Theoretical Implications of the Second Time Derivative of the Period of the Pulsar NP0532*

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ABSTRACT

The expected value of the second time derivative of the period of NP0532 is given and possible relations with existing magnetic dipole models are outlined. The amount of gravitational radiation to be expected is estimated.

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In a fundamental article[1] John A. Wheeler pointed out some possible relations between the physics of neutron stars,[2] the emission of gravitational waves,[3] the supernova explosions and the problem of the source of energy in supernova remnants. As an example he analyzed these problems in the case of the Crab Nebula; he showed how the residual energy of the neutron star could be, due to suitably shaped magnetic field, an important source of power. He also analyzed as a prime emitter of gravitational radiation the quadrupole modes of oscillations as well as the coupling between the radial and quadrupole modes of vibrations due to the oblateness of the neutron star. The amount of gravitational radiation emitted by such mechanism, he showed, could be as high as 10^4 erg/cm^2/sec passing the Earth even supposing the source to be 10^4 pc. away.

The recent discovery of pulsars[4] and later the astonishing observations of NP0532[5] in the Crab Nebula have given new vigor to theoretical work in this field.

Among the questions that are often asked, few are more interesting than the following: 1) What keeps the period of the pulsars so sharply well defined? 2) Which is the cause of the slight variation of this period? 3) Are the pulsars emitting gravitational waves and, if so, how much? 4) How useful would be an accurate knowledge not only of the period and its first derivative, but also of the second derivative? 5) Could this new information discriminate between existing theories?

The generally accepted answer to the first question was proposed by

Thomas Gold.[6] He suggested that the great stability of the intrinsic period could be explained by assuming that the pulsar emission mechanism is related to the rotation of a neutron star. Based on this hypothesis, many models have been advanced; among others F. Pacini[7] and J. Gunn and J. Ostriker[8] have developed a particularly simple one which gives some well defined predictions that may be tested observationally. Their work assumes for the neutron star the oblique rotator model developed by Deutsch[9] (which he applied to the case of magnetic stars).

In the model of Pacini the hypotheses are the following: a) A rotating neutron star; b) time independent oblique dipole magnetic field; c) the loss of rotational energy is only due to electromagnetic radiation emitted by the dipole field. It is then easy to compute the value of the second derivative of the period for the Crab Nebula pulsar today.

We assume for the period of the Crab pulsar $P = 0.033090$ sec and for its first derivative $\dot{P} = 4.2257 \times 10^{-13}$.[10] The rotational energy loss is

$$\frac{d}{dt}\left(\frac{1}{2} I\omega^2\right) = -\frac{\alpha I}{2}\omega^4 \qquad (1)$$

where $\alpha = 4m^2/3c^3$, m is the component of the magnetic moment perpendicular to the rotation axis, and I is the moment of inertia. We obtain:

$$\ddot{P} = -\frac{\dot{P}^2}{P} = -\ 0.014703 \text{ nsec/yr/day} \quad .$$

From the age of the Crab, $\tau = 915$ yr, we can even compute the initial value of the angular velocity:

$$\omega_{IN} = \omega_0(1-\alpha\omega_0^2\tau)^{-\frac{1}{2}} = 370.159 \text{ rad/sec} \quad , \qquad (2)$$

where ω_0 is the angular velocity today.

The model of J. Gunn and J. Ostriker can be considered a generalization of that given above. Their main hypotheses are the following: a) a rotating neutron star; b) an oblique dipole magnetic field, eventually weakly time dependent; c) a mass quadrupole moment due to an eccentricity ϵ in the equatorial plane of the star constant in time; d) the loss of rotational energy is due to electromagnetic and gravitational wave emission. We have now new degrees of freedom and we can correspondingly determine a range of possible values for the second derivative of the period.

Let us consider first the case of a constant magnetic dipole moment. In the particular case of pure electromagnetic radiation ($\epsilon = 0$) we have as before $\ddot{P} = -\ 0.014703$ nsec/yr/day. Let us analyze the opposite limiting case--namely null magnetic dipole--and see if we can still explain the slowing down of the period. From Landau and Lifschitz [11] we have the formula:

$$\frac{dE}{dt} = -\frac{G}{45c^5}\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta} \quad , \tag{3}$$

recalling that for an ellipsoid with principal axes a, b, c we have:

$$D^0_{\alpha\beta} = \frac{M}{5}\begin{pmatrix} 2a^2-b^2-c^2 & 0 & 0 \\ 0 & 2b^2-a^2-c^2 & 0 \\ 0 & 0 & 2c^2-a^2-b^2 \end{pmatrix} \tag{4}$$

Applying the rotation operator $R(z)$ around the z-axis and differentiating three times with respect to time, we obtain:

$$\dddot{D}_{\alpha\beta}\dddot{D}^{\alpha\beta} = \frac{288}{25} M^2(a^2-b^2)^2\omega^6 \tag{5}$$

For a ~ b:

$$\dddot{D}_{\alpha\beta}\,\dddot{D}^{\alpha\beta} = 288\, I^2 \epsilon^2 \omega^6 \qquad (6)$$

where I is the moment of inertia of the ellipsoid around the z-axis and $\epsilon = (a-b)/\sqrt{ab}$.[12]

Assuming now that the loss of rotational energy is only due to gravitational radiation, it follows:

$$\frac{d}{dt}\left(\tfrac{1}{2} I\omega^2\right) = -\frac{\beta I}{2}\,\omega^6 \qquad (7)$$

where $\beta = \dfrac{576\, GI\epsilon^2}{45c^5}$. From equation (7) and the known values of P and $\dot{P}$ we obtain:

$$\ddot{P} = -3\frac{\dot{P}^2}{P} = -0.044111 \text{ nsec/yr/day} \quad . \qquad (8)$$

This value for the second derivative of the period is perfectly justified, if we assume that ϵ be constant at the present (but not necessarily for all time). We can fix the absolute upper limit to the possible amount of gravitational energy emitted by rotation from the Crab pulsar today. Namely

$$\frac{dE}{dt} = -4.599 \times 10^{-7}\ I \text{ ergs/sec} \quad .$$

If we make the more restrictive assumption, as in Gunn and Ostriker,[8] of ϵ being constant we would be forced to reject this value of $\ddot{P}$. In fact from the formula for the age:

$$\tau = \frac{1}{4(\dot{P}/P)_0}\left[1 - \left(\frac{\omega_0}{\omega_{1N}}\right)^4\right] \qquad (9)$$

it is clear that even for $\omega_{1N} = \infty$ we could only obtain an age of about 621 years for NP0532, that is ~ 300 years less than the required value! However, we would like to emphasize that there is no physical reason whatsoever that forces ϵ be constant in time.

Let us study now the case in which both electromagnetic and gravitational radiation are present. We will assume m and ϵ constant in time. The decay of rotational energy is given by the following expression:

$$\frac{d}{dt}\left(\frac{1}{2} I\omega^2\right) = -\frac{\alpha I}{2}\omega^4 - \frac{\beta I}{2}\omega^6 \qquad (10)$$

from which we find:

$$\tau = \frac{1}{\alpha}\left[\frac{1}{\omega_0^2} - \frac{1}{\omega_{1N}^2} + \frac{\beta}{\alpha}\,\mathrm{lg}\left(\frac{\alpha\omega_0^2 + \beta\omega_0^2\omega_{1N}^2}{\alpha\omega_{1N}^2 + \beta\omega_0^2\omega_{1N}^2}\right)\right] \qquad (11.1)$$

$$\left(\frac{\dot{P}}{P}\right)_0 = \frac{\alpha}{2}\omega_0^2 + \frac{\beta}{2}\omega_0^4 \qquad (11.2)$$

$$\left(\frac{\ddot{P}}{P}\right)_0 = -\left(\frac{\dot{P}}{P}\right)_0\left[\left(\frac{\dot{P}}{P}\right)_0 + \beta\omega_0^4\right] \qquad (11.3)$$

The known quantities are the present angular velocity ω_0, the period P_0 and its first derivative $\dot{P}_0$, as well as the age τ. We have determined by computer analysis the range of variations of α, β, $\ddot{P}_0$ and ω_{1N} (the initial angular velocity of the rotating neutron star). The numerical results are reported in Table I and Fig. 1. The limits found for the second derivative are:

$$-\,0.014703 \geq \ddot{P} \geq -\,0.021038 \ \text{nsec/day/year}\,. \qquad (12)$$

The upper limit corresponds to an absence of gravitational radiation, the lower limit to the maximum flux of gravitational radiation allowed in this model. This maximum value is $(\dot{E}_{grav}/\dot{E}_{em})_0 = 0.2758$. Therefore a measurement of the second derivative in the range given by (12) would certainly support this model and also would allow us to evaluate the amount of gravitational, as well as electromagnetic energy emitted by the pulsar.

However, this would give us neither the strength of the magnetic field nor the value of the eccentricity. In fact for this purpose it would be necessary to know the radius and the moment of inertia of the star. Unfortunately very little is known about the equilibrium property of a rotating neutron star. With the simplifying assumption of uniform, axially symmetric configuration (therefore no emission of gravitational radiation) we have a first post-Newtonian solution due to Hartle[13] and Hartle and Thorne.[14] To give the order of magnitude of the magnetic field and of the eccentricity we have adopted the rotating models of Hartle and Thorne as reported in Table II. We have used these figures in the region:

$$\omega \leq \left(\frac{GM}{R^3}\right)^{\frac{1}{2}} , \tag{13}$$

i.e. when the post-Newtonian approximation applies.

In Figs. 2 and 3 we give the values of ϵ and B both for the Harrison-Thorne-Wakano-Wheeler and the Tsuruta-Cameron rotating equilibrium configurations.

How good is the approximation of assuming the magnetic field constant during all the lifetime of the Crab? L. Woltjer[15] and J. Ostriker and J. Gunn[16] pointed out that the initially present magnetic field would be expected to decay on a time scale of $\sim 10^6$ years. If we believe that the resistive decays of the field be the only mechanism for change, then this approximation can be considered quite good. On the contrary the assumption of taking the eccentricity ϵ constant in time seems to be a much more critical hypothesis and, moreover, we do not see a satisfactory physical reason to explain its presence. If we want in fact to explain even an eccentricity as small as 10^{-4} by the effect of the anisotropic magnetic pressure, we would need $B \sim 10^{15}$ gauss! With this field strength certainly other phenomena take place (vacuum polarization, geon-like structure, etc.). A possible alternative explanation of the eccentricity could be found in a more profound and fully relativistic study of the equilibrium configurations of massive condensed rotating objects.[17] In conclusion, the forthcoming experimental result of the measure of the second derivative of the period of NP0532 could be:

a) $\ddot{P}_0 > -0.014703$ nsec/day/year; in this case an explanation on the basis of an oblique rotator model will be impossible.

b) $\ddot{P}_0 = -0.014703$ nsec/day/year; this value would support the model of F. Pacini and of J. Gunn and J. Ostriker in the case of absence of gravitational radiation.

c) $-0.021038 \leq \ddot{P}_0 < -0.014703$ nsec/day/year; this result would defnitely support the J. Gunn and J. Ostriker model with gravitational radiation present and would give at the same time an estimate of its strength.

d) $\ddot{P}_0 < -0.021038$ nsec/day/year; in this case any theory based on the oblique rotator would not be reliable.

e) $\ddot{P}_0 = -0.044111$ nsec/day/year; in this case assuming a time dependent eccentricity in the past, we could explain the slowing down as being due to the emission of only gravitational radiation.

Finally, we would like to point out that all the models usually considered are nonrelativistic (apart from the use of formula (3)). We are convinced that this is a very poor approximation and that any future reliable pulsar model should take into serious account contributions due to both special and general relativity.

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17. See, i. e., a series of papers by Chandrasekhar and Lebovitz in Astrophysical Journal 1960-1966. A general relativistic study of the classical Poincare results (see i. e. figures d'equilibre d'une masse fluide, Paris, Gauthier-Villars, 1902) is highly desirable today for a better understanding of the Pulsar. Of course, such a study should also take into account the emission of gravitational waves. The change between different configurations of equilibrium could account for sudden changes in the period of pulsars as observed in PSR0833-45. Different configurations of equilibrium would have different amount of emission of gravitational radiation and therefore different rate in the charge of period.

TABLE I

Ω_{1N} (rad/sec)	α ($\times 10^{-15}$ sec/rad^4)	β ($\times 10^{-20}$ sec^3/rad^6)	$\ddot{P}$ (nsec/day/year)	$(\dot{E}_{grav}/\dot{E}_{em})_{today}$
370.159	0.70775	0.0	-0.014703	0.0
390.159	0.67846	0.08125	-0.015904	0.04318
410.159	0.65610	0.14327	-0.016832	0.07873
430.159	0.63873	0.19144	-0.017553	0.10807
450.159	0.62503	0.22944	-0.018122	0.13236
470.159	0.61407	0.25982	-0.018576	0.15255
570.159	0.58293	0.34642	-0.019869	0.21413
670.159	0.56990	0.38233	-0.020410	0.24189
.....				
1070.159	0.55723	0.41748	-0.020936	0.27013
2070.159	0.55493	0.42386	-0.021031	0.27539
.....				
5070.159	0.55475	0.42436	-0.0210386	0.27580

TABLE II

Rotating Models of Harrison-Wakano-Wheeler
-from J. Hartle and K. Thorne-

$M/M_\odot$	ρ_c g/cm^3	$M_{rot}/M_\odot$	R_{rot} km	I moment of inertia x 10^{43} g cm^2
0.266	1.0 10^{14}	0.279	44.8	49.8
0.405	3.0 10^{14}	0.457	24.9	44.6
0.554	1.0 10^{15}	0.644	16.8	43.9
0.661	3.0 10^{15}	0.768	11.8	36.4
0.684	6.0 10^{15}	0.786	9.7	25.6

Rotating Models of Tsuruta-Cameron
-from J. Hartle and K. Thorne-

$M/M_\odot$	ρ_c g/cm^3	$M_{rot}/M_\odot$	R_{rot} km	I moment of inertia x 10^{43} g cm^2
0.202	2.4 10^{14}	0.221	21.7	12.8
0.648	5.0 10^{14}	0.859	14.7	51.9
1.400	1.0 10^{15}	1.840	13.3	139.1
1.820	1.7 10^{15}	2.286	11.9	181.0
1.950	3.0 10^{15}	2.340	10.2	170.2

FIGURE CAPTIONS

Fig. 1: We consider the J. Gunn and J. Ostriker model. As a function of the initial angular velocity of the neutron star are given $\ddot{P}$, α and β, determining the electromagnetic and gravitational outputs, and the ratio γ of the gravitational to electromagnetic energy emitted today.

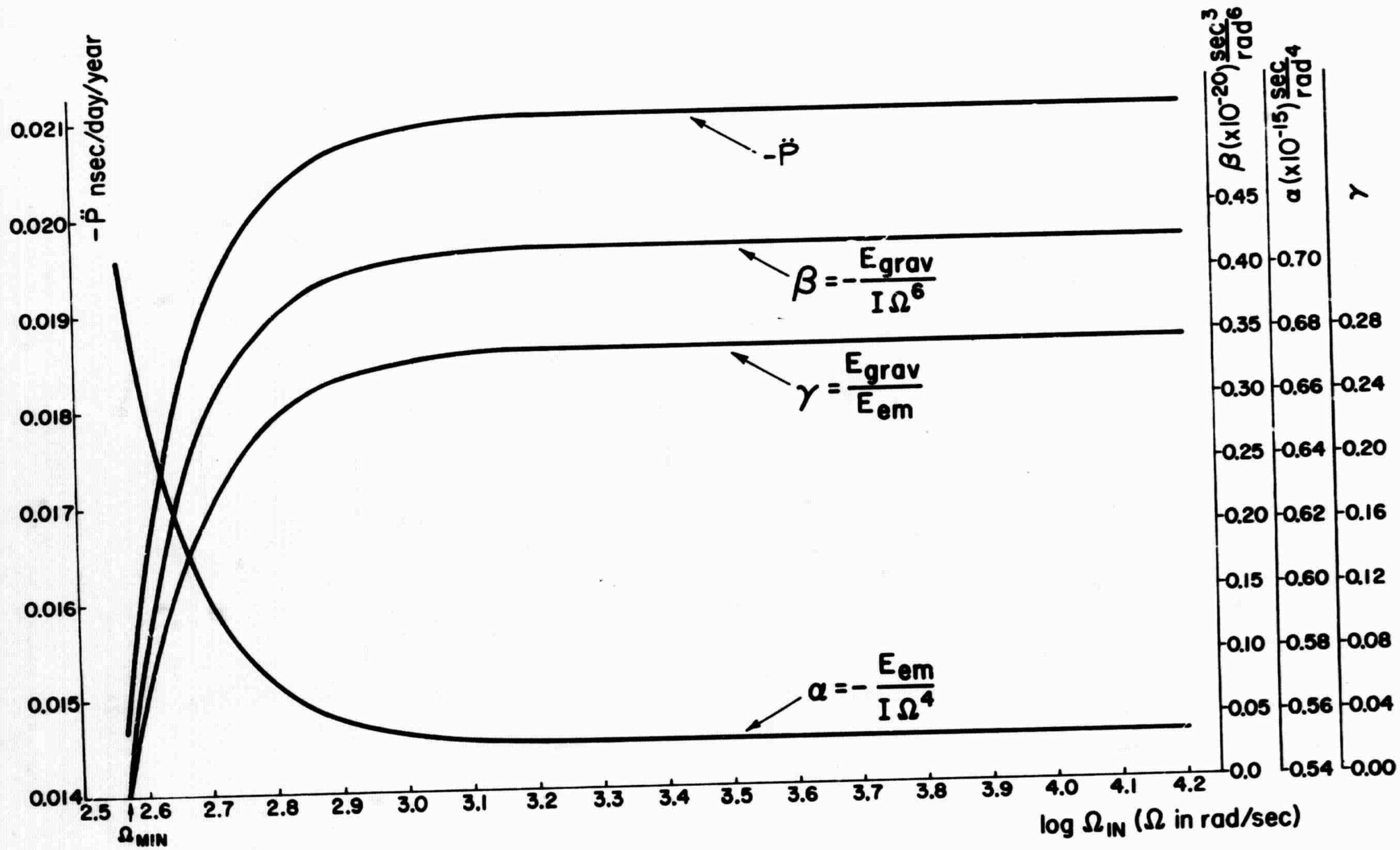
Fig. 2: The eccentricities to be expected in the Gunn-Ostriker model for different masses and radii are plotted as functions of the initial angular velocity.

Fig. 3: The magnetic fields to be expected in the Gunn-Ostriker model for different values of the mass are plotted as functions of the initial angular velocity.

TABLE CAPTIONS

Table I: The possible range of values of $\ddot{P}$, the corresponding second derivative of the period of NP0532, is given in the Gunn-Ostriker model. Correspondingly, the initial angular velocity of the rotating neutron star, α and β, determining the electromagnetic and gravitational outputs, and their ratio, are given.

Table II: The figures used for mass, radius and moment of inertia of the rotating neutron star are given. See J. Hartle and K. Thorne in Ref. (14).



$-\ddot{P}$ nsec/day/year
0.021
0.020
0.019
0.018
0.017
0.016
0.015
0.014
2.5
2.6
2.7
2.8
2.9
3.0
3.1
3.2
3.3
3.4
3.5
3.6
3.7
3.8
3.9
4.0
4.1
4.2
Ω_{MIN}
$\log \Omega_{IN}$ (Ω in rad/sec)
$-\ddot{P}$
$\beta = -\frac{E_{grav}}{I\Omega^6}$
$\gamma = \frac{E_{grav}}{E_{em}}$
$\alpha = -\frac{E_{em}}{I\Omega^4}$
$\beta (\times 10^{-20}) \frac{sec^3}{rad^6}$
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.0
$\alpha (\times 10^{-15}) \frac{sec}{rad^4}$
0.70
0.68
0.66
0.64
0.62
0.60
0.58
0.56
0.54
γ
0.28
0.24
0.20
0.16
0.12
0.08
0.04
0.00

Figure 1.

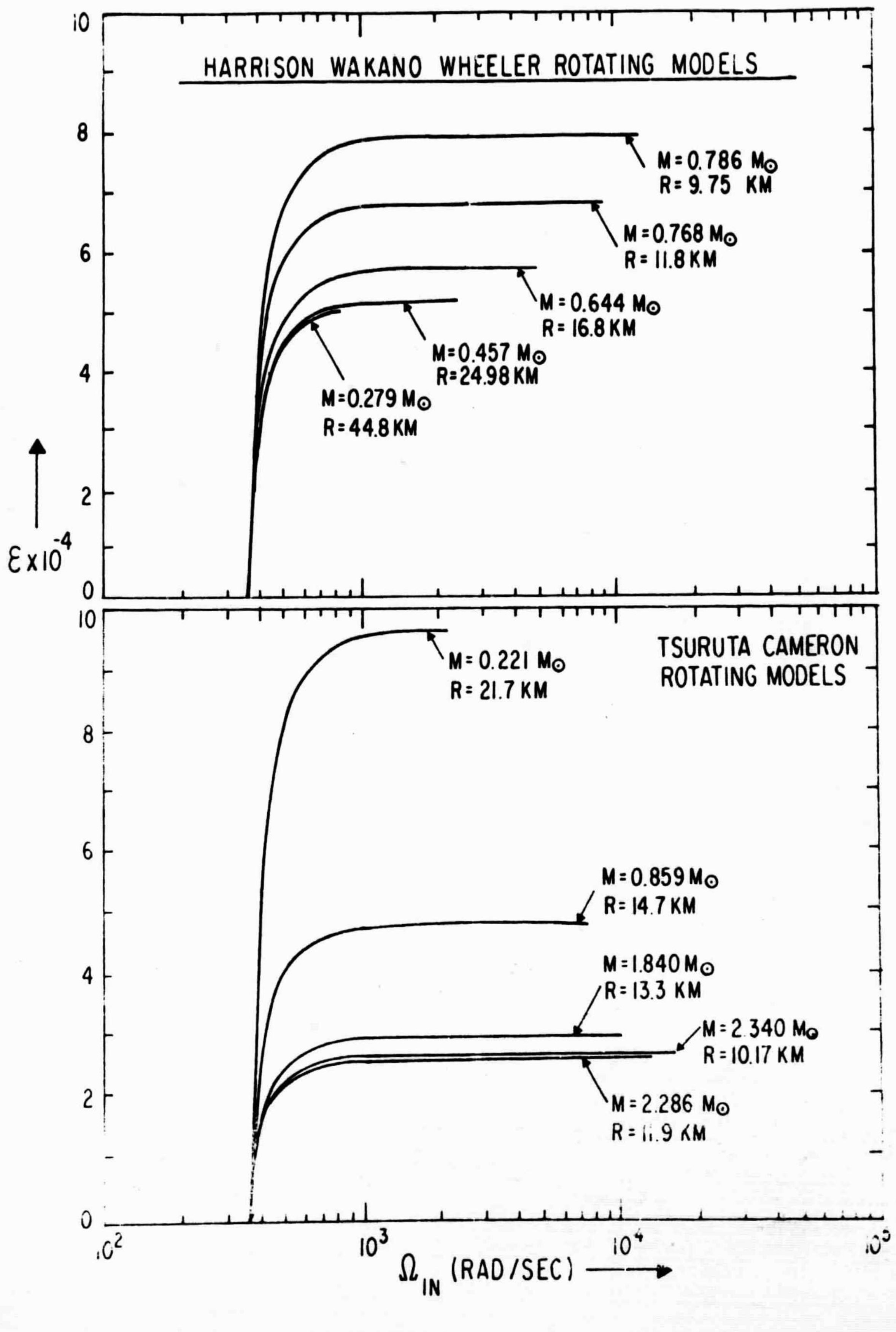

HARRISON WAKANO WHEELER ROTATING MODELS
M = 0.786 M⊙ R = 9.75 KM
M = 0.768 M⊙ R = 11.8 KM
M = 0.644 M⊙ R = 16.8 KM
M = 0.457 M⊙ R = 24.98 KM
M = 0.279 M⊙ R = 44.8 KM
TSURUTA CAMERON ROTATING MODELS
M = 0.221 M⊙ R = 21.7 KM
M = 0.859 M⊙ R = 14.7 KM
M = 1.840 M⊙ R = 13.3 KM
M = 2.340 M⊙ R = 10.17 KM
M = 2.286 M⊙ R = 11.9 KM
ε x 10^-4
Ω_IN (RAD/SEC)

Figure 2.

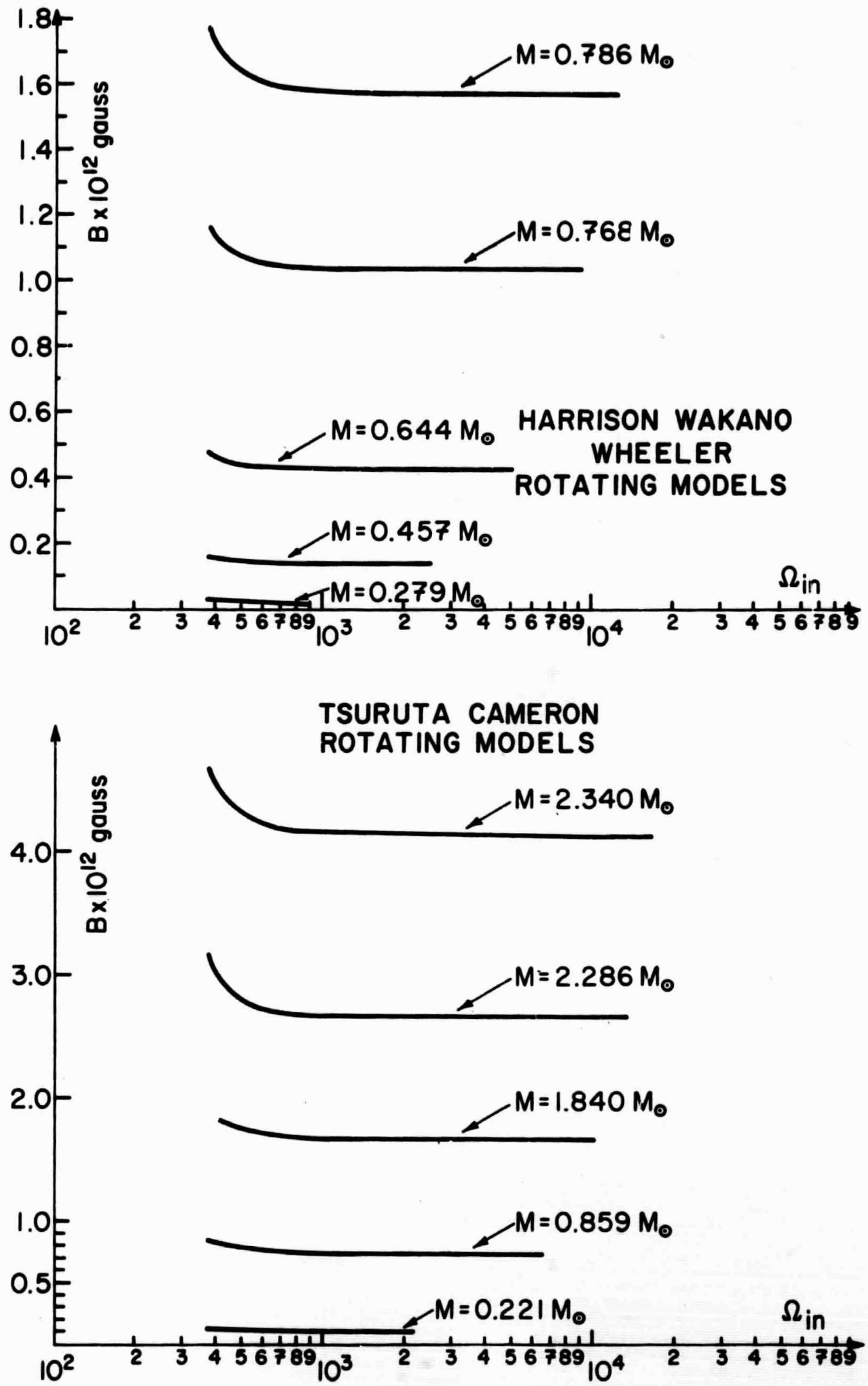

1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
B x 10^12 gauss
M=0.786 M☉
M=0.768 M☉
M=0.644 M☉
M=0.457 M☉
M=0.279 M☉
HARRISON WAKANO
WHEELER
ROTATING MODELS
Ωin
10^2 2 3 4 5 6 7 8 9 10^3 2 3 4 5 6 7 8 9 10^4 2 3 4 5 6 7 8 9
TSURUTA CAMERON
ROTATING MODELS
B x 10^12 gauss
4.0
3.0
2.0
1.0
0.5
M=2.340 M☉
M=2.286 M☉
M=1.840 M☉
M=0.859 M☉
M=0.221 M☉
Ωin
10^2 2 3 4 5 6 7 8 9 10^3 2 3 4 5 6 7 8 9 10^4 2 3 4 5 6 7 8 9

Figure 3.