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COLLISION INTEGRALS FOR NONELASTIC PROCESSES IN PLASMA KINETIC THEORY

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16. Abstract

Often in the treatment of plasma dynamic problems the effects of nonelastic encounters must be included. In this note collision integrals for nonelastic processes that are important in plasma kinetic theory are developed. The following nonelastic interactions associated with the electron Boltzmann equation are presented in detail.

(1) Inelastic and superelastic encounters between electrons and neutral species,

(2) Ionization and three-body recombination encounters in which the third body is an electron, and

(3) Photoionization and two-body recombination encounters.

In addition, collision integrals are given for some of the processes that affect the transfer equations of heavy particles (ions and neutrals) and photons. Included here are radiative excitation and de-excitation encounters. The compatibility of these collision integrals with the forms associated with the usual macroscopic rate equations are demonstrated.

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CONTENTS

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nonelastic encounters must be included. In this note collision integrals for
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II. INTRODUCTION

Often in the treatment of plasma dynamic problems the effects of
nonelastic encounters must be included. Such effects are frequently of primary
importance in the calculation of electron distribution functions and in the
evaluation of electron continuity and energy equations (refs. 1-13). To a
lesser extent they can also play a role in transport property calculations
(refs. 1, 7, 13).

The purpose of this note is to present, under one report, in a consistent
form, collision integrals for nonelastic processes that are important in
plasma gasdynamic situations. These collision integrals will be presented in
the format usually associated with elastic collision integrals (ref. 14). It
is hoped that this format will lend itself to quick assimilation by the reader.

This paper represents some modifications and extensions of the work of
many authors. In addition to some of the previously cited references, the
work of Fowler (ref. 15) was found to be particularly helpful. Some of the

material presented in Section IV has also been published in reference 16. However, for convenience, it seemed appropriate to reproduce the material here.

This paper is organized as follows: The first part presents generalized nonelastic collision integrals for electrons that account for

(1) Inelastic encounters between electrons and neutral species,

(2) Ionization and three-body recombination encounters in which the third body is an electron, and

(3) Photoionization and two-body recombination encounters.

Next the nonelastic collision integrals that affect the distribution of neutral species, ionic species, and photons are treated briefly. Included are radiative excitation and de-excitation encounters. In the last section the compatibility of these collision integrals with the forms associated with the macroscopic rate equations is demonstrated.

III. COLLISION INTEGRALS FOR NONELASTIC PROCESSES ASSOCIATED WITH THE ELECTRON BOLTZMANN EQUATION

In a collision-dominated plasma composed of electrons, neutrals, and ions, the rate of change of the electron distribution function[1] $f(\vec{v}_s)$ as a result of nonelastic encounters can be represented as

$$\left(\frac{\partial_e f_s}{\partial t}\right)_{NE} = \left(\frac{\partial_e f_s}{\partial t}\right)_{Ex} + \left(\frac{\partial_e f_s}{\partial t}\right)_{Ion} + \left(\frac{\partial_e f_s}{\partial t}\right)_{Ph} \qquad (1)$$

where $f_s \equiv f(\vec{v}_s)$. The three collision terms on the right-hand side of equation (1) represent the nonelastic interactions considered most important in collision-dominated plasmas. They, respectively, account for

(1) Inelastic (excitation) and superelastic (de-excitation) encounters between electrons and neutral species,

(2) Ionization and three-body recombination encounters in which the third body is an electron, and

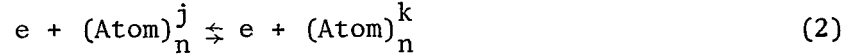(3) Photoionization and two-body recombination encounters.

In this section the collision integrals for these terms will be developed. The state of the heavy particles will be indicated by the subscript j, k, or

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[1]As is common practice in the treatment of collisions, the spatial and temporal dependence of the distribution functions are not shown explicitly.

ℓ added to the species subscript. These states are considered to be specified by the principal and total angular-momentum quantum numbers and degeneracies.

Inelastic and Superelastic Collisions

In this section collision integrals are developed for the conservation of electrons in the set $\vec{v}_s$, d^3v_s as a result of inelastic and superelastic encounters that cause neutrals to undergo excitation and de-excitation reactions, respectively. A typical interaction can be represented by

$$e + (Atom)_n^j \rightleftharpoons e + (Atom)_n^k \qquad (2)$$

where the subscript n stands for the type of neutral atoms and the superscripts j and k (k > j) correspond to the state of the neutral particle.

This type of encounter can be distinguished by whether the reaction causes the electron to be subtracted from or added to the set. When an excitation (j → k) of a neutral atom is initiated by an electron with velocity $\vec{v}_s$, the energy equation is

$$(1/2)m_e v_s^2 + (1/2)m_n v_{n_j}^2 = (1/2)m_e v_o^2 + (1/2)m_n v_{n_k}^2 + \Delta_{jk} \qquad (3)$$

Here $\vec{v}_o$ is the electron velocity after the collision, $\vec{v}_{n_j}$ and $\vec{v}_{n_k}$ are the velocities of the neutral particle before and after the encounter, respectively (the subscripts j and k serve the dual role of distinguishing between the neutral before and after collision and representing its state), and $\Delta_{jk} \equiv h\nu_{jk}$ is the excitation energy for the interaction. When an excitation results in an electron being added to the set, the energy equation is

$$(1/2)m_e v_1^2 + (1/2)m_n v_{n_j}^2 = (1/2)m_e v_s^2 + (1/2)m_n v_{n_k}^2 + \Delta_{jk} \qquad (4)$$

where $\vec{v}_1$ is the velocity of the electron before the inelastic collision. The inverse (superelastic) encounters, which also contribute to the number of electrons in the set, are described energetically by equations (3) and (4).

The number of electrons lost to the set $\vec{v}_s$, d^3v_s in unit time per unit volume as a result of inelastic collisions with neutrals of velocity range $\vec{v}_{n_j}$, $d^3v_{n_j}$, such that the neutrals are placed in the set $\vec{v}_{n_k}$, $d^3v_{n_k}$ and the electrons are scattered through an angle χ_{os}, $d\Omega_{os}$ measured relative to $\vec{v}_s - \vec{v}_{n_j}$ in the center-of-mass frame, with the velocity $\vec{v}_o$, d^3v_o is

3

$$f_s \, d^3v_s \, f_{n_j} \, d^3v_{n_j} \, g_{sj} \, \sigma^k_{sj}(g_{sj}, \chi_{os}) d\Omega_{os} \qquad (5)$$

This type of transition is written:

$$\left(\vec{v}_s, \vec{v}_{n_j} \rightarrow \vec{v}_o, \vec{v}_{n_k} \right)$$

In the preceding inelastic loss expression, $\sigma^k_{sj}(g_{sj}, \chi_{os})d\Omega_{os}$ is the differential scattering cross section for the excitation collision in question. The relative velocity g_{sj} is defined by $g_{sj} \equiv |\vec{v}_s - \vec{v}_{n_j}|$, $f_{n_j} \equiv f_n(\vec{v}_{n_j})$ is the velocity distribution function for the species n neutral in the state j normalized on the number density for this state.

In writing g_{sj} and χ_{os} as arguments of σ^k_{sj}, we have addressed ourselves to an examination of excitation collisions which cause heavy particle state changes that can be classified by the principal and total angular-momentum quantum numbers and degeneracies. The cross section should have enough information specified in its arguments to determine, along with the conservation equations, the velocities of both particles in the appropriate reference frame after the collision. In the dynamics of inelastic collisions in the center-of-mass coordinates, changes in the component contributions to the angular-momentum vector of the system will cause the plane of the particles after the encounter to differ from the plane of the particles before the encounter. If no strong external magnetic fields exist, the direction of the total orbital angular-momentum vector of the heavy particles can be taken to be arbitrary both before and after the collision. Only the magnitude of this vector is of interest. Then by symmetry the azimuthal dependence of the orientation of the relative velocity vector after the collision $\vec{g}_{ok}$ with respect to $\vec{g}_{sj}$ can be neglected. Thus only the angle χ_{os} between $\vec{g}_{ok}$ and $\vec{g}_{sj}$ needs to be specified to describe adequately our inelastic collisions.

This argument can also be illustrated by the interaction between two monoenergetic streams, one of which is taken to be the set of electrons having the velocity range $\vec{v}_s$, d^3v_s; the other is a stream of neutrals in the set $\vec{v}_{n_j}$, $d^3v_{n_j}$. As a result of the averaging effects of the streams the sum of the total angular-momentum vectors of the heavy particles can be expected to be zero. This initial symmetry would be preserved after the encounter allowing us to neglect the azimuthal dependence of $\vec{g}_{ok}$ relative to $\vec{g}_{sj}$. Thus, only the single deflection angle χ_{os} is needed as a parameter of the collision. Knowing χ_{os}, g_{sj} and using symmetry considerations one can find the velocity of the particles after the collision in the center-of-mass reference frame.

These same arguments also apply for the other inelastic (superelastic) collisions discussed in this paper.

4

The number of electrons lost to the set $\vec{v}_s$, $d^3 v_s$ in unit time per unit volume as a result of superelastic collisions described by $\left(\vec{v}_s, \vec{v}_{n_k} \rightarrow \vec{v}_1, \vec{v}_{n_j}\right)$ is

$$f_s \; d^3 v_s f_{n_k} d^3 v_{n_k} g_{sk} \sigma_{sk}^j (g_{sk}, \chi_{1s}) d\Omega_{1s} \qquad (6)$$

Here $\sigma_{sk}^j (g_{sk}, \chi_{1s}) d\Omega_{1s}$ is the differential cross section for this super-elastic encounter. The angles are measured in the center-of-mass reference frame relative to $\vec{g}_{sk}$. The number of electrons lost to the set per unit time per unit volume is then the sum of (5) and (6).

Similarly, the number of electrons gained by the set per unit volume per unit time as a result of the inverse encounters represented by $\left(\vec{v}_0, \vec{v}_{n_k} \rightarrow \vec{v}_s, \vec{v}_{n_j}\right)$ and $\left(\vec{v}_1, \vec{v}_{n_j} \rightarrow \vec{v}_s, \vec{v}_{n_k}\right)$ is

$$f_0 \; d^3 v_0 f_{n_k} d^3 v_{n_k} g_{ok} \sigma_{ok}^j (g_{ok}, \chi_{os}) d\Omega_{os} \qquad (7)$$

and

$$f_1 \; d^3 v_1 f_{n_j} d^3 v_{n_j} g_{1j} \sigma_{1j}^k (g_{1j}, \chi_{1s}) d\Omega_{1s} \qquad (8)^2$$

Expression (7) is the (superelastic) inverse of (5). Expression (8) represents the inverse of (6). Typically the relative velocity g_{ok} is defined by $g_{ok} \equiv |\vec{v}_0 - \vec{v}_{n_k}|$.

A simplified expression is obtained for the net gain of electrons to the set per unit volume for the reaction given by (2) if the principle of detailed balancing is applied to the inverses among expressions (5), (6), (7), and (8) before the integrations are made over scattering angles and particle velocities. Applying this principle, we equate (5) and its inverse (7) at thermodynamic equilibrium to obtain

$$\left(f_s f_{n_j}\right)_{Eq} g_{sj} \sigma_{sj}^k (g_{sj}, \chi_{os}) d^3 v_s \; d^3 v_{n_j} = \left(f_0 f_{n_k}\right)_{Eq} g_{ok} \sigma_{ok}^j (g_{ok}, \chi_{os}) d^3 v_0 \; d^3 v_{n_k} \qquad (9)$$

The equilibrium distribution function for f_s is given as

$$\left(f_s\right)_{Eq} = \left(n_e\right)_{Eq} \left(\frac{m_e}{2\pi kT}\right)^{3/2} e^{-m_e v_s^2 / 2kT} \qquad (10)$$

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[2]To be consistent these angles should be labeled χ_{so}, $d\Omega_{so}$ and χ_{s1}, $d\Omega_{s1}$; however, it is easy to deduce that for inverse collisions as described here $\vec{\chi}_{so} = \vec{\chi}_{os}$ and $\vec{\chi}_{s1} = \vec{\chi}_{1s}$.

where T is the equilibrium temperature, with corresponding expressions for f_0, f_{n_k}, and f_{n_j}. The relative populations of various excited levels of an atom at equilibrium are given by the Boltzmann distribution

$$\left(\frac{n_{n_k}}{n_{n_j}}\right)_{Eq} = \frac{\omega_k}{\omega_j} e^{-\Delta_{jk}/kT} \tag{11}$$

where ω_k and ω_j are the atomic degeneracies associated with the states j and k of the neutral particles. The differential velocity elements in equation (9) can be related via their Jacobian as

$$d^3v_{n_k} d^3v_o = \left(\frac{g_{ok}}{g_{sj}}\right) d^3v_{n_j} d^3v_s \tag{12}$$

Then the combination of equations (10), (11), (12), and (3) with (9) yields the following detailed balancing result:

$$g_{ok}^2 \sigma_{ok}^j (g_{ok}, \chi_{os}) = \frac{\omega_j}{\omega_k} g_{sj}^2 \sigma_{sj}^k (g_{sj}, \chi_{os}) \tag{13}$$

A similar detailed balancing analysis with (6) and (8) yields

$$g_{sk}^2 \sigma_{sk}^j (g_{sk}, \chi_{1s}) = \frac{\omega_j}{\omega_k} g_{1j}^2 \sigma_{1j}^k (g_{1j}, \chi_{1s}) \tag{14}$$

The differential velocity elements for this encounter are related by

$$d^3v_{n_k} d^3v_s = \left(\frac{g_{sk}}{g_{1j}}\right) d^3v_{n_j} d^3v_1 \tag{15}$$

If the electron velocities before a superelastic and an inelastic encounter are denoted by $\vec{v}_S$ and $\vec{v}_I$, respectively, it is easy to show that equations (13) and (14), relating the differential cross sections, can both be written as the following general detailed balancing result:

$$g_{Sk}^2 \sigma_{Sk}^j (g_{Sk}, \chi) = \frac{\omega_j}{\omega_k} g_{Ij}^2 \sigma_{Ij}^k (g_{Ij}, \chi) \tag{16}$$

The terms g_{Sk} and g_{Ij} in equation (16) are defined as

$$g_{Sk} \equiv |\vec{v}_S - \vec{v}_{n_k}|$$

and

6

$$g_{Ij} \equiv |\vec{v}_I - \vec{v}_{n_j}|$$

The velocity elements in this general notation are related by

$$d^3v_{n_k} d^3v_S = \frac{g_{Sk}}{g_{Ij}} d^3v_{n_j} d^3v_I \tag{17}$$

Equations (12), (13), (14), and (15) can be combined with (5), (6), (7), and (8) to yield an expression for the net number of electrons gained by the set $\vec{v}_s$, d^3v_s by the reactions represented by $(\vec{v}_o, \vec{v}_{n_k} \leftrightarrow \vec{v}_s, \vec{v}_{n_j})$ and $(\vec{v}_1, \vec{v}_{n_j} \leftrightarrow \vec{v}_s, \vec{v}_{n_k})$. When the resulting expression is then integrated over all possible scattering angles and heavy particle velocities, the following expression is obtained for the net gain of electrons to the set per unit volume per unit time for the reaction given by expression (2):

$$\left(\frac{\partial_e f_s}{\partial t}\right)_{\substack{Ex \\ j \leftrightarrow k}} d^3v_s = d^3v_s \left[\iint \left(f_o f_{n_k} \frac{\omega_j}{\omega_k} - f_s f_{n_j} \right) g_{sj} \sigma_{sj}^k \, d\Omega_{os} \, d^3v_{n_j} \right.$$

$$\left. + \iint \left(f_1 f_{n_j} \frac{\omega_k}{\omega_j} - f_s f_{n_k} \right) g_{sk} \sigma_{sk}^j \, d\Omega_{1s} \, d^3v_{n_k} \right] \tag{18}$$
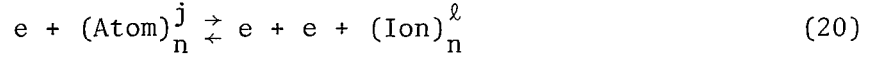
The sum of (18) over the various neutral species and states of these neutral species divided by d^3v_s is

$$\left(\frac{\partial_e f_s}{\partial t}\right)_{Ex} = \sum_{n,k,j} \left[\iint \left(f_o f_{n_k} \frac{\omega_j}{\omega_k} - f_s f_{n_j} \right) g_{sj} \sigma_{sj}^k \, d\Omega_{os} \, d^3v_{n_j} \right.$$

$$\left. + \iint \left(f_1 f_{n_j} \frac{\omega_k}{\omega_j} - f_s f_{n_k} \right) g_{sk} \sigma_{sk}^j \, d\Omega_{1s} \, d^3v_{n_k} \right] \tag{19}$$

In this equation the first collision operator represents the superelastic gain and the inelastic loss to the electron set $\vec{v}_s$, d^3v_s for all possible species n and for all possible values of j and k. The second collision operator represents the inelastic gain and the superelastic loss to this set for all possible species n and for all possible values of j and k.

Ionization and Three-Body Recombination Encounters

In this section collision integrals are developed for the conservation of electrons in the set $\vec{v}_s, d^3v_s$ during collisional ionization and three-body recombination encounters. A typical reaction in this case can be represented by

$$e + (Atom)_n^j \; \underset{\leftarrow}{\rightarrow} \; e + e + (Ion)_n^\ell \qquad (20)$$

Here, as previously, the subscript n corresponds to the type of neutral atom, and the superscripts j and ℓ signify the state of the neutral atom and its related ion, respectively.

When the way that electrons enter or leave the set $\vec{v}_s, d^3v_s$ is of concern, the energy equations for this type of encounter become:

$$\frac{1}{2}m_e v_s^2 + \frac{1}{2}m_n v_{n_j}^2 = \frac{1}{2}m_e v_2^2 + \frac{1}{2}m_e v_4^2 + \frac{1}{2}m_i v_{i_\ell}^2 + \Delta_{j\ell} \qquad (21)$$

when the ionization is caused by a v_s electron and

$$\frac{1}{2}m_e v_1^2 + \frac{1}{2}m_n v_{n_j}^2 = \frac{1}{2}m_e v_s^2 + \frac{1}{2}m_e v_3^2 + \frac{1}{2}m_i v_{i_\ell}^2 + \Delta_{j\ell} \qquad (22)$$

when the ionization results in an electron in the set. In these energy equations m_i and $\vec{v}_{i_\ell}$ are the mass and velocity, respectively, of the ionized type n neutral atom in the state distinguished by the subscript ℓ, and $\Delta_{j\ell}$ is the ionization potential of the atom from its jth state of excitation to its related ion in the ℓth state. In equation (21) $\vec{v}_2$ and $\vec{v}_4$ are the velocities of the two electrons resulting from the ionization. Making use of the principle of indistinguishability (ref. 15) we will not distinguish between bound and unbound electrons during an ionization or three-body recombination encounter. Thus, the electron that corresponded to the ionizing electron after the ionization will not be specified; it will merely be stated that two electrons result from the ionization. Likewise in a recombination encounter, no attempt is made to identify which of the two free electrons becomes bound and which remains free. Similarly, in equation (22) a $\vec{v}_1$ electron caused an ionization which resulted in two electrons, one with velocity $\vec{v}_s$ and the other with velocity $\vec{v}_3$. It should be noted that the $\vec{v}_1$ appearing in this section is not the $\vec{v}_1$ that appears in Section III (p. 3). They are distinguished by the type interactions being considered.

The number of electrons lost to the set $\vec{v}_s d^3v_s$ in unit time per unit volume as a result of ionization encounters with neutrals, such that the following transition occurs

$$(\vec{v}_s, \vec{v}_{n_j} \rightarrow \vec{v}_2, \vec{v}_4, \vec{v}_{i_\ell})$$

8

is

$$f_s \, d^3v_s f_{n_j} \, d^3v_{n_j} g_{sj} \sigma^\ell_{sj} \left(g_{sj}; \chi_{2s}, \vec{v}_{42_{CM}} \right) d^3v_{42_{CM}} \, d\Omega_{2s} \tag{23}$$

Here $\sigma^\ell_{sj}(g_{sj}; \chi_{2s}, \vec{v}_{42_{CM}}) d^3v_{42_{CM}} \, d\Omega_{2s}$ is defined as the differential "cross section" for an ionization encounter between electrons in the set $\vec{v}_s, d^3v_s$ and neutrals in the set $\vec{v}_{n_j}, d^3v_{n_j}$ resulting in two electrons, their veloc-ities given by $\vec{v}_2, d^3v_2$ and $\vec{v}_4, d^3v_4$, and an ion in the state ℓ with its velocity range and state signified by $\vec{v}_{i_\ell}, d^3v_{i_\ell}$. Here the semicolon in the argument of the cross section separates before collision parameters from after collision parameters. The angle χ_{2s} is measured relative to the direction of $\vec{g}_{sj}$ in the center-of-mass frame. The velocity $\vec{v}_{42_{CM}}$ is defined by $(v_{4_{CM}}, \vec{\chi}_{42_{CM}})$, where $\vec{\chi}_{42_{CM}}$ is the direction of v_4 measured relative to v_2 in the center-of-mass frame. The center-of-mass subscript CM is used here to avoid confusion with non-CM velocities.

Again symmetry arguments have been used to arrive at the above choice of cross-section parameters. As presented, sufficient information has been specified that the velocities of the particles after the encounter can be determined when symmetry considerations are included with the conservation equations and the initial velocities. There is no single "correct" way to designate a differential cross section of this complexity. For example, let us examine qualitatively the dynamics of an ionizing collision as regards a choice of parameters for the cross section. Consider the interaction of a monoenergetic beam of electrons with a monoenergetic beam of neutrals which results in ionization of the neutrals. Before the ionization encounter between an electron and a neutral particle, the two particles lie in a single plane in their center-of-mass reference frame. After the encounter, however, the resulting three particles are not restricted to move in a single plane in this reference frame. Averaging all such encounters in the beams will still give a symmetry of sorts about the relative velocity vector $\vec{g}_{sj}$; that is, the resultant momentum vector of any two of the scattered particles (say the elec-trons) from a single collision can be found equally likely in any azimuth about $\vec{g}_{sj}$ as will the momentum vector of the third particle. These momentum vec-tors, that of the above resultant and that of the third particle, are equal, antiparallel, and lie in a plane which has the same symmetry about $\vec{g}_{sj}$ as the particles on page 4 after an excitation collision. This symmetry can be represented here if any of the three velocity vectors is allowed to be arbi-trarily located in an axial sense with respect to $\vec{g}_{sj}$. We choose $\vec{v}_{2_{CM}}$ to have this symmetric character and thus only specify χ_{2s} in σ^ℓ_{sj}. The collision is then completely described by specifying the velocity of one of the remaining particles relative to $\vec{v}_{2_{CM}}$. We picked $\vec{v}_4$ for this distinc-tion. It is important to note the arbitrariness of these choices. Thus some other choice of parameters could have been made. It may appear from the above choice that the "created" electron must have the velocity $\vec{v}_4$. This is not the case. Consistent with the principle of indistinguishability we have

9

merely stated the velocity of one of the resulting particles without specifying which particle it was prior to the encounter.

The number of electrons lost to the set $\vec{v}_s$, d^3v_s in unit time per unit volume as a result of three-body recombination encounters with ions and other electrons such that the $(\vec{v}_s, \vec{v}_3, \vec{v}_{i_\ell} \rightarrow \vec{v}_1, \vec{v}_{n_j})$ transition occurs is (ref. 15)

$$f_s\, d^3v_s f_{i_\ell}\, d^3v_{i_\ell} f_3\, d^3v_3 g_{s\ell} g_{3\ell} \left[\sigma_{s3\ell}^j\left(g_{s\ell}, g_{3\ell}, \vec{\chi}_{g_3 g_s}; \chi_{1s}\right) d\Omega_{1s}\right.$$

$$\left. + \sigma_{s3\ell}^j\left(g_{3\ell}, g_{s\ell}, \vec{\chi}_{g_s g_3}; \chi_{13}\right) d\Omega_{13}\right] \left(1 + \delta_{\vec{v}_3, \vec{v}_s}\right) \qquad (24)$$

The relative velocities are defined as

$$g_{s\ell} \equiv \left|\vec{v}_s - \vec{v}_{i_\ell}\right|$$

and

$$g_{3\ell} \equiv \left|\vec{v}_3 - \vec{v}_{i_\ell}\right|$$

Two cross sections appear in expression (24) because there are two ways in which electrons can be lost to the set $\vec{v}_s$, d^3v_s by the $(\vec{v}_s, \vec{v}_3, \vec{v}_{i_\ell} \rightarrow \vec{v}_1, \vec{v}_{n_j})$ reaction. That is, on momentarily distinguishing between electrons, either the v_s electron can become bound to the ion and the v_3 electron becomes the v_1 electron or the v_s electron can become the v_1 electron while the v_3 electron becomes bound to the ion. Thus $\sigma_{s3\ell}^j\, d\Omega_{1s}$ is the "differential cross section" for the recombination encounter in which $s \rightarrow 1$ and $\sigma_{3s\ell}^j\, d\Omega_{13}$ is the "differential cross section" for the recombination encounter in which $3 \rightarrow 1$. From the principle of indistinguishability we can say

$$\sigma_{s3\ell}^j\, d\Omega_{1s} = \sigma_{3s\ell}^j\, d\Omega_{13}$$

Thus, expression (24) becomes

$$2f_s\, d^3v_s f_3\, d^3v_3 f_{i_\ell}\, d^3v_{i_\ell} g_{s\ell} g_{3\ell} \sigma_{s3\ell}^j\left(g_{s\ell}, g_{3\ell}, \vec{\chi}_{g_3 g_s}; \chi_{1s}\right) d\Omega_{1s}\left(1 + \delta_{\vec{v}_3, \vec{v}_s}\right) \qquad (25)$$

We have taken $\sigma_{s3\ell}^j\left(g_{s\ell}, g_{3\ell}, \vec{\chi}_{g_3 g_s}; \chi_{1s}\right) d\Omega_{1s}$ as the "differential cross section" for this recombination collision. After this encounter the remaining free electron is scattered into the angle χ_{1s}, $d\Omega_{1s}$ measured relative to $\vec{v}_{s_{CM}}$. The angle between $\vec{g}_{3\ell}$ and $\vec{g}_{s\ell}$ is given by $\vec{\chi}_{g_3 g_s}$. Again, the choice of argument for the recombination cross section is not unique, and also we do not

10

specify which of the two free electrons becomes bound (the cross section is symmetric in the subscripts s and 3). The delta function $\delta_{\vec{v}_3,\vec{v}_s}$ is included in (25) to account for the case when both electrons participating in the recombination encounter are in the set $\vec{v}_s$, d^3v_s. This function is defined as follows:

$$\delta_{\vec{v}_3,\vec{v}_s} = \begin{cases} 0 \; ; & \vec{v}_3 \neq \vec{v}_s \\ 1 \; ; & \vec{v}_3 = \vec{v}_s \end{cases}$$

Thus, for any smooth finite function $K(\vec{v}_3)$

$$\int K(\vec{v}_3)\delta_{\vec{v}_3,\vec{v}_s} \, d^3v_3 = 0$$

so that for practical purposes the delta function can be omitted. In an analogous manner, the number of electrons *gained* by the set per unit volume per unit time as a result of the inverse recombination $(\vec{v}_2,\vec{v}_4,\vec{v}_{i\ell} \rightarrow \vec{v}_s,\vec{v}_{n_j})$ and ionization $(\vec{v}_1,\vec{v}_{n_j} \rightarrow \vec{v}_s,\vec{v}_3,\vec{v}_{i\ell})$ interactions are, respectively,

$$f_2 \; d^3v_2 f_4 \; d^3v_4 f_{i_\ell} \; d^3v_{i_\ell} g_{2\ell} g_{4\ell} \sigma^j_{24\ell}\left(g_{2\ell},g_{4\ell},\vec{\chi}_{g_4g_2};\chi_{s2}\right)d\Omega_{s2} \tag{26}$$

and

$$2f_1 \; d^3v_1 f_{n_j} \; d^3v_{n_j} g_{1j} \sigma^\ell_{1j}\left(g_{1j};\chi_{s1},\vec{v}_{3s_{CM}}\right)d\Omega_{s1} \; d^3v_{3s_{CM}}\left(1 + \delta_{\vec{v}_3,\vec{v}_s}\right) \tag{27}$$

In expression (27) the coefficient 2 accounts for the two ways in which electrons can enter the set via the $(\vec{v}_1,\vec{v}_{n_j} \rightarrow \vec{v}_s,\vec{v}_3,\vec{v}_{i\ell})$ interaction, that is, either $\vec{v}_1 \rightarrow \vec{v}_s$ and the bound electron becomes $\vec{v}_3$, or $\vec{v}_1 \rightarrow \vec{v}_3$ and the bound electron becomes $\vec{v}_s$, and the cross sections for these reactions, which are symmetric in the subscripts s and 3, have been set equal on the basis of indistinguishability arguments. As a result of the collisions being inverses, and the symmetry discussion following expression (23), the scattering angles in (26) and (27) are the same as those for the direct collisions and will subsequently be treated accordingly. Note that the angles are measured relative to velocity sets and not relative to particular particles. The delta function in (27) is to account for the possibility of the ionization resulting in two electrons entering the set.

As in the previous section we will derive a simplified expression for the net gain of electrons to the set per unit volume for the reaction given by (20) through the principle of detailed balancing. Applying this principle expression (23) and its inverse (26) are equated at equilibrium to obtain

$$\left(f_s f_{n_j}\right)_{Eq} g_{sj} \sigma^\ell_{sj} \; d^3v_{42_{CM}} \; d^3v_s \; d^3v_{n_j} = \left(f_2 f_4 f_{i_\ell}\right)_{Eq} g_{2\ell} g_{4\ell} \sigma^j_{24\ell} \; d^3v_2 \; d^3v_4 \; d^3v_{i_\ell} \tag{28}$$

11

At equilibrium the distribution functions are Maxwellian:

$$(f_m)_{Eq} = (n_e)_{Eq} \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v_m^2}, \qquad m = s, 2, 4 \tag{29}$$

where $\beta \equiv m_e/2kT$, with corresponding expressions for f_{n_j} and f_{i_ℓ}. The number densities of the ions, the neutrals and the electrons are related at equilibrium via the Saha equation, which can be written

$$\left(\frac{n_{i_\ell} n_e}{n_{n_j}}\right)_{Eq} = \frac{m_e^3}{h^3} \left(\frac{\pi}{\beta}\right)^{3/2} \frac{2\omega_\ell}{\omega_j} e^{-\Delta_{j\ell}/kT} \tag{30}[3]$$

In equation (30) ω_j and ω_ℓ are the degeneracies for the type n atom in its jth excitation level and its related ion in the ℓth excitation level. Combining equations (21), (29), and (30) with (28) the following detailed balancing result is obtained:

$$g_{2\ell} g_{4\ell} \sigma_{24\ell}^j \, d^3 v_2 \, d^3 v_4 \, d^3 v_{i_\ell} = H g_{sj} \sigma_{sj}^\ell \, d^3 v_{42_{CM}} \, d^3 v_s \, d^3 v_{n_j} \tag{31}$$

where

$$H \equiv \frac{h^3}{m_e^3} \frac{\omega_j}{2\omega_\ell}$$

A similar detailed balancing analysis with expressions (25) and (27) yields

$$g_{s\ell} g_{3\ell} \sigma_{s3\ell}^j \, d^3 v_s \, d^3 v_3 \, d^3 v_\ell = H g_{1j} \sigma_{1j}^\ell \, d^3 v_{3s_{CM}} \, d^3 v_1 \, d^3 v_j \tag{32}$$

As for inelastic and superelastic encounters, equations (31) and (32) can both be included in a single expression. If subscript Z denotes the electron causing the ionization and subscripts R and B denote the electrons resulting from the ionization, equations (31) and (32) can, consistent with previous notation, be included in the following equation:

$$g_{R\ell} g_{B\ell} \sigma_{RB\ell}^j \, d^3 v_R \, d^3 v_B \, d^3 v_{i_\ell} = H g_{Zj} \sigma_{Zj}^\ell \, d^3 v_{BR_{CM}} \, d^3 v_Z \, d^3 v_{n_j} \tag{33}$$

Since the following relations between the differential velocity elements hold:

$$d^3 v_R \, d^3 v_B \, d^3 v_{i_\ell} = d^3 G \, d^3 g_{R\ell} \, d^3 g_{B\ell}$$

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[3]As written equation (30) includes the approximation that $m_i = m_n$.

and

$$d^3v_Z \, d^3v_{n_j} = d^3G \, d^3g_{Zj}$$

where $\vec{G}$ is the velocity of the center of mass, equation (33) can be written as

$$g_{R\ell} g_{B\ell} \sigma_{RB\ell}^j \, d^3g_{R\ell} \, d^3g_{B\ell} = H g_{Zj} \sigma_{Zj}^{\ell} \, d^3v_{BR_{CM}} \, d^3g_{Zj} \tag{34}$$

Equations (31) and (32) can be combined with (23), (25), (26), and (27) to yield an expression for the net number of electrons gained by the set $\vec{v}_s$, d^3v_s as a result of the interactions represented by $(\vec{v}_s, \vec{v}_{n_j} \leftrightarrow \vec{v}_2, \vec{v}_4, \vec{v}_{i_\ell})$ and $(\vec{v}_s, \vec{v}_3, \vec{v}_{i_\ell} \leftrightarrow \vec{v}_1, \vec{v}_{n_j})$. This resulting expression can then be integrated over all possible scattering angles and heavy particle velocities to obtain

$$\left(\frac{\partial_e f_s}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}} d^3v_s = d^3v_s \left[\iiint \left(f_2 f_4 f_{i_\ell} H - f_s f_{n_j}\right) g_{sj} \sigma_{sj}^{\ell} \, d\Omega_{2s} \, d^3v_{42_{CM}} \, d^3v_{n_j} \right.$$

$$\left. + 2 \iiint \left(f_1 f_{n_j} H^{-1} - f_s f_3 f_{i_\ell}\right) g_{s\ell} g_{3\ell} \sigma_{s3\ell}^j \, d\Omega_{1s} \, d^3v_3 \, d^3v_{i_\ell} \right] \tag{35}$$

This equation represents the net gain of electrons to the set per unit volume per unit time for the reaction given by expression (20).
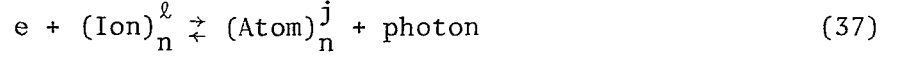
Now, on summing over the relevant neutral species and the excited states of the neutral species and its ion and dividing by d^3v_s, we obtain the following collision integral for the net gain of electrons to the set per unit volume of phase space per unit time as a result of collisional ionization and three-body recombination encounters:

$$\left(\frac{\partial_e f_s}{\partial t}\right)_{Ion} = \sum_{n,\ell,j} \left[\iiint \left(f_2 f_4 f_{i_{n\ell}} H - f_s f_{n_j}\right) g_{sj} \sigma_{sj}^{\ell} \, d\Omega_{2s} \, d^3v_{42_{CM}} \, d^3v_{n_j} \right.$$

$$\left. + 2 \iiint \left(f_1 f_{n_j} H^{-1} - f_s f_3 f_{i_{n\ell}}\right) g_{s\ell} g_{3\ell} \sigma_{s3\ell}^j \, d\Omega_{1s} \, d^3v_3 \, d^3v_{i_\ell} \right] \tag{36}$$

Here the first collision integral represents the gain by three-body recombination and the loss by ionization to the set $\vec{v}_s$, d^3v_s; the second collision integral represents the gain by ionization and the loss by three-body recombination to the set.

13

Photoionization and Two-Body Recombination Encounters

The remaining nonelastic collision term is that associated with photoionization. Physically, a photon of energy $h\nu$ is absorbed by an atom in the state j, resulting in an ion in the state ℓ and a free electron. The reverse reaction is when a free electron and an ion combine with the emission of radiation. A typical reaction for this case can be represented by

$$e + (\text{Ion})_n^\ell \rightleftarrows (\text{Atom})_n^j + \text{photon} \qquad (37)$$

In terms of the relative velocity $\vec{g}_{s\ell}$ the energy equation for this encounter is

$$h\nu = \frac{1}{2}\mu_i g_{s\ell}^2 + \Delta_{j\ell} \qquad (38)$$

where μ_i is the reduced mass defined by $m_e m_i/(m_e + m_i)$. We consider only nonrelativistic electrons in this analysis and neglect the momentum of the photons relative to the momentum of the electrons or the heavy particles. Then in the center-of-mass reference frame the direct encounter (photoionization) will result in an electron and an ion moving in an antiparallel direction arbitrarily oriented relative to the direction of the incoming photon. If we consider the interaction between a monoenergetic photon beam and a cloud of neutrals in the state j in the center-of-mass frame which are not polarized by an external magnetic field, then by symmetry arguments we can conclude that the collision probabilities will be independent of the "scattering" angle of the recoiling particles. Thus, in any direct collision ν would be the only parameter needed, in addition to the conservation equations, to determine the velocities of the resulting ion and electron in a particle direction.

Let $Q_j^\ell(\nu)d\Omega_s$ represent the differential cross section for the ionization of a neutral atom in state j and set $\vec{v}_{n_j} d^3v_{n_j}$ by the absorption of radiation of frequency ν resulting in an electron being emitted into the angle $d\Omega_s$ measured from some arbitrary reference axis with the speed v_s, dv_s. The resulting ion will be in the state ℓ with velocity $\vec{v}_{i_\ell}$, $d^3v_{i_\ell}$. The solid angle associated with the ion motion will be $-d\Omega_s$ in the center-of-mass frame. The number of electrons gained by the set $\vec{v}_s$, d^3v_s per unit time per unit volume as a result of photoionization can then be represented by

$$f_{n_j} d^3v_{n_j} \bar{f}^R(\nu)d\nu c Q_j^\ell d\Omega_s \qquad (39)$$

In (39) $\bar{f}^R$ is the photon distribution function per unit frequency per unit volume; it is normalized on the photon number density and is related to $\bar{I}(\nu)$, the specific intensity of ν radiation integrated over all solid angles, by (ref. 17)

$$\bar{f}^R(\nu) = \frac{\bar{I}(\nu)}{ch\nu} \qquad (40)$$

$\overline{I}(\nu)$ is proportional to the radiant energy density at this frequency. Here, and in the previous expression, c is the speed of light. It plays the role of the relative speed in expression (39). Note that we could have included explicitly here the direction of propagation of photons $\vec{\chi}_\nu$ and worked with $f^R(\nu, \vec{\chi}_\nu) d\nu \, d\Omega_\nu$ rather than $\overline{f}^R(\nu) d\nu$. This will be done in the discussion on radiative excitation and de-excitation in the following section. These distribution functions are related by

$$\overline{f}^R \equiv \int_0^{4\pi} f^R(\nu, \vec{\chi}_\nu) d\Omega_\nu$$

where $d\Omega_\nu$ is the solid angle centered about $\vec{\chi}_\nu$.

Two-body recombination, the inverse of the above encounter, results in a depletion of electrons from the set. An examination of the dynamical equations in the center-of-mass frame will reveal that the only parameter needed to determine the frequency of the radiant energy is the relative speed $g_{s\ell}$. Then, defining $\overline{P}_\ell^j(g_{s\ell})$ as the differential cross section for radiative capture (two-body recombination) of an electron in the velocity range $\vec{v}_s$, d^3v_s by an ion in the set $\vec{v}_{i_\ell}$, $d^3v_{i_\ell}$ which results in the emission of a quantum of radiation $h\nu$ and a neutral atom in the set $\vec{v}_{n_j}$, $d^3v_{n_j}$, an expression for the number of electrons lost by radiative capture for the reaction represented by (37) can be written as

$$f_s \, d^3v_s f_{i_\ell} \, d^3v_{i_\ell} g_{s\ell} \overline{P}_\ell^j(g_{s\ell}) \tag{41}$$

Electron captures result in both spontaneous and stimulated emission. Thus $\overline{P}_\ell^j$ can be written

$$\overline{P}_\ell^j(g_{s\ell}) = 4\pi\alpha_\ell^j(g_{s\ell}) + \overline{I}(\nu)\beta_\ell^j(g_{s\ell}) \tag{42}$$

where the first term on the right-hand side is the contribution from spontaneous capture and the remaining term is the contribution from induced capture.

In order to obtain a simplified expression for the collision integral associated with (37) we apply the principle of detailed balancing and equate (39) and (41) at equilibrium. Utilizing (42) and (40) with this equality we obtain

$$(f_s f_{i_\ell})_{Eq} \, g_{s\ell} \left\{ 4\pi\alpha_\ell^j + [\overline{I}(\nu)]_{Eq} \, \beta_\ell^j \right\} d^3v_s \, d^3v_{i_\ell} = \frac{[f_{n_j}\overline{I}(\nu)]_{Eq}}{h\nu} \, Q_j^\ell(\nu) d^3v_{n_j} \, d\nu \, d\Omega_s \tag{43}$$

At equilibrium the distribution functions are Maxwellian and are given by (29). The equilibrium specific radiation intensity, given below, is Planck's black body intensity:

$$[\overline{I}(\nu)]_{Eq} = \frac{8\pi h\nu^3}{c^2}\left(\frac{1}{e^{h\nu/kT} - 1}\right) \qquad (44)$$

The Saha equation (30) relates the number densities in equation (43). Combining equations (29), (30), and (44) with (43), we obtain

$$H^{-1}g_{s\ell}\left[\alpha_\ell^j\left(1 - e^{-h\nu/kT}\right) + \left(\frac{2h\nu^3}{c^2}\right)e^{-h\nu/kT}\beta_\ell^j\right]d^3v_s\ d^3v_{i_\ell} = Q_j^\ell(\nu)\ \frac{2\nu^2}{c^2}\ d^3v_{n_j}\ d\nu\ d\Omega_s \qquad (45)$$

Since this equation must be independent of the temperature, the following relations must hold:

$$\alpha_\ell^j(g_{s\ell}) = \frac{2h\nu^3}{c^2}\ \beta_\ell^j(g_{s\ell}) \qquad (46)$$

and

$$Q_j^\ell(\nu)d\Omega_s\ d\nu\ d^3v_{n_j} = H^{-1}\ \frac{c^2}{2\nu^2}\ g_{s\ell}\alpha_\ell^j(g_{s\ell})d^3v_s\ d^3v_{i_\ell} \qquad (47)$$

The differential elements in equation (47) can be related through the Jacobian of the following transformation:

$$d\Omega_s\ d\nu\ d^3v_{n_j} = |J|d^3v_s\ d^3v_{i_\ell} \qquad (48)$$

To evaluate $|J|$ we make use of the following differential relationship, which holds for our photoionizing model:

$$d^3v_s\ d^3v_{i_\ell} = d^3g_{s\ell}\ d^3v_{n_j}$$

When this relation is combined with (48) the result can be reduced to

$$d\Omega_s\ d\nu = |J|d^3g_{s\ell} \qquad (49)$$

Since $d^3g_{s\ell} = g_{s\ell}^2\ dg_{s\ell}\ d\Omega_s$, equation (49) can be written as

$$d\nu = |J|g_{s\ell}^2\ dg_{s\ell}$$

so that

$$|J| = \frac{1}{g_{s\ell}^2}\ \frac{\partial\nu}{\partial g_{s\ell}}$$

Then using (38) we can find the following relationship

$$\frac{\partial \nu}{\partial g_{s\ell}} = \frac{\mu_i g_{s\ell}}{h}$$

Thus the Jacobian becomes

$$|J| = \frac{\mu_i}{h g_{s\ell}}$$

and (48) can then be written

$$d\Omega_s \; d\nu \; d^3 v_{n_j} = \frac{\mu_i}{h g_{s\ell}} \; d^3 v_s \; d^3 v_{i_\ell} \tag{50}$$

With equation (50) and the definition of H, equation (47) can be reduced to

$$Q_j^\ell(\nu) = \frac{m_e c^2}{h^2 \nu^2} \left(\frac{2\omega_\ell}{\omega_j}\right)\left(\frac{m_e g_{s\ell}^2}{2}\right) \alpha_\ell^j(g_{s\ell}) \tag{51}$$

where μ_i has been replaced by m_e. The variable $g_{s\ell}$ on the right-hand side of (51) is related to ν by equation (38). Equations (46) and (51) are the detailed balancing relations for this photoionization encounter.

When these expressions are introduced into the gain expression (39) and the result combined is with the loss expression (41) and then integrated over all possible ion velocities,

$$\left(\frac{\partial_e f_s}{\partial t}\right)_{\substack{Ph \\ j \leftrightarrow \ell}} d^3 v_s = d^3 v_s \iint \left(f_{n_j} \overline{f}^R \frac{c^3 \alpha_\ell^j}{2H\nu^2 \overline{P}_\ell^j} - f_s f_{i_\ell}\right) g_{s\ell} \overline{P}_\ell^j \; d^3 v_{i_\ell} \tag{52}$$

With the aid of equations (40) and (46) we can rewrite (52) as

$$\left(\frac{\partial_e f_s}{\partial t}\right)_{\substack{Ph \\ j \leftrightarrow \ell}} d^3 v_s = d^3 v_s \iint \left(f_{n_j} \frac{\overline{I}\beta_\ell^j}{H \overline{P}_\ell^j} - f_s f_{i_\ell}\right) g_{s\ell} \overline{P}_\ell^j \; d^3 v_{i_\ell} \tag{53}$$

This equation represents the net gain of electrons to the set $\vec{v}_s$, $d^3 v_s$ per unit volume per unit time for the reaction represented by (37). On summing over all possible species and states, we obtain the following collision integral for photoionization:

$$\left(\frac{\partial_e f_s}{\partial t}\right)_{Ph} = \sum_{n,\ell,j} \iint \left(f_{n_j} \frac{\overline{I}\beta_\ell^j}{H \overline{P}_\ell^j} - f_s f_{i_\ell}\right) g_{s\ell} \overline{P}_\ell^j \; d^3 v_{i_\ell} \tag{54}$$

17

IV.   COLLISION INTEGRALS FOR NONELASTIC PROCESSES AFFECTING
THE TRANSFER EQUATIONS OF THE HEAVY PARTICLES
AND THE PHOTONS

In an energetic plasma the heavy particles as well as the electrons are influenced by nonelastic collisions. Although this effect on the heavy particles manifests itself primarily as a change in species concentration, momentum, and energy density rather than a change in the heavy particle distribution function itself, it will prove useful in the section on rate expressions to have at hand a convenient collection of nonelastic collision operators for the heavy particles as well as for the photons and the electrons.

We will consider here radiative excitation and de-excitation reactions (indicated by subscript Ra) in addition to those reactions considered in the previous section. This reaction proves important in any calculation of number densities and energy transfers for the heavy particles and the photons and thus affects the electron distribution function in a coupled problem (ref. 9).

The collision operators we will be considering in this section are indicated symbolically as follows:

(1) Neutral species:

$$\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{NE} = \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ex} + \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ion} + \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ra} + \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ph} \qquad (55)$$

(2) Ionic species:[4]

$$\left(\frac{\partial_e f_{i_{n\ell}}}{\partial t}\right)_{NE} = \left(\frac{\partial_e f_{i_{n\ell}}}{\partial t}\right)_{Ion} + \left(\frac{\partial_e f_{i_{n\ell}}}{\partial t}\right)_{Ph} \qquad (56)$$

(3) Photons:

$$\left(\frac{\partial_e \overline{f}^R}{\partial t}\right)_{NE} = \left(\frac{\partial_e \overline{f}^R}{\partial t}\right)_{Ra} + \left(\frac{\partial_e \overline{f}^R}{\partial t}\right)_{Ph} \qquad (57)$$

In this section, except for the development of the radiative excitation and de-excitation terms, we shall rely on the previous section for details and merely apply those results to the present formulation.

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[4]Collision integrals for ionic excitation or ionization can be deduced directly from the appropriate terms in equation (55) and will not be presented here.

18

Neutral Species

As equation (55) indicates, the nonelastic collisions being considered that involve neutral species are: collisional excitation and de-excitation, collisional ionization and three-body recombination, radiative excitation and de-excitation, and photoionization and two-body recombination. All of these interactions except for radiative excitation and its inverse have been considered in detail in the previous section. Hence by inspection or by following a development similar to that outlined in the previous section we can write the excitation and de-excitation collision operator for a neutral species in state j as

$$\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ex} = \sum_{\substack{k \\ k \neq j}} \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ex} = \sum_{\substack{k \\ k \neq j}} \iint \left(f_s' f_{n_k} \frac{\omega_j}{\omega_k} - f_s f_{n_j}\right) g_{sj} \sigma_{sj}^k \, d\Omega_{s's} \, d^3 v_s \tag{58}$$

Note that in this expression k is not restricted to being greater than j and the subscript s' is used for the electron involved in the gain term in (58). Of course (58) could have been written with two collision operators on the right-hand side: one for $k > j$ and one for $k < j$, for which s' would be replaced by a 0 and a 1, respectively, consistent with equations (3) and (4) with subscripts n_j and n_k interchanged.

We can also follow a development similar to that outlined in the previous section and write for the collisional ionization and three-body encounter:

$$\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ion} = \sum_\ell \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}} = \sum_\ell \iiint \left(f_2 f_4 f_{i_\ell} H - f_s f_{n_j}\right) g_{sj} \sigma_{sj}^\ell \, d^3 v_{42CM} \, d\Omega_{2s} \, d^3 v_s \tag{59}$$
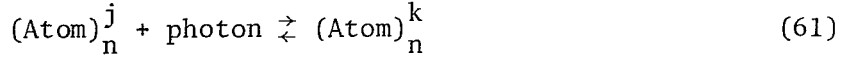
From Section III (p. 17) we can also deduce

$$\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ph} = \sum_\ell \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ph \\ j \leftrightarrow \ell}} = \sum_\ell \iint \left(f_s f_{i_\ell} - f_{n_j} \frac{\overline{I} \beta_\ell^j}{H \overline{P}_\ell^j}\right) \overline{P}_\ell^j \frac{g_{s\ell}^2 h}{\mu_i} \, d\nu \, d\Omega_s \tag{60}$$

for the photoionization and two-body recombination encounters.

Radiative excitation and de-excitation.- For the radiative excitation and de-excitation process we shall follow the development associated with photoionization and two-body recombination encounters (pp. 14-17), accounting for the differences between the "discrete" spectrum associated with this bound-bound transition and the continuous (above a limit) spectrum associated with the bound-free photoionization transition. Here, however, the development will be carried out with f^R rather than $\overline{f}^R$ for the photon distribution

function. A typical reaction for this case can be represented by

$$(\text{Atom})_n^j + \text{photon} \rightleftarrows (\text{Atom})_n^k \qquad (61)$$

The energy equation for this encounter is

$$\frac{1}{2} m_n v_{n_j}^2 + h\nu = \frac{1}{2} m_n v_{n_k}^2 + \Delta_{jk} \qquad . \qquad (62)$$

Let $Q_j^k(\nu)$ represent the cross section for the radiative excitation of a neutral atom from state j to state k as indicated in equations (61) and (62). The number of atoms in the set $\vec{v}_{n_j}$, $d^3v_{n_j}$ which are excited to $\vec{v}_{n_k}$, $d^3v_{n_k}$ atoms per unit time per unit volume as a result of radiative excitation can be represented by

$$f_{n_j} \, d^3v_{n_j} f^R(\nu, \vec{\chi}_\nu) d\nu \, d\Omega_\nu \psi_j^k(\nu) c Q_j^k(\nu) \qquad (63)$$

In equation (63) $\psi_j^k(\nu)$ is a line-shape factor, which accounts for the fact that the frequency over which the absorption occurs is not precise but varies over a narrow range. The number of atoms in the set $\vec{v}_{n_k}$, $d^3v_{n_k}$ that relax by radiative de-excitation to $\vec{v}_{n_j}$, $d^3v_{n_j}$ atoms per unit time per unit volume can be represented by

$$f_{n_k} \, d^3v_{n_k} P_k^j \psi_k^j(\nu) d\nu \, d\Omega_\nu \qquad (64)$$

As in expression (41) the differential cross section for relaxation $P_k^j(\nu, \vec{\chi}_\nu) \psi_k^j(\nu) d\nu \, d\Omega_\nu$ has spontaneous and stimulated components. Thus we can write

$$P_k^j = A_k^j(\nu) + I(\nu, \vec{\chi}_\nu) B_k^j(\nu) \qquad (65)$$

where the first term on the right corresponds to spontaneous relaxation and the remaining term corresponds to the induced relaxation. The differential elements $d\nu \, d\Omega_\nu$ represent the range and direction of the emitted photon. The line-shape factor for emission ψ_k^j will be assumed to be the same as the line-shape factor for absorption;[5] thus $\psi_k^j = \psi_j^k$.

Applying the principle of detailed balancing we can equate (63) with (64) and (65) at equilibrium and obtain

$$(f_{n_k})_{Eq} \left[A_k^j + (I)_{Eq} B_k^j \right] = (f_{n_j})_{Eq} (I)_{Eq} \frac{Q_j^k}{h\nu} \qquad (66)$$

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[5]This assumption is strictly true for Local Thermodynamic Equilibrium (L.T.E.) and usually is assumed to hold for situations in which collisions contribute predominately to the excited state population distribution (ref. 18).

In obtaining (66) equation (40) was used to replace f^R by I in (63). Also used was the differential volume element relation $d^3v_{n_j} = d^3v_{n_k}$ which follows directly from a momentum balance and the fact that the momentum of the photons is neglected. Now using

$$(I)_{Eq} = \frac{2h\nu^3}{c^2} \frac{1}{(e^{h\nu/kT} - 1)}$$

and the relations given by (10) and (11) of Section III (p. 3) we find from (66)

$$A_k^j = \frac{2h\nu^3}{c^2} B_k^j \tag{67}$$

and

$$Q_j^k = \frac{\omega_k}{\omega_j} \frac{c^2}{2\nu^2} A_k^j \tag{68}$$

The cross section Q_j^k in (63) could be replaced by an Einstein coefficient for absorption B_j^k such that (63) would be written as

$$f_{n_j} d^3v_{n_j} B_j^k \psi_j^k I \, d\nu \, d\Omega_\nu$$

This suggests the following relationships

$$B_j^k = \frac{Q_j^k}{h\nu}$$

and

$$B_j^k = \frac{\omega_k}{\omega_j} B_k^j$$

the latter being well known.

When these detailed balance expressions are introduced to the appropriate gain expressions and the result combined with the loss expression and integrated over the photon range and direction, one obtains

$$\left(\frac{\partial_e f_{n_k}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}} d^3v_{n_k} = d^3v_{n_k} \iint \left(f_{n_j} \frac{IB_j^k}{P_k^j} - f_{n_k}\right) P_k^j \psi_k^j \, d\Omega_\nu \, d\nu \tag{69}$$

or

$$\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}} d^3v_{n_j} = d^3v_{n_j} \iint \left(f_{n_k} - f_{n_j} \frac{IB_j^k}{P_k^j}\right) P_k^j \psi_j^k \, d\Omega_\nu \, d\nu \tag{70}$$

21

On carrying out the integration over the solid angle and summing over states we obtain

$$\left(\frac{\partial_e f_{n_k}}{\partial t}\right)_{Ra} = \underset{\substack{k > j \\ j < k}}{\sum_{j}}{}' \left(\frac{\partial_e f_{n_k}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}} = \underset{\substack{j \\ j < k}}{\sum}{}' \int \left(f_{n_j} \frac{\overline{IB}_j^k}{\overline{P}_k^j} - f_{n_k}\right) \overline{P}_k^j \psi_k^j \, dv \qquad (71)$$

or

$$\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ra} = \underset{\substack{j < k \\ j < k}}{\sum_{k}}{}' \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}} = \underset{\substack{k \\ j < k}}{\sum}{}' \int\!\!\int \left(f_{n_k} - f_{n_j} \frac{\overline{IB}_j^k}{\overline{P}_k^j}\right) \overline{P}_k^j \psi_j^k \, dv \qquad (72)$$

Note that by definition $\int \psi_k^j \, dv = 1$ (ref. 17).

If we want to account for either excitation or de-excitation from a single state we combine (71) and (72) and write

$$\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ra} = \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ra \\ j < k}} + \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ra \\ j > k}} \qquad (73)$$

For the second term on the right-hand side of (73) we would use (71) with k and j interchanged.

Ionic Species

The two interactions indicated by (56) have been considered in detail in Section III (pp. 8-17); thus by a slight intuitive extension of that work we can write directly:

$$\left(\frac{\partial_e f_{i_\ell}}{\partial t}\right)_{Ion} = \sum_j{}' \left(\frac{\partial_e f_{i_\ell}}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}}$$

$$= \sum_j{}' \int\!\!\int\!\!\int \left(f_1 f_{n_j} H^{-1} - f_s f_3 f_{i_\ell}\right) g_{s\ell} g_{3\ell} \sigma_{s3\ell}^j \, d\Omega_{1s} \, d^3 v_s \, d^3 v_3 \qquad (74)$$

for the rate of change of the distribution function for ions of species n in state ℓ as a result of collisional ionization and three-body recombination, and

$$\left(\frac{\partial_e f_{i_\ell}}{\partial t}\right)_{Ph} = \sum_{j} \left(\frac{\partial_e f_{i_\ell}}{\partial t}\right)_{\substack{Ph \\ j \leftrightarrow \ell}} = \sum_{j} \int\!\!\!\int \left(f_{n_j} \frac{\overline{I}\beta_\ell^j}{H\overline{P}_\ell^j} - f_s f_{i_\ell}\right)\overline{P}_\ell^j g_{s\ell} \; d^3 v_s \qquad (75)$$

for the rate of change of this ionic distribution function as a result of photoionization and its inverse.

It is fairly easy to establish upon a change of variables of integration between (60) and (75) that

$$\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ph \\ j \leftrightarrow \ell}} d^3 v_{n_j} = -\left(\frac{\partial_e f_{i_\ell}}{\partial t}\right)_{\substack{Ph \\ j \leftrightarrow \ell}} d^3 v_{i_\ell} \qquad (76)$$

This fact will prove useful in Section V.

Photons

In a direct manner one can deduce from Section III (pp. 14-17) that the net gain to the photon distribution function per unit volume per unit time per unit frequency range as a result of photoionization encounters is

$$\left(\frac{\partial_e \overline{f}^R}{\partial t}\right)_{Ph} = \sum_{\ell,j} \left(\frac{\partial_e \overline{f}^R}{\partial t}\right)_{\substack{Ph \\ j \leftrightarrow \ell}} = \sum_{\ell,j} \int\!\!\!\int\!\!\!\int \left(f_s f_{i_\ell} \frac{h g_{s\ell}^2 \overline{P}_\ell^j}{\mu_i c Q_j^\ell} - f_n \overline{f}^R\right) c Q_j^\ell \; d\Omega_s \; d^3 v_{n_j}$$

$$= \sum_{\ell,j} \int\!\!\!\int\!\!\!\int \left(f_s f_{i_\ell} - f_{n_j} \frac{\overline{I}\beta_\ell^j}{H\overline{P}_\ell^j}\right)\overline{P}_\ell^j \frac{g_{s\ell}^2 h}{\mu_i} \; d\Omega_s \; d^3 v_{n_j} \qquad (77)$$

From the preceding work in Section IV (pp. 19-22) we can write

$$\left(\frac{\partial_e \overline{f}^R}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k \\ k > j}} = \int \left(f_{n_k} - f_{n_j} \frac{IB_j^k}{P_k^j}\right) P_k^j \psi_j^k \; d^3 v_{n_j}$$

from which we find

$$\left(\frac{\partial_e \overline{f}^R}{\partial t}\right)_{Ra} = \sum_{\substack{j,k \\ k > j}} \left(\frac{\partial_e \overline{f}^R}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}} = \sum_{\substack{j,k \\ k > j}} \int\!\!\!\int \left(f_{n_k} - f_{n_j} \frac{\overline{I}B_j^k}{\overline{P}_k^j}\right)\overline{P}_k^j \psi_j^k \; d^3 v_{n_j} \qquad (78)$$

23

for the rate of change of $\overline{f}^R$ per unit volume per unit frequency as a result of radiative excitation and its inverse.

It should be noted here that to the extent that the internal energy state of a particle is independent of its translational energy we can integrate over the heavy particle velocity in equation (78) directly and obtain

$$\left(\frac{\partial_e \overline{f}^R}{\partial t}\right)_{Ra} = \sum_{\substack{j,k \\ k > j}} \left(n_k \overline{P}_k^j \psi_j^k - n_j \overline{IB}_j^k \psi_j^k\right) \tag{79}$$

V. RATE EXPRESSIONS

The rate of change of any electron property $\phi(\vec{v}_s)$ as a result of non-elastic collisions can be designated as

$$\int \phi(\vec{v}_s) \left(\frac{\partial_e f_s}{\partial t}\right)_{NE} d^3 v_s \equiv \left[\frac{\partial_e (n_e \overline{\phi})}{\partial t}\right]_{NE} \tag{80}$$

where $\overline{\phi}$ is the mean value of ϕ. Similarly, the rate of change of any property of the neutral species $\phi(\vec{v}_{n_j})$, the ions $\phi(\vec{v}_{i_\ell})$, or the photons $\phi^R(\nu)$ as a result of nonelastic collisions can, respectively, be written as

$$\int \phi(\vec{v}_{n_j}) \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{NE} d^3 v_{n_j} \equiv \left[\frac{\partial_e (n_{n_j} \overline{\phi}_{n_j})}{\partial t}\right]_{NE} \tag{81}$$

with corresponding expressions for the ions and the photons. When equations (1), (55), (56), and (57) are combined with the above expressions,

$$\left[\frac{\partial_e (n_e \overline{\phi})}{\partial t}\right]_{NE} = \left[\frac{\partial_e (n_e \overline{\phi})}{\partial t}\right]_{Ex} + \left[\frac{\partial_e (n_e \overline{\phi})}{\partial t}\right]_{Ion} + \left[\frac{\partial_e (n_e \overline{\phi})}{\partial t}\right]_{Ph} \tag{82}$$

$$\left[\frac{\partial_e (n_{n_j} \overline{\phi}_{n_j})}{\partial t}\right]_{NE} = \left[\frac{\partial_e (n_{n_j} \overline{\phi}_{n_j})}{\partial t}\right]_{Ex} + \left[\frac{\partial_e (n_{n_j} \overline{\phi}_{n_j})}{\partial t}\right]_{Ion} + \left[\frac{\partial_e (n_{n_j} \overline{\phi}_{n_j})}{\partial t}\right]_{Ra} + \left[\frac{\partial_e (n_{n_j} \overline{\phi}_{n_j})}{\partial t}\right]_{Ph} \tag{83}$$

$$\left[\frac{\partial_e (n_{i_\ell} \overline{\phi}_{i_\ell})}{\partial t}\right]_{NE} = \left[\frac{\partial_e (n_{i_\ell} \overline{\phi}_{i_\ell})}{\partial t}\right]_{Ion} + \left[\frac{\partial_e (n_{i_\ell} \overline{\phi}_{i_\ell})}{\partial t}\right]_{Ph} \tag{84}$$

$$\left[\frac{\partial_e (n^R \overline{\phi}^R)}{\partial t}\right]_{NE} = \left[\frac{\partial_e (n^R \overline{\phi}^R)}{\partial t}\right]_{Ra} + \left[\frac{\partial_e (n^R \overline{\phi}^R)}{\partial t}\right]_{Ph} \tag{85}$$

24

In regard to equations (83) and (84) we can sum over states and obtain

$$\left[\frac{\partial_e(n_n\overline{\phi}_n)}{\partial t}\right]_{NE} = \sum_j \left[\frac{\partial_e(n_{n_j}\overline{\phi}_{n_j})}{\partial t}\right]_{NE} \tag{86}$$

and

$$\left[\frac{\partial_e(n_{i_n}\overline{\phi}_{i_n})}{\partial t}\right]_{NE} = \sum_\ell \left[\frac{\partial_e(n_{i_\ell}\overline{\phi}_{i_\ell})}{\partial t}\right]_{NE} \tag{87}$$

To check the consistency of the formulation of the collision integrals, in this section the terms on the right-hand side of equations (82)-(85) will be evaluated for various forms of the property functions. It will be demonstrated that equations (82)-(87) with the collision operators developed in the preceding sections will yield nonelastic expressions that agree with the usual conservation equations. In particular, for the reactions considered herein, if $\phi(\vec{v}_s) = 1$ it will be shown that

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{Ex} = 0$$

and

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{NE} = \left(\frac{\partial_e n_e}{\partial t}\right)_{Ion} + \left(\frac{\partial_e n_e}{\partial t}\right)_{Ph} = \sum_n \left(\frac{\partial_e n_{i_n}}{\partial t}\right)_{NE} = -\sum_n \left(\frac{\partial_e n_n}{\partial t}\right)_{NE} \tag{88}$$

The last two terms in the above equality are the rate of creation and destruction of ions and atoms, respectively. Equations (88) state that excitation and de-excitation type collisions do not result in any net creation of electrons but that ionization and photoionization and their inverses are the only terms contributing to a net production of electrons. In addition, of course, the net rate of increase of free electrons equals the net rate of creation of ions which also equals the rate of destruction of neutral particles. It will also be demonstrated that if $\phi(\vec{v}_s) = (1/2)m_e v_s^2$,

$$\left[\frac{\partial_e\left(\frac{1}{2}n_e m_e v_e^2\right)}{\partial t}\right]_{NE} = \sum_{n,j,k} \Delta_{jk}\left(\frac{\partial_e n_j}{\partial t}\right)_{Ex \atop j\leftrightarrow k} + \sum_{n,j,\ell} \Delta_{j\ell}\left[\left(\frac{\partial_e n_j}{\partial t}\right)_{Ion \atop j\leftrightarrow \ell} + \left(\frac{\partial_e n_j}{\partial t}\right)_{Ph \atop j\leftrightarrow \ell}\right]$$

$$- \left\{\begin{array}{l}\text{energy gained by neutrals, ions, and photons}\\\text{for the Ex, Ion, and Ph reactions}\end{array}\right\} \tag{89}$$

where $(1/2)m_e\overline{v_e^2}$ is the mean translatory kinetic energy of the electrons. This expression shows that the net gain of thermal energy by the electrons during the encounters considered are a result of the following:

(a) The net relaxation of excited states (that is, $(\partial_e n_{n_j}/\partial t)_{Ex,j\leftrightarrow k}$ is the net production of state-j species from state-k species (k > j) and $(\partial_e n_{n_j}/\partial t)_{Ex,j\leftrightarrow k} = -(\partial_e n_{n_k}/\partial t)_{Ex,j\leftrightarrow k})$,

(b) The net production of neutral species, at the expense of the ions,

(c) The net gain of photon energy, as a result of neutral species production, $(\partial_e n_{n_j}/\partial t)_{Ph,j\leftrightarrow k}$, and

(d) The net loss of thermal and radiant energy, respectively, of the heavy species and photons involved in the encounters.

The case when $\phi(\vec{v}_s) = m_e\vec{v}_s$ is not as interesting as the preceding cases and is briefly treated separately in the momentum transfer part of Section V.

Inelastic and Superelastic Collisions

Equation (19) can be used to write

$$\left[\frac{\partial_e(n_e\overline{\phi})}{\partial t}\right]_{Ex} = \int \phi(\vec{v}_s)\left(\frac{\partial_e f_s}{\partial t}\right)_{Ex} d^3v_s$$

$$= \sum_{n,j,k}\left[\iiint \phi(\vec{v}_s)\left(f_0 f_{n_k}\frac{\omega_j}{\omega_k} - f_s f_{n_j}\right)g_{sj}\sigma_{sj}^k \, d\Omega_{os} \, d^3v_{n_j} \, d^3v_s\right.$$

$$\left. + \iiint \phi(\vec{v}_s)\left(f_1 f_{n_j}\frac{\omega_k}{\omega_j} - f_s f_{n_k}\right)g_{sk}\sigma_{sk}^j \, d\Omega_{1s} \, d^3v_{n_k} \, d^3v_s\right] \qquad (90)$$

For $\phi(\vec{v}_s) = 1$ a typical term of equation (90) can be expanded:

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{Ex}_{j \leftrightarrow k} = \iiint f_o f_{n_k} \frac{\omega_j}{\omega_k} g_{sj} \sigma^k_{sj} \, d\Omega_{os} \, d^3 v_{n_j} \, d^3 v_s$$

$$- \iiint f_s f_{n_j} g_{sj} \sigma^k_{sj} \, d\Omega_{os} \, d^3 v_{n_j} \, d^3 v_s$$

$$+ \iiint f_1 f_{n_j} \frac{\omega_k}{\omega_j} g_{sk} \sigma^j_{sk} \, d\Omega_{1s} \, d^3 v_{n_k} \, d^3 v_s$$

$$- \iiint f_s f_{n_k} g_{sk} \sigma^j_{sk} \, d\Omega_{1s} \, d^3 v_{n_k} \, d^3 v_s \qquad (91)$$

Combining the detailed balance relations, (13) and (14), and the relations between the differential velocity elements, (12) and (15), of Section III with this expression yields

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{Ex}_{j \leftrightarrow k} = \iiint f_o f_{n_k} g_{ok} \sigma^j_{ok} \, d\Omega_{os} \, d^3 v_{n_k} \, d^3 v_o$$

$$- \iiint f_s f_{n_j} g_{sj} \sigma^k_{sj} \, d\Omega_{os} \, d^3 v_{n_j} \, d^3 v_s$$

$$+ \iiint f_1 f_{n_j} g_{1j} \sigma^k_{1j} \, d\Omega_{1s} \, d^3 v_{n_j} \, d^3 v_1$$

$$- \iiint f_s f_{n_k} g_{sk} \sigma^j_{sk} \, d\Omega_{1s} \, d^3 v_{n_k} \, d^3 v_s \qquad (92)$$

By this change of variables $(v_{n_j}, v_s \rightarrow v_{n_k}, v_o$ and $v_{n_k}, v_s \rightarrow v_{n_j}, v_1)$ in the first and third terms of (91), respectively, the threshold energy dependence of the first set of integrals on the right-hand side of (91) has been removed and a threshold energy dependent cross section has been introduced into the third set of integrals of this equation. Now, since the subscripts associated with the electrons can be changed without affecting the value of the integrals, the first and second terms in equation (92) will cancel with the fourth and third terms, respectively. That is, if in the first and second terms of equation (92) we consistently change $s \rightarrow 1$, $0 \rightarrow s$ and use the fact that $d\Omega_{s1} = d\Omega_{1s}$ we will recreate the negative of the fourth and third

27

integrals.[6] Thus, when $\phi(\vec{v}_s) = 1$, $(\partial_e n_e/\partial t)_{Ex,j\leftrightarrow k} = 0$ which implies:

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{Ex} = 0 \tag{93}$$

This is as expected since excitation and de-excitation encounters do not create or destroy electrons.

For $\phi(\vec{v}_s) = (1/2)m_e v_s^2$ a typical term of (90) can be expanded by making use of detailed balancing expressions, energy equations, and variable transformations to obtain

$$\left[\frac{\overline{\partial_e\left(\frac{1}{2}\,n_e m_e v_e^2\right)}}{\partial t}\right]_{Ex}_{j\leftrightarrow k}$$

$$= \iiint\left[\frac{1}{2}\,m_e v_e^2 + \frac{1}{2}\,m_n\left(v_{n_k}^2 - v_{n_j}^2\right) + \Delta_{jk}\right]f_o f_{n_k} g_{ok}\,\sigma_{ok}^j\,d\Omega_{os}\,d^3 v_{n_k}\,d^3 v_o$$

$$- \iiint\frac{1}{2}\,m_e v_s^2 f_s f_{n_j} g_{sj}\sigma_{sj}^k\,d\Omega_{os}\,d^3 v_{n_j}\,d^3 v_s$$

$$+ \iiint\left[\frac{1}{2}\,m_e v_1^2 + \frac{1}{2}\,m_n\left(v_{n_j}^2 - v_{n_k}^2\right) - \Delta_{jk}\right]f_1 f_{n_j} g_{1j}\sigma_{1j}^k\,d\Omega_{1s}\,d^3 v_{n_j}\,d^3 v_1$$

$$- \iiint\frac{1}{2}\,m_e v_s^2 f_s f_{n_k} g_{sk}\sigma_{sk}^j\,d\Omega_{1s}\,d^3 v_{n_k}\,d^3 v_s \tag{94}$$

In this equation, in a manner analogous to that leading to equation (93), the first term in the bracket of the first set of integrals cancels with the last set of integrals, and the second set of integrals cancels with the first term in the bracket of the third set of integrals. Thus,

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[6]A change of variables of integration to a center of mass and relative velocity dependence will explicitly illustrate this equality.

$$\left[\frac{\partial_e\left(\frac{1}{2}\ n_e m_e \overline{v_e^2}\right)}{\partial t}\right]_{Ex \atop j\leftrightarrow k} = \Delta_{jk}\left[\iiint f_o f_{n_k} g_{ok}\sigma_{ok}^j\ d\Omega_{os}\ d^3v_{n_k}\ d^3v_o\right.$$

$$\left.- \iiint f_1 f_{n_j} g_{1j}\sigma_{1j}^k\ d\Omega_{1s}\ d^3v_{n_j}\ d^3v_1\right]$$

$$+ \iiint \frac{1}{2}\ m_n v_{n_k}^2 f_o f_{n_k} g_{ok}\sigma_{ok}^j\ d\Omega_{os}\ d^3v_{n_k}\ d^3v_o$$

$$- \iiint \frac{1}{2}\ m_n v_{n_k}^2 f_1 f_{n_j} g_{1j}\sigma_{1j}^k\ d\Omega_{1s}\ d^3v_{n_j}\ d^3v_1$$

$$+ \iiint \frac{1}{2}\ m_n v_{n_j}^2 f_1 f_{n_j} g_{1j}\sigma_{1j}^k\ d\Omega_{1s}\ d^3v_{n_j}\ d^3v_1$$

$$- \iiint \frac{1}{2}\ m_n v_{n_j}^2 f_o f_{n_k} g_{ok}\sigma_{ok}^j\ d\Omega_{os}\ d^3v_{n_k}\ d^3v_o \qquad (95)$$

Again if the variables and dummy subscripts are changed and the results of detailed balancing are used this expression can be written as

$$\left[\frac{\partial_e\left(\frac{1}{2}\ n_e m_e \overline{v_e^2}\right)}{\partial t}\right]_{Ex \atop j\leftrightarrow k} = \Delta_{jk}\left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{Ex \atop j\leftrightarrow k} - \left[\int \frac{1}{2}\ m_n v_{n_k}^2 \left(\frac{\partial_e f_{n_k}}{\partial t}\right)_{Ex \atop j\leftrightarrow k} d^3v_{n_k}\right.$$

$$\left.+ \int \frac{1}{2}\ m_n v_{n_j}^2 \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ex \atop j\leftrightarrow k} d^3v_{n_j}\right] \qquad (96)$$

where here we recall that $k > j$. The first term on the right-hand side of (96) is obtained from equations (58), (81), and (83) when $\phi_{n_j} = 1$ for a two-state atom (say j and k). It is easy to show, via detailed balance relationships and dummy variable changes, that for this particular $j \leftrightarrow k$ type excitation and de-excitation reaction

$$\left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{Ex \atop j\leftrightarrow k} = - \left(\frac{\partial_e n_{n_k}}{\partial t}\right)_{Ex \atop j\leftrightarrow k} \qquad (97)$$

as expected. The bracketed terms on the right-hand side of equation (96) are obtained from (58), when $k > j$, and the last four terms in equation (95). If now the states and species are summed and equations (58) and (86) are used,

$$\left[\frac{\partial_e\left(\frac{1}{2} n_e m_e \overline{v_e^2}\right)}{\partial t}\right]_{Ex} = \sum_{\substack{j,k,n \\ k > j}} \Delta_{jk}\left(\frac{\partial_e^n n_j}{\partial t}\right)_{\substack{Ex \\ j \leftrightarrow k}} - \sum_{\substack{j,k,n \\ k \neq j}} \left[\frac{\partial_e\left(\frac{1}{2} n_{n_j} m_n \overline{v_{n_j}^2}\right)}{\partial t}\right]_{\substack{Ex \\ j \leftrightarrow k}}$$

$$= \sum_{\substack{j,k,n \\ k > j}} \Delta_{jk}\left(\frac{\partial_e^n n_j}{\partial t}\right)_{\substack{Ex \\ j \leftrightarrow k}} - \left[\frac{\partial_e\left(\frac{1}{2} n_N m_N \overline{v_N^2}\right)}{\partial t}\right]_{Ex} \qquad (98)$$

Equation (98) is interpreted as follows: The net gain of kinetic energy by the electron gas as a result of inelastic-superelastic encounters is equal to the gain of energy from a net de-excitation of excited species minus the net gain of kinetic energy by the neutral particles involved.

The last term on the right-hand side of (98) is obtained via the following sequence of equalities:

$$\sum_{\substack{n,j,k \\ k \neq j}} \left[\frac{\partial_e\left(\frac{1}{2} n_{n_j} m_n \overline{v_{n_j}^2}\right)}{\partial t}\right]_{\substack{Ex \\ j \leftrightarrow k}} = \sum_{n,j}\left\{\sum_{\substack{k \\ k \neq j}} \left[\frac{\partial_e\left(\frac{1}{2} n_{n_j} m_n \overline{v_{n_j}^2}\right)}{\partial t}\right]_{\substack{Ex \\ j \leftrightarrow k}}\right.$$

$$= \sum_{n,j} \left[\frac{\partial_e\left(\frac{1}{2} n_{n_j} m_n \overline{v_{n_j}^2}\right)}{\partial t}\right]_{Ex}$$

$$= \sum_{n} \left[\frac{\partial_e\left(\frac{1}{2} n_n m_n \overline{v_n^2}\right)}{\partial t}\right]_{Ex}$$

$$\left. = \left[\frac{\partial_e\left(\frac{1}{2} n_N m_N \overline{v_N^2}\right)}{\partial t}\right]_{Ex}\right\}$$

30

where

$$n_N \equiv \sum_n n_n$$

$$m_N \equiv \frac{1}{n_N} \sum_n n_n m_n$$

and

$$\overline{v_N^2} \equiv \frac{1}{n_N m_N} \sum_n n_n m_n \overline{v_n^2}$$

Ionization and Three-Body Recombination Encounters

The rate of change of an electron property $\phi(\vec{v}_s)$ as a result of collisional ionization and three-body recombination encounters can be expressed as

$$\left[\frac{\partial_e(n_e\overline{\phi})}{\partial t}\right]_{Ion} = \int \phi(\vec{v}_s)\left(\frac{\partial_e f_s}{\partial t}\right)_{Ion} d^3v_s$$

$$= \sum_{n,\ell,j}\left[\iiiint \phi(\vec{v}_s)\left(f_2 f_4 f_{i_\ell}H - f_s f_{n_j}\right)g_{sj}\sigma_{sj}^\ell \, d\Omega_{2s} \, d^3v_{4_{2CM}} \, d^3v_{n_j} \, d^3v_s\right.$$

$$\left.+ 2\iiiint \phi(\vec{v}_s)\left(f_1 f_{n_j}H^{-1} - f_s f_{i_\ell} f_3\right)g_{s\ell} g_{3\ell}\sigma_{s3\ell}^j \, d\Omega_{1s} \, d^3v_3 \, d^3v_{i_\ell} \, d^3v_s\right]$$

$$(99)$$

Before equation (99) is evaluated for various values of $\phi(\vec{v}_s)$, it will prove helpful to have at hand expressions for the collisional rate of change of number density of the heavy particles involved as a result of this type of collision. If we set $\phi_{n_j} = \phi_{i_\ell} = \phi_n = \phi_{i_n} = 1$ in equations (83)-(87) (and make use of (55), (56), (59), and (74)), we find the following expected results:

$$-\left(\frac{\partial_e^n n_j}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}} = \left(\frac{\partial_e^n i_\ell}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}} \qquad (100)$$

31

and

$$-\left(\frac{\partial_e n_n}{\partial t}\right)_{Ion} = \left(\frac{\partial_e n_i}{\partial t}\right)_{Ion} \qquad (101)$$

That is, the net rate of creation of ions is equal to the net rate of destruction of neutral atoms. These results follow directly once the variables are changed in equation (59) by the following detailed balance result

$$H g_{sj} \sigma^{\ell}_{sj} \, d^3 v_{42_{CM}} \, d^3 v_s \, d^3 v_{n_j} = g_{2\ell} g_{4\ell} \sigma^j_{24\ell} \, d^3 v_2 \, d^3 v_4 \, d^3 v_{i_\ell}$$

and the dummy subscripts s, 2, 4 are changed in the resulting equation to 1, s, and 3, respectively, and use is made of the fact that $d\Omega_{1s} = d\Omega_{s1}$.

For $\phi(\vec{v}_s) = 1$ it is fairly easy to show by equations (59), (74), (81) (eq. (81) applied to the ions as well as to the neutrals), (55), and (56) that from a typical term of (99) we obtain

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}} = \left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}} + 2\left(\frac{\partial_e n_{i_\ell}}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}} \qquad (102)$$

Equation (102) in combination with (100) yields the expected

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}} = \left(\frac{\partial_e n_{i_\ell}}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}} = -\left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}} \qquad (103)$$

Similarly, on summing over all states and species, we find

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{Ion} = \sum_n{}' \left(\frac{\partial_e n_i}{\partial t}\right)_{Ion} = -\sum_n{}' \left(\frac{\partial_e n_n}{\partial t}\right)_{Ion} \qquad (104)$$

Equations (103) and (104) illustrate that for this type of encounter, the rate of creation of electrons equals the rate of creation of ions or the rate of destruction of neutral particles.

For $\phi(\vec{v}_s) = (1/2)m_e v_s^2$ a typical term of equation (99) can be expanded by making use of equations (21), (22), (31), and (32) to obtain

$$\left[\frac{\partial_e\left(\frac{1}{2}\,n_e m_e \overline{v_e^2}\right)}{\partial t}\right]_{\substack{Ion \\ j\leftrightarrow\ell}}$$

$$=\iiint\left[\frac{1}{2}\,m_e(v_2^2+v_4^2)+\Delta_{j\ell}+\frac{m_i v_{i_\ell}^2}{2}-\frac{m_n v_{n_j}^2}{2}\right]f_2 f_4 f_{i_\ell}g_{2\ell}g_{4\ell}\sigma_{24\ell}^j\,d\Omega_{2s}\,d^3v_2\,d^3v_4\,d^3v_{i_\ell}$$

$$-\iiint\frac{1}{2}\,m_e v_s^2 f_s f_{n_j}g_{sj}\sigma_{sj}^\ell\,d\Omega_{2s}\,d^3v_{42_{CM}}\,d^3v_{n_j}\,d^3v_s$$

$$+2\iiint\left[\frac{1}{2}\,m_e(v_1^2-v_3^2)+\frac{1}{2}\,m_n v_{n_j}^2-\Delta_{j\ell}-\frac{m_i v_{i_\ell}^2}{2}\right]f_1 f_{n_j}g_{1j}\sigma_{1j}^\ell\,d\Omega_{1s}\,d^3v_{3s_{CM}}\,d^3v_1\,d^3v_{n_j}$$

$$-2\iiint\frac{1}{2}\,m_e v_s^2 f_s f_3 f_{i_\ell}g_{s\ell}g_{3\ell}\sigma_{s3\ell}^j\,d\Omega_{1s}\,d^3v_3\,d^3v_s \tag{105}$$

Close inspection of equation (105) when the dummy character of the electron subscripts are accounted for reveals that the sets of integrals containing $(1/2)m_e(v_2^2+v_4^2)$ and $-2[(1/2)m_e v_s^2]$ are equal in absolute value and therefore cancel each other.

Using the following equality

$$2\left[\frac{1}{2}\,m_e(v_1^2-v_3^2)+\frac{1}{2}\,m_n v_{n_j}^2-\Delta_{j\ell}-\frac{m_i v_{i_\ell}^2}{2}\right]$$

$$=\frac{1}{2}\,m_e(v_1^2-v_3^2)+\frac{1}{2}\,m_n v_{n_j}^2-\Delta_{j\ell}-\frac{m_i v_{i_\ell}^2}{2}+\frac{1}{2}\,m_e v_s^2$$

in the third set of integrals in (105) and recalling that the ionization cross section is symmetric in the subscripts s and 3 we find that the remaining terms in (105) containing electron kinetic energies cancel each other. We are finally left with

$$\left[\frac{\partial_e\left(\frac{1}{2}\,n_e m_e \overline{v_e^2}\right)}{\partial t}\right]_{\substack{\text{Ion}\\ j\leftrightarrow\ell}} = \Delta_{j\ell}\left(\iiiint f_2 f_4 f_{i_\ell} g_{2\ell} g_{4\ell}\,\sigma^j_{24\ell}\;d\Omega_{2s}\;d^3 v_2\;d^3 v_4\;d^3 v_{i_\ell}\right.$$

$$\left.-\iiiint f_1 n_j g_{1j}\,\sigma^\ell_{1j}\;d\Omega_{1s}\;d^3 v_{3s\,CM}\;d^3 v_1\;d^3 v_{n_j}\right)$$

$$+\left(\iiiint \frac{m_i v_{i_\ell}^2}{2}\,f_2 f_4 f_{i_\ell} g_{2\ell} g_{4\ell}\,\sigma^j_{24\ell}\;d\Omega_{2s}\;d^3 v_2\;d^3 v_4\;d^3 v_{i_\ell}\right.$$

$$\left.-\iiiint \frac{m_i v_{i_\ell}^2}{2}\,f_1 f_{n_j} g_{1j}\,\sigma^\ell_{1j}\;d\Omega_{1s}\;d^3 v_{3s\,CM}\;d^3 v_1\;d^3 v_{n_j}\right)$$

$$+\left(\iiiint \frac{m_n v_{n_j}^2}{2}\,f_1 f_{n_j} g_{1j}\,\sigma^\ell_{1j}\;d\Omega_{1s}\;d^3 v_{3s\,CM}\;d^3 v_1\;d^3 v_{n_j}\right.$$

$$\left.-\iiiint \frac{m_n v_{n_j}^2}{2}\,f_2 f_4 f_{i_\ell} g_{2\ell} g_{4\ell}\,\sigma^j_{24\ell}\;d\Omega_{2s}\;d^3 v_2\;d^3 v_4\;d^3 v_{i_\ell}\right)$$

$$\tag{106}$$

If the electron subscripts in the first set of integrals in the first of the above bracketed terms are changed from 2, 4, and s to s, 3, and 1, respectively, and the detailed balancing result given by equation (32) is used along with $d\Omega_{1s} = d\Omega_{s1}$, this bracketed term becomes equal to $-(\partial_e n_{i_\ell}/\partial t)_{\text{Ion},j\leftrightarrow\ell}$ or $-(\partial_e n_e/\partial t)_{\text{Ion},j\leftrightarrow\ell}$. In a similar manner the second bracketed term in (106) can be shown to equal $-\left\{\partial_e\left[(1/2)n_{i_\ell} m_i \overline{v_{i_\ell}^2}\right]/\partial t\right\}_{\text{Ion},j\leftrightarrow\ell}$. The third bracketed term in equation (106) can be shown to equal $-\left\{\partial_e\left[(1/2)n_{n_j} m_n \overline{v_{n_j}^2}\right]/\partial t\right\}_{\text{Ion},j\leftrightarrow\ell}$ when the subscripts in the first part of this bracketed term are changed from 1, s, 3 to s, 2, 4 and the appropriate variable changes through equation (31) are made. Thus equation (106) can be written as

$$\left[\frac{\partial_e\left(\frac{1}{2}\,n_e m_e \overline{v_e^2}\right)}{\partial t}\right]_{\substack{\text{Ion}\\ j\leftrightarrow\ell}} = -\Delta_{j\ell}\left(\frac{\partial_e n_e}{\partial t}\right)_{\substack{\text{Ion}\\ j\leftrightarrow\ell}} - \left[\frac{\partial_e\left(\frac{1}{2}\,n_{i_\ell} m_i \overline{v_{i_\ell}^2}\right)}{\partial t}\right]_{\substack{\text{Ion}\\ j\leftrightarrow\ell}}$$

$$-\left[\frac{\partial_e\left(\frac{1}{2}\,n_{n_j} m_n \overline{v_{n_j}^2}\right)}{\partial t}\right]_{\substack{\text{Ion}\\ j\leftrightarrow\ell}} \tag{107}$$

34

On summing over states and species and using (103) we obtain

$$\left[\frac{\partial_e\left(\frac{1}{2} n_e m_e \overline{v_e^2}\right)}{\partial t}\right]_{Ion} = \sum_{j,\ell,n} \Delta_{j\ell} \left(\frac{\partial_e n_j}{\partial t}\right)_{\substack{Ion \\ j \leftrightarrow \ell}} - \left[\frac{\partial_e\left(\frac{1}{2} n_I m_I \overline{v_I^2}\right)}{\partial t}\right]_{Ion}$$

$$- \left[\frac{\partial_e\left(\frac{1}{2} n_N m_N \overline{v_N^2}\right)}{\partial t}\right]_{Ion} \qquad (108)$$

where

$$n_I \equiv \sum_n n_i$$

$$m_I \equiv \frac{1}{n_I} \sum_n n_i m_i$$

and

$$\overline{v_I^2} \equiv \frac{1}{n_I m_I} \sum_n n_i m_i \overline{v_i^2}$$

The sum over n here is interpreted as a sum over the ionic species of type n. Equation (108) illustrates that the net gain of kinetic energy by the electron gas as a result of collisional ionization and three-body recombination encounters is equal to the net gain of ionizational energy due to recombination minus the net gain of kinetic energy by the heavy particles involved.

Photoionization and Two-Body Recombination Encounters

As a result of photoionization and two-body recombination encounters, changes in an electron property $\phi(\vec{v}_s)$ can be accounted for on a rate basis by the following equation:

$$\left[\frac{\partial_e(n_e\bar{\phi})}{\partial t}\right]_{Ph} = \int \phi(\vec{v}_s)\left(\frac{\partial_e f_s}{\partial t}\right)_{Ph} d^3v_s$$

$$= \sum_{n,\ell,j}' \iint \phi(\vec{v}_s)\left(f_{n_j}\frac{\bar{I}\beta_\ell^j}{H\bar{P}_\ell^j} - f_s f_{i_\ell}\right)g_{s\ell}\bar{P}_\ell^j d^3v_{i_\ell} d^3v_s \qquad (109)$$

Again it is convenient to have expressions at hand for the rate of change of number density of the heavy species and the photons involved in this reaction. From equation (60), (75)-(77), and (81) (eq. (81) applied to ions and photons as well as to neutrals), it is immediately apparent that when $\phi_{n_j} = \phi_{i_\ell} = \phi^R = 1$ the following results are obtained:

$$\left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{Ph \atop j\leftrightarrow\ell} = -\left(\frac{\partial_e n_{i_\ell}}{\partial t}\right)_{Ph \atop j\leftrightarrow\ell} = \left(\frac{\partial_e n^R}{\partial t}\right)_{Ph \atop j\leftrightarrow\ell} \qquad (110)$$

which obviously extends to:

$$\left(\frac{\partial_e n_n}{\partial t}\right)_{Ph} = -\left(\frac{\partial_e n_{i_n}}{\partial t}\right)_{Ph} = \left(\frac{\partial_e n^R}{\partial t}\right)_{Ph} \qquad (111)$$

That is, photons and neutrals are created at the same rate by radiative capture.

For $\phi(\vec{v}_s) = 1$ a typical term of equation (109) when compared with (75), (81) (eq. (81) applied to ions) and (110), will yield

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{Ph \atop j\leftrightarrow\ell} = \left(\frac{\partial_e n_{i_\ell}}{\partial t}\right)_{Ph \atop j\leftrightarrow\ell} = -\left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{Ph \atop j\leftrightarrow\ell} = -\left(\frac{\partial_e n^R}{\partial t}\right)_{Ph \atop j\leftrightarrow\ell} \qquad (112)$$

which extends to

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{Ph} = \left(\frac{\partial_e n_i}{\partial t}\right)_{Ph} = -\left(\frac{\partial_e n_n}{\partial t}\right)_{Ph} = -\left(\frac{\partial_e n^R}{\partial t}\right)_{Ph} \qquad (113)$$

This expression can be summed over states to yield equalities similar to equation (104).

36

For $\phi(\vec{v}_s) = (1/2)m_e v_s^2$ in the form of the energy equation for this encounter, that is,

$$\frac{1}{2} m_e v_s^2 = h\nu + \frac{1}{2} m_n v_{n_j}^2 - \frac{1}{2} m_i v_{i_\ell}^2 - \Delta_{j\ell}$$

a typical term of (109) yields

$$\left[\frac{\partial_e\left(\frac{1}{2} n_e m_e \overline{v_e^2}\right)}{\partial t}\right]_{\substack{Ph \\ j \leftrightarrow \ell}} = \iint h\nu \left(f_{n_j} \frac{\overline{I\beta}_\ell^j}{\overline{HP}_\ell^j} - f_s f_{i_\ell}\right) g_{s\ell} \overline{P}_\ell^j \, d^3 v_{i_\ell} \, d^3 v_s$$

$$+ \iint \frac{1}{2} m_n v_{n_j}^2 \left(f_{n_j} \frac{\overline{I\beta}_\ell^j}{\overline{HP}_\ell^j} - f_s f_{i_\ell}\right) g_{s\ell} \overline{P}_\ell^j \, d^3 v_{i_\ell} \, d^3 v_s$$

$$- \iint \frac{1}{2} m_i v_{i_\ell}^2 \left(f_{n_j} \frac{\overline{I\beta}_\ell^j}{\overline{HP}_\ell^j} - f_s f_{i_\ell}\right) g_{s\ell} \overline{P}_\ell^j \, d^3 v_{i_\ell} \, d^3 v_s$$

$$- \Delta_{j\ell} \left(\frac{\partial_e n_e}{\partial t}\right)_{\substack{Ph \\ j \leftrightarrow \ell}} \tag{114}$$

When equation (50) is substituted into the first set of integrals in equation (114) and the result is compared with (77) and (81) (eq. (81) applied to photons), we see that this set of integrals is equal to

$$- \left[\frac{\partial_e(n^R \overline{h\nu})}{\partial t}\right]_{\substack{Ph \\ j \leftrightarrow \ell}} \equiv - \left(\frac{\partial_e u^R}{\partial t}\right)_{\substack{Ph \\ j \leftrightarrow \ell}} \tag{115}$$

which is defined as the negative of the rate of change of radiant energy per unit volume, u^R, resulting from this interaction. The radiant energy density of the photon gas is defined by (ref. 17) $u^R \equiv \int h\nu \overline{f}^R \, d\nu$.

Similar comparisons of equations (50), (59), (75), and (81) (eq. (81) applied to ions and neutrals), with the second and third sets of integrals of equation (114), reveal these integral sets to be

$$- \left[\frac{\partial_e\left(\frac{1}{2} n_{n_j} m_n \overline{v_{n_j}^2}\right)}{\partial t}\right]_{\substack{Ph \\ j \leftrightarrow \ell}}$$

and

$$-\left[\frac{\partial_e\left(\frac{1}{2}\,n_{i_\ell}m_i\overline{v_{i_\ell}^2}\right)}{\partial t}\right]_{\substack{Ph\\j\leftrightarrow\ell}}$$

Thus equation (114) reduces to

$$\left[\frac{\partial_e\left(\frac{1}{2}\,n_e m_e\overline{v_e^2}\right)}{\partial t}\right]_{\substack{Ph\\j\leftrightarrow\ell}} = -\left(\frac{\partial_e u^R}{\partial t}\right)_{\substack{Ph\\j\leftrightarrow\ell}} - \left[\frac{\partial_e\left(\frac{1}{2}\,n_{n_j}m_n\overline{v_{n_j}^2}\right)}{\partial t}\right]_{\substack{Ph\\j\leftrightarrow\ell}}$$

$$-\left[\frac{\partial_e\left(\frac{1}{2}\,n_{i_\ell}m_i\overline{v_{i_\ell}^2}\right)}{\partial t}\right]_{\substack{Ph\\j\leftrightarrow\ell}} - \Delta_{j\ell}\left(\frac{\partial_e n_e}{\partial t}\right)_{\substack{Ph\\j\leftrightarrow\ell}} \qquad (116)$$

This expression can be interpreted for this interaction as: The net rate of loss of electron kinetic energy per unit volume is equal to the sum of net gain of energy by the photons, the neutrals, the ions plus the ionization potential times the net rate of production of electrons. This statement may appear to be misleading unless it is noted that when the net production rate of electrons is positive, the net rate of change of radiant energy will be negative.

On summing over states and species we can write for equation (101)

$$\left[\frac{\partial_e\left(\frac{1}{2}\,n_e m_e\overline{v_e^2}\right)}{\partial t}\right]_{Ph} = -\left(\frac{\partial_e u^R}{\partial t}\right)_{Ph} - \left[\frac{\partial_e\left(\frac{1}{2}\,n_N m_N\overline{v_N^2}\right)}{\partial t}\right]_{Ph}$$

$$-\left[\frac{\partial_e\left(\frac{1}{2}\,n_I m_I\overline{v_I^2}\right)}{\partial t}\right]_{Ph} - \sum_{j,\ell,n}\Delta_{j\ell}\left(\frac{\partial_e n_e}{\partial t}\right)_{\substack{Ph\\j\leftrightarrow\ell}} \qquad (117)$$

Radiative Excitation and De-excitation

The rate of change of a property of the neutral species as a result of radiative excitation and its inverse can be expressed as (see eqs. (86), (81), (71)-(73)):

$$\left[\frac{\partial_e(n_n\overline{\phi}_n)}{\partial t}\right]_{Ra} = \sum_j \left[\frac{\partial_e(n_{n_j}\phi_{n_j})}{\partial t}\right]_{Ra} = \sum_j \int \phi_{n_j}\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ra} d^3v_{n_j}$$

$$= \sum_j \int \phi_{n_j}\left[\sum_{\substack{k\\j<k}} \int\int\left(f_{n_k} - f_{n_j}\frac{\overline{IB}_j^k}{\overline{P}_k^j}\right)\overline{P}_k^j\psi_j^k\,d\nu\right.$$

$$\left. + \sum_{\substack{k\\k<j}}\int\int\left(f_{n_k}\frac{\overline{IB}_k^j}{\overline{P}_j^k} - f_{n_j}\right)\overline{P}_j^k\psi_j^k\,d\nu\right]d^3v_{n_j} \tag{118}$$

For $\phi_{n_j} = 1$ we obtain from a typical term in (118)

$$\left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{Ra} = \int\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{Ra} d^3v_{n_j} = \int\left[\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ra\\j<k}} + \left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ra\\j>k}}\right]d^3v_{n_j}$$

$$= \int\left[\sum_{\substack{k\\j<k}}\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ra\\j\leftrightarrow k}} + \sum_{\substack{k\\j>k}}\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ra\\k\leftrightarrow j}}\right]d^3v_{n_j}$$

$$= \int\sum_{\substack{k\\j<k}}\int\left(f_{n_k} - f_{n_j}\frac{\overline{IB}_j^k}{\overline{P}_k^j}\right)\overline{P}_k^j\psi_j^k\,d\nu\,d^3v_{n_j}$$

$$+ \int\sum_{\substack{k\\j>k}}\int\left(f_{n_k}\frac{\overline{IB}_k^j}{\overline{P}_j^k} - f_{n_j}\right)\overline{P}_j^k\psi_j^k\,d\nu\,d^3v_{n_j} \tag{119}$$

From a typical term in equation (119) or from equations (69) and (70), we can deduce

$$\left(\frac{\partial_e f_{n_j}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}} = -\left(\frac{\partial_e f_{n_k}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}}$$

since

$$d^3 v_{n_j} = d^3 v_{n_k}$$

This leads directly to the following simple conservation of species expression:

$$\left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}} = -\left(\frac{\partial_e n_{n_k}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}} \tag{120}$$

Applying these expressions to equation (119) yields

$$\left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{Ra} = \sum_{\substack{k \\ j < k}}' \left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}} + \sum_{\substack{k \\ j > k}} \left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{\substack{Ra \\ k \leftrightarrow j}} \tag{121}$$

On summing over all states j we now obtain

$$\left(\frac{\partial_e n_n}{\partial t}\right)_{Ra} = \sum_{j}' \left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{Ra} = 0 \tag{122}$$

That is, the total number density of a species does not change as a result of radiative excitation and de-excitation type encounters. In obtaining equation (122) we made use of $d^3 v_{n_j} = d^3 v_{n_k}$, $\psi_j^k = \psi_k^j$, and expressions like (120) for both $j < k$ and $k < j$. (Since k and j range over the same number of states, there is always an even number of terms in (122) that will cancel in pairs according to equation (120).)

If $\phi^R = 1$ in the first term on the right-hand side of equation (85) is combined with (78), and the result is compared with the first moment of equations (71) and (72),

$$\left(\frac{\partial_e n^R}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k \\ k > j}} = \left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k \\ k > j}} = -\left(\frac{\partial_e n_{n_k}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k \\ k > j}}$$

40

which extends to

$$\left(\frac{\partial_e n^R}{\partial t}\right)_{Ra} = \sum_{\substack{j,k \\ k>j}} \left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}} = -\sum_{\substack{j,k \\ k>j}} \left(\frac{\partial_e n_{n_k}}{\partial t}\right)_{\substack{Ra \\ j \leftrightarrow k}}$$

That is, the net rate at which photons are created is related to the net rate at which excited particles are destroyed.

For $\phi_{n_j} = m_n \vec{v}_{n_j}$ a typical term in (118) can be written

$$\left[\frac{\partial_e\left(n_{n_j} m_n \overline{\vec{v}_{n_j}}\right)}{\partial t}\right]_{Ra} = \int \left[\sum_{\substack{k \\ j<k}} \int m_n \vec{v}_{n_j} \left(f_{n_k} - f_{n_j} \frac{\overline{IB}_j^k}{\overline{P}_k^j}\right) \overline{P}_k^j \psi_j^k \, d\nu \right.$$

$$\left. + \sum_{\substack{k \\ j>k}} \int m_n \vec{v}_{n_j} \left(f_{n_k} \frac{\overline{IB}_k^j}{\overline{P}_j^k} - f_{n_j}\right) \overline{P}_j^k \psi_j^k \, d\nu \right] d^3 v_{n_j} \qquad (123)$$

For a two-state encounter, making use of the momentum balance $m_n \vec{v}_{n_j} = m_n \vec{v}_{n_k}$ yields

$$\left[\frac{\partial_e\left(n_{n_j} m_n \overline{\vec{v}_{n_j}}\right)}{\partial t}\right]_{\substack{Ra \\ j \leftrightarrow k}} = -\left[\frac{\partial_e\left(n_{n_k} m_n \overline{\vec{v}_{n_k}}\right)}{\partial t}\right]_{\substack{Ra \\ j \leftrightarrow k}} \qquad (124)$$

On summing over all states j, making use of (124), the momentum balance for each specific encounter, and the relations used in obtaining (122) we conclude that the momentum of the neutral species does not change as a result of these nonelastic encounters; that is,

$$\left[\frac{\partial_e\left(n_n m_n \overline{\vec{v}_n}\right)}{\partial t}\right]_{Ra} \equiv \sum_j \left[\frac{\partial_e\left(n_{n_j} m_n \overline{\vec{v}_{n_j}}\right)}{\partial t}\right]_{Ra} = 0 \qquad (125)$$

For $\phi_{n_j} = m_n v_{n_j}^2/2$ we can use the energy equation (62) and write a typical term in equation (118) as

$$\left[\frac{\partial_e\left(n_{n_j}\frac{1}{2}m_n\overline{v_{n_j}^2}\right)}{\partial t}\right]_{Ra} = \int\left[\sum_{\substack{k\\j<k}}\int\left(\frac{1}{2}m_nv_{n_k}^2 + \Delta_{jk} - h\nu\right)\left(f_{n_k} - f_{n_j}\frac{\overline{IB}_j^k}{\overline{P}_k^j}\right)\overline{P}_k^j\psi_j^k\,d\nu\right.$$

$$\left. + \sum_{\substack{k\\j>k}}\int\left(\frac{1}{2}m_nv_{n_k}^2 + \Delta_{jk} + h\nu\right)\left(f_{n_k}\frac{\overline{IB}_k^j}{\overline{P}_j^k} - f_{n_j}\right)\overline{P}_j^k\psi_j^k\,d\nu\right]d^3v_{n_j}$$

<div align="right">(126)</div>

From equations (78), (81) (eq. (81) applied to photons), and (115)

$$\left(\frac{\partial_e u^R}{\partial t}\right)_{\substack{Ra\\j\leftrightarrow k\\j<k}} = \iint h\nu\left(f_{n_k} - f_{n_j}\frac{\overline{IB}_j^k}{\overline{P}_k^j}\right)\overline{P}_k^j\psi_j^k\,d\nu\,d^3v_{n_j}$$

<div align="right">(127)</div>

and

$$\left(\frac{\partial_e u^R}{\partial t}\right)_{\substack{Ra\\k\leftrightarrow j\\j>k}} = \iint h\nu\left(f_{n_j} - f_{n_k}\frac{\overline{IB}_k^j}{\overline{P}_j^k}\right)\overline{P}_j^k\psi_k^j\,d\nu\,d^3v_{n_k}$$

<div align="right">(128)</div>

Also using equations (71), (72), (120), and $d^3v_{n_j} = d^3v_{n_k}$ we have

$$\Delta_{jk}\left(\frac{\partial_e n_{n_k}}{\partial t}\right)_{\substack{Ra\\j\leftrightarrow k\\j<k}} = \iint\Delta_{jk}\left(f_{n_j}\frac{\overline{IB}_j^k}{\overline{P}_k^j} - f_{n_k}\right)\overline{P}_k^j\psi_k^j\,d\nu\,d^3v_{n_k}$$

<div align="right">(129)</div>

and

$$\Delta_{jk}\left(\frac{\partial_e n_n}{\partial t}\right)_{\substack{Ra\\k\leftrightarrow j\\j>k}} = \iint\Delta_{jk}\left(f_{n_k}\frac{\overline{IB}_k^j}{\overline{P}_j^k} - f_{n_j}\right)\overline{P}_j^k\psi_j^k\,d\nu\,d^3v_{n_k}$$

<div align="right">(130)</div>

42

Substituting these expressions into equation (126) yields

$$\left[\frac{\partial_e\left(n_{n_j}\,\frac{1}{2}\,m_n\overline{v_{n_j}^2}\right)}{\partial t}\right]_{Ra} = \int\left[\sum_{\substack{k\\j<k}}\int\frac{1}{2}\,m_nv_{n_k}^2\left(f_{n_k} - f_{n_j}\,\frac{\overline{IB}_j^k}{\overline{P}_k^j}\right)\overline{P}_k^j\psi_j^k\,d\nu\right.$$

$$+ \sum_{\substack{k\\j>k}}\int\frac{1}{2}\,m_nv_{n_k}^2\left(f_{n_k}\,\frac{\overline{IB}_k^j}{\overline{P}_j^k} - f_{n_j}\right)\overline{P}_j^k\psi_j^k\,d\nu\Bigg]d^3v_{n_j}$$

$$- \sum_{\substack{k\\j<k}}\left[\left(\frac{\partial_e u^R}{\partial t}\right)_{\substack{Ra\\j\leftrightarrow k}} + \Delta_{jk}\left(\frac{\partial_e n_{n_k}}{\partial t}\right)_{\substack{Ra\\j\leftrightarrow k}}\right]$$

$$- \sum_{\substack{k\\j>k}}\left[\left(\frac{\partial_e u^R}{\partial t}\right)_{\substack{Ra\\k\leftrightarrow j}} + \Delta_{jk}\left(\frac{\partial_e n_{n_k}}{\partial t}\right)_{\substack{Ra\\k\leftrightarrow j}}\right]\tag{131}$$

Now on summing over states j the last two sets of sums on the right-hand side are symmetric and

$$\sum_{\substack{k,j\\j<k}}\left(\frac{\partial_e u^R}{\partial t}\right)_{\substack{Ra\\j\leftrightarrow k}} = \sum_{\substack{k,j\\j>k}}\left(\frac{\partial_e u^R}{\partial t}\right)_{\substack{Ra\\k\leftrightarrow j}} \equiv \left(\frac{\partial_e u^R}{\partial t}\right)_{Ra}$$

and

$$\sum_{\substack{k,j\\j<k}}\Delta_{jk}\left(\frac{\partial_e n_{n_k}}{\partial t}\right)_{\substack{Ra\\j\leftrightarrow k}} = \sum_{\substack{k,j\\j>k}}\Delta_{jk}\left(\frac{\partial_e n_{n_k}}{\partial t}\right)_{\substack{Ra\\k\leftrightarrow j}}$$

43

With these expressions, equation (131) when summed over the j states becomes

$$\left[\frac{\partial e\left(n_n \frac{1}{2} m_n \overline{v_n^2}\right)}{\partial t}\right]_{Ra} = \sum_j \left[\frac{\partial e\left(n_{n_j} \frac{1}{2} m_n \overline{v_{n_j}^2}\right)}{\partial t}\right]_{Ra}$$

$$= \sum_{\substack{j,k \\ j<k}} \iint \frac{1}{2} m_n v_{n_k}^2 \left(f_{n_k} - f_{n_j} \frac{\overline{IB}_j^k}{\overline{P}_k^j}\right) \overline{P}_k^j \psi_j^k \, d\nu \, d^3v_{n_j}$$

$$+ \sum_{\substack{j,k \\ j>k}} \iint \frac{1}{2} m_n v_{n_k}^2 \left(f_{n_k} \frac{\overline{IB}_k^j}{\overline{P}_j^k} - f_{n_j}\right) \overline{P}_j^k \psi_j^k \, d\nu \, d^3v_{n_j} + 2\left(\frac{\partial e^{u^R}}{\partial t}\right)_{Ra}$$

$$- 2 \sum_{\substack{k,j \\ j<k}} \Delta_{jk} \left(\frac{\partial e^{n_{n_k}}}{\partial t}\right)_{\substack{Ra \\ j\leftrightarrow k}} \tag{132}$$

The first two sums on the right-hand side of (132) can, on comparison with (71) and (72), be written as

$$-\sum_{\substack{k \\ j<k}} \left[\frac{\partial e\left(n_{n_k} \frac{1}{2} m_n \overline{v_{n_k}^2}\right)}{\partial t}\right]_{Ra}$$

and

$$-\sum_{\substack{k \\ j>k}} \left[\frac{\partial e\left(n_{n_k} \frac{1}{2} m_n \overline{v_{n_k}^2}\right)}{\partial t}\right]_{Ra}$$

44

These two terms can be combined, via equation (73), to obtain

$$\sum_k \left[\frac{\partial_e \left(n_{n_k} \frac{1}{2} m_n \overline{v_{n_k}^2} \right)}{\partial t} \right]_{Ra} = \sum_k \left\{ \left[\frac{\partial_e \left(n_{n_k} \frac{1}{2} m_n \overline{v_{n_k}^2} \right)}{\partial t} \right]_{\substack{Ra \\ j<k}} + \left[\frac{\partial_e \left(n_{n_k} \frac{1}{2} m_n \overline{v_{n_k}^2} \right)}{\partial t} \right]_{\substack{Ra \\ j>k}} \right\}$$

$$\equiv \left[\frac{\partial_e \left(n_n \frac{1}{2} m_n \overline{v_n^2} \right)}{\partial t} \right]_{Ra}$$

This result can be combined with equation (132) to yield

$$\left[\frac{\partial_e \left(n_n \frac{1}{2} m_n \overline{v_n^2} \right)}{\partial t} \right]_{Ra} = - \left(\frac{\partial_e u^R}{\partial t} \right)_{Ra} - \sum_{\substack{j,k \\ j<k}} \Delta_{jk} \left(\frac{\partial_e n_{n_k}}{\partial t} \right)_{\substack{Ra \\ j \leftrightarrow k}} \tag{133}$$

Equation (133) shows that the net gain of kinetic energy by the neutral species is equal to the net loss of energy by the photon gas and the net loss of internal energy from the upper excited states.

Momentum Transfer

If $\phi(\vec{v}_s) = m_e \vec{v}_s$ in the above rate expressions, and the appropriate momentum equation is used for each interaction, it is relatively easy to show that the momentum lost by the electrons in each type of nonelastic collision is gained by the heavy particles involved. We can summarize this result briefly as follows:

Inelastic and superelastic encounters.- The momentum expressions for the reaction given by equation (2) which correspond to equations (3) and (4), respectively, are:

$$m_e \vec{v}_s + m_n \vec{v}_{n_j} = m_e \vec{v}_o + m_n \vec{v}_{n_k}$$

or

$$m_e \vec{v}_1 + m_n \vec{v}_{n_j} = m_e \vec{v}_s + m_n \vec{v}_{n_k}$$

When these equations are substituted into the rate equation (90) with $\phi(\vec{v}_s) = m_e \vec{v}_s$ and appropriate variable changes are made,

$$\left[\frac{\partial_e\left(n_e m_e \overline{\vec{v}_e}\right)}{\partial t}\right]_{Ex} = -\left[\frac{\partial_e\left(n_N m_N \overline{\vec{v}_N}\right)}{\partial t}\right]_{Ex} \tag{134}$$

where $\overline{\vec{v}_N} \equiv (1/m_N) \sum_n m_n \overline{\vec{v}_n}$. This expression illustrates the momentum exchange between the electrons and the neutral particles.

Ionization and three-body recombination.- Here the relevant momentum equations are

$$m_e \vec{v}_s + m_n \vec{v}_{n_j} = n_e \vec{v}_2 + m_e \vec{v}_4 + m_i \vec{v}_{i\ell}$$

$$m_e \vec{v}_1 + m_n \vec{v}_{n_j} = m_e \vec{v}_s + m_e \vec{v}_3 + m_i \vec{v}_{i\ell}$$

When these expressions are combined with the momentum form of equation (99) and appropriate variable changes are made,

$$\left[\frac{\partial_e\left(n_e m_e \overline{\vec{v}_e}\right)}{\partial t}\right]_{Ion} = -\left[\frac{\partial_e\left(n_I m_I \overline{\vec{v}_I}\right)}{\partial t}\right]_{Ion} - \left[\frac{\partial_e\left(n_N m_N \overline{\vec{v}_N}\right)}{\partial t}\right]_{Ion} \tag{135}$$

where $\overline{\vec{v}_I} \equiv (1/m_I) \sum_n m_i \overline{\vec{v}_i}$. The electron has exchanged momentum with both the neutrals and the ions.

Photoionization and two-body recombination.- When the momentum of the photons is neglected, the momentum equation for this reaction is

$$m_n \vec{v}_{n_j} = m_i \vec{v}_{i\ell} + m_e \vec{v}_s$$

On substitution into the momentum form of equation (109) and changing variables there remains

$$\left[\frac{\partial_e\left(n_e m_e \overline{\vec{v}_e}\right)}{\partial t}\right]_{Ph} = -\left[\frac{\partial_e\left(n_I m_I \overline{\vec{v}_I}\right)}{\partial t}\right]_{Ph} - \left[\frac{\partial_e\left(n_N m_N \overline{\vec{v}_N}\right)}{\partial t}\right]_{Ph} \tag{136}$$

Again the electrons have exchanged momentum with both heavy species.

Nonelastic Terms in the Equations of Motion

Thus far in this section the rate expressions associated with various nonelastic collision operators have been developed. Here the preceding results will be summarized with the following relations between nonelastic terms that might appear in the equations of motion of a plasma composed of electrons, ions, neutral species and photons:

Species conservation:

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{Ex} = \left(\frac{\partial_e n_N}{\partial t}\right)_{Ex} = \left(\frac{\partial_e n_N}{\partial t}\right)_{Ra} = 0$$

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{NE} = \left(\frac{\partial_e n_e}{\partial t}\right)_{Ion} + \left(\frac{\partial_e n_e}{\partial t}\right)_{Ph} = \left(\frac{\partial_e n_I}{\partial t}\right)_{NE} = -\left(\frac{\partial_e n_N}{\partial t}\right)_{\substack{Ion,\\Ph}}$$

$$(137)\,[7]$$

Mass conservation ($\rho_e \equiv n_e m_e$, etc.):

$$\left(\frac{\partial_e \rho_N}{\partial t}\right)_{NE} = -\left[\left(\frac{\partial_e \rho_e}{\partial t}\right)_{NE} + \left(\frac{\partial_e \rho_I}{\partial t}\right)_{NE}\right] \qquad (138)$$

Momentum conservation (neglect photon momentum):

$$\left[\frac{\partial_e(\rho_e \vec{v}_e)}{\partial t}\right]_{NE} = -\left[\frac{\partial_e(\rho_N \vec{v}_N)}{\partial t}\right]_{\substack{Ex,\\Ion,\\Ph}} - \left[\frac{\partial_e(\rho_I \vec{v}_I)}{\partial t}\right]_{\substack{Ion,\\Ph}} \qquad (139)$$

Energy conservation:

$$\left[\frac{\partial_e\left(\frac{1}{2}\rho_e \overline{v_e^2}\right)}{\partial t}\right]_{NE} = \sum_{j,k,n} \Delta_{jk}\left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{\substack{Ex\\j\leftrightarrow k}} + \sum_{j,\ell,n} \Delta_{j\ell}\left[\left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{\substack{Ion\\j\leftrightarrow \ell}} + \left(\frac{\partial_e n_{n_j}}{\partial t}\right)_{\substack{Ph\\j\leftrightarrow \ell}}\right]$$

$$-\left[\frac{\partial_e\left(\frac{1}{2}\rho_N \overline{v_N^2}\right)}{\partial t}\right]_{\substack{Ex,\\Ion,\\Ph}} - \left[\frac{\partial_e\left(\frac{1}{2}\rho_I \overline{v_I^2}\right)}{\partial t}\right]_{NE} - \left(\frac{\partial_e u^R}{\partial t}\right)_{Ph} \qquad (140)$$

$$\left[\frac{\partial_e\left(\frac{1}{2}\rho_N \overline{v_N^2}\right)}{\partial t}\right]_{NE} = \left[\frac{\partial_e\left(\frac{1}{2}\rho_N \overline{v_N^2}\right)}{\partial t}\right]_{\substack{Ex,\\Ion,\\Ph}} - \left(\frac{\partial_e u^R}{\partial t}\right)_{Ra} - \sum_{j,k,n} \Delta_{jk}\left(\frac{\partial_e n_{n_k}}{\partial t}\right)_{\substack{Ra\\j\leftrightarrow k}} \qquad (141)$$

---

[7] The last term in equation (137) represents $(\partial_e n_N/\partial t)_{Ion} + (\partial_e n_N/\partial t)_{Ph}$. This sum can also be represented by $(\partial_e n_N/\partial t)_{NE}$ (used in eq. (138)) since $(\partial_e n_N/\partial t)_{Ex,Ra} = 0$.

$$\left(\frac{\partial_e u^R}{\partial t}\right)_{NE} = \left(\frac{\partial_e u^R}{\partial t}\right)_{Ph} + \left(\frac{\partial_e u^R}{\partial t}\right)_{Ra} \tag{142}$$

These expressions agree with the usual macroscopic species conservation equations (ref. 19).

If we sum over species such that

$$n \equiv n_N + n_e + n_I$$

$$\rho \equiv \rho_N + \rho_e + \rho_I$$

$$\rho\overline{\vec{v}} \equiv \rho_N\overline{\vec{v}_N} + \rho_I\overline{\vec{v}_I} + \rho_e\overline{\vec{v}_e}$$

and

$$\rho\overline{v^2} \equiv \rho_N\overline{v_N^2} + \rho_I\overline{v_I^2} + \rho_e\overline{v_e^2} + u^R$$

we find

$$\left(\frac{\partial_e n}{\partial t}\right)_{NE} = \left(\frac{\partial_e n_e}{\partial t}\right)_{NE}$$

$$\left(\frac{\partial_e \rho}{\partial t}\right)_{NE} = 0$$

$$\left[\frac{\partial_e(\rho\overline{\vec{v}})}{\partial t}\right]_{NE} = 0$$

$$\left[\frac{\partial_e\left(\frac{1}{2}\rho\overline{v^2}\right)}{\partial t}\right]_{NE} = \sum_{j,k,n} \Delta_{jk}\left(\frac{\partial_e n_j}{\partial t}\right)_{\substack{Ex,Ra \\ j\leftrightarrow k}} + \sum_{j,\ell,n} \Delta_{j\ell}\left(\frac{\partial_e n_j}{\partial t}\right)_{\substack{Ion,Ph \\ j\leftrightarrow\ell}}$$

$$\left.\vphantom{\sum}\right\} \tag{143}$$

Equations (143) also agree with the usual macroscopic (ref. 19) conservation equations when the right-hand side is identified as the nonelastic rate of change of the average internal energy of the gas mixture.

48

Radiative Energy Loss

A common phenomenon accompanying nonelastic collisions in plasmas is the loss of radiative energy to the boundaries of the plasma. The amount of radiative energy lost to the plasma depends on many factors such as the plasma density, its optical characteristics, its geometrical characteristics and the type of transition. This energy loss mechanism affects, in a coupled fashion, the electron distribution function and the species number densities as well as the plasma thermal energy.

Physically, in a plasma in which the electrons are redistributing their energy among the other species the following might be expected:

(1) Energetic electrons collisionally excite or ionize heavy particles.

(2) These heavy particles in turn either undergo the inverse to the preceding encounters or relax radiatively, emitting photons. (The method of relaxation depends upon the relaxation rates for each process. This will be discussed in the next section.)

(3) The resulting photons are then either absorbed by or lost to the plasma, depending upon the opacity of the system to the frequency of the emitted radiation.

A loss of photons manifests itself in at least two significant ways:

(1) The excited states of the heavy particles, from which the radiative relaxation occurs, tend to become depopulated.

(2) The electron gas looses energy equivalent to the volumetric radiant energy lost from the plasma.

Analytically the expressions developed in the preceding and subsequent sections remain valid. However, in application one must solve the equation of radiative transfer, coupled with the appropriate equations of motion, to find the local rate at which photons and the radiant energy leave the plasma.

The rate at which photons are lost from the plasma can be related to the rate at which excited states of the heavy particles are depopulated as can be seen from the radiative transfer equation (ref. 17) and the results of pages 35 to 45. In addition the energy moment of the radiative transfer equation yields a relation between the temporal and volumetric changes in the photon energy density and the collisional term given by equation (142).

Detailed accounting of radiative losses is a formidable task. To circumvent some of the difficulties Holstein (ref. 20) introduced and others (ref. 9) used local, frequency-dependent, energy loss factors. Holstein, in particular, evaluated radiation-escape factors for resonance radiation for a few geometric shapes.

These local radiation-escape parameters are defined for each transition as unity minus the ratio of photon absorption to photon emission. Thus, the

difference between the emission and absorption terms appearing in the preceding expressions would be replaced by the product of the particular radiation-escape parameter and the appropriate emission term. The value of these parameters (between 0 and 1) determines the local opacity of the plasma to a particular frequency of radiation.

Reaction Rate Coefficients

The usual reaction rate coefficients for a reaction which proceeds in a forward K_f or backward K_b direction can be obtained directly from the preceding results. For example, consider the collisional ionization-three-body recombination reaction described by equation (20): The local net rate of change in electron number density as a result of this reaction is determined by an equation of the form

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{Ion} = K_{Ion} n_e n_N - K_{Rec} n_e^2 n_I \tag{144}$$

From equations (74) and (104) we can deduce that

$$\left(\frac{\partial_e n_e}{\partial t}\right)_{Ion} = \sum_{j,\ell,n} \iiiint (f_1 f_{n_j} H^{-1} - f_s f_3 f_{i_\ell}) g_{s\ell} g_{3\ell} \sigma_{s3\ell}^j \, d\Omega_{1s} \, d^3 v_3 \, d^3 v_s \, d^3 v_{i_\ell} \tag{145}$$

A comparison of (144) and (145) yields

$$K_{Ion} = \frac{1}{n_e n_N} \sum_{j,\ell,n} \iiiint f_1 f_{n_j} H^{-1} g_{s\ell} g_{3\ell} \sigma_{s3\ell}^j \, d\Omega_{1s} \, d^3 v_3 \, d^3 v_s \, d^3 v_{i_\ell}$$

and

$$K_{Rec} = \frac{1}{n_e^2 n_I} \sum_{j,\ell,n} \iiiint f_s f_3 f_{i_\ell} g_{s\ell} g_{3\ell} \sigma_{s3\ell}^j \, d\Omega_{1s} \, d^3 v_3 \, d^3 v_s \, d^3 v_{i_\ell}$$

If desired the detailed balancing expression given by equation (32) can be used to obtain K_f and/or K_b in terms of ionization cross sections rather than recombination cross sections. These expressions clearly show the dependence of the reaction rate coefficients upon the distribution functions

and the cross sections. In addition one can deduce that the macroscopic detailed balance result, $K_{Ion}/K_{Rec} = (n_e n_I/n_N)_{Eq}$, does not hold in general but does hold at equilibrium.

In a like manner rate coefficients for the other reactions considered herein can be obtained. Such coefficients for unidirectional reactions are particularly useful when competing reactions have dominant terms. In these cases one needs to consider only the rate coefficients for the dominant processes rather than the complete nonelastic collision integrals.

VI. CONCLUDING REMARKS

Collision integrals have been developed for some nonelastic processes that are important in plasma dynamic problems. On taking moments of these collision integrals, we have also demonstrated compatibility between these results and the nonelastic terms of the macroscopic equations of motion.

The collision integrals developed here may prove to be unwieldy in their present form for many plasma dynamic analyses. Many simplifications are possible to make the collision integrals more tractable. The particular simplification employed depends upon the problem at hand. A few of the more successful simplifications, that is, the small electron mass approximation, the spherical harmonic expansion of the electron distribution function, the use of Maxwellian distribution functions, and the use of isotropic cross sections for nonelastic processes, are treated in some detail in the first 16 references.

It is hoped that the development presented herein allows a clearer understanding of the physics implicit in many less detailed treatments of nonelastic collision integrals.

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NOMENCLATURE

This is a partial alphabetical listing of the present nomenclature. All symbols are defined locally in the text where they are first used.

$\left.\begin{array}{c} A_k^j \\ \\ B_k^j \end{array}\right\}$ coefficients for radiative de-excitation and excitation (eqs. (65)-(69))

c speed of light

e electron

f_m electron velocity distribution function, $f(\vec{v}_m)$

$\left.\begin{array}{c} f^R(\nu, \chi_\nu), \\ \\ \bar{f}^R(\nu) \end{array}\right\}$ photon distribution functions (eq. (39))

f_h velocity distribution function for heavy particle h

g_{mh} relative speed $|\vec{v}_m - \vec{v}_h|$

H parameter associated with Saha equation $\dfrac{h^3}{m_e^3} \dfrac{\omega_j}{2\omega_\ell}$

h Planck's constant

$\bar{I}(\nu)$ specific intensity of ν-radiation integrated over all solid angles

$|J|$ Jacobian of a transformation, usually defined locally

K_{Ion}, K_{Rec} reaction rate coefficients defined on page 50

k Boltzmann's constant

m_e electron mass

m_h mass of heavy particle

n_e, n_h number densities: electrons, heavy particles

$\bar{P}_\ell^j(g_{s\ell})$ differential cross section for radiative capture (eqs. (41) and (42))

$\overline{P}_k^j(\nu)$ coefficient of differential cross section for radiative
de-excitation (eqs. (64) and (65))

$\left.\begin{array}{l} Q_j^\ell \\ \\ Q_j^k \end{array}\right\}$ distribution functions associated with photionization and radiative
excitation cross section (eqs. (39) and (63))

T equilibrium temperature

u^R radiant energy density

$\vec{v}_m, \vec{v}_h$ velocity of electrons, velocity of heavy particles

$\vec{v}_{mm'}$ electron velocity defined by $(v_m, \vec{\chi}_{mm'})$

$\left.\begin{array}{l} \alpha_\ell^j \\ \\ \\ \beta_\ell^j \end{array}\right\}$ coefficients for spontaneous and induced radiative capture (eq. (42))

β parameter defined by, $\dfrac{m_e}{2kT}$

$\Delta_{jk}, \Delta_{j\ell}$ excitation or ionization potential associated with a particular
transition (eqs. (3), (21), (38), and (62))

$\delta_{\vec{v}_3, \vec{v}_s}$ delta function defined on pages 8-13

μ_i reduced mass, $\dfrac{m_e m_i}{m_e + m_i}$

ν radiation frequency

$\left.\begin{array}{l} \rho, \rho_e, \\ \\ \rho_N, \rho_I \end{array}\right\}$ mass densities defined on pages 46-48 ($\rho_e = n_e m_e$, etc.)

$\sigma_{mj}^k, \sigma_{mk}^j$ angular distribution functions associated with inelastic and
superelastic differential cross sections (eqs. (5) and (6))

$\left.\begin{array}{l} \sigma_{mh}^{h'} \\ \\ \sigma_{mm'h}^{h'} \end{array}\right\}$ distribution functions associated with ionization and three-body
recombination cross sections (eqs. (23) and (24))

ϕ property of electron gas, $\phi(\vec{v}_m)$

ϕ_h property of heavy particles, $\phi(\vec{v}_h)$

53

ϕ^R property of photon gas, $\phi^R(\nu)$

$\chi_{mm'}$ angle of "deflection" of electrons

$\vec{\chi}_{g_s g_3}$ angle between relative velocity vectors $\vec{g}_{s\ell}$ and $\vec{g}_{3\ell}$

ψ_j^k, ψ_k^j line shape factors for absorption and emission

Ω solid angle used in describing particle direction

$\omega_k, \omega_j, \omega_\ell$ degeneracies associated with states of an atom (ω_j, ω_k) and its ion (ω_ℓ)

Superscripts

$(\bar{\ })$ averaged quantity; velocity averaged or averaged over angles

j, k, ℓ see subscripts j, k, ℓ

Subscripts

m   association with electron of speed v_m (m = s, 0, 1, 2, 3, 4)

m'   association with electron of speed v_m'

n   neutral species of type n

n_j, n_k neutral particles of type n in states j, k

i   ion associated with type n neutral

i_ℓ ion in state ℓ associated with type n neutral

h   heavy particle, i, n, n_j, n_k, i_ℓ, I, N, and states j, k, ℓ

I   ion mixture

CM   center-of-mass coordinates

Eq   equilibrium

NE   quantities associated with nonelastic encounters

Ex, Ion, Ra, Ph   association with collisional excitation and de-excitation encounters, collisional ionization and three-body recombination encounters, radiative excitation and de-excitation encounters, and photo-ionization and two-body recombination encounters

$\left.\begin{array}{l} (\ )_{Ex} \\ \quad _{j \leftrightarrow k} \\ (\ )_{Ion} \\ \quad _{j \leftrightarrow \ell} \\ (\ )_{Ra} \\ \quad _{j \leftrightarrow k} \\ (\ )_{Ph} \\ \quad _{j \leftrightarrow \ell} \end{array}\right\}$ particular nonelastic encounter and its inverse which results in the heavy particle changing between a neutral in state j to a neutral in state k or an ion in state ℓ

j, k, ℓ states of a neutral particle (j,k) and its ion (ℓ)

Miscellaneous

$(Atom)^j_n$ atom of type n in state j

$\dfrac{\partial_e}{\partial t}$ rate of change of quantity per unit volume in phase space as a result of collisions

$\left.\begin{array}{l} d^3 v_m, \\ d^3 v_h \end{array}\right\}$ volume elements in velocity space

55

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