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ANALYSIS OF ATMOSPHERICALLY INDUCED ERRORS IN AN OPTICAL TRACKING SYSTEM

by

William E. Webb, Project Director

1st Quarterly Report on Contract NAS8-30507
for the period
October 14, 1968 to January 14, 1969

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This report was prepared by the University of Alabama under Contract Number NAS8-30507, Study of Atmospheric Effects on Optical Communication and Optical Systems, for the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration. This work was administered under the technical direction of the Astrionics Laboratory, George C. Marshall Space Flight Center.

During this quarterly reporting period we have been engaged in an analysis of the errors introduced into a laser tracking system by atmospheric turbulence. The system being investigated is identical to those being used by Marshall Space Flight Center personnel to investigate atmospheric turbulence and is similar to systems which are proposed for early launch tracking of the Saturn V vehicle and for pointing deep space optical communication systems. We have derived theoretical expressions for the magnitude of the root mean square tracking errors (angle of arrival fluctuation) as a function of the aperture of the receiving optics. Two assumptions have been made in deriving these expressions viz; (1) that the tracking system effectively measures the centroid of illuminance in the focal plane of a well corrected lens, and (2) that the fluctuations in the amplitude of the incoming wave may be neglected compared to fluctuations in its phase. A detailed presentation of the results which have been obtained at this time is appended to and made part of this report.

The project director has devoted approximately 120 man hours to the project this quarter. It was originally intended that he would be assisted by a graduate student and one or more undergraduate assistants. Since this project was not initiated until after the beginning of the semester, no student was available. A suitable graduate student, Mr. Kermit George, has accepted a research assistantship for the Spring Semester and will be employed on this project beginning this month and continuing until the project terminates.

During the next reporting period we will continue our analysis of atmospherically induced tracking errors by investigating the effects of amplitude fluctuations across the aperture on the apparent angle of arrival. Our approach will be to simulate the effect of the atmosphere by using sets of quasi-random number having appropriate statistical properties to represent the amplitude of an atmospherically distorted wave. We will also investigate the tracking errors in a receiver having several small apertures in place of a single large aperture.

An optical tracking system which was developed for Marshall Space Flight Center by Sylvania Electronics Corporation under contract NAS8-20841 is currently being used by Marshall Space Flight Center personnel to investigate the effects of atmospheric turbulence. This system has been fully described in the literature [1]. Without becoming involved in the details of the operation of this tracker we may state that systems consist of receiving optics which focus an incoming laser beam onto the face of an image dissector tube. The image dissector measures the position of the focused spot as it moves across the face of the tube due to atmospheric turbulence. The output signal from the system is proportional to the x and y corrdinates of this spot. This data is then analyzed to yield, among other things, the root mean square deviation of the apparent angle of arrival of the laser beam in terms of the azimuth and elevation angles. We shall derive a theoretical expression for the RMS deviation in the azimuthal angle. A similar analysis holds for the elevation.

The problem of defining the angle of arrival of distorted electromagnetic waves is discussed by Beckman [2]. Clearly for practical purposes

[1] R. E. Johnson, P. E. Wiess, Applied Optics 7, p. 1095 (1968.

[2] P. Beckman, RADIO SCIENCE Journal of Research NBS/USNC-UTRS, vol. 69D No. 4, p. 629 (1965).

one must take as the definition of the angle of arrival the quantity which is actually measured by the receiving system. In order to determine this quantity a careful analysis of the operation of the tracking system is required. If, for example, the system in question employs a quadrant detector with the output taken as the difference in signal from opposite quadrants then the apparent angle of arrival of the incoming beam as measured by the tracker will be proportional to the difference in illuminance on the two quadrants of the detector. For the case of a tracking system using an image dissector tube the situation is somewhat more complicated since the output will depend to a large extent on how the signal is processed. We will assume that the tracker measures the instantaneous position of the centroid of illuminance in the plane of the detector. While this assumption has not been fully verified by a detailed analysis of the particular tracking system in question it is felt that it should be sufficiently accurate for our purposes.

Centroid of Illuminance

Let x and y be Cartesian coordinates in the plane of the detector, x being taken in the horizontal direction and y vertically. With the assumptions discussed above the instantaneous azimuthal tracking error may clearly be expressed in terms of the moments of the intensity distribution on the face of the image dissector tube by $\theta_x(t) = \bar{x}/f$ where $\bar{x}$ is x-coordinate of the centroid of the illuminance $I(x,y)$ anf f is the focal length of the system.

$$\theta_x(t) = \frac{1}{f} \frac{\iint xI(x,y)dydx}{\iint I(x,y)dydx} \qquad (1)$$

We have assumed that in the absence of atmospheric disturbance the centroid of illumination at the origin the limits of integration will be assumed to be over the entire plane otherwise stated. $I(x,y)$ may be expressed as the absolute square of the complex electric field quantity $u(x,y)$.

$$I(x,y) = u(x,y)u^*(x,y) \tag{2}$$

For a well corrected optical system $u(x,y)$ is given in terms of the electric field in the entrance pupil $U(\zeta,\eta)$ by the well known relation

$$u(x,y) = A \int U(\zeta,\eta)\exp\left\{ - \frac{2\pi i}{\lambda f}(x\zeta + y\eta) \right\} d\zeta d\eta \tag{3}$$

A quadratic phase factor has been suppressed in equation (3) as it will not effect the illuminance. The integration extends over the aperture of the system. We may define $U(\zeta,\eta)$ to be zero everywhere except in the aperture and extend the limits of integration over the entire ζ-η plane.

We now let $V = x/\lambda f$ and $W = y/\lambda f$ and combine equations (1) and (2).

$$\theta_x(t) = \lambda \frac{\iint v|u(v,w)|^2 dvdw}{\iint |u(v,w)|^2 dvdw} \tag{4}$$

From equation (3) we have

$$\iint u(v,w)^2 dvdw = \iint\Big[A\iint U(\zeta,\eta)e^{-2\pi i(v\zeta+w\eta)}d\zeta d\eta \times A^*\int U^*(\zeta',\eta')e^{+2\pi i(v\zeta'+w\eta')}d\zeta' d\eta'\Big] dvdw \tag{5}$$

which may be rewritten

$$\iint |u(v,w)|^2 dvdw = |A|^2 \iiiint\!\!\iint U(\zeta,\eta)U^*(\zeta',\eta') \times \exp\left\{ -2\pi i\left[v(\zeta-\zeta') + w(\eta-\eta')\right]\right\} dvdwd\eta d\zeta d\eta' d\zeta' \tag{6}$$

Performing the integration on v and w we obtain

$$\iint |u(b,w)|^2 dvdw = |A|^2 \iiiint U(\zeta,\eta)U^*(\zeta',\eta') \times \delta(\zeta-\zeta')\delta(\eta-\eta')d\zeta d\eta d\zeta' d\eta' \tag{7}$$

so that the integration on ζ' and η' become trivial

$$\iint |u(v,w)|^2 dvdw = |A|^2 \iint U(\zeta,\eta)U^*(\zeta',\eta')d\zeta d\eta \tag{8}$$

Likewise the other integral in equation (3) is

$$\iint v|u(v,w)|^2 dvdw = |A|^2 \iiiint\!\!\iint U(\zeta,\eta)U^*(\zeta',\eta') \times v \exp\left\{ -2\pi i \left[v(\zeta-\zeta') + w(\eta-\eta')\right] \right\} dvdwd\zeta d\eta d\zeta' d\eta' \tag{9}$$

On interchanging the order of integration this becomes

$$\iint v|u(v,w)|^2 dvdw = |A|^2 \iiiint U(\zeta,\eta)U^*(\zeta',\eta') \times \left[\int \exp\left\{ -2\pi i w(\eta-\eta')\right\} dw\right]\left[\int \exp\left\{-2\pi i v(\zeta-\zeta')\right\} vdv \right] d\zeta d\zeta' d\eta d\eta' \tag{10}$$

The integration on w leads to a delta function in the usual way. To perform the integration on v we note that

$$\delta(\chi-\chi') = \int e^{-2\pi i(\chi-\chi')v}\, dv \tag{11}$$

Differentiating with respect to χ we have

$$\frac{\partial\delta(\chi-\chi')}{\partial\chi} = -2\pi i \int e^{-2\pi iv(\chi-\chi')} v dv \tag{12}$$

or

$$\int e^{-2\pi iv(\chi-\chi')} v dv = \frac{i}{2\pi} \frac{\partial\delta(\chi-\chi')}{\partial\chi} \tag{13}$$

Making use of this result, equation (10) becomes

$$\iint v|u(v,w)|^2 dvdw = \frac{i}{2\pi}|A|^2 \iiiint U(\chi,\eta)U^*(\chi',\eta') \times \delta(\eta-\eta') \frac{\partial\delta(\chi-\chi')}{\partial\chi} d\eta\, d\eta'\, d\chi\, d\chi' \tag{14}$$

The integration of η is easily performed and the integration on χ may be carried out by parts.

$$\iint v|u(v,w)|^2 dvdw = \frac{i}{2\pi}|A|^2 \iiint U(\chi,\eta)U^*(\chi',\eta') \frac{\partial\delta'(\chi-\chi')}{\partial\chi} d\chi d\chi' d\eta \tag{15}$$

We take

$$u = U(\chi,\eta') \qquad dv = \frac{\partial\delta(\chi-\chi')}{\partial\chi} d\chi$$

$$du = \frac{\partial U}{\partial\chi} \qquad v = \delta(\chi-\chi') \tag{16}$$

then

$$\int U(\zeta,\eta')\,\frac{\partial\delta(\zeta-\zeta')}{\partial\zeta}d\zeta = U(\zeta,\eta')\Big|_{\zeta_1}^{\zeta_2} - \int \frac{\partial U(\zeta,\eta')}{\partial\zeta}\cdot\delta(\zeta-\zeta')d\zeta \qquad (17)$$

$$\int U(\zeta,\eta')\,\frac{\partial\delta(\zeta-\zeta')}{\partial\zeta}d\zeta = U(\zeta,\eta')\Big|_{\zeta_1}^{\zeta_2} - \frac{\partial U(\zeta',\eta')}{\partial\zeta'} \qquad (18)$$

here ζ_2 and ζ_1 are the values of ζ at the edge of the aperture. In general for any aperture other than a rectangular one ζ_1 and ζ_2 will be functions of η. Substituting equation (18) into equation (15) we have

$$\iint v|u(v,w)|^2 dvdw = \frac{i|A|^2}{2\pi}\iint U^*(\zeta',\eta')$$
$$\times\left[U(\zeta_2,\eta')\delta(\zeta_2-\zeta') - U(\zeta_1,\eta')\delta(\zeta_1-\zeta') - \frac{\partial U(\zeta',\eta')}{\partial\zeta'}\right]d\zeta' d\eta' \qquad (19)$$

hence

$$\iint v|u(v,w)|^2 dvdw = \frac{i}{2\pi}|A|^2\int\Big[\frac{1}{2}|U(\zeta_2,\eta)|^2 - \frac{1}{2}|U(\zeta_1,\eta)|^2$$
$$-\int U^*(\zeta,\eta)\,\frac{\partial U(\zeta,\eta)}{\partial\zeta}d\zeta\Big]d\eta \qquad (20)$$

The factor 1/2 in each of the first two terms arises from the fact that $U(\zeta,\eta)$ is discontinuous at ζ_1 and ζ_2 since it has been defined to be zero for all ζ and η outside the aperture. Since integrals of the form $\int f(x)\delta(x)dx$ are properly defined only if $f(x)$ is continuous at the point where the argument of the delta function vanishes we must exercise care in evaluating the integrals. First we perform a transformation of coordinate by letting $\zeta = \zeta - \zeta_1$ so that the argument of the delta function is zero at the origin. The limits of integration then extend from zero to infinity. We may now define $U(-\zeta) = U(\zeta)$ rather than zero. This does not effect the value of the integral since $U(-\zeta)$ is outside the range of integration. This integral may then be written as one-half the integral from minus infinity to plus infinity since the integrand is symmetrical. Hence the origin of the factor of one-half

Now we write U as $U e^{i\phi}$ so that the last term in equation (20) becomes

$$\int |U| e^{-i\varphi} \frac{\partial}{\partial \zeta} \left[|U| e^{+i\phi} \right] d\zeta \tag{21}$$

which may be integrated by parts to yield

$$i \int |U(\zeta,\eta)|^2 \frac{\partial \varphi}{\partial \zeta} \, d\zeta + \frac{1}{2} |U(\zeta_2,\eta)|^2 - \frac{1}{2} |U(\zeta_1,\eta)|^2 \tag{22}$$

Substituting into (20) we have

$$\iint v\, |u(v,w)|^2 dv dw = \frac{A^2}{2\pi} \iint |U(\zeta,\eta)|^2 \frac{\partial \varphi}{\partial \zeta} \, d\zeta d\eta \tag{23}$$

so that equation (4) becomes

$$\theta_x(t) = \frac{\lambda}{2\pi} \frac{\iint_\Sigma U(\zeta,\eta)^2 \frac{\partial \varphi}{\partial \zeta} \, d\zeta d\eta}{\iint_\Sigma U(\zeta,\eta) d\zeta d\eta} \tag{24}$$

Equation (24) is a quite general expression for the instantaneous tracking error. By squaring and taking an ensemble average over all physical realizations of the incoming wave one would obtain the RMS angular fluctuation of θx. Unfortunately, the resulting expressions cannot be readily evaluated in terms of the statistical functions which are normally used to describe the fluctuation of the atmosphere. In one important case, viz,..if the amplitude fluctuation can be neglected in comparison to variations in the phase, then equation (24) may be evaluated. We shall first turn our attention to this case and later discuss methods of evaluating equation (24) when this approximation is not valid.

The Ray Optics Approximation

Let us assume that the random fluctuations in the amplitude of the field may be neglected in comparison to the fluctuation in the phase. This is essentially the ray optics approximation. The validity of this approximation has been widely discussed in the literature [3-5]. With this approximation $U(\zeta,\eta)^2$ is a constant and may be taken outside of the integral sign in equation (24). We then have

$$\theta_x(t) = \frac{\lambda}{2\pi} \frac{\iint \frac{\partial \delta}{\partial \zeta} \, d\zeta d\eta}{\iint d\zeta d\eta} \tag{25}$$

where the integration extends over the aperture of the system. The integral in the denominator is just the area of the aperture, hence

$$\partial_x(t) = \frac{\lambda}{2\pi A} \int_{\eta_2}^{\eta_1} \int_{\zeta_2}^{\zeta_1} \frac{\partial \varphi}{\partial \zeta} \, d\zeta d\eta \tag{26}$$

$$\theta_x(t) = \frac{\lambda}{2\pi A} \int_{\eta_2}^{\eta_1} \left[\varphi(\zeta_1,\eta) - \varphi(\zeta_2,\eta) \right] d\eta \tag{27}$$

[3] John W. Strohbern, Proc. IEEE 56, 1301 (1968).
[4] Lenord S. Taylor, J. Opt. Soc. Am. 58, 57 (1968).
[5] Lenord S. Taylor, J. Opt. Soc. Am. 58, 705 (1968).

where ζ_1, ζ_2, η_1 and η_2 are the appropriate limits of integration. We shall consider three cases: (a) a narrow slit of length a; (b) a rectangular aperture of length a and width b; and (c) a circular aperture of diameter D.

(a) Narrow slit aperture: For this case equation (27) reduces to

$$\theta_x(t) = \frac{\lambda}{2\pi a}\left[\varphi(\frac{a}{2}) - \varphi(-\frac{a}{2})\right] \tag{28}$$

Squaring θ_x and taking the ensemble average we obtain

$$\theta_x^2 = (\frac{\lambda}{2\pi a})^2 \left\langle\left[\varphi(\frac{a}{2}) - \varphi(-\frac{a}{2})\right]^2\right\rangle \tag{29}$$

We recognize the quantity on the right as the phase structure function; hence the root mean square tracking error is

$$\theta_{RMS} = \frac{\lambda}{2\pi a}\sqrt{D_\varphi(a)} \tag{30}$$

Following Tartarski [6] we take $D_\varphi(a)$ for a plane wave as

$$D_\varphi(r) = \sqrt{6.88}\left|\frac{r}{r_o}\right|^{5/3} \tag{31}$$

for values of r much larger than the inner scale of turbulence l_o. Then

$$\theta_{RMS} = \frac{\lambda}{2\pi}\sqrt{6.88}\; r_o^{-5/6}\; a^{-1/6} \tag{32}$$

[6] V. A. Tartarski, <u>Wave Propagation Through a Turbulent Media</u>, McGraw-Hill, New York, (1961).

For $a \ll \ell_o$ D_ϕ is proportional to r^2 [7]. Clearly θ_{RMS} must be independent of the length for very short slits. From Tartarski's equation 7.101 we have after some manipulation

$$D_\varphi = 4.21\ \ell_o^{-1/3}\ r_o^{-5/3}\ a^2 \tag{33}$$

which yields a value

$$\theta_{RMS} = .326\lambda r_o^{-5/6}\ \ell_o^{-1/6} \tag{34}$$

From equation (32) we find the value of (a) which will give this limit to be $5.2\ell_o$. Hence we may write (32) and (34) as

$$\theta_{rms} = (.429\lambda r_o^{-5/6})\ a^{-1/6} \qquad a > 5\ell_o \tag{35a}$$

$$\theta_{rms} = .326\lambda r_o^{-5/6}\ \ell_o^{-1/6} \qquad a < 5\ell_o \tag{35b}$$

(b) A rectangular aperture: Now consider a rectangular aperture of length a and height b. If we take the origin of our coordinate system at the geometrical center of the aperture, then $\zeta_1 = \frac{1}{2}b$, $\zeta_2 = -\frac{1}{2}b$, $\eta_1 = \frac{1}{2}a$ and $\eta_2 = \frac{1}{2}a$. Equation (27) then becomes

$$\theta_x(t) = \frac{\lambda}{2\pi ab}\int_{\eta_1}^{\eta_2}\left[\phi(\zeta_1,\eta) - \phi(\zeta_2,\eta)\right]\,d\eta \tag{36}$$

[7] Tartarski, Op. Cit. p. 152.

Squaring θ_x and writing the square of the integral as a double integral over η and η' we obtain

$$\theta_x^2 = (\frac{\lambda}{2\pi ab})^2 \int_{\eta_1}^{\eta_2}\int_{\eta_1}^{\eta_2} \left[\phi(\zeta_1,\eta)-\varphi(\zeta_2,\eta)\right]\cdot\left[\varphi(\zeta_1,\eta') - \varphi(\zeta_2,\eta')\right] d\eta d\eta' \tag{37}$$

or

$$\theta_x^2 = (\frac{\lambda}{2\pi ab})^2 \int_{\eta_1}^{\eta_2}\int_{\eta_1}^{\eta_2} \left[\varphi(\zeta_1,\eta)\varphi(\zeta_1\eta') + \varphi(\zeta_2,\eta)\varphi(\zeta_2,\eta') - \varphi(\zeta_1,\eta)\varphi(\zeta_2,\eta') - \varphi(\zeta_2,\eta)\varphi(\zeta_1,\eta')\right] d\eta d\eta' \tag{38}$$

We now take the ensemble average of θ_x^2 in equation (38). Interchanging the linear operations of averaging and integration we note that the resulting quantities in the right hand side of equation (38) are phase covariance functions.

$$C_\varphi(\vec{r}_1 - \vec{r}_2) = \left\langle \phi(\vec{r}_1)\phi(\vec{r}_2) \right\rangle \tag{39}$$

Making use of the fact that ϕ is a homogeneous and isotropic random variable, equation (38) can be written as

$$\theta_x^2 = (\frac{\lambda}{2\pi ab})^2 \int_{\eta_1}^{\eta_2}\int_{\eta_1}^{\eta_2} \left[2C_\phi(\eta-\eta') - 2C_\phi(\sqrt{(\zeta_1-\zeta_2)^2+(\eta-\eta')^2})\right] d\eta d\eta' \tag{40}$$

By the well known relation between the covariance and structure function, i.e.,

$$2C_{\varphi}(r) = D_{\phi}(\infty) - D_{\phi}(r) \tag{41}$$

equation (40) may be put in the form

$$\left\langle\theta_x^2\right\rangle = \left(\frac{\lambda}{2\pi ab}\right)^2 \int_{\eta_1}^{\eta_2}\int_{\eta_1}^{\eta_2} \left[D_{\varphi}\left(\sqrt{(\xi_1-\xi_2)^2 + (\eta-\eta')^2}\right) - D_{\phi}(\eta\ \eta')\right] d\eta d\eta' \tag{42}$$

$$\left\langle\theta_x^2\right\rangle = \left(\frac{\lambda}{2\pi ab}\right)^2 6.88 r_o^{-5/3} \int_{\eta_1}^{\eta_2}\int_{\eta_1}^{\eta_2} \left[\left((\xi_1-\xi_2)^2 + (\eta-\eta')^2\right)^{5/6} - |\eta-\eta'|^{5/3} \right] d\eta d\eta' \tag{43}$$

We first remove the absolute value signs from equation (43). The integrand in the first term is everywhere positive so that it does not present a problem. In the second term we may make a change of variables by letting $x = \eta + a/2$ and $y = \eta' + a/2$. Noting that $\eta_1 = a/2$ and $\eta_2 = -a/2$, the second integral becomes

$$\int_o^a \int_o^a |x-y|^{5/3}\, dydx \tag{44}$$

Since the integrand is symmetrical about the line x = y and is positive for all $x > y$ we may write it in the following form which may be directly integrated.

$$2\int_0^a\int_0^x (x-y)^{5/3}\,dy\,dx = \frac{9}{44}\,a^{11/3} \tag{45}$$

The first term in equation (43) may be evaluated by expanding the integrand in a Taylor's series and integrating term by term. Noting that $\zeta_2-\zeta_1 = b$, we have

$$\iint\left[(\zeta_1-\zeta_2)^2+(\eta-\eta')^2\right]^{5/6} d\eta\, d\eta' = b^{5/3}\int_{-a/2}^{a/2}\int \left(1+\left(\frac{\eta-\eta'}{b}\right)^2\right)^{5/6} d\eta\, d\eta'$$

$$= b^{5/3}\int_{-a/2}^{+a/2}\int \left\{1+\frac{5}{6}\left(\frac{\eta-\eta'}{b}\right)^2 - \frac{5}{6^2 2!}\left(\frac{\eta-\eta'}{b}\right)^4 + \frac{5.7}{6^3 3!}\left(\frac{\eta-\eta'}{b}\right)^6 - \frac{5.7.13}{6^4 4!}\left(\frac{\eta-\eta'}{b}\right)^8 + ---\right\} d\eta\, d\eta' \tag{46}$$

From equations (44), (46) and (43), we obtain

$$\left\langle \theta_x^2 \right\rangle = \frac{\sqrt{6.88}}{r_o^{5/3}}\left(\frac{\lambda}{2\pi}\right)^2 a^{-2} b^{-2}\left\{-\frac{9}{44}\,a^{11/3} + a^2 b^{5/3} + 10\sum_{n=1}^{\infty}(-1)^n \frac{C_n}{6^n n!\,(2n+1)(2n+2)}\,a^{2n+2}\,b^{5/3-2n}\right\} \tag{47}$$

Where C_n = 1; 1; 7 x 13; 7 x 13 x 19; 7 x 13 x 19 x 25; etc. for N = 1, 2, 3, ... Equation (47) is valid for a less than b since for a greater than b the expansion in equation (46) does not converge. Now if we consider a rectangular aperture of length (b) in the direction along which the component of the tracking error is being measured and width a perpendicular to this direction, we define a shape parameter σ for the aperture as the ratio of a to b. Clearly σ must lie between zero and one for the series expansion to converge. Writing equation (47) in terms of σ and taking the square root we obtain for θ_{rms}.

$$\theta_{rms} = \frac{\lambda}{2\pi} \sqrt{6.88}\; r_o^{-5/6} b^{-1/6} \left\{ 1 - \frac{9}{44}\sigma^{5/3} + 10 \sum_{1}^{\infty} (-1)^n \frac{C_n \sigma^{2n}}{6^n n!(2n+1)(2n+2)} \right\}^{1/2} \tag{48}$$

We see that for a rectangular aperture the RMS tracking error varies with the b dimension (i.e., the direction along which the component of angle is measured) like the reciprocal one-sixth power. The dependence on the other dimension is very slight since the term in braces varies from unity for σ equal to zero (a narrow slit) to 0.96405 for σ equal to one (a square aperture). Clearly equation (48) agrees with our previous results for the case of a narrow slit.

We have evaluated the numerical coefficient in equation (48) for several intermediate values of the shape parameter. These results are given in Figure 1.

(c) Circular Aperture: For a circular aperture of radius R equation (38) becomes

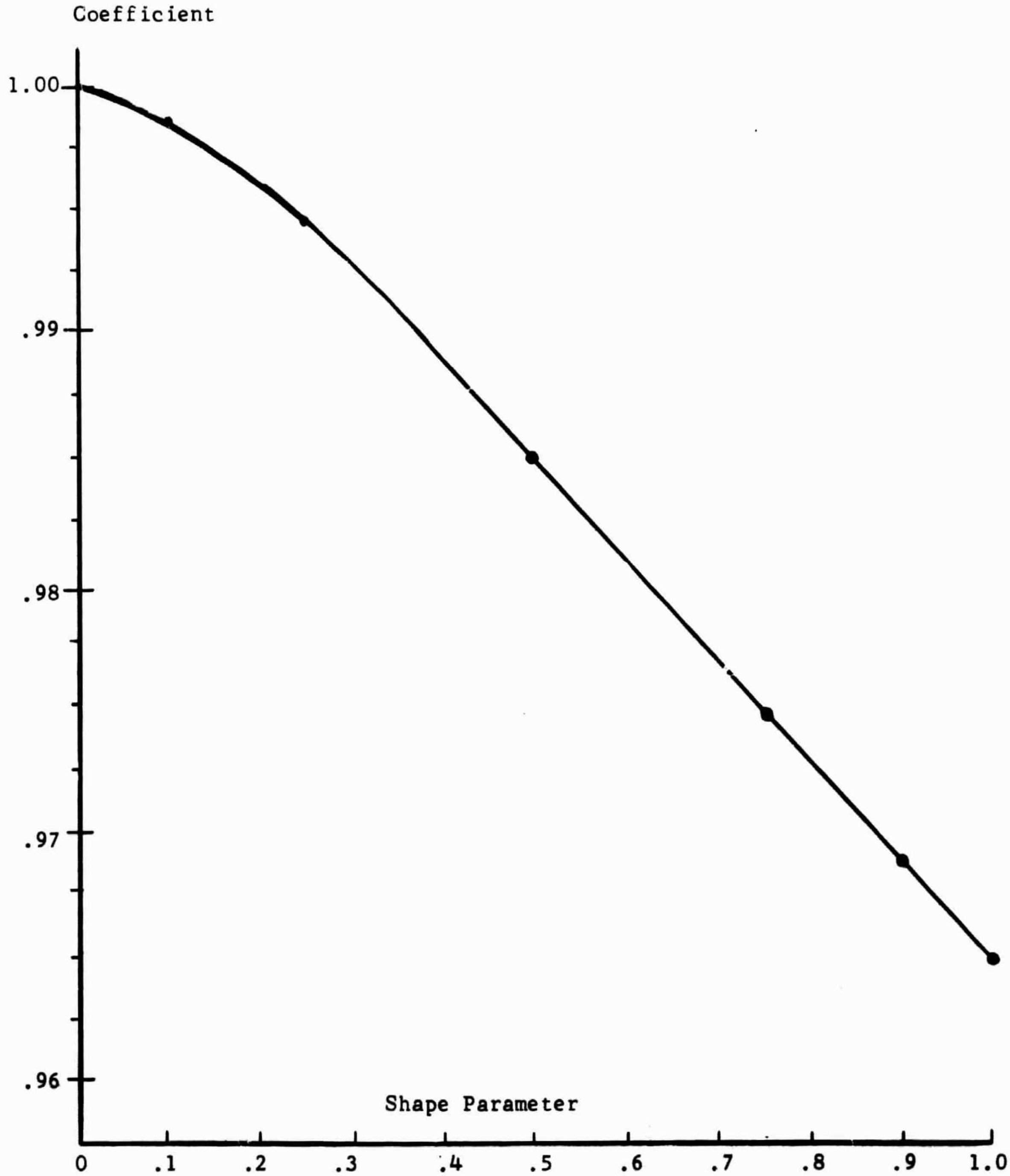

Figure 1. Numerical coefficient of equation (48) for a square aperture versus the shape parameter $\sigma = a/b$.

$$\theta_x^2 = \left(\frac{\lambda}{2\pi A}\right)^2 \iint_{-R}^{R} \Big[\varphi(\zeta_1,\eta)\varphi(\zeta_1',\eta')+\phi(\zeta_2,\eta)\varphi(\zeta_2',\eta')$$

$$- \varphi(\zeta_1,\eta)\varphi(\zeta_2',\eta') - \phi(\zeta_2,\eta)\phi(\zeta_1',\eta') \Big] \, d\eta d\eta' \tag{49}$$

Where $A = \pi R^2$ is the area of the aperture and

$$\zeta_1 = (R^2-\eta^2)^{1/2} \qquad \zeta_2 = -(R^2-\eta^2)^{1/2}$$

$$\zeta_1' = (R^2-\eta'^2)^{1/2} \qquad \zeta_2' = -(R^2-\eta'^2)^{1/2} \tag{50}$$

We now take the ensemble average of equation (49), making use of equations (31), (39), (41) and (50), and the homogenity and isotropy of the structure function. We introduce new variables $x = R\eta$ and $y = R\eta'$ and obtain, after considerable manipulation

$$\theta_x^2 = \left(\frac{\lambda}{2\pi R^2}\right)^2 \frac{6.88}{r_o^{5/3}} R^{11/3} \int_{-1}^{+1}\int_{-1}^{+1} \Bigg\{ \Big[(\sqrt{1-x^2}+\sqrt{1-y^2})^2 + (x-y)^2 \Big]^{5/6}$$

$$- \Big[(\sqrt{1-x^2}-\sqrt{1-y^2})^2 + (x-y)^2 \Big]^{5/6} \Bigg\} \, dxdy \tag{51}$$

The root mean square tracking error is easily obtained by taking the square root of equation (51). Introducing $D = 2R$, the aperture diameter, and simplifying somewhat we obtain:

$$\theta_{rms} = \frac{\sqrt{6.88}}{r_o^{5/6}} \frac{\lambda}{2\pi} \left\{ \frac{2^{7/12}}{\pi} (J^- - J^+) \right\}^{1/2} D^{-1/6} \tag{52}$$

where

$$J^{\pm} = \int_{-1}^{+1} \int_{-1}^{+1} \left[1 - xy \pm \sqrt{(1-x^2)(1-y^2)} \right]^{5/6} dydx \tag{53}$$

is a numerical coefficient which exactly corresponds to the shape coefficient which was introduced for the square aperture. The integrals of equation (53) were programmed for the IBM 360 computer. Using a 200 point Simpson's rule integration this yielded a value of .986 for the shape coefficient (the term in equation (52) contained in curly braces). Thus we see that the tracking error for a circular aperture corresponds very closely to that for a rectangular aperture with length to width ratio of 1/2.

Circular Aperture of Very Small Diameter

For apertures which are small compared to the inner scale of turbulence the structure function is given by equation (33) instead of equation (31). Thus for small apertures equation (51) becomes

$$\left\langle \theta_x^2 \right\rangle = \left(\frac{\lambda}{2\pi}\right)^2 \frac{4.21}{r_o^{5/3} \ell_o^{1/3}} R^4 \int_{-1}^{+1} \int_{-1}^{+1} \left\{ \left[\left(\sqrt{1-x^2} + \sqrt{1-y^2}\right)^2 + (x-y)^2 \right]^2 - \left[\left(\sqrt{1-x^2} - \sqrt{1-y^2}\right)^2 + (x-y)^2 \right]^2 \right\} dydx \tag{54}$$

which reduces to

$$\left\langle \theta_x^2 \right\rangle = \left(\frac{\lambda}{2\pi}\right)^2 \frac{4.21}{r_o^{5/3}\,\ell_o^{1/3}} \cdot 16 \int_{-1}^{+1}\int_{-1}^{+1} (1-xy)\sqrt{(1-x^2)(1-y^2)}\,dy\,dx \quad (55)$$

Since the x and y integrations may be separated we write equation (55) as

$$\left\langle \theta_x^2 \right\rangle = \left(\frac{\lambda}{2\pi}\right)^2 \frac{67.36}{r_o^{5/3}\,\ell_o^{1/3}} \left\{ \left[\int_{-1}^{+1} \left(\sqrt{1-x^2}\right) dx \right]^2 - \left[\int_{-1}^{1} x\sqrt{1-x^2}\, dx \right]^2 \right\} \quad (56)$$

The integrals in equation (56) may be evaluated directly yielding values of 2 and zero for the first and second terms respectively. Taking the square root to obtain the root mean square angular fluctuation we have

$$\theta_{rms} = \frac{\lambda}{\pi} \frac{\sqrt{67.36}}{r_o^{5/6}\,\ell_o^{1/6}} \quad (57)$$

Thus we see that for a circular aperture the angular fluctuation will be independent of aperture size for apertures which are small compared to the inner scale of turbulence.

Comparison with Calculations Based on Average Wave Front Tilt

Fried [8] had developed expressions for the average tilt of a wave front after passing through a turbulent atmosphere, by expanding the phase ϕ in a series of Modified Zernike Polynomials - i.e.,

[8] D. L. Fried, J. Am. Opt. Soc. 55, 1427 (1965).

$$\varphi(x_1 y) = \quad a_n F_n(x_1 y) \tag{58}$$

where

$$F_1 = (\pi R^2)^{-1/2} \qquad F_4 = (\pi R^6/12)(x^2+y^2-R^2/2)$$

$$F_2 = (\pi R^4/4)^{-1/2} x \qquad F_5 = (\pi R^6/6)(x^2-y^2) \tag{59}$$

$$F_3 = (\pi R^4/4)^{-1/2} y \qquad F_6 = (\pi R^6/24)xy$$

Here a_1 represents the average phase shift of the wave front, a_2 and a_3 its average tilt, a_4 the spherical deformation, a_5 and a_6 the astigmatic deformation, etc. The average tilt $\langle a_L^2 \rangle$ is given in terms of a_1 and a_2 by

$$\langle a_L^2 \rangle = \langle a_1^2 \rangle + \langle a_2^2 \rangle \tag{60}$$

Fried has derived an expression for $\langle a_L^2 \rangle$ by requiring that the polynomial expansion of φ (equation (58)) gives a best fit to the actual phase in a least mean square sense. We have found that Fried's work contains an error in that he has omitted a factor of $(\pi R^2)^{-1}$ in his equation 4.6a. Following through Fried's work we find that the correct expression for $\langle a_L^2 \rangle$ is

$$\langle a_L^2 \rangle = .883\pi R^2 \left(\frac{D}{r_o}\right)^{5/3} \tag{61}$$

The reader should compare this expression with equation 7.8a in Fried's paper.

With this correction in hand, we may proceed to calculate the apparent angle of arrival of the beam. Consider a plane wave traveling in approximately the Z direction. Neglecting deformation of the wave front we may write its phase as

$$\varphi = a_2 F_2 + a_3 F_3 \tag{62}$$

But the phase is also $(\frac{2\pi z}{\lambda})$ so that the equation of the isophase surface is

$$\frac{2\pi z}{\lambda} + a_2F_2(x) + a_3F_3(y) = \text{constant} \tag{63}$$

We let $\vec{K}$ be a vector normal to the isophase surface and $\vec{k}$ be the unit vector in the Z direction. Substituting the values of F from equation (59) into equation (63) we may write $\vec{K}$ as

$$\vec{K} = \overrightarrow{\text{GRAD}}\left[\ z + \frac{\lambda}{\pi^{3/2}R^2}\ (a_2x+a_3y)\ \right] \tag{64}$$

Since

$$\left| K \right| \cos\theta_T = \vec{K}_1 \cdot \vec{K} \tag{65}$$

a straightforward calculation yields

$$\cos\theta_T = \left[\frac{\lambda^2}{\pi^3R^4}\ (a_2{}^2 + a_3{}^2) + 1\right]^{-1/2} \tag{66}$$

Squaring both sides and noting that for small angles

$$\cos\theta_T = 1 - \frac{\theta_T^2}{2} + \dots \tag{67}$$

and

$$\left[1 + \frac{\lambda^2}{\pi^3 R^4}(a_2^2 + a_3^2)\right]^{-1/2} = 1 - \frac{\lambda^2}{\pi^3 R^4}(a_2^2 + a_3^2) + \cdots \tag{68}$$

We obtain

$$\theta_{RMS} = \sqrt{\left\langle \theta_T^2 \right\rangle} = \frac{\lambda}{\pi^{3/2} R^2} \sqrt{\left\langle a_L^2 \right\rangle} \tag{69}$$

Substituting from equation (61)

$$\theta_{RMS} = .414 \frac{\lambda}{r_o^{5/6}} D^{-1/6} \tag{70}$$

Chase [9] has noted that the constant (.883) which in equation (61) should properly be .828 since there were numerical errors in Fried's calculations which were later corrected by Chase.

Comparison with Experimental Results

Wyman [10] has reported experimental measurements of the RMS tracking errors as a function of receiver aperture over 3.2 and 6.4 Km paths. Figure 2 shows the root mean square angular fluctuations for both path lengths and several aperture sizes. Theoretical curves of angular fluctuation versus aperture size from equation (52) are also plotted in Figure 2. Here the correlation distance r_o has been chosen to produce

[9] D. M. Chase, J. Am. Opt. Soc. 56, page 33 (1966).
[10] C. A. Wyman, Master's Thesis, University of Alabama (1968).

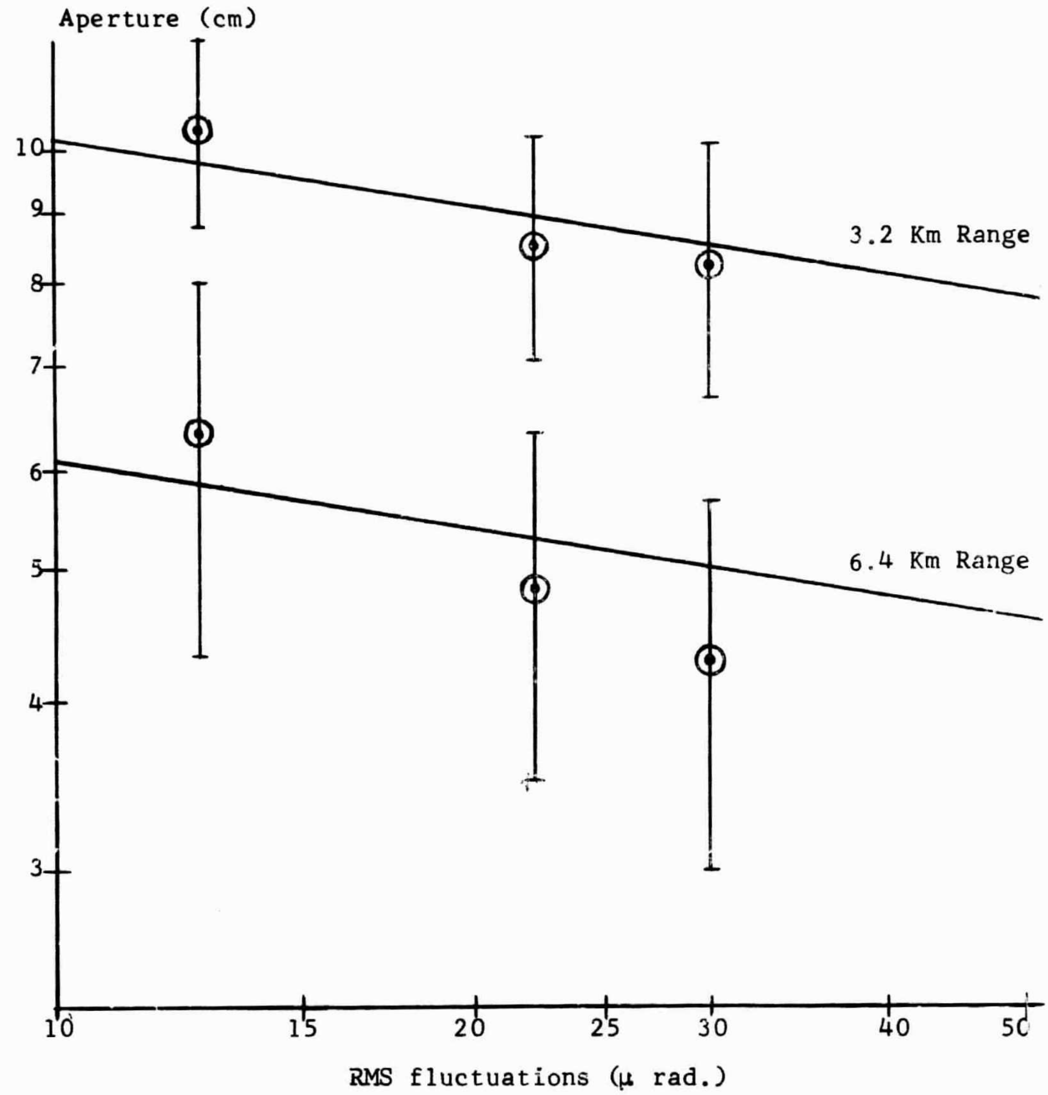

Figure 2. RMS angular fluctuations in microradians as a function of receiving aperture diameter for 3.2 and 6.4 kilometer paths.

the best fit with the experimental data.

The amount of data available is very minimal so that the theory cannot be said to be fully verified. Nevertheless the available data does fit a $D^{-1/6}$ power law to within the experimental error.