In this paper we study the behavior of certain solutions of the quadratic matrix equation

\[ A'K + KA - KBB'K = -\rho C'C \]  

as a function of a real variable \( \rho \). Our main result is a new a priori bound on the solutions. The methods we use draw freely on the variational interpretation of the associated Riccati equation

\[ \dot{K} = A'K + KA - KBB'K + \rho C'C \]

as well as the use of transform techniques and an elementary version of Parseval's formula.

1. Preliminaries

Let \( A, B, \) and \( C \) be real, constant matrices of dimensions \( n \) by \( n \), \( n \) by \( m \) and \( q \) by \( n \) respectively. By a linear system we mean a pair of equations

\[ \dot{x} = Ax + Bu ; \ y = Cx \]  

we also refer to the triple \([A,B,C]\) as a linear system with the understanding that \( A, B \) and \( C \) are the matrices appearing in equation (2). If the conditions

i) \( \text{rank} \ (B, AB, \ldots, A^{n-1}B) = n \)

and

ii) \( \text{rank} \ (C; CA; \ldots; CA^{n-1}) = n \)

where "\( ; \)" indicates column partition and "\( ; \)" a row partition are satisfied, we call \([A,B,C]\) a minimal linear system. Let \( I \) be the identity matrix. We define the spectral norm of a linear system as the minimum value of \( r > 0 \) such that the Hermetian matrix inequality

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The linear system \([A,B,C]\) will be considered together with a functional

\[ \eta = \int_0^\infty u'u + py'y \, dt \]  

whose minimization is to be considered. We take as known, the facts that under the hypothesis that \([A,B,C]\) is a minimal linear system there is,

1) at most one solution such that \(A-BB'K\) has its eigenvalues in \(\text{Res} < 0\)

2) \(\min_u \int_0^\infty u'u + py'y \, dt\) exists for \(p > 0\)

3) if the minimum exists, \(\min_u \int_0^\infty u'u + py'y \, dt = x'(0)K_1x(0)\)

where \(K_1\) is a solution of equation (1).

Items ii) and iii) are widely known since Kalman [1]; for a proof of i) see [2].

A few additional preliminary results will be required.

**Lemma 1**: Let \(u\) be given by \(u(t) = He^t g\). Let \(y\) be given by

\[ \dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t); \quad x(0) = 0 \]

Assume that the eigenvalues of \(A\) and \(F\) lie in the half-plane \(\text{Re} \, s < 0\). Then

\[ \int_0^\infty y'(t)y(t) \, dt \leq r^2 \int_0^\infty u'(t)u(t) \, dt \]

where \(r\) is the spectral norm of \([A,B,C]\).

**Proof**: Since \(x(0)\) is zero the Laplace transform of \(y\) is \(\hat{y} = RG\). Using Parseval's relation and the definition of the spectral norm we have

\[ \int_0^\infty y'(t)y(t) \, dt = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} u'(-i\omega)G'(-i\omega)G(i\omega)u(i\omega) \, d\omega \]

\[ \leq \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} u(-i\omega)(r^2I)u(i\omega) \, d\omega \]

\[ = r^2 \int_0^\infty u'(t)u(t) \, dt \]

**Lemma 2**: If \([A,B,C]\) is a linear system with a finite spectral norm then equation (1) has no real solution unless \(p > -r^2\).
Proof: We assume the contrary and look for a contradiction. Let \( K_1 \) be a real solution. Manipulation of equation (1) in the style of Yacubovich, Kalman et al. [3] gives

\[
(-Is-A')K_1 + K_1(Is-A) + K_1BB'K_1 = \rho C'C
\]

Pre- and post-multiplying by \((-Is-A')^{-1}\) and \((Is-A)^{-1}\) respectively gives

\[
K_1(Is-A)^{-1} + (-Is-A')^{-1}K_1 + (-Is-A')^{-1}K_1BB'K_1(Is-A)^{-1} = \rho(-Is-A')^{-1}C'C(Is-A)^{-1}
\]

Now if we pre and post multiply by \( B' \) and \( B \) respectively and add \( I \) to each side we obtain

\[
[I + R_1(-s)][I+R_1(s)] = I + \rho R'(-s)R(s)
\]

where \( R(s) = C(Is-A)^{-1}B \) and \( R_1(s) = B'K_1(Is-A)^{-1}B \). Since the left side is non-negative for \( s = \text{i} \omega \) the right must be also. Hence unless \( \rho \) is greater than \(-r^2\) we can have no solution.

Lemma 3: The quadratic equation

\[
A'K + KA - KBB'K = 0 \tag{5}
\]

has an invertible solution if and only if there exists an invertible solution of the linear equation

\[
L(Q) = QA' + AQ = BB' \tag{6}
\]

If the solution of the linear equation is unique and invertible then it is the only invertible solution of the quadratic equation.

Proof: If the quadratic equation has an invertible solution \( K_1 \) then pre and post multiplication of \( A'K_1 + K_1A - K_1BB'K_1 \) by \( K_1^{-1} \) gives \( K_1^{-1}A^{-1}K_1 = BB' \).

On the other hand, if \( Q \) is an invertible solution of the linear equation then its inverse satisfies the quadratic equation. Uniqueness follows by the same reasoning.

2. The Case where \( A \) has its Eigenvalues in \( \text{Re} \ s < 0 \)

Using these results it is possible to investigate the solutions of the quadratic matrix equation

\[
KA + A'K - KBB'K = \rho C'C
\]

and to relate them to the solution of the linear equation
Our notation will be as follows. By $K_+(p)$ we mean the (unique) solution of equation (1) having the property that the eigenvalues of $A-BB'K$ lie in the half-plane $Re s < 0$. Likewise, we let $K_-(p)$ be the solution of equation (1) having the property that all the eigenvalues of $A-BB'K$ lie in the half plane $Re s > 0$.

**Lemma 4**: Let $[A,B,C]$ be a minimal linear system with finite spectral norm $r$. Assume there exists solutions $K_+$ and $K_-$ described above. Then $K_+(p) - K_-(p)$ is positive definite for $p > -r^2$ and

$$K_+(p) - K_-(p) = \left[ \int_0^\infty e^{BB'e} \left[ A-BB'K_+(p) \right]^t \left[ A-BB'K_+(p) \right]^t dt \right]^{-1}$$

or alternatively

$$K_+(p) - K_-(p) = \left[ \int_0^\infty e^{BB'e} \left[ A-BB'K_-(p) \right]^t \left[ A-BB'K_-(p) \right]^t dt \right]^{-1}$$

**Proof**: Direct manipulation shows that $K_+(p) - K_-(p)$ satisfies

$$[K_+(p) - K_-(p)][A-BB'K_+(p)] + [A-BB'K_-(p)]'[K_+(p) - K_-(p)] = -[K_+(p) - K_-(p)]BB'[K_+(p) - K_-(p)]$$

From Lemma 3 we see that if there exists an invertible solution $[K_+(p) - K_-(p)]$ it must satisfy equation (6) and conversely. However, it is easily seen that the given expressions for $K_+(p) - K_-(p)$ are well defined and invertible as long as $A-BB'K(p)$ has its eigenvalues in $Re s < 0$ using standard results from controllability theory [2].

The following theorem gives a bound on the solution of equation (1) in terms of the solution of $KA + A'K = -C'C$ and the spectral norm $r$.

**Theorem 1**: Let $[A,B,C]$ be a minimal linear system with spectral norm $r$. Assume that the eigenvalues of $A$ lie in the half-plane $Re s < 0$. Then for $p > -r^2$ there exists a solution of $KA + A'K - KBBK = -\rho C'C$ which has the property that $A-BB'K_+(p)$ has its eigenvalues in $Re s < 0$ and

$$K_1\rho > K_+(p) > K_1\rho / (1+r^2\rho)$$

where $K_1$ is the solution of $AK_1 + K_1A = -C'C$. Moreover, there are no other solutions of $KA + A'K - KBB'K = -\rho C'C$ which have the property that $A-BBK$ has its eigenvalues in $Re s < 0$. 
Proof: First of all observe that the upper bound on \( K_+(\rho) \) is obvious from the variational interpretation of \( K_+(\rho) \) since by letting \( u \) be zero we obtain

\[
\int_0^\infty u^2 + \rho y^2 \, dt = \rho x'(0)K_+x(0)
\]

We know that for the minimal linear system (2) we have

\[
\min_u \int_0^\infty u'(t)u(t) + \rho y'(t)y(t) \, dt = x'(0)K_+(\rho)x(0)
\]

provided \( A-BB'K_+(\rho) \) has its eigenvalues in \( \text{Re } s < 0 \). If \( u_o \) and \( y_o \) denote the optimal control and the optimal response then

\[
y_o = Ce^{[A-BB'K_+(\rho)]}x(0) ; \quad u_o = [A-BB'K_+(\rho)]^{-1}x(0)
\]

Moreover, \( y_o \) can be expressed using transforms as the sum of an initial condition term and the effect of \( u_o \), i.e.

\[
y_o = C(I-A)^{-1}x(0) + R(s)u_o(s) = y_1(s) + y_2(s)
\]

In terms of this notation

\[
x'(0)K_+(\rho)x(0) = \int_0^\infty \rho[y_1'(t) + y_2'(t)][y_1(t) + y_2(t)] + u'(t)u(t) \, dt
\]

Using the preceding lemma we have

\[
r^2 \int_0^\infty u'(t)u(t) \, dt \geq \int_0^\infty y_1'(t)y_1(t) \, dt
\]

Also, from the known relationship between \( KA + A'K = -C'C \) and quadratic integrals we have

\[
x'(0)K_1x(0) = \int_0^\infty y'_1(t)y_1(t) \, dt
\]

Denote this last quantity by \( \mu^2 \) and let \( \nu^2 \) be defined by

\[
\nu^2 = \int_0^\infty y_2'(t)y_2(t) \, dt
\]

Combining these results we have

\[
x'(0)K_+(\rho)x(0) \geq \rho \mu^2 - 2 \rho |\rho| \int_0^\infty y_1'(t)y_2(t) \, dt + (\rho + r^{-2})\nu^2
\]

Now use the Schwartz inequality
\[ | \int_0^1 y'_1(t)y_2(t)dt | \geq \sqrt{\int_0^1 y'_1(t)y_1(t)dt} \sqrt{\int_0^1 y'_2(t)y_2(t)dt} \]

to obtain

\[ x'(0)K_+(\rho)x(0) \geq \rho \mu^2 - 2|\rho|\nu + (\rho + r^2)\nu^2 \]

Considering this as a function of \( \nu \), it has a minimum at \( \nu = \frac{|\rho|}{(\rho + r^2)} \) and the minimum value is \( \rho \mu^2 (1 + r^2) \). Therefore it is clear that for \( \rho > -r^2 \) the inequalities

\[ x'(0)K_1x(0) \geq x'(0)K_+(\rho)x(0) \geq x'(0)K_1x(0)\rho/(1 + r^2) \]

hold. The matrix inequality follows immediately.

To study existence we observe that a solution exists for \( \rho > 0 \) and by differentiation

\[ \left[ \frac{d}{d\rho} K_+(\rho) \right] [A-BB'K_+(\rho)] + [A-BB'K_+(\rho)]' \left[ \frac{d}{d\rho} K_+(\rho) \right] = -C'C \]

or

\[ \frac{d}{d\rho} K_+(\rho) = \int_0^\infty [A-BB'K_+(\rho)]t \ C'C \ e^{-C'Ct} dt \]

This differential equation can be integrated in the direction of decreasing \( \rho \) until \( A-BB'K_+(\rho) \) has an eigenvalue with a zero real part. In view of inequality (8), a solution \( K_+(\rho) \) will therefore exist for \( \rho > -r^2 \). To show that it also exists for \( \rho > r^2 \). Note that \( K_+(\rho) \) is monotone decreasing for \( \rho \) decreasing. By lemma 4 \( K_+(\rho) \) is bounded from below for \( \rho > -r^2 \) hence

\[ \lim_{\rho \to -r^2} K_+(\rho) = \overline{K} \]

exists and by continuity \( \overline{K} \) satisfies equation (1) with \( \rho = r^2 \).

Notice that the spectral norm of \([-A, B, C]\) is the same as that of \([A, B, C]\) and hence that there also exists a solution of

\[ (-A')K + K(-A) - KBB'K = -\rho C'C \]

which puts the eigenvalues of \(-A-BB'K\) in \( \text{Re } s < 0 \). The negative of this solution is \( K_-(\rho) \).
3. The Case where the Spectral Norm is Finite

We now extend the results of the previous section to a wider class of systems. The main result, Theorem 2, includes Theorem 1 as a special case but the proof makes a full case of Theorem 1.

We need the following lemma to reduce the general case to Theorem 1.

Lemma 5: If \( K_0 = K' \) is any solution of
\[
K_0 A + A'K_0 - K_0 BB'K_0 = 0
\]
And if \( K(\rho) \) is any solution of
\[
A'K(\rho) + K(\rho)A - K(\rho)BB'K(\rho) = -\rho CC'
\]
then
\[
[K(\rho) - K_0][A - BB'K_0] + [A - BB'K_0]'[K(\rho) - K_0]
+ [K(\rho) - K_0]BB'[K(\rho) + K_0] = -\rho CC'
\]

Proof: The proof is just a matter of expanding and using the definitions. The details are omitted.

As we have seen, equation (5) can have at most one invertible solution but it can have numerous non-invertible ones. In particular 0 is always a solution as are \( K_+(0) \) and \( K_-(0) \). However, the particular solution \( K_+(0) \) satisfies
\[
K_+(0)[A - BB'K_+(0)] + [A - BB'K_+(0)]'K_+(0) = -K_+(0)BB_+(0)
\]
Since \( A - BB'K_+(0) \) has its eigenvalues in \( \text{Re } s < 0 \) for \( \rho > 0 \) it will follow that its eigenvalues lie in \( \text{Re } s < 0 \) for \( \rho = 0 \) unless the spectral norm of
\[ [A - BB'K_+(0), B, C] \]
is infinite. This makes the following lemma of interest.

Lemma 6: If \( K_0 = K' \) is a solution of
\[
K_0 A + A'K_0 - K_0 BB'K_0 = 0
\]
such that \( A - BB'K_0 \) has its eigenvalues in the half-plane \( \text{Re } s < 0 \) then the spectral norms of \( [A, B, C] \) and \( [A - BB'K_0, B, C] \) are the same.

Proof: From lemma 5 see that if \( K_0 \) satisfies the hypothesis then
\[
K(A - BB'K_0) + (A - BB'K_0)'K - KBB'K = -\rho CC'
\]
has a solution if and only if there exists a solution of
Combining Lemma 4 and Theorem 1 we see that this equation has a solution if and only if \( \rho > r^2 \) where \( r \) is the spectral norm of \([A, B, C]\). Since the same is true for equation (11) the spectral norm of \([A-BB'K_0, B, C]\) must also be \( r^2 \).

Putting these lemmas together with Theorem 1 gives the following generalization of Theorem 1.

**Theorem 2**: Let \([A, B, C]\) be a minimal linear system with spectral norm \( r \).

Then for \( \rho > r^2 \) there exists a solution of equation (1) and

\[
K_1 \rho > K_+(\rho) - K_+(0) > K_1 \rho/(1+r^2 \rho)
\]

where \( K_1 \) is the solution of \([A-BB'K_+(0)]K_1+K_1[A-BB'K_+(0)] = -CC\). Moreover, there are no other solutions of \( KA + A'K - KBB'K = -\rho C'C \) which have the property that \( A-BB'K \) has its eigenvalues in \( \text{Re } s < 0 \).

4. Additional Comments

The results given here give the following (still incomplete) picture of the solutions of equation (1) under the hypothesis that \( A \) has no eigenvalue with zero real part.

i) There exist real solutions if and only if \( \rho > r^2 \)

ii) For \( \rho > r^2 \) there is exactly one solution such that \( A-BB'K \) has its eigenvalues in \( \text{Re } s < 0 \) and exactly one solution such that \( A-BB'K \) its eigenvalues in \( \text{Re } s > 0 \)

iii) \( K_+(\rho) - K_-(\rho) > 0 \)

Figure 1 suggests the main qualitative features and illustrates the bounds. Of course similar bounds hold for \( K_-(\rho) \).

We note that our results provide a new proof of certain important theorems on the absence of conjugate points [5]. Moreover, our proof does not use any results on spectral factorization of rational matrices.

Additional refinements of these ideas can be found in Canales' thesis [4].
Figure 1: A suggestive picture of the general behavior of $K_+(\rho)$ and $K_-(\rho)$.

If $K$ is one dimensional then $K_+(\rho)$ and $K_-(\rho)$ join at $\rho = -r^2$.

5. References


