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NUMERICAL ANALYSIS OF SHELLS
Volume I
Unsymmetric Analysis of Orthotropic Reinforced Shells of Revolution

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16. ABSTRACT

This report describes a general theory for the analysis of thin orthotropic shells of revolution subjected to unsymmetric loading. The various shell geometries that can be considered are: ellipsoid, ogive, paraboloid, cylinder, cone, or any selected special geometry. The theory also treats a wide variety of shell reinforcing including eccentric and rotated waffle construction. A digital computing system based on this theory is formulated utilizing a Fourier series expansion in the circumferential direction and numerical integration in the meridional direction. The ensuing program requires simple, yet general, input procedures and produces readily usable output data.

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FOREWORD

This report was prepared by the Grumman Aircraft Engineering Corporation, Bethpage, New York, under contract NAS8-2113, entitled, "Computer Program for Analysis of Shell Structures". The work was performed by the Structural Mechanics Section of Engineering and the Digital Computing Section of Management Information Systems.

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This volume is devoted to a presentation of the theory and numerical techniques developed for implementation as a digital computer program. The user's information for the actual program is presented in two separate volumes: "Numerical Analysis of Shells, Vol II: A Users Manual for STARS II - Shell Theory Automated for Rotational Structures II - Digital Computer Program", by V. Svalbonas and N. Angrisano, and "Numerical Analysis of Shells, Vol III: Engineer's Program Manual for STARS II - Shell Theory Automated for Rotational Structures II - Digital Computer Program", by N. Angrisano, F. Hughes and V. Svalbonas.

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INTRODUCTION

The STARS II digital computer program is an automated procedure for the analysis of thin orthotropic shells of revolution, reinforced in various ways, and subjected to unsymmetric loads. The program can treat shells having multiply-connected joints, walls of sandwich construction, and thermal variations through the walls.

The theory presented in this report and the techniques used for the unsymmetric case are the outgrowth of work that began at Grumman in the early sixties (References 1 and 5). Much of this report involves modification of relationships that appear in Reference 5 to include orthotropic effects (Section 5 and parts of Section 6 are taken directly from that work). The basic shell theory is based upon the work of J. Kempner (References 2 and 3).

The partial differential equations are derived for the general unsymmetric case. They are then reduced to ordinary differential equations by a Fourier series expansion in the circumferential coordinate. These equations are specialized to several convenient coordinate systems, and rederived to represent various reinforcement cases.

The shell is divided into segments of common analytical form, cylinders, cones, ellipsoids, ogives, parabolas, or any special function desired. Influence coefficients are calculated for each segment by combining unit solutions obtained by forward integration using a Runge-Kutta procedure. Since influence coefficients cannot be accurately computed for segments that are too large, the size of these segments is limited by the accuracy of calculation desired. This accuracy is determined by checking the symmetry of the resulting stiffness matrices. The segments are then elastically coupled by conventional matrix methods into regions. These regions are determined by shell branch points and concentrated line loadings. The region matrices thus obtained are first reduced, and then coupled elastically to form structure stiffness matrices. These matrices are inverted to get flexibility matrices which are used to obtain deformation conditions at the ends of each region.

The deformations are used as initial conditions for the final forward integration through each segment, thus yielding the displacements at points throughout the structure. Finally, the stress distributions are obtained, using the total displacement patterns.

The required input data is relatively simple, consisting of specification of geometry, material properties, loads and support conditions. The output is in a form directly usable by the stress analyst, that is, stresses and displacements at various points on the shell.

SYMBOLS

LOWER CASE LATIN

a	semi-diameter in ellipsoid (in.); index on applied loads
b	semi-height in ellipsoid (in.); index on boundary conditions
c	offset in ogive (in.); cosine function
d	differential
f	distributed load in local coordinates (lb/in.^{2}); stress resultant matrix in local coordinates; parabolic geometry constant
h	thickness of face sheet in honeycomb shell
i	index: beginning edge of shell segment; independent joint of kinematic link; subscript "inside"
j	index: ending edge of shell segment; dependent joint of kinematic link
k	segment stiffness matrix; lineal forces due to displacements; curvature
ℓ	length (in.); load matrix
m	mass (slugs); distributed moment (in.-lb/in.^{2}); index on nodes
n	dimension of matrix; index on harmonic
o	subscript "outside"; subscript "reference surface"
p	distributed load in global coordinates (lb/in.^{2}); index on distributed loads
q	number of degrees of freedom
r	radius
s	index on segment; sine function; meridional coordinate in cylinder or cone; arc distance
t	index on topological arrangement; core thickness in honeycomb shell
u	circumferential displacement, positive by right-hand rule about Z axis (in.)
v	meridional displacement, positive in direction of increasing ϕ (in.)
w	normal displacement, positive inward (in.)

SYMBOLS (Cont)

UPPER CASE LATIN

A	number of applied loads
B	number of different sets of boundary conditions
C	constant; bending-membrane interaction stiffness
D	shell flexural stiffness (in.-lb)
E	Young's modulus (lb/in.^{2})
F	subscript "free;" lineal force (lb/in. of circumference); distributed load in nonlinear cases
G	shear modulus (lb/in.^{2})
H	total shell thickness (in.)
I	moment of inertia (in.^{4})
J	effective transverse shear stress resultant (lb/in.)
K	shell extensional stiffness (lb/in.); shell stiffness matrix
L	load matrix; fixed end forces due to distributed load
M	bending moment on shell (in.-lb/in.); number of nodes
N	membrane force (lb/in.); number of harmonics
P	number of distributed loads
Q	transverse shear stress resultant (lb/in.)
R	radius vector; subscript "region"
S	number of segments
T	effective membrane shear (lb/in.); number of topological arrangements; subscript total; temperature
U	amplitude of sinusoidally varying u
V	amplitude of cosinusoidally varying v
W	amplitude of cosinusoidally varying w
X	Cartesian coordinate, $\theta = 0$ at X axis; matrix defined in Section 5
Y	Cartesian coordinate; matrix defined in Section 5
Z	Cartesian coordinate, coincides with axis of revolution

SYMBOLS (Cont)

GREEK

α	coefficient of thermal expansion (deg^{-1})
β	b/a - ratio of semi-height to semi-diameter of ellipsoid
γ	shear strain
δ	matrix of displacements in local coordinates
ε	extensional strain
ζ	normal coordinate, positive inward
θ	circumferential angular coordinate (rad)
ν	Poisson's ratio
ρ	dimensionless radius; position vector of shell relative to inertial frame (in.)
σ	normal stress ($lb/in.^2$)
τ	shear stress ($lb/in.^2$)
ϕ	meridional angular coordinate (rad)
ω	rotational displacement (rad)
Δ	displacements in global system
Ω	amplitude of cosinusoidally varying ω

MISCELLANEOUS

$\wedge$	denotes total forces, $2\pi r_o$ times lineal force
,	denotes partial differentiation
*	pre-superscript, denotes nonlinear terms

SUBSCRIPTS

eq	equivalent

NOTE

Other symbols are defined in the text where they appear.

SECTION 1

FORMULATION OF SHELL EQUATIONS

EQUILIBRIUM EQUATIONS

The equilibrium equations which are derived in general form in Reference 2 by means of the variational principle, are of the accuracy of Love's first approximation as modified by E. Reissner. The reduction of these general equations to the special case of shells of revolution is given in Reference 3, page 8. They are repeated here as Equations 1-1a through 1-1e. It should be noted that the equilibrium relations have been written in the undeformed coordinate system. See Figures 1-1, 1-2, and 1-3.

$$\sum F_\theta = 0 \; : \; r_1 N_{\theta,\theta} + \frac{1}{r_0}\left(N_{\phi\theta} r_0^2\right)_{,\phi} - Q_\theta r_1 \sin\phi = -r_1 r_0 f_\theta \qquad (1\text{-}1a)$$

$$\sum F_\phi = 0 \; : \; \left(N_\phi r_0\right)_{,\phi} + r_1 N_{\phi\theta,\theta} - N_\theta r_1 \cos\phi - r_0 Q_\phi = -r_1 r_0 f_\phi \qquad (1\text{-}1b)$$

$$\sum F_\zeta = 0 \; : \; \left(Q_\phi r_0\right)_{,\phi} + r_1 Q_{\theta,\theta} + r_0 N_\phi + N_\theta r_1 \sin\phi = -r_1 r_0 f_\zeta \qquad (1\text{-}1c)$$

$$\sum M_\theta = 0 \; : \; -r_1 M_{\phi\theta,\theta} - \left(M_\phi r_0\right)_{,\phi} + M_\theta r_1 \cos\phi + r_1 r_0 Q_\phi = -r_1 r_0 m_\theta \qquad (1\text{-}1d)$$

$$\sum M_\phi = 0 \; : \; -\left(M_{\phi\theta} r_0\right)_{,\phi} - r_1 M_{\theta,\theta} - M_{\phi\theta} r_1 \cos\phi + r_1 r_0 Q_\theta = -r_1 r_0 m_\phi \qquad (1\text{-}1e)$$

where commas denote partial differentiation, e.g.,

$$N_{\theta,\theta} \equiv \frac{\partial N_\theta}{\partial \theta} \; ; \qquad \left(N_{\phi\theta} r_0{}^2\right)_{,\phi} \equiv \frac{\partial}{\partial \phi}\left(N_{\phi\theta} r_0{}^2\right)$$

The distributed loading terms, "f" and "m", are dimensionally in terms of force/unit area of middle surface, and (force x length)/unit area of middle surface. Distributed moments occur, for example, in threaded connections in pressure vessels or fittings where they are associated with tangential loads applied away from the middle surface.

It should be noted that the sixth equilibrium equation:

$$\sum M_\zeta = 0 : \qquad N_{\theta\phi} - N_{\phi\theta} + \frac{M_{\theta\phi}}{r_2} + \frac{M_{\phi\theta}}{r_1} = 0 \qquad (1\text{-}2)$$

is not included with the set of equilibrium equations since it will only be satisfied when ζ/r is not neglected in comparison with unity in expressions for stress-resultants and strains (Reference 4, pages 5, 6, and 317). As will be seen later, for the case of $\zeta \ll r_1, r_2$ the physical definitions of the stress resultants lead to $N_{\theta\phi} = N_{\phi\theta}$ and $M_{\theta\phi} = -M_{\phi\theta}$. In this approximate theory then, the equation is identically satisfied in the special case when $r_1 = r_2$ (sphere). Otherwise, the equation is violated.

STRAIN-DISPLACEMENT RELATIONS

The strain-displacement relations (Reference 3, pages 10, 34, and 39) are presented as follows:

$$\varepsilon_{\theta_o} = \frac{1}{r_0}\left(u,_\theta + v \cos\phi - w \sin\phi\right) \qquad (1\text{-}3a)$$

$$\varepsilon_{\theta_o} = \frac{1}{r_1}\left(v,_\phi - w\right) \qquad (1\text{-}3b)$$

$$\gamma_{\phi\theta_o} = \frac{v,_\theta}{r_0} + \frac{r_0}{r_1}\left[\frac{u}{r_0}\right],_\phi = \frac{v,_\theta - u\cos\phi}{r_0} + \frac{u,_\phi}{r_1} \qquad (1\text{-}3c)$$

$$\omega_\theta = \frac{1}{r_1}\left(w,_\phi + v\right) \qquad (1\text{-}3d)$$

$$\omega_\phi = -\left(\frac{w,_\theta}{r_0} + \frac{u}{r_2}\right) = -\frac{1}{r_0}\left(w,_\theta + u \sin\phi\right) \qquad (1\text{-}3e)$$

$$k_\theta = -\frac{1}{r_0}\left(\omega_{\phi,\theta} - \omega_\theta \cos\phi\right) \qquad (1\text{-}3f)$$

$$k_\phi = \frac{1}{r_1}\,\omega_{\theta,\phi} \qquad (1\text{-}3g)$$

$$k_{\phi\theta} = k_{\theta\phi} = \frac{1}{2r_0}\left[\omega_{\theta,\theta} - \frac{r_0}{r_1}\omega_{\phi,\phi} + \omega_\phi \cos\phi\right] \qquad (1\text{-}3h)$$

where the geometric relations

$$r_{0,\phi} = r_1 \cos\phi \qquad r_0 = r_2 \sin\phi$$

have been used.

STRESS-STRAIN RELATIONS

The stress-strain relations (Reference 3, page 32) are obtained by employing Hooke's Laws and assuming that dimensions "ζ", normal to the middle surface, are much smaller than the radius of the shell. In this case, orthotropic relations will be used to increase the usefulness and applicability of the analysis.

$$\sigma_\theta = \frac{E_\theta}{1 - \nu_{\phi\theta}\nu_{\theta\phi}}\left[\varepsilon_{\theta_o} + \nu_{\theta\phi}\varepsilon_{\phi_o} - \zeta(k_\theta + \nu_{\theta\phi}k_\phi) - (\alpha_\theta + \nu_{\theta\phi}\alpha_\phi)T\right] \qquad (1\text{-}4a)$$

$$\sigma_\phi = \frac{E_\phi}{1 - \nu_{\phi\theta}\nu_{\theta\phi}}\left[\varepsilon_{\phi_o} + \nu_{\phi\theta}\varepsilon_{\theta_o} - \zeta(k_\phi + \nu_{\phi\theta}k_\theta) - (\alpha_\phi + \nu_{\phi\theta}\alpha_\theta)T\right] \qquad (1\text{-}4b)$$

$$\tau_{\phi\theta} = G_{\phi\theta}\left[\gamma_{\phi\theta_o} - 2\zeta k_{\phi\theta}\right] \qquad (1\text{-}4c)$$

where ε, k and γ are functions of θ and ϕ only. But α, T, E and ν may, in general, also be arbitrary functions of ζ. Thus, the stresses, σ and τ, are arbitrary functions of the thickness coordinate. These equations assume that the thickness is negligible compared to the radii of curvature, r_1 and r_2. If this assumption is not made, but normals do remain straight and normal, then strains (and stresses) are not linear functions of ζ, even with constant α, E and ν. This is the same phenomenon that occurs in curved beams. (Refer to Reference 2, page 29 and Reference 4, page 316.) For present purposes, only T will be allowed to vary in the ζ direction.

STRESS RESULTANTS

Again neglecting thickness in comparison with the radii of curvature (Reference 3, pages 33, 34),

$$N_\theta = \int \sigma_\theta \, d\zeta \qquad M_\theta = \int \sigma_\theta \zeta \, d\zeta \tag{1-5}$$

$$N_\phi = \int \sigma_\phi \, d\zeta \qquad M_\phi = \int \sigma_\phi \zeta \, d\zeta$$

$$N_{\phi\theta} = +N_{\theta\phi} = \int \tau_{\phi\theta} \, d\zeta \qquad M_{\phi\theta} = -M_{\theta\phi} = +\int \tau_{\phi\theta} \zeta \, d\zeta$$

$$N_{T\theta} = \int \frac{E_\theta \, (\alpha_\theta + \nu_{\theta\phi}\alpha_\phi)T}{1 - \nu_{\phi\theta}\nu_{\theta\phi}} d\zeta \qquad M_{T\theta} = \int \frac{E_\theta \, (\alpha_\theta + \nu_{\theta\phi}\alpha_\phi)T}{1 - \nu_{\phi\theta}\nu_{\theta\phi}} \zeta d\zeta \tag{1-6}$$

$$N_{T\phi} = \int \frac{E_\phi \, (\alpha_\phi + \nu_{\phi\theta}\alpha_\theta)T}{1 - \nu_{\phi\theta}\nu_{\theta\phi}} d\zeta \qquad M_{T\phi} = \int \frac{E_\phi \, (\alpha_\phi + \nu_{\phi\theta}\alpha_\theta)T}{1 - \nu_{\phi\theta}\nu_{\theta\phi}} \zeta d\zeta$$

where the integrals are taken over the entire thickness. Neglecting "ζ" compared to "r" ignores the fact that, for an elemental length ds along the median surface, the lamella inside and outside the median surface are shorter or longer than ds. (Refer to Reference 4, pages 4 and 5.) If this effect is considered, and the radii associated with the ϕ and θ directions are unequal ($r_1 \neq r_2$), then $N_{\theta\phi} \neq N_{\phi\theta}$ and $M_{\theta\phi} \neq -M_{\phi\theta}$.

The following general definitions of extensional, bending, and in-plane shear stiffnesses are introduced:

$$K_{11} = \frac{E_\theta \, h}{1 - \nu_{\phi\theta}\nu_{\theta\phi}} \qquad K_{22} = \frac{E_\phi h}{1 - \nu_{\phi\theta}\nu_{\theta\phi}} \tag{1-7a}$$

$$D_{11} = \frac{E_\theta h^3}{12(1 - \nu_{\phi\theta}\nu_{\theta\phi})} \qquad D_{22} = \frac{E_\phi h^3}{12(1 - \nu_{\phi\theta}\nu_{\theta\phi})} \tag{1-7b}$$

$$K_{33} = G_{\phi\theta} h \qquad D_{33} = \frac{G_{\phi\theta} h^3}{12} \tag{1-7c}$$

The definitions for rigid-core sandwich configurations of various construction are presented in Figure 1-4.

Combining Equations 1-4 through 1-7 yields the stress resultants as functions of the strains, curvatures, and temperatures.

$$N_\theta = K_{11}\left[\varepsilon_{\theta_o} + \nu_{\theta\phi}\,\varepsilon_{\phi_o}\right] - N_{T\theta} \tag{1-8a}$$

$$N_\phi = K_{22}\left[\varepsilon_{\phi_o} + \nu_{\phi\theta}\,\varepsilon_{\theta_o}\right] - N_{T\phi} \tag{1-8b}$$

$$N_{\phi\theta} = N_{\theta\phi} = K_{33}\,\gamma_{\phi\theta_o} \tag{1-8c}$$

$$M_\theta = -D_{11}\left[k_\theta + \nu_{\theta\phi}\,k_\phi\right] - M_{T\theta} \tag{1-8d}$$

$$M_\phi = -D_{22}\left[k_\phi + \nu_{\phi\theta}\,k_\theta\right] - M_{T\phi} \tag{1-8e}$$

$$M_{\phi\theta} = -M_{\theta\phi} = -2\,D_{33}\,k_{\phi\theta} \tag{1-8f}$$

BOUNDARY CONDITIONS

The required boundary conditions at a plane of constant ϕ are obtained as a result of the variational procedure (Reference 3, page 28). It is necessary to specify either displacements <u>or</u> corresponding stress resultants.

$$u \text{ or } \left[N_{\phi\theta} - \frac{M_{\phi\theta}}{r_0}\sin\phi\right] \tag{1-9a}$$

$$v \text{ or } N_\phi \tag{1-9b}$$

$$w \text{ or } \left[Q_\phi + \frac{M_{\phi\theta,\theta}}{r_0}\right] \tag{1-9c}$$

$$\omega_\theta \text{ or } M_\phi \tag{1-9d}$$

In view of the form of the boundary conditions (Equation 1-9), the equilibrium equations are re-formulated in terms of the "effective stress resultants", $T_{\phi\theta}$ N_ϕ, J_ϕ and M_ϕ where,

$$T_{\phi\theta} = N_{\phi\theta} - M_{\phi\theta}\frac{\sin\phi}{r_0} \tag{1-10a}$$

$$N_\phi = N_\phi \tag{1-10b}$$

$$J_\phi = Q_\phi + \frac{M_{\phi\theta,\theta}}{r_0} \tag{1-10c}$$

$$M_\phi = M_\phi \tag{1-10d}$$

The partial derivatives of $T_{\phi\theta}$ and J_ϕ, which will be required later, are:

$$\frac{T_{\phi\theta,\phi}}{r_1} = \frac{N_{\phi\theta,\phi}}{r_1} - M_{\phi\theta,\phi}\frac{\sin\phi}{r_0 r_1} + M_{\phi\theta}\frac{\sin\phi\cos\phi}{r_0^2} - M_{\phi\theta}\frac{\cos\phi}{r_0 r_1} \tag{1-11a}$$

$$\frac{J_{\phi,\phi}}{r_1} = \frac{Q_{\phi,\phi}}{r_1} + \frac{M_{\phi\theta,\theta\phi}}{r_0 r_1} - M_{\phi\theta,\theta}\frac{\cos\phi}{r_0^2} \tag{1-11b}$$

FINAL EQUATIONS

By eliminating strains and curvatures and utilizing the orthotropic identity $\nu_{\theta\phi}E_\theta = \nu_{\phi\theta}E_\phi$, each stress resultant is expressed in terms of displacements and other stress resultants. From Equations 1-3 and 1-8:

$$\begin{aligned} N_\theta &= \nu_{\phi\theta} N_\phi + (K_{11} - \nu_{\phi\theta}^2 K_{22})\,\varepsilon_{\theta_0} - N_{T\theta} + \nu_{\phi\theta} N_{T\phi} \\ &= \nu_{\phi\theta} N_\phi + (K_{11} - \nu_{\phi\theta}^2 K_{22})\left[\frac{u_{,\theta} + v\cos\phi - w\sin\phi}{r_0}\right] - N_{T\theta} + \nu_{\phi\theta} N_{T\phi} \end{aligned} \tag{1-12a}$$

$$N_\phi = \nu_{\theta\phi} N_\theta + (K_{22} - \nu_{\theta\phi}{}^2 K_{11})\, \varepsilon_{\phi_o} + \nu_{\theta\phi} N_{T\theta} - N_{T\phi}$$

$$= \nu_{\theta\phi} N_\theta + (K_{22} - \nu_{\theta\phi}{}^2 K_{11}) \left[\frac{v,_\phi - w}{r_1}\right] - N_{T\phi} + \nu_{\theta\phi} N_{T\theta} \qquad (1\text{-}12b)$$

$$N_{\phi\theta} = N_{\theta\phi} = K_{33}\, \gamma_{\phi\theta_o} = K_{33}\left[\frac{v,_\theta - u\cos\phi}{r_0} + \frac{u,_\phi}{r_1}\right] \qquad (1\text{-}12c)$$

$$M_\theta = \nu_{\phi\theta} M_\phi - (D_{11} - \nu_{\phi\theta}{}^2 D_{22})\, k_\theta - M_{T\theta} + \nu_{\phi\theta} M_{T\phi}$$

$$= \nu_{\phi\theta} M_\phi - \frac{(D_{11} - \nu_{\phi\theta}{}^2 D_{22})}{r_0}\left[\frac{w,_{\theta\theta} + u,_\theta \sin\phi}{r_0} + \omega_\theta \cos\phi\right]$$

$$- M_{T\theta} + \nu_{\phi\theta} M_{T\phi} \qquad (1\text{-}13a)$$

$$M_\phi = \nu_{\theta\phi} M_\theta - (D_{22} - \nu_{\theta\phi}{}^2 D_{11})\, k_\phi + \nu_{\theta\phi} M_{T\theta} - M_{T\phi}$$

$$= \nu_{\theta\phi} M_\theta - (D_{22} - \nu_{\theta\phi}{}^2 D_{11})\left[\frac{\omega_{\theta,\phi}}{r_1}\right] - M_{T\phi} + \nu_{\theta\phi} M_{T\theta} \qquad (1\text{-}13b)$$

$$M_{\phi\theta} = -\frac{D_{33}}{r_0}\left[\omega_{\theta,\theta} - \frac{r_0}{r_1}\omega_{\phi,\phi} + \omega_\phi \cos\phi\right]$$

Using Equations 1-3d, 1-10a, and 1-12c

$$M_{\phi\theta} = -\frac{D_{33}}{r_0}\left\{\omega_{\theta,\theta} + \left[(\omega_{\theta,\theta} - \frac{v_{,\theta}}{r_1}) + \left(\frac{u\cos\phi\sin\phi}{r_0} - \frac{v,_\theta \sin\phi}{r_0} + \frac{T_{\phi\theta}\sin\phi}{K_{33}}\right.\right.\right.$$

$$\left.\left.\left. + \frac{M_{\phi\theta}\sin^2\phi}{K_{33}\, r_0}\right) + \frac{u\cos\phi}{r_1}\right] - \frac{2\cos\phi}{r_0}(w,_\theta + u\sin\phi)\right\}$$

$$M_{\phi\theta} = \left[\frac{-1}{\dfrac{r_0}{D_{33}} + \dfrac{\sin^2\phi}{r_0 K_{33}}}\right]\left\{2\,\omega_{\theta,\theta} + u\left(\frac{\cos\phi}{r_1} - \frac{\cos\phi\sin\phi}{r_0}\right) - v,_\theta\left(\frac{\sin\phi}{r_0} + \frac{1}{r_1}\right)\right.$$

$$\left. - 2\, w,_\theta \frac{\cos\phi}{r_0} + \frac{T_{\phi\theta}}{K_{33}}\sin\phi\right\} \qquad (1\text{-}13c)$$

Expansion of the equilibrium equations yields:

$$N_{\theta,\theta} + 2N_{\phi\theta} \cos\phi + N_{\phi\theta,\phi} \frac{r_0}{r_1} - Q_\theta \sin\phi + r_0 f_\theta = 0 \qquad (1\text{-}14a)$$

$$N_{\phi,\phi} \frac{r_0}{r_1} + N_\phi \cos\phi + N_{\phi\theta,\theta} - N_\theta \cos\phi - Q_\phi \frac{r_0}{r_1} + r_0 f_\phi = 0 \qquad (1\text{-}14b)$$

$$Q_{\phi,\phi} \frac{r_0}{r_1} + Q_\phi \cos\phi + Q_{\theta,\theta} + N_\phi \frac{r_0}{r_1} + N_\theta \sin\phi + r_0 f_\zeta = 0 \qquad (1\text{-}14c)$$

$$-M_{\phi\theta,\theta} - M_{\phi,\phi} \frac{r_0}{r_1} - M_\phi \cos\phi + M_\theta \cos\phi + Q_\phi r_0 = - r_0 m_\theta \qquad (1\text{-}14d)$$

$$-M_{\phi\theta,\theta} \frac{r_0}{r_1} - 2M_{\phi\theta} \cos\phi - M_{\theta,\theta} + Q_\theta r_0 = - r_0 m_\phi \qquad (1\text{-}14e)$$

A set of eight partial differential equations of first order in the independent variable, ϕ, will now be obtained by appropriate substitution of the previous equations. The first four of these equations result directly from the equilibrium equations. Combining Equations 1-14a through e with Equations 1-10 and 1-11 yields:

$$\frac{T_{\phi\theta,\phi}}{r_1} = -2T_{\phi\theta} \frac{\cos\phi}{r_0} - \frac{N_{\theta,\theta}}{r_0} + M_{\theta,\theta} \frac{\sin\phi}{r_0^2} - M_{\phi\theta} \frac{\cos\phi}{r_0} \left[\frac{1}{r_1} - \frac{\sin\phi}{r_0}\right]$$

$$- f_\theta - m_\phi \frac{\sin\phi}{r_0} \qquad (1\text{-}15a)$$

$$\frac{N_{\phi,\phi}}{r_1} = -N_\phi \frac{\cos\phi}{r_0} + N_\theta \frac{\cos\phi}{r_0} - \frac{T_{\phi\theta,\theta}}{r_0} - M_{\phi\theta,\theta} \left[\frac{\sin\phi}{r_0^2} + \frac{1}{r_0 r_1}\right]$$

$$+ \frac{J_\phi}{r_1} - f_\phi \qquad (1\text{-}15b)$$

$$\frac{J_{\phi,\phi}}{r_1} = -J_\phi \frac{\cos\phi}{r_0} - N_\theta \frac{\sin\phi}{r_0} - \frac{N_\phi}{r_1} - \frac{M_{\theta,\theta\theta}}{r_0^2} - 2M_{\phi\theta,\theta} \frac{\cos\phi}{r_0^2} - f_\zeta + \frac{m_{\phi,\theta}}{r_0} \tag{1-15c}$$

$$\frac{M_{\phi,\phi}}{r_1} = M_\theta \frac{\cos\phi}{r_0} - M_\phi \frac{\cos\phi}{r_0} - 2\frac{M_{\phi\theta,\theta}}{r_0} + J_\phi + m_\theta \tag{1-15d}$$

The remaining four equations involve differentiation with respect to ϕ and are obtained from Equations 1-3d, 1-10a, 1-12b and c, and 1-13b.

$$\frac{u_{,\phi}}{r_1} = \frac{u\cos\phi}{r_0} - \frac{v_{,\theta}}{r_0} + \frac{T_{\phi\theta}}{K_{33}} + \frac{M_{\phi\theta}\sin\phi}{r_0 K_{33}} \tag{1-16a}$$

$$\frac{v_{,\phi}}{r_1} = \frac{w}{r_1} + \left(K_{22} - \nu_{\theta\phi}^2 K_{11}\right)^{-1} \left\{N_\phi - \nu_{\theta\phi}N_\theta + N_{T\phi} - \nu_{\theta\phi}N_{T\theta}\right\} \tag{1-16b}$$

$$\frac{w_{,\phi}}{r_1} = \omega_\theta - \frac{v}{r_1} \tag{1-16c}$$

$$\frac{\omega_{\theta,\phi}}{r_1} = \left(D_{22} - \nu_{\theta\phi}^2 D_{11}\right)^{-1} \left\{-M_\phi + \nu_{\theta\phi}M_\theta - M_{T\phi} + \nu_{\theta\phi}M_{T\theta}\right\} \tag{1-16d}$$

In order to obtain a complete set of equations, the following auxiliary equations are necessary:

$$N_\theta = \nu_{\phi\theta} N_\phi + (K_{11} - \nu_{\phi\theta}^2 K_{22}) \left[\frac{u_{,\theta} + v\cos\phi - w\sin\phi}{r_0}\right] - N_{T\theta} + \nu_{\phi\theta} N_{T\phi} \tag{1-17a}$$

$$M_\theta = \nu_{\phi\theta} M_\phi - \frac{(D_{11} - \nu_{\phi\theta}{}^2 D_{22})}{r_0}\left[\frac{w,_{\theta\theta} + u,_\theta \sin\phi}{r_0} + \omega_\theta \cos\phi\right]$$

$$- M_{T\theta} + \nu_{\phi\theta} M_{T\phi} \qquad \text{(1-17b)}$$

$$M_{\phi\theta} = \left[\frac{-1}{\dfrac{r_0}{D_{33}} + \dfrac{\sin^2\phi}{r_0\, K_{33}}}\right]\left\{2\omega_{\theta,\theta} + u\left(\frac{\cos\phi}{r_1} - \frac{\cos\phi\,\sin\phi}{r_0}\right) - v,_\theta\left(\frac{\sin\phi}{r_0} + \frac{1}{r_1}\right)\right.$$

$$\left. - 2w,_\theta \frac{\cos\phi}{r_0} + \frac{T_{\phi\theta}}{K_{33}} \sin\phi\right\} \qquad \text{(1-17c)}$$

$$N_{\phi\theta} = T_{\phi\theta} + \frac{M_{\phi\theta}}{r_0}\sin\phi \qquad \text{(1-17d)}$$

$$\omega_\phi = -\frac{w,_\theta}{r_0} - \frac{u\sin\phi}{r_0} \qquad \text{(1-17e)}$$

$$Q_\phi = J_\phi - \frac{M_{\phi\theta,\theta}}{r_0} \qquad \text{(1-17f)}$$

$$Q_\theta = \left\{\frac{3\cos\phi}{r_0} - \frac{2\cos\phi\left(r_0\,K_{33} + D_{33}\dfrac{\sin\phi}{r_1}\right)}{r_0{}^2 K_{33} + D_{33}\sin^2\phi}\right\} M_{\phi\theta} + \left[\frac{-\dfrac{1}{r_1}}{\dfrac{r_0}{D_{33}} + \dfrac{\sin^2\phi}{r_0 K_{33}}}\right]\left\{2\,\omega_{\theta,\theta\phi}\right.$$

$$+ u,_\phi\left(\frac{\cos\phi}{r_1} - \frac{\cos\phi\sin\phi}{r_0}\right) + u\left(\frac{\sin^2\phi}{r_0} - \frac{\cos^2\phi}{r_0} - \frac{\sin\phi}{r_1} - \frac{r_{1,\phi}\cos\phi}{r_1{}^2}\right.$$

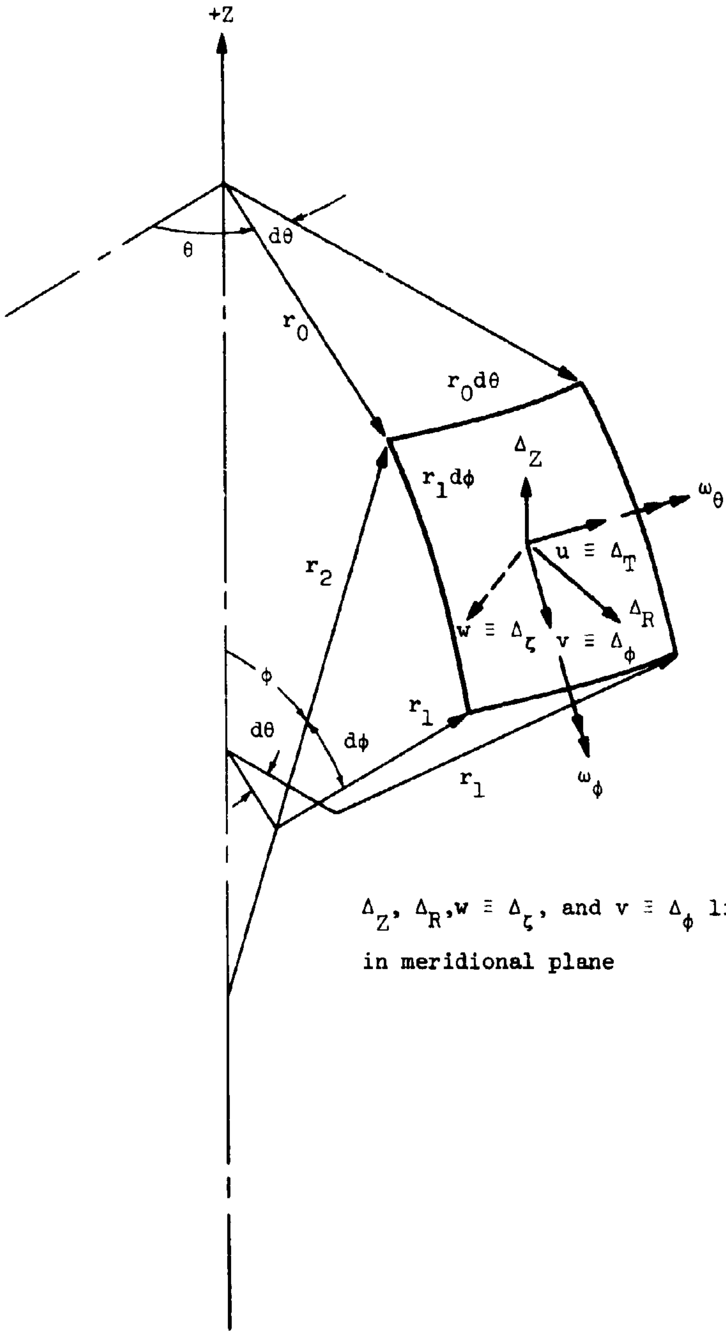
$$
+ \frac{r_1 \cos^2\phi \sin\phi}{r_0^{\;2}}\Bigg) - v_{,\theta\phi}\left(\frac{\sin\phi}{r_0} + \frac{1}{r_1}\right) - v_{,\theta}\Bigg(\frac{\cos\phi}{r_0} - \frac{r_1 \sin\phi\cos\phi}{r_0^{\;2}}
$$

$$
- \frac{r_{1,\phi}}{r_1^{\;2}}\Bigg) - 2w_{,\theta\phi}\frac{\cos\phi}{r_0} + 2w_{,\theta}\left(\frac{\sin\phi}{r_0} + \frac{r_1 \cos^2\phi}{r_0^{\;2}}\right)
$$

$$
+ T_{\phi\theta,\phi}\frac{\sin\phi}{K_{33}} + T_{\phi\theta}\frac{\cos\phi}{K_{33}}\Bigg\} + \frac{M_{\theta,\theta}}{r_0} - m_\phi \qquad (1\text{-}17g)
$$

These auxiliary equations could be included in the eight partial differential equations by direct substitution, and indeed in the case of Equations 1-17d through g this has already been done. However these quantities are also of technical interest and computing them separately is desirable.

The equations presented above constitute a complete formulation of a consistent first-order thin shell theory. Techniques for the solution of this set of equations are given in the following sections.



Δ_Z, Δ_R, $w \equiv \Delta_\zeta$, and $v \equiv \Delta_\phi$ lie in meridional plane

Figure 1-1. Shell Element Geometry and Displacements

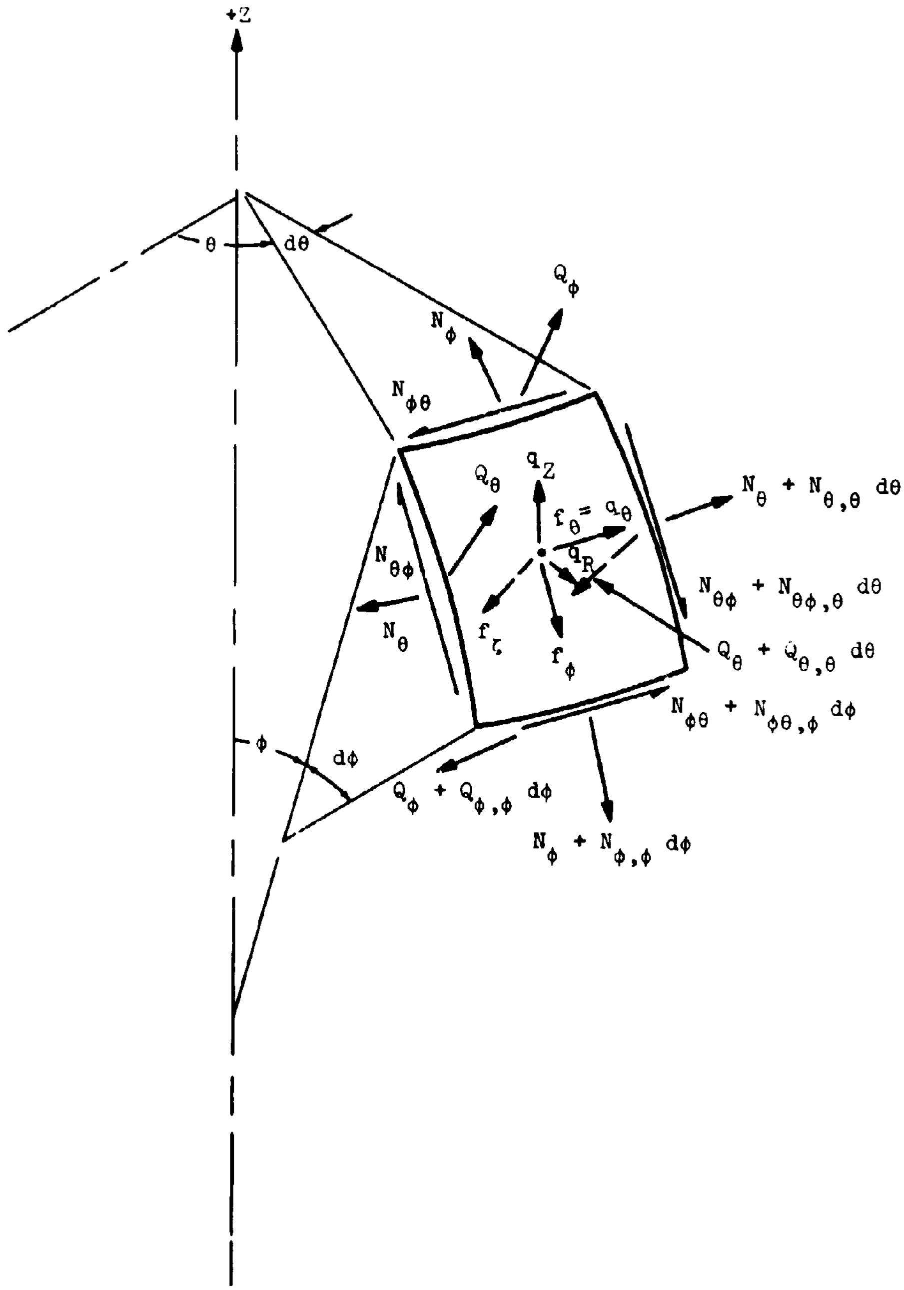


Figure 1-2. Forces On Shell Element

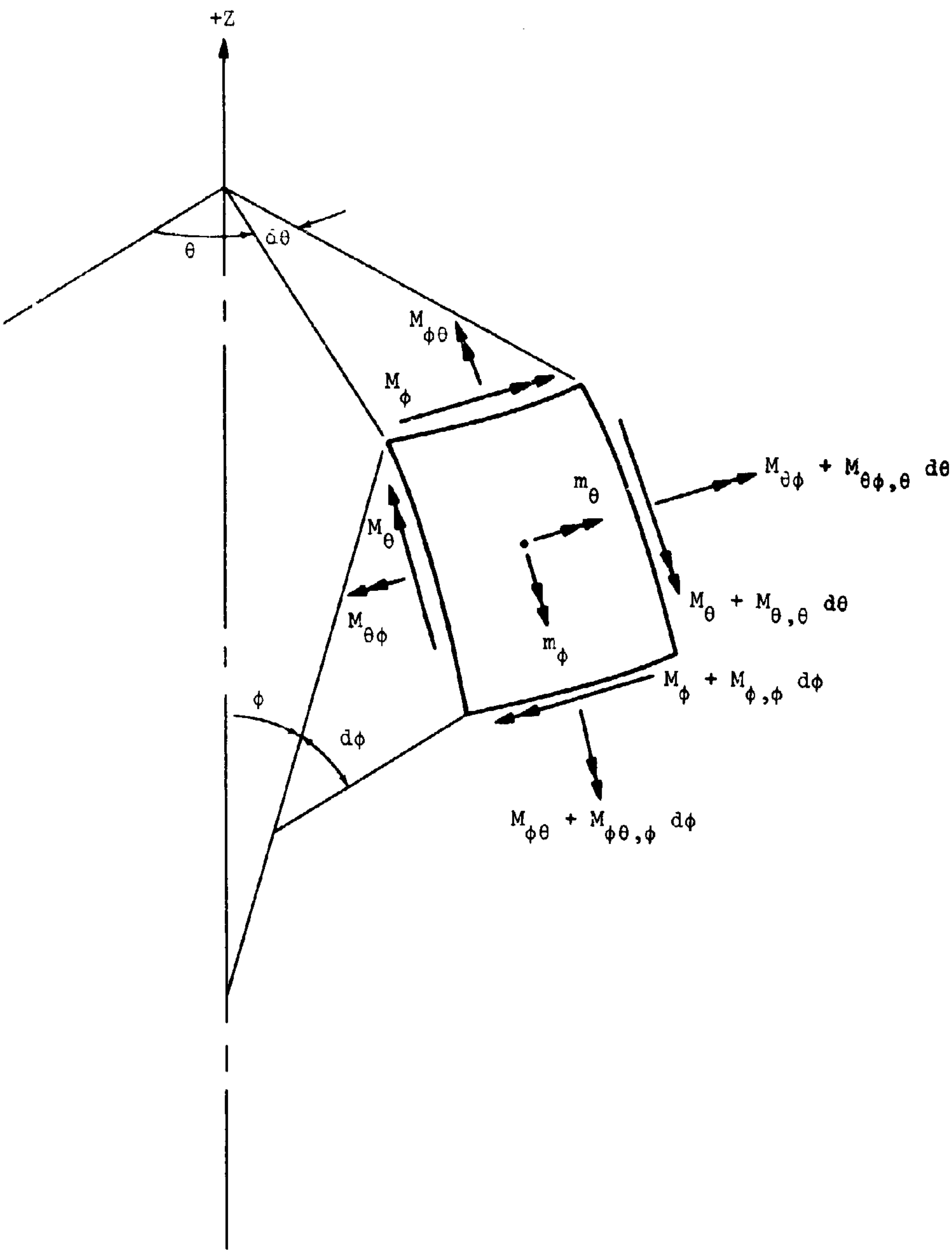
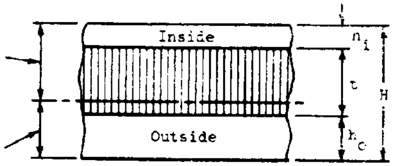


Figure 1-3. Moments On Shell Element

$$\bar{\zeta}_{in} = \frac{n_i^2 + h_o^2 + 2h_ih_o + 2h_ot}{2(h_i + h_o)}$$

$$\bar{\zeta}_{out} = \frac{h_i^2 + h_c^2 + 2h_ih_o + 2h_it}{2(h_i + h_o)}$$



E, ν, Constant through thickness

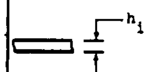
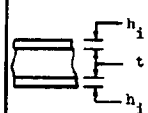
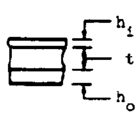
Configuration	Extensional Stiffnesses	Flexural Stiffnesses	Shear Stiffnesses
 h_i Ortho-tropic	$K_{11} = \frac{E_\theta h_i}{1 - \nu_{\phi\theta}\nu_{\theta\phi}}$ $K_{22} = \frac{E_\phi h_i}{1 - \nu_{\phi\theta}\nu_{\theta\phi}}$	$D_{11} = \frac{E_\theta h_i^3}{12(1 - \nu_{\phi\theta}\nu_{\theta\phi})}$ $D_{22} = \frac{E_\phi h_i^3}{12(1 - \nu_{\phi\theta}\nu_{\theta\phi})}$	$K_{33} = G_{\phi\theta}h_i$ $D_{33} = \frac{G_{\phi\theta}h_i^3}{12}$
 h_i t h_i Equal Face Sheets	$K_{11} = \frac{2E_\theta h_i}{1 - \nu_{\phi\theta}\nu_{\theta\phi}}$ $K_{22} = \frac{2E_\phi h_i}{1 - \nu_{\phi\theta}\nu_{\theta\phi}}$	$D_{11} = \frac{E_\theta h_i\left[4h_i^2 + 6h_it + 3t^2\right]}{6(1 - \nu_{\phi\theta}\nu_{\theta\phi})}$ $D_{22} = \frac{E_\phi h_i\left[4h_i^2 + 6h_it + 3t^2\right]}{6(1 - \nu_{\phi\theta}\nu_{\theta\phi})}$	$K_{33} = 2\,G_{\phi\theta}h_i$ $D_{33} = \frac{G_{\phi\theta}h_i\left[4h_i^2 + 6h_it + 3t^2\right]}{6}$
 h_i t h_o Unequal Face Sheets	$K_{11} = \frac{E_\theta(h_i + h_o)}{1 - \nu_{\phi\theta}\nu_{\theta\phi}}$ $K_{22} = \frac{E_\phi(h_i + h_o)}{1 - \nu_{\phi\theta}\nu_{\theta\phi}}$	$D_{11} = E_\theta\left[\frac{(h_i + h_o)^4 + 12h_ih_ot(h_i + h_o + t)}{12(h_i + h_o)(1 - \nu_{\phi\theta}\nu_{\theta\phi})}\right]$ $D_{22} = E_\phi\left[\frac{(h_i + h_o)^4 + 12h_ih_ct(h_i + h_o + t)}{12(h_i + h_o)(1 - \nu_{\phi\theta}\nu_{\theta\phi})}\right]$	$K_{33} = \varepsilon_{\phi\theta}(h_i + h_o)$ $D_{33} = G_{\phi\theta}\left[\frac{(h_i + h_o)^4 + 12h_ih_ot(h_i + h_o + t)}{12(h_i + h_o)}\right]$

Figure 1-4. Shell Section Properties

SECTION 2

FOURIER ANALYSIS

Efficient techniques for the numerical solution of partial differential equations are not readily available. However, by assuming a Fourier series distribution in the circular coordinate, θ, it is possible to reduce the solution to "N" sets of ordinary differential equations. Usually the number of sets (harmonics) that are required to solve practical problems is rather limited. The actual number will depend upon the type of load distribution being investigated and the degree of accuracy demanded. By restricting consideration to cases symmetric about $\theta = 0$, only "one-half" of the general expansion is needed. By physical reasoning, (or complete expansion of the series), the appropriate function (sine or cosine) may be chosen. The choices are verified when the trigonometric functions may be factored out of the differential equations.

The reduction of the system of partial differential equations to sets of ordinary differential equations is most convenient since these equations may now be solved by employing a standard numerical integrating procedure such as Runge-Kutta.

The expansion of the previously developed partial differential equations into sets of ordinary differential equations will now be discussed. The appropriate series expansions for the quantities of interest are:

$$u = U^{(0)} + \sum_{n=1}^{\infty} U^{(n)} \sin n\theta \qquad N_\theta = \sum_{n=0}^{\infty} N_\theta^{(n)} \cos n\theta$$

$$v = \sum_{n=0}^{\infty} V^{(n)} \cos n\theta \qquad N_\phi = \sum_{n=0}^{\infty} N_\phi^{(n)} \cos n\theta$$

$$w = \sum_{n=0}^{\infty} W^{(n)} \cos n\theta \qquad N_{\phi\theta} = N_{\phi\theta}^{(0)} + \sum_{n=1}^{\infty} N_{\phi\theta}^{(n)} \sin n\theta$$

$$\omega_\theta = \sum_{n=0}^{\infty} \Omega_\theta^{(n)} \cos n\theta \qquad M_\theta = \sum_{n=0}^{\infty} M_\theta^{(n)} \cos n\theta$$

$$\omega_\phi = \Omega_\phi^{(0)} + \sum_{n=1}^{\infty} \Omega_\phi^{(n)} \sin n\theta$$

$$f_\theta = f_\theta^{(0)} + \sum_{n=1}^{\infty} f_\theta^{(n)} \sin n\theta$$

$$f_\phi = \sum_{n=0}^{\infty} f_\phi^{(n)} \cos n\theta$$

$$f_\zeta = \sum_{n=0}^{\infty} f_\zeta^{(n)} \cos n\theta$$

$$m_\theta = \sum_{n=0}^{\infty} m_\theta^{(n)} \cos n\theta$$

$$m_\phi = m_\phi^{(0)} + \sum_{n=1}^{\infty} m_\phi^{(n)} \sin n\theta$$

$$N_{T\theta} = \sum_{n=0}^{\infty} N_{T\theta}^{(n)} \cos n\theta$$

$$N_{T\phi} = \sum_{n=0}^{\infty} N_{T\phi}^{(n)} \cos n\theta$$

$$T = \sum_{n=0}^{\infty} T^{(n)} \cos n\theta$$

$$M_\phi = \sum_{n=0}^{\infty} M_\phi^{(n)} \cos n\theta$$

$$M_{\phi\theta} = M_{\phi\theta}^{(0)} + \sum_{n=1}^{\infty} M_{\phi\theta}^{(n)} \sin n\theta$$

$$Q_\theta = Q_\theta^{(0)} + \sum_{n=1}^{\infty} Q_\theta^{(n)} \sin n\theta$$

$$Q_\phi = \sum_{n=0}^{\infty} Q_\phi^{(n)} \cos n\theta$$

$$T_{\phi\theta} = T_{\phi\theta}^{(0)} + \sum_{n=1}^{\infty} T_{\phi\theta}^{(n)} \sin n\theta$$

$$J_\phi = \sum_{n=0}^{\infty} J_\phi^{(n)} \cos n\theta$$

$$M_{T\theta} = \sum_{n=0}^{\infty} M_{T\theta}^{(n)} \cos n\theta$$

$$M_{T\phi} = \sum_{n=0}^{\infty} M_{T\phi}^{(n)} \cos n\theta \tag{2-1}$$

FINAL SET OF EQUATIONS

Substituting these equations into the sets of partial differential equations (Equations 1-15, 1-16, and 1-17) will yield the final harmonic form of the equations required.

Differential Equations

When n = 0, 1, 2..., then

$$\frac{T_{\phi\theta,\phi}^{(n)}}{r_1} = -2T_{\phi\theta}^{(n)} \frac{\cos\phi}{r_0} + n\frac{N_\theta^{(n)}}{r_0} - nM_\theta^{(n)} \frac{\sin\phi}{r_0^2} - M_{\phi\theta}^{(n)} \frac{\cos\phi}{r_0}\left[\frac{1}{r_1} - \frac{\sin\phi}{r_0}\right]$$

$$- f_\theta^{(n)} - m_\phi^{(n)} \frac{\sin\phi}{r_0} \tag{2-2}$$

$$\frac{N_{\phi,\phi}^{(n)}}{r_1} = - N_{\phi}^{(n)} \frac{\cos\phi}{r_0} + N_{\theta}^{(n)} \frac{\cos\phi}{r_0} - n\frac{T_{\phi\theta}^{(n)}}{r_0} - nM_{\phi\theta}^{(n)}\left[\frac{\sin\phi}{r_0^{\,2}} + \frac{1}{r_0 r_1}\right] + \frac{J_{\phi}^{(n)}}{r_1} - f_{\phi}^{(n)}$$

$$\frac{J_{\phi,\phi}^{(n)}}{r_1} = - J_{\phi}^{(n)} \frac{\cos\phi}{r_0} - N_{\theta}^{(n)} \frac{\sin\phi}{r_0} - \frac{N_{\phi}^{(n)}}{r_1} + n^2 \frac{M_{\theta}^{(n)}}{r_0^{\,2}} - 2n\, M_{\phi\theta}^{(n)} \frac{\cos\phi}{r_0^{\,2}} - f_{\zeta}^{(n)} + \frac{nm_{\phi}^{(n)}}{r_0}$$

$$\frac{M_{\phi,\phi}^{(n)}}{r_1} = M_{\theta}^{(n)} \frac{\cos\phi}{r_0} - M_{\phi}^{(n)} \frac{\cos\phi}{r_0} - 2n \frac{M_{\phi\theta}^{(n)}}{r_0} + J_{\phi}^{(n)} + m_{\theta}^{(n)}$$

$$\frac{U_{,\phi}^{(n)}}{r_1} = U^{(n)} \frac{\cos\phi}{r_0} + n\frac{V^{(n)}}{r_0} + \frac{T_{\phi\theta}^{(n)}}{K_{33}} + \frac{M_{\phi\theta}^{(n)}\sin\phi}{r_0 K_{33}}$$

$$\frac{V_{,\phi}^{(n)}}{r_1} = \frac{W^{(n)}}{r_1} + \left(K_{22} - \nu_{\theta\phi}^{\,2} K_{11}\right)^{-1} \left\{ N_{\phi}^{(n)} - \nu_{\theta\phi} N_{\theta}^{(n)} + N_{T\phi}^{(n)} - \nu_{\theta\phi} N_{T\theta}^{(n)} \right\}$$

$$\frac{W_{,\phi}^{(n)}}{r_1} = \Omega_{\theta}^{(n)} - \frac{V^{(n)}}{r_1}$$

$$\frac{\Omega_{\theta,\phi}^{(n)}}{r_1} = \left(D_{22} - \nu_{\theta\phi}^{\,2} D_{11}\right)^{-1} \left\{ -M_{\phi}^{(n)} + \nu_{\theta\phi} M_{\theta}^{(n)} - M_{T\phi}^{(n)} + \nu_{\theta\phi} M_{T\theta}^{(n)} \right\}$$

Auxiliary Equations

When $n = 0, 1, 2 \ldots$, then

$$N_\theta^{(n)} = \nu_{\phi\theta} N_\phi^{(n)} + (K_{11} - \nu_{\phi\theta}^2 K_{22}) \left[\frac{nU^{(n)} + V^{(n)} \cos\phi - W^{(n)} \sin\phi}{r_0} \right]$$

$$- N_{T\theta}^{(n)} + \nu_{\phi\theta} N_{T\phi}^{(n)} \tag{2-3}$$

$$M_\theta^{(n)} = \nu_{\phi\theta} M_\phi^{(n)} - \frac{(D_{11} - \nu_{\phi\theta}^2 D_{22})}{r_0} \left[\frac{nU^{(n)} \sin\phi - n^2 W^{(n)}}{r_0} + \Omega_\theta^{(n)} \cos\phi \right]$$

$$- M_{T\theta}^{(n)} + \nu_{\phi\theta} M_{T\phi}^{(n)}$$

$$M_{\phi\theta}^{(n)} = \left[\frac{-1}{\frac{r_0}{D_{33}} + \frac{\sin^2\phi}{r_0 K_{33}}} \right] \left\{ -2n\Omega_\theta^{(n)} + U^{(n)} \left(\frac{\cos\phi}{r_1} - \frac{\cos\phi \sin\phi}{r_0} \right) \right.$$

$$\left. + nV^{(n)} \left(\frac{\sin\phi}{r_0} + \frac{1}{r_1} \right) + 2nW^{(n)} \frac{\cos\phi}{r_0} + T_{\phi\theta}^{(n)} \frac{\sin\phi}{K_{33}} \right\}$$

$$N_{\phi\theta}^{(n)} = T_{\phi\theta}^{(n)} + M_{\phi\theta}^{(n)} \frac{\sin\phi}{r_0}$$

$$\Omega_\phi^{(n)} = \frac{nW^{(n)}}{r_0} - U^{(n)} \frac{\sin\phi}{r_0}$$

$$Q_\phi^{(n)} = J_\phi^{(n)} - \frac{nM_{\phi\theta}^{(n)}}{r_0}$$

$$
\begin{aligned}
Q_\theta^{(n)} = & \left\{ \frac{3\cos\phi}{r_0} - \frac{2\cos\phi\left(r_0 K_{33} + D_{33}\dfrac{\sin\phi}{r_1}\right)}{r_0^{2} K_{33} + D_{33}\sin^2\phi} \right\} M_{\phi\theta}^{(n)} + \left[\frac{\dfrac{-1}{r_1}}{\dfrac{r_0}{D_{33}} + \dfrac{\sin^2\phi}{r_0 K_{33}}} \right] \Bigg\{ -2n\Omega_{\theta,\phi}^{(n)} \\
& + U_{,\phi}^{(n)} \left(\frac{\cos\phi}{r_1} - \frac{\cos\phi \sin\phi}{r_0} \right) + U^{(n)} \left(\frac{\sin^2\phi}{r_0} - \frac{\cos^2\phi}{r_0} \right. \\
& \left. - \frac{\sin\phi}{r_1} - \frac{r_{1,\phi}\cos\phi}{r_1^{2}} + \frac{r_1 \cos^2\phi \sin\phi}{r_0^{2}} \right) + nV_{,\phi}^{(n)} \left(\frac{\sin\phi}{r_0} + \frac{1}{r_1} \right) \\
& + nV^{(n)} \left(\frac{\cos\phi}{r_0} - \frac{r_1 \sin\phi\cos\phi}{r_0^{2}} - \frac{r_{1,\phi}}{r_1^{2}} \right) \\
& + 2n\, W_{,\phi}^{(n)} \frac{\cos\phi}{r_0} - 2nW^{(n)} \left(\frac{\sin\phi}{r_0} + \frac{r_1\cos^2\phi}{r_0^{2}} \right) \\
& + T_{\phi\theta,\phi}^{(n)} \frac{\sin\phi}{K_{33}} + T_{\phi\theta}^{(n)} \frac{\cos\phi}{K_{33}} \Bigg\} - \frac{nM_\theta^{(n)}}{r_0} - m_\phi^{(n)}
\end{aligned}
$$

If $m_\phi^{(0)} = f_\theta^{(0)} = 0$, then $Q_\theta^{(0)} = 0$.

For the axisymmetric case, the problem is defined by only one harmonic, $n = 0$. For an unsymmetric problem, the required harmonics may be superimposed, and again, only one set of ordinary differential equations need be solved at a time.

SECTION 3

PROGRAMMED EQUATIONS FOR VARIOUS CONFIGURATIONS

The differential equations actually programmed are given below for shell shapes using the coordinate angle ϕ, and for the cylinder and cone, which use the meridional distance coordinate s.

INDEPENDENT VARIABLE, ϕ

$$\frac{T_{\phi\theta,\phi}^{(n)}}{r_1} = -2T_{\phi\theta}^{(n)} \frac{\cos\phi}{r_0} + n\frac{N_\theta^{(n)}}{r_0} - nM_\theta^{(n)} \frac{\sin\phi}{r_0^2} - M_{\phi\theta}^{(n)} \frac{\cos\phi}{r_0}\left[\frac{1}{r_1} - \frac{\sin\phi}{r_0}\right]$$

$$- f_\theta^{(n)} - m_\phi^{(n)} \frac{\sin\phi}{r_0} \qquad (3\text{-}1)$$

$$\frac{N_{\phi,\phi}^{(n)}}{r_1} = -N_\phi^{(n)} \frac{\cos\phi}{r_0} + N_\theta^{(n)} \frac{\cos\phi}{r_0} - n\frac{T_{\phi\theta}^{(n)}}{r_0}$$

$$- nM_{\phi\theta}^{(n)}\left[\frac{\sin\phi}{r_0^2} + \frac{1}{r_0 r_1}\right] + \frac{J_\phi^{(n)}}{r_1} - f_\phi^{(n)}$$

$$\frac{J_{\phi,\phi}^{(n)}}{r_1} = -J_\phi^{(n)} \frac{\cos\phi}{r_0} - N_\theta^{(n)} \frac{\sin\phi}{r_0} - \frac{N_\phi^{(n)}}{r_1} + n^2\frac{M_\theta^{(n)}}{r_0^2}$$

$$- 2n\, M_{\phi\theta}^{(n)} \frac{\cos\phi}{r_0^2} - f_\zeta^{(n)} + \frac{nm_\phi^{(n)}}{r_0}$$

$$\frac{M_{\phi,\phi}^{(n)}}{r_1} = M_\theta^{(n)} \frac{\cos\phi}{r_0} - M_\phi^{(n)} \frac{\cos\phi}{r_0} - 2n\frac{M_{\phi\theta}^{(n)}}{r_0} + J_\phi^{(n)} + m_\theta^{(n)}$$

$$\frac{U_{,\phi}^{(n)}}{r_1} = U^{(n)} \frac{\cos\phi}{r_0} + n\frac{V^{(n)}}{r_0} + \frac{T_{\phi\theta}^{(n)}}{K_{33}} + \frac{M_{\phi\theta}^{(n)} \sin\phi}{r_0 K_{33}}$$

$$\frac{V_{,\phi}^{(n)}}{r_1} = \frac{W^{(n)}}{r_1} + \left(K_{22} - \nu_{\theta\phi}^{2} K_{11}\right)^{-1} \left\{ N_{\phi}^{(n)} - \nu_{\theta\phi} N_{\theta}^{(n)} + N_{T\phi}^{(n)} - \nu_{\theta\phi} N_{T\theta}^{(n)} \right\}$$

$$\frac{W_{,\phi}^{(n)}}{r_1} = \Omega_{\theta}^{(n)} - \frac{V^{(n)}}{r_1}$$

$$\frac{\Omega_{\theta,\phi}^{(n)}}{r_1} = \left(D_{22} - \nu_{\theta\phi}^{2} D_{11}\right)^{-1} \left\{ -M_{\phi}^{(n)} + \nu_{\theta\phi} M_{\theta}^{(n)} - M_{T\phi}^{(n)} + \nu_{\theta\phi} M_{T\theta}^{(n)} \right\}$$

$$N_{\theta}^{(n)} = \nu_{\phi\theta} N_{\phi}^{(n)} + \left(K_{11} - \nu_{\phi\theta}^{2} K_{22}\right) \left[\frac{nU^{(n)} + V^{(n)} \cos\phi - W^{(n)} \sin\phi}{r_0} \right]$$

$$- N_{T\theta}^{(n)} + \nu_{\phi\theta} N_{T\phi}^{(n)}$$

$$M_{\theta}^{(n)} = \nu_{\phi\theta} M_{\phi}^{(n)} - \frac{\left(D_{11} - \nu_{\phi\theta}^{2} D_{22}\right)}{r_0} \left[\frac{nU^{(n)} \sin\phi - n^2 W^{(n)}}{r_0} + \Omega_{\theta}^{(n)} \cos\phi \right]$$

$$- M_{T\theta}^{(n)} + \nu_{\phi\theta} M_{T\phi}^{(n)}$$

$$M_{\phi\theta}^{(n)} = \left[\frac{-1}{\frac{r_0}{D_{33}} + \frac{\sin^2\phi}{r_0 K_{33}}} \right] \left\{ -2n\Omega_{\theta}^{(n)} + U^{(n)} \left(\frac{\cos\phi}{r_1} - \frac{\cos\phi \sin\phi}{r_0} \right) \right.$$

$$\left. + nV^{(n)} \left(\frac{\sin\phi}{r_0} + \frac{1}{r_1} \right) + 2nW^{(n)} \frac{\cos\phi}{r_0} + T_{\phi\theta}^{(n)} \frac{\sin\phi}{K_{33}} \right\}$$

$$N_{\phi\theta}^{(n)} = T_{\phi\theta}^{(n)} + M_{\phi\theta}^{(n)} \frac{\sin\phi}{r_0}$$

$$\Omega_{\phi}^{(n)} = \frac{nW^{(n)}}{r_0} - U^{(n)} \frac{\sin\phi}{r_0}$$

$$Q_{\phi}^{(n)} = J_{\phi}^{(n)} - \frac{nM_{\phi\theta}^{(n)}}{r_0}$$

$$\begin{aligned}
Q_{\theta}^{(n)} = & \left\{ \frac{3\cos\phi}{r_0} - \frac{2\cos\phi\left(r_0 K_{33} + D_{33}\dfrac{\sin\phi}{r_1}\right)}{r_0^{2} K_{33} + D_{33}\sin^2\phi} \right\} M_{\phi\theta}^{(n)} \\
& + \left[\frac{\dfrac{-1}{r_1}}{\dfrac{r_0}{D_{33}} + \dfrac{\sin^2\phi}{r_0 K_{33}}} \right] \Bigg\{ -2n\Omega_{\theta,\phi}^{(n)} + U_{,\phi}^{(n)} \left(\frac{\cos\phi}{r_1} - \frac{\cos\phi\sin\phi}{r_0} \right) \\
& + U^{(n)} \left(\frac{\sin^2\phi}{r_0} - \frac{\cos^2\phi}{r_0} - \frac{\sin\phi}{r_1} - \frac{r_{1,\phi}\cos\phi}{r_1^{2}} + \frac{r_1\cos^2\phi\sin\phi}{r_0^{2}} \right) \\
& + nV_{,\phi}^{(n)} \left(\frac{\sin\phi}{r_0} + \frac{1}{r_1} \right) + nV^{(n)} \left(\frac{\cos\phi}{r_0} - \frac{r_1\sin\phi\cos\phi}{r_0^{2}} - \frac{r_{1,\phi}}{r_1^{2}} \right) \\
& + 2nW_{,\phi}^{(n)} \frac{\cos\phi}{r_0} - 2nW^{(n)} \left(\frac{\sin\phi}{r_0} + \frac{r_1\cos^2\phi}{r_0^{2}} \right) \\
& + T_{\phi\theta,\phi}^{(n)} \frac{\sin\phi}{K_{33}} + T_{\phi\theta}^{(n)} \frac{\cos\phi}{K_{33}} \Bigg\} - \frac{nM_{\theta}^{(n)}}{r_0} - m_{\phi}^{(n)}
\end{aligned}$$

If $m_{\phi}^{(0)} = f_{\theta}^{(0)} = 0$, then $Q_{\theta}^{(0)} = 0$.

When using Equations 3-1, it is necessary to specify the functions $r_0(\phi)$, $r_1(\phi)$, and $r_1(\phi)_{,\phi}$. For completeness, these functions and $r_2(\phi)$ are given for the various analytic forms.

Ellipsoidal (Figure 3-1): When $\beta = b/a$, $\beta = 1$ for a sphere, and "a" and "β" are given data, then:

$$r_2 = a(\sin^2\phi + \beta^2\cos^2\phi)^{-\frac{1}{2}} \tag{3-2}$$

$$r_0 = r_2 \sin\phi$$

$$r_1 = \frac{\beta^2}{a^2}\, r_2{}^3$$

$$r_{1,\phi} = 3\left(\frac{r_2\beta}{a}\right)^2 \frac{\cos\phi}{\sin^2\phi}\,(r_1\sin\phi - r_0)$$

$$r_{1,\phi} = 0 \text{ for sphere}$$

Ogival (Figure 3-2): When "r_1" and "c" are given data, then:

$$r_2 = r_1 - \frac{c}{\sin\phi} \tag{3-3}$$

$$r_0 = r_2 \sin\phi$$

$$r_{1,\phi} = 0$$

"Modified" Ellipse Shape (Figure 3-3): When "n" and "a" are given data, and the range is $1 \geq n > -1$, then:

$$r_2 = a\left(\frac{2}{1 + \sin^{(1+n)}\phi}\right)^{\frac{1}{1+n}} \tag{3-4}$$

$$r_0 = r_2 \sin\phi$$

$$r_1 = \frac{a}{2}\left(\frac{r_2}{a}\right)^{2+n}$$

$$r_{1,\phi} = -(2+n)\,\frac{a}{4}\sin^{(n)}\phi\,\cos\phi\left(\frac{2}{1+\sin^{(1+n)}\phi}\right)^{\frac{3+n}{1+n}}$$

for

$n = 1$	$b/a = 0.707$
$n = 0$	$b/a = 0.666$
$n = -\frac{1}{2}$	$b/a = 0.639$
$n \to -1$	$b/a \to 0.618$

Parabolic (Figure 3-4): When the given parabolic equation is $z = f_1 + f_2 r + f_3 r^2$ and "f_1", "f_2", and "f_3" are given input data, then:

$$r_0 = \frac{-\tan\phi + f_2}{2f_3} \tag{3-5}$$

$$r_2 = \frac{r_0}{\sin\phi}$$

$$r_1 = -\frac{\sec^3\phi}{2f_3}$$

$$r_{1,\phi} = -\frac{3\sec^4\phi\,\sin\phi}{2f_3}$$

CYLINDRICAL, s MEASURED OPPOSITE TO GLOBAL COORDINATE Z (Figure 3-5)

$$\frac{dT_{\phi\theta}^{(n)}}{ds} = +n\,\frac{N_\theta^{(n)}}{r_0} - n\,\frac{M_\theta^{(n)}}{r_0^{\,2}} - f_\theta^{(n)} - \frac{1}{r_0}\,m_\phi^{(n)} \tag{3-6}$$

$$\frac{dN_\phi^{(n)}}{ds} = -n\,\frac{T_{\phi\theta}^{(n)}}{r_0} - n\,\frac{M_{\phi\theta}^{(n)}}{r_0^{\,2}} - f_\phi^{(n)}$$

$$\frac{dQ_\zeta^{(n)}}{ds} = -\frac{N_\theta^{(n)}}{r_0} + n^2\left[\frac{M_\theta^{(n)}}{r_0^{\,2}}\right] - f_\zeta^{(n)} + n\,\frac{m_\phi^{(n)}}{r_0}$$

$$\frac{dN_{\phi}^{(n)}}{ds} = -\frac{2n\, M_{\phi\theta}^{(n)}}{r_0} + J_{\phi}^{(n)} + m_{\theta}^{(n)}$$

$$\frac{dU^{(n)}}{ds} = \frac{nV^{(n)}}{r_0} + \frac{T_{\phi\theta}^{(n)}}{K_{33}} + \frac{M_{\phi\theta}^{(n)}}{r_0\, K_{33}}$$

$$\frac{dV^{(n)}}{ds} = \left(K_{22} - \nu_{\theta\phi}^{\;2} K_{11}\right)^{-1} \left\{ N_{\phi}^{(n)} - \nu_{\theta\phi} N_{\theta}^{(n)} + N_{T\phi}^{(n)} - \nu_{\theta\phi} N_{T\theta}^{(n)} \right\}$$

$$\frac{dW^{(n)}}{ds} = \Omega_{\theta}^{(n)}$$

$$\frac{d\Omega_{\theta}^{(n)}}{ds} = \left(D_{22} - \nu_{\theta\phi}^{\;2} D_{11}\right)^{-1} \left\{ -M_{\phi}^{(n)} + \nu_{\theta\phi} M_{\theta}^{(n)} - M_{T\phi}^{(n)} + \nu_{\theta\phi} M_{T\theta}^{(n)} \right\}$$

$$N_{\theta}^{(n)} = \nu_{\phi\theta} N_{\phi}^{(n)} + \left(K_{11} - \nu_{\phi\theta}^{\;2} K_{22}\right)\left[\frac{nU^{(n)} - W^{(n)}}{r_0}\right] - N_{T\theta}^{(n)} + \nu_{\phi\theta} N_{T\phi}^{(n)}$$

$$M_{\theta}^{(n)} = \nu_{\phi\theta} M_{\phi}^{(n)} - \frac{\left(D_{11} - \nu_{\phi\theta}^{\;2} D_{22}\right)}{r_0} \left[\frac{nU^{(n)} - n^2 W^{(n)}}{r_0}\right] - M_{T\theta}^{(n)}$$

$$+ \nu_{\phi\theta} M_{T\phi}^{(n)}$$

$$M_{\phi\theta}^{(n)} = \left[\frac{-1}{\dfrac{r_0}{D_{33}} + \dfrac{1}{r_0\, K_{33}}}\right]\left\{-2n\Omega_{\theta}^{(n)} + \frac{nV^{(n)}}{r_0} + \frac{T_{\phi\theta}^{(n)}}{K_{33}}\right\}$$

$$N_{\phi\theta}^{(n)} = T_{\phi\theta}^{(n)} + \frac{M_{\phi\theta}^{(n)}}{r_0}$$

$$\Omega_{z}^{(n)} = \frac{nW^{(n)}}{r_0} - \frac{U^{(n)}}{r_0}$$

$$Q_\phi^{(n)} = J_\phi^{(n)} - \frac{nM_{\phi\theta}^{(n)}}{r_0}$$

$$Q_\theta^{(n)} = \left[\frac{-1}{\frac{r_0}{D_{33}} + \frac{1}{r_0 K_{33}}}\right]\left\{-2n\frac{d\Omega_\theta^{(n)}}{ds} + \frac{n}{r_0}\frac{dV^{(n)}}{ds} + \frac{1}{K_{33}}\frac{dT_{\phi\theta}^{(n)}}{ds}\right\} - \frac{nM_\theta^{(n)}}{r_0}$$

$$- m_\phi^{(n)}$$

If $m_\phi^{(0)} = f_\theta^{(0)} = 0$, then $Q_\theta^{(0)} = 0$.

CONICAL, s MEASURED ALONG MERIDIAN FROM APEX (Figure 3-6)

$$\frac{dT_{\phi\theta}^{(n)}}{ds} = -\frac{2T_{\phi\theta}^{(n)}}{s} + \frac{nN_\theta^{(n)}}{s\cos\phi} - \frac{nM_\theta^{(n)}\sin\phi}{s^2\cos^2\phi} + \frac{M_{\phi\theta}^{(n)}\tan\phi}{s^2} - f_\theta^{(n)}$$

$$- \frac{m_\phi^{(n)}\tan\phi}{s} \tag{3-7}$$

$$\frac{dN_\phi^{(n)}}{ds} = -\frac{N_\phi^{(n)}}{s} + \frac{N_\theta^{(n)}}{s} - \frac{nT_{\phi\theta}^{(n)}}{s\cos\phi} - \frac{nM_{\phi\theta}^{(n)}\sin\phi}{s^2\cos^2\phi} - f_\phi^{(n)}$$

$$\frac{dJ_\phi^{(n)}}{ds} = -\frac{J_\phi^{(n)}}{s} - \frac{N_\theta^{(n)}\tan\phi}{s} + \frac{n^2M_\theta^{(n)}}{s^2\cos^2\phi} - \frac{2nM_{\phi\theta}^{(n)}}{s^2\cos\phi}$$

$$- f_\zeta^{(n)} + \frac{nm_\phi^{(n)}}{s\cos\phi}$$

$$\frac{dM_\phi^{(n)}}{ds} = \frac{M_\theta^{(n)}}{s} - \frac{M_\phi^{(n)}}{s} - \frac{2nM_{\phi\theta}^{(n)}}{s\cos\phi} + J_\phi^{(n)} + m_\theta^{(n)}$$

$$\frac{dU^{(n)}}{ds} = \frac{U^{(n)}}{s} + \frac{nV^{(n)}}{s\cos\phi} + \frac{T_{\phi\theta}{}^{(n)}}{K_{33}} + \frac{M_{\phi\theta}{}^{(n)}\tan\phi}{K_{33}s}$$

$$\frac{dV^{(n)}}{ds} = \left(K_{22} - \nu_{\theta\phi}{}^{2}\, K_{11}\right)^{-1} \left\{ N_{\phi}{}^{(n)} - \nu_{\theta\phi}N_{\theta}{}^{(n)} + N_{T\phi}{}^{(n)} - \nu_{\theta\phi}N_{T\theta}{}^{(n)} \right\}$$

$$\frac{dW^{(n)}}{ds} = \Omega_{\theta}{}^{(n)}$$

$$\frac{d\Omega_{\theta}{}^{(n)}}{ds} = \left(D_{22} - \nu_{\theta\phi}{}^{2}\, D_{11}\right)^{-1} \left\{ -M_{\phi}{}^{(n)} + \nu_{\theta\phi}M_{\theta}{}^{(n)} - M_{T\phi}{}^{(n)} + \nu_{\theta\phi}M_{T\theta}{}^{(n)} \right\}$$

$$N_{\theta}{}^{(n)} = \nu_{\phi\theta}N_{\phi}{}^{(n)} + \left(K_{11} - \nu_{\phi\theta}{}^{2}\, K_{22}\right)\left[\frac{nU^{(n)} + V^{(n)}\cos\phi - W^{(n)}\sin\phi}{s\cos\phi}\right]$$
$$- N_{T\theta}{}^{(n)} + \nu_{\phi\theta}N_{T\phi}{}^{(n)}$$

$$M_{\theta}{}^{(n)} = \nu_{\phi\theta}M_{\phi}{}^{(n)} - \left(\frac{D_{11} - \nu_{\phi\theta}{}^{2}\, D_{22}}{s\cos\phi}\right)\left[\frac{nU^{(n)}\sin\phi - n^{2}W^{(n)}}{s\cos\phi} + \Omega_{\theta}{}^{(n)}\cos\phi\right]$$
$$- M_{T\theta}{}^{(n)} + \nu_{\phi\theta}M_{T\phi}{}^{(n)}$$

$$M_{\phi\theta}{}^{(n)} = \left[\frac{-1}{\dfrac{s\cos\phi}{D_{33}} + \dfrac{\sin^{2}\phi}{K_{33}s\cos\phi}}\right]\left\{ -2n\Omega_{\theta}{}^{(n)} - \frac{U^{(n)}\sin\phi}{s} + \frac{nV^{(n)}\tan\phi}{s} + \frac{2nW^{(n)}}{s}\right.$$
$$\left. + \frac{T_{\phi\theta}{}^{(n)}\sin\phi}{K_{33}} \right\}$$

$$N_{\phi\theta}{}^{(n)} = T_{\phi\theta}{}^{(n)} + \frac{M_{\phi\theta}{}^{(n)}}{s}\tan\phi$$

$$\Omega_{\phi}{}^{(n)} = \frac{nW^{(n)}}{s\cos\phi} - \frac{U^{(n)}}{s}\tan\phi$$

$$Q_{\phi}{}^{(n)} = J_{\phi}{}^{(n)} - \frac{nM_{\phi\theta}{}^{(n)}}{s\cos\phi}$$

$$Q_\theta^{(n)} = \left\{ \frac{3}{s} - \frac{2K_{33} s \cos^2\phi}{K_{33} s^2 \cos^2\phi + D_{33} \sin^2\phi} \right\} M_{\phi\theta}^{(n)} + \left[\frac{-1}{\frac{s \cos\phi}{D_{33}} + \frac{\sin^2\phi}{K_{33} s \cos\phi}} \right] \left\{ -2n \frac{d\Omega_\theta^{(n)}}{ds} \right.$$

$$- \frac{dU^{(n)}}{ds} \frac{\sin\phi}{s} + U^{(n)} \frac{\sin\phi}{s^2} + n \frac{dV^{(n)}}{ds} \frac{\tan\phi}{s} - \frac{nV^{(n)} \tan\phi}{s^2}$$

$$\left. + \frac{2n}{s} \frac{dW^{(n)}}{ds} - \frac{2nW^{(n)}}{s^2} + \frac{dT_{\phi\theta}^{(n)}}{ds} \frac{\sin\phi}{K_{33}} \right\} - \frac{nM_\theta^{(n)}}{s \cos\phi} - m_\phi^{(n)}$$

If $m_\phi^{(0)} = f_\theta^{(0)} = 0$, then $Q_\theta^{(0)} = 0$.

All the above equations are written in terms of stiffness parameters (K and D) rather than explicit geometry. This is due to the fact that a variety of crossection geometries are to be considered, specifically those described in Figure 1-4. However, one more option is available (described in detail in Section 4 and Reference 7): that of inputing the K's and D's representing any shell wall construction directly into the equations. With this option there becomes available an analysis for a great multitude of shell wall constructions. (In this respect refer to Section 4.) With all the geometries available, it becomes necessary to calculate thermal resultants separately.

As noted in Section 1, Equation 1-6, the definitions of the thermal resultants are:

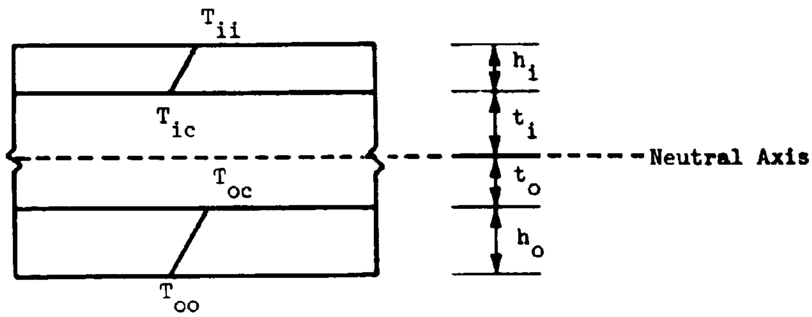
$$N_{T\phi} = \int \frac{E_\phi (\alpha_\phi + \nu_{\phi\theta} \alpha_\theta) T}{1 - \nu_{\phi\theta} \nu_{\theta\phi}} d\zeta \qquad M_{T\phi} = \int \frac{E_\phi (\alpha_\phi + \nu_{\phi\theta} \alpha_\theta) T}{1 - \nu_{\phi\theta} \nu_{\theta\phi}} \zeta d\zeta$$

$$N_{T\theta} = \int \frac{E_\epsilon (\alpha_\theta + \nu_{\theta\phi} \alpha_\phi) T}{1 - \nu_{\phi\theta} \nu_{\theta\phi}} d\zeta \qquad M_{T\theta} = \int \frac{E_\theta (\alpha_\theta + \nu_{\theta\phi} \alpha_\phi) T}{1 - \nu_{\phi\theta} \nu_{\theta\phi}} \zeta d\zeta$$

The temperature is defined to vary linearly as follows:

$$T_i = (T_{ii} - T_{ic}) \left(\frac{\zeta - t_i}{h_i} \right) + T_{ic} - \bar{T} \qquad (3\text{-}8)$$

$$T_o = (T_{oc} - T_{oo}) \left(\frac{\zeta + t_o}{h_o} \right) + T_{oc} - \bar{T}$$



where $\bar{T}$ is the stress-free temperature. Combining Equations 3-8 and 1-6, the necessary thermal resultants are obtained.

ORTHOTROPIC SINGLE LAYER

$$N_{T\theta} = \frac{E_\theta \, (\alpha_\theta + \nu_{\theta\phi} \, \alpha_\phi)}{1 - \nu_{\phi\theta} \, \nu_{\theta\phi}} \left(\frac{h_i}{4} \right) \left[T_{ii} + T_{ic} + T_{oc} + T_{oo} - 4\bar{T} \right] \qquad (3\text{-}9)$$

$$N_{T\phi} = \frac{E_\phi \, (\alpha_\phi + \nu_{\phi\theta} \, \alpha_\theta)}{1 - \nu_{\phi\theta} \, \nu_{\theta\phi}} \left(\frac{h_i}{4} \right) \left[T_{ii} + T_{ic} + T_{oc} + T_{oo} - 4\bar{T} \right]$$

$$M_{T\theta} = \frac{E_\theta \, (\alpha_\theta + \nu_{\theta\phi} \, \alpha_\phi)}{1 - \nu_{\phi\theta} \, \nu_{\theta\phi}} \left(\frac{h_i^2}{24} \right) \left[2T_{ii} + T_{ic} - T_{oc} - 2T_{oo} \right]$$

$$M_{T\phi} = \frac{E_\phi \, (\alpha_\phi + \nu_{\phi\theta} \, \alpha_\theta)}{1 - \nu_{\phi\theta} \, \nu_{\theta\phi}} \left(\frac{h_i^2}{24} \right) \left[2T_{ii} + T_{ic} - T_{oc} - 2T_{oo} \right]$$

EQUAL FACE SHEET SANDWICH

$$N_{T\theta} = \frac{E_\theta\ (\alpha_\theta + \nu_{\theta\phi}\ \alpha_\phi)}{1 - \nu_{\phi\theta}\ \nu_{\theta\phi}} \left(\frac{h_i}{2}\right) \left[T_{ii} + T_{ic} + T_{oc} + T_{oo} - 4\bar{T}\right] \tag{3-10}$$

$$N_{T\phi} = \frac{E_\phi\ (\alpha_\phi + \nu_{\phi\theta}\ \alpha_\theta)}{1 - \nu_{\phi\theta}\ \nu_{\theta\phi}} \left(\frac{h_i}{2}\right) \left[T_{ii} + T_{ic} + T_{oc} + T_{oo} - 4\bar{T}\right]$$

$$M_{T\theta} = \frac{E_\theta\ (\alpha_\theta + \nu_{\theta\phi}\ \alpha_\phi)}{1 - \nu_{\phi\theta}\ \nu_{\theta\phi}} \left(\frac{h_i}{2}\right) \left[\frac{h_i}{3} \left(2\ T_{ii} + T_{ic} - T_{oc} - 2T_{oo}\right) + \frac{t}{2} \left(T_{ii} + T_{ic} - T_{oc} - T_{oo}\right)\right]$$

$$M_{T\phi} = \frac{E_\phi\ (\alpha_\phi + \nu_{\phi\theta}\ \alpha_\theta)}{1 - \nu_{\phi\theta}\ \nu_{\theta\phi}} \left(\frac{h_i}{2}\right) \left[\frac{h_i}{3} \left(2T_{ii} + T_{ic} - T_{oc} - 2T_{oo}\right) + \frac{t}{2} \left(T_{ii} + T_{ic} - T_{oc} - T_{oo}\right)\right]$$

UNEQUAL FACE SHEET SANDWICH

$$N_{T\theta} = \frac{E_\theta\ (\alpha_\theta + \nu_{\theta\phi}\ \alpha_\phi)}{1 - \nu_{\phi\theta}\ \nu_{\theta\phi}} \left[\frac{h_i}{2} (T_{ii} + T_{ic} - 2\bar{T}) + \frac{h_o}{2} (T_{oc} + T_{oo} - 2\bar{T})\right] \tag{3-11}$$

$$N_{T\phi} = \frac{E_\phi\ (\alpha_\phi + \nu_{\phi\theta}\ \alpha_\theta)}{1 - \nu_{\phi\theta}\ \nu_{\theta\phi}} \left[\frac{h_i}{2} (T_{ii} + T_{ic} - 2\bar{T}) + \frac{h_o}{2} (T_{oc} + T_{oo} - 2\bar{T})\right]$$

$$M_{T\theta} = \frac{E_\theta\ (\alpha_\theta + \nu_{\theta\phi}\ \alpha_\phi)}{1 - \nu_{\phi\theta}\ \nu_{\theta\phi}} \left[\frac{h_i^2}{6} (2T_{ii} + T_{ic} - 3\bar{T}) - \frac{h_o^2}{6} (2T_{oo} + T_{oc} - 3\bar{T}) + \frac{t_i h_i}{2} (T_{ii} + T_{ic} - 2\bar{T}) - \frac{t_o h_o}{2} (T_{oo} + T_{oc} - 2\bar{T})\right]$$

$$M_{T\phi} = \frac{E_\phi\ (\alpha_\phi + \nu_{\phi\theta}\ \alpha_\theta)}{1 - \nu_{\phi\theta}\nu_{\theta\phi}} \left[\frac{h_i^2}{6} (2T_{ii} + T_{ic} - 3\bar{T}) - \frac{h_o^2}{6} (2T_{oo} + T_{oc} - 3\bar{T}) + \frac{t_i h_i}{2} (T_{ii} + T_{ic} - 2\bar{T}) - \frac{t_o h_o}{2} (T_{oo} + T_{oc} - 2\bar{T})\right]$$

where:

$$t_i = \bar{\zeta}_i - h_i = \frac{h_o^2 - h_i^2 + 2h_o t}{2\,(h_i + h_o)}$$

$$t_o = \bar{\zeta}_o - h_o = \frac{h_i^2 - h_o^2 + 2h_i t}{2\,(h_i + h_o)}$$

ARBITRARY STIFFNESS PARAMETERS (K AND D)

Since the geometry is not known in this case, certain assumptions are necessary in order to calculate the thermal resultants. The given stiffnesses are set equal to equivalent single sheet stiffnesses:

$$K_{11} = \frac{E_{\theta\ eq}\, h_{\theta\ eq}}{1 - \nu_{\phi\theta}\,\nu_{\theta\phi}} \qquad D_{11} = \frac{-E_{\theta\ eq}\, h_{\theta\ eq}^3}{12\,(1 - \nu_{\phi\theta}\,\nu_{\theta\phi})}$$

In this way $E_{\theta\ eq}$ and $h_{\theta\ eq}$ and similarly in the ϕ direction, $E_{\phi\ eq}$ and $h_{\phi\ eq}$ can be calculated. Substituting the values thus obtained into Equation 3-9:

$$N_{T\theta} = \frac{K_{11}\,(\alpha_\theta + \nu_{\theta\phi}\,\alpha_\phi)}{4}\left[T_{ii} + T_{ic} + T_{oc} + T_{oo} - 4\bar{T}\right] \qquad (3\text{-}12)$$

$$N_{T\phi} = \frac{K_{22}\,(\alpha_\phi + \nu_{\phi\theta}\,\alpha_\theta)}{4}\left[T_{ii} + T_{ic} + T_{oc} + T_{oo} - 4\bar{T}\right]$$

$$M_{T\theta} = \frac{(-K_{11}D_{11})^{\frac{1}{2}}\,(\alpha_\theta + \nu_{\theta\phi}\,\alpha_\phi)}{4\sqrt{3}}\left[2T_{ii} + T_{ic} - T_{oc} - 2T_{oo}\right]$$

$$M_{T\phi} = \frac{(-K_{22}D_{22})^{\frac{1}{2}}\,(\alpha_\phi + \nu_{\phi\theta}\,\alpha_\theta)}{4\sqrt{3}}\left[2T_{ii} + T_{ic} - T_{oc} - 2T_{oo}\right]$$

Thus, in this case the thermal resultants to be applied are obtained on the basis of equivalent stiffness sections, and are then applied to the original structure. In inputing the stiffness parameters (K and D) one must remember that they are functions of material properties, and thus functions of temperature. The negative definition of the bending stiffness, D, is used to be consistant with Section 4 and Appendix A.

STRESS CALCULATIONS

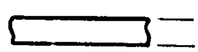
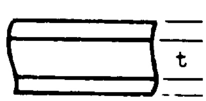
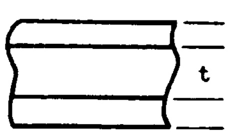
The stress formulas to be used are given below:

$$\sigma_{\theta_{in}} = \frac{N_\theta}{\mathcal{A}} + \frac{M_\theta}{\mathcal{J}}\,\bar{\zeta}_{in} \qquad \sigma_{\theta_{out}} = \frac{N_\theta}{\mathcal{A}} - \frac{M_\theta}{\mathcal{J}}\,\bar{\zeta}_{out} \tag{3-13a}$$

$$\sigma_{\phi_{in}} = \frac{N_\phi}{\mathcal{A}} + \frac{M_\phi}{\mathcal{J}}\,\bar{\zeta}_{in} \qquad \sigma_{\phi_{out}} = \frac{N_\phi}{\mathcal{A}} - \frac{M_\phi}{\mathcal{J}}\,\bar{\zeta}_{out}$$

$$\tau_{\phi\theta_{in}} = \frac{N_{\phi\theta}}{\mathcal{A}} + \frac{M_{\phi\theta}}{\mathcal{J}}\,\bar{\zeta}_{in} \qquad \tau_{\phi\theta_{out}} = \frac{N_{\phi\theta}}{\mathcal{A}} - \frac{M_{\phi\theta}}{\mathcal{J}}\,\bar{\zeta}_{out}$$

$$\mathcal{A} = \frac{(1 - \nu_{\phi\theta}\,\nu_{\theta\phi})\,K_{11}}{E_\theta} \qquad \mathcal{J} = \frac{(1 - \nu_{\phi\theta}\,\nu_{\theta\phi})\,D_{11}}{E_\theta} \tag{3-13b}$$

Configuration	$\bar{\zeta}_{in}$	$\bar{\zeta}_{out}$
 Orthotropic	$\frac{h_i}{2}$	$\frac{h_i}{2}$
 equal face sheets	$\frac{t}{2} + h_i$	$\frac{t}{2} + h_i$
 unequal face sheets	$\frac{h_i^2 + h_o^2 + 2h_ih_o + 2h_ot}{2(h_i + h_o)}$	$\frac{h_i^2 + h_o^2 + 2h_ih_o + 2h_it}{2(h_i + h_o)}$

In addition, the Huber-von Mises-Hencky effective stresses will also be calculated.

$$\sigma_{F_{in}} = \sqrt{\sigma_{\theta_{in}}^{2} - \sigma_{\theta_{in}} \sigma_{\phi_{in}} + \sigma_{\phi_{in}}^{2} + 3\tau_{\phi\theta_{in}}^{2}} \qquad (3\text{-}14)$$

$$\sigma_{F_{out}} = \sqrt{\sigma_{\theta_{out}}^{2} - \sigma_{\theta_{out}} \sigma_{\phi_{out}} + \sigma_{\phi_{out}}^{2} + 3\tau_{\phi\theta_{out}}^{2}}$$

These stresses are not useful for design or failure criteria, since such criteria are material dependent. However, they are useful for comparison, since they combine all stress components in a consistent manner characterized by a single number. Stresses for the arbitrary stiffness parameter (K and D) case are calculated using the appropriate Hooke's Laws of shell, ring, or stringer.

Approximations to core transverse shear stresses in a sandwich shell will be calculated as follows:

$$\tau_{\zeta\phi} = \frac{Q_\phi}{t} \qquad (3\text{-}15)$$

$$\tau_{\zeta\theta} = \frac{Q_\theta}{t}$$

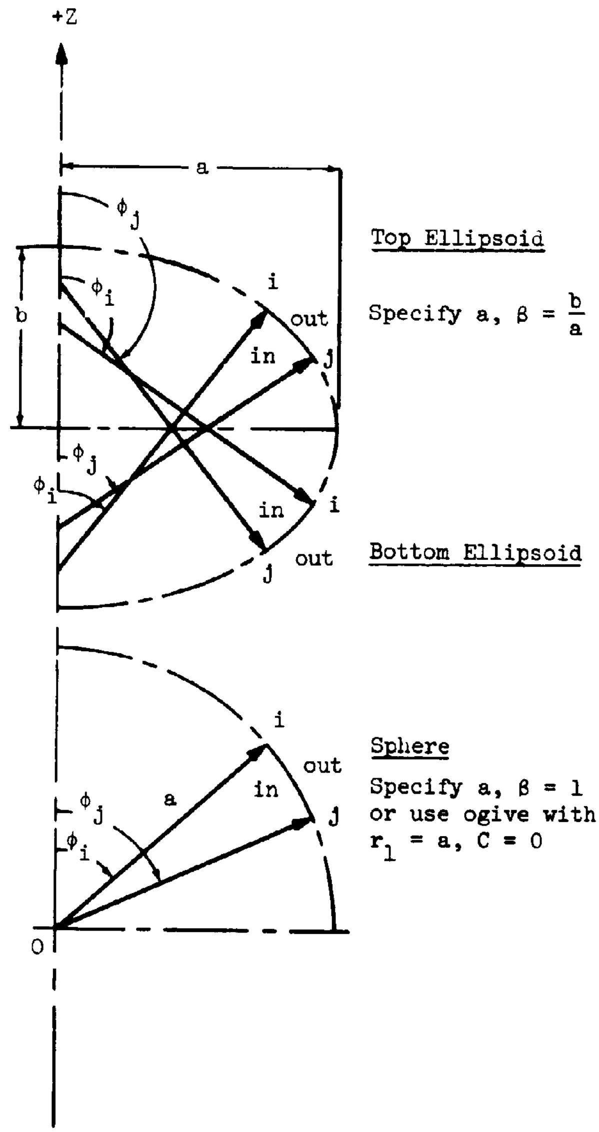


Figure 3-1. Ellipsoid

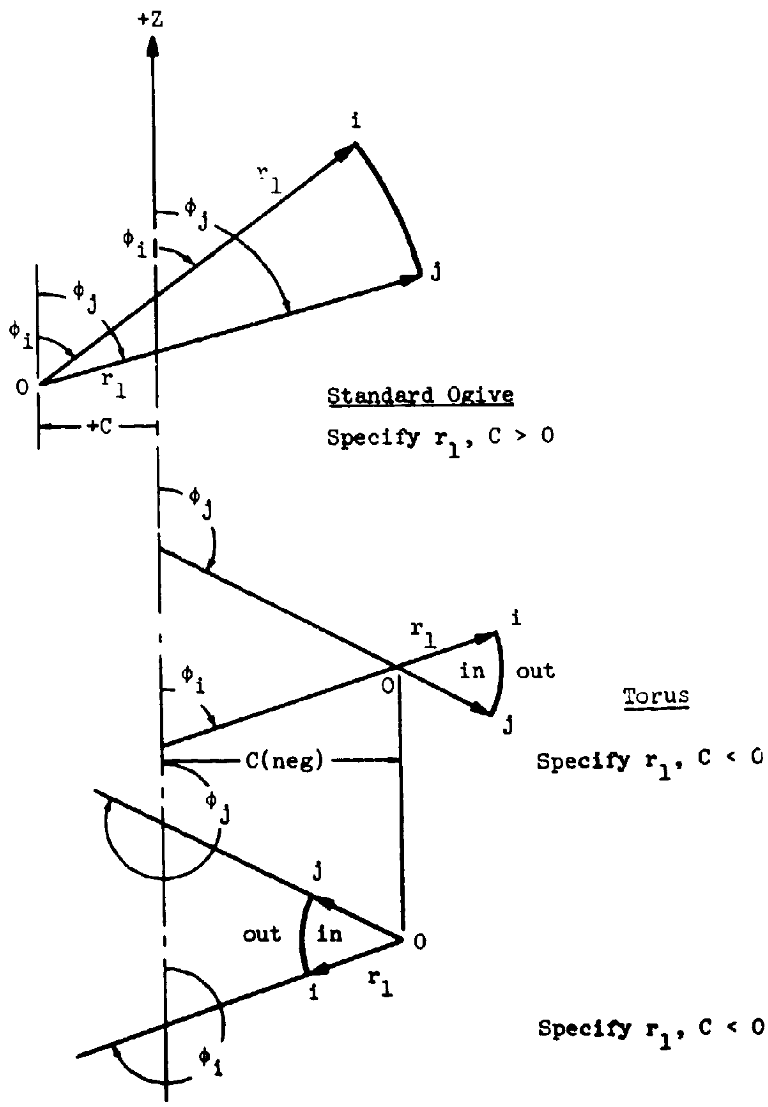


Figure 3-2. Ogive

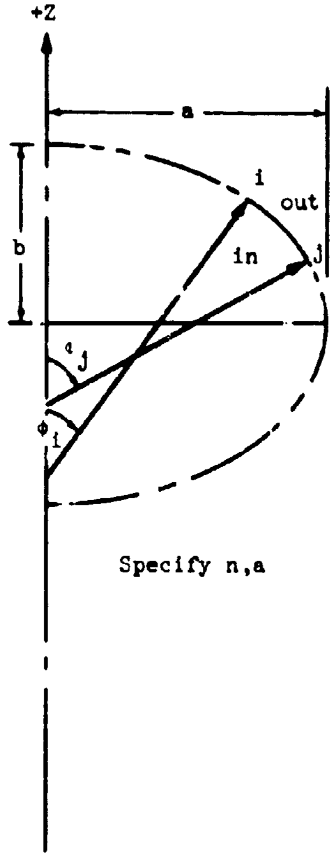


Figure 3-3. Modified Ellipsoid

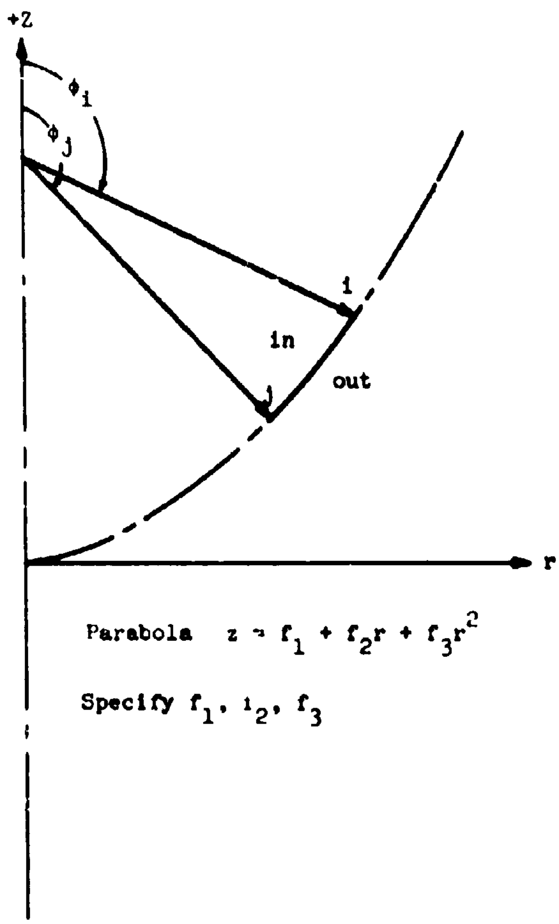


Figure 3-4. Paraboloid

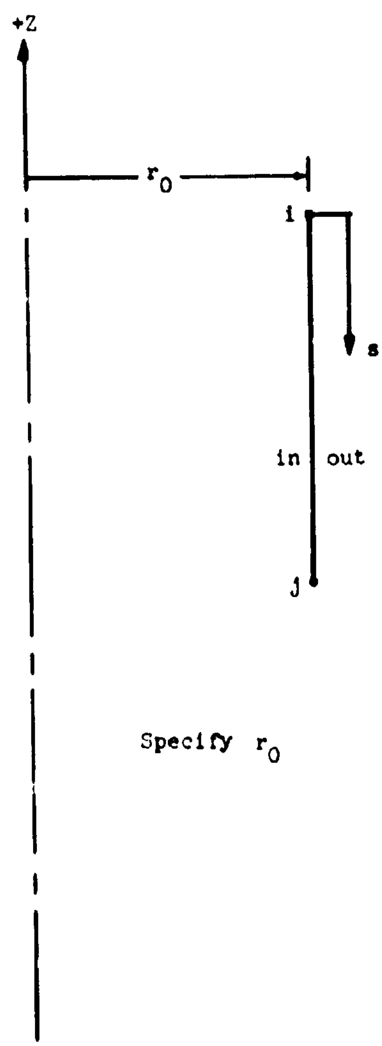


Figure 3-5. Cylinder

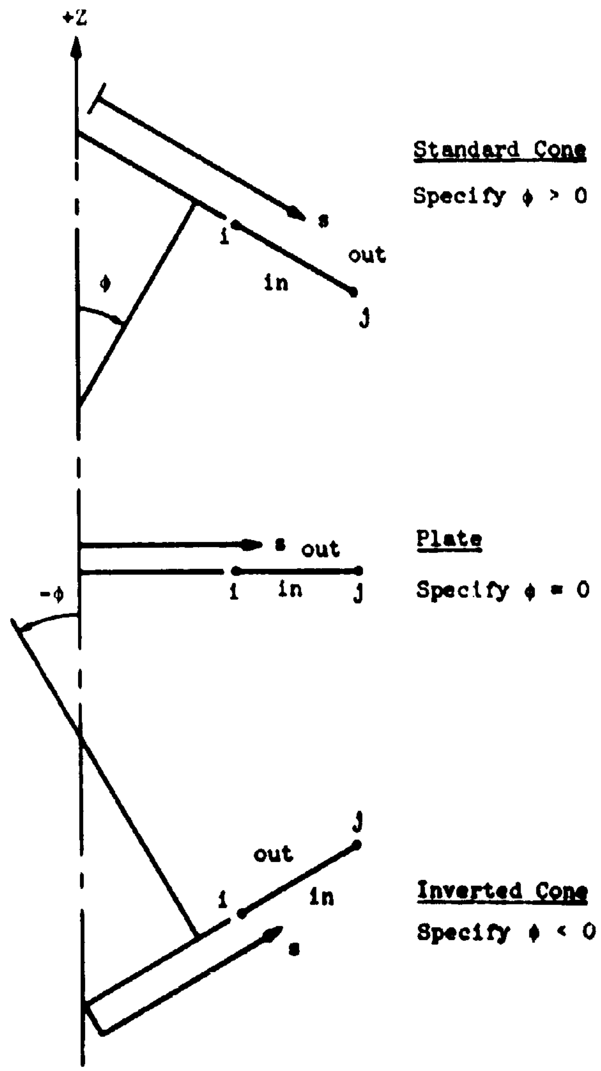


Figure 3-6. Cone

SECTION 4

EQUATIONS FOR ORTHOTROPICALLY REINFORCED SHELLS

As mentioned in Section 3, the differential equations are written in terms of stiffness parameters since a variety of wall crossection geometries are to be considered. There will also be cases allowed in the program, where the stiffness parameters are input directly. Since this option will most often be used to describe reinforced shells, the eccentricity of any reinforcement must be taken into consideration. To this end, some of the differential equations derived in Section 1 must be revised, since new stress-resultant to strain relationships must be used. In the case of orthotropic shell reinforcement, this relationship becomes as follows*:

$$N_\theta = K_{11}\epsilon_{\theta_o} + K_{12}\epsilon_{\phi_o} - C_{11}k_\theta - N_{T\theta} \tag{4-1a}$$

$$N_\phi = K_{22}\epsilon_{\phi_o} + K_{21}\epsilon_{\theta_o} - C_{22}k_\phi - N_{T\phi} \tag{4-1b}$$

$$N_{\phi\theta} = N_{\theta\phi} = K_{33}\gamma_{\phi\theta_o} \tag{4-1c}$$

$$M_\theta = D_{11}k_\theta + D_{12}k_\phi + C_{11}\epsilon_{\theta_o} - M_{T\theta} \tag{4-1d}$$

$$M_\phi = D_{22}k_\phi + D_{21}k_\theta + C_{22}\epsilon_{\phi_o} - M_{T\phi} \tag{4-1e}$$

$$M_{\phi\theta} = -M_{\theta\phi} = -2D_{33}k_{\phi\theta} \tag{4-1f}$$

As can be seen, due to the eccentric reinforcement, N_θ and N_ϕ are functions of curvatures, and M_θ and M_ϕ also depend upon membrane strain. This was not the case in Equation 1-8.

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* Refer to Appendix A for derivation and definitions of K, C, and D parameters for several cases.

The new Equation 4-1 thus requires a change in four of the original differential and auxiliary equations formed in Section 1. These changed equations will now be obtained. By combining Equations 4-1b and 4-1e first to eliminate k_ϕ and then again to eliminate ϵ_{ϕ_0}, the following differential equations can be obtained:

$$\frac{v_{,\phi}}{r_1} = \frac{w}{r_1} + \left(K_{22} + \frac{C_{22}^2}{D_{22}}\right)^{-1} \left\{ N_\phi + N_{T\phi} + \frac{C_{22}}{D_{22}} \left(M_\phi + M_{T\phi}\right) - \frac{K_{21}}{r_0} \left(u_{,\theta} + v \cos\phi \right.\right.$$

$$\left.\left. - w \sin\phi\right) - \frac{C_{22} D_{21}}{D_{22}} \left[\frac{1}{r_0^2}\left\{v_{,\theta\theta} + u_{,\theta} \sin\phi\right\} + \omega_\theta \frac{\cos\phi}{r_0}\right]\right\} \qquad (4\text{-}2a)$$

$$\frac{\omega_{\theta,\phi}}{r_1} = -\left(C_{22} + \frac{K_{22} D_{22}}{C_{22}}\right)^{-1} \left\{ N_\phi + N_{T\phi} - \frac{K_{22}}{C_{22}} \left(M_\phi + M_{T\phi}\right) - \frac{K_{21}}{r_0} \left(u_{,\theta} + v \cos\phi \right.\right.$$

$$\left.\left. - w \sin\phi\right) + \frac{K_{22} D_{21}}{C_{22}} \left[\frac{1}{r_0^2}\left\{v_{,\theta\theta} + u_{,\theta} \sin\phi\right\} + \omega_\theta \frac{\cos\phi}{r_0}\right]\right\} \qquad (4\text{-}2b)$$

Utilizing the above solutions in Equations 4-1a and 4-1d, the new auxiliary equations are obtained.

$$N_\theta = K_{12} \left(K_{22} + \frac{C_{22}^2}{D_{22}}\right)^{-1} \left\{ N_\phi + N_{T\phi} + \frac{C_{22}}{D_{22}} (M_\phi + M_{T\phi}) \right\} - N_{T\theta}$$

$$+ \left(\frac{K_{11}}{r_0} - \frac{K_{12} K_{21}}{r_0} \left[K_{22} + \frac{C_{22}^2}{D_{22}}\right]^{-1}\right)\left(u_{,\theta} + v \cos\phi - w \sin\phi\right)$$

$$- \left(C_{11} + \frac{K_{12} C_{22} D_{21}}{D_{22}} \left[K_{22} + \frac{C_{22}^2}{D_{22}}\right]^{-1}\right)\left(\frac{1}{r_0^2}\left\{v_{,\theta\theta}\right.\right.$$

$$\left.\left. + u_{,\theta} \sin\phi\right\} + \omega_\theta \frac{\cos\phi}{r_0}\right) \qquad (4\text{-}3a)$$

$$M_\theta = -D_{12}\left(C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right)^{-1}\left\{N_\phi + N_{T\phi} - \frac{K_{22}}{C_{22}}(M_\phi + M_{T\phi})\right\} - M_{T\theta}$$

$$+\left(\frac{C_{11}}{r_0} + \frac{D_{12}K_{21}}{r_0}\left[C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right]^{-1}\right)\left(u_{,\theta} + v\cos\phi - w\sin\phi\right)$$

$$+\left(D_{11} - \frac{D_{12}K_{22}D_{21}}{C_{22}}\left[C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right]^{-1}\right)\left(\frac{1}{r_0^2}\left\{w_{,\theta\theta} + u_{,\theta}\sin\phi\right\} + \omega_\theta\frac{\cos\phi}{r_0}\right) \tag{4-3b}$$

The Equations 4-2 and 4-3 are the four equations which must be substituted for their counterparts in the set of equations developed in Section 1, to adapt the earlier analysis to handle eccentric reinforcement. These revised equations are expanded in Fourier Series as described in Section 2, and presented for the necessary geometries below.

ϕ COORDINATE

$$\frac{V_{,\phi}^{(n)}}{r_1} = \frac{W^{(n)}}{r_1} + \left(K_{22} + \frac{C_{22}^2}{D_{22}}\right)^{-1}\left\{N_\phi^{(n)} + N_{T\phi}^{(n)} + \frac{C_{22}}{D_{22}}\left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right)\right.$$

$$- \frac{K_{21}}{r_0}\left(nU^{(n)} + V^{(n)}\cos\phi - W^{(n)}\sin\phi\right)$$

$$\left.- \frac{C_{22}D_{21}}{D_{22}}\left[\frac{1}{r_0^2}\left\{nU^{(n)}\sin\phi - n^2W^{(n)}\right\} + \Omega_\theta^{(n)}\frac{\cos\phi}{r_0}\right]\right\} \tag{4-4a}$$

$$\frac{\Omega_{\theta,\phi}^{(n)}}{r_1} = -\left(C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right)^{-1} \left\{ N_\phi^{(n)} + N_{T\phi}^{(n)} - \frac{K_{22}}{C_{22}} \left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right) \right.$$

$$- \frac{K_{21}}{r_0} \left(nU^{(n)} + V^{(n)} \cos\phi - W^{(n)} \sin\phi\right)$$

$$\left. + \frac{K_{22}D_{21}}{C_{22}} \left[\frac{1}{r_0^2} \left\{ nU^{(n)} \sin\phi - n^2 W^{(n)} \right\} + \Omega_\theta^{(n)} \frac{\cos\phi}{r_0} \right] \right\} \quad (4\text{-}4b)$$

$$N_\theta^{(n)} = K_{12}\left(K_{22} + \frac{C_{22}^2}{D_{22}}\right)^{-1} \left\{ N_\phi^{(n)} + N_{T\phi}^{(n)} + \frac{C_{22}}{D_{22}} \left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right) \right\}$$

$$- N_{T\theta}^{(n)} + \left(\frac{K_{11}}{r_0} - \frac{K_{12}K_{21}}{r_0} \left[K_{22} + \frac{C_{22}^2}{D_{22}} \right]^{-1} \right) \left(nU^{(n)} \right.$$

$$\left. + V^{(n)} \cos\phi - W^{(n)} \sin\phi \right) - \left(C_{11} + \frac{K_{12}C_{22}D_{21}}{D_{22}} \left[K_{22} \right. \right.$$

$$\left. \left. + \frac{C_{22}^2}{D_{22}} \right]^{-1} \right) \left(\frac{1}{r_0^2} \left\{ nU^{(n)} \sin\phi - n^2 W^{(n)} \right\} + \Omega_\theta^{(n)} \frac{\cos\phi}{r_0} \right) \quad (4\text{-}4c)$$

$$M_\theta^{(n)} = -D_{12} \left(C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right)^{-1} \left\{ N_\phi^{(n)} + N_{T\phi}^{(n)} - \frac{K_{22}}{C_{22}} \left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right) \right\} - M_{T\theta}^{(n)}$$

$$+ \left(\frac{C_{11}}{r_0} + \frac{D_{12}K_{21}}{r_0} \left[C_{22} + \frac{K_{22}D_{22}}{C_{22}} \right]^{-1} \right) \left(nU^{(n)} + V^{(n)} \cos\phi - W^{(n)} \sin\phi \right)$$

$$+ \left(D_{11} - \frac{D_{12}K_{22}D_{21}}{C_{22}} \left[C_{22} + \frac{K_{22}D_{22}}{C_{22}} \right]^{-1} \right) \left(\frac{1}{r_0^2} \left\{ nU^{(n)} \sin\phi \right. \right.$$

$$\left. \left. - n^2 W^{(n)} \right\} + \Omega_\theta^{(n)} \frac{\cos\phi}{r_0} \right) \quad (4\text{-}4d)$$

CYLINDER

$$\frac{dV^{(n)}}{ds} = \left(K_{22} + \frac{C_{22}^2}{D_{22}}\right)^{-1}\left\{N_\phi{}^{(n)} + N_{T\phi}{}^{(n)} + \frac{C_{22}}{D_{22}}\left(M_\phi{}^{(n)} + M_{T\phi}{}^{(n)}\right)\right.$$

$$\left. - \frac{K_{21}}{r_0}\left(nU^{(n)} - W^{(n)}\right) - \frac{C_{22}D_{21}}{D_{22}}\left[\frac{1}{r_0{}^2}\left\{nU^{(n)} - n^2W^{(n)}\right\}\right]\right\} \quad (4\text{-}5a)$$

$$\frac{d\Omega_\theta{}^{(n)}}{ds} = -\left(C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right)^{-1}\left\{N_\phi{}^{(n)} + N_{T\phi}{}^{(n)} - \frac{K_{22}}{C_{22}}\left(M_\phi{}^{(n)} + M_{T\phi}{}^{(n)}\right)\right.$$

$$\left. - \frac{K_{21}}{r_0}\left(nU^{(n)} - W^{(n)}\right) + \frac{K_{22}D_{21}}{C_{22}}\left[\frac{1}{r_0{}^2}\left\{nU^{(n)} - n^2W^{(n)}\right\}\right]\right\} \quad (4\text{-}5b)$$

$$N_\theta{}^{(n)} = K_{12}\left(K_{22} + \frac{C_{22}^2}{D_{22}}\right)^{-1}\left\{N_\phi{}^{(n)} + N_{T\phi}{}^{(n)} + \frac{C_{22}}{D_{22}}\left(M_\phi{}^{(n)} + M_{T\phi}{}^{(n)}\right)\right\} - N_{T\theta}{}^{(n)}$$

$$+ \left(\frac{K_{11}}{r_0} - \frac{K_{12}K_{21}}{r_0}\left[K_{22} + \frac{C_{22}^2}{D_{22}}\right]^{-1}\right)\left(nU^{(n)} - W^{(n)}\right)$$

$$- \left(C_{11} + \frac{K_{12}C_{22}D_{21}}{D_{22}}\left[K_{22} + \frac{C_{22}^2}{D_{22}}\right]^{-1}\right)\left(\frac{1}{r_0{}^2}\left\{nU^{(n)} - n^2W^{(n)}\right\}\right)$$

(4-5c)

$$M_\theta{}^{(n)} = -D_{12}\left(C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right)^{-1}\left\{N_\phi{}^{(n)} + N_{T\phi}{}^{(n)} - \frac{K_{22}}{C_{22}}\left(M_\phi{}^{(n)} + M_{T\phi}{}^{(n)}\right)\right\}$$

$$- M_{T\theta}{}^{(n)} + \left(\frac{C_{11}}{r_0} + \frac{D_{12}K_{21}}{r_0}\left[C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right]^{-1}\right)\left(nU^{(n)} - W^{(n)}\right)$$

$$+ \left(D_{11} - \frac{D_{12}K_{22}D_{21}}{C_{22}}\left[C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right]^{-1}\right)\left(\frac{1}{r_0{}^2}\left\{nU^{(n)} - n^2W^{(n)}\right\}\right)$$

(4-5d)

CONE

$$\frac{dV^{(n)}}{ds} = \left(K_{22} + \frac{C_{22}^{2}}{D_{22}}\right)^{-1} \left\{ N_{\phi}^{(n)} + N_{T\phi}^{(n)} + \frac{C_{22}}{D_{22}}\left(M_{\phi}^{(n)} + M_{T\phi}^{(n)}\right) \right.$$

$$- \frac{K_{21}}{s \cos\phi}\left(nU^{(n)} + V^{(n)} \cos\phi - W^{(n)} \sin\phi\right)$$

$$\left. - \frac{C_{22}D_{21}}{D_{22}}\left[\frac{1}{s^{2}\cos^{2}\phi}\left\{nU^{(n)} \sin\phi - n^{2}W^{(n)}\right\} + \frac{\Omega_{\theta}^{(n)}}{s}\right]\right\} \qquad (4\text{-}6a)$$

$$\frac{d\Omega_{\theta}^{(n)}}{ds} = -\left(C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right)^{-1} \left\{ N_{\phi}^{(n)} + N_{T\phi}^{(n)} - \frac{K_{22}}{C_{22}}\left(M_{\phi}^{(n)} + M_{T\phi}^{(n)}\right) \right.$$

$$- \frac{K_{21}}{s \cos\phi}\left(nU^{(n)} + V^{(n)} \cos\phi - W^{(n)} \sin\phi\right)$$

$$\left. + \frac{K_{22}D_{21}}{C_{22}}\left[\frac{1}{s^{2}\cos^{2}\phi}\left\{nU^{(n)} \sin\phi - n^{2}W^{(n)}\right\} + \frac{\Omega_{\theta}^{(n)}}{s}\right]\right\} \qquad (4\text{-}6b)$$

$$N_{\theta}^{(n)} = K_{12}\left(K_{22} + \frac{C_{22}^{2}}{D_{22}}\right)^{-1} \left\{ N_{\phi}^{(n)} + N_{T\phi}^{(n)} + \frac{C_{22}}{D_{22}}\left(M_{\phi}^{(n)} + M_{T\phi}^{(n)}\right)\right\} - N_{T\theta}^{(n)}$$

$$+ \left(\frac{K_{11}}{s \cos\phi} - \frac{K_{12}K_{21}}{s \cos\phi}\left[K_{22} + \frac{C_{22}^{2}}{D_{22}}\right]^{-1}\right)\left(nU^{(n)} + V^{(n)} \cos\phi - W^{(n)} \sin\phi\right)$$

$$- \left(C_{11} + \frac{K_{12}D_{21}C_{22}}{D_{22}}\left[K_{22} + \frac{C_{22}^{2}}{D_{22}}\right]^{-1}\right)\left(\frac{1}{s^{2}\cos^{2}\phi}\left\{nU^{(n)} \sin\phi\right.\right.$$

$$\left.\left. - n^{2}W^{(n)}\right\} + \frac{\Omega_{\theta}^{(n)}}{s}\right) \qquad (4\text{-}6c)$$

$$M_\theta^{(n)} = -D_{12}\left(C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right)^{-1}\left\{N_\phi^{(n)} + N_{T\phi}^{(n)} - \frac{K_{22}}{C_{22}}\left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right)\right\}$$

$$- M_{T\theta}^{(n)} + \left(\frac{C_{11}}{s \cos\phi} + \frac{D_{12}K_{21}}{s \cos\phi}\left[C_{22} + \frac{K_{22}D_{22}}{C_{22}}\right]^{-1}\right)\left(nU^{(n)}\right.$$

$$\left. + V^{(n)} \cos\phi - W^{(n)} \sin\phi\right) + \left(D_{11} - \frac{D_{12}K_{22}D_{21}}{C_{22}}\left[C_{22}\right.\right.$$

$$\left.\left. + \frac{K_{22}D_{22}}{C_{22}}\right]^{-1}\right)\left(\frac{1}{s^2 \cos^2\phi}\left\{nU^{(n)} \sin\phi - n^2W^{(n)}\right\} + \frac{\Omega_\theta^{(n)}}{s}\right) \quad (4\text{-}6d)$$

Although the stress-resultant to strain relationships (Equations 4-1) were derived in Appendix A on the physical basis of eccentric orthotropic shell reinforcing, they can be used to describe other shell wall construction as well. This is possible in the program since the parameters K, C, and D are direct input. It is only necessary to use the proper formulas in place of those given in Appendix A to calculate the stiffness parameters which are to be input. Although in Equations 4-1 membrane forces are dependent upon curvature and moments upon extensional strain, these equations are not completely general, and thus cannot be used for arbitrary layered shells. Fully general equations, adapted specifically for layered shells, can be obtained in Reference 8.

When the reinforcement is not eccentric, simpler equations than Equations 4-4, 4-5, and 4-6, can be applied. The necessary equations for this case can be derived as in Section 1, and they are presented below*:

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* In this case the Hooke's Laws contain no coupling between bending and membrane, therefore $C_{11} = C_{22} \equiv 0$.

ϕ COORDINATES

$$\frac{V_{,\phi}^{(n)}}{r_1} = \frac{W^{(n)}}{r_1} + \left(K_{22} - \frac{K_{21}K_{12}}{K_{11}}\right)^{-1}\left\{N_\phi^{(n)} + N_{T\phi}^{(n)} - \frac{K_{21}}{K_{11}}\left(N_\theta^{(n)} + N_{T\theta}^{(n)}\right)\right\} \tag{4-7a}$$

$$\frac{\Omega_{\theta,\phi}^{(n)}}{r_1} = \left(D_{22} - \frac{D_{12}D_{21}}{D_{11}}\right)^{-1}\left\{+ M_\phi^{(n)} + M_{T\phi}^{(n)} - \frac{D_{21}}{D_{11}}\left(M_\theta^{(n)} + M_{T\theta}^{(n)}\right)\right\} \tag{4-7b}$$

$$N_\theta^{(n)} = \frac{K_{12}}{K_{22}}\left(N_\phi^{(n)} + N_{T\phi}^{(n)}\right) - N_{T\theta}^{(n)}$$

$$+ \left(K_{11} - \frac{K_{12}K_{21}}{K_{22}}\right)\left[\frac{nU^{(n)} + V^{(n)}\cos\phi - W^{(n)}\sin\phi}{r_0}\right] \tag{4-7c}$$

$$M_\theta^{(n)} = \frac{D_{12}}{D_{22}}\left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right) - M_{T\theta}^{(n)} + \left(D_{11} - \frac{D_{12}D_{21}}{D_{22}}\right)\left[\frac{nU^{(n)}\sin\phi - n^2W^{(n)}}{r_0}\right.$$

$$\left. + \Omega_\theta^{(n)}\cos\phi\right]\frac{1}{r_0} \tag{4-7d}$$

CYLINDER

$$\frac{dV^{(n)}}{ds} = \left(K_{22} - \frac{K_{21}K_{12}}{K_{11}}\right)^{-1}\left\{N_\phi^{(n)} + N_{T\phi}^{(n)} - \frac{K_{21}}{K_{11}}\left(N_\theta^{(n)} + N_{T\theta}^{(n)}\right)\right\} \tag{4-8a}$$

$$\frac{d\Omega_\theta^{(n)}}{ds} = \left(D_{22} - \frac{D_{12}D_{21}}{D_{11}}\right)^{-1}\left\{M_\phi^{(n)} + M_{T\phi}^{(n)} - \frac{D_{21}}{D_{11}}\left(M_\theta^{(n)} + M_{T\theta}^{(n)}\right)\right\} \tag{4-8b}$$

$$N_\theta^{(n)} = \frac{K_{12}}{K_{22}}\left(N_\phi^{(n)} + N_{T\phi}^{(n)}\right) - N_{T\theta}^{(n)} + \left(K_{11} - \frac{K_{12}K_{21}}{K_{22}}\right)\left[\frac{nU^{(n)} - W^{(n)}}{r_0}\right] \tag{4-8c}$$

$$M_\theta^{(n)} = \frac{D_{12}}{D_{22}}\left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right) - M_{T\theta}^{(n)} + \left(D_{11} - \frac{D_{12}D_{21}}{D_{22}}\right)\left[\frac{nU^{(n)} - n^2W^{(n)}}{r_0^2}\right] \tag{4-8d}$$

CONE

$$\frac{dV^{(n)}}{ds} = \left(K_{22} - \frac{K_{21}K_{12}}{K_{11}}\right)^{-1} \left\{N_{\phi}^{(n)} + N_{T\phi}^{(n)} - \frac{K_{21}}{K_{11}}\left(N_{\theta}^{(n)} + N_{T\theta}^{(n)}\right)\right\} \qquad (4\text{-}9a)$$

$$\frac{d\Omega_{\theta}^{(n)}}{ds} = \left(D_{22} - \frac{D_{12}D_{21}}{D_{11}}\right)^{-1} \left\{M_{\phi}^{(n)} + M_{T\phi}^{(n)} - \frac{D_{21}}{D_{11}}\left(M_{\theta}^{(n)} + M_{T\theta}^{(n)}\right)\right\} \qquad (4\text{-}9b)$$

$$N_{\theta}^{(n)} = \frac{K_{12}}{K_{22}}\left(N_{\phi}^{(n)} + N_{T\phi}^{(n)}\right) - N_{T\theta}^{(n)}$$

$$+ \left(K_{11} - \frac{K_{12}K_{21}}{K_{22}}\right)\left[\frac{nU^{(n)} + V^{(n)}\cos\phi - W^{(n)}\sin\phi}{s\cos\phi}\right] \qquad (4\text{-}9c)$$

$$M_{\theta}^{(n)} = \frac{D_{12}}{D_{22}}\left(M_{\phi}^{(n)} + M_{T\phi}^{(n)}\right) - M_{T\theta}^{(n)}$$

$$+ \left(D_{11} - \frac{D_{12}D_{21}}{D_{22}}\right)\left[\frac{nU^{(n)} - n^2W^{(n)}}{s^2\cos^2\phi} + \frac{\Omega_{\theta}^{(n)}}{s}\right] \qquad (4\text{-}9d)$$

The specific type of reinforcement which can be analyzed by all the previous equations has to be placed coincident with the coordinate axes of the shell (θ and ϕ or s). More complex reinforcement, such as waffle construction rotated 45 deg from the coordinate axes, must also be analyzed on the Saturn SII stage. The specific integrated Hooke's Law relationships for this case are given below (derivations and stiffness parameter definitions can be found in Appendix A):

$$N_{\theta} = K_{11}\,\varepsilon_{\theta_o} + K_{12}\,\varepsilon_{\phi_o} - C_{11}\,(k_{\theta} + k_{\phi}) - N_{T\theta} \qquad (4\text{-}10a)$$

$$N_{\phi} = K_{22}\,\varepsilon_{\phi_o} + K_{21}\,\varepsilon_{\theta_o} - C_{11}\,(k_{\theta} + k_{\phi}) - N_{T\phi} \qquad (4\text{-}10b)$$

$$N_{\phi\theta} = K_{33}\,\gamma_{\phi\theta_o} - 2\,C_{11}\,k_{\phi\theta} \qquad (4\text{-}10c)$$

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$$M_\theta = D_{11}k_\theta + D_{12}k_\phi + C_{11}\left(\varepsilon_{\theta_o} + \varepsilon_{\phi_o}\right) - M_{T\theta} \tag{4-10d}$$

$$M_\phi = D_{22}k_\phi + D_{21}k_\theta + C_{11}\left(\varepsilon_{\theta_o} + \varepsilon_{\phi_o}\right) - M_{T\phi} \tag{4-10e}$$

$$M_{\phi\theta} = -2D_{33}k_{\phi\theta} + C_{11}\,\gamma_{\phi\theta_o} \tag{4-10f}$$

Equations 4-10 will also necessitate a change in some of the auxiliary and differential equations derived in Section 1. These revised equations are given below. Eliminating k_ϕ and ε_{ϕ_o} successively from Equations 4-10b and 4-10e we obtain equations for $v,_\phi/r_1$ and $\omega_{\theta,\phi}/r_1$. Substituting these equations respectively into the equations obtained by subtracting 4-10b from 4-10a and 4-10e from 4-10d we arrive at equations for N_θ and M_θ. Finally, equations for $u,_\phi/r_1$ and $M_{\phi\theta}$ can be obtained from 4-10c and 4-10f, and Q_θ is obtained from equilibrium. Thus the revised equations are provided below, already expanded in Fourier Series, for the necessary coordinate systems:

ϕ COORDINATE

$$\frac{V,_\phi^{(n)}}{r_1} = \frac{W^{(n)}}{r_1} + \left(K_{22} + \frac{C_{11}^2}{D_{22}}\right)^{-1} \left\{ N_\phi^{(n)} + N_{T\phi}^{(n)} + \frac{C_{11}}{D_{22}}\left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right) - \left(K_{12} + \frac{C_{11}^2}{D_{22}}\right)\frac{1}{r_0}\left(nU^{(n)} + V^{(n)}\cos\phi - W^{(n)}\sin\phi\right) - \left(\frac{D_{12}C_{11}}{D_{22}} - C_{11}\right)\frac{1}{r_0^2}\left(nU^{(n)}\sin\phi - n^2W^{(n)} + r_0\Omega_\theta^{(n)}\cos\phi\right)\right\} \tag{4-11a}$$

$$\frac{\Omega_{\theta,\phi}^{(n)}}{r_1} = \left(C_{11} + \frac{D_{22}K_{22}}{C_{11}}\right)^{-1} \left\{ \frac{K_{22}}{C_{11}}\left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right) - N_\phi^{(n)} - N_{T\phi}^{(n)} + \left(K_{12} - K_{22}\right)\frac{1}{r_0}\left(nU^{(n)} + V^{(n)}\cos\phi - W^{(n)}\sin\phi\right)\right.$$

$$- \left(C_{11} + \frac{D_{12}K_{22}}{C_{11}}\right) \frac{1}{r_0^{\,2}} \left(nU^{(n)} \sin \phi - n^2W^{(n)} + r_0\Omega_\theta^{(n)}\cos \phi\right)\Bigg\}$$

(4-11b)

$$\frac{U,_\phi^{(n)}}{r_1} = \frac{U^{(n)}\cos \phi}{r_0} + \frac{nV^{(n)}}{r_0} + \left(K_{33} - \frac{C_{11}^{\,2}}{D_{33}}\right)^{-1} \left\{T_{\phi\theta}^{(n)} + M_{\phi\theta}^{(n)} \left(\frac{\sin \phi}{r_0} - \frac{C_{11}}{D_{33}}\right)\right\}$$

(4-11c)

$$N_\theta^{(n)} = \left(N_\phi^{(n)} + N_{T\phi}^{(n)}\right) \left\{1 + (K_{12} - K_{22})\left(K_{22} + \frac{C_{11}^{\,2}}{D_{22}}\right)^{-1}\right\} - N_{T\theta}^{(n)}$$

$$+ (K_{12} - K_{22}) \left(\frac{C_{11}}{K_{22}D_{22} + C_{11}^{\,2}}\right) \left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right)$$

$$- (K_{12} - K_{22}) \left(\frac{D_{12}C_{11} - C_{11}D_{22}}{K_{22}D_{22} + C_{11}^{\,2}}\right) \frac{1}{r_0^{\,2}} \left(nU^{(n)}\sin \phi - n^2W^{(n)}\right.$$

$$\left.+ r_0\Omega_\theta^{(n)} \cos \phi\right) + \frac{1}{r_0} \left(nU^{(n)} + V^{(n)}\cos \phi\right.$$

$$\left.- W^{(n)}\sin \phi\right) \left\{K_{11} - K_{12} - (K_{12} - K_{22}) \left(\frac{K_{12}D_{22} + C_{11}^{\,2}}{K_{22}D_{22} + C_{11}^{\,2}}\right)\right\} \quad (4\text{-}11d)$$

$$M_\theta^{(n)} = \left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right) \left\{1 + (D_{12} - D_{22}) \left(\frac{K_{22}}{C_{11}^{\,2} + D_{22}K_{22}}\right)\right\}$$

$$- M_{T\theta}^{(n)} - \left(N_\phi^{(n)} + N_{T\phi}^{(n)}\right) (D_{12} - D_{22}) \left(\frac{C_{11}}{C_{11}^{\,2} + D_{22}K_{22}}\right)$$

$$+ (D_{12} - D_{22})(K_{12} - K_{22}) \left(\frac{C_{11}}{C_{11}^{2} + D_{22}K_{22}}\right)\frac{1}{r_0}\left(nU^{(n)} + V^{(n)}\cos\phi\right.$$

$$\left.- W^{(n)}\sin\phi\right) + \frac{1}{r_0^{2}}\left(nU^{(n)}\sin\phi - n^2W^{(n)}\right.$$

$$\left.+ r_0\Omega_\theta^{(n)}\cos\phi\right)\left\{D_{11} - D_{12} - (D_{12} - D_{22})\left(\frac{C_{11}^{2} + D_{12}K_{22}}{C_{11}^{2} + D_{22}K_{22}}\right)\right\} \qquad (4\text{-}11e)$$

$$M_{\phi\theta}^{(n)} = \left\{\frac{\sin\phi}{r_0} - \frac{K_{33}}{C_{11}} - \frac{\sin\phi}{r_0}\left(\frac{D_{33}\sin\phi}{C_{11}r_0} - 1\right)\right\}^{-1}\left\{\left(\frac{K_{33}D_{33}}{C_{11}} - C_{11}\right)\frac{1}{r_0}\left[-2n\Omega_\theta^{(n)}\right.\right.$$

$$+ U^{(n)}\left(\frac{\cos\phi}{r_1} - \frac{\sin\phi\cos\phi}{r_0}\right) + nV^{(n)}\left(\frac{1}{r_1} + \frac{\sin\phi}{r_0}\right)$$

$$\left.\left.+ 2nW^{(n)}\frac{\cos\phi}{r_0}\right] + T_{\phi\theta}^{(n)}\left[\frac{D_{33}\sin\phi}{C_{11}r_0} - 1\right]\right\} \qquad (4\text{-}11f)$$

$$Q_\theta^{(n)} = \left[\frac{4\cos\phi}{r_0} - \frac{2\cos\phi\left(C_{11}\frac{r_0}{r_1} + C_{11}\sin\phi - r_0K_{33} - \sin\phi\frac{D_{33}}{r_1}\right)}{2r_0C_{11}\sin\phi - r_0^{2}K_{33} - D_{33}\sin^2\phi}\right]M_{\phi\theta}^{(n)}$$

$$+\left\{\frac{\frac{1}{r_1}}{\frac{2\sin\phi}{r_0} - \frac{K_{33}}{C_{11}} - \frac{D_{33}\sin^2\phi}{r_0^{2}C_{11}}}\right\}\left\{\left(\frac{K_{33}D_{33}}{C_{11}} - C_{11}\right)\left[\frac{-2n\,\Omega_{\theta,\phi}^{(n)}}{r_0}\right.\right.$$

$$+ \frac{2n\,\Omega_\theta^{(n)}\,r_1\cos\phi}{r_0^{2}} + U_{,\phi}^{(n)}\left(\frac{\cos\phi}{r_1r_0} - \frac{\cos\phi\sin\phi}{r_0^{2}}\right)$$

$$+ U^{(n)} \left(- \frac{\sin \phi}{r_1 r_0} - \frac{\cos \phi \left[r_0 r_{1,\phi} + r_1^{\,2} \cos \phi \right]}{r_1^{\,2} r_0^{\,2}} \right.$$

$$- \frac{\left(\cos^2\phi - \sin^2\phi\right)}{r_0^{\,2}} + \left. \frac{2r_1 \cos^2\phi \sin \phi}{r_0^{\,3}} \right) + nV,_{\phi}^{(n)} \left(\frac{1}{r_1 r_0} \right.$$

$$+ \left. \frac{\sin \phi}{r_0^{\,2}} \right) + nV^{(n)} \left(\frac{-2r_1 \cos \phi \sin \phi}{r_0^{\,3}} - \frac{r_{1,\phi}}{r_1^{\,2} r_0} \right)$$

$$\left. + 2nW,_{\phi}^{(n)} \frac{\cos \phi}{r_0^{\,2}} - 2nW^{(n)} \left(\frac{\sin \phi}{r_0^{\,2}} + \frac{2r_1 \cos^2 \phi}{r_0^{\,3}} \right) \right]$$

$$+ T_{\phi\theta,\phi}^{(n)} \left[\frac{D_{33} \sin \phi}{C_{11} r_0} - 1 \right] + T_{\phi\theta}^{(n)} \left[\frac{D_{33} \cos \phi}{C_{11} r_0} \right.$$

$$\left. \left. - \frac{D_{33} r_1 \cos \phi \sin \phi}{C_{11} r_0^{\,2}} \right] \right\} - \frac{nM_{\theta}^{(n)}}{r_0} - m_{\phi}^{(n)} \qquad (4\text{-}11g)$$

If $m_{\phi}^{(0)} = f_{\theta}^{(0)} = 0$, then $Q_{\theta}^{(0)} = 0$.

<u>CONE</u>

$$\frac{dV^{(n)}}{ds} = \left(K_{22} + \frac{C_{11}^{\,2}}{D_{22}} \right)^{-1} \left\{ N_{\phi}^{(n)} + N_{T\phi}^{(n)} + \frac{C_{11}}{D_{22}} \left(M_{\phi}^{(n)} + M_{T\phi}^{(n)} \right) \right.$$

$$- \left(K_{12} + \frac{C_{11}^{\,2}}{D_{22}} \right) \frac{1}{s \cos \phi} \left(nU^{(n)} + V^{(n)} \cos \phi - W^{(n)} \sin \phi \right)$$

$$\left. - \left(\frac{D_{12} C_{11}}{D_{22}} - C_{11} \right) \left[\frac{1}{s^2 \cos^2 \phi} \left(nU^{(n)} \sin \phi - n^2 W^{(n)} \right) + \frac{\Omega_{\theta}^{(n)}}{s} \right] \right\} \qquad (4\text{-}12a)$$

$$\frac{d\Omega_\theta^{(n)}}{ds} = \left(C_{11} + \frac{D_{22}K_{22}}{C_{11}}\right)^{-1} \Bigg\{ \frac{K_{22}}{C_{11}} \left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right) - N_\phi^{(n)} - N_{T\phi}^{(n)}$$

$$+ (K_{12} - K_{22}) \frac{1}{s \cos \phi} \left(nU^{(n)} + V^{(n)} \cos \phi - W^{(n)} \sin \phi\right)$$

$$- \left(C_{11} + \frac{D_{12}K_{22}}{C_{11}}\right)\left[\frac{1}{s^2 \cos^2\phi} \left(nU^{(n)} \sin \phi - n^2W^{(n)}\right) + \frac{\Omega_\theta^{(n)}}{s}\right]\Bigg\} \tag{4-12b}$$

$$\frac{dU^{(n)}}{ds} = \frac{U^{(n)}}{s} + \frac{nV^{(n)}}{s \cos \phi} + \left(K_{33} - \frac{C_{11}^2}{D_{33}}\right)^{-1} \left\{ T_{\phi\theta}^{(n)} + M_{\phi\theta}^{(n)} \left(\frac{\tan \phi}{s} - \frac{C_{11}}{D_{33}}\right)\right\} \tag{4-12c}$$

$$N_\theta^{(n)} = \left(N_\phi^{(n)} + N_{T\phi}^{(n)}\right)\left\{1 + (K_{12} - K_{22}) \left(K_{22} + \frac{C_{11}^2}{D_{22}}\right)^{-1}\right\} - N_{T\theta}^{(n)}$$

$$+ (K_{12}-K_{22}) \left(\frac{C_{11}}{K_{22}D_{22} + C_{11}^2}\right) \left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right)$$

$$- (K_{12} - K_{22}) \left(\frac{D_{12}C_{11} - C_{11}D_{22}}{K_{22}D_{22} + C_{11}^2}\right) \left[\frac{1}{s^2 \cos^2\phi} \left(nU^{(n)}\sin \phi - n^2W^{(n)}\right)\right.$$

$$\left.+ \frac{\Omega_\theta^{(n)}}{s}\right] + \frac{1}{s \cos \phi} \left(nU^{(n)} + V^{(n)}\cos \phi - W^{(n)}\sin \phi\right) \Bigg\{ K_{11}$$

$$- K_{12} - (K_{12} - K_{22}) \left(\frac{K_{12}D_{22} + C_{11}^2}{K_{22}D_{22} + C_{11}^2}\right) \Bigg\} \tag{4-12d}$$

$$M_\theta^{(n)} = \left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right)\left\{1 + (D_{12} - D_{22})\left(\frac{K_{22}}{C_{11}^2 + D_{22}K_{22}}\right)\right\} - M_{T\theta}^{(n)}$$

$$- \left(N_\phi^{(n)} + N_{T\phi}^{(n)}\right)(D_{12} - D_{22})\left(C_{11} + \frac{D_{22}K_{22}}{C_{11}}\right)^{-1}$$

$$+ (D_{12} - D_{22})\ (K_{12} - K_{22})\ \left(C_{11} + \frac{D_{22}K_{22}}{C_{11}}\right)^{-1}\left(\frac{1}{s\ \cos\ \phi}\right)\Big(nU^{(n)}$$

$$+ V^{(n)}\cos\ \phi - W^{(n)}\sin\ \phi\Big) + \left[\frac{1}{s^2\ \cos^2\phi}\ \left(nU^{(n)}\ \sin\ \phi - n^2W^{(n)}\right)\right.$$

$$\left.+ \frac{\Omega_\theta^{(n)}}{s}\right]\left\{D_{11} - D_{12} - (D_{12} - D_{22})\left(\frac{C_{11}^2 + D_{12}K_{22}}{C_{11}^2 + D_{22}K_{22}}\right)\right\} \qquad (4\text{-}12e)$$

$$M_{\phi\theta}^{(n)} = \left\{\frac{\tan\ \phi}{s} - \frac{K_{33}}{C_{11}} - \frac{\tan\ \phi}{s}\left(\frac{D_{33}\ \tan\ \phi}{C_{11}s} - 1\right)\right\}^{-1}\left\{\left(\frac{K_{33}D_{33}}{C_{11}}\right.\right.$$

$$\left.- C_{11}\right)\left(\frac{1}{s\ \cos\ \phi}\right)\left[-2n\Omega_\theta^{(n)} - \frac{U^{(n)}\sin\ \phi}{s} + nV^{(n)}\ \frac{\tan\ \phi}{s} + \frac{2nW^{(n)}}{s}\right]$$

$$\left.+ T_{\phi\theta}^{(n)}\left[\frac{D_{33}\ \tan\ \phi}{C_{11}s} - 1\right]\right\} \qquad (4\text{-}12f)$$

$$Q_\theta^{(n)} = \left[\frac{4}{s} - \frac{2\cos\phi\,(C_{11}\sin\phi - sK_{33}\cos\phi)}{2sC_{11}\cos\phi\sin\phi - s^2K_{33}\cos^2\phi - D_{33}\sin^2\phi}\right] M_{\phi\theta}^{(n)}$$

$$+ \left\{\frac{1}{\frac{2\tan\phi}{s} - \frac{K_{33}}{C_{11}} - \frac{D_{33}\tan^2\phi}{s^2C_{11}}}\right\} \left\{\left(\frac{K_{33}D_{33}}{C_{11}} - C_{11}\right)\left[\frac{-2n}{s\cos\phi}\frac{d\Omega_\theta^{(n)}}{ds}\right.\right.$$

$$+ \frac{2n\Omega_\theta^{(n)}}{s^2\cos\phi} - \frac{dU^{(n)}}{ds}\left(\frac{\tan\phi}{s^2}\right) + U^{(n)}\left(\frac{2\tan\phi}{s^3}\right) + n\frac{dV^{(n)}}{ds}\left(\frac{\tan\phi}{s^2\cos\phi}\right)$$

$$\left. -nV^{(n)}\left(\frac{2\tan\phi}{s^3\cos\phi}\right) + \frac{dW^{(n)}}{ds}\left(\frac{2n}{s^2\cos\phi}\right) - 2nW^{(n)}\left(\frac{2}{s^3\cos\phi}\right)\right]$$

$$\left.\left. + \frac{dT_{\phi\theta}^{(n)}}{ds}\left[\frac{D_{33}\tan\phi}{C_{11}s} - 1\right] - T_{\phi\theta}^{(n)}\left(\frac{D_{33}\tan\phi}{C_{11}s^2}\right)\right\}\right\}$$

$$- \frac{nM_\theta^{(n)}}{s\cos\phi} - m_\phi^{(n)} \qquad (4\text{-}12g)$$

If $m_\phi^{(0)} = f_\theta^{(0)} = 0$, then $Q_\theta^{(0)} = 0$.

CYLINDER

$$\frac{dV^{(n)}}{ds} = \left(K_{22} + \frac{C_{11}^2}{D_{22}}\right)^{-1}\left\{N_\phi^{(n)} + N_{T\phi}^{(n)} + \frac{C_{11}}{D_{22}}\left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right)\right.$$

$$- \left(K_{12} + \frac{C_{11}^2}{D_{22}}\right)\frac{1}{r_0}\left(nU^{(n)} - W^{(n)}\right) - \left(\frac{D_{12}C_{11}}{D_{22}}\right.$$

$$\left.\left. - C_{11}\right)\frac{1}{r_0^2}\left(nU^{(n)} - n^2W^{(n)}\right)\right\} \qquad (4\text{-}13a)$$

$$\frac{d\Omega_\theta^{(n)}}{ds} = \left(C_{11} + \frac{D_{22}K_{22}}{C_{11}}\right)^{-1}\left\{\frac{K_{22}}{C_{11}}\left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right) - N_\phi^{(n)} - N_{T\phi}^{(n)}\right.$$

$$+ (K_{12} - K_{22})\frac{1}{r_0}\left(nU^{(n)} - W^{(n)}\right) - \left(C_{11}\right.$$

$$\left.\left. + \frac{D_{12}K_{22}}{C_{11}}\right)\frac{1}{r_0^{\,2}}\left(nU^{(n)} - n^2W^{(n)}\right)\right\} \qquad (4\text{-}13b)$$

$$\frac{dU^{(n)}}{ds} = \frac{nV^{(n)}}{r_0} + \left(K_{33} - \frac{C_{11}^{\,2}}{D_{33}}\right)^{-1}\left\{T_{\phi\theta}^{(n)} + M_{\phi\theta}^{(n)}\left(\frac{1}{r_0} - \frac{C_{11}}{D_{33}}\right)\right\} \qquad (4\text{-}13c)$$

$$N_\theta^{(n)} = \left(N_\phi^{(n)} + N_{T\phi}^{(n)}\right)\left\{1 + (K_{12} - K_{22})\left(K_{22} + \frac{C_{11}^{\,2}}{D_{22}}\right)^{-1}\right\} - N_{T\theta}^{(n)}$$

$$+ (K_{12} - K_{22})\left(\frac{C_{11}}{K_{22}D_{22} + C_{11}^{\,2}}\right)\left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right)$$

$$- (K_{12} - K_{22})\left(\frac{D_{12}C_{11} - C_{11}D_{22}}{K_{22}D_{22} + C_{11}^{\,2}}\right)\frac{1}{r_0^{\,2}}\left(nU^{(n)} - n^2W^{(n)}\right)$$

$$+ \frac{1}{r_0}\left(nU^{(n)} - W^{(n)}\right)\left\{K_{11} - K_{12} - (K_{12} - K_{22})\left(\frac{K_{12}D_{22} + C_{11}^{\,2}}{K_{22}D_{22} + C_{11}^{\,2}}\right)\right\}$$

(4-13d)

$$M_\theta^{(n)} = \left(M_\phi^{(n)} + M_{T\phi}^{(n)}\right)\left\{1 + (D_{12} - D_{22})\left(\frac{K_{22}}{C_{11}^{\;2} + D_{22}K_{22}}\right)\right\} - M_{T\theta}^{(n)}$$

$$- \left(N_\phi^{(n)} + N_{T\phi}^{(n)}\right)(D_{12} - D_{22})\left(C_{11} + \frac{D_{22}K_{22}}{C_{11}}\right)^{-1}$$

$$+ (D_{12} - D_{22})(K_{12} - K_{22})\left(C_{11} + \frac{D_{22}K_{22}}{C_{11}}\right)^{-1}\frac{1}{r_0}\left(nU^{(n)} - W^{(n)}\right)$$

$$+ \frac{1}{r_0^{\;2}}\left(nU^{(n)} - n^2W^{(n)}\right)\left\{D_{11} - D_{12} - (D_{12} - D_{22})\left(\frac{C_{11}^{\;2} + D_{12}K_{22}}{C_{11}^{\;2} + D_{22}K_{22}}\right)\right\}$$

(4-13e)

$$M_{\phi\theta}^{(n)} = \left\{\frac{1}{r_0} - \frac{K_{33}}{C_{11}} - \frac{1}{r_0}\left(\frac{D_{33}}{C_{11}r_0} - 1\right)\right\}^{-1}\left\{\left(\frac{K_{33}D_{33}}{C_{11}} - C_{11}\right)\frac{1}{r_0}\left[-2n\Omega_\theta^{(n)} + \frac{nV^{(n)}}{r_0}\right] + T_{\phi\theta}^{(n)}\left[\frac{D_{33}}{C_{11}r_0} - 1\right]\right\} \qquad (4\text{-}13f)$$

$$Q_\theta^{(n)} = \left\{\frac{1}{\frac{2}{r_0} - \frac{K_{33}}{C_{11}} - \frac{D_{33}}{r_0^{\;2}C_{11}}}\right\}\left\{\left(\frac{K_{33}D_{33}}{C_{11}} - C_{11}\right)\left[-\frac{2n}{r_0}\frac{d\Omega_\theta^{(n)}}{ds} + \frac{n}{r_0^{\;2}}\frac{dV^{(n)}}{ds}\right] + \frac{dT_{\phi\theta}^{(n)}}{ds}\left[\frac{D_{33}}{r_0C_{11}} - 1\right]\right\} - \frac{nM_\theta^{(n)}}{r_0} - m_\phi^{(n)} \qquad (4\text{-}13g)$$

If $m_\phi^{(0)} = f_\theta^{(0)} = 0$, then $Q_\theta^{(0)} = 0$.

SECTION 5

SEGMENT STIFFNESS MATRICES

For each segment of the shell, we will require a relationship between the edge displacements and edge forces in global coordinates. These are obtained as stiffness matrices which are to be used in calculating the elastic interaction of all segments making up the structure. The global components in terms of the local coordinates are, in accordance with Figures 1-1, 5-1 and 5-2.

$$\left\{F(i)\right\} = \left[IFT\right]\left\{f(i)\right\} \qquad \begin{Bmatrix} F_T \\ F_Z \\ F_R \\ M \end{Bmatrix}(i) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +si & +ci & 0 \\ 0 & -ci & +si & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \begin{Bmatrix} T_{\phi\theta} \\ N_\phi \\ J_\phi \\ M_\phi \end{Bmatrix}(i) \qquad (5\text{-}1)$$

$$\left\{F(j)\right\} = \left[JFT\right]\left\{f(j)\right\} \qquad \begin{Bmatrix} F_T \\ F_Z \\ F_R \\ M \end{Bmatrix}(j) = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -sj & -cj & 0 \\ 0 & +cj & -sj & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} T_{\phi\theta} \\ N_\phi \\ J_\phi \\ M_\phi \end{Bmatrix}(j) \qquad (5\text{-}2)$$

and

$$\left\{\Delta(i)\right\} = \left[IDT\right]\left\{\delta(i)\right\} \qquad \begin{Bmatrix} \Delta_T \\ \Delta_Z \\ \Delta_R \\ \Omega_\theta \end{Bmatrix}(i) = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -si & -ci & 0 \\ 0 & +ci & -si & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \omega_\theta \end{Bmatrix}(i) \qquad (5\text{-}3)$$

$$\{\Delta(j)\} = [JDT]\{\delta(j)\} \qquad \begin{Bmatrix} \Delta_T \\ \Delta_Z \\ \Delta_R \\ \Omega_\theta \end{Bmatrix}(j) = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -sj & -cj & 0 \\ 0 & +cj & -sj & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \\ \omega_\theta \end{Bmatrix}(j) \qquad (5\text{-}4)$$

Where the letters in the I Force Transformation and J Force Transformation matrices are sines or cosines of ϕ_i or ϕ_j, the meridional coordinates of the beginning and end of the segment. (Similarly for the Displacements.) These transformations hold for functions $F(\theta,\phi_{i \text{ or } j})$ and $\Delta(\theta,\phi_{i \text{ or } j})$ as well as for the amplitudes of harmonics, $F^{(n)}(\phi_{i \text{ or } j})$ and $\Delta^{(n)}(\phi_{i \text{ or } j})$. Only one harmonic at a time is considered. Thus, the transformation matrices are 4x4.

The set of influence coefficients represents the general solution of the boundary value problem for conditions imposed on the edges of the segment. In addition, there is a solution corresponding to each distributed loading on the segment. Since the differential equations are linear, we expect a linear relationship between segment edge forces and displacements. Further, the edge forces for zero displacements are linear functions of the loading for each problem. Thus we seek matrices $[k]$ and $[\ell]$ such that

$$\underset{8xP}{\begin{bmatrix} F(ip) \\ \hline F(jp) \end{bmatrix}} = \underset{8x8}{\left[\begin{array}{c|c} k_{ii} & k_{ij} \\ \hline k_{ji} & k_{jj} \end{array}\right]} \underset{8xP}{\begin{bmatrix} \Delta(ip) \\ \hline \Delta(jp) \end{bmatrix}} + \underset{8xP}{\begin{bmatrix} \ell(ip) \\ \hline \ell(jp) \end{bmatrix}} \qquad (5\text{-}5)$$

The quantities $[\ell]$ are the "fixed end forces" due to unit value distributed loads; that is, the forces at the ends when displacements are zero and the distributed loads are applied. The indices i and j indicate the beginning and end points of the segment. When automating the theory, P will be set equal to 10 in order to obtain the capability of analyzing a single structure under 10 consecutive loading conditions within one machine submission.

The differential equations are solved by Runge-Kutta forward integration for different sets of initial conditions. Since there are four boundary conditions for each edge, we would expect eight separate solutions to be required to construct the 8x8 stiffness matrix. Additional solutions are required for the distributed load. For simplicity, these solutions are obtained by assuming a unit value of a force or displacement. The process is schematically outlined in Figure 5-3. In columns 1 through 4, successive unit values of local displacement with zero forces are assumed. In columns 5 through 8, the unit forces are applied and the displacements are taken as zero. In further runs, the initial forces and displacements are both set to zero and the distributed load cases are applied. The solutions are obtained simultaneously by applying all the initial conditions at once as an 8x18 (maximum) matrix into the Runge-Kutta integration scheme. Each initial and load condition then will become an equation as the matrix proceeds through the Runge-Kutta integration to the final values. It should be noted that these assumptions, choosing both forces and displacements at one boundary, are not inconsistent. It is permissible to specify four on each edge, or to give all eight at one edge (or any intermediate combination). The forces f(j) arising at the j^{th} edge due to initial displacements, 1, and forces, 2, at the i^{th} edge, and distributed loading, 3, are recorded in the matrix

$$\underset{4x(8+P)}{\left[X_1 \;\vdots\; X_2 \;\vdots\; X_3\right]}$$

Similarly, the displacements δ(j) are recorded in the matrix.

$$\underset{4x(8+P)}{\left[Y_1 \;\vdots\; Y_2 \;\vdots\; Y_3\right]}$$

where P is the number of external loading conditions (maximum = 10). Thus, the forces and displacements at the j^{th} edge may be expressed in terms of forces and displacements at the i^{th} edge. For simplicity, the maximum case of P = 10 is used, therefore:

$$\underset{4x1}{\begin{Bmatrix} F(j) \end{Bmatrix}} = \underset{4x4}{\begin{bmatrix} JFT \end{bmatrix}} \underset{4x1}{\begin{Bmatrix} f(j) \end{Bmatrix}} = \underset{4x4}{\begin{bmatrix} JFT \end{bmatrix}} \underset{4x18}{\begin{bmatrix} X_1 & \vdots & X_2 & \vdots & X_3 \end{bmatrix}} \underset{18x1}{\begin{Bmatrix} \delta(i) \\ f(i) \\ \ell \end{Bmatrix}} \tag{5-6}$$

$$\underset{4x1}{\begin{Bmatrix} \Delta(j) \end{Bmatrix}} = \underset{4x4}{\begin{bmatrix} JDT \end{bmatrix}} \underset{4x1}{\begin{Bmatrix} \delta(j) \end{Bmatrix}} = \underset{4x4}{\begin{bmatrix} JDT \end{bmatrix}} \underset{4x18}{\begin{bmatrix} Y_1 & \vdots & Y_2 & \vdots & Y_3 \end{bmatrix}} \underset{18x1}{\begin{Bmatrix} \delta(i) \\ f(i) \\ \ell \end{Bmatrix}} \tag{5-7}$$

From Equations 5-1 and 5-7,

$$\{F(i)\} = [IFT]\ [Y_2]^{-1} \left([JDT]^T \ \{\Delta(j)\} - [Y_1]\ \{\delta(i)\} - \{Y_3\}\ \{\ell\} \right) \tag{5-8}$$

From Equations 5-6 and 5-8,

$$\{F(j)\} = [JFT]\ [X_1]\ \{\delta(i)\} + [JFT]\ [X_2]\ [Y_2]^{-1} \Big([JDT]^T\ \{\Delta(j)\}$$
$$- [Y_1]\ \{\delta(i)\} - \{Y_3\}\{\ell\} \Big) + [JFT]\ \{X_3\}\{\ell\} \tag{5-9}$$

Using Equations 5-5, 5-8, and 5-9, and transforming the (δ) matrices to (Δ) matrices by Equations 5-3 and 5-4, where necessary, we obtain for "P" loading conditions:

$$\underset{8x(8+P)}{\begin{bmatrix} k & \vdots & \ell \end{bmatrix}} \equiv \begin{bmatrix} k(ii) & k(ij) & \ell(ip) \\ \hline k(ji) & k(jj) & \ell(jp) \end{bmatrix} \equiv$$

$$\left[\begin{array}{c|c} IFT & 0 \\ \hline 0 & JFT \end{array}\right]\left[\begin{array}{c|c|c} 0 & I_4 & 0 \\ \hline X_1 & X_2 & X_3 \end{array}\right]\left[\begin{array}{c|c|c} I_4 & 0 & 0 \\ \hline 0 & Y_2^{-1} & 0 \\ \hline 0 & 0 & I_P \end{array}\right]\left[\begin{array}{c|c|c} I_4 & 0 & 0 \\ \hline -Y_1 & JDT^T & -Y_3 \\ \hline 0 & 0 & I_P \end{array}\right]\left[\begin{array}{c|c|c} IDT^T & 0 & 0 \\ \hline 0 & I_4 & 0 \\ \hline 0 & 0 & I_P \end{array}\right] \quad (5\text{-}10)$$

As can be seen, it is necessary only to invert the 4x4 matrix $\left[Y_2\right]$ in order to compute the matrix of stiffness $[k]$ and fixed end forces $[\ell]$. Equation 5-10 is actually a more compact matrix formulation of Equations 5-8 and 5-9.

SOME STIFFNESS MATRIX [k] PROPERTIES

In the reciprocity law, the integral of forces must be used, thus, in the linear case, the stiffness matrix $[k]$, is symmetric in the sense that

$$[k(ii)] = [k(ii)]^T, \quad [k(jj)] = [k(jj)]^T$$

and

$$[k(ji)] = \frac{r_0(i)}{r_0(j)} \quad [k(ij)]^T$$

We shall find it more convenient to utilize the fully symmetric matrix

$$[\hat{k}] \equiv \left[\begin{array}{c|c} \ddots \; 2\pi r_0(i) \; \ddots & \\ \hline & \ddots \; 2\pi r_0(j) \; \ddots \end{array}\right] [\,k\,]$$

and

$$[\hat{\ell}] = \left[\begin{array}{c|c} \ddots \; 2\pi r_0(i) \; \ddots & \\ \hline & \ddots \; 2\pi r_0(j) \; \ddots \end{array}\right] \left[\begin{array}{c} \ell(ip) \\ \hline \ell(jp) \end{array}\right]$$

so that $\left[\hat{F}\right] = \left[\hat{k}\right]\left[\Delta\right] + \left[\hat{\ell}\right]$ (5-11)

where
$$\left\{\hat{F}\right\} \equiv \widehat{\left\{\begin{matrix} F(i) \\ \hline F(j)\end{matrix}\right\}} \equiv \left\{\begin{matrix} 2\pi r_0(i)\ F(i) \\ \hline 2\pi r_0(j)\ F(j)\end{matrix}\right\}$$

Where "F" is in units of force/unit length, and "$\hat{F}$" is measured in units of force.

IDENTIFICATION OF SEGMENT PROPERTIES

In order to identify data in the subsequent discussions and in calculations, the following notation is introduced.

$${}_s\left[\hat{F}\right]^{(n)} = {}_s\left[\hat{k}\right]^{(n)}\ {}_s\left[\Delta\right]^{(n)} + {}_s\left[\hat{\ell}\right]^{(n)}$$

or, in greater detail,

$$\underset{8xP}{{}_s\widehat{\begin{bmatrix} F(ip) \\ \hline F(jp)\end{bmatrix}}^{(n)}} = \underset{8x8}{{}_s\widehat{\left[\begin{array}{c|c} k(ii) & k(ij) \\ \hline k(ji) & k(jj)\end{array}\right]}^{(n)}}\ \underset{8xP}{{}_s\begin{bmatrix} \Delta(ip) \\ \hline \Delta(jp)\end{bmatrix}^{(n)}} + \underset{8xP}{{}_s\widehat{\begin{bmatrix} \ell(ip) \\ \hline \ell(jp)\end{bmatrix}}^{(n)}} \quad (5\text{-}12)$$

where the following symbols are used:

s,i,j — The s^{th} segment connects joints i and j. The i and j appear in parentheses next to the main symbol. Right-hand subscripts are reserved for component directions or total and reduced stiffnesses.

n — Fourier harmonic

$\wedge$ — Denotes total force

HARMONIC JOINT LOAD PHYSICAL CHARACTERISTICS

The expansion in a Fourier Series results in a separation into physically distinct effects. The 0^{th} term ($n = 0$) is the axisymmetric case. It includes net axial forces and net torsion. The first harmonic ($n = 1$) is the antisymmetric case. It includes net lateral loads in the X direction, and net moments about the Y axis. The remaining harmonics, $n = 2, 3...$, are self-equilibrating systems.

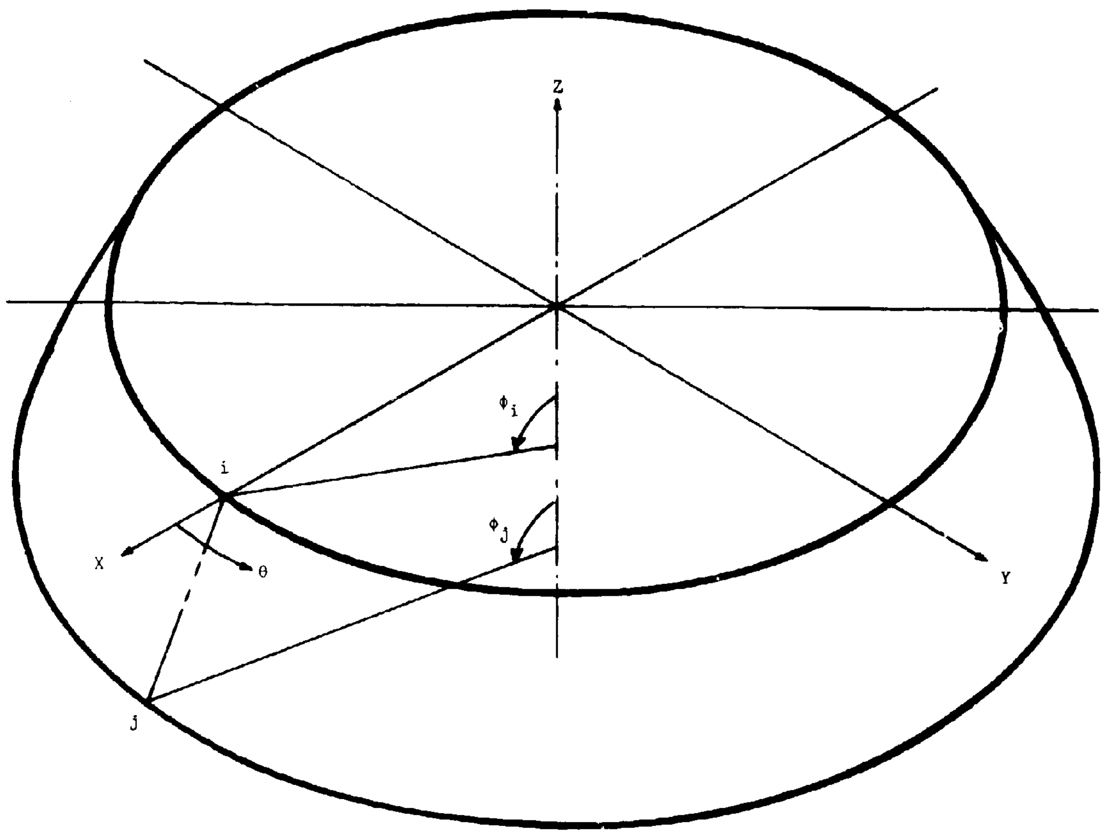

Figure 5-1. Typical Shell Segment

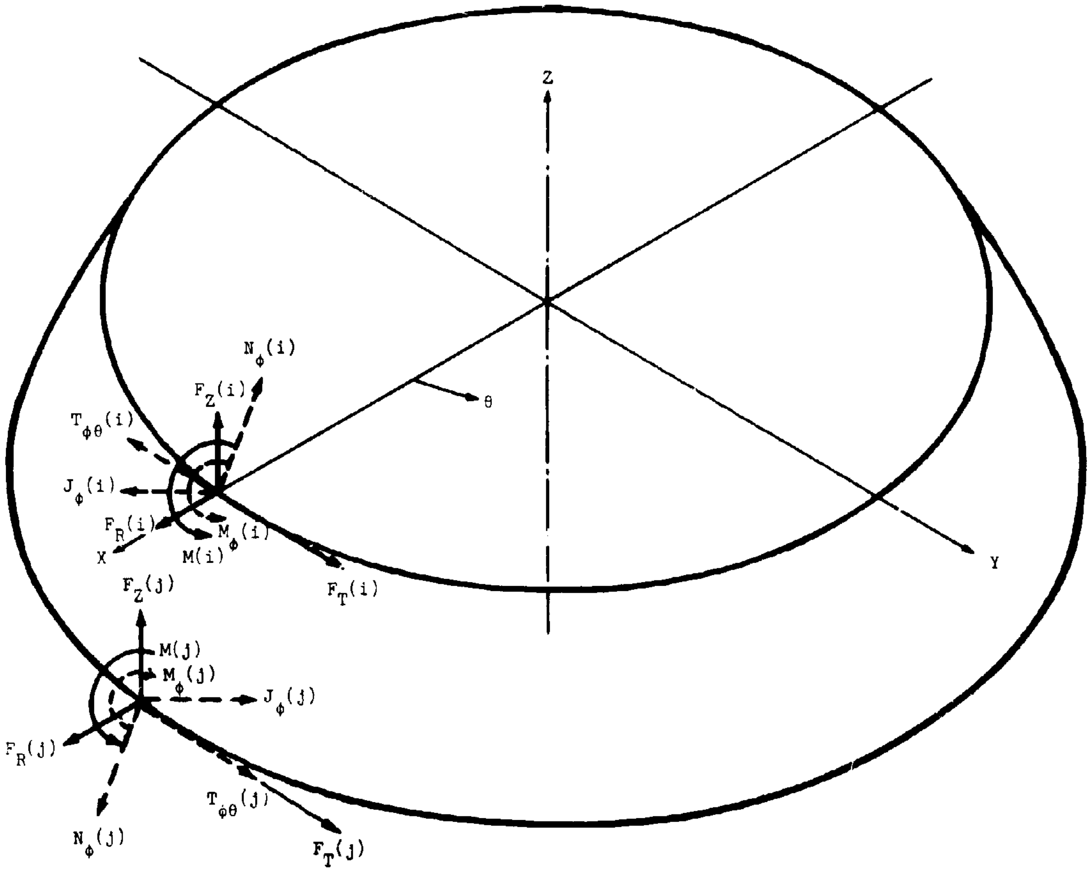

Figure 5-2. Forces on Typical Shell Segment

			Unit Displacements Applied	Unit Forces Applied	Distributed Load Applied (10 possible loading cases)
Column Number			1 2 3 4	5 6 7 8	9 10 •••••••• 18
Initial Conditions	$\{f(i)\}$	$T_{\phi\theta}(i)$ $N_\phi\ (i)$ $J_\phi\ (i)$ $M_\phi\ (i)$	$\left[\ \mathbf{0}\ \right]$	$\left[\ \mathbf{I}_4\ \right]$	$\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$
	$\{\delta(i)\}$	$U\ (i)$ $V\ (i)$ $W\ (i)$ $\Omega_\theta\ (i)$	$\left[\ \mathbf{I}_4\ \right]$	$\left[\ \mathbf{0}\ \right]$	$\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$
Final Conditions	$\{f(j)\}$	$T_{\phi\theta}(j)$ $N_\phi\ (j)$ $J_\phi\ (j)$ $M_\phi\ (j)$	$\left[\ \mathbf{X}_1\ \right]$	$\left[\ \mathbf{X}_2\ \right]$	$\left[\ \mathbf{X}_3\ \right]$
	$\{\delta(j)\}$	$U\ (j)$ $V\ (j)$ $W\ (j)$ $\Omega_\theta\ (j)$	$\left[\ \mathbf{Y}_1\ \right]$	$\left[\ \mathbf{Y}_2\ \right]$	$\left[\ \mathbf{Y}_3\ \right]$

Figure 5-3. Calculations for Stiffness and Load Matrices

SECTION 6

STRUCTURE MATRICES AND STIFFNESS ANALYSIS

In order to calculate the interaction of the segments comprising the structure, the "direct stiffness method" is used (References 5 and 6). Familiarity with this method will be assumed in the following discussion.

To increase the capacity of the program, the shell segments will first be coupled into regions. These regions are defined as singly-connected shells with no internal concentrated line loadings (Figure 6-1). The next step is to construct the region stiffness matrix $\left[\hat{K}_R\right]$ and the matrix of fixed-end forces $\left[\hat{L}_R\right]$. This requires splitting each segment's $\left[\hat{k}\right]$ matrix into its four 4x4 matrices and inserting the portions into the region stiffness matrix in accordance with the topological arrangement. The $\left[\hat{\ell}\right]$ matrix is similarly split into two 4xP matrices. Thus, in addition to the geometric description of each segment, its position in the assembly must be specified. To this end, all segments begin (i) and end (j) at a joint. The s^{th} segment is said to connect the i^{th} and j^{th} joints. (Not the j^{th} and i^{th} joints, since direction of increasing coordinate within the segment must be from i to j). To allow for the possibility of discontinuous centerlines within a region, kinematic links must be included. These links are rigid pieces which relate displacements across a discontinuity. Thus a kinematic link matrix $\left[SKL\right]$ must also be formed. Due to the topology and line-load requirements for regions, the equations of the coupled segments will be the following:

$$\widehat{\left[\begin{array}{c} F_{iR} \\ F_{jR} \\ \hline (8xP) \\ 0 \end{array}\right]} = \widehat{\left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline (8x8) & \\ K_{21} & K_{22} \end{array}\right]} \left[\begin{array}{c} \Delta_{iR} \\ \Delta_{jR} \\ \hline (8xP) \\ \Delta \end{array}\right] + \widehat{\left[\begin{array}{c} L_{iR1} \\ L_{jR1} \\ \hline (8xP) \\ L \end{array}\right]} \qquad (6\text{-}1)$$

where

$$\widehat{\left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right]} = \left[\ SKL\ \right]^T \widehat{\left[\begin{array}{c|c} K'_{11} & K'_{12} \\ \hline K'_{21} & K'_{22} \end{array}\right]} \left[\begin{array}{c|c} SKL_{11} & SKL_{12} \\ \hline SKL_{21} & SKL_{22} \end{array}\right]$$

$$\widehat{\left[\begin{array}{c} L_{iR1} \\ L_{jR1} \\ \hline L \end{array}\right]} = \left[\ SKL\ \right]^T \widehat{\left[\begin{array}{c} L'_{iR} \\ L'_{jR} \\ \hline L' \end{array}\right]}$$

and where iR, jR refer to the region initial and final points, and the $[\Delta]$, $[K']$, and $[L']$ are the deflection, stiffness, and load matrices of internal segments. If there are no internal kinematic links, $[SKL]$ will be an identity matrix. Partitioning Equation 6-1 will yield:

$$\left[\widehat{F_R}\right] = \left[\widehat{K_{11}}\right]\left[\Delta_R\right] + \left[\widehat{K_{12}}\right]\left[\Delta\right] + \left[\widehat{L_{R1}}\right] \qquad (6\text{-}2a)$$

$$\left[0\right] = \left[\widehat{K_{21}}\right]\left[\Delta_R\right] + \left[\widehat{K_{22}}\right]\left[\Delta\right] + \left[\widehat{L}\right] \qquad (6\text{-}2b)$$

Solving Equation 6-2b for $[\Delta]$ and substituting into Equation 6-2a yields:

$$\underset{8xP}{\left[\widehat{F_R}\right]} = \underset{8x8}{\left[\widehat{K_R}\right]}\,\underset{8xP}{\left[\Delta_R\right]} + \underset{8xP}{\left[\widehat{L_R}\right]} \qquad (6\text{-}3)$$

where

$$\left[\widehat{K_R}\right] = \left[\widehat{K_{11}}\right] - \left(\left[\widehat{K_{12}}\right]\left[\widehat{K_{22}}\right]^{-1}\left[\widehat{K_{21}}\right]\right)$$

$$\left[\widehat{L_R}\right] = \left[\widehat{L_{R1}}\right] - \left(\left[\widehat{K_{12}}\right]\left[\widehat{K_{22}}\right]^{-1}\left[\widehat{L}\right]\right)$$

The next step is to construct the total structure stiffness matrix $[\hat{K}]_T$ and the matrix of fixed-end forces $[\hat{L}]_T$. This again requires splitting each region's $[\hat{K}_R]$ matrix into its four 4x4 matrices and inserting the portions into the total stiffness matrix in accordance with the topological arrangement of the structure. The $\left[\hat{L}_R\right]$ matrix is similarly split into two 4xP matrices. Thus, in addition to the geometric description of each region, its position in the structure must be specified. To this end, again all regions begin (i) and end (j) at a joint. The s^{th} region is said to connect the i^{th} and j^{th} joints. (Not the j^{th} and i^{th} joints, since direction of increasing coordinate within the region must be from i to j). If there are M joints, the total stiffness matrix is 4Mx4M, since there are four degrees of freedom at each joint. Hence for equilibrium at all the joints,

$$\underset{4M\times P}{\left[\hat{F}\right]_T} = \underset{4M\times 4M}{\left[\hat{K}\right]_T} \; \underset{4M\times P}{\left[\Delta\right]_T} + \underset{4M\times P}{\left[\hat{L}\right]_T} \tag{6-4}$$

where M is the number of joints, and subscript T denotes Total. This equation characterizes a structure free in space. The singularity of the matrix $\left[\hat{K}\right]_T$ for axisymmetric and antisymmetric (n = 0, 1) cases may be physically interpreted. The stiffness matrix permits calculation of all forces due to all displacements; thus the inverse would relate displacements to forces. But the displacements are not unique, and one valid solution may differ from another by rigid body motion. Hence, we cannot expect such a relationship to exist; the mathematical manifestation is singularity. However, the total stiffness matrix of a complete shell of revolution need not be singular. For harmonics greater than unity, the forces are self-equilibrating systems and, since the displacements follow the same pattern, there can be no rigid-body motion.

REDUCED STIFFNESS

It is necessary to specify restrictions on the displacements such that rigid-body motions are prevented or specific support conditions are met. This is done by means of a "Boundary Condition" matrix $\left[BC(b)\right]$ for the b^{th} boundary condition,

$$\underset{4M\times 1}{\left\{\Delta(b)\right\}_T} = \underset{4M\times q}{\left[BC(b)\right]} \; \underset{q\times 1}{\left\{\Delta(b)\right\}_F} \qquad \text{for } q \leq 4M \tag{6-5}$$

where q is the number of degrees of freedom, subscripts T and F denote Total and Free. The displacements $\{\Delta(b)\}_T$ are in the global coordinates, T, Z, R, Ω_θ. The free, (non-zero) displacements may be in the global system, or they may be rotated through some angle. This may be done, for example, to provide a roller-on-a-ramp restraint for an edge. The [BC(b)] matrix consists merely of an identity matrix without the columns corresponding to fixed coordinates. If, however, there are some rotated coordinates, trigonometric functions will appear. In addition, there may be specified relations between the displacements of one joint and another. This occurs (for example) at discontinuities in thickness or at multiple discontinuities. The [BC] matrix is developed for the simple structure of Figure 6-2a. We have an assembly of conical shells with a discontinuity in thickness. At the discontinuity, the median surface shifts and a kinematic link (2-3) is needed, as shown in Figure 6-2b. The upper edge is attached to a very heavy boss. The lower edge is to be supported by membrane forces. Let us assume that the load is distributed axisymmetric pressure.

The displacements at each of the free joints are given in Figure 6-3. Joint 3 is kinematically dependent on Joint 2 as indicated by algebraic relations. Joint 5 is a support point where membrane stresses are assumed to exist. The coordinate system is, therefore, rotated so that the edge is supported on "rollers". Instead of Δ_Z and Δ_R, we have Δ_ϕ and Δ_ζ, the meridional and normal components, where

$$\begin{Bmatrix} v \\ w \end{Bmatrix} \equiv \begin{Bmatrix} \Delta_\phi \\ \Delta_\zeta \end{Bmatrix} = \begin{bmatrix} -\sin\phi & +\cos\phi \\ -\cos\phi & -\sin\phi \end{bmatrix} \begin{Bmatrix} \Delta_Z \\ \Delta_R \end{Bmatrix} \tag{6-6}$$

At Joint 5, $\phi = \beta$, and we set $\Delta_\phi = 0$ for support. We see by inspection that the structure is statically supported for all possible loadings. If this is not true, the reduced stiffness matrix, which relates forces to free displacements, will be singular.

Now let us construct the boundary condition matrix [BC] for this structure. Assume that the tangential displacement is zero at Joint 5 but that others are free to move (although they will not displace under pressure loading).

Recall that

$$\begin{Bmatrix} \Delta_Z \\ \Delta_R \end{Bmatrix} = \begin{bmatrix} -\sin\phi & -\cos\phi \\ +\cos\phi & -\sin\phi \end{bmatrix} \begin{Bmatrix} \Delta_\phi \\ \Delta_\zeta \end{Bmatrix} \tag{6-7}$$

Then we have

$$\underset{4M\text{x}1}{\{\Delta\}_T} = \underset{4M\text{x}q}{[BC]} \; \underset{q\text{x}1}{\{\Delta\}_F}$$

and in greater detail:

$$\begin{Bmatrix} \Delta_T(1) \\ \Delta_Z(1) \\ \Delta_R(1) \\ \Omega_\theta(1) \\ \Delta_T(2) \\ \Delta_Z(2) \\ \Delta_R(2) \\ \Omega_\theta(2) \\ \Delta_T(3) \\ \Delta_Z(3) \\ \Delta_R(3) \\ \Omega_\theta(3) \\ \Delta_T(4) \\ \Delta_Z(4) \\ \Delta_R(4) \\ \Omega_\theta(4) \\ \Delta_T(5) \\ \Delta_Z(5) \\ \Delta_R(5) \\ \Omega_\theta(5) \end{Bmatrix} = \begin{bmatrix} 1 & & & & & & & & & & & \\ & 1 & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & 1 & & & & & & & & & \\ & & & 1 & & & & & & & & \\ & & & & 1 & & & & & & & \\ & & & & & 1 & & & & & & \\ & & \frac{r_0(3)}{r_0(2)} & & & & & & & & & \\ & & & 1 & & r_0(2)-r_0(3) & & & & & & \\ & & & & 1 & z_0(3)-z_0(2) & & & & & & \\ & & & & & 1 & & & & & & \\ & & & & & & 1 & & & & & \\ & & & & & & & 1 & & & & \\ & & & & & & & & 1 & & & \\ & & & & & & & & & 1 & & \\ & & & & & & & & & & & \\ & & & & & & & & & & -c\phi & \\ & & & & & & & & & & -s\phi & \\ & & & & & & & & & & & 1 \end{bmatrix} \begin{Bmatrix} \Delta_T(1) \\ \Delta_Z(1) \\ \Delta_T(2) \\ \Delta_Z(2) \\ \Delta_R(2) \\ \Omega_\theta(2) \\ \Delta_T(4) \\ \Delta_Z(4) \\ \Delta_R(4) \\ \Omega_\theta(4) \\ \Delta_\zeta(5) \\ \Omega_\theta(5) \end{Bmatrix} \tag{6-8}$$

NOTE

There is a blank row for each displacement specified as zero (fixed). There are no components for the dependent Joint 3 in the right-hand side. The kinematic relations are given in the $[BC]$ matrix. The meridional component $\Delta_\phi(5)$ does not appear since it is fixed. But the perpendicular component $\Delta_\zeta(5)$ contributes to both $\Delta_Z(5)$ and $\Delta_R(5)$.

By a similar procedure, it may be shown that the forces in the directions of free displacements may be expressed in terms of the total forces. This relationship is

$$\{\hat{F}\}_F = [BC]^T \{\hat{F}\}_T \tag{6-9}$$

Corresponding to Δ_ϕ and Δ_ζ, there are free forces $\hat{F}_\phi$ and $\hat{F}_\zeta$. In the example, they are equal to $2\pi r_0(j)\ N_\phi(j)$ and $2\pi r_0(j)\ J_\phi(j)$. If applied at the i^{th} edge, there would be a change in sign. In general though, the direction denoted by ϕ and ζ need not be related to the shell. It is specified separately.

Thus, from Equations 6-4 through 6-6 and Equation 6-9:

$$\{\hat{F}\}_F = [BC]^T [\hat{K}]_T [BC] \{\Delta\}_F + [BC]^T \{\hat{L}\}_T \tag{6-10}$$

$$\{\hat{F}\}_F = [\hat{K}]_F \{\Delta\}_F + \{\hat{L}\}_F \tag{6-11}$$

$$\{\Delta\}_F = [\hat{A}]_F \left(\{\hat{F}\}_F - \{\hat{L}\}_F\right) \tag{6-12}$$

where

$$[\hat{A}]_F \equiv [\hat{K}]_F^{-1} \tag{6-13}$$

The total displacements are then calculated from Equation 6-5

$$\{\Delta\}_T = [BC] \{\Delta\}_F \tag{6-14}$$

Thus for the region ends, combining Equations 6-14 and 6-12

$$\{\Delta_R\} = [BC]\,[\hat{A}]_F\left(\{\hat{F}\}_F - \{\hat{L}\}_F\right) \tag{6-15}$$

and in the interior of each region, for each segment

$$\{\Delta\} = [SKL_{22}]\left\{-[\hat{K}_{22}]^{-1}\left([\hat{K}_{21}]\,\{\Delta_R\} + [\hat{L}]\right)\right\} \tag{6-16}$$

FINAL STRESS DISTRIBUTION

We must obtain the forces and displacements at the i^{th} edge of each segment to use as initial values in the integration of the differential equations. Since the variables in the differential equations are given in the local coordinate system, from Equation 5-3, we seek

$$\{\delta(i)\} = [IDT]^T\,\{\Delta(i)\}$$

From Equations 5-1 and 5-5,

$$\{f(i)\} = [IFT]^T\left(\left[k_{ii}\;\vdots\;k_{ij}\right]\left\{\begin{matrix}\Delta(i)\\ \hdashline \Delta(j)\end{matrix}\right\} \quad \{\ell(i)\}\right) \tag{6-17}$$

where the $\Delta(i)$ and $\Delta(j)$ are obtained from Equation 6-16.

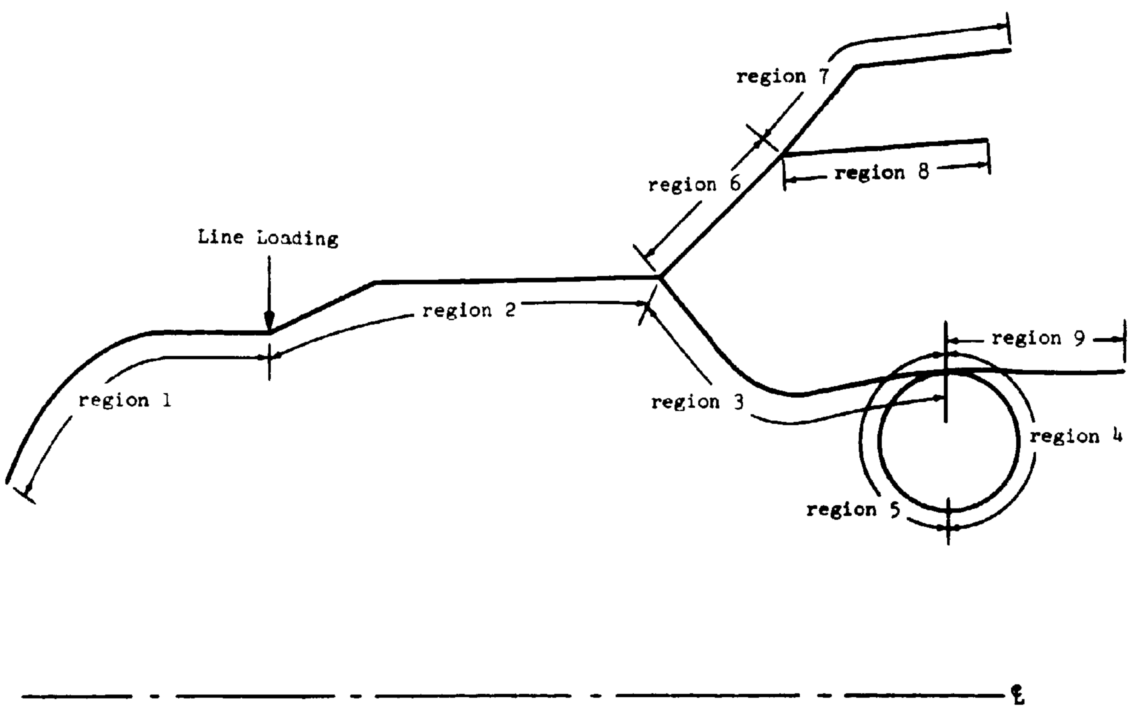

Figure 6-1. Example of Region Topology

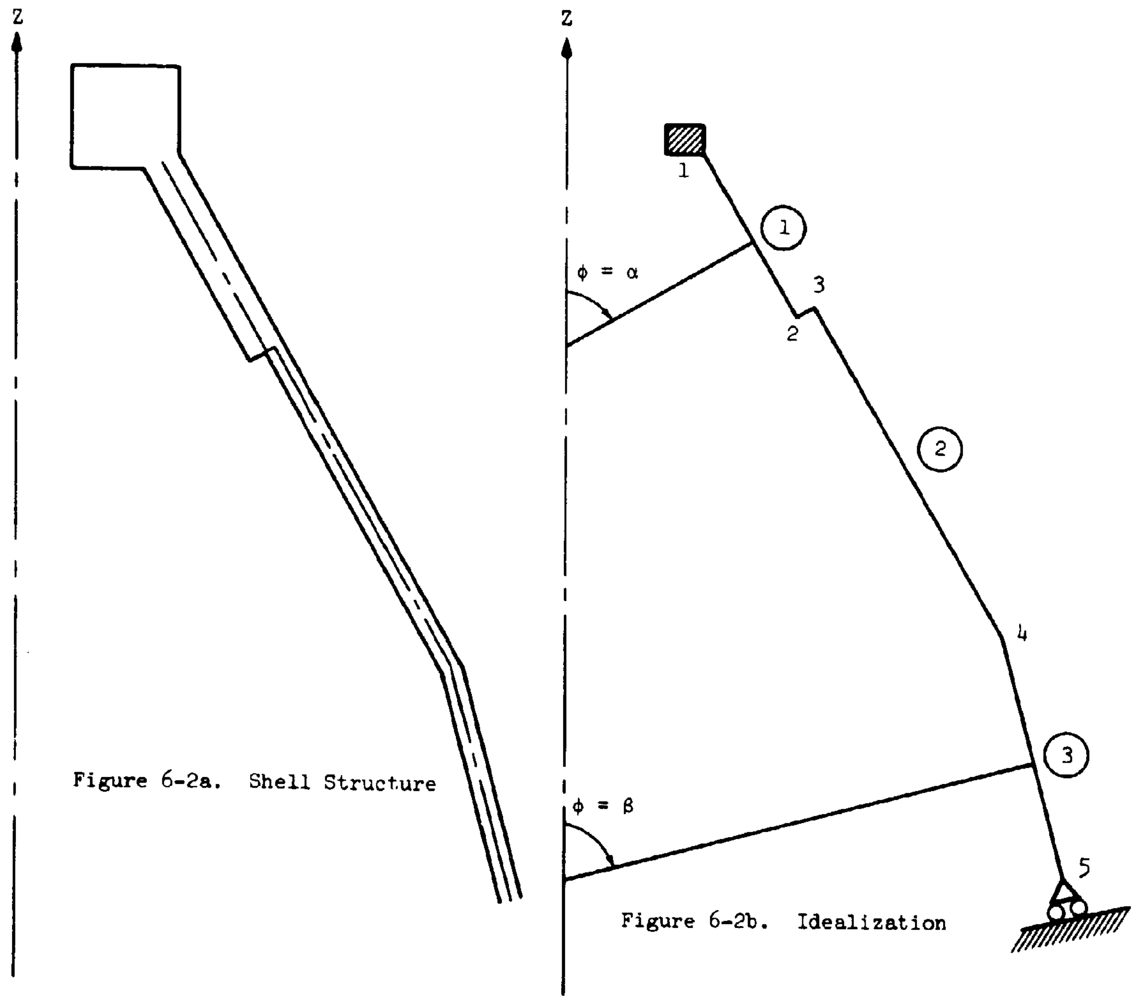

Figure 6-2a. Shell Structure

Figure 6-2b. Idealization

Figure 6-2. Shell Structure and Idealization

JOINT DISPLACEMENTS

Joint	Δ_T	Δ_Z	Δ_R	Ω_θ	Rotation Angle
1	*	Free: Rigid-body motion of boss	0: Built-in to stiff boss	0: Built-in to stiff boss	---
2	*	Free	Free	Free	---
3	$\frac{+r_0(3)}{r_0(2)}\Delta_T(2)$	$\Delta_Z(2)-\left[r_0(3)-r_0(2)\right]\Omega_\theta(2)$	$\Delta_R(2)+\left[z_0(3)-z_0(2)\right]\Omega_\theta(2)$	$\Omega_\theta(2)$	---
4	*	Free	Free	Free	---
5	*	$\Delta_\phi = 0$	Δ_ζ Free	Free	β

*Although tangential loads are not considered in this case, the system must have reactions for all possible loads of the harmonic. Therefore, at least one of the Δ_T must be 0 to prevent rigid body rotation. Other displacement components are specified as 0 (fixed) or 1 (free).

Figure 6-3. Joint Displacements

SECTION 7

AXISYMMETRIC NON-LINEAR ANALYSIS

The previous sections all dealt with linear shell analysis. Geometric non-linear effects in the axisymmetric case can be included by the use of a simple iterative analysis and some additions to the shell equilibrium equations. The revised axisymmetric (n = 0) equilibrium equations are as follows:

INDEPENDENT VARIABLE, ϕ

$$\frac{T_{\phi\theta,\phi}^{(0)}}{r_1} = -2T_{\phi\theta}^{(0)} \frac{\cos\phi}{r_0} - M_{\phi\theta}^{(0)} \frac{\cos\phi}{r_0}\left[\frac{1}{r_1} - \frac{\sin\phi}{r_0}\right] - f_\theta^{(0)} - m_\phi^{(0)} \frac{\sin\phi}{r_0} \qquad (7\text{-}1)$$

$$\frac{N_{\phi,\phi}^{(0)}}{r_1} = -N_\phi^{(0)} \frac{\cos\phi}{r_0} + N_\theta^{(0)} \frac{\cos\phi}{r_0} + \frac{{}^*J_\phi^{(0)}}{r_1} - f_\phi^{(0)}$$

$$\frac{{}^*J_{\phi,\phi}^{(0)}}{r_1} = -J_\phi^{(0)} \frac{\cos\phi}{r_0} - N_\theta^{(0)} \frac{\sin\phi}{r_0} - \frac{N_\phi^{(0)}}{r_1} - f_\zeta^{(0)} - {}^{**}f_\zeta^{(0)}$$

$$\frac{M_{\phi,\phi}^{(0)}}{r_1} = M_\theta^{(0)} \frac{\cos\phi}{r_0} - M_\phi^{(0)} \frac{\cos\phi}{r_0} + J_\phi^{(0)} + m_\theta^{(0)}$$

where the nonlinear terms are defined as follows:

$${}^{**}f_\zeta^{(0)} = \frac{\cos\phi}{r_0}\left[\overline{N}_\phi^{(0)}\ \Omega_\theta^{(0)}\right] \qquad (7\text{-}2)$$

$$J_\phi^{(0)} = {}^*J_\phi^{(0)} - \overline{N}_\phi^{(0)}\ \Omega_\theta^{(0)}$$

$$f_\theta^{(0)} = F_\theta^{(0)}$$

$$f_\phi^{(0)} = F_\phi^{(0)}\left(1 + \frac{V^{(0)}\cos\phi}{r_0} - W^{(0)}\left[\frac{1}{r_1} + \frac{\sin\phi}{r_0}\right] + \frac{V,_\phi^{(0)}}{r_1}\right)$$

$$- F_\zeta^{(0)}\,\Omega_\theta^{(0)}$$

$$f_\zeta^{(0)} = F_\zeta^{(0)}\left(1 + \frac{V^{(0)}\cos\phi}{r_0} - W^{(0)}\left[\frac{1}{r_1} + \frac{\sin\phi}{r_0}\right] + \frac{V,_\phi^{(0)}}{r_1}\right)$$

$$+ F_\phi^{(0)}\,\Omega_\theta^{(0)}$$

These terms are essentially components of the membrane forces and the pressure loads in directions normal to their original lines of action (Figure 7-1). Similar components of the transverse shear forces, Q_θ and Q_ϕ, also exist, but for thin shells they are usually neglected (Reference 3).

The equations specialized for other geometries are:

CYLINDER

$$\frac{dT_{\phi\theta}^{(0)}}{ds} = -f_\theta^{(0)} - \frac{1}{r_0}m_\phi^{(0)} \qquad (7\text{-}3)$$

$$\frac{dN_\phi^{(0)}}{ds} = -f_\phi^{(0)}$$

$$\frac{d^*J_\phi^{(0)}}{ds} = -\frac{N_\theta^{(0)}}{r_0} - f_\zeta^{(0)}$$

$$\frac{dM_\phi^{(0)}}{ds} = J_\phi^{(0)} + m_\theta^{(0)}$$

where the nonlinear terms are defined as follows:

$$J_\phi^{(0)} = {}^*J_\phi^{(0)} - \overline{N}_\phi^{(0)}\ \Omega_\theta^{(0)} \tag{7-4}$$

$$f_\theta^{(0)} = F_\theta^{(0)}$$

$$f_\phi^{(0)} = F_\phi^{(0)}\left(1 - \frac{W^{(0)}}{r_0} + \frac{dV^{(0)}}{ds}\right) - F_\zeta^{(0)}\ \Omega_\theta^{(0)}$$

$$f_\zeta^{(0)} = F_\zeta^{(0)}\left(1 - \frac{W^{(0)}}{r_0} + \frac{dV^{(0)}}{ds}\right) + F_\phi^{(0)}\ \Omega_\theta^{(0)}$$

<u>CONE</u>

$$\frac{dT_{\phi\theta}^{(0)}}{ds} = -\frac{2T_{\phi\theta}^{(0)}}{s} + \frac{M_{\phi\theta}^{(0)}\tan\phi}{s^2} - f_\theta^{(0)} - \frac{m_\phi^{(0)}\tan\phi}{s} \tag{7-5}$$

$$\frac{dN_\phi^{(0)}}{ds} = -\frac{N_\phi^{(0)}}{s} + \frac{N_\theta^{(0)}}{s} - f_\phi^{(0)}$$

$$\frac{d\,{}^*J_\phi^{(0)}}{ds} = -\frac{J_\phi^{(0)}}{s} - \frac{N_\theta^{(0)}\tan\phi}{s} - f_\zeta^{(0)} - {}^{**}f_\zeta^{(0)}$$

$$\frac{dM_\phi^{(0)}}{ds} = \frac{M_\theta^{(0)}}{s} - \frac{M_\phi^{(0)}}{s} + J_\phi^{(0)} + m_\theta^{(0)}$$

where the nonlinear terms are defined as follows:

$${}^{**}f_\zeta^{(0)} = \frac{1}{s}\left\{\overline{N}_\phi^{(0)}\ \Omega_\theta^{(0)}\right\} \tag{7-6}$$

$$J_\phi^{(0)} = {}^*J_\phi^{(0)} - \overline{N}_\phi^{(0)}\ \Omega_\theta^{(0)}$$

$$f_\theta^{(0)} = F_\theta^{(0)}$$

$$f_\phi^{(0)} = F_\phi^{(0)}\left(1 + \frac{v^{(0)}}{s} - \frac{w^{(0)}\tan\phi}{s} + \frac{dv^{(0)}}{ds}\right) - F_\zeta^{(0)}\,\Omega_\theta^{(0)}$$

$$f_\zeta^{(0)} = F_\zeta^{(0)}\left(1 + \frac{v^{(0)}}{s} - \frac{w^{(0)}\tan\phi}{s} + \frac{dv^{(0)}}{ds}\right) + F_\phi^{(0)}\,\Omega_\theta^{(0)}$$

The capital F's in all the above equations are the actual distributed loads applied to the shell structure.

In this presentation, the capability for axisymmetric non-linear torsion analysis is omitted.

The numerical procedure used to solve these differential equations is entirely analogous to the linear case described in earlier sections. The equilibrium equations given herein are coupled with equations obtained from Hooke's Laws and the strain-displacement relations; for instance the latter portion of Equations 3-1 for unreinforced shells. The matrix procedures described in Sections 5 and 6 are then utilized as previously. First, however, the equations are linearized by assuming a value for $\overline{N}_\phi$. This will also destroy the symmetry of the "stiffness" matrix, since Maxwell's Law of reciprocity holds only for linear systems. In the mathematical sense, we may now regard this as an arbitrary linear boundary value problem, since the equations have been linearized by the assumption of $\overline{N}_\phi$. Thus the "stiffness" matrix [k] now simply represents the quantities {F} as a linear combination of the quantities {Δ}. Of course, it is understood that the relationship is linear only for the special loading case which produces the particular assumed $\overline{N}_\phi$ value. Thus for the non-linear case, use of the matrices [k] in Sections 5 and 6 may be regarded only as elements in matrix algebra, and no assumption of symmetry is required. However, the physical interpretation still exists in the linear cases.

The iteration procedure for a non-linear analysis would then operate as follows:

1. A value of $\overline{N}_\phi$ is obtained for a first approximation. This

value may be obtained by a hand computation utilizing membrane theory or a full linear bending solution of the problem.

2. Utilizing this value of $\overline{N}_\phi$ as a constant, a solution is obtained for the linearized system of equations.

3. The solution for N_ϕ from step 2 is compared with the assumed $\overline{N}_\phi$. If the agreement is not close enough, the N_ϕ value is used as the new assumed $\overline{N}_\phi$.

4. Steps 2 and 3 are repeated until the agreement of N_ϕ with $\overline{N}_\phi$ is within the accuracy desired.

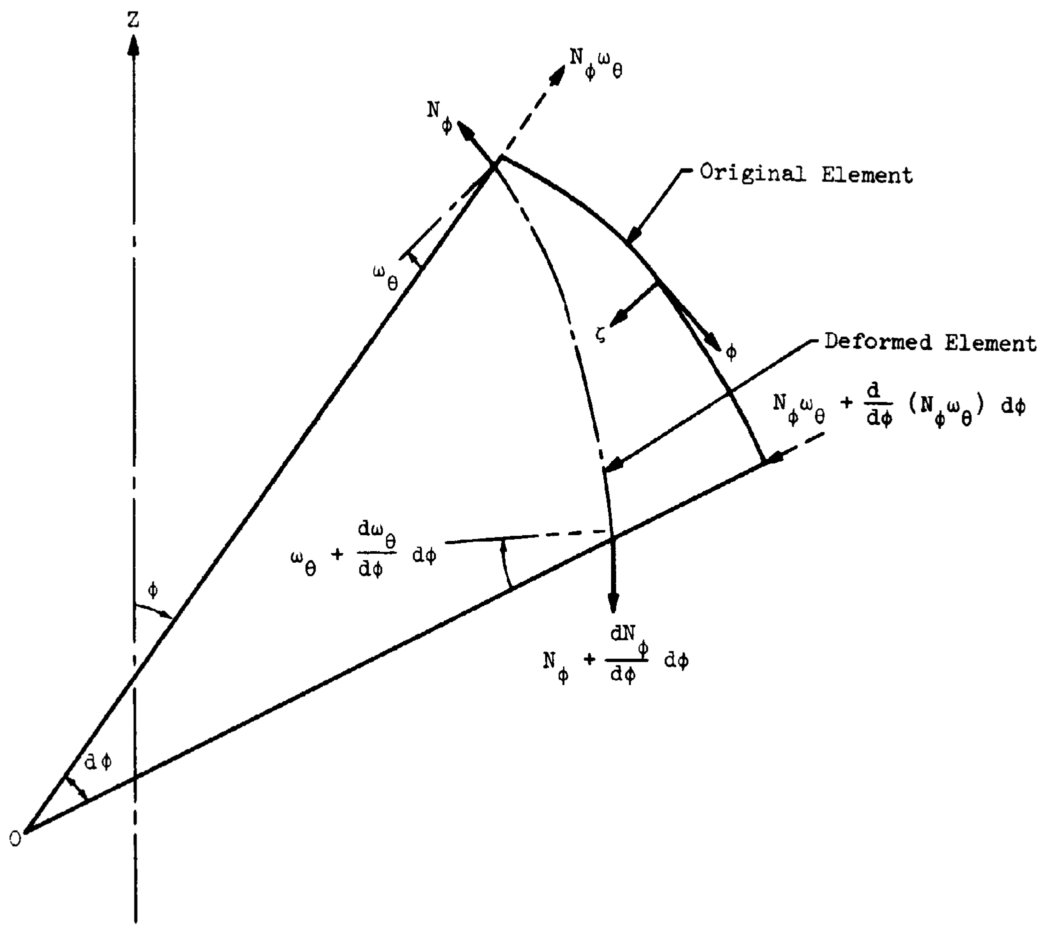

Figure 7-1. Non-Linear Effects

SECTION 8

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APPENDIX A

DERIVATIONS OF INTEGRATED HOOKE'S LAWS

One method of obtaining the necessary relationships between stress-resultants and strains for eccentrically reinforced shells is basically an "equivalent energy" approach. First we obtain the energy of the composite system in terms of stress-resultants and strains, and then equate it to an equivalent shell energy expression. Since the necessary equations for an orthotropic sheet are known (Equations 1-7 and 1-8), only the equations for the eccentric stiffeners have to be cast in the proper form.

Consider the strain energy of a circumferential rib:

$$U_\theta = \sum_{k=1}^{n} \left(\int_0^{2\pi r_0} \int_{A_\theta} \frac{E_\theta}{2} \varepsilon_{\theta_T}^{\ 2} \, dA_\theta d\theta + \frac{G_\theta J_\theta}{2} \int_0^{2\pi r_0} k_{\theta\phi}^{\ 2} d\theta \right)_k \qquad \text{(A-1)}$$

where n is the number of circumferential stiffeners, and the rib is assumed to have the same twist as the shell. If the stiffeners are spaced a distance S_θ apart, Equation A-1 can be rewritten as:

$$U_\theta = \frac{1}{S_\theta} \int_0^{S} \left[\int_0^{2\pi r_0} \int_{A_\theta} \frac{E_\theta}{2} \varepsilon_{\theta_T}^{\ 2} \, dA_\theta d\theta + \frac{G_\theta J_\theta}{2} \int_0^{2\pi r_0} k_{\theta\phi}^{\ 2} d\theta \right] d\phi \quad \text{(A-2)}$$

Substituting for the total strain in terms of centroidal strain and curvature, and using the assumption that the stiffeners are integrally connected to the shell and remain so after deformation, we obtain

$$U_\theta = \frac{1}{S_\theta} \int_0^{S} \left[\int_0^{2\pi r_0} \int_{A_\theta} \frac{E_\theta}{2} (\varepsilon_{\theta_o} - \zeta k_\theta)^2 dA_\theta d\theta + \frac{G_\theta J_\theta}{2} \int_0^{2\pi r_0} k_{\theta\phi}^{\ 2} d\theta \right] d\phi$$

Simplifying:

$$U_\theta = \frac{1}{S_\theta} \int_0^S \int_0^{2\pi r_0} \left(\frac{E_\theta A_\theta}{2} \epsilon_{\theta_o}^{\;2} - E_\theta C_\theta A_\theta \epsilon_{\theta_o} k_\theta + \frac{E_\theta}{2} I_\theta k_\theta^{\;2} + \frac{G_\theta J_\theta}{2} k_{\theta\phi}^{\;2} \right) d\theta d\phi \tag{A-3}$$

where:

$$\int_{A_\theta} dA_\theta = A_\theta \qquad \int_{A_\theta} \zeta dA_\theta = C_\theta A_\theta \qquad \int_{A_\theta} \zeta^2 dA_\theta = I_\theta$$

Similarly, the stiffeners in the meridional direction have a total strain energy of:

$$U_\phi = \frac{1}{S_\phi} \int_0^S \int_0^{2\pi r_0} \left(\frac{E_\phi A_\phi}{2} \epsilon_{\phi_o}^{\;2} - E_\phi C_\phi A_\phi \epsilon_{\phi_o} k_\phi + \frac{E_\phi}{2} I_\phi k_\phi^{\;2} + \frac{G_\phi J_\phi}{2} k_{\phi\theta}^{\;2} \right) d\theta d\phi \tag{A-4}$$

The strain energy expression of a particular shell sheet is

$$U_S = \frac{1}{2} \int_0^{2\pi r_0} \int_0^S \left(N_\theta \epsilon_{\theta_o} + N_\phi \epsilon_{\phi_o} + N_{\phi\theta} \gamma_{\phi\theta_o} - M_\theta k_\theta - M_\phi k_\phi - 2M_{\phi\theta} k_{\phi\theta} \right) d\theta d\phi \tag{A-5}$$

Equating like terms of Equations A-3 and A-4 to A-5 we can obtain the additional terms to expand Equations 1-8 which were written as Equations 4-1 in Section 4.

$$N_\theta = \frac{E_\theta h}{1-\nu_{\phi\theta}\nu_{\theta\phi}}\left[\epsilon_{\theta_o} + \nu_{\theta\phi}\epsilon_{\phi_o}\right] + \frac{E_{\theta_R}A_\theta}{S_\theta}\epsilon_{\theta_o} - \frac{E_{\theta_R}C_\theta A_\theta}{S_\theta}k_\theta - N_{T\theta} \qquad (A\text{-}6)$$

$$N_\phi = \frac{E_\phi h}{1-\nu_{\phi\theta}\nu_{\theta\phi}}\left[\epsilon_{\phi_o} + \nu_{\phi\theta}\epsilon_{\theta_o}\right] + \frac{E_{\phi_R}A_\phi}{S_\phi}\epsilon_{\phi_o} - \frac{E_{\phi_R}C_\phi A_\phi}{S_\phi}k_\phi - N_{T\phi}$$

$$N_{\phi\theta} = G_{\phi\theta} h\, \gamma_{\phi\theta_o}$$

$$M_\theta = \frac{-E_\theta h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})}\left[k_\theta + \nu_{\theta\phi}k_\phi\right] - \frac{E_{\theta_R}I_\theta}{S_\theta}k_\theta + \frac{E_{\theta_R}C_\theta A_\theta}{S_\theta}\epsilon_{\theta_o} - M_{T\theta}$$

$$M_\phi = \frac{-E_\phi h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})}\left[k_\phi + \nu_{\phi\theta}k_\theta\right] - \frac{E_{\phi_R}I_\phi}{S_\phi}k_\phi + \frac{E_{\phi_R}C_\phi A_\phi}{S_\phi}\epsilon_{\phi_o} - M_{T\phi}$$

$$M_{\phi\theta} = \frac{-G_{\phi\theta}h^3}{6}k_{\phi\theta} - \frac{G_{\phi_R}J_\phi}{2S_\phi}k_{\phi\theta} - \frac{G_{\theta_R}J_\theta}{2S_\theta}k_{\phi\theta}$$

where A = area of reinforcement

C = eccentricity of reinforcement from shell middle surface (inwards positive)

I = moment of inertia of reinforcement about basic shell centroidal axis

J = crossection twist constant of reinforcement

S = spacing of reinforcement

subscripts θ or ϕ indicate coordinate directions, and subscript R refers to reinforcement properties.

Thus the stiffness parameters in Equation 4-1 are defined as follows:

$$K_{11} = \frac{E_\theta h}{1-\nu_{\phi\theta}\nu_{\theta\phi}} + \frac{E_{\theta_R}A_\theta}{S_\theta} \qquad K_{22} = \frac{E_\phi h}{1-\nu_{\phi\theta}\nu_{\theta\phi}} + \frac{E_{\phi_R}A_\phi}{S_\phi} \qquad \text{(A-7)}$$

$$K_{12} = \frac{\nu_{\theta\phi}E_\theta h}{1-\nu_{\phi\theta}\nu_{\theta\phi}} \qquad K_{21} = \frac{\nu_{\phi\theta}E_\phi h}{1-\nu_{\phi\theta}\nu_{\theta\phi}}$$

$$C_{11} = \frac{E_{\theta_R}C_\theta A_\theta}{S_\theta} \qquad C_{22} = \frac{E_{\phi_R}C_\phi A_\phi}{S_\phi}$$

$$K_{33} = G_{\phi\theta}h \qquad D_{22} = \frac{-E_\phi h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})} - \frac{E_{\phi_R}I_\phi}{S_\phi}$$

$$D_{11} = \frac{-E_\theta h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})} - \frac{E_{\theta_R}I_\theta}{S_\theta} \qquad D_{21} = \frac{-\nu_{\phi\theta}E_\phi h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})}$$

$$D_{12} = \frac{-\nu_{\theta\phi}E_\theta h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})} \qquad D_{33} = \frac{G_{\phi\theta}h^3}{12} + \frac{G_{\phi_R}J_\phi}{4S_\phi} + \frac{G_{\theta_R}J_\theta}{4S_\theta}$$

Utilizing the orthotropic identity, $K_{12} = K_{21}$ and $D_{12} = D_{21}$.

The Equations A-6 are somewhat approximate. Firstly, since the reinforcement properties are "smeared", the equations will not be accurate where the reinforcement is widely spaced. To this end, when loading a shell with a high circumferential harmonic (n) loading, one should check if the load peaks attain closer spacing than the reinforcement. If this is so, the Equations A-6 are not applicable. The load patterns should similarly be checked in the meridional direction. Secondly, due to our first order theory assumption that $M_{\phi\theta} = -M_{\theta\phi}$, the torsional constant is only approximate in cases where reinforcement properties GJ/S are not equal in the two coordinate directions.

The above derivations are valid when the reinforcement coincides with the coordinate directions of the shell. If the reinforcement is rotated, such as in a waffle oriented at 45 degrees, extensive revisions are necessary. Such equations, applicable only to a 45-degree waffle construction are derived in a different manner below (Reference 9).

The rotated rib grid stresses are:

$$\sigma_\theta = \frac{tE_R}{2S}\left[(\varepsilon_{\theta_o} + \varepsilon_{\phi_o}) - \zeta\,(k_\phi + k_\theta)\right] = \sigma_\phi$$

$$\tau_{\phi\theta} = \frac{tE_R}{2S}\left[\gamma_{\phi\theta_o} - 2\zeta k_{\phi\theta}\right] = \tau_{\theta\phi} \qquad \text{(A-8)}$$

When the above equations are integrated and added to the sheet stress resultant Equations 1-8, the final set is obtained.

$$N_\theta = \frac{E_\theta h}{1-\nu_{\phi\theta}\nu_{\theta\phi}}\left[\varepsilon_{\theta_o} + \nu_{\theta\phi}\varepsilon_{\phi_o}\right] + \frac{E_R A}{2S}(\varepsilon_{\theta_o} + \varepsilon_{\phi_o}) - \frac{E_R AC}{2S}(k_\theta + k_\phi) - N_{T\theta}$$

(A-9)

$$N_\phi = \frac{E_\phi h}{1-\nu_{\phi\theta}\nu_{\theta\phi}}\left[\varepsilon_{\phi_o} + \nu_{\phi\theta}\varepsilon_{\theta_o}\right] + \frac{E_R A}{2S}(\varepsilon_{\theta_o} + \varepsilon_{\phi_o}) - \frac{E_R AC}{2S}(k_\theta + k_\phi) - N_{T\phi}$$

$$N_{\phi\theta} = G_{\phi\theta} h\gamma_{\phi\theta_o} + \frac{E_R A}{2S}\gamma_{\phi\theta_o} - \frac{E_R AC}{S}k_{\phi\theta}$$

$$M_\theta = \frac{-E_\theta h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})}\left[k_\theta + \nu_{\theta\phi}k_\phi\right] - \frac{E_R I}{2S}(k_\theta + k_\phi) + \frac{E_R AC}{2S}(\varepsilon_{\theta_o} + \varepsilon_{\phi_o}) - M_{T\theta}$$

$$M_\phi = \frac{-E_\phi h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})}\left[k_\phi + \nu_{\phi\theta}k_\theta\right] - \frac{E_R I}{2S}(k_\phi + k_\theta) + \frac{E_R AC}{2S}(\varepsilon_{\theta_o} + \varepsilon_{\phi_o}) - M_{T\phi}$$

$$M_{\phi\theta} = \frac{-G_{\phi\theta}h^3}{6}k_{\phi\theta} - \frac{E_R I}{S}k_{\phi\theta} + \frac{E_R AC}{2S}\gamma_{\phi\theta_o}$$

where A = area of reinforcement

C = eccentricity of reinforcement from shell middle surface (inwards positive)

I = moment of inertia of reinforcement about basic shell centroidal axis

S = spacing of reinforcement

and subscript R refers to reinforcement properties.

Thus the stiffness parameters in Equation 4-10 are defined as follows:

$$K_{11} = \frac{E_\theta h}{1-\nu_{\phi\theta}\nu_{\theta\phi}} + \frac{E_R A}{2S} \qquad K_{22} = \frac{E_\phi h}{1-\nu_{\phi\theta}\nu_{\theta\phi}} + \frac{E_R A}{2S} \qquad \text{(A-10)}$$

$$K_{12} = \frac{\nu_{\theta\phi} E_\theta h}{1-\nu_{\phi\theta}\nu_{\theta\phi}} + \frac{E_R A}{2S} \qquad K_{21} = \frac{\nu_{\phi\theta} E_\phi h}{1-\nu_{\phi\theta}\nu_{\theta\phi}} + \frac{E_R A}{2S}$$

$$C_{11} = \frac{E_R AC}{2S} \qquad K_{33} = G_{\phi\theta} h + \frac{E_R A}{2S}$$

$$D_{11} = \frac{-E_\theta h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})} - \frac{E_R I}{2S} \qquad D_{22} = \frac{-E_\phi h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})} - \frac{E_R I}{2S}$$

$$D_{12} = \frac{-\nu_{\theta\phi} E_\theta h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})} - \frac{E_R I}{2S} \qquad D_{21} = \frac{-\nu_{\phi\theta} E_\phi h^3}{12(1-\nu_{\phi\theta}\nu_{\theta\phi})} - \frac{E_R I}{2S}$$

$$D_{33} = \frac{G_{\phi\theta} h^3}{12} + \frac{E_R I}{2S}$$

Utilizing the orthotropic identity, $K_{12} = K_{21}$ and $D_{12} = D_{21}$.

APPENDIX B

APPLIED EDGE LOADS

In utilizing the program, it is frequently necessary to apply forces at the edge of the structure. We first note some relations between the internal stress resultants and global forces in the rotated system. Referring to Figure 5-2, we see that it is necessary to distinguish between the i^{th} and j^{th} edges of the loaded segment. For any harmonic n,

$$\begin{Bmatrix} F_T\binom{i}{j} \\ F_\phi\binom{i}{j} \\ F_\zeta\binom{i}{j} \\ M\binom{i}{j} \end{Bmatrix}^{(n)} = \pm \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \begin{Bmatrix} T_{\phi\theta}\binom{i}{j} \\ N_\phi\binom{i}{j} \\ J_\phi\binom{i}{j} \\ M_\phi\binom{i}{j} \end{Bmatrix}^{(n)} \qquad (B-\tfrac{1}{2})$$

where the ± corresponds to the $\genfrac{}{}{0pt}{}{i^{th}}{j^{th}}$ edges.

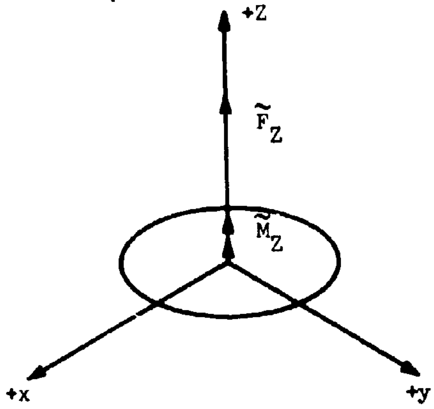
As in Equation 6-7 for rotated displacements, the rotated forces are given by

$$\begin{Bmatrix} F_T \\ F_Z \\ F_R \\ M \end{Bmatrix}^{(n)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -s\phi & -c\phi & 0 \\ 0 & +c\phi & -s\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_T \\ F_\phi \\ F_\zeta \\ M \end{Bmatrix}^{(n)} \qquad (B-3)$$

We now relate the resultant external loads (~) to the magnitudes of distributed loads. These will involve only the 0^{th} and 1^{st} harmonics.

AXISYMMETRIC LOADS (n = 0)

We may write directly



$$\begin{Bmatrix} \tilde{F}_Z \\ \tilde{M}_Z \end{Bmatrix} = 2\pi r_0 \begin{bmatrix} 0 & 1 & 0 & 0 \\ r_0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} F_T \\ F_Z \\ F_R \\ M \end{Bmatrix}^{(0)} \qquad \text{(B-4)}$$

ANTISYMMETRIC LOAD (n = 1)

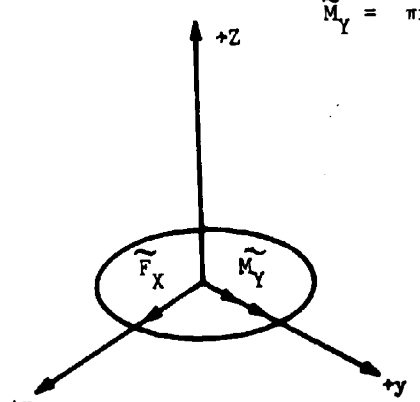
Here, the integration of the distributed forces is not obvious.

$$\tilde{F}_X = \int_0^{2\pi} \left(-F_T^{(1)} r_0 d\theta\right) \sin\theta + \int_0^{2\pi} \left(F_R^{(1)} r_0 d\theta\right) \cos\theta$$

$$\tilde{F}_X = \pi r_0 \left[-F_T^{(1)} + F_R^{(1)}\right]$$

$$\tilde{M}_Y = \int_0^{2\pi} \left(- F_Z^{(1)} r_0 d\theta\right) (r_0 \cos\theta) + \int_0^{2\pi} \left(M^{(1)} r_0 d\theta\right) \cos\theta$$

$$\tilde{M}_Y = \pi r_0 \left[-r_0 F_Z^{(1)} + M^{(1)}\right]$$



$$\begin{Bmatrix} \tilde{F}_X \\ \tilde{M}_Y \end{Bmatrix} = \pi r_0 \begin{bmatrix} -1 & 0 & +1 & 0 \\ 0 & -r_0 & 0 & +1 \end{bmatrix} \begin{Bmatrix} F_T \\ F_Z \\ F_R \\ M \end{Bmatrix}^{(1)} \qquad \text{(B-5)}$$

SPECIFYING LOADS

When the loaded edge has standard coordinates, Equations B-4 or B-5 are used to determine the forces F_T, F_Z, F_R, and M in terms of the net applied loads. In the axisymmetric case, this is straight forward; contributions to $\widetilde{F}_Z$ are made only by $F_Z^{(0)}$ and there is a similar relation between $\widetilde{M}_Z$ and $F_T^{(0)}$.

However, in the antisymmetric case, there are four unknowns and only two equations. Thus, additional data is required. Often, these loads are applied in a region of assumed membrane stress; then $F_R^{(1)} = 0$ (cylinder) or $F_Z^{(1)} = 0$ (plate) and $M^{(1)} = 0$, since these are transverse shear and bending stress resultants.

When the loaded edge has rotated coordinates, use of Equation B-3 with B-4 and B-5 yields:

$$\begin{Bmatrix} \widetilde{F}_Z \\ \widetilde{M}_Z \end{Bmatrix} = 2\pi r_0 \begin{bmatrix} 0 & -s\phi & -c\phi & 0 \\ r_0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} F_T \\ F_\phi \\ F_\zeta \\ M \end{pmatrix}^{(0)} \qquad \text{(B-6)}$$

and

$$\begin{Bmatrix} \widetilde{F}_X \\ \widetilde{M}_Y \end{Bmatrix} = \pi r_0 \begin{bmatrix} -1 & +c\phi & -s\phi & 0 \\ 0 & r_0 s\phi & r_0 c\phi & +1 \end{bmatrix} \begin{pmatrix} F_T \\ F_\phi \\ F_\zeta \\ M \end{pmatrix}^{(1)} \qquad \text{(B-7)}$$

In the axisymmetric case, $\widetilde{F}_Z$ now depends on F_ϕ and F_ζ. For membrane support, $F_\zeta^{(0)} = 0$. Then $F_T^{(0)}$ and $F_\psi^{(0)}$ are uniquely determined. In the antisymmetric case, membrane support implies $F_\zeta^{(1)} = M^{(1)} = 0$. Then

$$\begin{Bmatrix} \widetilde{F}_X \\ \\ \widetilde{M}_Y \end{Bmatrix} = \pi r_0 \begin{bmatrix} -1 & +c\phi \\ & \\ 0 & +r_0 s\phi \end{bmatrix} \begin{Bmatrix} F_T \\ \\ F_\phi \end{Bmatrix}^{(1)} \qquad \text{(B-7')}$$

which permits evaluation of $F_T^{(1)}$ and $F_\phi^{(1)}$.

CHECKING RESULTS

It is frequently desirable to be able to calculate net loads at a cut section, or a built-in edge.

From Equations B-1, B-2, B-6 and B-1, B-2, B-7 respectively we have

$$\begin{Bmatrix} \widetilde{F}_Z\binom{i}{j} \\ \\ \widetilde{M}_Z\binom{i}{j} \end{Bmatrix} = \pm 2\pi r_0 \begin{bmatrix} 0 & +s\phi & +c\phi & 0 \\ & & & \\ -r_0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} T_{\phi\theta}\binom{i}{j} \\ N_\phi\binom{i}{j} \\ J_\phi\binom{i}{j} \\ M_\phi\binom{i}{j} \end{pmatrix}^{(0)} \qquad \left(\text{B-}{}^{8}_{9}\right)$$

and

$$\begin{Bmatrix} \widetilde{F}_X\binom{i}{j} \\ \\ \widetilde{M}_Y\binom{i}{j} \end{Bmatrix} = \pm \pi r_0 \begin{bmatrix} +1 & -c\phi & +s\phi & 0 \\ & & & \\ 0 & -r_0 s\phi & -r_0 c\phi & +1 \end{bmatrix} \begin{pmatrix} T_{\phi\theta}\binom{i}{j} \\ N_\phi\binom{i}{j} \\ J_\phi\binom{i}{j} \\ M_\phi\binom{i}{j} \end{pmatrix}^{(1)} \qquad \left(\text{B-}{}^{10}_{11}\right)$$

where the sign is chosen to correspond with the edge, i or j, on which the applied force is desired.

END

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