singular control problems. This provides a further hint of a close connection between state-constrained and singular control problems which has been suggested elsewhere [25].

8. A third order problem

Third and fourth order state constrained problems are illustrated. The third order problem confirms the result of Section 7, as all optimal trajectories do not stay on the constraint boundary for any nonzero length of time.

Consider the following problem:

$$\underset{u}{\text{Minimize}} \int_0^1 \frac{u^2}{2}\, dt \tag{94}$$

subject to

$$\begin{cases} \dot{x}_1 = x_2 & x_1(0) = 0 = x_1(1) \\ \dot{x}_2 = x_3 & x_2(0) = 1 = -x_2(1) \\ \dot{x}_3 = u & x_3(0) = 2 = x_3(1) \end{cases} \tag{95}$$

and the constraint

$$x_1(t) - \ell \leq 0 \qquad t \in [0, 1] \tag{96}$$

where ℓ ranges as

$$\frac{3}{8} \geq \ell \geq 0 \quad . \tag{97}$$

The solution to the unconstrained problem is obtained first. The Hamiltonian H is

$$H = \frac{u^2}{2} + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 u \quad . \tag{98}$$

The adjoint equations are

$$\begin{cases} \dot{\lambda}_1 = 0 & \lambda_1(1) = \text{constant} \\ \dot{\lambda}_2 = -\lambda_1 & \lambda_2(1) = \text{constant} \\ \dot{\lambda}_3 = -\lambda_2 & \lambda_3(1) = \text{constant} \end{cases} \tag{99}$$

Minimizing the Hamiltonian gives the optimal control

$$H_u = 0 = u + \lambda_3 \Longrightarrow u = -\lambda_3 \tag{100}$$

The solution to the problem is:

Optimal control is u^o

$$u^o = 48(t - \frac{1}{2}) \quad . \tag{101}$$

Optimal trajectory

$$\begin{cases} x_1^o = 2t^4 - 4t^3 + t^2 + t \\ x_2^o = 8t^3 - 12t^2 + 2t + 1 \\ x_3^o = 24t^2 - 24t + 2 \end{cases} \tag{102}$$

Adjoint variable histories

$$\begin{cases} \lambda_1^o = 0 \\ \lambda_2^o = 48 \\ \lambda_3^o = 48(\frac{1}{2} - t) \end{cases} \tag{103}$$

Note that the constraint is not effective for $\ell > \frac{3}{8}$.

The solution to the constrained problem consists of two parts.

For ℓ in the range $\frac{9}{40} < \ell \leqslant \frac{3}{8}$ there is only one point of contact with the constraint boundary at $t = \frac{1}{2}$. The complete solution is

$$u^o = \begin{cases} at^2 + bt + c & 0 \leqslant t \leqslant \frac{1}{2} \\ -[a(1-t)^2 + b(1-t) + c] & \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \tag{104}$$

$$x_1^o = \begin{cases} \frac{at^5}{60} + \frac{bt^4}{24} + \frac{ct^3}{6} + t^2 + t & 0 \leqslant t \leqslant \frac{1}{2} \\ \frac{a(1-t)^5}{60} + \frac{b(1-t)^4}{24} + \frac{c(1-t)^3}{6} + (1-t)^2 + (1-t) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \tag{105}$$

$$x_2^o = \begin{cases} \frac{at^4}{12} + \frac{bt^3}{6} + \frac{ct^2}{2} + 2t + 1 & 0 \leqslant t \leqslant \frac{1}{2} \\ -[\frac{a(1-t)^4}{12} + \frac{b(1-t)^3}{6} + \frac{c(1-t)^2}{2} + 2(1-t) + 1] & \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \tag{106}$$

$$x_3^o = \begin{cases} \dfrac{at^3}{3} + \dfrac{bt^2}{2} + ct + 2 & 0 \leqslant t \leqslant \dfrac{1}{2} \\ \dfrac{a(1-t)^3}{3} + \dfrac{b(1-t)^2}{2} + c(1-t) + 2 & \dfrac{1}{2} \leqslant t \leqslant 1 \end{cases} \tag{107}$$

and the adjoint variable history is

$$\lambda_1 = \begin{cases} -2a & 0 \leqslant t \leqslant \dfrac{1}{2} \\ 2a & \dfrac{1}{2} \leqslant t \leqslant 1 \end{cases} \tag{108}$$

$$\lambda_2 = \begin{cases} 2at + b & 0 \leqslant t \leqslant \dfrac{1}{2} \\ 2a(1-t) + b & \dfrac{1}{2} \leqslant t \leqslant 1 \end{cases} \tag{109}$$

and

$$\lambda_3 = \begin{cases} -at^2 - bt - c & 0 \leqslant t \leqslant \dfrac{1}{2} \\ -a(1-t)^2 + b(1-t) + c & \dfrac{1}{2} \leqslant t \leqslant 1 \end{cases} \tag{110}$$

where

$$\begin{cases} a = 5120(\ell - \frac{3}{8}) \\ b = -3200(\ell - \frac{39}{100}) \\ c = 320(\ell - \frac{9}{20}) \end{cases} \tag{111}$$

When $\ell < \frac{9}{40}$, the single point of contact splits into two, instead of a boundary arc as would occur for the same constraint on $x_2(1)$ [27]. The solution is a trifle more complicated and consists of four parts: one prior to the first point of contact with the constraint boundary at time t_1 $(< \frac{1}{2})$, one for $t \in [t_1, \frac{1}{2}]$, one for $t \in [\frac{1}{2}, 1 - t_1]$ and one for $t \in [1 - t_1, 1]$. The times of contact are symmetric about $t = \frac{1}{2}$.

The optimal control u is:

$$u = \begin{cases} at^2 + bt + c & 0 \leq t < t_1 \\ dt + e & t_1 < t \leq \frac{1}{2} \\ -d(1 - t) - e & \frac{1}{2} < t \leq 1 - t_1 \\ -a(1 - t)^2 - b(1 - t) - c & 1 - t_1 < t \leq 1 \end{cases} \tag{112}$$

The optimal trajectory is

$$x_1 = \begin{cases} \frac{at^5}{60} + \frac{bt^4}{24} + \frac{ct^3}{6} + t^2 + t & 0 \leq t \leq t_1 \\ \frac{dt^4}{24} + \frac{et^3}{6} + \frac{At^2}{2} + Bt + C & t_1 \leq t \leq \frac{1}{2} \\ \frac{d(1 - t)^4}{24} + \frac{e(1 - t)^3}{6} + \frac{A(1 - t)^2}{2} + B(1 - t) + C & \frac{1}{2} \leq t \leq 1 - t_1 \\ \frac{a(1 - t)^5}{60} + \frac{b(1 - t)^4}{24} + \frac{c(1 - t)^3}{6} + (1 - t)^2 + (1 - t) & 1 - t_1 \leq t \leq 1 \end{cases} \tag{113}$$

$$x_2 = \begin{cases} \frac{at^4}{12} + \frac{bt^3}{6} + \frac{ct^2}{2} + 2t + 1 & 0 \leq t \leq t_1 \\ \frac{dt^3}{6} + \frac{et^2}{2} + At + B & t_1 \leq t \leq \frac{1}{2} \\ -\frac{d(1 - t)^3}{6} - \frac{e(1 - t)^2}{2} - A(1 - t) - B & \frac{1}{2} \leq t \leq 1 - t_1 \\ -\frac{a(1 - t)^4}{12} - \frac{b(1 - t)^3}{6} - \frac{c(1 - t)^2}{2} - 2(1 - t) - 1 & 1 - t_1 \leq t \leq 1 \end{cases} \tag{114}$$

and

$$x_3 = \begin{cases} \frac{at^3}{3} + \frac{bt^2}{2} + ct + 2 \\ \frac{dt^2}{2} + et + A \\ \frac{d(1 - t)^2}{2} + e(1 - t) + A \\ \frac{a(1 - t)^3}{3} + \frac{b(1 - t)^2}{2} + c(1 - t) + 2 \end{cases} \tag{115}$$

where

$$A \equiv \frac{dt_1^3}{3} + \frac{bt_1^2}{2} + ct_1 + 2 - \frac{dt_1^2}{2} - et_1 \tag{116}$$

$$B \equiv \frac{dt_1^4}{12} + \frac{bt_1^3}{6} + \frac{ct_1^2}{2} + 2t_1 + 1 - \frac{dt_1^3}{6} - \frac{et_1^2}{2} - At_1 \tag{117}$$

and

$$C \equiv \frac{at_1^5}{60} + \frac{bt_1^4}{24} + \frac{ct_1^3}{6} + t_1^2 + t_1 - \frac{dt_1^4}{24} - \frac{et_1^3}{6} - \frac{At_1^2}{2} - Bt_1 \quad . \tag{118}$$

The adjoint variable histories are

$$\lambda_1 = \begin{cases} -2a & 0 \leqslant t \leqslant t_1 \\ 0 & t_1 \leqslant t \leqslant 1 - t_1 \\ 2a & 1 - t_1 \leqslant t \leqslant 1 \end{cases} \tag{119}$$

$$\lambda_2 = \begin{cases} 2at + b & 0 \leqslant t \leqslant t_1 \\ d & t_1 \leqslant t \leqslant 1 - t_1 \\ +2a(1 - t) + b & 1 - t_1 \leqslant t \leqslant 1 \end{cases} \tag{120}$$

and

$$\lambda_3 = \begin{cases} -at^2 - bt - c & 0 \leqslant t \leqslant t_1 \\ -dt - e & t_1 \leqslant t \leqslant \frac{1}{2} \\ d(1 - t) + e & \frac{1}{2} \leqslant t \leqslant 1 - t_1 \\ a(1 - t)^2 + b(1 - t) + c & 1 - t_1 \leqslant t \leqslant 1 \end{cases} \tag{121}$$

The constants a, b, c, d, e and t_1 are related to ℓ by the following set of equations:

$$\begin{cases} d + 2e = 0 \\ at_1^2 + bt_1 + e - dt_1 - e = 0 \\ \frac{d}{48} + \frac{e}{8} + \frac{A}{2} + B = 0 \\ \frac{dt_1^4}{12} + \frac{bt_1^3}{6} + \frac{ct_1^2}{2} + 2t_1 + 1 = 0 \\ \frac{dt_1^5}{60} + \frac{bt_1^4}{24} + \frac{ct_1^3}{6} + t_1^2 + t_1 = \ell \\ 2at_1 + b - d = 0 \end{cases} \tag{122}$$

The equations (122) were treated as a set of linear equations in a, b, c, d, e and ℓ, and solutions found for t_1 in the range $[0, \frac{1}{2}]$. The times of contact are plotted vs ℓ in Fig. 1 which shows $x_1(\cdot)$ for various values of ℓ. Fig. 2 shows the corresponding control history. It is worth noting that, as the Hamiltonian is regular, the control u is continuous, as also is $\dot{u}$ but $\ddot{u}$ is discontinuous at t_1 and $1 - t_1$. For this problem, $\hat{\eta}(\cdot) = 0$.

9. A fourth order problem

We have set up this problem to demonstrate that a nonextremal can satisfy the necessary conditions of Bryson, Denham and Dreyfus [3] and Speyer [7].

Consider the following fourth order problem

$$\min_{u} \int_0^{10} \frac{u^2}{2}\, dt \tag{123}$$

subject to

$$\begin{cases} \dot{x}_1 = x_2 & x_1(0) = 0 = x_1(10) \\ \dot{x}_2 = x_3 & x_2(0) = -x_2(10) = \frac{15}{12} \\ \dot{x}_3 = x_4 & x_3(0) = x_3(10) = -\frac{15}{12} \\ \dot{x}_4 = u & x_4(0) = -x_4(10) = \frac{15}{16} \end{cases} \tag{124}$$

and the constraint

$$x_1(t) - 1 \leqslant 0 \qquad t \in [0, 10] \quad . \tag{125}$$

The following trajectory, consisting of a boundary arc between t = 4 and t = 6 and two interior arcs satisfies all the necessary conditions given in [7].

$$u_1(t) = \begin{cases} -\frac{15}{128}(4 - t) & 0 \leqslant t \leqslant 4 \\ 0 & 4 \leqslant t \leqslant 6 \\ \frac{15}{128}(6 - t) & 6 \leqslant t \leqslant 10 \end{cases} \tag{126}$$

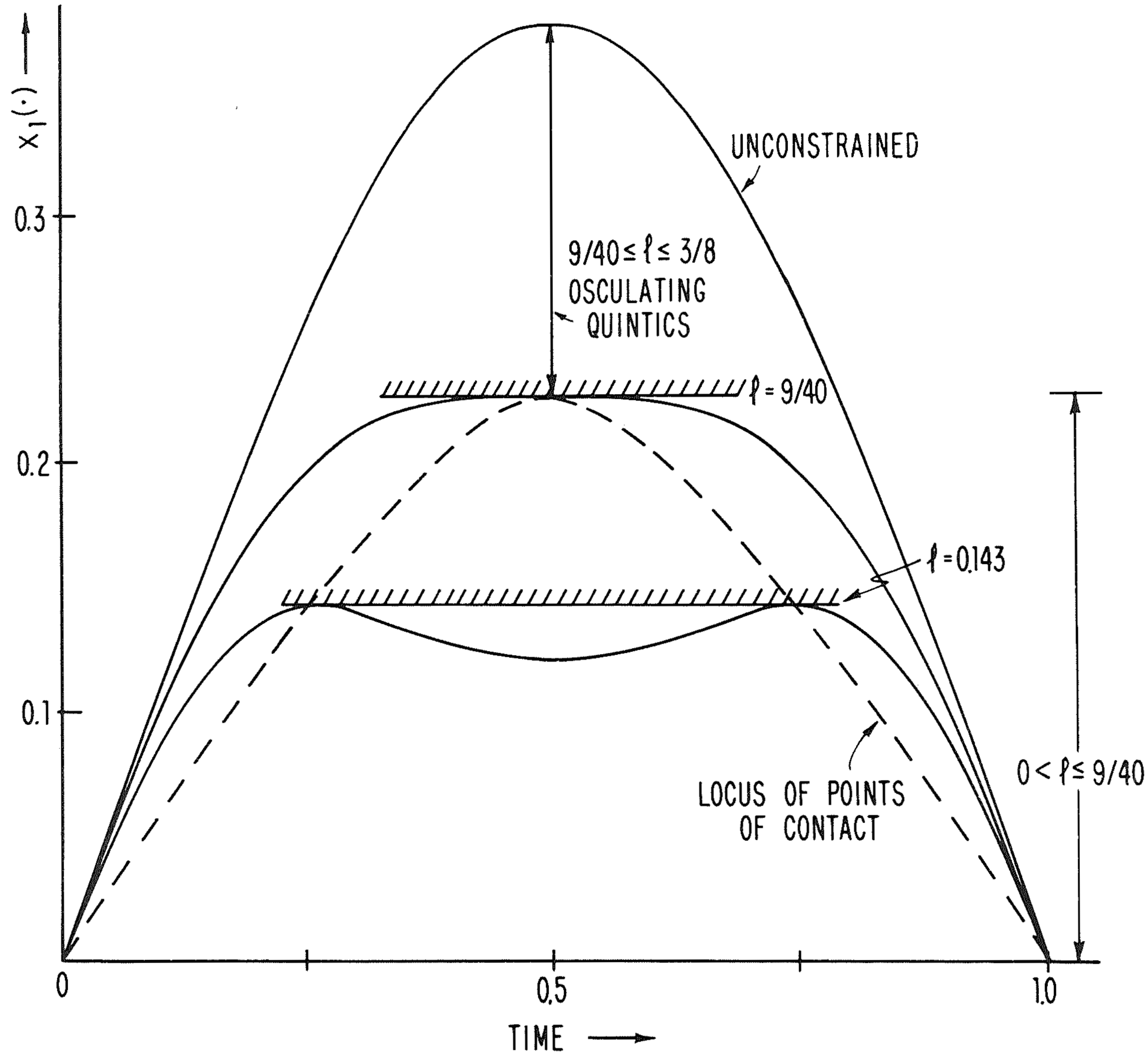

FIG. 1 THIRD ORDER PROBLEM $x_1(\cdot)$ vs. TIME

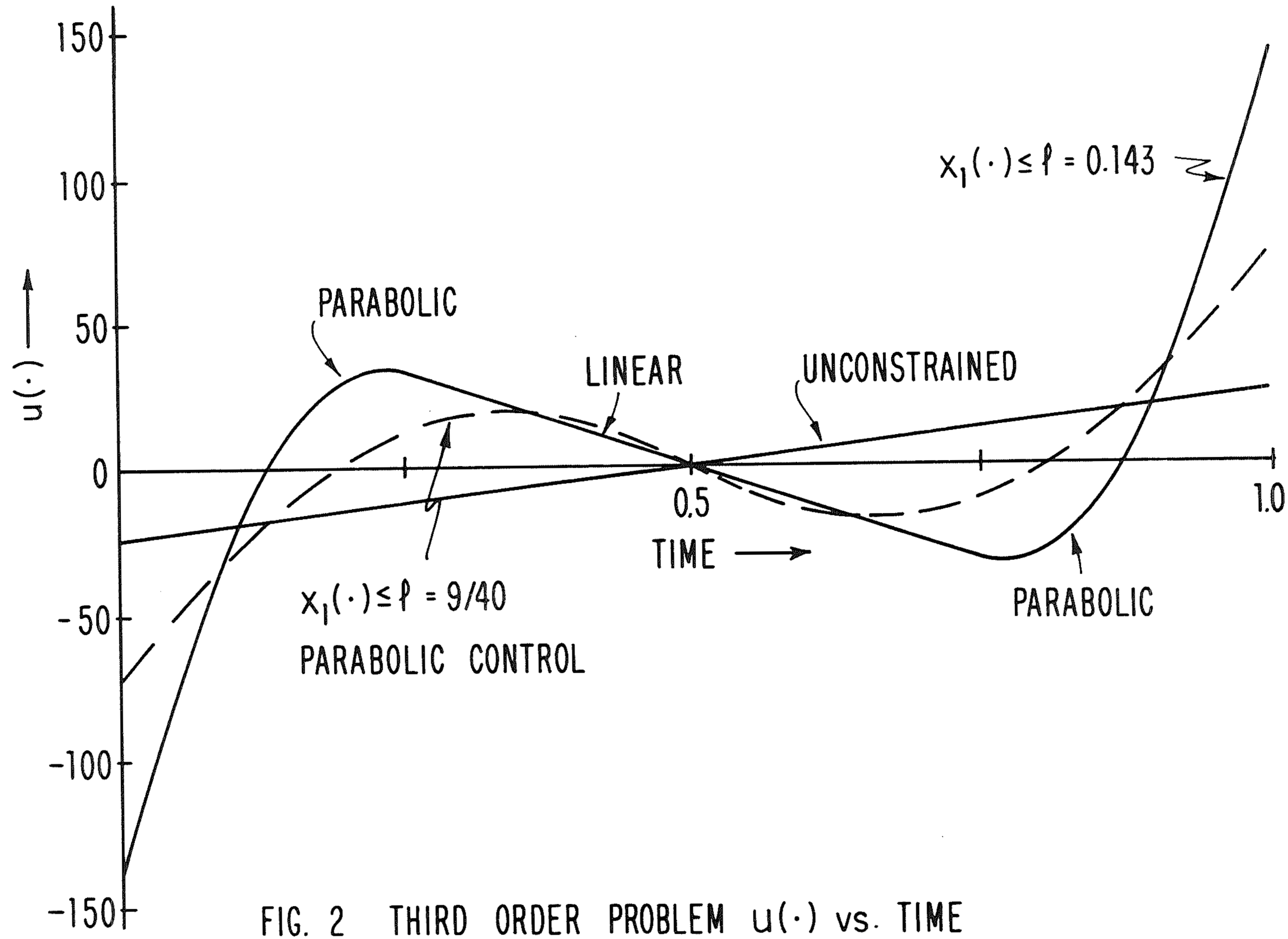

FIG. 2 THIRD ORDER PROBLEM u(·) vs. TIME

$$x_1(t) = \begin{cases} \ell - \dfrac{\alpha(4-t)^5}{120} & 0 \leqslant t \leqslant 4 \\ \ell & 4 \leqslant t \leqslant 6 \\ \ell + \dfrac{\alpha(6-t)^5}{120} & 6 \leqslant t \leqslant 10 \end{cases} \tag{127}$$

$$x_2(t) = \begin{cases} \dfrac{\alpha(4-t)^4}{24} & 0 \leqslant t \leqslant 4 \\ 0 & 4 \leqslant t \leqslant 6 \\ -\dfrac{\alpha(6-t)^4}{24} & 6 \leqslant t \leqslant 10 \end{cases} \tag{128}$$

$$x_3(t) = \begin{cases} -\dfrac{\alpha(4-t)^3}{6} & 0 \leqslant t \leqslant 4 \\ 0 & 4 \leqslant t \leqslant 6 \\ \dfrac{\alpha(6-t)^3}{6} & 6 \leqslant t \leqslant 10 \end{cases} \tag{129}$$

$$x_4(t) = \begin{cases} \dfrac{\alpha(4-t)^2}{2} & 0 \leqslant t \leqslant 4 \\ 0 & 4 \leqslant t \leqslant 6 \\ -\dfrac{\alpha(6-t)^2}{2} & 6 \leqslant t \leqslant 10 \end{cases} \tag{130}$$

where $\alpha = \frac{15}{128}$. The adjoint variables are

$$\lambda_1 = 0 \qquad 0 \leqslant t \leqslant 10 \tag{131}$$

$$\lambda_2 = 0 \tag{132}$$

$$\lambda_3 = \begin{cases} \dfrac{15}{128} & 0 \leqslant t \leqslant 4 \\ 0 & 4 \leqslant t \leqslant 6 \\ -\dfrac{15}{128} & 6 \leqslant t \leqslant 10 \end{cases} \tag{133}$$

and

$$\lambda_4 = -u \quad . \tag{134}$$

Here $\mu(\cdot) = 0$, $\nu_{S_1} = 0$, $\nu_{S_2} = 0$, $\nu_{S_3} = \frac{15}{128}$, and $\nu_{S_4} = 0$ at entry and exit. All of Speyer's optimality conditions are satisfied and the value of the cost functional is 0.293. Also, Bryson, Denham and Dreyfus' necessary conditions are satisfied with

$$\nu_{b_1} = 0, \quad \nu_{b_2} = 0, \quad \nu_{b_3} = \frac{30}{128}, \quad \nu_{b_4} = \frac{30}{128}$$

and

$$\gamma(\cdot) = 0 \qquad 0 \leq t \leq 4, \quad 6 \leq t \leq 10$$

$$\gamma(t) = \frac{15}{128}(6 - t), \qquad 4 \leq t \leq 6 \quad .$$

However, the unconstrained optimal trajectory given below, turns out to be feasible,† and gives a cost of 0.2897. Other stationary trajectories are ruled out as the problem is convex. This implies that the necessary conditions of Bryson, Denham and Dreyfus and Speyer have yielded a spurious extremal.

Unconstrained trajectory

$$u = bt^2 + ct + d \tag{135}$$

$$x_1 = \frac{bt^6}{360} + \frac{ct^5}{120} + \frac{dt^4}{24} + \frac{15t^3}{96} - \frac{15t^2}{24} + \frac{15t}{12} \tag{136}$$

$$x_2 = \frac{bt^5}{60} + \frac{ct^4}{24} + \frac{dt^3}{6} + \frac{15t^2}{32} - \frac{15t}{12} + \frac{15}{12} \tag{137}$$

$$x_3 = \frac{bt^4}{12} + \frac{ct^3}{6} + \frac{dt^2}{2} + \frac{15t}{16} - \frac{15}{12} \tag{138}$$

$$x_4 = \frac{bt^3}{3} + \frac{ct^2}{2} + dt + \frac{15}{16} \tag{139}$$

where $b = -0.02025$, $c = 0.2025$, $d = -0.525$. Fig. 3 shows x_1 for both cases.

10. Conclusions

We have considered the question of necessary conditions for optimality of state-constrained control problems. Two approaches were used. In the

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† Note that for certain initial conditions optimal trajectories will lie along the constraint boundary; expression (89) suggests this.

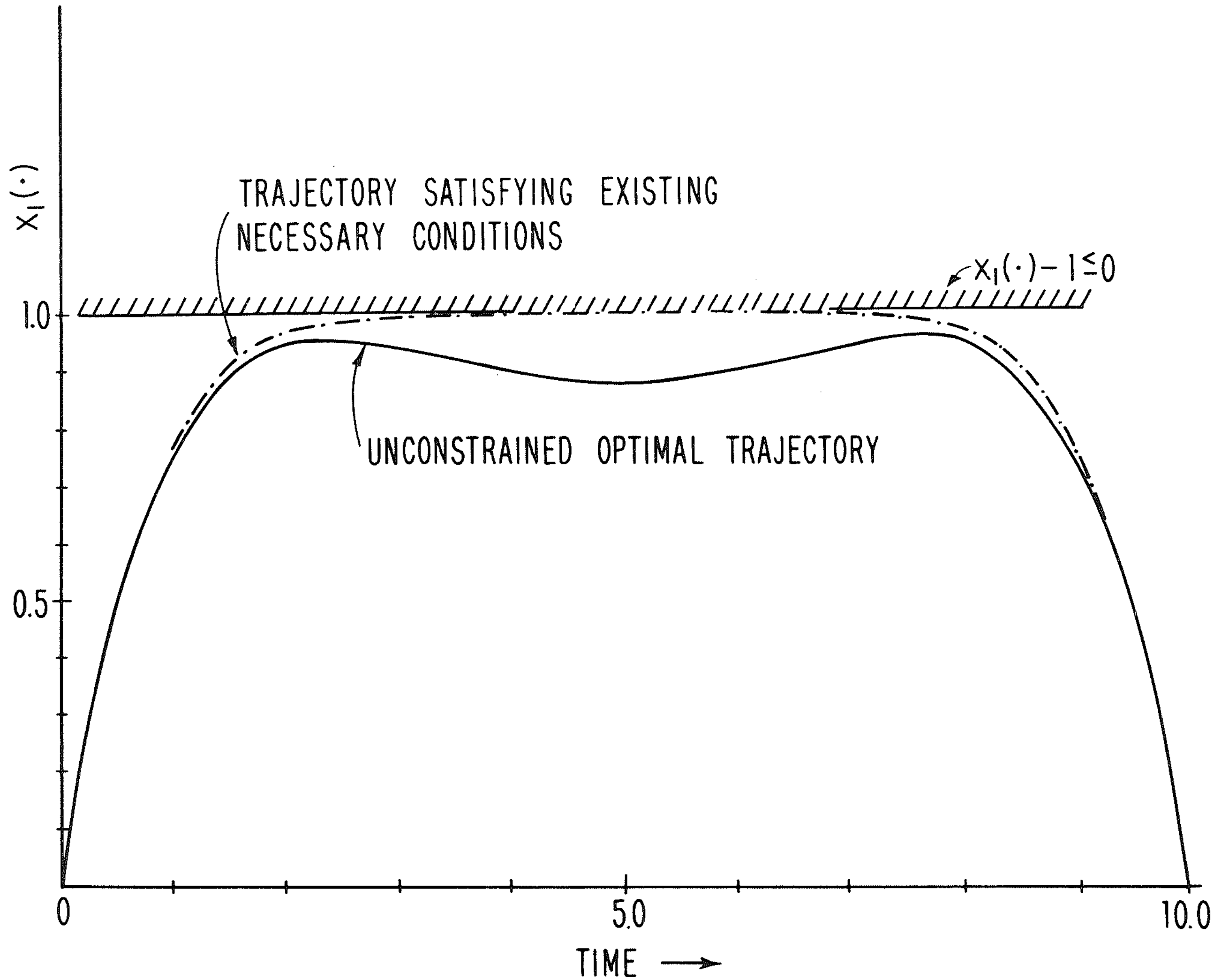

FIG. 3 FOURTH ORDER PROBLEM : $x_1(\cdot)$ vs. TIME

first, necessary conditions were obtained by limiting arguments based on the well-known Kelley penalty technique. The second approach utilized functional analysis and variational theory; the significant difference was that, unlike [3] and [7] no a priori constraints were imposed to ensure feasibility. Our necessary conditions yield a considerable simplification in the junction conditions on the influence functions over those obtained by previous researchers. We do not imply that the necessary conditions obtained by previous workers are incorrect, but rather, that, inasmuch as they overspecify the conditions at the junction, there exists the possibility of non-stationary solutions satisfying these conditions as shown in Section 9. Thus misleading results may be obtained using the existing necessary conditions. Our necessary conditions yield extremals.

For the regular case, we have discovered that, if $\overset{p-1}{(u)}{}^{-} \neq \overset{p-1}{(u)}{}^{+}$, problems with odd-ordered constraints do not have boundary arcs, (as opposed to boundary points). We feel that this result has a two-fold significance; first, it yields further insight into the structure of solutions of state constrained problems, and second, it provides one more clue towards the connection between state-constrained and singular problems, which has been speculated upon elsewhere [25].

The comparatively simple form of the new necessary conditions should stimulate research into new, efficient techniques for solving state constrained optimal control problems.

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13. ABSTRACT

Necessary conditions of optimality for state-variable inequality constrained problems are derived by examining the limiting behavior of the Kelley penalty function technique. The conditions so obtained differ from those presently known, with regard to the behavior of the adjoint variables at junctions of interior and boundary arcs. A second, rigorous, derivation is given; this confirms the necessary conditions obtained by the limiting argument. These conditions are related to those known earlier; in particular, it is shown that the earlier conditions over-specify the behavior of the adjoint variables at the junctions. An example is used to demonstrate that the earlier conditions may yield non-stationary trajectories.

For the regular case, it is shown that, under certain conditions, only boundary points, as opposed to boundary arcs, are possible. An analytic example illustrates this behavior.

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14. KEY WORDS	LINK A		LINK B		LINK C	
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Phase Variable Constraints						
Necessary Conditions						
Optimal Control Theory						
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