STRUCTURAL PROPERTIES OF THE EQUILIBRIUM SOLUTIONS OF RICCATI EQUATIONS*

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In this paper we study the behavior of certain solutions of the quadratic matrix equation

$$A'K + KA - KBB'K = -\rho C'C \tag{1}$$

as a function of a real variable ρ. Our main result is a new a priori bound on the solutions. The methods we use draw freely on the variational interpretation of the associated Riccati equation

$$-\dot{K} = A'K + KA - KBB'K + \rho C'C$$

as well as the use of transform techniques and an elementary version of Parseval's formula.

1. Preliminaries

Let A, B, and C be real, constant matrices of dimensions n by n, n by m and q by n respectively. By a <u>linear system</u> we mean a pair of equations

$$\dot{x} = Ax + Bu \; ; \; y = Cx \tag{2}$$

we also refer to the triple [A,B,C] as a linear system with the understanding that A, B and C are the matrices appearing in equation (2). If the conditions

i) rank $(B, AB, \ldots, A^{n-1}B) = n$

and

ii) rank $(C; CA; \ldots; CA^{n-1}) = n$

where "," indicates column partition and ";" a row partition are satisfied, we call [A,B,C] a <u>minimal linear system</u>. Let I be the identity matrix. We define the <u>spectral norm</u> of a linear system as the minimum value of $r > 0$ such that the Hermetian matrix inequality

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$$Ir^2 - B'(-Ii\omega - A')^{-1}C'C(Ii\omega - A)^{-1}B \geq 0 \quad ; \quad i = \sqrt{-1} \tag{3}$$

holds for all real ω. If no such r exists the spectral norm is said to be infinite.

The linear system [A,B,C] will be considered together with a functional

$$\eta = \int_0^\infty u'u + \rho y'y dt \tag{4}$$

whose minimization is to be considered. We take as known, the facts that under the hypothesis that [A,B,C] is a minimal linear system there is,

i) at most one solution such that $A-BB'K$ has its eigenvalues in Res < 0

ii) $\min\limits_u \int_0^\infty u'u + \rho y'y dt$ exists for $\rho \geq 0$

iii) if the minimum exists, $\min\limits_u \int_0^\infty u'u + \rho y'y dt = x'(0)K_1 x(0)$

where K_1 is a solution of equation (1).

Items ii) and iii) are widely known since Kalman [1]; for a proof of i) see [2].

A few additional preliminary results will be required.

Lemma 1 : Let u be given by $u(t) = He^{Ft}g$. Let y be given by

$$\dot{x}(t) = Ax(t) + Bu(t) \ ; \ y(t) = Cx(t) \ ; \ x(0) = 0$$

Assume that the eigenvalues of A and F lie in the half-plane Re s < 0. Then

$$\int_0^\infty y'(t)y(t)dt \leq r^2 \int_0^\infty u'(t)u(t)dt$$

where r is the spectral norm of [A,B,C].

Proof : Since $x(0)$ is zero the Laplace transform of y is $\hat{y} = R\hat{u}$. Using Parseval's relation and the definition of the spectral norm we have

$$\begin{aligned}\int_0^\infty y'(t)y(t)dt &= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} u'(-i\omega)G'(-i\omega)G(i\omega)u(i\omega)d\omega \\ &\leq \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} u(-i\omega)(r^2 I)u(i\omega)d\omega \\ &= r^2 \int_0^\infty u'(t)u(t)dt\end{aligned}$$

Lemma 2 : If [A,B,C] is a linear system with a finite spectral norm then equation (1) has no real solution unless $\rho \geq -r^2$.

Proof : We assume the contrary and look for a contradiction. Let K_1 be a real solution. Manipulation of equation (1) in the style of Yacubovich, Kalman et al. [3] gives

$$(-Is-A')K_1 + K_1(Is-A) + K_1BB'K_1 = \rho C'C$$

Pre- and post-multiplying by $(-Is-A')^{-1}$ and $(Is-A)^{-1}$ respectively gives

$$K_1(Is-A)^{-1} + (-Is-A')^{-1}K_1 + (-Is-A')^{-1}K_1BB'K_1(Is-A)^{-1} = \rho(-Is-A')^{-1}C'C(Is-A)^{-1}$$

Now if we pre and post multiply by B' and B respectively and add I to each side we obtain

$$[I + R_1(-s)]'[I+R_1(s)] = I + \rho R'(-s)R(s)$$

where $R(s) = C(Is-A)^{-1}B$ and $R_1(s) = B'K_1(Is-A)^{-1}B$. Since the left side is non-negative for $s = i\omega$ the right must be also. Hence unless ρ is greater than $-r^2$ we can have no solution.

Lemma 3 : The quadratic equation

$$A'K + KA - KBB'K = 0 \tag{5}$$

has an invertible solution if and only if there exists an invertible solution of the linear equation

$$L(Q) = QA' + AQ = BB' \tag{6}$$

If the solution of the linear equation is unique and invertible then it is the only invertible solution of the quadratic equation.

Proof : If the quadratic equation has an invertible solution K_1 then pre and post multiplication of $A'K_1+K_1A-K_1BB'K_1$ by K_1^{-1} gives $A K_1^{-1}+K_1^{-1}A' = BB'$. On the other hand, if Q is an invertible solution of the linear equation then its inverse satisfies the quadratic equation. Uniqueness follows by the same reasoning.

2. The Case where A has its Eigenvalues in Re s < 0

Using these results it is possible to investigate the solutions of the quadratic matrix equation

$$KA + A'K - KBB'K = \rho C'C$$

and to relate them to the solution of the linear equation

$$KA + A'K = -C'C$$

Our notation will be as follows. By $K_+(\rho)$ we mean the (unique) solution of equation (1) having the property that the eigenvalues of $A-BB'K$ lie in the half-plane Re $s < 0$. Likewise we let $K_-(\rho)$ be the solution of equation (1) having the property that all the eigenvalues of $A-BB'K$ lie in the half plane Re $s > 0$.

Lemma 4 : Let $[A,B,C]$ be a minimal linear system with finite spectral norm r. Assume there exists solutions K_+ and K_- described above. Then $K_+(\rho) - K_-(\rho)$ is positive definite for $\rho > -r^{-2}$ and

$$K_+(\rho) - K_-(\rho) = [\int_0^\infty e^{[A-BB'K_+(\rho)]t} BB'e^{[A-BB'K_+(\rho)]'t} dt]^{-1}$$

or alternatively

$$K_+(\rho) - K_-(\rho) = [\int_{-\infty}^0 e^{[A-BB'K_-(\rho)]t} BB'e^{[A-BB'K_-(\rho)]'t} dt]^{-1}$$

Proof : Direct manipulation shows that $K_+(\rho)-K_-(\rho)$ satisfies

$$[K_+(\rho)-K_-(\rho)][A-BB'K_+(\rho)] + [A-BB'K_+(\rho)]'[K_+(\rho)-K_-(\rho)] =$$

$$- [K_+(\rho)-K_-(\rho)]BB'[K_+(\rho)-K_-(\rho)] \quad (7)$$

From Lemma 3 we see that if there exists an invertible solution $[K_+(\rho)-K_-(\rho)]$ it must satisfy equation (6) and conversely. However, it is easily seen that the given expressions for $K_+(\rho)-K_-(\rho)$ are well defined and invertible as long as $A-BB'K(\rho)$ has its eigenvalues in Re $s < 0$ using standard results from controllability theory [2].

The following theorem gives a bound on the solution of equation (1) in terms of the solution of $KA + A'K = -C'C$ and the spectral norm r.

Theorem 1 : Let $[A,B,C]$ be a minimal linear system with spectral norm r. Assume that the eigenvalues of A lie in the half-plane Re $s < 0$. Then for $\rho \geq -r^{-2}$ there exists a solution of $KA+A'K-KBBK = -\rho C'C$ which has the property that $A-BB'K_+(\rho)$ has its eigenvalues in Re $s < 0$ and

$$K_1\rho \geq K_+(\rho) \geq K_1\rho/(1+r^2\rho) \quad (8)$$

where K_1 is the solution of $AK_1 + K_1A = -C'C$. Moreover, there are no other solutions of $KA + A'K - KBB'K = -\rho C'C$ which have the property that $A-BBK$ has its eigenvalues in Re $s < 0$.

Proof : First of all observe that the upper bound on $K_+(\rho)$ is obvious from the variational interpretation of $K_+(\rho)$ since by letting u be zero we obtain

$$\int_0^\infty u^2 + \rho y^2 dt = \rho x'(0)K_1 x(0)$$

We know that for the minimal linear system (2) we have

$$\min_u \int_0^\infty u'(t)u(t) + \rho y'(t)y(t)dt = x'(0)K_+(\rho)x(0)$$

provided $A-BB'K_+(\rho)$ has its eigenvalues in Re $s < 0$. If u_o and y_o denote the optimal control and the optimal response then

$$y_o = Ce^{[A-BB'K_+(\rho)]}x(0) \; ; \; u_o = -B'K_+(\rho)^{[A-BB'K_+(\rho)]}x(0)$$

Moreover, y_o can be expressed using transforms as the sum of an initial condition term and the effect of u_o, i.e.

$$y_o = C(Is-A)^{-1}x(0) + R(s)u_o(s) \overset{def}{=} y_1(s) + y_2(s)$$

In terms of this notation

$$x'(0)K_+(\rho)x(0) = \int_0^\infty \rho[y_1'(t)+y_2'(t)][y_1(t)+y_2(t)] + u'(t)u(t)dt$$

Using the preceeding lemma we have

$$r^2 \int_0^\infty u'(t)u(t)dt \geq \int_0^\infty y_2'(t)y_2(t)dt$$

Also, from the known relationship between $KA + A'K = -C'C$ and quadratic integrals we have

$$x'(0)K_1 x(0) = \int_0^\infty y_1'(t)y_1(t)dt$$

Denote this last quantity by μ^2 and let ν^2 be defined by

$$\nu^2 = \int_0^\infty y_2'(t)y_2(t)dt$$

Combining these results we have

$$x'(0)K_+(\rho)x(0) \geq \rho\mu^2 - 2|\rho|.\left|\int_0^\infty y_1'(t)y_2(t)dt\right| + (\rho + r^{-2})\nu^2$$

Now use the Schwartz inequality

$$|\int_0^\infty y_1'(t)y_2(t)dt| \geq \sqrt{\int_0^\infty y_1'(t)y_1(t)dt}\ \sqrt{\int_0^\infty y_2'(t)y_2(t)dt}$$

to obtain

$$x'(0)K_+(\rho)x(0) \geq \rho\mu^2 - 2|\rho|\mu\nu + (\rho+r^{-2})\nu^2$$

Considering this as a function of ν, it has a minimum at $\nu=\mu|\rho|/(\rho+r^{-2})$ and the minimum value is $\rho\mu^2(1+\rho r^2)$. Therefore it is clear that for $\rho > -r^2$ the inequalities

$$x'(0)K_1x(0) \geq x'(0)K_+(\rho)x(0) \geq x'(0)K_1x(0)\rho/(1+\rho r^2)$$

hold. The matrix inequality follows immediately.

To study existance we observe that a solution exists for $\rho > 0$ and by differentiation

$$[\frac{d}{d\rho}K_+(\rho)][A-BB'K_+(\rho)] + [A-BB'K_+(\rho)]'[\frac{d}{d\rho}K_+(\rho)] = -C'C$$

or

$$\frac{d}{d\rho}K_+(\rho) = \int_0^\infty e^{[A-BB'K_+(\rho)]t}C'Ce^{[A-BB'K_+(\rho)]'t}dt$$

This differential equation can be integrated in the direction of decreasing ρ until $A-BB'K_+(\rho)$ has an eigenvalue with a zero real part. In view of inequality (8), a solution $K_+(\rho)$ will therefore exist for $\rho > -r^2$. To show that it also exists for $\rho \geq r^2$. Note that $K_+(\rho)$ is monotone decreasing for ρ decreasing. By lemma 4 $K_+(\rho)$ is bounded from below for $\rho > -r^2$ hence

$$\lim_{\rho\to -r^2} K_+(\rho) = \overline{K}$$

exists and by continuity $\overline{K}$ satisfies equation (1) with $\rho = r^2$.

Notice that the spectral norm of [-A,B,C] is the same as that of [A,B,C] and hence that there also exists a solution of

$$(-A')K + K(-A) - KBB'K = -\rho C'C$$

which puts the eigenvalues of $-A-BB'K$ in Re $s < 0$. The negative of this solution is $K_-(\rho)$.

3. The Case where the Spectral Norm is Finite

We now extend the results of the previous section to a wider class of systems. The main result, Theorem 2, includes Theorem 1 as a special case but the proof makes a full case of Theorem 1.

We need the following lemma to reduce the general case to Theorem 1.

Lemma 5 : If $K_o = K_o'$ is any solution of

$$K_oA + A'K_o - K_oBB'K_o = 0 \tag{9}$$

And if $K(\rho)$ is any solution of

$$A'K(\rho) + K(\rho)A - K(\rho)BB'K(\rho) = -\rho CC'$$

then

$$[K(\rho) - K_o][A-BB'K_o] + [A-BB'K_o]'[K(\rho)-K_o] + [K(\rho) - K_o]BB'[K(\rho) + K_o] = -\rho C'C \tag{10}$$

Proof : The proof is just a matter of expanding and using the definitions. The details are omitted.

As we have seen, equation (5) can have at most one invertible solution but it can have numerous non-invertible ones. In particular 0 is always a solution as are $K_+(0)$ and $K_-(0)$. However, the particular solution $K_+(0)$ satisfies

$$K_+(0)[A-BB'K_+(0)] + [A-BB'K_+(0)]'K_+(0) = - K_+(0)BBK_+(0)$$

Since $A-BB'K_+(\rho)$ has its eigenvalues in Re $s < 0$ for $\rho > 0$ it will follow that its eigenvalues lie in Re $s < 0$ for $\rho = 0$ unless the spectral norm of $[A-BB'K_+(0),B,C]$ is infinite. This makes the following lemma of interest.

Lemma 6 : If $K_o = K_o'$ is a solution of

$$K_oA + A'K_o - K_oBB'K_o = 0$$

such that $A-BB'K_o$ has its eigenvalues in the half-plane Re $s < 0$ then the spectral norms of $[A,B,C]$ and $[A-BB'K_o,B,C]$ are the same.

Proof : From lemma 5 see that if K_o satisfies the hypothesis then

$$K(A-BBK_o) + (A-BB'K_o)'K - KBB'K = -\rho C'C \tag{11}$$

has a solution if and only if there exists a solution of

$$KA + A'K - KBB'K = -\rho C'C$$

Combining Lemma 4 and Theorem 1 we see that this equation has a solution if and only if $\rho \geq -r^2$ where r is the spectral norm of [A,B,C]. Since the same is true for equation (11) the spectral norm of $[A-BB'K_o,B,C]$ must also be r^2.

Putting these lemmas together with Theorem 1 gives the following generalization of Theorem 1.

Theorem 2 : Let [A,B,C] be a minimal linear system with spectral norm r. Then for $\rho \geq r^{-2}$ there exists a solution of equation (1) and

$$K_1\rho \geq K_+(\rho) - K_+(0) \geq K_1\rho/(1+r^2\rho)$$

where K_1 is the solution of $[A-BB'K_+(0)]'K_1+K_1[A-BB'K_+(0)] = -CC$. Moreover, there are no other solutions of $KA + A'K - KBB'K = -\rho C'C$ which have the property that $A-BB'K$ has its eigenvalues in $\mathrm{Re}\ s < 0$.

4. Additional Comments

The results given here give the following (still incomplete) picture of the solutions of equation (1) under the hypothesis that A has no eigenvalue with zero real part.

i) There exist real solutions if and only if $\rho \geq r^{-2}$

ii) For $\rho > r^2$ there is exactly one solution such that $A-BB'K$ has its eigenvalues in $\mathrm{Re}\ s < 0$ and exactly one solution such that $A-BB'K$ its eigenvalues in $\mathrm{Re}\ s > 0$

iii) $K_+(\rho) - K_-(\rho) \geq 0$

Figure 1 suggests the main qualitative features and illustrates the bounds. Of course similar bounds hold for $K_-(\rho)$.

We note that our results provide a new proof of certain important theorems on the absence of conjugate points [5]. Moreover, our proof does not use any results on spectral factorization of rational matrices.

Additional refinements of these ideas can be found in Canales' thesis [4].

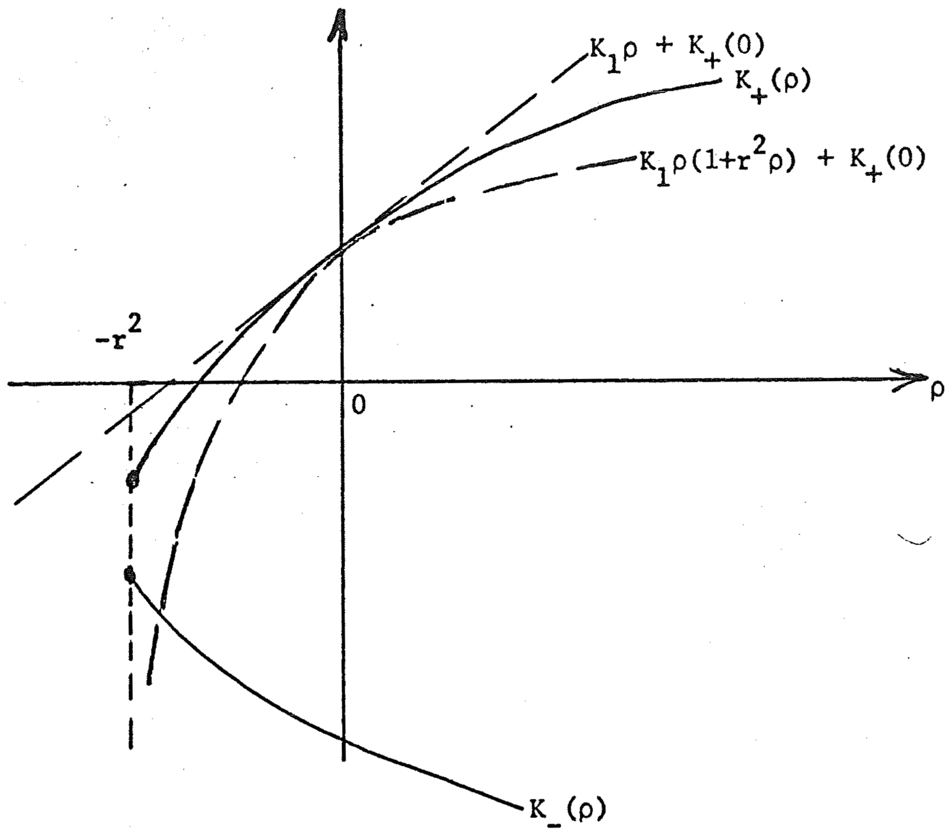

Figure 1 : A suggestive picture of the general behavior of $K_+(\rho)$ and $K_-(\rho)$. If K is one dimensional then $K_+(\rho)$ and $K_-(\rho)$ join at $\rho = -r^2$.

5. References

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