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TRANSIENT CONFORMAL MAPPING METHOD FOR TWO-DIMENSIONAL SOLIDIFICATION OF FLOWING LIQUID ONTO A COLD SURFACE

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16. Abstract

A method was developed to analyze the transient formation of a two-dimensional solidified region. The transient shapes of the frozen region are found by mapping the region into a potential plane and then determining the time varying conformal transformations between the potential and physical planes. The method is applied to analyze solidification on a cold plate of finite width immersed in flowing liquid having a bulk temperature above the freezing point. The transient results are compared with a quasi-steady solution in which the frozen region boundaries pass instantaneously through steady-state shapes.

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TRANSIENT CONFORMAL MAPPING METHOD FOR TWO-DIMENSIONAL SOLIDIFICATION OF FLOWING LIQUID ONTO A COLD SURFACE

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SUMMARY

A transient conformal mapping method has been developed to determine the configuration of a frozen region formed during two-dimensional solidification. The transient shapes of this region are found by mapping the solidified region into a potential plane and then determining the time varying conformal transformations between the potential and physical planes. The method is applied to the case where solidification is taking place on a cold plate of finite width immersed in flowing liquid having a bulk temperature above the freezing point. During the transient, the combined energy resulting from convection to the interface and latent heat of fusion is conducted through the region from the moving interface to the plate. The transient results are compared with a simpler quasi-steady solution obtained by letting the growth of the frozen region pass instantaneously through a succession of steady-state profile shapes.

INTRODUCTION

The present report is concerned with developing a method for solving two-dimensional transient solidification problems with a convective boundary condition at the moving interface. If a flowing warm liquid is placed in contact with a cold surface which is at a temperature below the freezing point of the liquid, a frozen region will grow on the plate. The flowing liquid supplies energy by convection to the interface formed between the solid and liquid phases. This energy along with the latent heat of fusion released during the transient growth must be removed by conduction through the solidified region to the cooled plate. There is also internal energy removed as the frozen material is subcooled below its freezing point. The temperature at the solid-liquid interface is specified throughout the transient growth by the fact that the interface is within a fraction of a degree of being isothermal at the equilibrium freezing temperature even when the interface is moving.

To obtain a solution to the transient solidification problem, it is necessary to solve the heat conduction equation within the solidifying layer in order to determine the growth of the solidified region. A shape of this region must be found which will result in an isothermal interface and at the same time allow the convective, fusion, and subcooling energies to be conducted through the frozen layer to the cooled plate. Since the rate of freezing is in general nonuniform along the interface, the temperature derivative (i.e., the heat flow) in the solid normal to the interface is an unknown function of position and time. This is contrasted with the condition at steady state where the only heat flow at the interface is by convection from the liquid. The convection is specified in the problem, so in the steady-state case the normal derivative at the interface is known. In the transient case the interface boundary condition will be a differential equation relating the temperature derivative at the interface to the local rate of freezing. An inherent difficulty in the solution of freezing problems is the nonlinearity introduced by the fact that the interface is moving. As a result, it is not possible to utilize superposition to construct solutions for time varying conditions.

In the present problem the energy for subcooling solidified material below the freezing point will be neglected. This is a common assumption in freezing problems since the subcooling energy is usually small compared with the heat of fusion that is liberated at the interface and then conducted through the frozen region. For freezing in a flowing liquid the convective energy supplied must also be conducted from the interface through the frozen region, and this can be a large heat flow compared with the energy for subcooling. Hence, it is usually the case that practically all of the energy flow in the frozen region arises from that entering the region at the solid-liquid interface. There are only some limited cases, for example, where cryogenic coolants are used, for which the subcooling can become important. The solutions in references 1 and 2 show that the subcooling can be neglected within several percent error if the quantity $C_p(t_f - t_w)/\lambda$ is less than about 1.

With heat capacity neglected, the heat flow in the solidified region is governed by the two-dimensional steady-state heat conduction equation (Laplace equation) which must be satisfied at each instant within the frozen region. A numerical solution is difficult since the shape of the frozen region is unknown and changing with time. To solve Laplace's equation in two dimensions, conformal mapping can be utilized. The mapping method will be developed here for a transient situation, that is, the mapping functions will be allowed to vary with time. The method will be demonstrated by analyzing the solidification on a cooled plate of finite width immersed in flowing liquid. The transient results will be compared with quasi-steady results obtained by having the frozen region pass through a series of instantaneous steady-state frozen configurations during the transient growth. The steady-state profiles which form the basis for the quasi-steady solution are the same as those in reference 3 where the steady-state mapping method was developed for two-dimensional solidification.

There has been little analytical work done for two-dimensional solidification. One example is reference 4 where an analysis is given for freezing inside a square prism with the liquid always at the fusion temperature. The transient frozen-layer configurations were determined approximately. In this instance the liquid was initially at the freezing temperature so there was no convective energy transfer to the frozen interface. References 5 to 8 consider steady and transient solidified layers in flow channels. Since the channels are of symmetric cross section, being either a tube or gap between parallel plates, there is only one cross-sectional coordinate. In references 5 to 7, the frozen layers are two-dimensional as they change in thickness along the channel length. The analyses, however, neglect axial conduction within the layers so that the solidification portion of the analysis is locally one-dimensional.

SYMBOLS

A	dimensionless half width of plate, $\frac{ha}{k}\frac{t_l - t_f}{t_f - t_w}$
A^-	value of A for times before start of transient
A_n	time dependent coefficients in mapping
a	half width of plate
b	time dependent parameter in mapping
$b_{initial}$	initial value of b at start of transient
C_n	function of β and b defined in eq. (27)
C_p	specific heat of solid
E	complete elliptic integral of second kind; quantity defined by eq. (38)
h	heat transfer coefficient from flowing liquid to frozen interface
I_Θ	frozen region in Z-plane
J_Θ	frozen region in W-plane
K	complete elliptic integral of first kind
K_n	definite integrals defined in eq. (53)
k	thermal conductivity of solidified material

L_Θ	frozen region in ζ-plane
$\hat{n}$	unit outward normal
p	integer, index in eq. (57)
Q	heat flow rate through frozen layer per unit length of plate
$\vec{R}_S$	dimensionless position vector, $A\vec{r}_s/a$
$\vec{r}_s$	position vector to frozen interface
S	dimensionless frozen-layer-liquid interface
s	frozen-layer-liquid interface
T	dimensionless temperature, $(t - t_w)/(t_f - t_w)$
t	temperature
t_f	freezing temperature
t_l	liquid temperature
t_w	surface temperature of cold plate
U	intermediate mapping plane
W	analytic function, $\varphi + i\psi$
X, Y	dimensionless coordinates, $A\frac{x}{a}$, $A\frac{y}{a}$
x, y	Cartesian coordinates in physical plane
Z	dimensionless complex physical plane, X + iY
α_n	time dependent coefficients in mapping eq. (23)
β_n	time dependent coefficients defined in eq. (26)
Γ	frozen region in Ω-plane
γ	length scale parameter, $\frac{k}{h}\frac{t_f - t_w}{t_l - t_f}$
$\tilde{\nabla}$	dimensionless gradient operator, $\frac{a}{A}\nabla$
ζ	$\left(-\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y}\right)^{-1}$
Θ	dimensionless time, $\frac{h^2}{k\rho\lambda}\frac{\left(t_l - t_f\right)^2}{(t_f - t_w)}\theta$

θ time

λ latent heat of fusion

ρ density of solidified material

φ real part of W

ψ imaginary part of W

Ω intermediate mapping plane

ω argument in Ω-plane

Subscript:

s on frozen interface

Superscripts:

ss steady state

$\bar{}$ (overbar) complex conjugate

GENERAL TRANSIENT ANALYSIS

To help fix the general ideas of the analysis, the following physical problem will be considered during the development of the analytical method. Consider a liquid at constant temperature t_l flowing over an infinitely long flat plate of width 2a as shown in figure 1. Suppose that both vertical sides of the plate are insulated and that the plate surface is maintained at a uniform temperature t_w which is below the freezing temperature t_f of the liquid. Then a frozen region will grow on the plate until the shape and size of the region are such that the heat transferred to the frozen interface by the flowing liquid is exactly balanced by the heat transferred through the frozen region to the plate. If the direction of flow of the liquid is parallel to the long dimension of the plate, then the heat transfer coefficient h on the surface of the frozen layer can be assumed essentially constant. A procedure will be developed here that predicts the transient shape and size of this frozen layer as it grows from some initial state to its final equilibrium configuration.

Before proceeding with the analysis in detail, it is helpful to briefly outline some aspects of the general method. In reference 3 a conformal mapping method was devised to determine the shapes of steady two-dimensional frozen regions. By applying those ideas to the present situation, it is found that the physical coordinates of the frozen region can be obtained at each instant in time by carrying out the integral

$$Z = \int \zeta \, dW$$

where ζ is a temperature derivative function

$$\frac{1}{\zeta} = -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y}$$

and W is the complex potential having negative temperature as the real part and lines of constant ψ in the direction of heat flow as the imaginary part, that is,

$$W = -T + i\psi$$

The functions ζ and W must be related to each other before the integration for determining Z can be performed.

For freezing with convective heat transfer at the interface, the boundary conditions provide information about the shape of the image of the frozen region in the ζ and W complex planes. For example, the moving interface is always an isotherm at the freezing temperature, and hence it will be a vertical line in the W-plane. The frozen region is then drawn in the ζ- and W-planes, some of the region boundaries, such as the moving interface in the ζ-plane, being unknown functions of time. Then to relate ζ and W, both the ζ and W regions are mapped into a common fixed intermediate region in a plane designated as the Ω-plane. The mappings will involve unknown functions of time since the region in the Ω-plane is chosen to be nontime-varying.

The integral for determining Z is then carried out in terms of the common variable Ω. Thus

$$Z(\Omega) = \int \zeta(\Omega)\frac{dW}{d\Omega}(\Omega)\, d\Omega$$

The resulting expression for Z contains the unknown time functions resulting from the mappings of the time varying frozen region in the ζ- and W-planes into the fixed region in the Ω-plane. These unknown functions are found by substituting this expression for Z (evaluated at the moving boundary) into the heat flow boundary condition at the moving interface.

Each of the previous steps will now be carried out. First consider the boundary conditions in detail since these will be needed to represent the frozen region in the temperature derivative and potential planes.

Specification of Physical Boundary Conditions

Since the problem is two-dimensional, let x and y denote coordinates of an arbitrary point in some fixed cross-sectional plane with the origin of the coordinate system as shown in figure 1. Let s denote the two-dimensional free surface of the frozen layer in this plane and $\hat{n}$ denote the unit normal to s directed outward from the frozen layer. The position vector of an arbitrary point of s is denoted by $\vec{r}_s$.

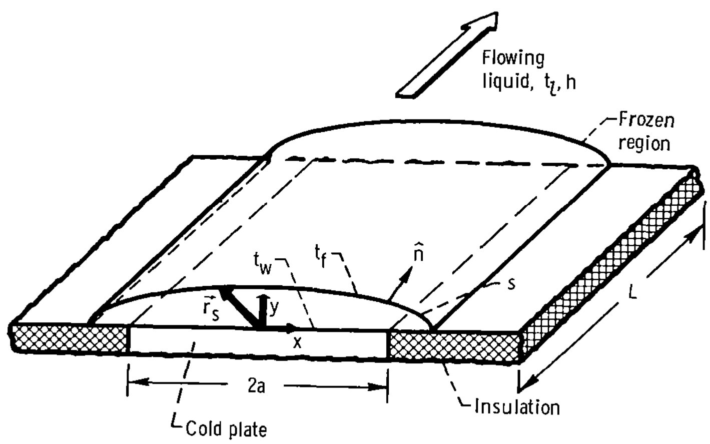


Figure 1. - Two-dimensional solidified layer formed on cold plate.

At the surface s of the frozen layer, the temperature is constant and equal to the freezing temperature t_f of the liquid. The local rate at which heat of fusion is being liberated at the freezing interface per unit area is equal to

$$\rho\lambda\hat{n} \cdot \frac{\partial \vec{r}_s}{\partial \theta} \tag{1}$$

Heat is being supplied to the frozen interface by convective heat transfer from the warm liquid at the rate

$$h(t_l - t_f) \tag{2}$$

per unit interface area. Both of the heat fluxes in expressions (1) and (2) must be balanced by a conduction heat flux away from the interface given by

$$k\hat{n} \cdot \nabla t \tag{3}$$

In view of these considerations, the boundary conditions at the freezing interface can be written as

$$t(\vec{r}_s, \theta) = t_f \tag{4a}$$

$$k\hat{n} \cdot \nabla t - h(t_l - t_f) = \rho\lambda\hat{n} \cdot \frac{\partial \vec{r}_s}{\partial \theta} \tag{4b}$$

The remaining boundary conditions are along the plane $y = 0$ at the surface of the cooled plate. For $\theta > 0$ the plate is maintained at constant temperature. Thus

$$t(x, 0, \theta) = t_w \qquad -a < x < a \tag{5a}$$

For the insulated region on either side of the plate there is no heat flow which implies

$$\frac{\partial t}{\partial y} = 0 \qquad y = 0 \text{ and } x < -a,\ x > a \tag{5b}$$

It is convenient to introduce the following dimensionless quantities:

$$A = \left(\frac{t_l - t_f}{t_f - t_w}\right)\frac{h}{k}\,a$$

$$T = \frac{t - t_w}{t_f - t_w}$$

$$X = A\,\frac{x}{a}, \qquad Y = A\,\frac{y}{a}, \qquad \vec{R}_S = A\,\frac{\vec{r}_s}{a}$$

$$\tilde{\nabla} = \frac{a}{A}\,\nabla$$

$$\Theta = \frac{Ah(t_l - t_f)}{\rho\lambda a}\,\theta = \frac{h^2}{k\rho\lambda}\,\frac{(t_l - t_f)^2}{(t_f - t_w)}\,\theta$$

With these definitions the boundary conditions, equations (4) and (5), have the normalized form

$$T(\vec{R}_S, \Theta) = 1 \tag{6a}$$

$$\hat{n} \cdot \tilde{\nabla} T - \hat{n} \cdot \frac{\partial \vec{R}_S}{\partial \Theta} = 1 \tag{6b}$$

$$T(X, 0, \Theta) = 0 \qquad -A < X < A \tag{7a}$$

$$\frac{\partial T}{\partial Y} = 0 \qquad Y = 0 \text{ and } X < -A, \ X > A \tag{7b}$$

Specification of Boundary Conditions in Terms of Complex Quantities

It has been shown (refs. 1 and 2) that the effect of the heat capacity of the frozen layer is negligible in most instances since the subcooling energy is small compared with the latent heat removed from the moving interface. Hence, for the purposes of this analysis the heat capacity will be neglected; therefore, the heat flow in the frozen region is governed by the Laplace equation. Thus the temperature is a harmonic function of position within the frozen layer. Let φ denote the harmonic function $-T$, and let ψ be the harmonic conjugate to φ. The complex variable $X + iY$ will be denoted by Z, and the complex variable $\varphi + i\psi$ will be denoted by W. The symbol I_Θ will be used to designate the instantaneous dimensionless region occupied by the frozen material in the Z-plane.

Within the frozen layer the potential function W is a function of time Θ and a holomorphic function of position Z. Then at each fixed time, $\Theta = \Theta_c$, the function

$$W(Z, \Theta = \Theta_c) \qquad \text{for all } Z \in I_{\Theta_c}$$

is a holomorphic function of the complex variable Z. In view of this the notation $\partial W/\partial Z$ will be used to signify the ordinary derivative $dW(Z, \Theta = \Theta_c)/dZ$. If the convention is adopted whereby the real and imaginary parts of a complex number are identified respectively with the X and Y components of a vector, then

$$\tilde{\nabla} T = \frac{\partial T}{\partial X} + i \frac{\partial T}{\partial Y}$$

The Cauchy-Riemann equation $\partial\psi/\partial X = \partial T/\partial Y$ implies

$$\frac{\partial W}{\partial Z} = -\frac{\partial T}{\partial X} + i\frac{\partial \psi}{\partial X} = -\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y}$$

Hence,

$$\tilde{\nabla} T = -\overline{\frac{\partial W}{\partial Z}} \tag{8}$$

Equation (8) will now be used to express the boundary condition, equation (6b), in terms of complex quantities that will be needed in the conformal mapping method.

Since the surface S of the frozen layer is isothermal, the temperature gradient must be normal to it. Since the temperature is increasing in the direction of the outward drawn normal, the unit normal vector is given by

$$\hat{n} = \frac{\tilde{\nabla} T}{|\tilde{\nabla} T|}$$

Using equation (8) yields

$$\hat{n} = -\frac{\overline{\frac{\partial W}{\partial Z}}}{\left|\frac{\partial W}{\partial Z}\right|} = -\frac{\overline{\frac{\partial W}{\partial Z}}\,\frac{\partial W}{\partial Z}}{\left|\frac{\partial W}{\partial Z}\right|\frac{\partial W}{\partial Z}} = -\frac{\left|\frac{\partial W}{\partial Z}\right|^2}{\left|\frac{\partial W}{\partial Z}\right|\frac{\partial W}{\partial Z}} = -\frac{\frac{\partial Z}{\partial W}}{\left|\frac{\partial Z}{\partial W}\right|} \tag{9}$$

The first term in equation (6b) can be written as

$$\hat{n} \cdot \tilde{\nabla} T = \frac{\tilde{\nabla} T}{|\tilde{\nabla} T|} \cdot \tilde{\nabla} T$$

Although

$$\tilde{\nabla} T = \frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y}$$

in carrying out the dot product the i is an ordinary unit vector and, hence, the number -1 does not appear in front of $(\partial T/\partial Y)^2$.* Then

*For two vectors written as complex numbers, $z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 + y_1y_2 = \mathcal{R}e\,(x_1 + iy_1)(x_2 - iy_2) = \mathcal{R}e\,z_1\bar{z}_2$.

$$\hat{n} \cdot \tilde{\nabla} T = \frac{|\tilde{\nabla} T|^2}{|\tilde{\nabla} T|} = |\tilde{\nabla} T| = \left|\frac{\partial W}{\partial Z}\right| \tag{10}$$

The second term in equation (6b) becomes

$$\hat{n} \cdot \frac{\partial \vec{R}_S}{\partial \Theta} = - \frac{\dfrac{\partial Z}{\partial W}}{\left|\dfrac{\partial Z}{\partial W}\right|} \cdot \left.\frac{\partial Z}{\partial \Theta}\right|_S = - \frac{1}{\left|\dfrac{\partial Z}{\partial W}\right|} \mathcal{R}e \, \frac{\partial Z}{\partial \Theta} \overline{\frac{\partial Z}{\partial W}} \tag{11}$$

evaluated on the interface.

Equations (10) and (11) are then inserted into the boundary condition, equation (6b), to give

$$1 + \mathcal{R}e \, \frac{\partial Z}{\partial \Theta} \overline{\frac{\partial Z}{\partial W}} = \left|\frac{\partial Z}{\partial W}\right| \tag{12}$$

Mapping of Frozen Region in Z-Plane into Rectangle in W-Plane

The boundary conditions on the frozen region provide information from which the shape of the region in the potential W-plane can be found. The instantaneous frozen region I_Θ is shown in the physical Z-plane in figure 2. It is clear from the boundary conditions (eqs. (6a) and (7a)) specifying the temperatures at the frozen interface and plate that, since the real part of W is -T,

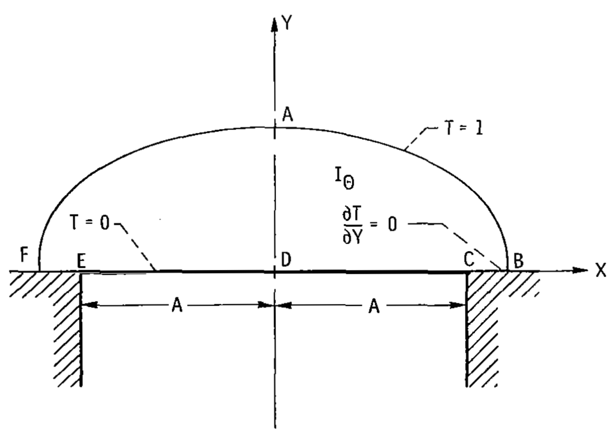


Figure 2. - Dimensionless physical plane, Z = X + iY.

$$-T(X, Y, \Theta) = \mathcal{Re}\ W(Z, \Theta) = -1 \qquad Z \in \widehat{FAB} \tag{13}$$

$$-T(X, Y, \Theta) = \mathcal{Re}\ W(Z, \Theta) = 0 \qquad Z \in \widehat{EDC} \tag{14}$$

where the notation $\widehat{FAB}$ represents the set of points along the boundary FAB. The boundary condition (eq. (7b)) together with the Cauchy-Riemann condition

$$\frac{\partial T}{\partial Y} = \frac{\partial \psi}{\partial X}$$

shows that at any instant of time

$$\frac{\partial \psi}{\partial X} = 0 \qquad \text{for} \begin{cases} Z \in \widehat{FE} \\ \\ Z \in \widehat{CB} \end{cases}$$

Hence, since ψ is the imaginary part of W,

$$\mathcal{Im}\ W(Z, \Theta) = \text{const} \qquad Z \in \widehat{FE} \tag{15}$$

$$\mathcal{Im}\ W(Z, \Theta) = \text{const} \qquad Z \in \widehat{CB} \tag{16}$$

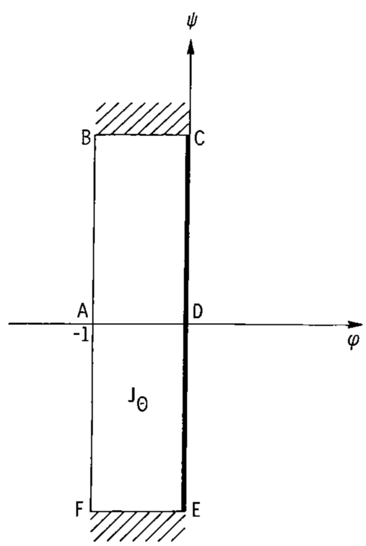


Figure 3. - Potential plane, $W = \varphi + i\psi$.

Equations (13) to (16) show that at each instant of time the region I_Θ in the physical plane maps into a rectangular region J_Θ in the W-plane as shown in figure 3. The height of the rectangle J_Θ varies with time since the height is related to the heat flow through the frozen region (this will be shown in connection with eq. (59)) and must be determined from the solution to the problem. The symmetry of the rectangle with respect to the φ-axis follows from the symmetry of the problem with respect to the imaginary axis in the physical Z-plane.

Mapping of Frozen Region in Z-Plane into ζ-Plane

Now consider the complex variable ζ defined by

$$\zeta = \frac{\partial Z}{\partial W} \tag{17}$$

Then, it follows from the relation for $\partial W/\partial Z$ immediately preceding equation (8) that

$$\frac{1}{\zeta} = \frac{\partial W}{\partial Z} = -\frac{\partial T}{\partial X} + i\,\frac{\partial T}{\partial Y}$$

which together with the temperature derivative boundary conditions can be used to represent the frozen region in the ζ-plane. Along the insulated portion of the surface, equation (7b) shows that

$$\mathcal{I}m\,\frac{1}{\zeta} = 0 \qquad \text{for} \begin{cases} Z \in \widehat{CB} \\ Z \in \widehat{EF} \end{cases}$$

This is shown in figure 4. It follows from the constant temperature boundary condition (eq. (7a)) along the plate that

$$\frac{\partial T}{\partial X} = 0 \qquad \text{for } -A < X < A$$

Hence,

$$\mathcal{R}e\,\frac{1}{\zeta} = 0 \qquad Z \in \widehat{EDC}$$

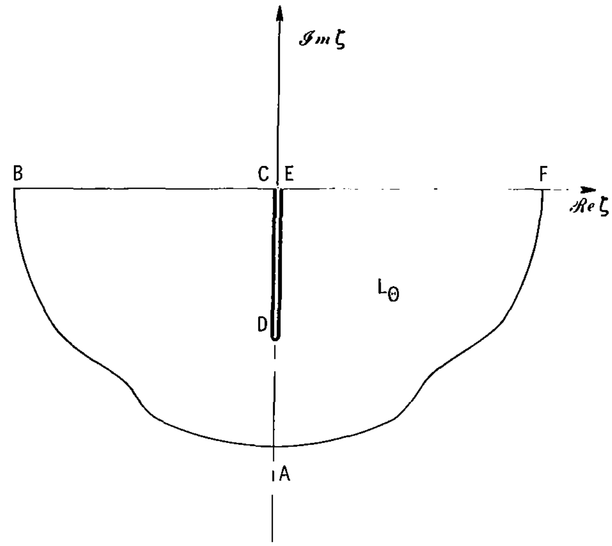


Figure 4. - Temperature derivative plane, $\zeta = \left(\frac{\partial T}{\partial X} + i\,\frac{\partial T}{\partial Y}\right)^{-1}$

The symmetry of the problem shows that

$$\frac{\partial T}{\partial X} = 0 \qquad \text{for} \quad X = 0$$

Hence,

$$\mathscr{Re}\ \frac{1}{\zeta} = 0 \qquad Z \in \widehat{DA}$$

Since

$$\frac{1}{\zeta} = \frac{1}{|\zeta|^2}\,\bar{\zeta}$$

we conclude from these conditions that

$$\left.\begin{array}{ll} \mathcal{I}m\ \zeta = 0 & \left\{\begin{array}{l} Z \in \widehat{CB} \\ Z \in \widehat{EF} \end{array}\right. \\ \mathcal{R}e\ \zeta = 0 & \left\{\begin{array}{l} Z \in \widehat{EDC} \\ Z \in \widehat{DA} \end{array}\right. \end{array}\right\} \tag{18}$$

This is indicated in figure 4. Equation (9) and the definition (eq. (17)) show that the outward drawn normal to the surface $\widehat{FAB}$ is in the $-\zeta$ direction. These considerations are sufficient to show that the region I_Θ in the physical plane must map into the region L_Θ in the ζ-plane shown in figure 4. The shape of the curve $\widehat{BAF}$ and the length of the line $\widehat{BCEF}$ depend on the magnitudes of the temperature derivatives and hence the heat flow in the solid at the moving interface. These quantities vary with time and are not known at this stage of the solution.

Integral Relating W and ζ to Physical Coordinates

In order to obtain the frozen layer shape, it is convenient to introduce a parametric (intermediate) complex variable Ω which will be used to relate W and ζ. Consider the region Γ in the Ω-plane depicted in figure 5. This region is chosen to not change with time and is always bounded by a unit semicircle and the real axis. As will be seen later, however, it is necessary to allow the length of the line $\widehat{DE}$ to vary with time. Suppose that the mapping $\Omega \rightarrow W$ is known which takes at every instant of time the fixed region Γ in the Ω-plane into the time variable region J_Θ of the W-plane in the manner indicated by figures 5 and 3. Also suppose that the mapping $\Omega \rightarrow \zeta$ which takes Γ

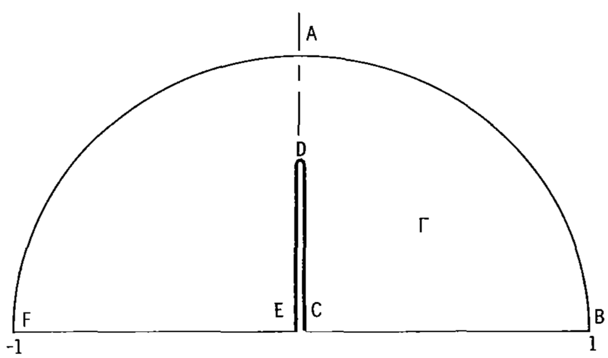


Figure 5. - Intermediate Ω-plane.

into the variable region L_Θ in the ζ-plane in the manner indicated by figures 5 and 4 is known. The locations of the various points in the physical plane can then be found in terms of the parametric variable Ω from the formula obtained by integrating equation (17), which is

$$Z = \int \zeta \frac{\partial W}{\partial \Omega} d\Omega + \text{function of } \Theta \tag{19}$$

In particular by letting Ω be along the semicircle in figure 5, the shape of the freezing surface is found at each instant of time. Since W is a known function of Ω, and Z is known as a function of Ω from equation (19), it is possible to compute the temperature at each point of the physical plane. Thus the solution to the problem can be obtained by finding the mappings $\Omega \rightarrow W$ and $\Omega \rightarrow \zeta$ at each instant of time and then performing the integration in equation (19).

Determination of the Mapping $\Omega \rightarrow W$

To determine this mapping, it is convenient to introduce an intermediate variable U and to map the rectangular region J_Θ in the W-plane into the upper half of the U-plane in the manner indicated in figure 6. An application of the Schwarz-Christoffel transformation shows that this mapping is defined by

$$\frac{\partial W}{\partial U} = \frac{i}{K\left(\sqrt{1 - b^2}\right)\sqrt{U^2 - b^2}\sqrt{U^2 - 1}} \tag{20}$$

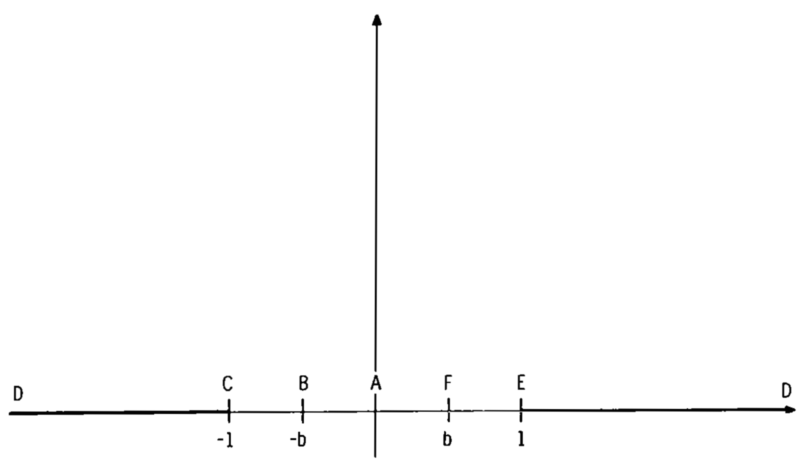


Figure 6. - Intermediate U-plane.

The parameter b appearing in this mapping is a function of time.

The mapping which takes the semicircular region Γ in the Ω-plane into the upper half of the U-plane in the manner indicated in the figures is defined by

$$U = -\frac{b(1+\Omega^2)}{\sqrt{(1+\Omega^2)^2 - (1-b^2)(1-\Omega^2)^2}} \qquad \Omega \in \Gamma \tag{21}$$

Substituting equation (21) into equation (20) to eliminate U reveals that the mapping which takes Γ into J_Θ in the manner indicated by figures 5 and 3 is defined by

$$\frac{\partial W}{\partial \Omega} = \frac{\partial W}{\partial U}\frac{\partial U}{\partial \Omega} = -\frac{2}{K\left(\sqrt{1-b^2}\right)\sqrt{(1+\Omega^2)^2 - (1-b^2)(1-\Omega^2)^2}} \qquad \Omega \in \Gamma \tag{22}$$

Determination of the Mapping $\Omega \to \zeta$

The mapping which takes the region Γ into L_Θ is, of course, holomorphic in the interior of Γ since there are no singularities within the region. An examination of figures 4 and 5 indicates that this mapping may be expected to be continuous on the boundaries of Γ since there are no singularities that occur there. Hence, the mapping has an analytic continuation to a region which includes the boundaries of Γ. Since the function $\Omega \to \zeta$ is real on the real axis of the Ω-plane, it is known from the Schwarz reflection principle that the function $\Omega \to \zeta$ can be analytically continued to the region that is the mirror image of Γ with respect to the real axis. Hence, ζ can be extended analytically to a function that is holomorphic in the interior of a unit circle in the Ω-plane and is continuous on the boundary of the circle. In view of this, the function $\Omega \to \zeta$ has a Taylor series expansion about the origin of the Ω-plane that converges in a circle that contains the closure of Γ. Since $\zeta(\Omega)$ is real for Ω on the real axis in the Ω-plane, it follows that the coefficients in the series expansion must be real. Since $\zeta(\Omega)$ is pure imaginary for Ω on the imaginary axis in the Ω-plane, it follows (since even powers of i would be real) that only odd powers of Ω can appear in this expansion. Hence, ζ can be represented by the convergent series

$$\zeta(\Omega, \Theta) = -K\left(\sqrt{1-b^2}\right)\sum_{n=0}^{\infty} \alpha_n \Omega^{2n+1} \tag{23}$$

for all Ω in the closure of Γ with α_n real for $n = 0, 1, 2, 3, \ldots$. Each coefficient α_n in this expansion is, in general, a function of time. Thus the mapping $\Omega \rightarrow \zeta$ will be completely determined if the sequence of real functions of time $\{\alpha_n\}$ can be found.

To better understand equation (23), introduce polar coordinates $|\Omega|$ and ω for the Ω-plane by letting

$$\Omega = |\Omega| e^{i\omega}$$

Then equation (23) becomes

$$\zeta(\Omega, \Theta) = -K\left(\sqrt{1 - b^2}\right) \sum_{n=0}^{\infty} \alpha_n |\Omega|^{2n+1} e^{i(2n+1)\omega}$$

$$= -K\left(\sqrt{1 - b^2}\right) \sum_{n=0}^{\infty} \alpha_n |\Omega|^{2n+1} \left[\cos (2n + 1)\omega + i \sin (2n + 1)\omega\right]$$

Since on the solid-liquid interface $|\Omega| = 1$, this becomes on the interface

$$\zeta(\Omega, \Theta) = -K\left(\sqrt{1 - b^2}\right) \sum_{n=0}^{\infty} \alpha_n \left[\cos (2n + 1)\omega + i \sin (2n + 1)\omega\right] \qquad \zeta \in \widehat{FAB}$$

Thus the interface of unknown shape in figure 4 has been expressed as a series of cosine and sine harmonics.

The functions α_n will be found by requiring that ζ satisfy the boundary condition (eq. (12)) on the free surface $\widehat{FAB}$, that is, on the unit semicircle in the Ω-plane. By introducing the definition of ζ given by equation (17), the boundary condition (eq. (12)) can be written as

$$1 + \mathcal{R}e \left[\overline{\zeta(\Omega, \Theta)}\, \frac{\partial Z(\Omega, \Theta)}{\partial \Theta}\right] = |\zeta(\Omega, \Theta)| \qquad \Omega \in \widehat{FAB} \tag{24a}$$

Using polar coordinates, on the boundary $\Omega = e^{i\omega}$, results in equation (24a) becoming

$$1 + \mathcal{R}e \left[\overline{\zeta(e^{i\omega}, \Theta)}\, \frac{\partial Z(e^{i\omega}, \Theta)}{\partial \Theta}\right] = |\zeta(e^{i\omega}, \Theta)| \qquad 0 \leq \omega \leq \pi \tag{24b}$$

Integration to Obtain Physical Coordinates in Terms of Ω

Before the α_n can be found by use of equation (24b), it is necessary to calculate the complex variable Z in equation (24a) in terms of Ω and the coefficients in the expansion (eq. (23)) in order to obtain the $\partial Z/\partial\Theta$ term. If the origin of the coordinate system is chosen in the physical plane to be at the point D, then equation (19) shows that

$$Z(\Omega, \Theta) - A = \int_0^{\Omega} \zeta \frac{\partial W}{\partial \Omega} d\Omega$$

where A is the dimensionless length in figure 2. Substituting equations (22) and (23) yields

$$Z(\Omega, \Theta) = \int_0^{\Omega} \left[\frac{2}{\sqrt{(1 + \Omega^2)^2 - (1 - b^2)(1 - \Omega^2)^2}} \sum_{n=0}^{\infty} \alpha_n \Omega^{2n+1}\right] d\Omega + A$$

Putting $\gamma = \Omega^2$ yields

$$Z(\Omega, \Theta) = \sum_{n=0}^{\infty} \alpha_n \int_0^{\Omega^2} \frac{\gamma^n \, d\gamma}{\sqrt{(1 + \gamma)^2 - (1 - b^2)(1 - \gamma)^2}} + A$$

Upon carrying out the indicated integration, it is found that (see the appendix for details of the integration)

$$Z(\Omega, \Theta) - A = \ln\left[\frac{b\sqrt{X(\Omega)} + (1 + \Omega^2) + (1 - b^2)(1 - \Omega^2)}{2}\right] \sum_{n=0}^{\infty} \alpha_n C_n$$

$$+ \sqrt{X(\Omega)} \sum_{n=1}^{\infty} \alpha_n \sum_{r=0}^{n-1} \beta_r^n \Omega^{2r} - b \sum_{n=1}^{\infty} \alpha_n \beta_0^n \qquad (25a)$$

where

$$X(\Omega) \equiv (1 + \Omega^2)^2 - (1 - b^2)(1 - \Omega^2)^2 \tag{25b}$$

and the β_r^n are defined recursively by

$$\left.\begin{aligned} \beta_{n-1}^n &= \frac{1}{b^2 n} \\ \beta_{n-2}^n &= -\frac{\left[1 + (1 - b^2)\right](2n - 1)}{b^4(n - 1)n} \\ \beta_{r-1}^n &= -\frac{\left[1 + (1 - b^2)\right]}{b^2} \frac{(2r + 1)}{r} \beta_r^n - \frac{(r + 1)}{r} \beta_{r+1}^n \qquad 0 \le r \le n - 2 \end{aligned}\right\} \tag{26}$$

and the C_n are defined in terms of the β_r^n by

$$\left.\begin{aligned} C_0 &= \frac{1}{b} \\ C_n &= -\frac{\left(\beta_0^n + \beta_1^n\right) + (1 - b^2)\left(\beta_0^n - \beta_1^n\right)}{b} \qquad n = 1,\ 2,\ 3,\ \ldots \end{aligned}\right\} \tag{27}$$

It is necessary to impose the restriction that the distance $\widehat{DC}$ remain constant (and equal to A) with time. To accomplish this, notice that point D in figure 6 is the point at infinity. This can only occur when the denominator of equation (21) is zero. Now equating the denominator to zero gives a quadratic equation for Ω^2 which is solved to show that at point D

$$\Omega^2 = \frac{-2 + b^2 \pm 2\sqrt{1 - b^2}}{b^2} = -\left(\frac{-\sqrt{1 - b^2} \pm 1}{b}\right)^2$$

Then the point D maps into the point

$$\Omega = i\left(\frac{1 - \sqrt{1 - b^2}}{b}\right)$$

in the Ω plane, where the $+$ sign is chosen so that D will be in the upper half plane. This is also equivalent to

$$\Omega = i\left(\frac{1 - \sqrt{1 - b^2}}{1 + \sqrt{1 - b^2}}\right)^{1/2} \tag{28}$$

as can be shown by multiplying both the numerator and denominator by

$$\left(1 - \sqrt{1 - b^2}\right)^{1/2}$$

Since D is at the origin of the physical plane, it follows that

$$Z\left(i\sqrt{\frac{1 - \sqrt{1 - b^2}}{1 + \sqrt{1 - b^2}}}\right) = 0 \tag{29}$$

Substituting equations (28) and (29) in equation (25a) yields after some algebraic manipulation

$$A = b\sum_{n=1}^{\infty} \alpha_n\beta_0^n - \ln\sqrt{1 - b^2}\sum_{n=0}^{\infty} \alpha_n C_n \tag{30}$$

Substituting this into equation (25a) yields

$$Z(\Omega, \Theta) = \left(\frac{b\sum\limits_{n=1}^{\infty} \alpha_n\beta_0^n - A}{\ln\sqrt{1 - b^2}}\right)\ln\left[\frac{b\sqrt{X(\Omega)} + (1 + \Omega^2) + (1 - b^2)(1 - \Omega^2)}{2\sqrt{1 - b^2}}\right]$$

$$+ \sqrt{X(\Omega)}\sum_{n=1}^{\infty} \alpha_n \sum_{r=0}^{n-1} \beta_r^n\Omega^{2r} \tag{31}$$

To reduce the double series in equation (31) to a more convenient single sum, notice that (see ref. 9, p. 57)

$$\sum_{n=1}^{\infty} \alpha_n \sum_{r=0}^{n-1} \beta_r^n \Omega^{2r} = \sum_{n=0}^{\infty} \alpha_{n+1} \sum_{r=0}^{n} \beta_r^{n+1} \Omega^{2r} = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \alpha_{n+1+r} \beta_r^{n+1+r} \Omega^{2r}$$

$$= \sum_{r=0}^{\infty} \Omega^{2r} \sum_{n=0}^{\infty} \alpha_{n+1+r} \beta_r^{n+1+r} = \sum_{r=0}^{\infty} \Omega^{2r} \sum_{p=r+1}^{\infty} \alpha_p \beta_r^p$$

Hence upon defining A_r by

$$A_r \equiv \sum_{p=r+1}^{\infty} \alpha_p \beta_r^p \qquad r = 0, 1, 2, \ldots \tag{32}$$

equation (31) becomes

$$Z(\Omega, \Theta) = \frac{bA_0 - A}{\ln \sqrt{1 - b^2}} \ln \left[\frac{b\sqrt{X(\Omega)} + (1 + \Omega^2) + (1 - b^2)(1 - \Omega^2)}{2\sqrt{1 - b^2}} \right] + \sqrt{X(\Omega)} \sum_{n=0}^{\infty} \Omega^{2n} A_n \tag{33}$$

which is the expression for Z that will be used to substitute into the boundary condition (eq. (24b)).

Relation for α_n Explicitly in Terms of the A_n

The expression for Z given by equation (33) is in terms of the A_n while the expression for ζ given by equation (23) is in terms of the α_n. Hence, before substituting equation (23) into boundary condition (24b), it is desirable to express the α_n in terms of the A_n. This can be done through the use of equation (32). To this end, equation (27) is used to give

$$\sum_{n=0}^{\infty} \alpha_n C_n = \alpha_0 C_0 + \sum_{n=1}^{\infty} \alpha_n C_n = \frac{\alpha_0}{b} - \frac{1}{b} \left[\sum_{n=1}^{\infty} \left(\beta_0^n - \beta_1^n\right) \alpha_n + \frac{(1 - b^2)}{b} \sum_{n=1}^{\infty} \left(\beta_0^n - \beta_1^n\right) \alpha_n \right]$$

In the definitions of β (eq. (26)), the β_1^1 does not appear, that is, $\beta_1^1 = 0$. Then

$$\sum_{n=0}^{\infty} \alpha_n C_n = -\frac{1}{b}\left[\sum_{n=1}^{\infty} \alpha_n \beta_0^n - \sum_{n=2}^{\infty} \alpha_n \beta_1^n + (1 - b^2)\left(\sum_{n=1}^{\infty} \alpha_n \beta_0^n - \sum_{n=2}^{\infty} \alpha_n \beta_1^n\right) - \alpha_0\right]$$

Now use the definition of A_n from equation (32) to give

$$\sum_{n=0}^{\infty} \alpha_n C_n = \frac{1}{b}\left[\alpha_0 - (A_0 + A_1) - (1 - b^2)(A_0 - A_1)\right]$$

Then by the additional use of equation (32) to obtain A_0, equation (30) becomes

$$A = bA_0 - \frac{\ln \sqrt{1 - b^2}}{b}\left[\alpha_0 - (A_0 + A_1) - (1 - b^2)(A_0 - A_1)\right]$$

which is solved for α_0 to obtain

$$\alpha_0 = \frac{(bA_0 - A)b}{\ln \sqrt{1 - b^2}} + A_0 + A_1 + (1 - b^2)(A_0 - A_1) \tag{34}$$

Now it follows from definitions (26) and (32) that for $r \geq 1$

$$A_{r+1} = \sum_{p=r+2}^{\infty} \alpha_p \beta_{r+1}^p = -\frac{\left[1 + (1 - b^2)\right]}{b^2} \frac{(2r + 1)}{r + 1} \sum_{p=r+2}^{\infty} \alpha_p \beta_r^p - \frac{r}{r + 1} \sum_{p=r+2}^{\infty} \alpha_p \beta_{r-1}^p \tag{35}$$

From equations (32) and (26) the two summations are found to be

$$\sum_{p=r+2}^{\infty} \alpha_p \beta_r^p = \sum_{p=r+1}^{\infty} \alpha_p \beta_r^p - \alpha_{r+1}\beta_r^{r+1} = A_r - \alpha_{r+1}\beta_r^{r+1} = A_r - \alpha_{r+1} \frac{1}{b^2(r+1)}$$

$$\sum_{p=r+2}^{\infty} \alpha_p \beta_{r-1}^p = \sum_{p=r}^{\infty} \alpha_p \beta_{r-1}^p - \alpha_r \beta_{r-1}^r - \alpha_{r+1}\beta_{r-1}^{r+1}$$

$$= A_{r-1} - \alpha_r \frac{1}{b^2 r} + \alpha_{r+1} \frac{(2r+1)\left[1 + (1 - b^2)\right]}{b^4 r(r+1)}$$

Substituting into equation (35) gives

$$A_{r+1} = \frac{\left[1 + (1 - b^2)\right]}{b^2} \frac{(2r+1)}{r+1} \alpha_{r+1} \frac{1}{b^2(r+1)} - \frac{\left[1 + (1 - b^2)\right]}{b^2} \frac{(2r+1)}{r+1} A_r + \frac{r}{r+1} \frac{1}{b^2 r} \alpha_r$$

$$- \frac{r}{r+1} \frac{\left[1 + (1 - b^2)\right]}{b^2} \frac{(2r+1)}{r(r+1)b^2} \alpha_{r+1} - \frac{r}{r+1} A_{r-1}$$

Cancelling terms and solving for α_r gives

$$\alpha_r = b^2(r+1)A_{r+1} + \left[1 + (1 - b^2)\right](2r+1)A_r + b^2 r A_{r-1} \qquad r = 1, 2, \ldots \qquad (36)$$

Upon combining equations (34) and (36), the α_n's can be written as

$$\alpha_n = \sum_{j=0}^{2} \binom{2}{j} \left[1 - (-1)^j (1 - b^2)\right] \left(n + 1 - \frac{j}{2}\right) A_{n+1-j} + Eb\delta_{n,0} \qquad n = 0, 1, 2, \ldots$$

(37)

where

$$E = \frac{bA_0 - A}{\ln \sqrt{1 - b^2}} \qquad (38)$$

the symbol $\binom{2}{j}$ denotes the binomial coefficients whose values are 1, 2, and 1 for $j = 0$, 1, and 2, respectively, and $\delta_{n,0}$ is the Kronecker delta. Equation (37) allows us to replace the variables α_n in equation (23) by the variables A_n and so, in view of equation (33), to completely formulate the boundary condition (eq. (24b)) in terms of the A_n.

Terms in Interface Boundary Condition Needed to Determine the A_n and b

The interface boundary condition as given by equation (24a) or (24b) will provide a differential equation which determines the A_n. Each of the terms to be substituted into this condition will now be derived.

The term $\underline{\mathcal{R}e\left(\overline{\zeta}\,\frac{\partial Z}{\partial \Theta}\right)}$. - In order to insert an expression for $\partial Z/\partial\Theta$ into the boundary condition given by equations (24), equation (33) is differentiated with respect to time. After collecting terms, this gives

$$\frac{\partial Z(\Omega, \Theta)}{\partial \Theta} = \frac{1}{\sqrt{X(\Omega)}}\left\{\frac{(bA_0 - A)\dot{b}}{(1 - b^2)\ln\sqrt{1 - b^2}}\left[(1 + \Omega^2) - (1 - b^2)(1 - \Omega^2)\right]\right.$$
$$\left. + b\dot{b}(1 - \Omega^2)^2 \sum_{n=0}^{\infty} A_n \Omega^{2n} + X(\Omega) \sum_{n=0}^{\infty} \dot{A}_n \Omega^{2n}\right\}$$
$$+ \left[\frac{(bA_0 - A)b\dot{b}}{(1 - b^2)\left(\ln\sqrt{1 - b^2}\right)^2} + \frac{A_0\dot{b} + b\dot{A}_0}{\ln\sqrt{1 - b^2}}\right]\ln\left[\frac{b\sqrt{X(\Omega)} + (1 + \Omega^2) + (1 - b^2)(1 - \Omega^2)}{2\sqrt{1 - b^2}}\right]$$

where the dot denotes differentiation with respect to time. By using equation (38), this equation can also be written as

$$\frac{\partial Z(\Omega, \Theta)}{\partial \Theta} = \frac{1}{\sqrt{X(\Omega)}} \left\{ \frac{E\dot{b}}{(1 - b^2)} \sum_{j=0}^{1} \left[1 - (-1)^j (1 - b^2)\right] \Omega^{2j} \right.$$

$$\left. + b\dot{b} \sum_{n=0}^{\infty} A_n \sum_{j=0}^{2} \binom{2}{j} (-1)^j \Omega^{2(n+j)} + \sum_{n=0}^{\infty} \dot{A}_n \sum_{j=0}^{2} \binom{2}{j} \left[1 - (-1)^j (1 - b^2)\right] \Omega^{2(n+j)} \right\}$$

$$+ \dot{E} \ln \left[\frac{b\sqrt{X(\Omega)} + (1 + \Omega^2) + (1 - b^2)(1 - \Omega^2)}{2\sqrt{1 - b^2}} \right] \tag{39}$$

where it was found by differentiating equation (38) that

$$\dot{E} = \frac{b\dot{b}(bA_0 - A)}{(1 - b^2)\left(\ln\sqrt{1 - b^2}\right)^2} + \frac{A_0\dot{b} + b\dot{A}_0}{\ln\sqrt{1 - b^2}} \tag{40}$$

It follows from equation (25b) that on the interface where $\Omega = e^{i\omega}$ for $0 \leq \omega \leq \pi$

$$X(e^{i\omega}) = \left(1 + e^{2i\omega}\right)^2 - (1 - b^2)\left(1 - e^{2i\omega}\right)^2$$

$$= 4e^{2i\omega}\left[\cos^2 \omega + (1 - b^2) \sin^2 \omega\right] = 4e^{2i\omega}(1 - b^2 \sin^2 \omega)$$

Hence,

$$\sqrt{X(e^{i\omega})} = 2e^{i\omega}\sqrt{1 - b^2 \sin^2 \omega} \tag{41}$$

which will be used in equation (39). On the interface the ln term in equation (39) becomes

$$\ln\left[\frac{b\sqrt{X(\Omega)} + (1 + \Omega^2) + (1 - b^2)(1 - \Omega^2)}{2\sqrt{1 - b^2}}\right]$$

$$= i\omega + \ln\left[\frac{b\sqrt{1 - b^2 \sin^2 \omega} + \cos \omega - i(1 - b^2) \sin \omega}{\sqrt{1 - b^2}}\right]$$

$$= \ln\left[\frac{b^2(1 - b^2 \sin^2 \omega) + \cos^2 \omega + 2b\sqrt{1 - b^2 \sin^2 \omega} \cos \omega + (1 - b^2)^2 \sin^2 \omega}{1 - b^2}\right]^{1/2}$$

$$+ i\omega + i \tan^{-1}\left[\frac{-(1 - b^2) \sin \omega}{b\sqrt{1 - b^2 \sin^2 \omega} + \cos \omega}\right]$$

$$= \ln\left[\frac{b \cos \omega + \sqrt{1 - b^2 \sin^2 \omega}}{\sqrt{1 - b^2}}\right] + i\left\{\omega + \tan^{-1}\left[\frac{-(1 - b^2) \sin \omega}{b\sqrt{1 - b^2 \sin^2 \omega} + \cos \omega}\right]\right\} \qquad (42)$$

Combining equations (39), (41), and (42) shows that

$$\frac{\partial Z(e^{i\omega}, \Theta)}{\partial \Theta} = \frac{1}{2\sqrt{1 - b^2 \sin^2 \omega}} \left\{ \frac{E\dot{b}}{(1 - b^2)} \sum_{j=0}^{1} \left[1 - (-1)^j (1 - b^2)\right] e^{i(2j-1)\omega} \right.$$

$$+ b\dot{b} \sum_{n=0}^{\infty} A_n \sum_{j=0}^{2} \binom{2}{j} (-1)^j e^{i[2(n+j)-1]\omega}$$

$$\left. + \sum_{n=0}^{\infty} \dot{A}_n \sum_{j=0}^{2} \binom{2}{j} \left[1 - (-1)^j (1 - b^2)\right] e^{i[2(n+j)-1]\omega} \right\}$$

$$+ \dot{E} \left(\ln \left(\frac{b \cos \omega + \sqrt{1 - b^2 \sin^2 \omega}}{\sqrt{1 - b^2}} \right) \right.$$

$$\left. + i \left\{ \omega + \tan^{-1} \left[\frac{-(1 - b^2) \sin \omega}{b\sqrt{1 - b^2 \sin^2 \omega} + \cos \omega} \right] \right\} \right) \tag{43}$$

Equation (23) shows that the expression for ζ on the interface is

$$\zeta(e^{i\omega}, \Theta) = -K \left(\sqrt{1 - b^2} \right) \sum_{n=0}^{\infty} \alpha_n e^{i(2n+1)\omega} \tag{44}$$

By taking the complex conjugate of equation (44), multiplying by equation (43), and then taking the real part, an expression is obtained for the term

$$\mathcal{Re} \left[\overline{\zeta(e^{i\omega}, \Theta)} \, \frac{\partial Z(e^{i\omega}, \Theta)}{\partial \Theta} \right]$$

which will be subsequently substituted into boundary condition (24b)

$$
\begin{aligned}
Re\left(\bar{\zeta}\frac{\partial Z}{\partial \Theta}\right)\Bigg|_{\Omega=e^{i\omega}} = -\frac{K\left(\sqrt{1-b^2}\right)}{2\sqrt{1-b^2\sin^2\omega}} & \Bigg\{\frac{\dot{E}b}{(1-b^2)}\sum_{n=0}^{\infty}\sum_{j=0}^{1}\left[1-(-1)^j(1-b^2)\right]\alpha_n\cos\left[2(j-n-1)\omega\right] \\
& + b\dot{b}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A_n\alpha_k\sum_{j=0}^{2}\binom{2}{j}(-1)^j\cos\left[2(j+n-k-1)\omega\right] \\
& + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\dot{A}_n\alpha_k\sum_{j=0}^{2}\binom{2}{j}\left[1-(-1)^j(1-b^2)\right]\cos\left[2(n+j-k-1)\omega\right]\Bigg\} \\
& - \dot{E}K\left(\sqrt{1-b^2}\right)\left(\ln\left(\frac{b\cos\omega+\sqrt{1-b^2\sin^2\omega}}{\sqrt{1-b^2}}\right)\sum_{n=0}^{\infty}\alpha_n\cos\left[(2n+1)\omega\right]\right. \\
& \left. + \left\{\omega+\tan^{-1}\left[\frac{-(1-b^2)\sin\omega}{b\sqrt{1-b^2\sin^2\omega}+\cos\omega}\right]\right\}\sum_{n=0}^{\infty}\alpha_n\sin\left[(2n+1)\omega\right]\right)
\end{aligned} \tag{45}
$$

The term $|\zeta(e^{i\omega},\Theta)|$. - An expression for the values of ζ on the interface is given by equation (44). However, the absolute value of this expression must be taken. This is difficult by ordinary means since it would require squaring the real and imaginary parts which are infinite series. As a preliminary step to finding the absolute value notice that if

$$
\sqrt{\sum_{n=0}^{\infty}a_n Z^n} = \sum_{n=0}^{\infty}b_n Z^n
$$

then in view of the Cauchy product rule (ref. 9, p. 162) squaring both sides shows that

$$
\sum_{n=0}^{\infty}a_n Z^n = \sum_{n=0}^{\infty}b_n Z^n\sum_{m=0}^{\infty}b_m Z^m = \sum_{n=0}^{\infty}Z^n\sum_{k=0}^{n}b_k b_{n-k}
$$

Hence, equating like powers of Z shows that

$$a_n = \sum_{k=0}^{n} b_k b_{n-k} \qquad n = 0,\ 1,\ 2,\ .\ .\ . \tag{46}$$

This set of equations can be solved recursively to determine the b_n in terms of the a_n. Equation (46) will be applied to find a series expansion for $\left|\zeta(e^{i\omega}, \Theta)\right|$ to be used in the boundary condition (eq. (24b)).

To this end, notice that equation (44) implies

$$\sqrt{\frac{\zeta(e^{i\omega}, \Theta)}{-K\left(\sqrt{1 - b^2}\right)}} = \sqrt{\sum_{n=0}^{\infty} \alpha_n e^{i(2n+1)\omega}} = e^{i\omega/2} \sqrt{\sum_{n=0}^{\infty} \alpha_n e^{i2n\omega}}$$

and

$$\sqrt{\frac{\overline{\zeta(e^{i\omega}, \Theta)}}{-K\left(\sqrt{1 - b^2}\right)}} = e^{-i\omega/2} \sqrt{\sum_{n=0}^{\infty} \alpha_n e^{-i2n\omega}}$$

Hence, if we let

$$\sqrt{\frac{\zeta(e^{i\omega}, \Theta)}{-K\left(\sqrt{1 - b^2}\right)}} = e^{i\omega/2} \sum_{n=0}^{\infty} \hat{\beta}_n e^{i2n\omega} \tag{47}$$

then

$$\sqrt{\frac{\overline{\zeta(e^{i\omega}, \Theta)}}{-K\left(\sqrt{1 - b^2}\right)}} = e^{-i\omega/2} \sum_{n=0}^{\infty} \hat{\beta}_n e^{-i2n\omega} \tag{48}$$

Using equation (46) shows that the $\hat{\beta}_n$ are determined in terms of the α_n by

$$\sum_{k=0}^{n} \hat{\beta}_k \hat{\beta}_{n-k} = \alpha_n \tag{49}$$

When equation (47) is multiplied by equation (48), all imaginary parts cancel out and the following is obtained:

$$
\begin{aligned}
\left|\zeta(e^{i\omega}, \Theta)\right| &= K\left(\sqrt{1 - b^2}\right)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\hat{\beta}_n\hat{\beta}_m e^{i2(n-m)\omega} \\
&= K\left(\sqrt{1 - b^2}\right)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\hat{\beta}_n\hat{\beta}_m \cos\left[2(n-m)\omega\right] \qquad (50)
\end{aligned}
$$

Equation (50) is the expression for the term which is to be substituted into the right side of the boundary condition (eq. (24b)).

Substitution into Boundary Condition and Integration over Cosine Harmonics

Equations (45) and (50) are to be substituted into equation (24b). This will give a relation containing A_n and b which are functions of time. The equation will also contain various sine and cosine harmonics of ω. In a way analogous to obtaining the coefficients in an ordinary Fourier series expansion, the equation will be multiplied through by $\cos 2p\omega$ and integrated over the interface. By letting $p = 0, 1, 2, 3, \ldots,$ a set of simultaneous ordinary differential equations will be obtained for the unknowns A_n and b.

Before carrying out this procedure, a few preliminary relations are developed. By differentiation it is found that, after considerable simplification,

$$
\frac{\partial}{\partial\omega}\left\{\omega + \tan^{-1}\left[\frac{-(1 - b^2)\sin\omega}{b\sqrt{1 - b^2\sin^2\omega} + \cos\omega}\right]\right\} = \frac{b\cos\omega}{\sqrt{1 - b^2\sin^2\omega}}
$$

and

$$
\frac{\partial}{\partial\omega}\left[\ln\left(\frac{b\cos\omega + \sqrt{1 - b^2\sin^2\omega}}{\sqrt{1 - b^2}}\right)\right] = -\frac{b\sin\omega}{\sqrt{1 - b^2\sin^2\omega}}
$$

It follows by integrating by parts that for $p = 0, 1, 2, \ldots$

$$\int_0^{\pi/2}\left(\ln\left(\frac{b\cos\omega+\sqrt{1-b^2\sin^2\omega}}{\sqrt{1-b^2}}\right)\cos\left[(2n+1)\omega\right]\cos 2p\omega\right.$$

$$\left.+\left\{\omega+\tan^{-1}\left[\frac{-(1-b^2)\sin\omega}{b\sqrt{1-b^2\sin^2\omega}+\cos\omega}\right]\right\}\sin\left[(2n+1)\omega\right]\cos 2p\omega\right)d\omega$$

$$=\left|\ln\left(\frac{b\cos\omega+\sqrt{1-b^2\sin^2\omega}}{\sqrt{1-b^2}}\right)\left\{\frac{\sin\left[2(n-p)+1\right]\omega}{2[2(n-p)+1]}+\frac{\sin\left[2(n+p)+1\right]\omega}{2[2(n+p)+1]}\right\}\right.$$

$$\left.-\left\{\omega+\tan^{-1}\left[\frac{-(1-b^2)\sin\omega}{b\sqrt{1-b^2\sin^2\omega}+\cos\omega}\right]\right\}\left\{\frac{\cos\left[2(n-p)+1\right]\omega}{2[2(n-p)+1]}+\frac{\cos\left[2(n+p)+1\right]\omega}{2[2(n+p)+1]}\right\}\right|_0^{\pi/2}$$

$$+\int_0^{\pi/2}\frac{b}{\sqrt{1-b^2\sin^2\omega}}\left\{\sin\omega\sin\left[2(n-p)+1\right]\omega+\cos\omega\cos\left[2(n-p)+1\right]\omega\right\}\frac{d\omega}{2[2(n-p)+1]}$$

$$+\int_0^{\pi/2}\frac{b}{\sqrt{1-b^2\sin^2\omega}}\left\{\sin\omega\sin\left[2(n+p)+1\right]\omega+\cos\omega\cos\left[2(n+p)+1\right]\omega\right\}\frac{d\omega}{2[2(n+1)+1]}$$

$$=b\int_0^{\pi/2}\frac{1}{\sqrt{1-b^2\sin^2\omega}}\left\{\frac{\cos\left[2(n-p)\omega\right]}{2[2(n-p)+1]}+\frac{\cos\left[2(n+p)\omega\right]}{2[2(n+p)+1]}\right\}d\omega$$

$$=\frac{b}{2}\left\{\frac{1}{2(n-p)+1}\int_0^{\pi/2}\frac{\cos\left[2(n-p)\omega\right]}{\sqrt{1-b^2\sin^2\omega}}d\omega+\frac{1}{2(n+p)+1}\int_0^{\pi/2}\frac{\cos\left[2(n+p)\omega\right]}{\sqrt{1-b^2\sin^2\omega}}d\omega\right\} \tag{51}$$

Equations (45) and (50) are now substituted into the boundary condition (eq. (24b)). The result is multiplied through by $\cos 2p\omega$ for $p = 0, 1, 2, 3, \ldots$ and integrated between $\omega = 0$ and $\pi/2$. Using the results in equation (51) leads to the following infinite set of first-order ordinary differential equations for the functions A_n and b:

$$
\begin{aligned}
&-\frac{2\pi\delta_{p,0}}{4K\left(\sqrt{1-b^2}\right)} + \frac{E\dot{b}}{4(1-b^2)} \sum_{n=0}^{\infty} \alpha_n \sum_{j=0}^{1} \left[1 - (-1)^j(1-b^2)\right](K_{j-n-1+p} + K_{j-n-1-p}) \\
&+ \frac{b\dot{b}}{4} \sum_{n=0}^{\infty} A_n \sum_{k=0}^{\infty} \alpha_k \sum_{j=0}^{2} \binom{2}{j} (-1)^j (K_{j+n-k-1+p} + K_{j+n-k-1-p}) \\
&+ \frac{1}{4} \sum_{n=0}^{\infty} \dot{A}_n \sum_{k=0}^{\infty} \alpha_k \sum_{j=0}^{2} \binom{2}{j} \left[1 - (-1)^j(1-b^2)\right] (K_{n+j-k-1+p} + K_{n+j-k-1-p}) \\
&+ \frac{\dot{E}b}{2} \sum_{n=0}^{\infty} \alpha_n \left[\frac{1}{2(n+p)+1} K_{n+p} + \frac{1}{2(n-p)+1} K_{n-p}\right] = -\frac{\pi}{4}(1 + \delta_{p,0}) \sum_{n=0}^{\infty} \hat{\beta}_n \hat{\beta}_{n+p}
\end{aligned}
\tag{52}
$$

where for $n = 0, 1, 2, \ldots$ the K_n are defined by

$$
K_n = \int_0^{\pi/2} \frac{\cos 2n\omega}{\sqrt{1 - b^2 \sin^2 \omega}} \, d\omega \tag{53}
$$

Equations (40) and (38) show that the derivative of E with respect to time is

$$
\dot{E} = \frac{\dot{b}A_0 + \dot{A}_0 b}{\ln\sqrt{1-b^2}} + \frac{E}{\ln\sqrt{1-b^2}} \frac{b\dot{b}}{1-b^2}
$$

This is substituted into equation (52) to eliminate $\dot{E}$, and the result is

$$
-\frac{2\pi\delta_{p,0}}{K\left(\sqrt{1-b^2}\right)} + \frac{E\dot{b}}{(1-b^2)}\sum_{n=0}^{\infty}\alpha_n\left\{\sum_{j=0}^{1}\left[1-(-1)^j(1-b^2)\right](K_{j-n-1+p}+K_{j-n-1-p})\right.
$$

$$
\left. + \frac{2b^2}{\ln\sqrt{1-b^2}}\left[\frac{1}{2(n+p)+1}K_{n+p} + \frac{1}{2(n-p)+1}K_{n-p}\right]\right\}
$$

$$
+ b\dot{b}\sum_{n=0}^{\infty}A_n\sum_{k=0}^{\infty}\alpha_k\left\{\sum_{j=0}^{2}\binom{2}{j}(-1)^j(K_{j+n-k-1+p}+K_{j+n-k-1-p})\right.
$$

$$
\left. + \frac{2\delta_{n,0}}{\ln\sqrt{1-b^2}}\left[\frac{1}{2(k+p)+1}K_{k+p} + \frac{1}{2(k-p)+1}K_{k-p}\right]\right\}
$$

$$
+ \sum_{n=0}^{\infty}\dot{A}_n\sum_{k=0}^{\infty}\alpha_k\left\{\sum_{j=0}^{2}\binom{2}{j}\left[1-(-1)^j(1-b^2)\right](K_{n+j-k-1+p}+K_{n+j-k-1-p})\right.
$$

$$
\left. + \frac{2\delta_{n,0}b^2}{\ln\sqrt{1-b^2}}\left[\frac{1}{2(k+p)+1}K_{k+p} + \frac{1}{2(k-p)+1}K_{k-p}\right]\right\} = -\pi(1+\delta_{p,0})\sum_{n=0}^{\infty}\hat{\beta}_n\hat{\beta}_{n+p} \tag{54}
$$

Now if we set

$$
\left.\begin{aligned}
H_{k,p}^{(0)} &= \frac{2b^2}{\ln\sqrt{1-b^2}}\left[\frac{K_{k+p}}{2(k+p)+1} + \frac{K_{k-p}}{2(k-p)+1}\right] \\
H_{k,p}^{(1)} &= K_{k+p} + K_{k-p}
\end{aligned}\right\}
\begin{aligned}
&p = 0,\ 1,\ 2,\ \ldots \text{ and} \\
&k = 0,\ \pm 1,\ \pm 2,\ \ldots
\end{aligned} \tag{55}
$$

and

$$J^{(r)}_{n,k,p}(b) = \sum_{j=0}^{\min(2,r)} \binom{2-\delta_{r,1}}{j} (-1)^j \left[(-1)^{rj} - (1 - b^2)\right] H^{(1)}_{n+j-k-1,p} + \delta_{n,0} H^{(0)}_{k,p}$$

$$r = 1,\ 2,\ 3 \text{ and } n,\ k,\ p = 0,\ 1,\ 2,\ \ldots \qquad (56)$$

then equation (54) can be put in the form

$$\dot{b}\left[\sum_{k=0}^{\infty} \alpha_k\left(\frac{E}{1 - b^2} J^{(1)}_{0,k,p} + \frac{1}{b}\sum_{n=0}^{\infty} A_n J^{(2)}_{n,k,p}\right)\right] + \sum_{n=0}^{\infty} \dot{A}_n \sum_{k=0}^{\infty} \alpha_k J^{(3)}_{n,k,p}$$

$$= -\pi(1 + \delta_{p,0}) \sum_{n=0}^{\infty} \hat{\beta}_n \hat{\beta}_{n+p} + \frac{2\pi\delta_{p,0}}{K\left(\sqrt{1 - b^2}\right)} \qquad p = 0,\ 1,\ 2,\ \ldots \qquad (57)$$

This infinite set of first-order differential equations completely determines the unknown functions of time, b and A_n, for $n = 0, 1, 2, 3, \ldots$ and so in view of equations (33) and (22) completely determines the solution to the problem.

Coordinates of the Solid-Liquid Interface

If equations (41) and (42) are substituted in equation (33) and the real and imaginary parts of the resulting expression are taken, the following parametric equations for the shape of the freezing surface are obtained:

$$
\left.
\begin{aligned}
X_s &= 2\sqrt{1 - b^2 \sin^2 \omega} \sum_{n=0}^{\infty} A_n \cos (2n + 1)\omega \\
&\quad + \frac{bA_0 - A}{\ln \sqrt{1 - b^2}} \left[\frac{b \cos \omega + \sqrt{1 - b^2 \sin^2 \omega}}{\sqrt{1 - b^2}} \right] \\
Y_s &= 2\sqrt{1 - b^2 \sin^2 \omega} \sum_{n=0}^{\infty} A_n \sin (2n + 1)\omega \\
&\quad + \frac{bA_0 - A}{\ln \sqrt{1 - b^2}} \left[\omega + \tan^{-1} \left(\frac{-(1 - b^2) \sin \omega}{b\sqrt{1 - b^2 \sin^2 \omega} + \cos \omega} \right) \right]
\end{aligned}
\right\}
\begin{aligned}
&0 \leq \omega \leq \frac{\pi}{2}, \\
&-\frac{\pi}{2} \leq \tan^{-1} \leq 0
\end{aligned}
\qquad (58)
$$

The evaluation of the set of differential equations to obtain the A_n and b as functions of time and the evaluation of the transient interface shapes will be given subsequently.

Heat Flow Through Frozen Region

An expression for the heat flow per unit plate length through the frozen layer can be obtained from the temperature gradient at the cold plate,

$$
Q = \int_{\text{plate}} k \frac{\partial t}{\partial y} \, dx
$$

or in dimensionless form

$$
\frac{Q}{2k(t_f - t_w)} = \int_{X_D}^{X_C} \frac{\partial T}{\partial Y} \, dX
$$

It has been pointed out in connection with equation (8) that $\partial T/\partial Y = \partial \psi/\partial X$. Hence,

$$
\frac{Q}{2k(t_f - t_w)} = \int_{\psi_D}^{\psi_C} \frac{\partial \psi}{\partial X} \, dX = \psi_C - \psi_D
$$

It can be seen from figure 3 that $\psi_C - \psi_D = \psi_B - \psi_A$. Using the coordinates in the U-plane (see fig. 6) gives

$$\frac{Q}{2k(t_f - t_w)} = \psi(-b + i0) - \psi(0) = \mathcal{I}m\left[W(-b + i0) - W(0)\right] = \mathcal{I}m \int_0^{-b} \frac{\partial W}{\partial U} dU$$

Substituting equation (20) for $\partial W/\partial U$ gives

$$\frac{Q}{2k(t_f - t_w)} = \frac{1}{K\left(\sqrt{1 - b^2}\right)} \int_0^{-b} \frac{dU}{\sqrt{U^2 - b^2}\sqrt{U^2 - 1}} = \frac{K(b)}{K\left(\sqrt{1 - b^2}\right)} \tag{59}$$

Some special cases will now be considered. These will aid in the interpretation of the transient results to be obtained later.

STEADY-STATE SOLUTION

The steady-state frozen layer shapes for the geometry considered here have been already given in reference 3. They can be obtained from the present analysis by letting all the time dependent coefficients A_n be zero. Then from equation (58)

$$\left.\begin{aligned} X_s^{ss} &= -\frac{A}{\ln\sqrt{1 - b^2}} \ln\left[\frac{b\cos\omega + \sqrt{1 - b^2\sin^2\omega}}{\sqrt{1 - b^2}}\right] \\ Y_s^{ss} &= -\frac{A}{\ln\sqrt{1 - b^2}}\left[\omega + \tan^{-1}\left(\frac{-(1 - b^2)\sin\omega}{b\sqrt{1 - b^2\sin^2\omega} + \cos\omega}\right)\right] \end{aligned}\right\} \begin{aligned} &0 \le \omega \le \frac{\pi}{2}, \\ &-\frac{\pi}{2} \le \tan^{-1} \le 0 \end{aligned} \tag{60}$$

The steady-state profiles are thus a function of A and b. The parameter A contains the physical quantities governing the frozen configuration

$$A = \frac{ha}{k} \frac{t_l - t_f}{t_f - t_w}$$

and hence a relation is needed between the mapping parameter b and A. To obtain this relation, note that at steady state, $\dot{b}$ and $\dot{A}$ are zero so equation (57) reduces to

$$(1 + \delta_{p,0}) \sum_{n=0}^{\infty} \hat{\beta}_n \hat{\beta}_{n+p} - \frac{2\delta_{p,0}}{K\left(\sqrt{1 - b^2}\right)} = 0 \tag{61}$$

With all the $A_n = 0$, equation (37) gives

$$\alpha_0 = - \frac{Ab}{\ln \sqrt{1 - b^2}} \qquad \alpha_n = 0 \text{ for } n \geq 1 \tag{62}$$

Then from equation (49)

$$\hat{\beta}_0 = \alpha_0^{1/2} = \sqrt{\frac{-Ab}{\ln \sqrt{1 - b^2}}} \qquad \hat{\beta}_n = 0 \text{ for } n \geq 1 \tag{63}$$

Equation (61) reduces to the following single relation for $p = 0$

$$- \frac{2Ab}{\ln \sqrt{1 - b^2}} - \frac{2}{K\left(\sqrt{1 - b^2}\right)} = 0$$

so that the relation between b and A at steady state is

$$A = - \frac{\ln \sqrt{1 - b^2}}{b\, K\left(\sqrt{1 - b^2}\right)} \tag{64}$$

This is the same as the result obtained in reference 3. This relation has been plotted in figure 7. For a given A as determined by the imposed temperatures and heat transfer

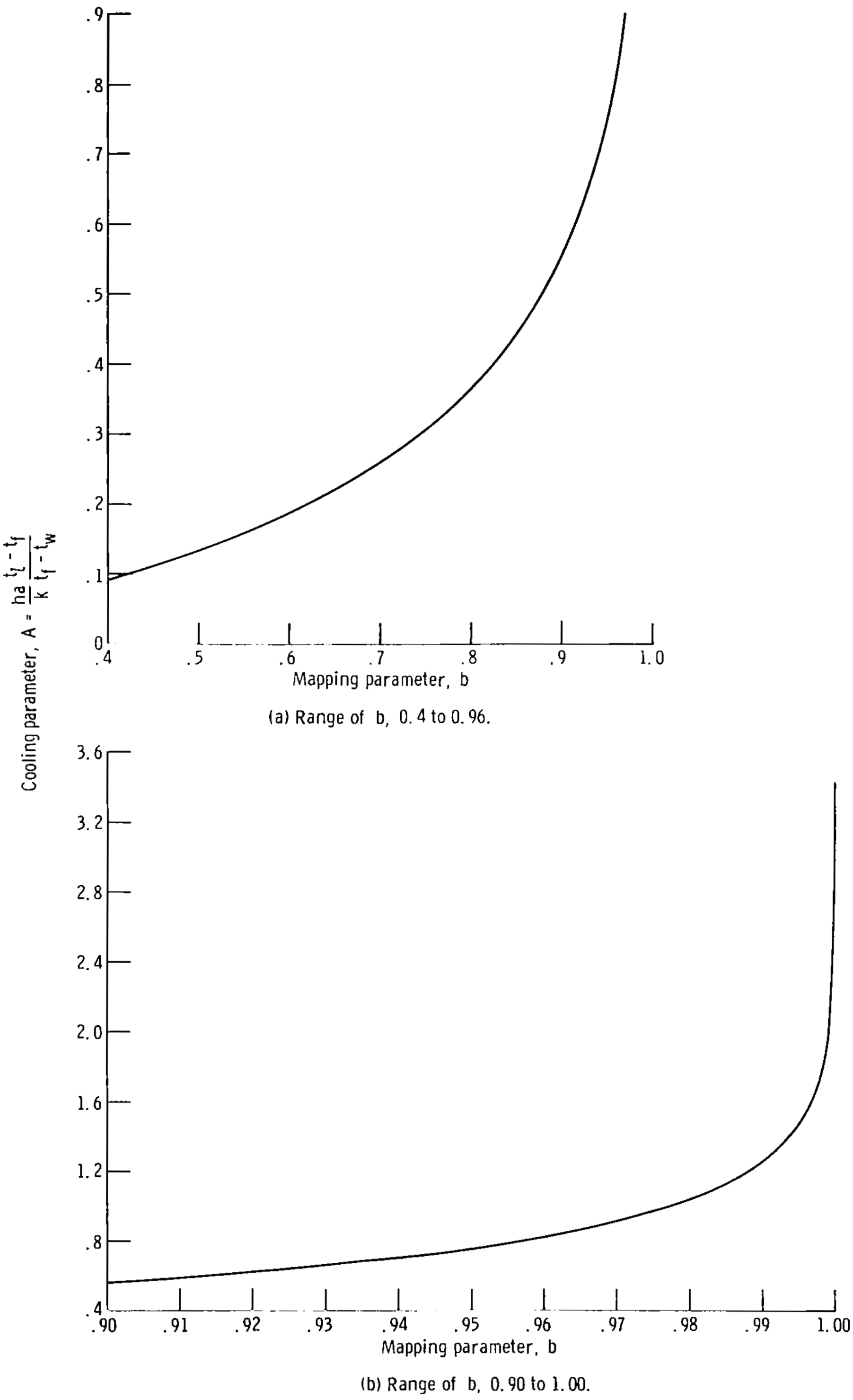
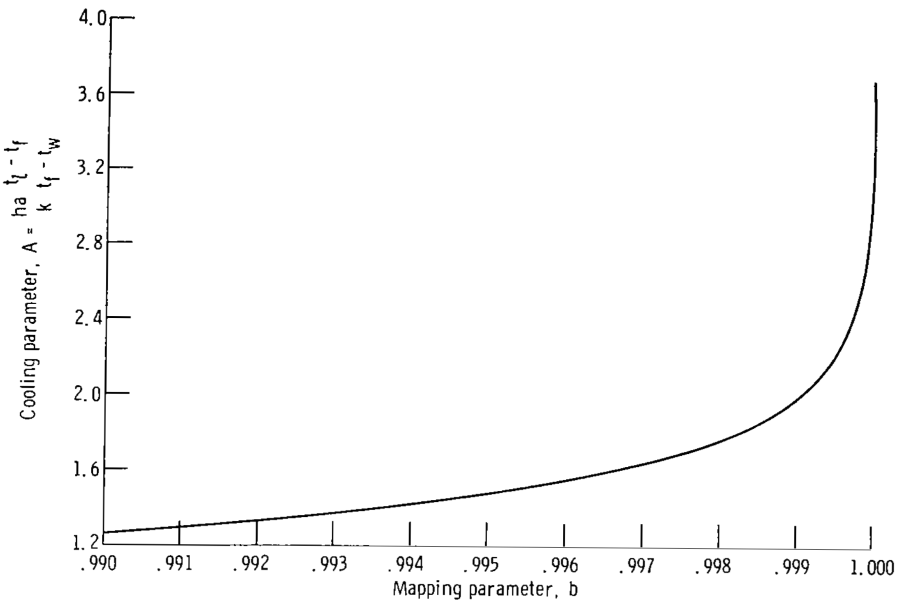


Figure 7. - Steady-state relation between cooling parameter A and mapping parameter b (eq. (64)).



(c) Range of b, 0.990 to 1.000.

Figure 7. - Concluded.

coefficient, b can then be found. Equations (60) are evaluated for various values of b, and these contours are given in figure 8. Hence, by using figures 7 and 8 the steady-state contour of the frozen region can be quickly found for a given A.

It follows from equation (59) that the heat flow through the layer is only a function of b. In fact, for both transient and steady conditions

$$\frac{Q}{2k(t_f - t_w)} = \frac{K(b)}{K\left(\sqrt{1 - b^2}\right)} \tag{65}$$

This relation between $Q/2k(t_f - t_w)$ and b is plotted in figure 9.

To summarize for the steady-state solution: for given physical conditions compute the value of A, and then determine b from figure 7. Figures 8 and 9 then give the contour of the frozen region and the heat flow through it.

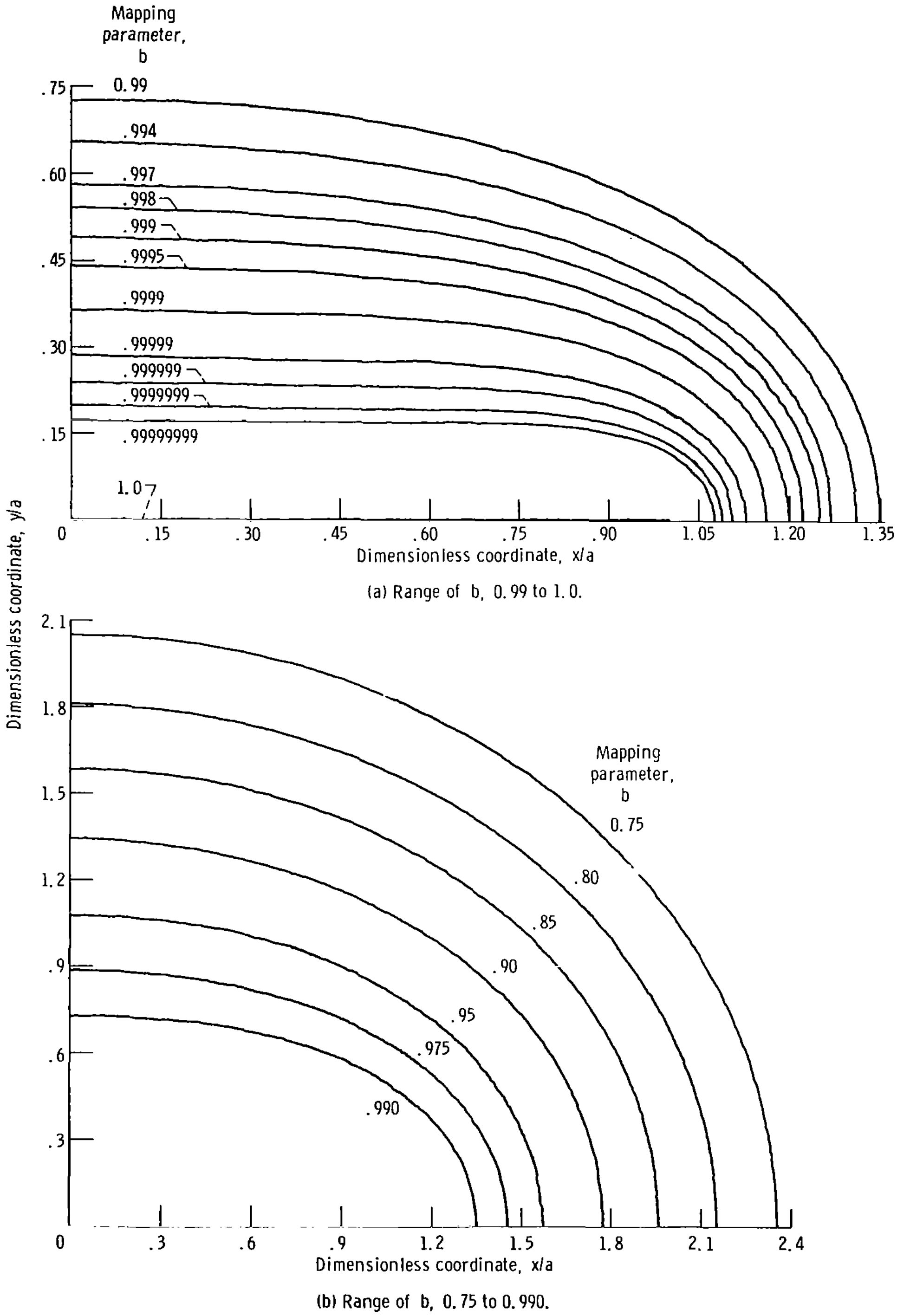
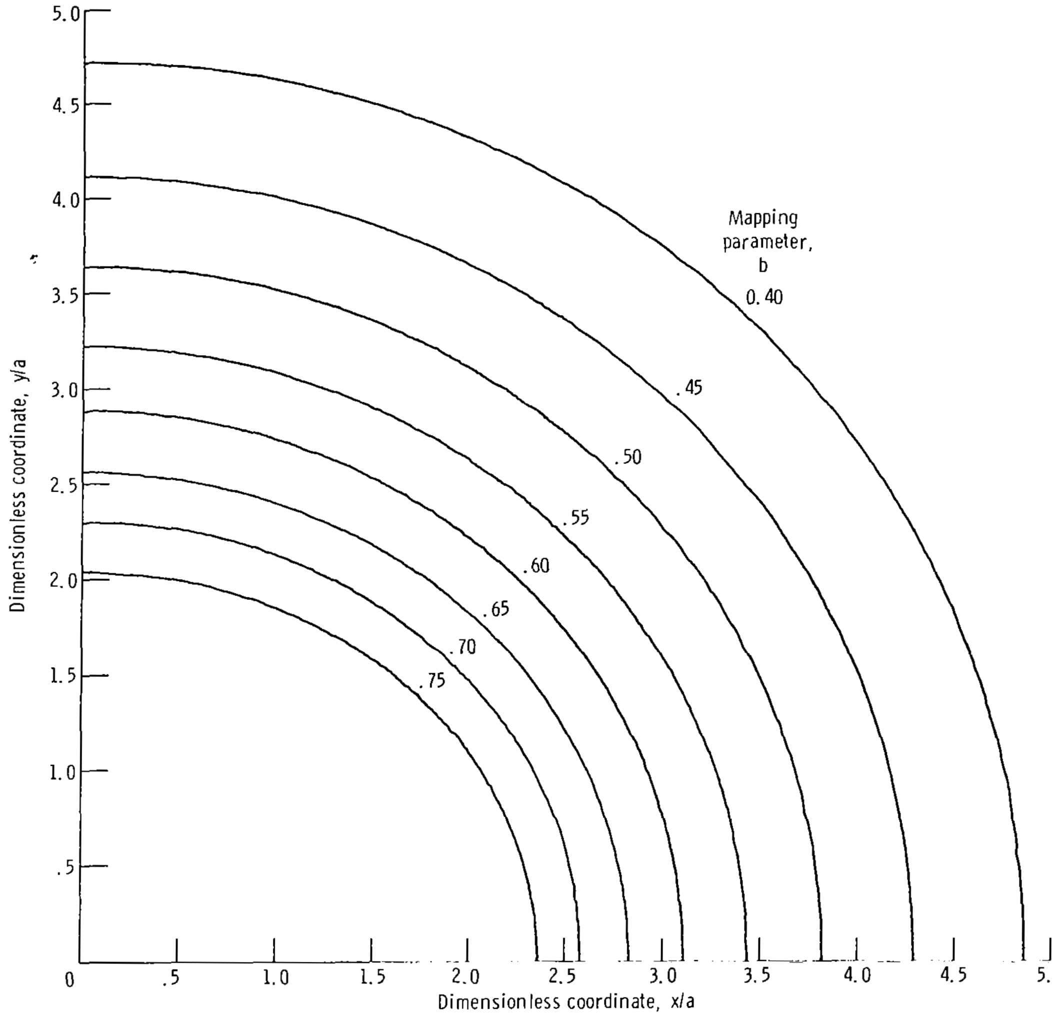


Figure 8. - Steady-state frozen layer profiles as a function of mapping parameter b (eq. (60)).



(c) Range of b, 0.40 to 0.75.

Figure 8. - Concluded.

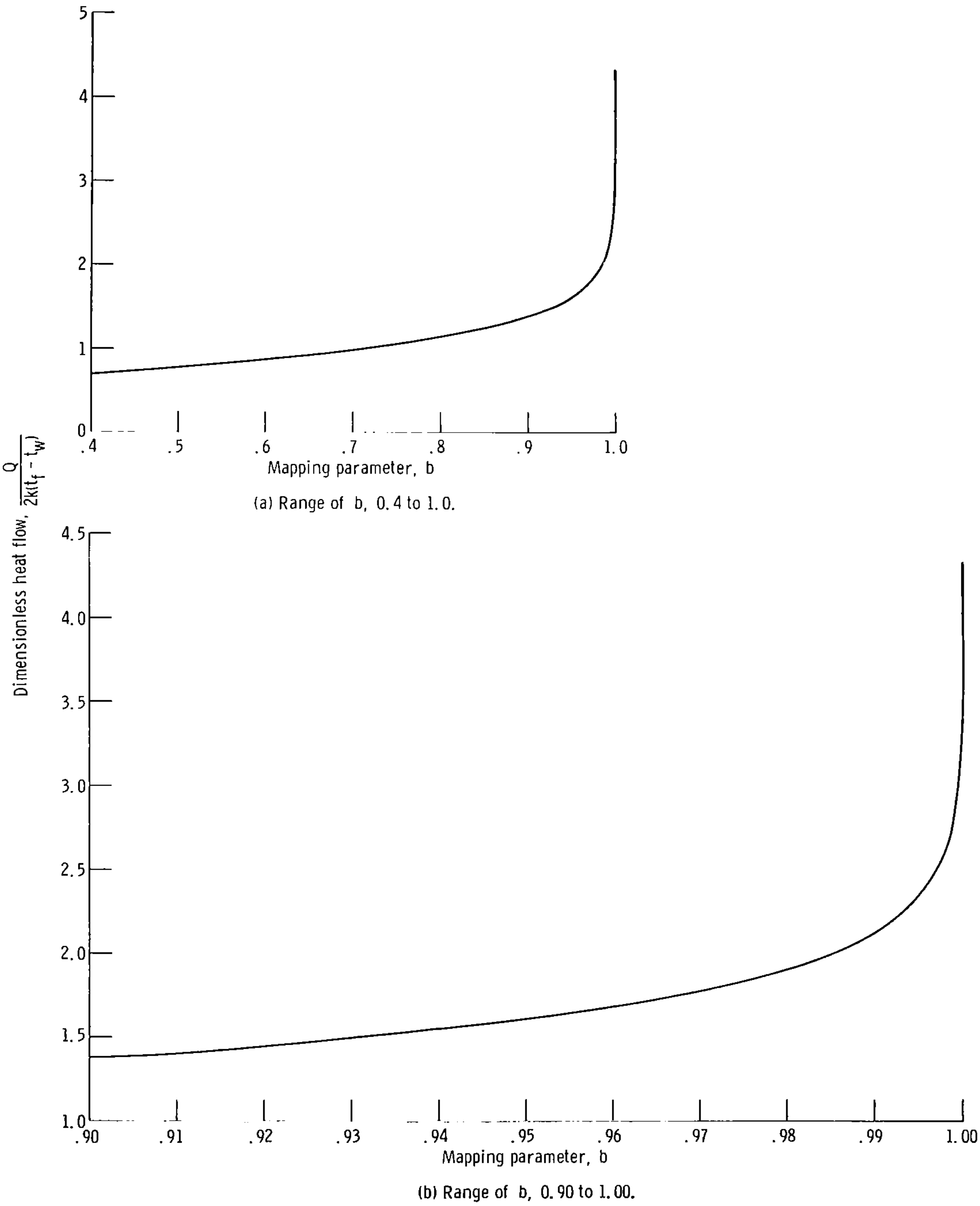
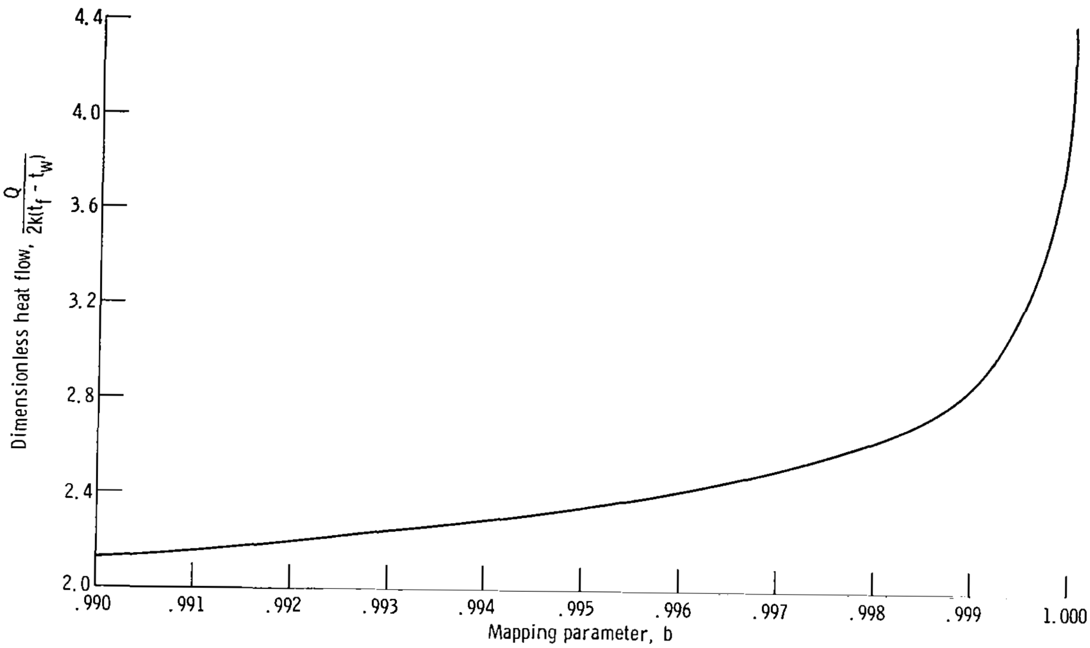


Figure 9. - Dimensionless heat flow through frozen layer as a function of mapping parameter b (eq. (65)).



(c) Range of b, 0.990 to 1.000.

Figure 9. - Concluded.

QUASI-STEADY SOLUTION

Analysis

If all the A_n are zero, then the only α_n that is not zero is α_0. In this case, the mapping function (eq. (23)) reduces to

$$\zeta = -K\left(\sqrt{1 - b^2}\right)\alpha_0 \Omega$$

On the frozen interface where $\Omega = e^{i\omega}$

$$\zeta = -K\left(\sqrt{1 - b^2}\right)\alpha_0 e^{i\omega} \qquad 0 \leq \omega \leq \pi$$

This is a semicircle which is the shape of the mapping for steady state. Thus, if all the A_n are zero and the b is allowed to vary with time, the transient solution will pass at each instant through a steady-state configuration corresponding to the instantaneous b.

This is a quasi-steady solution. What is required is to find the variation of b with time for a given set of imposed conditions. Then figures 8 and 9 can be used to obtain the instantaneous quasi-steady configuration and heat conducted through the frozen region.

The time variation of b is found from equation (57) by letting all the $A_n = 0$. This gives

$$\dot{b}\left[\sum_{k=0}^{\infty} \alpha_k\left(\frac{E}{1 - b^2} J_{0,k,p}^{(1)}\right)\right] = -\pi(1 + \delta_{p,0}) \sum_{n=0}^{\infty} \hat{\beta}_n\hat{\beta}_{n+p} + \frac{2\pi\delta_{p,0}}{K\left(\sqrt{1 - b^2}\right)}$$

Since only α_0 and $\hat{\beta}_0$ are nonzero for $A_n = 0$, their values from equations (62) and (63) are substituted to give

$$\frac{\dot{b}(-Ab)}{\ln\sqrt{1 - b^2}} \frac{E}{1 - b^2} J_{0,0,0}^{(1)} = \frac{2\pi Ab}{\ln\sqrt{1 - b^2}} + \frac{2\pi}{K\left(\sqrt{1 - b^2}\right)} \tag{66}$$

From equation (56)

$$J_{0,0,0}^{(1)} = \sum_{j=0}^{1}\binom{1}{j}(-1)^j\left[(-1)^j - (1 - b^2)\right] H_{j-1,0}^{(1)} + H_{0,0}^{(0)}$$

or

$$J_{0,0,0}^{(1)} = b^2 H_{-1,0}^{(1)} + (2 - b^2) H_{0,0}^{(1)} + H_{0,0}^{(0)}$$

The H's are found from equation (55) giving

$$J_{0,0,0}^{(1)} = b^2 2K_{-1} + (2 - b^2)2K_0 + \frac{2b^2}{\ln\sqrt{1 - b^2}} 2K_0$$

Substituting $J_{0,0,0}^{(1)}$ and E from equation (38) into equation (66) gives

$$\frac{db}{d\Theta} \quad \frac{2A^2 b}{(1 - b^2)\left(\ln\sqrt{1 - b^2}\right)^2}\left[b^2 K_{-1} + \left(2 - b^2 + \frac{2b^2}{\ln\sqrt{1 - b^2}}\right)K_0\right]$$

$$= 2\pi\left[\frac{Ab}{\ln\sqrt{1 - b^2}} + \frac{1}{K\left(\sqrt{1 - b^2}\right)}\right] \tag{67}$$

By virtue of equation (53) the K_n are only functions of b. Also K_{-1} is seen to equal K_1. Then the variables in equation (67) can be separated to give

$$\frac{A^2}{\pi} \frac{\dfrac{b}{(1 - b^2)\left(\ln\sqrt{1 - b^2}\right)^2}\left[b^2 K_1 + \left(2 - b^2 + \dfrac{2b^2}{\ln\sqrt{1 - b^2}}\right)K_0\right]}{\dfrac{Ab}{\ln\sqrt{1 - b^2}} + \dfrac{1}{K\left(\sqrt{1 - b^2}\right)}} db = d\Theta \tag{68}$$

Now note that

$$K_1 = \int_0^{\pi/2} \frac{1 - 2\sin^2\omega}{\sqrt{1 - b^2\sin^2\omega}} d\omega = \left(1 - \frac{2}{b^2}\right)K(b) + \frac{2}{b^2} E(b)$$

where

$$E(b) \equiv \int_0^{\pi/2} \sqrt{1 - b^2\sin^2\omega}\, d\omega$$

is the complete elliptic integral of the second kind. Also

$$K_0 = \int_0^{\pi/2} \frac{d\omega}{\sqrt{1 - b^2 \sin^2 \omega}} = K(b)$$

These relations are substituted into equation (68). The resulting equation is simplified and then integrated from $\Theta = 0$ to Θ to give

$$\int_{b_{initial}}^{b} \frac{k}{1 - k^2} \frac{K\left(\sqrt{1 - k^2}\right)}{\left(\ln\sqrt{1 - k^2}\right)^2} \left[\frac{E(k)\ln\sqrt{1 - k^2} + k^2 K(k)}{kA\, K\left(\sqrt{1 - k^2}\right) + \ln\sqrt{1 - k^2}}\right] dk = \frac{\pi}{2A^2} \Theta \tag{69}$$

The dummy variable k has been introduced on the left to avoid confusion with the running variable b.

The integration starts at $b_{initial}$ corresponding to any initial profile as shown in figure 8 resulting from imposed conditions A^-. At time zero the imposed conditions are changed from A^- to A. The denominator of the integrand goes to zero and hence $\Theta \rightarrow \infty$ when b is such that

$$A = -\frac{\ln\sqrt{1 - b^2}}{b\, K\left(\sqrt{1 - b^2}\right)}$$

But this is exactly the condition for the steady-state layer corresponding to A as given by equation (64). When the cold plate is initially at t_f and there is no frozen layer, then $b_{initial} = 1$.

If in equation (69) the integral is carried out from a lower limit of unity $\int_1^b$, then then the integral for any initial b can be found by subtraction, that is, $\int_{b_{initial}}^{b} = \int_1^b - \int_1^{b_{initial}}$. Hence by having integrated from 1 to arbitrary values of b, the quasi-steady time can be found for a layer to grow from any initial state to another state. The growth rate will of course be a function of A.

Results from Quasi-Steady Solution

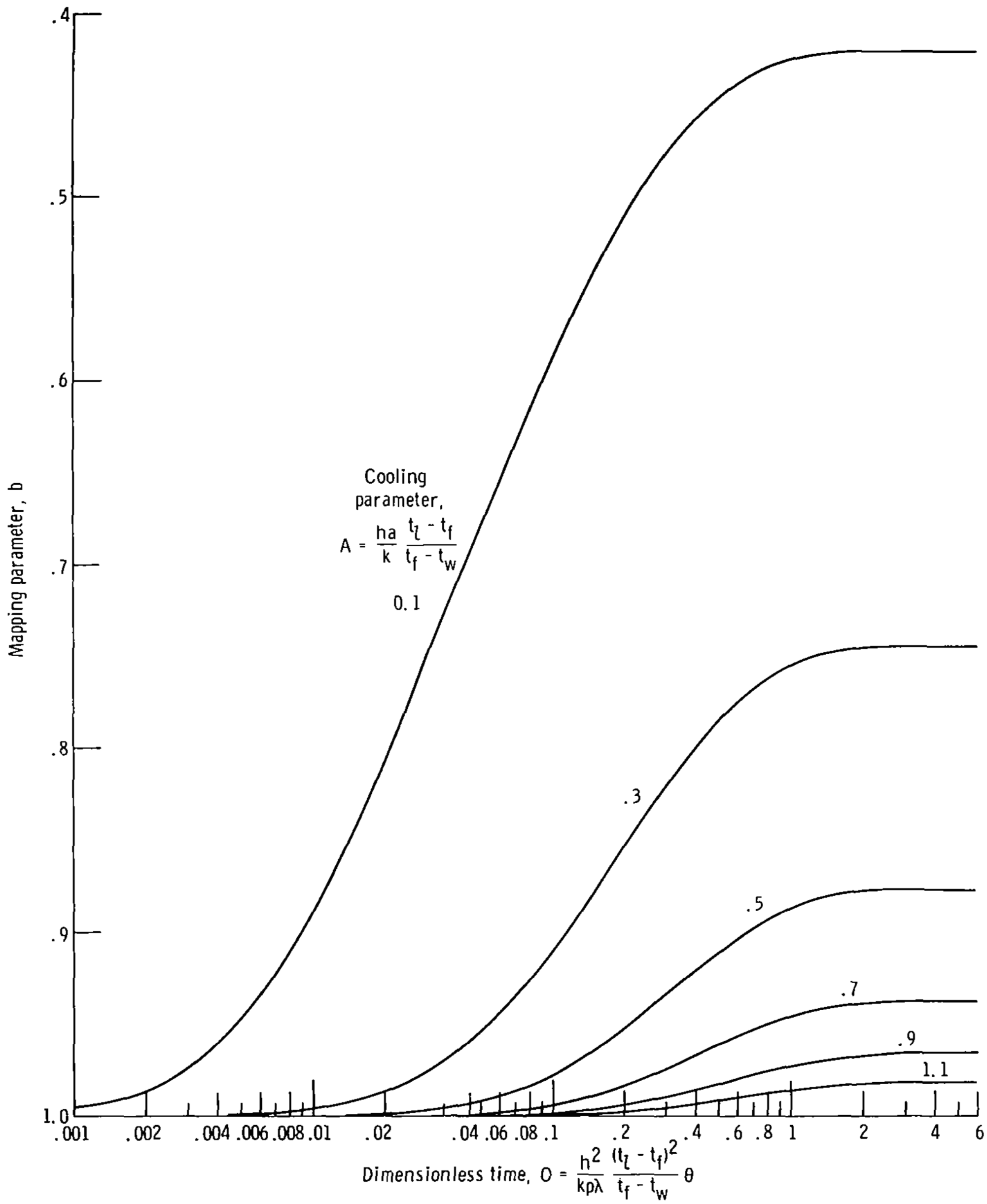
Figure 10 shows the variation of b as a function of dimensionless time for the quasi-steady solution. Curves are given for various values of A that are imposed at the beginning of the transient and maintained throughout the transient growth. As time proceeds, the b reaches a constant value corresponding to the steady-state frozen profile for the particular value of A that is imposed.

The sets of curves in figures 7 to 10 can be used as follows to obtain the quasi-steady solution. First consider the case where initially the wall is at the freezing temperature $t_w = t_f$. There will be no frozen layer on the plate and $A^- = \infty$ since $t_f - t_w$ is in the denominator of A. When $A^- \rightarrow \infty$, figure 7 shows that $b \rightarrow 1$. This is consistent with figure 8 which shows that for $b = 1$ there is no frozen layer on the plate. To initiate the transient, at time $\Theta = 0^+$ the physical conditions are changed so that

$$\frac{ha}{k} \frac{t_l - t_f}{t_f - t_w}$$

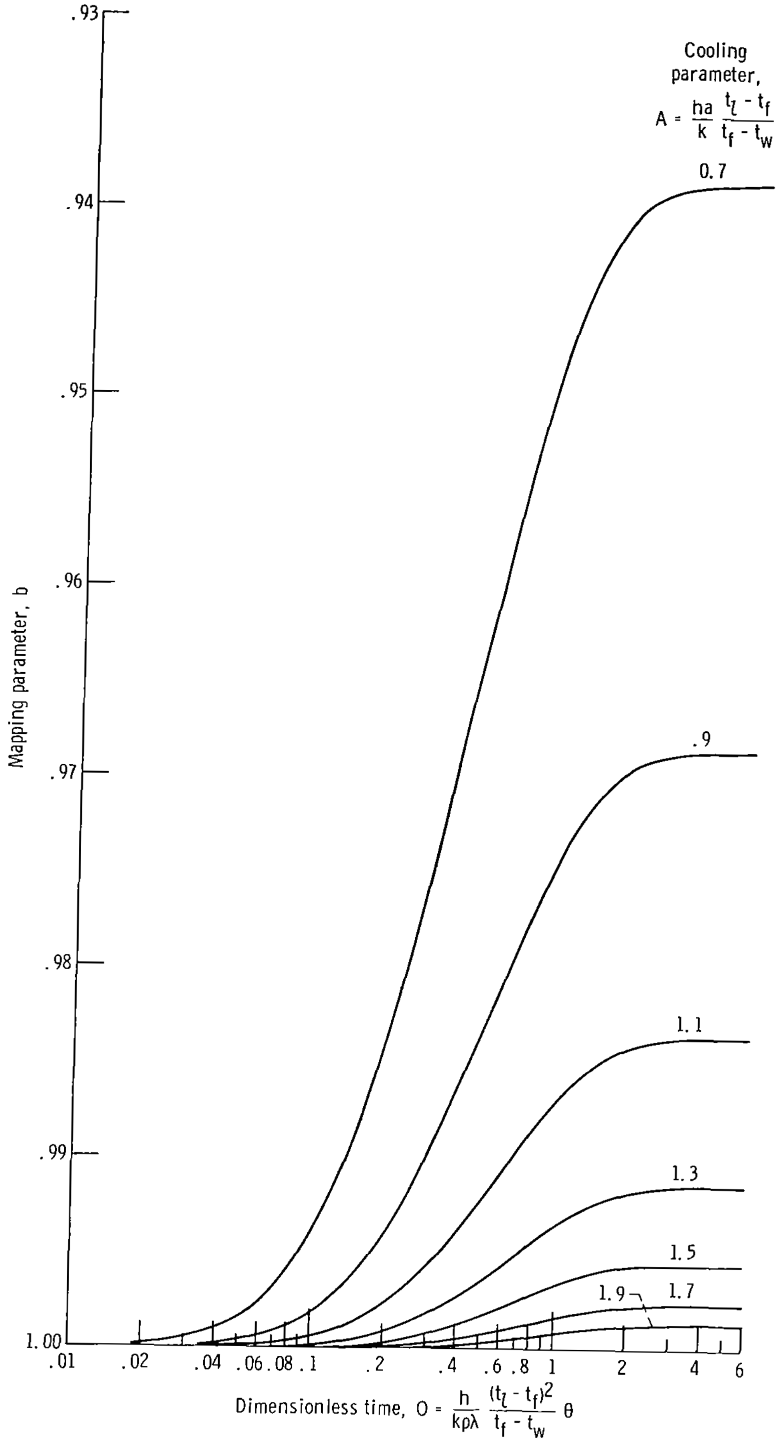
goes from $A^- = \infty$ to the constant value of A that is maintained throughout the transient. For that particular A the variation of b with Θ is found from figure 10. Then the time dependent profiles and heat flow during the transient are found from figures 8 and 9.

Now consider the case where A^- is finite, that is, there is initially a steady-state layer on the plate. Corresponding to the value A^- there is a $b_{initial}$ that can be found from figure 7, and an initial frozen profile from figure 8. To start the transient, a new value of A is imposed and the transient variation of b will follow the curve in figure 10 corresponding to the imposed A. To find the value of b after a given time lapse, the abscissa in figure 10 must be interpreted as being the actual Θ that has passed since the initiation of the transient plus the value of Θ corresponding to $b_{initial}$ as obtained on the curve for A in figure 10. With the b known as a function of time from the onset of the transient, the heat flow and profile shape can be found as a function of time from figures 9 and 8.



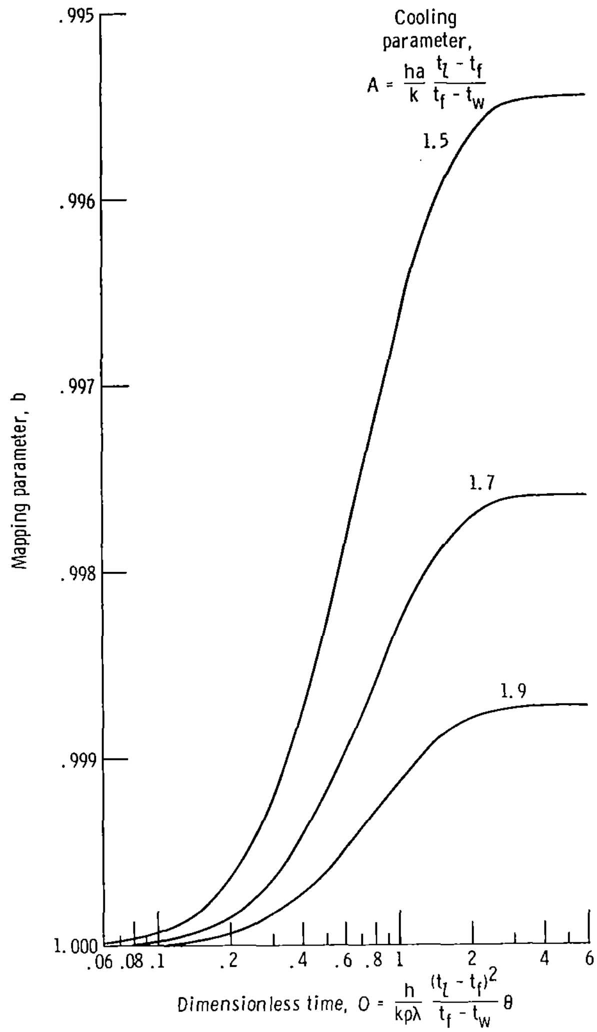
(a) Range of A, 0.1 to 1.1.

Figure 10. - Variation of mapping parameter b with dimensionless time for various A; quasi-steady solution (eq. (69)); $b_{initial} = 1.0$.



(b) Range of A, 0.7 to 1.9.

Figure 10. - Continued.



(c) Range of A, 1.5 to 1.9.

Figure 10. - Concluded.

COMPUTATION OF TRANSIENT SOLIDIFICATION

Having examined the steady and quasi-steady solutions, the general results will now be considered. The final result of the transient analysis is given by equation (57). This represents a set of equations, each equation corresponding to a different value of p. The mapping parameter b and the coefficients A_n are unknown functions of time which must be found to evaluate the profiles given by equations (58). Since the set of equations (57) involve the time derivatives $\dot{b}$ and $\dot{A}_n$ (n = 0, 1, 2, . . .), these are simultaneous first-order ordinary differential equations. Equations (58) for the X_s and Y_s coordinates of the interface each contain an infinite series. When the A_n are evaluated, they decrease in size as n is increased so that the series can be terminated after a finite number of terms.

To find the $b(\Theta)$ and $A_n(\Theta)$, the set of equations (57) is truncated at a value of p equal to 1 plus the number of A_n desired. For example, if only b and $A_0, \ldots, A_3$ are retained, the only equations of the set which are retained are those for which $p = 0, 1, \ldots, 4$. The series $\sum_{n=0}^{\infty}$ would extend only to $n = 3$. This set of five equations is solved using the Runge-Kutta method for simultaneous first-order differential equations. Since the series in A_n has been truncated, the values of b and A_n may not be accurate. A larger number of terms is then included, say $A_0, \ldots, A_8$ and $p = 0, 1, \ldots, 9$. The first four A_n are then checked against the A_n found by using only five simultaneous equations. This procedure is continued until increasing the number of equations does not change the values of $b(\Theta)$ and $A_n(\Theta)$. Also there must be a large enough number of A_n so that the series in equations (58) have converged.

The numerical computer program that was utilized could solve 20 simultaneous differential equations, so the present results are limited to cases that converged with 19 or less A_n. It was found that a larger number of A_n would be required when the transient solidification started from a thin initial layer. It will be shown subsequently, however, that the quasi-steady solution works very well for thin initial layers so that the present theory does provide a means for computing transients starting from an initial condition of any steady-state frozen region or a bare plate.

Now consider in more detail how equation (57) is written out for each p value. All of the quantities appearing in these equations such as α, E, and J must be replaced by their appropriate expressions in terms of b and A_n. The expression for E is given by equation (38), and the expressions for the α_n by equation (37). With the α_n known, equation (49) is used to find each of the $\hat{\beta}_n$. Each $\hat{\beta}_n$ can be found successively by writing out equation (49) for successive values of n starting with $n = 0$. For example with $n = 0$, $\hat{\beta}_0^2 = \alpha_0$; then for $n = 1$, $\hat{\beta}_0\hat{\beta}_1 + \hat{\beta}_1\hat{\beta}_0 = \alpha_1$ so that having found $\hat{\beta}_0$ the $\hat{\beta}_1$ can be found. Continuing, letting $n = 2$ gives an equation that can be solved for $\hat{\beta}_2$

$$\hat{\beta}_0\hat{\beta}_2 + \hat{\beta}_1\hat{\beta}_1 + \hat{\beta}_2\hat{\beta}_0 = \alpha_2$$

The $J_{n,k,p}^{(r)}$ given by equation (56) are only functions of b. They depend on the H functions that are defined in equations (55). These depend on the K_n which are the definite integrals defined in equation (53). It was necessary to develop a table of K_n values for various n and b by carrying out the definite integrals numerically.

To obtain a transient solidification solution, initial conditions must be specified. The transients computed here are all started from an initial steady-state layer. The magnitude of the physical quantity

$$\frac{ha}{k}\frac{t_l - t_f}{t_f - t_w}$$

that establishes this initial layer is called A^-. From figure 7 the value of A^- corresponds to a value of b which will be termed $b_{initial}$. The initial shape of the solidified region corresponding to this value of $b_{initial}$ can be found from figure 8. At time 0^+ a new physical condition specified by the value of A, is imposed and this remains fixed throughout the transient. For an initial steady-state layer, all the $A_n = 0$ at $\Theta = 0^+$. Hence the solution of equation (57) proceeds from the following initial conditions:

$$b = b_{initial} \text{ (corresponding to initial steady profile)}$$

$$A_n = 0$$

$$\frac{ha}{k}\frac{t_l - t_f}{t_f - t_w} = \begin{cases} A^- & \Theta < 0 \\ A & \Theta > 0 \end{cases}$$

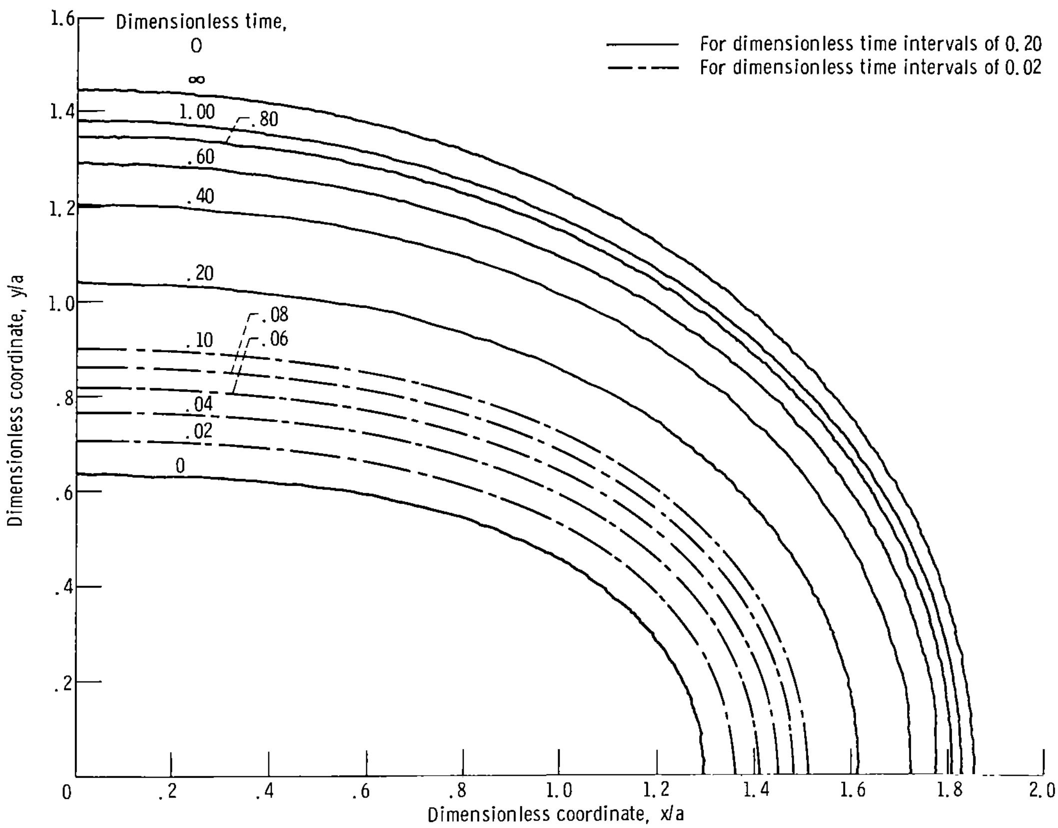
As time becomes large, the solidified layer approaches a steady state. The steady-state value of b is completely determined by the value of A from the curves of figure 7, and the steady-state profile can be found at this value of b by use of figure 8.

When the variations with time of b and the A_n are known, equations (58) are used to compute the transient shapes of the ice layer. The transient heat flow through the layer is found from equation (59).

RESULTS AND DISCUSSION OF TRANSIENT SOLUTION

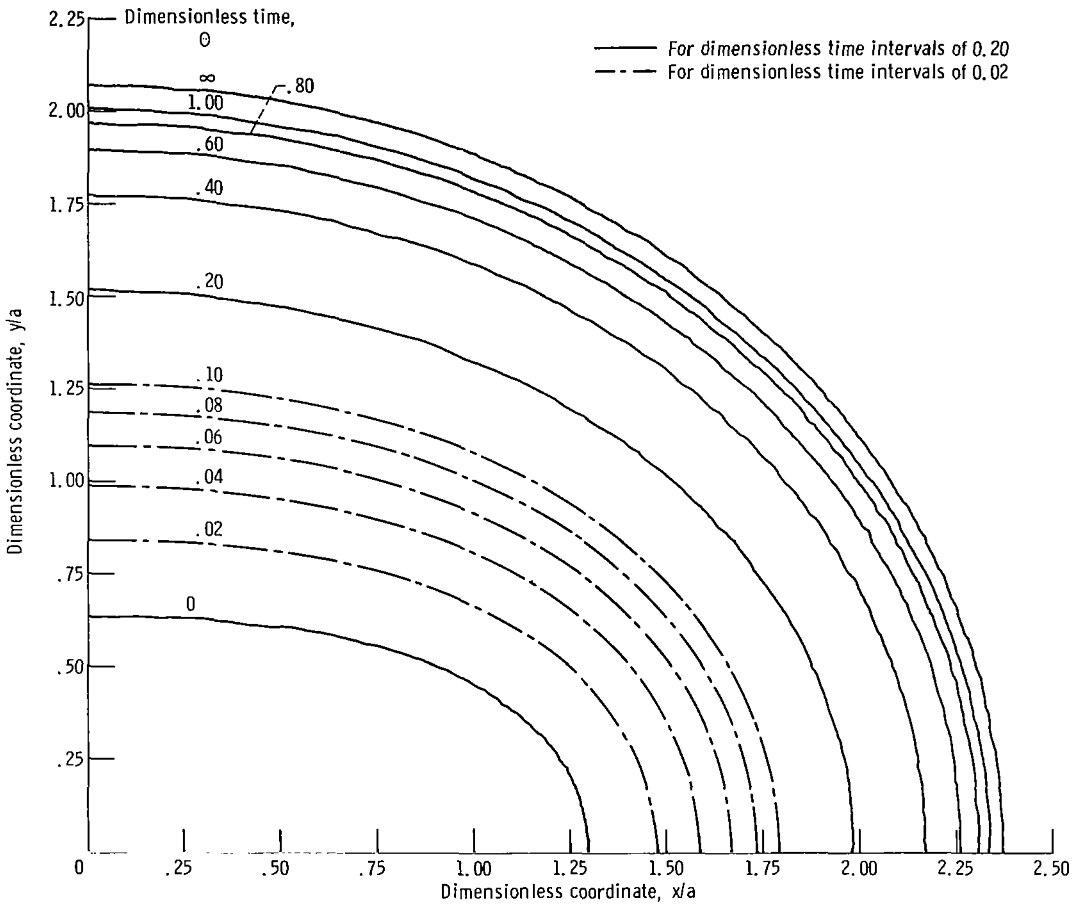
Solidification Computed from Transient Solution

Several groups of transient growth curves are shown in figures 11 and 12 to demonstrate the nature of the transient solution. In figure 11, before the transients begin there is an initial steady-state layer that has been established by having the cooling parameter equal to $A^- = 1.47$ (corresponding to the initial condition $b_{initial} = 0.995$). Then the cooling parameter is suddenly changed to another value, this being $A = 0.5$ for figure 11(a). The change in A could physically correspond to changing the liquid heat transfer coefficient, the liquid temperature, or the temperature of the cooled plate. The frozen layer then grows and the profiles are shown for various values of the dimensionless time until a new steady state is achieved. On figures 11(b) and (c) the values of A



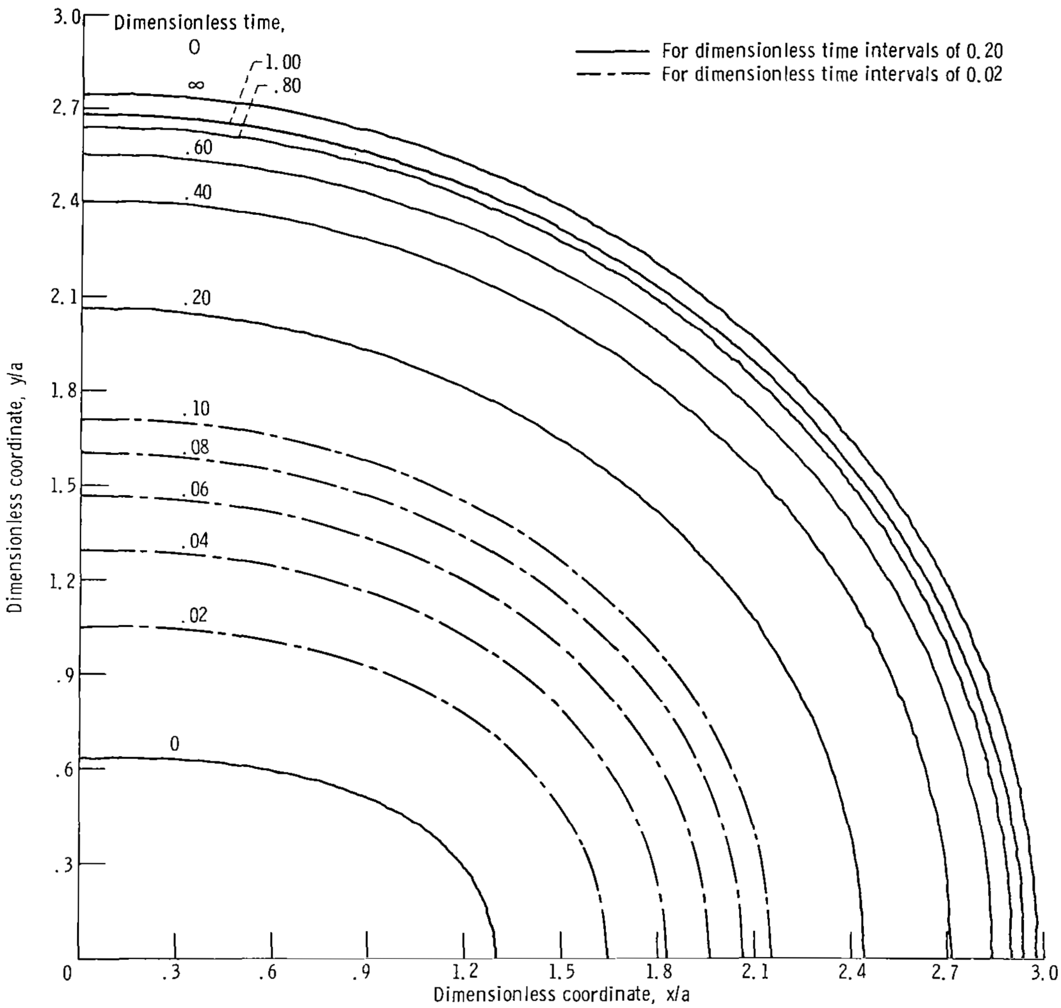
(a) A = 0.5.

Figure 11. - Effect of different cooling rates on solidification starting from same initial layer; $b_{initial} = 0.995$; $A^- = 1.47$.



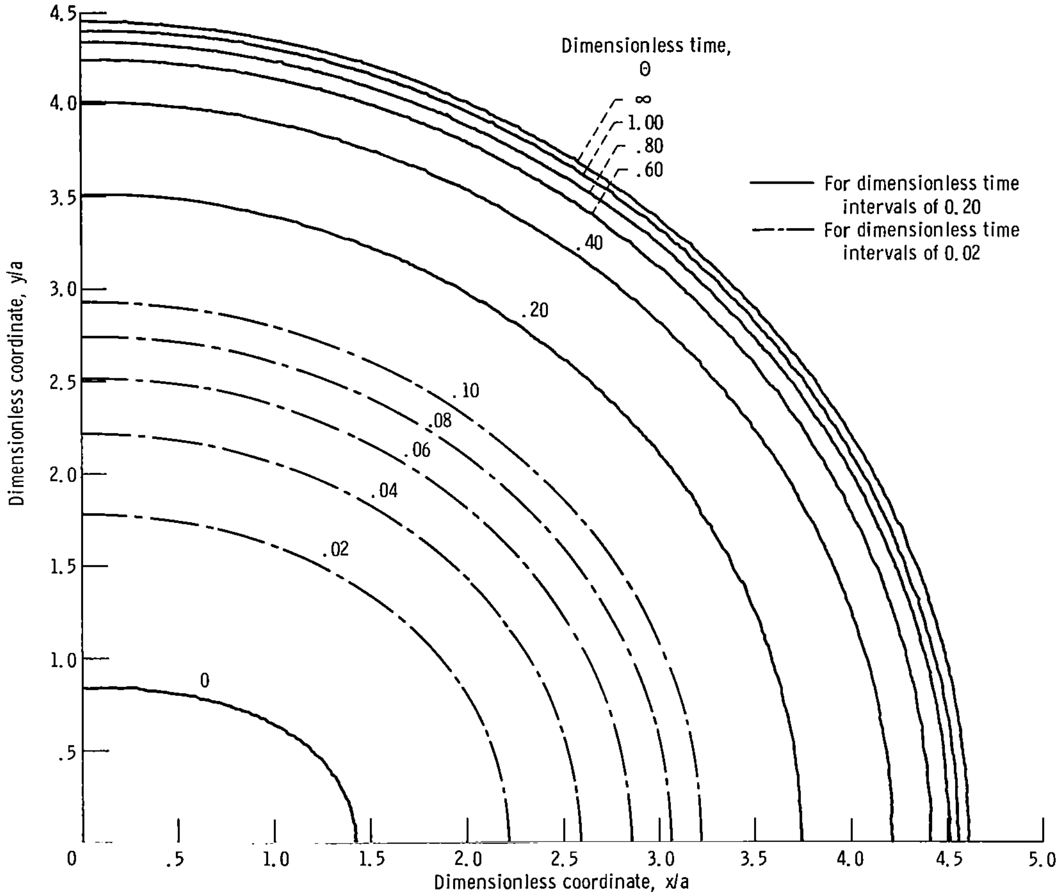
(b) A = 0.3.

Figure 11. - Continued.



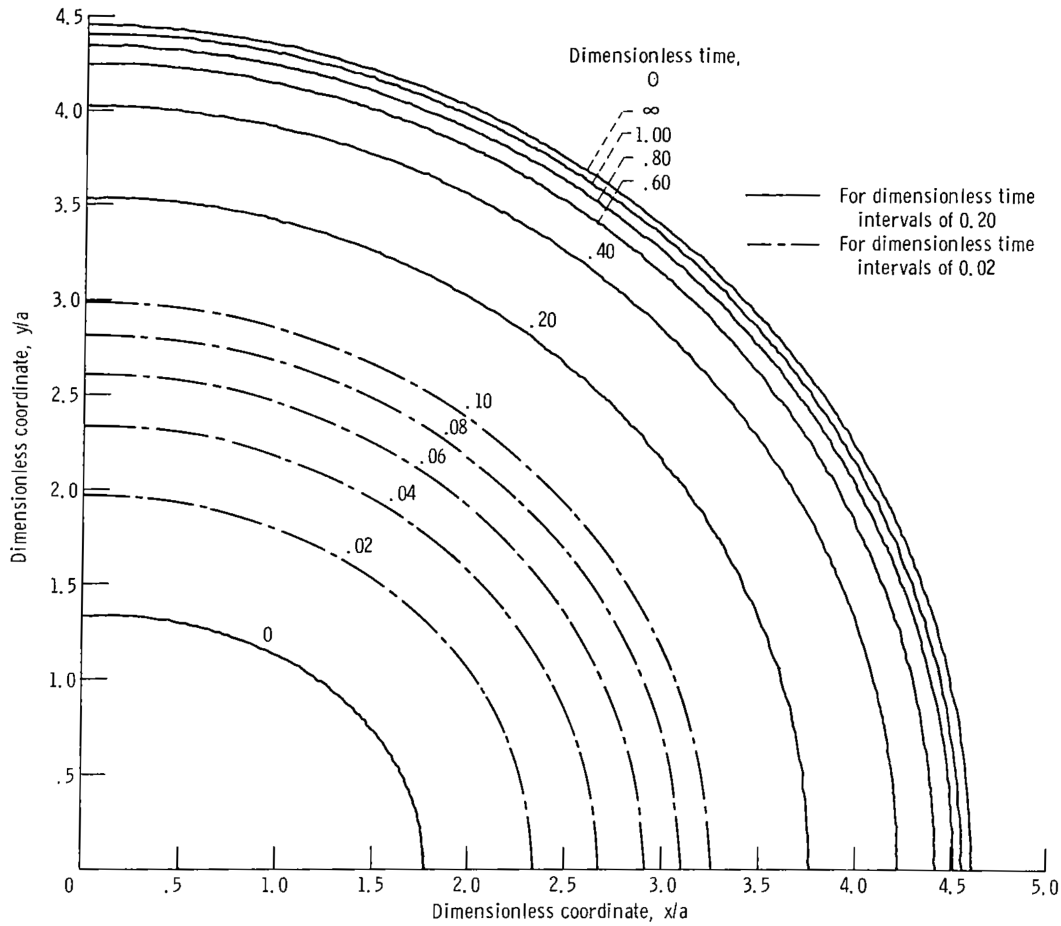
(c) A = 0.2.

Figure 11. - Concluded.



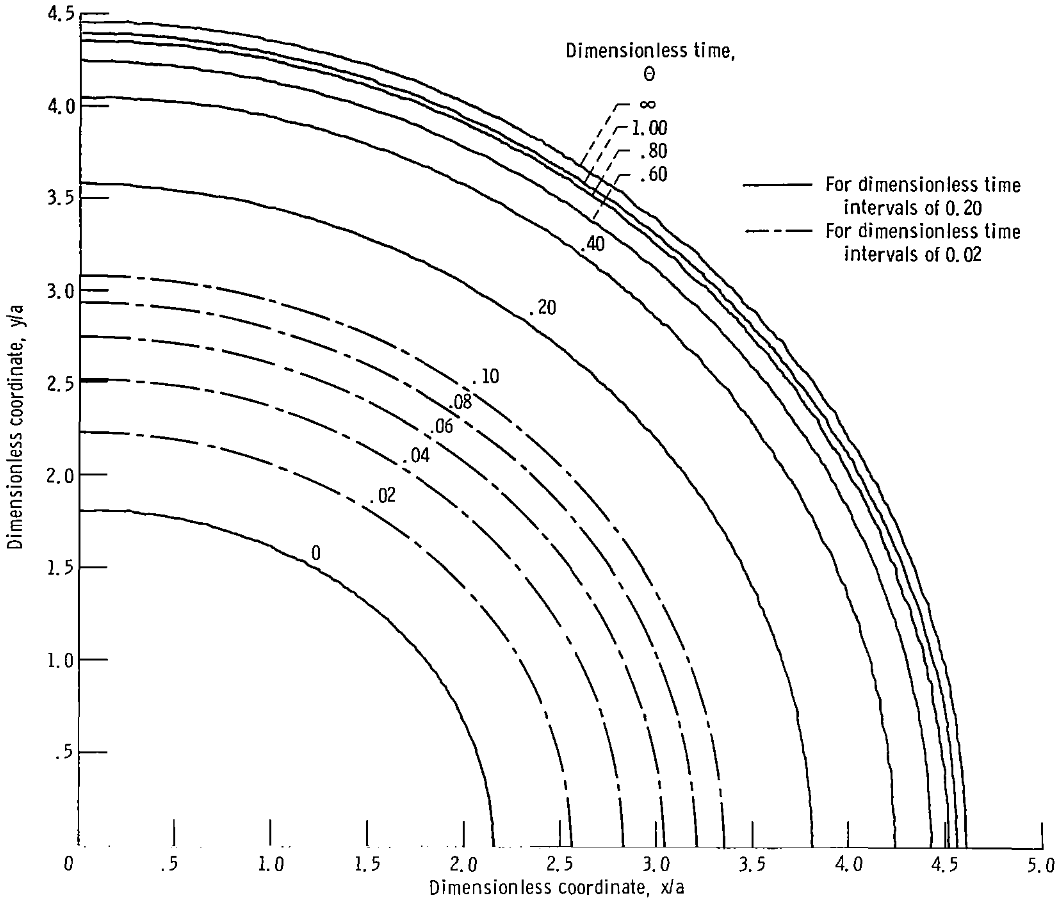
(a) $b_{initial} = 0.98$; $A^{-} = 1.04$.

Figure 12. - Transient solidification starting from three different initial layers and going to same final layer; A = 0.1.



(b) $b_{initial} = 0.9$; $A^{-} = 0.560$.

Figure 12. - Continued.



(c) $b_{initial} = 0.8$; $A^- = 0.366$.

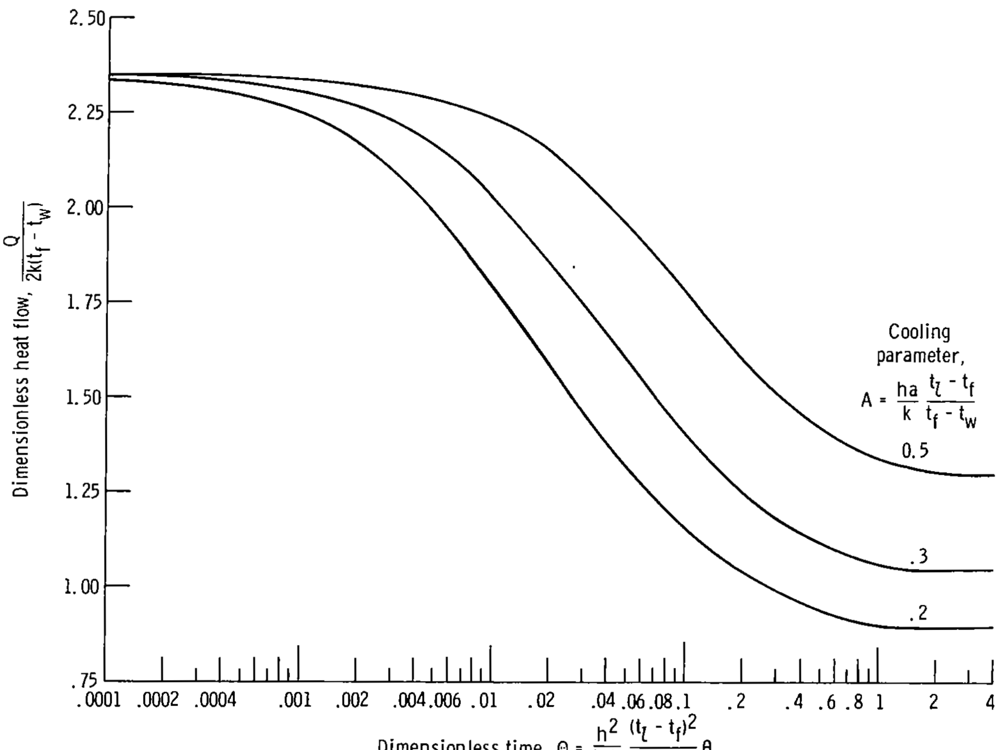
Figure 12. - Concluded.

are respectively 0.3 and 0.2. A smaller value of A corresponds to increased cooling (increased $t_f - t_w$) or a decreased convection of energy $h(t_l - t_f)$ to the frozen interface. Consequently, for smaller A the steady-state layer (at $\Theta \rightarrow \infty$) is larger.

In figure 12 the final cooling parameter A is equal to 0.1 for all three cases so that the steady-state layers are all the same. The transients begin, however, from various initial layers corresponding to cooling parameters prior to the transient A^- of 1.04, 0.560, and 0.366.

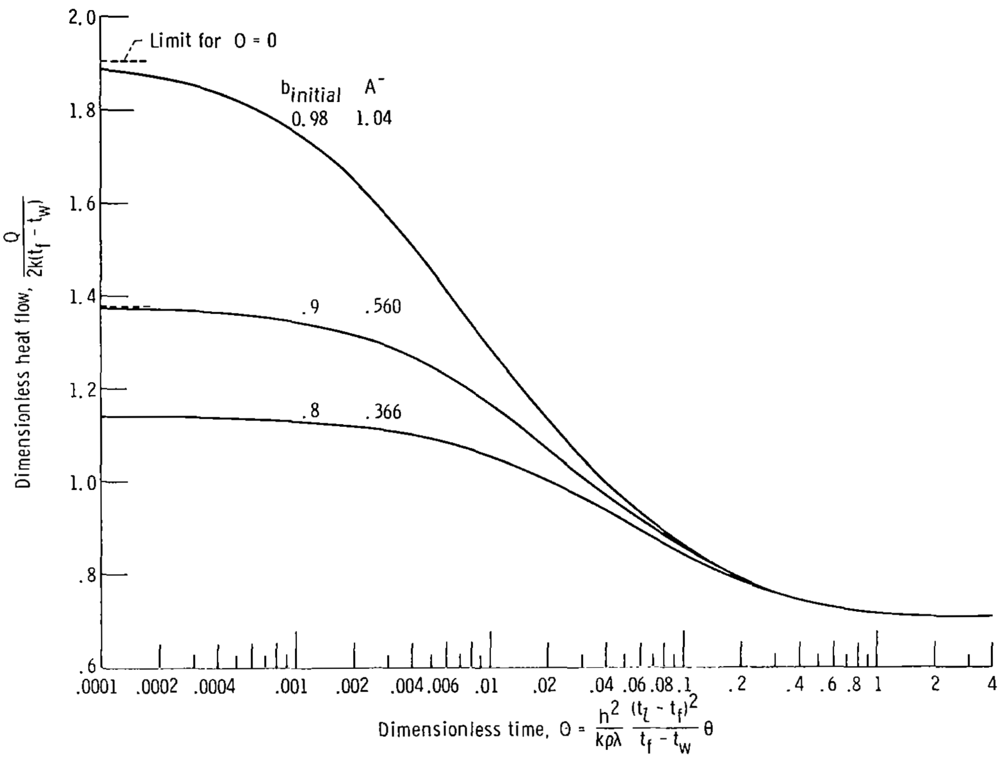
The results in figures 11 and 12 give the reader a quantitative idea of the rate at which the solidification occurs. The rate is of course most rapid at early times when the frozen region has the least thickness and hence the least resistance to heat flowing through it. As the frozen region becomes large compared with the width of the cooled plate (x/a = 1) the frozen shape becomes circular, tending toward the axisymmetric solution where the heat removal would be at a line sink at the center of the solidified region.

Figure 13 shows the heat flow through the solidified layer and into the wall for the six transients in figures 11 and 12. In figure 13(a) all the initial layers are the same and, hence, $Q/2k(t_f - t_w)$ starts from the same value. A small value of A corresponds to a large final solidified region and, consequently, a low steady-state heat flow. In figure 13(b), corresponding to the transients in figure 12, the A is the same for all cases and, hence, the same steady-state heat flow is reached. A large A^- corresponds to a thin initial layer and consequently a large heat flow at the beginning of the transient.



(a) Various cooling rates starting with same initial layer; $b_{initial} = 0.995$.

Figure 13. - Transient heat flow through frozen layer.



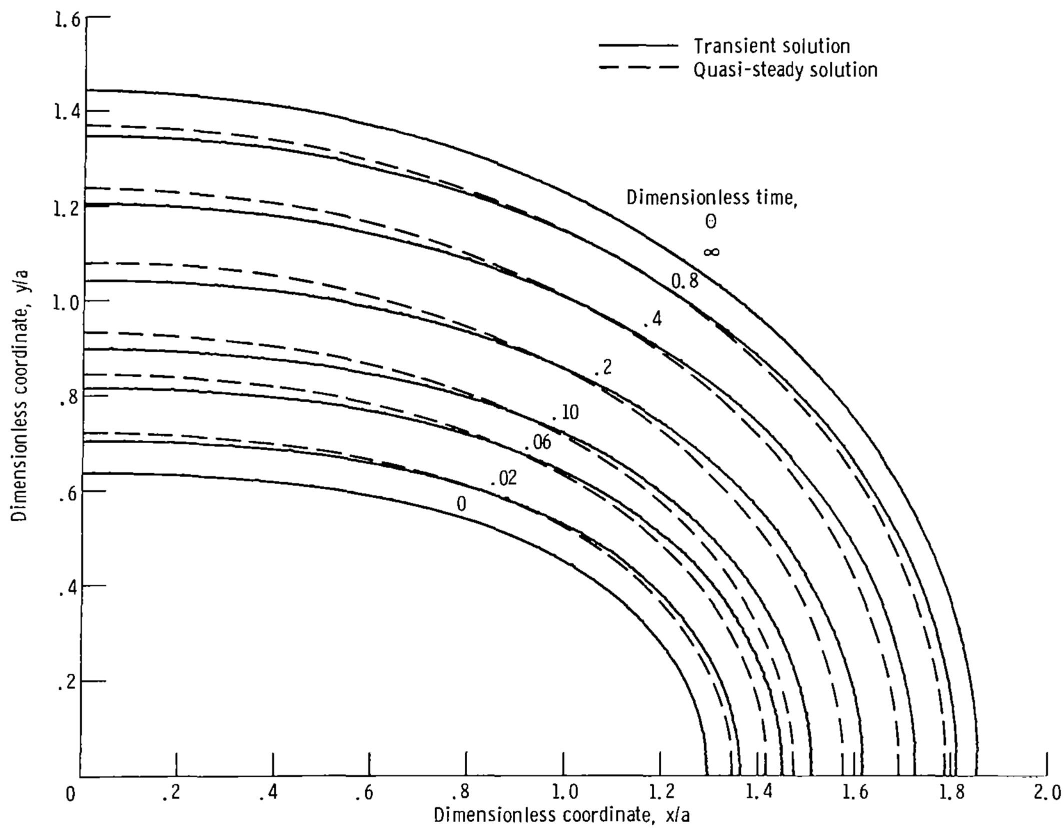
(b) Various initial layers with same cooling rate; A = 0.1.

Figure 13. - Concluded.

Comparison With Quasi-Steady Solution

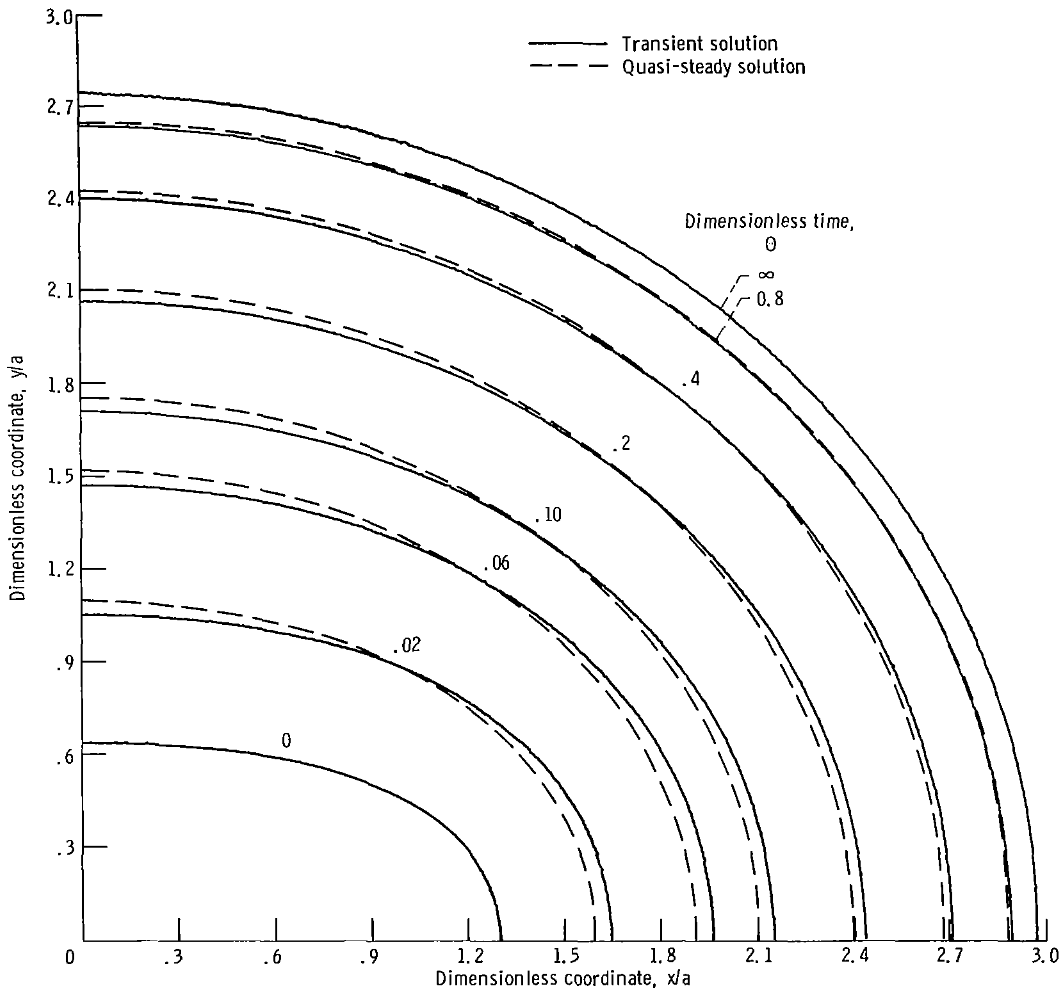
Figure 14 shows three different transient solutions and compares the layer shapes with those computed from the quasi-steady solution. These cases had the largest deviations that were obtained between the transient and quasi-steady solutions. It is evident that the transient profiles are always quite close to those predicted by the quasi-steady analysis.

As mentioned previously, the series in the transient solution converged less rapidly for thin initial layers, and hence transient solutions were not carried out for thinner initial layers than those shown in figure 11. However, thin layers as shown by figure 8 would be almost one-dimensional except for the curved portion of the interface near the edge of the plate $x/a \approx 1$. Hence, when starting a transient from a thin layer, the transient profiles must pass through a series of instantaneous steady states as both the transient and quasi-steady layers are very flat. Consequently, the quasi-steady solution



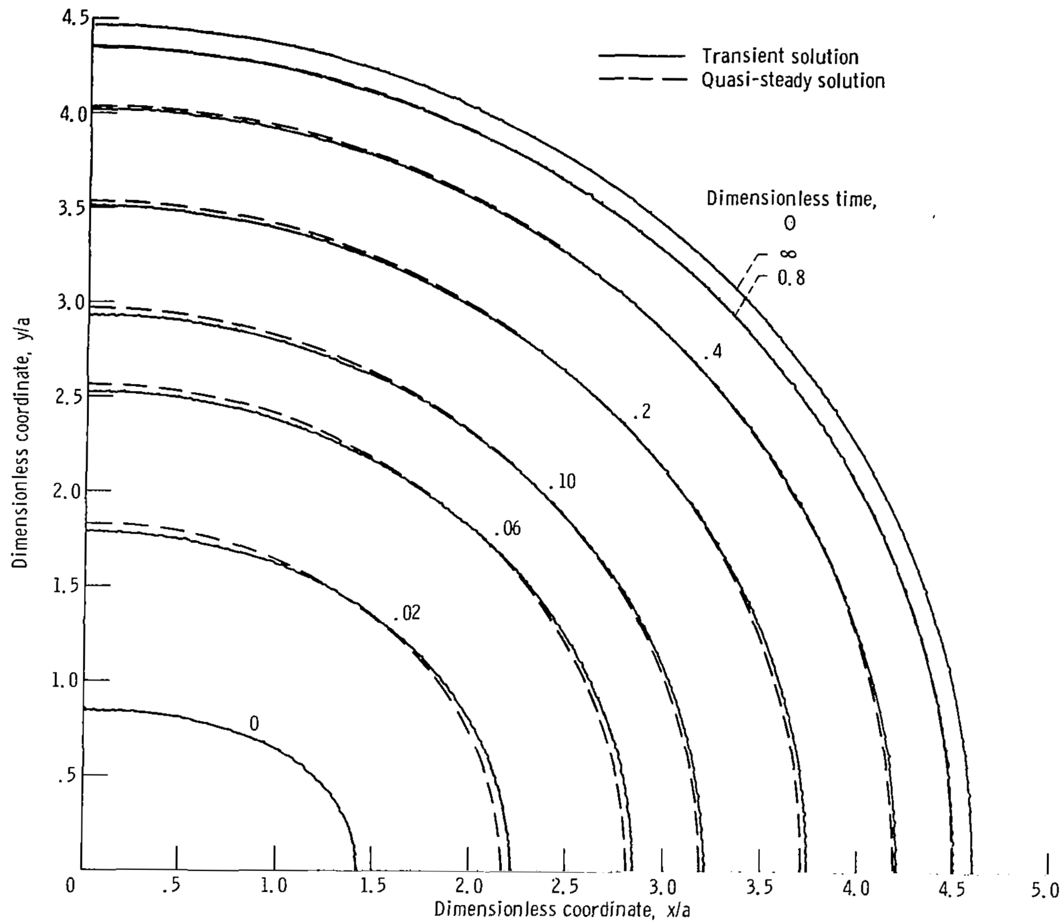
(a) Thin initial layer; $b_{initial} = 0.995$; small cooling; $A = 0.5$.

Figure 14. - Comparison of transient and quasi-steady solutions.



(b) Thin initial layer; $b_{initial} = 0.995$; moderate cooling; A = 0.2.

Figure 14. - Continued.



(c) Thin initial layer; $b_{initial} = 0.98$; large cooling; A = 0.1.

Figure 14. - Concluded.

should be a good engineering approximation for all cases. Thus for engineering calculations it is a simple procedure to obtain transient results by use of figures 7 to 10 as described in the section entitled QUASI-STEADY SOLUTION.

CONCLUSIONS

A conformal mapping method was developed and applied to a particular case of transient two-dimensional solidification. The configuration in the physical plane is obtained by a quadrature that involves the properties of the solidified region in a potential plane and in a temperature derivative plane. To perform the integration, the configurations in these planes are both mapped conformally into a fixed region of an intermediate plane. Since the regions in the potential and derivative planes depend on time, the mapping functions also contain time varying quantities. These quantities are found by satisfying the physical conditions at the moving interface.

A quasi-steady solution was also carried out. In this solution the profiles pass through a series of instantaneous steady-state shapes. The transient results from the general analysis were found to be predicted within engineering accuracy by the quasi-steady solution. The quasi-steady solution is provided in graphical form and can be used to quickly predict the solidification behavior.

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129-01.

APPENDIX - COMPUTATION OF THE INTEGRAL FOR $Z(\Omega, \Theta)$

To obtain the expression for $Z(\Omega, \Theta)$ given by equation (25), it is necessary to evaluate

$$Z - A = \sum_{n=0}^{\infty} \alpha_n \int_0^{\Omega^2} \frac{\gamma^n \, d\gamma}{\sqrt{(1+\gamma)^2 - (1-b^2)(1-\gamma)^2}} \tag{A1}$$

To this end, examine the form of the integral

$$\int \frac{\gamma^n \, d\gamma}{\sqrt{X}} \tag{A2}$$

where

$$\sqrt{X} = \sqrt{(1+\gamma)^2 - (1-b^2)(1-\gamma)^2}$$

for a few values of n. Any standard integral table shows that the results of these integrations are as follows:

$$\frac{1}{b} \ln\left(2 - b^2 + b^2\gamma + b\sqrt{X}\right) + \text{const} \qquad \text{for } n = 0 \tag{A3}$$

$$\frac{\sqrt{X}}{b^2} - \frac{4 - 2b^2}{2b^3} \ln\left(2 - b^2 + b^2\gamma + b\sqrt{X}\right) + \text{const} \qquad \text{for } n = 1 \tag{A4}$$

$$\left[\frac{2b^2\gamma - 6(2 - b^2)}{4b^4}\right]\sqrt{X} + \frac{3(4 - 2b^2)^2 - 4b^4}{8b^5} \ln\left(2 - b^2 + b^2\gamma + b\sqrt{X}\right) + \text{const} \qquad \text{for } n = 2 \tag{A5}$$

These integrations indicate that the n^{th} integral can be expressed in the form

$$\int \frac{\gamma^n}{\sqrt{X}}\, d\gamma = \sqrt{X} \sum_{r=0}^{n-1} \beta_r^n \gamma^r + C_n \ln\left(2 - b^2 + b^2\gamma + b\sqrt{X}\right) + \text{const} \qquad r \geq 0$$

(A6)

where the β_r^n and C_n are coefficients that will now be found. Differentiate equation (A6) with respect to γ (note that X contains γ)

$$\frac{\gamma^n}{\sqrt{X}} = \sqrt{X} \sum_{r=1}^{n-1} r\beta_r^n \gamma^{r-1} + \frac{2b^2\gamma + 4 - 2b^2}{2\sqrt{X}} \sum_{r=0}^{n-1} \beta_r^n \gamma^r$$

$$+ \frac{C_n}{2 - b^2 + b^2\gamma + b\sqrt{X}} \left[b^2 + \frac{b}{2\sqrt{X}} (2b^2\gamma + 4 - 2b^2)\right]$$

Multiply through by $\sqrt{X}$ and simplify the last term to obtain

$$\gamma^n = X \sum_{r=1}^{n-1} r\beta_r^n \gamma^{r-1} + (b^2\gamma + 2 - b^2) \sum_{r=0}^{n-1} \beta_r^n \gamma^r + C_n b$$

Substitute for $X = (1 + \gamma)^2 - (1 - b^2)(1 - \gamma)^2 = b^2\gamma^2 + (4 - 2b^2)\gamma + b^2$ to give

$$\gamma^n = \left[b^2\gamma^2 + (4 - 2b^2)\gamma + b^2\right] \sum_{r=1}^{n-1} r\beta_r^n \gamma^{r-1} + (b^2\gamma + 2 - b^2) \sum_{r=0}^{n-1} \beta_r^n \gamma^r + C_n b$$

$$= b^2 \sum_{r=0}^{n-1} (1 + r)\beta_r^n \gamma^{r+1} + (2 - b^2) \sum_{r=0}^{n-1} (1 + 2r)\beta_r^n \gamma^r + b^2 \sum_{r=0}^{n-1} r\beta_r^n \gamma^{r-1} + C_n b$$

(A7)

Terms in the same power of γ will be equated to determine the β_r^n and the C_n. To this end equation (A7) is rearranged further to obtain

$$\gamma^n = b^2 \sum_{r=1}^{n} r\beta_{r-1}^n \gamma^r + (2 - b^2)\beta_0^n + (2 - b^2) \sum_{r=1}^{n-1} (1 + 2r)\beta_r^n \gamma^r + b^2 \sum_{r=1}^{n-1} r\beta_r^n \gamma^{r-1} + C_n b$$

which can be put in the form

$$\gamma^n = (2 - b^2)\beta_0^n + b^2 n\beta_{n-1}^n\gamma^n + b^2(n - 1)\beta_{n-2}^n\gamma^{n-1} + (2 - b^2)[1 + 2(n - 1)]\beta_{n-1}^n\gamma^{n-1}$$

$$+ \sum_{r=1}^{n-2}\left[b^2 r\beta_{r-1}^n + (2 - b^2)(1 + 2r)\beta_r^n\right]\gamma^r + b^2\sum_{r=0}^{n-2}(r + 1)\beta_{r+1}^n\gamma^r + C_n b$$

$$\gamma^n = b^2 n\beta_{n-1}^n\gamma^n + \left[b^2(n - 1)\beta_{n-2}^n + (2 - b^2)(2n - 1)\beta_{n-1}^n\right]\gamma^{n-1}$$

$$+ \sum_{r=1}^{n-2}\left[b^2 r\beta_{r-1}^n + (2 - b^2)(1 + 2r)\beta_r^n + b^2(r + 1)\beta_{r+1}^n\right]\gamma^r + (2 - b^2)\beta_0^n + b^2\beta_1^n + C_n b \tag{A8}$$

Equating the terms in γ^0 gives the C_n for $n > 0$

$$C_n = -\frac{(2 - b^2)\beta_0^n + b^2\beta_1^n}{b} \qquad n = 1, 2, 3, \ldots$$

Comparing equations (A6) and (A3) gives the C_0 as

$$C_0 = \frac{1}{b}$$

Equating the coefficients of γ^n in equation (A8) gives

$$\beta_{n-1}^n = \frac{1}{b^2 n}$$

From the $(n - 1)^{th}$ order terms

$$b^2(n - 1)\beta_{n-2}^n + (2 - b^2)(2n - 1)\beta_{n-1}^n = 0$$

which gives

$$\beta^n_{n-2} = -\frac{2 - b^2}{b^4}\,\frac{(2n - 1)}{n(n - 1)}$$

For $1 \leq r \leq n - 2$

$$b^2 r\beta^n_{r-1} + (2 - b^2)(1 + 2r)\beta^n_r + b^2(r + 1)\beta^n_{r+1} = 0$$

which gives

$$\beta^n_{r-1} = -\frac{2 - b^2}{b^2}\left(\frac{2r + 1}{r}\right)\beta^n_r - \frac{r + 1}{r}\,\beta^n_{r+1}$$

This shows the way in which the coefficients in equations (26) and (27) were obtained.

To evaluate equation (A1), insert equation (A6) to obtain

$$Z - A = \sum_{n=0}^{\infty} \alpha_n \left[\sqrt{X} \sum_{r=0}^{n-1} \beta^n_r \gamma^r + C_n \ln\left(2 - b^2 + b^2\gamma + b\sqrt{X}\right)\right]_{\gamma=0}^{\Omega^2}$$

$$= \ln\left[\frac{2 - b^2 + b^2\Omega^2 + b\sqrt{X(\Omega)}}{2}\right]\sum_{n=0}^{\infty} \alpha_n C_n + \sqrt{X(\Omega)} \sum_{n=1}^{\infty} \alpha_n \sum_{r=0}^{n-1} \beta^n_r \Omega^{2r} - b \sum_{n=1}^{\infty} \alpha_n \beta^n_0 \qquad \text{(A9)}$$

This is the form of equation (25a).

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