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RESEARCH AND DEVELOPMENT PROGRAM ON COLD CATHODE MAGNETRON ULTRAHIGH VACUUM GAGE

by R. M. Oman

Prepared by
NORTON RESEARCH CORPORATION
Cambridge, Mass.
for Langley Research Center

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ABSTRACT

Theoretical models of the magnetron gage are developed by treating the electronic charge cloud as a rotating fluid. Calculations are given for the gage oscillation frequency and an expression for the relation between the pressure and ion current.

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Norton Research Corporation

SUMMARY

A theoretical model for the magnetron gauge was developed by treating the electronic charge cloud as a rotating fluid. This model accounts for the experimentally observed oscillations in the gauge and the linear relation between pressure and ion current. Expressions for the electric field and potential within the gauge structure were developed using this model. Diocotron waves were used to calculate the gauge oscillation frequency. These calculations are correct to within an order of magnitude and vary with 1/B in accord with experiment. Stability criteria for magnetron type devices, involving the size of the charge cloud and electron density have also been developed. If a constant number density and energy spectrum are assumed for electrons in the discharge, a linear relationship between pressure and ion current results over a wide pressure range. Calculations of the sensitivity are within a factor of four of measured values.

INTRODUCTION

Before discussing the solution to the magnetron gage problem an outline of the general approach to problems in field theory and in particular those problems and theoretical techniques which most closely conform to the magnetron is appropriate. This general picture of field theory will make it much easier to see where the magnetron problem fits in relation to the several other problems (general problems) that have been solved with field theoretical techniques. As will be seen later on, it is most appropriate to use this kind of an outline of problems with particles in fields because there are actually very few problems that have been successfully solved with field theoretical methods. Actually rather than specific problems, there are general problems that can be solved via field theory such as a single particle in a field, a beam of interacting particles in a field or a collection of particles sufficiently dense to drastically alter the initial field moving in a field.

The Magnetron Problem

There are many side problems associated with trying to solve the general problem of the magnetron. And many of these problems involve calculations the results of which are needed to decide which effects can be neglected and which need to be considered and what approximations are allowed. However, in this section we want to describe only the general features of the magnetron problem.

To place some restrictions on the problem consider that the magnetron is a smooth bore one, the charge cloud is entirely electronic, and the two major features that we would like to account for in any theoretical model are (1) the oscillations in the gage, and (2) the linearity, that is the linear relation between ion current and pressure. There are several other restrictions which can be placed on this particular magnetron problem but these are sufficient for out general discussion at least at this point.

The general situation regards the magnetron gage is shown in Fig. 1. The gage consists in concentric cylinders with a potential of 6000 volt between them and an axial magnetic field of strength 1000 gauss. The polarity of the voltage is such that ions are collected at the inner electrode. The utility of the device as a vacuum gage is illustrated in the accompanying graph of ion current to the cathode versus pressure. Over a wide pressure range and one in which hot filament type gages are unreliable primarily because of x-ray limitations (from 10^{-7} to 2×10^{-10} Torr for this operating point) there is a linear relation of current to pressure and below this point there is an exponential relationship. On this log-log plot the linear relation is shown as a straight line of slope unity and the non-linear relation as a straight line of slope different from unity. These relations are also shown in Fig. 1.

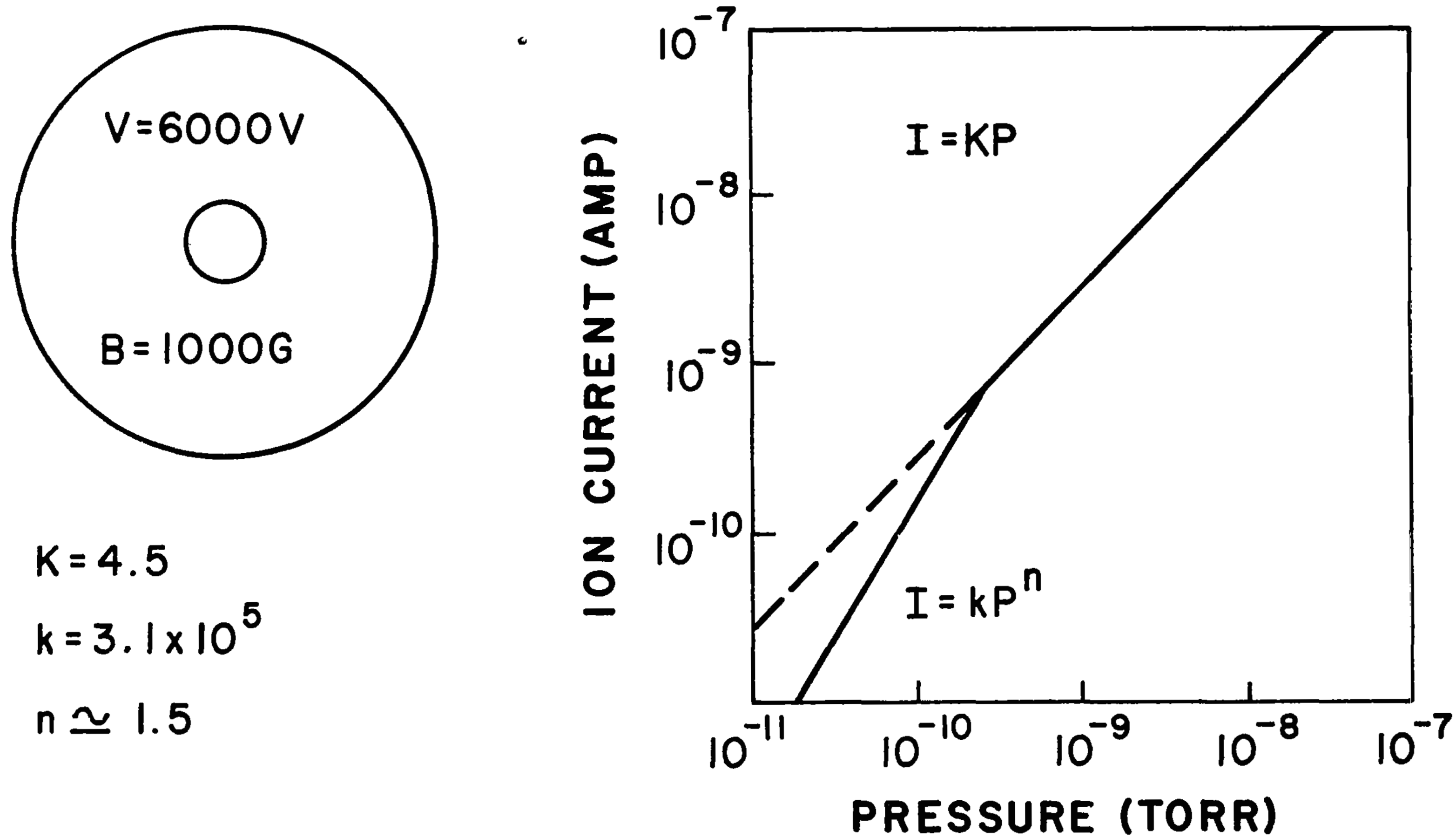

Fig. 1

General configuration of the Magnetron Gauge including the pressure-current relations.

Charged Particles In Electromagnetic Fields

In actual practice there are only three relatively narrow classifications of problem that can be solved by field theoretical techniques. Though this is clearly a somewhat arbitrary classification it is most appropriate for this discussion.

The simplest problem of particles in fields is one in which a single charged particle behaving in a classical manner is acted on by applied electric and magnetic fields. Next is the problem of the beam of charged particles interacting with electric and magnetic fields where the fields (outside the beam) are still those found for the charge free situation. And then there is that class of problems where the charge density of a beam of particles is so high and the beam is so large in extent that initial field conditions are drastically altered. This general classification of problems is shown in Fig. 2 where the charge and field situation is described and examples from the macroscopic and microscopic worlds are given.

Single Particle in a Field.- Now if we approach the problem of finding the trajectory of an electron in a field where we specify the initial conditions (position and velocity) and the field then the regular equations of classical physics, derived from simple Newton's laws relations uniquely determine the position and velocity of the particle in time. If we take the same field configuration, add another charged particle with its set of initial conditions we can again easily find the trajectories of both particles. If this process is continued for more and more particles each time, a point is very quickly

CHARGE & FIELD	MACROSCOPIC	MICROSCOPIC
E, q, m	SINGLE PARTICLE	HYDROGEN
	TWO PARTICLES	HELIUM
CHAOS		HARTREE-FOCH
BEAM	ELECTRON BEAMS	
E, Q, M		
STATISTICAL METHODS		SUPERCONDUCTORS
(DESTROY E)	(MAGNETIC FIELD)	
STREAM	PLASMA	
Q, M		

Fig. 2
General classification of problems in field theory.

reached when the interactions of the particles begin to influence the motion. If carried to extremes the interactions dominate in determining the motion. Except for special cases like the Helium atom problem we are unable to solve problems by continuous superposition of solutions to the problem of a single particle interacting with a field.

Beam of Particles in a Field.- The next most complicated approach for considering electrons in fields is to start with a beam of particles and have the beam enter the field with prescribed initial conditions. Then as with the single charged particle we can predict the history of the beam in time. After solving this problem we look inside the beam to determine the trajectories of the individual particles. This is the second general category of problems that can be solved with classical field theory. What we do in this case is break the problem up into two separate problems one similar to the problem of a single particle in a field and the other taking account of the interactions within the beam.

Stream of Particles in a Field.- The next most complicated problem that we can handle is one in which the beam of particles is sufficiently dense and sufficiently large in extent to drastically alter the initial applied electric field. In this situation the analysis is very complicated because the actual fields that the individual particles experience bears little resemblance to the initial conditions.

Comparison of Techniques.- Now we have three general categories of problems that can be solved (1) single particle in field, (2) beam of particles in field, (3) beam of particles sufficiently large and dense to essentially destroy the original field in a field. The discussion above of the situation of one particle in a field and then the addition of more and more individual particles (superposition of independent solutions) is really an illustration of the futility of attempting to superpose solutions neglecting particle interactions. This general observation is borne out by some examples from the macroscopic and microscopic worlds. In the general area of macroscopic problems, it is relatively easy to deal with single charged particles, either an ion or electron, one at a time. When we have to consider more than one electron it is necessary to add the further condition of noninteraction; that is, in real problems we can handle one particle in a field or two or three or four, etc. by treating each particle separately. But we very quickly reach the point of chaos in determining the interactions. Actually this chaos is due to the nonavailability of theoretical techniques. Again in the macroscopic world it is relatively easy to work with beams of charged particles such as in all of the electron beam devices by breaking the problem up into two parts that we can handle. We look at the beam and then inside of it separately. After the problem of dealing with the beam of charged particles we skip to the problem where the charged particle density is sufficiently high to destroy or very nearly destroy the applied fields such as generally encountered in plasma physics. Now in the microscopic world the realization of the futility of charged particle physics is more evident. For instance, it is possible to solve the hydrogen atom problem completely. The next most complicated atom however which consists of only two electrons in a central force field is much more difficult to solve and special techniques such as the Hartree-Foch have been developed to deal with many electron atoms. In the microscopic world the separation of particles is sufficiently small so that the problem of destroying the field is not regularly encountered except for shielding (electrons shielding the nucleus).

The Magnetron Problem in Relation to Field Theory

Looking at the three general categories of problems it is evident that the magnetron problem lies somewhere between the problem of the beam in a field and the stream being actually closer to the stream case since we would expect *a priori* the charge free field in a magnetron to be drastically altered by the charge cloud. In analysis of stream type problems techniques usually encountered in problems peculiar to the flow of rotating fluids are perhaps better suited than those encountered in particle physics. This is certainly not the only approach to the problem and may not eventually be the most successful one. However, the general problem of a fluid rotating between two concentric cylinders is sufficiently analogous to the magnetron problem where an electron cloud is rotating between two cylinders to justify an initial approach to the problem along these lines. Very roughly, the charge density in the cloud is analogous to the density of the liquid and the size and/or compressibility of the liquid is analogous to the constraining effect of the magnetic field on the individual charges.

General Configuration of the Fluid Flow Model

The particular situation which we set up is one where we have concentric cylinders. And assume that we have as we progress in the radial direction one liquid (a thin one), then another liquid which corresponds to the charge cloud and finally another liquid like the first one. This situation is a "liquid sandwich" in which a very dense liquid is between two very thin liquids. In this problem because of the magnetic field the more dense liquid will rotate with respect to the thin one. Actually we let the thin liquid have zero density. Problems of this general nature have been dealt with in the area of fluid flow and moving one fluid with respect to another results in a sine wave type disturbance at the fluid interface. In the cylindrical geometry cloud this disturbance, which is propagated around both the inner and outer edges of the charge cloud, corresponds to the diocotron wave or in the words peculiar to fluid flow the slipping stream instability. The general equation giving the frequency for this oscillation is usually very complicated and depends on the size of the chamber, the relative size of the charge cloud, the density in the cloud, and in this case the potential applied between the cylinders and the magnetic field. This general description of the magnetron problem, which is a close analogy to problems regularly encountered with fluids, warrants serious consideration. One of the drawbacks is that solutions are so complicated the aid of machine computers is essential. The first problem then in trying to match this fluid flow approach to the magnetron is to solve the problem, put in as many dimensions as possible and do a parameter study between applied potential, magnetic field, charge density and oscillation frequency to see if reasonable values for these parameters correspond to those experimentally observed in the gage. Parts of this exercise, presented in the next section, follow from the general description of diocotron waves by Levy (ref. 1).

DIOCOTRON INSTABILITIES IN THE MAGNETRON

Consider the geometry of Fig. 3. The wave equation relating the electric field to the charge density is

$$\frac{1}{r}\frac{d}{dr}(rE) = \frac{-n_o e}{\varepsilon_o} \tag{1}$$

where n_o the electron density (per unit length in the axial direction) is a function of r . In these calculations the sign is included in the equation so e is the specific electron charge. The charge per unit length residing on the central electrode is defined by Gauss' law. (This is for a line of charge.)

$$E(a) = \frac{\sigma}{2\pi\varepsilon_o a} \tag{2}$$

The value of σ is a measure of the potential between the cylinders and the total charge contained.

Now consider a single electron in this crossed field condition. If the equations are worked through (see Landau and Lifshitz (ref. 2) or Wouter Schuuman (ref. 3) the classical crossed

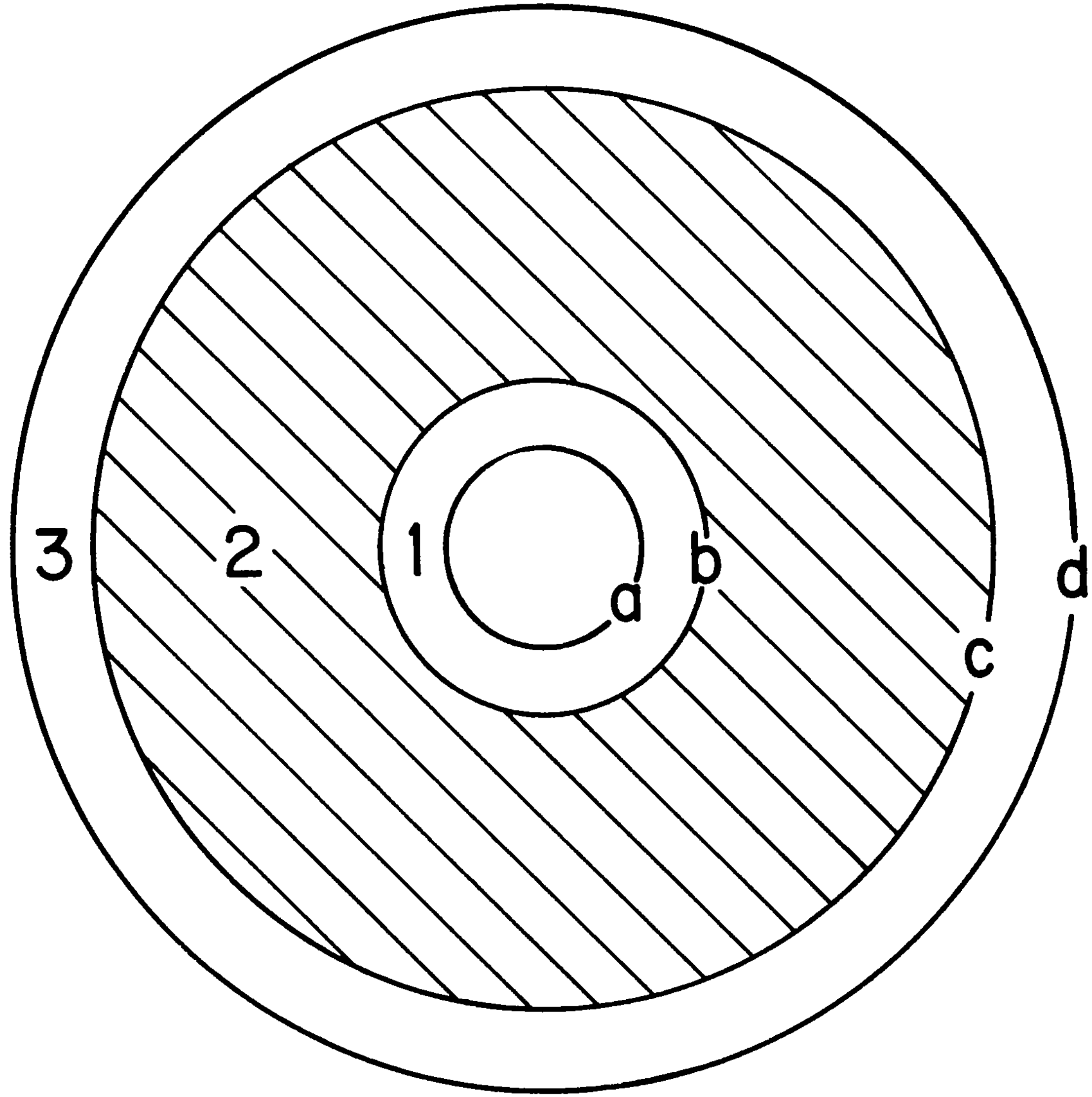

Fig. 3

Geometry of the gage cylinders and the electron cloud.

field mobility theory obtain. The result is that the electrons move in the aximuthal direction with a velocity $v_o = -E/B$. In a situation where there are collisions there is the possibility of migration in the direction of E . (If there were no collisions every electron leaving the central electrode would return.) This of course also assumes no perturbation to the applied electric field by the charge cloud. Thus we are able to write two general equations for the electrons

$$E_r = -vB \qquad E_\theta = uB \ , \tag{3}$$

where u is the radial velocity, v the aximuthal velocity and likewise for the electric field as indicated by the subscripts. This assumption of a continuity in the velocity implies the existence of potentials and in cylindrical coordinates we have

$$E_r = -\frac{\partial\phi}{\partial r} \qquad E_\theta = -\frac{1}{r}\frac{d\phi}{\partial\theta} \ . \tag{4}$$

If the circulating current is constant in numbers of electrons this corresponds to a constant electric and magnetic field so that the countinuity equation

$$\underline{\nabla}\cdot\rho\underline{v} = -\frac{\partial\rho}{\partial t} \tag{5}$$

for a rotating fluid is valid and if the fluid is incompressible ρ is a constant so $\underline{\nabla} \cdot v = 0$. In cylindrical co-ordinates (no z-component) then we have

$$\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} = 0 \tag{6}$$

as the conservation equation for an incompressible fluid.

The equation for conservation of the electrons, the conservation or transport equation for fluids is,

$$n\underline{\nabla} \cdot \underline{v} + \underline{v} \cdot \nabla_n = - \frac{\partial n}{\partial t} \tag{7}$$

but for an incompressible fluid $\underline{\nabla} \cdot \underline{v} = 0$ so

$$\underline{v} \cdot \underline{\nabla}_n + \frac{\partial n}{\partial t} = 0 \tag{8}$$

which is a simple statement that the density of any small volume of the flow is conserved as we follow it along or if we follow the density of a point in the flow the density of its environment is always constant. The $\underline{v} \cdot \nabla_n$ term is

$$u \frac{\partial n}{\partial r} + v \frac{1}{r} \frac{\partial n}{\partial \theta} \tag{9}$$

or

$$\frac{E_\theta}{B}\frac{\partial n}{\partial r} - \frac{E_r}{Br}\frac{\partial n}{\partial \theta} = \frac{1}{Br}\left(\frac{\partial \phi}{\partial r}\frac{\partial n}{\partial \theta} - \frac{\partial \phi}{\partial \theta}\frac{\partial n}{\partial r}\right) \tag{10}$$

The conservation equation then becomes

$$\frac{\partial n}{\partial t} + \frac{1}{Br}\left(\frac{\partial \phi}{\partial r}\frac{\partial n}{\partial \theta} - \frac{\partial \phi}{\partial \theta}\frac{\partial n}{\partial r}\right) = 0 \ . \tag{11}$$

Now we assume a potential and a density that contain a constant part plus a time and space (angular) varying part

$$\begin{aligned} \phi &= \phi_o(r) + \phi(r)\, e^{i(\ell\theta-\omega t)} \\ n &= n_o(r) + n(r)\, e^{i(\ell\theta-\omega t)} \end{aligned} \tag{12}$$

We can now substitute these assumed zero-order type profiles into the conservation equation. The essential partial derivatives are

$$\frac{\partial \phi}{\partial \theta} = i\ell\phi(r)\, e^{i(\ell\theta-\omega t)}$$

$$\frac{\partial n}{\partial \theta} = i\ell n(r)\, e^{i(\ell\theta-\omega t)}$$

$$\frac{\partial n}{\partial t} = - \omega i n(r)\, e^{i(\ell\theta-\omega t)}$$

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi_o}{\partial r} + e^{i(\ell\theta-\omega t)} \frac{d\phi(r)}{dr}$$

$$\frac{\partial n}{\partial r} = \frac{\partial n_o(r)}{\partial r} + \frac{\partial n(r)}{\partial r} e^{i(\ell\theta-\omega t)}$$

and on substituting into Eq. (11) we obtain

$$- i\omega n(r)\, e^{i(\ell\theta-\omega t)} + \frac{1}{Br}\left[\frac{\partial \phi_o(r)}{\partial r}\, i\ell n(r)\, e^{i(\ell\theta-\omega t)}\right] - \frac{1}{Br}\left[i\ell\phi(r)\, e^{i(\ell\theta-\omega t)} \frac{\partial n_o(r)}{\partial r}\right] \tag{13}$$

or

$$n(r)\left[-\omega + \frac{\ell}{Br}\frac{\partial \phi_o(r)}{\partial r}\right] = \frac{\ell}{Br}\,\phi(r)\,\frac{\partial n_o(r)}{\partial r} \tag{14}$$

And substituting $v_o = - \frac{E}{B} = \frac{1}{B}\frac{\partial \phi_o(r)}{\partial r}$ we obtain

$$n(r)\left[\omega - \frac{\ell v_o}{r}\right] = - \frac{\ell\phi(r)}{Br}\,\frac{dn_o(r)}{dr} \tag{15}$$

The Poisson's equation is $\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\varepsilon}$ and taking the divergence for cylindrical co-ordinates gives

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{1}{r}\frac{d^2\phi}{d\theta^2} = -\frac{ne}{\varepsilon_o} \qquad (16)$$

Now the ϕ and n which we put into this Poisson's equation are $\phi(r)$ and $n(r)$, Eq. (12), so $d^2\phi/r^2 d\theta^2$ is $-(\ell^2/r^2)\phi$ which is the form for any periodic potential. Putting the results of Eq. (15) into a Poisson's equation we have

$$\left[\omega - \frac{\ell v_o}{r}\right]\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \frac{\ell^2}{r^2}\phi\right] = -\frac{e\ell\phi}{\varepsilon Br}\frac{dn_o}{dr} \qquad (17)$$

Now we come to the question of the zero order profile for n. As a first approximation we take the profile defined in Eq. 18,

$$n_o = 0 \quad (a \le r < b : c < r \le d)$$
$$n_o = N \quad (b \le r \le c) \qquad (18)$$

This choice makes $dn_o/dr = 0$ in three separate regions both where the beam is and where it is not. Inside these three regions the working equation then reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \frac{\ell^2}{r^2} = 0 \tag{19}$$

There are two reasons for this choice. First looking over the work of Nedderman (ref. 4), Dow (ref. 5) and Reverdin (ref. 6) who have attempted to measure the electron density in magnetrons we see that this choice is not incompatible with measured electron profiles. Second this choice allows the working equation, Eq, (19), in a form so that the problem can be worked through. Other choices may turn out to be both easier to work with and closer to the actual situation but for a first attempt this choice is appropriate. This choice of profile (Eq. (18)) implies that the perturbed density $n(r)$ in Eq. (15) is zero inside the cloud so that the perturbation or wave we are interested in is at the edge of the charge cloud.

Now we can match solutions at the charge cloud boundary. Assume the potential to be continuous. To match the solutions we integrate Eq. (17) from $r=b-\delta$ to $r=b+\delta$ and look at the results as $\delta \to 0$

$$\int_{r=b-\delta}^{r=b+\delta} \left(\omega - \frac{\ell v_o}{r}\right)\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \frac{\ell^2}{r^2}\phi\right]dr = -\frac{e\ell}{\varepsilon_o B}\int_{r=b-\delta}^{r=b+\delta} \frac{\phi}{r}\frac{dn_o}{dr}dr \tag{20}$$

The term $\left(\omega - \frac{\ell v_o}{r}\right)$ is essentially a constant so with r=b can be taken out of the integration. The term $\frac{dn_o}{dr}$ can be considered a delta function and ϕ does not contribute to the integral so

$$\int_{r=b-\delta}^{r=b+\delta} \frac{\phi}{r} \frac{dn_o}{dr} dr = \frac{\phi(b)}{b} n_o \tag{21}$$

The equation for matching the potentials at r=b now looks like

$$\left(\omega - \frac{\ell v_o(b)}{b}\right) \int_{b-\delta}^{b+\delta} \left[\frac{1}{r} \frac{d}{dr} \left(r\frac{d\phi}{dr}\right) - \frac{\ell^2}{r^2}\phi\right] dr = - \frac{e\ell}{\varepsilon_o B} \frac{\phi(b)}{b} N \tag{22}$$

The only contribution to the integral is from the first term and to first order this is just the slope of ϕ at the boundary so

$$\left(\omega - \frac{\ell v_o(b)}{b}\right) \left[\left.\frac{d\phi}{dr}\right|_{b+} - \left.\frac{d\phi}{dr}\right|_{b-}\right] = - \frac{eN}{\varepsilon_o B} \frac{\ell\phi(b)}{b} \tag{23}$$

The quantity $Ne/\varepsilon_o B$ has the dimensions of a frequency and is equal to $\omega_p{}^2/\omega_c$ where ω_p is the plasma frequency and ω_c is the cyclotron frequency so that this equation can be written as

$$\left(\omega - \frac{\ell v_o(b)}{b}\right)\left[\frac{d\phi}{dr}\bigg|_{b+} - \frac{d\phi}{dr}\bigg|_{b-}\right] = -\frac{\omega_p{}^2}{\omega_c}\frac{\ell\phi(b)}{b} \tag{24}$$

Now we proceed to write down expressions for the field and the potential in each of the three regions. The field and potential are written down first in region 1. Then the field is written down for region 2 and the potential is written down with the constant of integration evaluated at r=b by comparison with the potential expression developed in region 1. The same procedure is applied at the outer cloud boundary (r=c) to find the potential in region 3. This procedure is shown in detail in Appendix A and the results are reproduced below.

In region 1:

$$\begin{aligned} E_o &= \frac{Q}{2\pi\varepsilon_o r} \\ \phi_o &= -\frac{Q}{2\pi\varepsilon_o}\ln\frac{r}{a} \end{aligned} \tag{25}$$

In region 2:

$$\begin{aligned} E_o &= \frac{Q}{2\pi\varepsilon_o r} - \frac{Neb}{\varepsilon_o}\left(\frac{r}{b} - \frac{b}{r}\right) \\ \phi_o &= -\frac{Q}{2\pi\varepsilon_o}\ln\frac{r}{a} + \frac{Neb^2}{4\varepsilon_o}\left[\frac{r^2}{b^2} - 1 - 2\ln\frac{r}{b}\right] \end{aligned} \tag{26}$$

In region 3:

$$E_o = \frac{Q}{2\pi\varepsilon_o r} - \frac{Ne}{2\varepsilon_o}\frac{(c^2-b^2)}{r}$$

$$\phi_o = \frac{Q}{2\pi\varepsilon_o}\ln\frac{r}{a} + \frac{Ne}{4\varepsilon_o}\left[(c^2-b^2) + 2c^2\ln\frac{r}{c} - 2b^2\ln\frac{r}{b}\right] \tag{27}$$

At this point the potential at the outer conductor can be written down,

$$\phi_o(d) = -\frac{Q}{2\pi\varepsilon_o}\ln\frac{d}{a} + \frac{Ne}{4\varepsilon_o}\left[(c^2-b^2) + 2c^2\ln\frac{d}{c} - 2b^2\ln\frac{d}{b}\right] \tag{28}$$

or rewriting Q in terms of N and the potential at the outer conductor

$$Q = \frac{\pi Ne}{2\ln\frac{d}{a}}\left[(c^2-b^2) + 2c^2\ln\frac{d}{c} - 2b^2\ln\frac{d}{b}\right] - \frac{2\pi\varepsilon_o\phi_o(d)}{\ln\frac{d}{a}} \tag{29}$$

In finding general solutions to the Poisson's equation (Eq. (19)) in each of the three regions it is convenient to start in region 2 where we assume a general solution of the form

$$\phi_2 = \beta r^{\ell} + \gamma r^{-\ell} \qquad (\ell \geq 1) \tag{30}$$

To make the solutions in regions 1 and 3 fit we must match them up at the boundaries, thus

$$\phi_1 = \left(\beta b^{2\ell} + \gamma\right)\left(r^{2\ell} - a^{2\ell}\right)\left(b^{2\ell} - a^{2\ell}\right)^{-1} r^{-\ell} \tag{31}$$

and

$$\phi_3 = (\beta c^{2\ell} + \gamma)(d^{2\ell} - r^{2\ell})(d^{2\ell} - c^{2\ell})^{-1} r^{-\ell} + \phi_o(d) d^{\ell}$$

$$(r^{2\ell} - c^{2\ell})(d^{2\ell} - c^{2\ell})^{-1} r^{-\ell} \tag{32}$$

In addition to matching at r=c, ϕ_3 must equal $\phi_o(d)$ at r=d and likewise ϕ_1 must vanish at r=a. These equations also must satisfy Eq. (19) which they do. According to Eq. (23) the condition for matching at the boundary b is

$$\left(\omega - \frac{\ell v_o(b)}{b}\right)\left[\frac{d\phi_2}{dr}\bigg|_b - \frac{d\phi_1}{dr}\bigg|_b\right] = - \frac{eN}{\varepsilon_o B}\,\frac{\ell\phi(b)}{b} \tag{33}$$

Expressions for ϕ and appropriate derivatives are given below the details of which are in Appendix B.

$$\phi_2(b) = \beta b^{\ell} + \gamma b^{-\ell}$$

$$\left.\frac{d\phi_2}{dr}\right|_b = \ell(\beta b^{\ell-1} - \gamma b^{-\ell-1})$$

$$\left.\frac{d\phi_1}{dr}\right|_b = \frac{(\beta b^{2\ell} + \gamma)}{(b^{2\ell} - a^{2\ell})}\, \ell b^{-\ell-1}(b^{2\ell} + a^{2\ell})$$

Substituting now in Eq. (33) we have

$$\left(\omega - \frac{\ell v_o(b)}{b}\right)\left[\ell b^{-\ell-1}(\beta b^{2\ell} - \gamma) - \frac{(\beta b^{2\ell} + \gamma)}{(b^{2\ell} - a^{2\ell})}(b^{2\ell} + a^{2\ell})\right] =$$

$$- \frac{eN}{\varepsilon_o B}\ell b^{-\ell 1}(\beta b^{2\ell} + \gamma)$$

$$\left(\omega - \frac{\ell v_o(b)}{b}\right)\left[(\beta b^{2\ell} - \gamma) - \frac{(\beta b^{2\ell} + \gamma)}{(b^{2\ell} - a^{2\ell})}(b^{2\ell} + a^{2\ell})\right] = - \frac{eN}{\varepsilon_o B}(\beta b^{2\ell}+\gamma)$$

$$\left(\omega - \frac{\ell v_o(b)}{b}\right)\left[(\beta-\gamma b^{-2\ell}) - \frac{(\beta+\gamma b^{-2\ell})}{(b^{2\ell}-a^{2\ell})}(b^{2\ell}+a^{2\ell})\right] = - \frac{eN}{\varepsilon_o B}(\beta+\gamma b^{-2\ell})$$

$$\left(\omega - \frac{\ell v_o(b)}{b}\right)\left[\frac{(\beta-\gamma b^{-2\ell})(b^{2\ell}-a^{2\ell})-(\beta+\gamma b^{-2\ell})(b^{2\ell}+a^{2\ell})}{(b^{2\ell} - a^{2\ell})}\right] = - \frac{eN}{\varepsilon_o B}(\beta+\gamma b^{-2\ell})$$

$$\left[\frac{(\beta b^{2\ell}-\beta a^{2\ell}-\gamma+\gamma a^{2\ell})-(\beta b^{2\ell}+\beta a^{2\ell}+\gamma+\gamma a^{2\ell})}{(b^{2\ell} - a^{2\ell})}\right]$$

$$\left(\omega - \frac{\ell v_o(b)}{b}\right)\left[2(\beta a^{2\ell} + \gamma)\right] = \frac{eN}{\varepsilon_o B}(\beta+\gamma b^{-2\ell})(b^{2\ell}-a^{2\ell})$$

Now $v_o(b) = -\frac{E_o(b)}{B} = -\frac{Q}{2\pi\varepsilon_o bB}$

so

$$\frac{\varepsilon_o B}{eN}\left(\omega + \frac{\ell Q}{2\pi\varepsilon_o b^2 B}\right)(2)\ (\beta a^{2\ell} + \gamma) = (\beta + \gamma b^{-2\ell})(b^{2\ell} - a^{2\ell})$$

$$2\left(\frac{\varepsilon_o B\omega}{eN} + \frac{Q\ell}{2\pi Neb^2}\right)(\beta a^{2\ell}+\gamma) = (\beta+\gamma b^{-2\ell})(b^{2\ell}-a^{2\ell}) \qquad (34)$$

Now we have to perform the same operation at the boundary r=c. And the equation we must satisfy is

$$\left(\omega - \frac{\ell v_o(c)}{c}\right)\left[\frac{d\phi_2}{dr}\bigg|_c - \frac{d\phi_3}{dr}\bigg|_c\right] = -\frac{eN}{\varepsilon_o B}\ell\ \frac{\phi(c)}{c} \qquad (35)$$

which is the same condition specified by the previous equation but for the boundary at r=c. Taking values for the derivatives, etc. from Appendix C, Eq. (35) becomes

$$\left(\omega - \frac{\ell v_o(c)}{c}\right)\left[\ell c^{\ell-1}\left[\frac{(\beta+\gamma c^{-2\ell})}{(d^{2\ell}-c^{2\ell})}(c^{2\ell}+d^{2\ell}) - \frac{2d^{\ell}c^{2\ell}\phi_o(d)}{(d^{2\ell}-c^{2\ell})}\right] + \ell c^{\ell-1}(\beta-\gamma c^{-2\ell})\right] = -\frac{eN}{\varepsilon_o B}\ell c^{-1}c^{\ell}(\beta+\gamma c^{-2\ell})$$

$$\left(\omega - \frac{\ell v_o(c)}{c}\right)\left[\frac{(\beta+\gamma c^{-2\ell})(c^{2\ell}+d^{2\ell})}{(d^{2\ell} - c^{2\ell})} - \frac{2d^{\ell}\phi_o(d)}{d^{2\ell}-c^{2\ell}} + (\beta-\gamma c^{-2\ell})\right] = -\frac{eN}{\varepsilon_o B}(\beta+\gamma c^{-2\ell})$$

$$\left(\omega - \frac{\ell v_o(c)}{c}\right)\left[(\beta+\gamma c^{-2\ell})(c^{2\ell}+d^{2\ell}) - 2d^{\ell}\phi_o(d) + (d^{2\ell}-c^{2\ell})(\beta-\gamma c^{-2\ell})\right] = -\frac{eN}{\varepsilon_o B}(\beta+\gamma c^{-2\ell})(d^{2\ell}-c^{2\ell})$$

$$\left(\omega - \frac{\ell v_o(c)}{c}\right)\left[2(\beta d^{2\ell}+\gamma-d^{\ell}\phi_o(d))\right] = -\frac{eN}{\varepsilon_o B}(\beta+\gamma c^{-2\ell})(d^{2\ell}-c^{2\ell})$$

now adding $v_o(c)$

$$2\left(\omega+\frac{\ell}{c}\left[\frac{Q-Ne\pi(c^2-b^2)}{2\pi\varepsilon_o cB}\right]\right)(\beta d^{2\ell}+\gamma-d^{\ell}\phi_o(d)) = -\frac{eN}{\varepsilon_o B}(\beta+\gamma c^{-2\ell})(d^{2\ell}-c^{2\ell})$$

or

$$2\left\{\frac{\varepsilon_o B}{eN}\omega + \frac{\ell}{c}\left[\frac{Q-Ne\pi(c^2-b^2)}{2\pi Nec}\right]\right\}\left(\beta d^{2\ell}+\gamma-d^{\ell}\phi_o(d)\right) = -\ (\beta+\gamma c^{-2\ell})(d^{2\ell}-c^{2\ell})$$

$$2\left\{\frac{\varepsilon_o B\omega}{eN} + \frac{Q\ell}{2\pi Nec^2} - \ell\frac{c^2-b^2}{c^2}\right\}\left(\beta d^{2\ell}+\gamma-d^{\ell}\phi_o(d)\right) = -(\beta+\gamma c^{-2\ell})(d^{2\ell}-c^{2\ell})$$

$$2\left\{\frac{\varepsilon_o B\omega}{eN} + \frac{Q\ell}{2\pi Nec^2} - \ell\left(1 - \frac{b^2}{c^2}\right)\right\}\left(\beta d^{2\ell}+\gamma-d^{\ell}\phi_o(d)\right) = -(\beta+\gamma c^{-2\ell})$$

$$(d^{2\ell}-c^{2\ell}) \qquad (36)$$

The particular term $\varepsilon_o B\omega/eN$ in both Eqs. (34 and (36) suggest a new variable $\zeta = \varepsilon_o B\omega/eN$ proportional to the frequency. This variable ζ is dimensionless. With this new variable we can proceed to rewrite Eqs. (34) and (36) in a more convenient form.

$$2\left(\zeta + \frac{Q\ell}{2\pi Neb^2}\right)(\beta a^{2\ell} + \gamma) = (\beta+\gamma b^{-2\ell})(b^{2\ell}-a^{2\ell}) \qquad (37)$$

and

$$\left[2\left(\zeta + \frac{Q\ell}{2\pi Nec^2}\right) - \ell\left(1 - \frac{b^2}{c^2}\right)\right]\left(\beta d^{2\ell}+\gamma-d^{\ell}\phi_o(d)\right) = -(\beta+\gamma c^{-2\ell})$$

$$(d^{2\ell}-c^{2\ell}) \qquad (38)$$

If we neglect the $-d^{\ell}\phi_o(d)$ term which is equivalent to setting $\phi_o(d) = 0$, these two equations can be rewritten in terms of the arbitrary constants β and γ. Then the expression for consistency of the solutions is obtained by setting the determinant of the characteristic equation equal to zero. This should give a dispersion relation in the frequency. This excercise is done in Appendix D. The result is shown below.

$$-4\zeta^2(d^{2\ell}-a^{2\ell})$$

$$+2\zeta\left[\ell(d^{2\ell}-a^{2\ell})\left\{\left(1 - \frac{b^2}{c^2}\right) - \frac{Q}{\pi Neb^2}\left(1 + \frac{b^2}{c^2}\right)\right\} + (b^{2\ell}c^{2\ell} - a^{2\ell}d^{2\ell})(c^{2\ell} - b^{2\ell})b^{-2\ell}c^{-2\ell}\right]$$

$$+\left[\frac{\ell^2 Q}{\pi Neb^2}\left(1 - \frac{b^2}{c^2} - \frac{Q}{\pi Nec^2}\right)(d^{2\ell}-a^{2\ell})\right.$$

$$-\frac{\ell Q}{\pi Neb^2}(c^{2\ell}-a^{2\ell})(d^{2\ell}-c^{2\ell})c^{-2\ell}$$

$$-\ell\left(1 - \frac{b^2}{c^2} - \frac{Q}{\pi Nec^2}\right)(d^{2\ell}-b^{2\ell})(b^{2\ell}-a^{2\ell})b^{-2\ell}$$

$$\left.+(c^{2\ell}-b^{2\ell})(d^{2\ell}-c^{2\ell})(b^{2\ell}-a^{2\ell})b^{-2\ell}c^{-2\ell}\right] = 0 \qquad (39)$$

The stability criteria for the oscillations in ζ is the reality of the roots of Eq. (39). This condition could be written down but for our purposes actual calculations on the computer can be used to give the situation for instability.

SOLVING THE DIOCOTRON EQUATIONS

The quadratic in ζ (Eq. (39)) can be solved by specifying a particular geometry a, d and taking a size factor K which determines the relative size of the electron cloud ($b = a + KL$, $c = d-KL$ where $L = d-a$) and values of N and Q . Thus in functional form we can write $\zeta = f\ (N, Q, K)$ for a specific geometry. We arbitrarily set $\ell = 1$ for this and subsequent calculations.

An expression for ζ is not obtainable in closed form. However none is absolutely necessary since solutions over any desired range can be found by machine computation. Indeed writing down these functional relationships does not imply that such a simple relationship exists but only that by specifying the parameters we can compute the ζ .

We can proceed to simplify the relationship for ζ by using Eq. (28) a relation between the applied voltage and Q,N. Thus we can relate Q to N through the applied voltage. Functionally $Q = f\ (V_0, N, K)$ so that we can write another functional relationship $\zeta = f\ (N, K, V_0)$. The actual observed oscillatory frequencies are $\omega = eN/\varepsilon_0 B\ \zeta$. In constructing a computer program then we must put in the proper initial values of a, d, V_o, B then vary N and K to get the observed frequency. Before computing the ζ's and the ω's we must first compute Q for specified N's and V_o's from Eq. (28). We have to have the values of Q before we can solve the quadiatic in ζ.

Before making even the first attempt to solve the quadratic for ζ we will work out some criteria for specifying the ranges of values of the parameters N, K .

STABILITY CRITERION

There is a stability criterion connected with the reality of the roots for ζ . Only those values of the parameters N, K, V_o that lead to real solutions to the quadratic in ζ are allowed. This is an extremely basic stability criterion and in this case one that is relatively easy to apply. The well known inequality for reality of the roots of quadratics of the general form $PX^2 + QX + R = 0$ is that the discriminant be greater than zero where the discriminant is $Q^2 - 4PR$. With this inequality we can proceed by machine compulation to calculate ζ for various values of N, K, V_o to put some bounds on these variables. The functional relationship for $\zeta = f\ (N, K, V_o)$ is the basis for the stability calculation.

The computer program sets the initial conditions of geometry and applied voltage then sets K and cycles through N by factors of 10 over the range 10^{12} to 10^{24} and prints out the value of N where the solutions become complex (imaginary roots). Along the way Q is calculated in terms of N, V_o. In a separate program the unstable regions were investigated beyond the point in N for complex roots with the result that a region close to $K = 0.50$ was found to be stable. This is however of little consequence.

The computer program, the results of the program and graphs showing the stable region are shown in the accompanying tables and figures (Table I, II; Fig. 4). The program was also done over for values of $V_0 = 2000, 10{,}000$ volt, the results of which are also in tabular and figure form, (Table III, IV; Fig. 5, 6,).

For an applied voltage of 6000 volt values of K of 0.220, 0.210, 0.200, 0.190, 0.180 show complex roots at $N = 10^{16}$. Further examination of this area shows that only a small region as shown in the appropriate diagram is unstable. The discovery of an unstable region smaller than the spacing between data points is fortuitous and serves as a reminder that there may be other unstable regions of a size smaller than the interval between data points.

TABLE I

```
1.01; STABILITY CRITERION FOR THE MAGNETRON
1.02 PAGE
1.10 PRINT"        ","STABILITY CRITERION FOR THE MAGNETRON"
1.11 TYPE #,#,#
1.20 TYPE "DISPERSION EQUN --P*X↑2+Q*X+R=0--"
1.30 PRINT"     ","A-IN CYL","         ","B-IN CLOUD","         "
1.31 PRINT"C-OUT CLOUD","         ","D-OUT CYL"
1.32 TYPE #
1.41 TYPE "K-F(B)-SIZE FACTOR FOR CLOUD-K=0,CLOUD FILLS CHAMBER"
1.50 PRINT"NUMBER DENSITY OF ELECTRONS (M↑3) - N"
1.51 TYPE #
1.52 PRINT"MAGNETIC FIELD (GAUSS)", BO FOR BO=1000
1.53 TYPE #
1.54 PRINT "APPLIED VOLTAGE (VOLT)", VO FOR VO=6000
1.55 TYPE #
1.56 PRINT"ELECTRON CHARGE (COULOMB)", EL FOR EL=1.6*10↑-19
1.57 TYPE #
1.58 PRINT"PERMITIVITY (FARAD/M)", EO FOR EO=8.8*10↑-12
1.59 TYPE #,#
1.60 TYPE A FOR A=((0.115/2)*2.54)*(1/100)
1.61 TYPE D FOR D=((1.180/2)*2.54)*(1/100)
1.71 L=D-A
1.72 TYPE #,#
1.99 DO PART 2

2.10 K=0.500
2.11 B=A+K*L
2.12 C=D-K*L
2.13 PRINT "K="
2.14 PRINT K IN FORM 1
2.15 PRINT "     "
2.21 N=10↑12
2.22 TO STEP 3.20 IF N>10↑24
2.31 Q5=($PI*N*EL)/(2*LN(D/A))
2.32 Q6=((C↑2-B↑2)+2*(C↑2)*LN(D/C)-2*(B↑2)*LN(D/B))
2.33 Q7=(2*$PI*EO*VO)/(LN(D/A))
2.34 Q4=Q5*Q6-Q7
2.35 E=(Q4)/($PI*N*EL)
2.41 P=-4*(D↑2-A↑2)
2.42 Q1=P/(-4)
2.43 R1=(E/B↑2)*(1-(B↑2/C↑2)-(E/C↑2))*(D↑2-A↑2)
2.44 Q2=(1-(B↑2/C↑2))-(E/B↑2)*(1+(B↑2/C↑2))
2.45 R2=(-E/B↑2)*(C↑2-A↑2)*(D↑2-C↑2)*C↑-2
```

TABLE I (CONT.)

```
2.46 Q3=(B↑2*C↑2-A↑2*D↑2)*(C↑2-B↑2)*(B↑-2)*(C↑-2)
2.47 R3=-(1-(B↑2/C↑2)-(E/C↑2))*(D↑2-B↑2)*(B↑2-A↑2)*B↑-2
2.48 R4=(C↑2-B↑2)*(D↑2-C↑2)*(B↑2-A↑2)*(B↑-2)*(C↑-2)
2.51 Q=2*(Q1*Q2+Q3)
2.52 R=R1+R2+R3+R4
2.53 DIS=Q↑2-4*P*R
2.60 TO STEP 2.71 IF DIS<0
2.61 TO STEP 3.10 IF DIS>0
2.71 PRINT"COMPLEX",N
2.72 TYPE #
2.81 TO STEP 3.20

3.10 SET N=10*N
3.11 TO STEP 2.22
3.20 SET K=K-0.010
3.21 TYPE #
3.22 STOP IF K<0
3.23 TO STEP 2.11
```

TABLE II

STABILITY CRITERION FOR THE MAGNETRON

DISPERSION EQUN --P*X↑2+Q*X+R=0--
A-IN CYL B-IN CLOUD C-OUT CLOUD D-OUT CYL
K-F(B)-SIZE FACTOR FOR CLOUD-K=0,CLOUD FILLS CHAMBER
NUMBER DENSITY OF ELECTRONS (M↑3) - N
MAGNETIC FIELD (GAUSS) BO= 1000
APPLIED VOLTAGE (VOLT) VO= 6000
ELECTRON CHARGE (COULOMB) EL= 1.6*10↑ -19
PERMITIVITY (FARAD/M) EO= 8.8*10↑ -12

A= 1.4605*10↑ -3
D= .014986

K= .500	COMPLEX	N=	1*10↑12
K= .490	COMPLEX	N=	1*10↑18
K= .480	COMPLEX	N=	1*10↑18
K= .470	COMPLEX	N=	1*10↑18
K= .460	COMPLEX	N=	1*10↑18
K= .450	COMPLEX	N=	1*10↑17
K= .440	COMPLEX	N=	1*10↑17
K= .430	COMPLEX	N=	1*10↑17
K= .420	COMPLEX	N=	1*10↑17
K= .410	COMPLEX	N=	1*10↑17
K= .400	COMPLEX	N=	1*10↑17
K= .390	COMPLEX	N=	1*10↑17
K= .380	COMPLEX	N=	1*10↑17
K= .370	COMPLEX	N=	1*10↑17

TABLE II (CONT.)

K= .360	COMPLEX	N=	1*10↑17
K= .350	COMPLEX	N=	1*10↑17
K= .340	COMPLEX	N=	1*10↑17
K= .330	COMPLEX	N=	1*10↑17
K= .320	COMPLEX	N=	1*10↑17
K= .310	COMPLEX	N=	1*10↑17
K= .300	COMPLEX	N=	1*10↑17
K= .290	COMPLEX	N=	1*10↑17
K= .280	COMPLEX	N=	1*10↑17
K= .270	COMPLEX	N=	1*10↑17
K= .260			
K= .250			
K= .240			
K= .230			
K= .220	COMPLEX	N=	1*10↑16
K= .210	COMPLEX	N=	1*10↑16
K= .200	COMPLEX	N=	1*10↑16
K= .190	COMPLEX	N=	1*10↑16
K= .180	COMPLEX	N=	1*10↑16
K= .170			
K= .160			
K= .150			
K= .140			
K= .130			
K= .120			
K= .110			
K= .100			
K= .090			

TABLE II (CONT.)

```
K= .080
K= .070
K= .060
K= .050
K= .040
K= .030
K= .020
K= .010
K= .000
060T  06/27/68 11:41:03 00.62 HRS
```

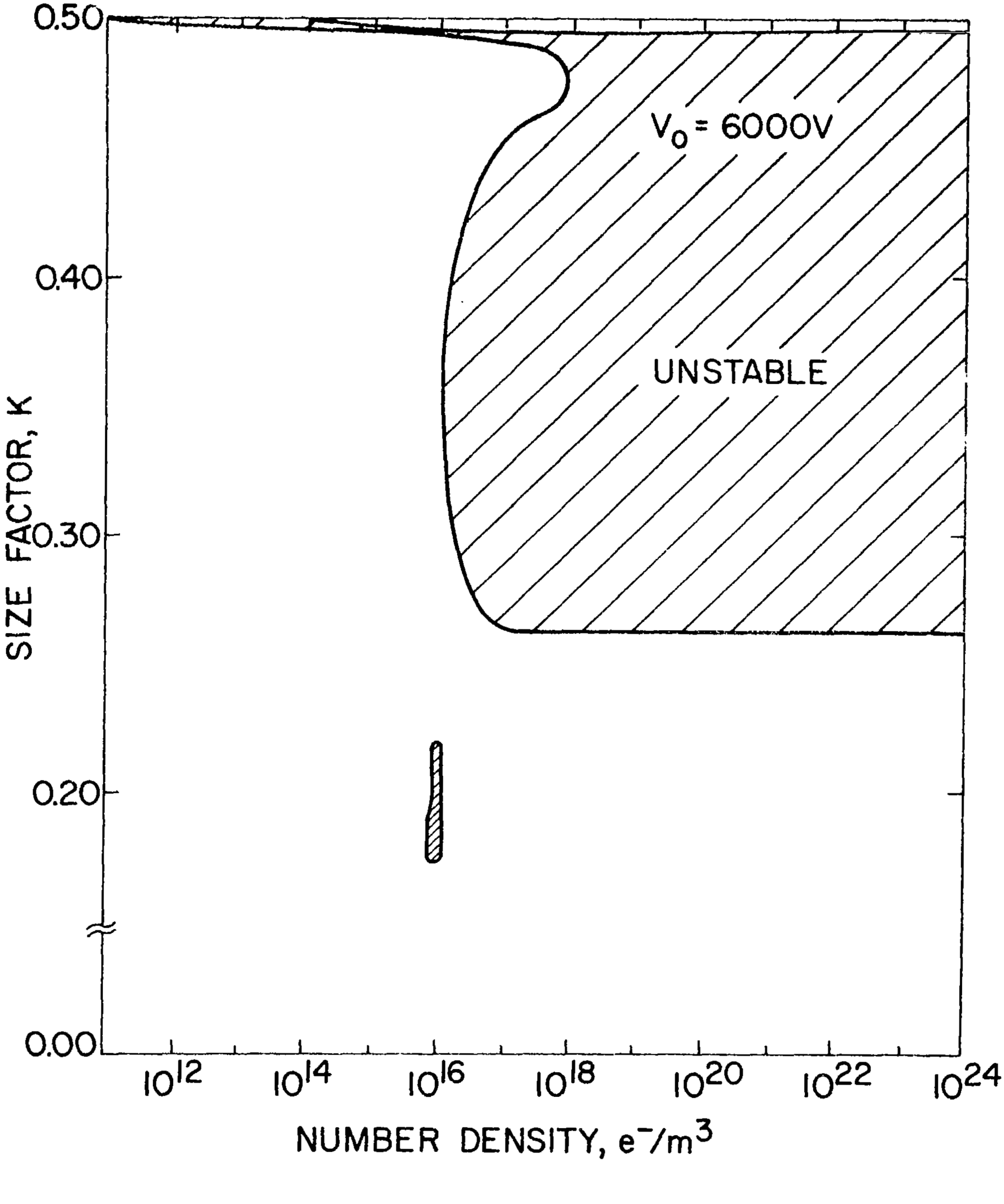

Fig. 4

Stability diagram for the magnetron, V_o = 6000V.

TABLE III

STABILITY CRITERION FOR THE MAGNETRON

```
DISPERSION EQUN --P*X↑2+Q*X+R=Ø--
    A-IN CYL          B-IN CLOUD          C-OUT CLOUD          D-OUT CYL
K-F(B)-SIZE FACTOR FOR CLOUD-K=Ø,CLOUD FILLS CHAMBER
NUMBER DENSITY OF ELECTRONS (M↑3) - N
MAGNETIC FIELD (GAUSS)              BO= 1ØØØ
APPLIED VOLTAGE (VOLT)              VO= 2ØØØ
ELECTRON CHARGE (COULOMB)              EL=     1.6*1Ø↑ -19
PERMITIVITY (FARAD/M)              EO=     8.8*1Ø↑ -12

          A=      1.46Ø5*1Ø↑ -3
          D=       .Ø14986
```

K= .5ØØ	COMPLEX	N=	1*1Ø↑15
K= .49Ø	COMPLEX	N=	1*1Ø↑18
K= .48Ø	COMPLEX	N=	1*1Ø↑17
K= .47Ø	COMPLEX	N=	1*1Ø↑17
K= .46Ø	COMPLEX	N=	1*1Ø↑17
K= .45Ø	COMPLEX	N=	1*1Ø↑17
K= .44Ø	COMPLEX	N=	1*1Ø↑17
K= .43Ø	COMPLEX	N=	1*1Ø↑17
K= .42Ø	COMPLEX	N=	1*1Ø↑17
K= .41Ø	COMPLEX	N=	1*1Ø↑17
K= .4ØØ	COMPLEX	N=	1*1Ø↑17
K= .39Ø	COMPLEX	N=	1*1Ø↑17
K= .38Ø	COMPLEX	N=	1*1Ø↑17
K= .37Ø	COMPLEX	N=	1*1Ø↑16

TABLE III (CONT.)

K= .360	COMPLEX	N=	1*10↑16
K= .350	COMPLEX	N=	1*10↑16
K= .340	COMPLEX	N=	1*10↑16
K= .330	COMPLEX	N=	1*10↑16
K= .320	COMPLEX	N=	1*10↑16
K= .310	COMPLEX	N=	1*10↑16
K= .300	COMPLEX	N=	1*10↑16
K= .290	COMPLEX	N=	1*10↑16
K= .280	COMPLEX	N=	1*10↑16
K= .270	COMPLEX	N=	1*10↑16
K= .260	COMPLEX	N=	1*10↑16
K= .250	COMPLEX	N=	1*10↑16
K= .240			
K= .230			
K= .220			
K= .210			
K= .200			
K= .190			
K= .180			
K= .170			
K= .160			
K= .150			
K= .140			
K= .130			
K= .120			
K= .110			
K= .100			
K= .090			
K= .080			
K= .070			
K= .060			

TABLE III (CONT.)

```
K= .050
K= .040
K= .030   .
K= .020
K= .010
K= .000
090T  06/26/68 13:29:47 02.09 HRS
```

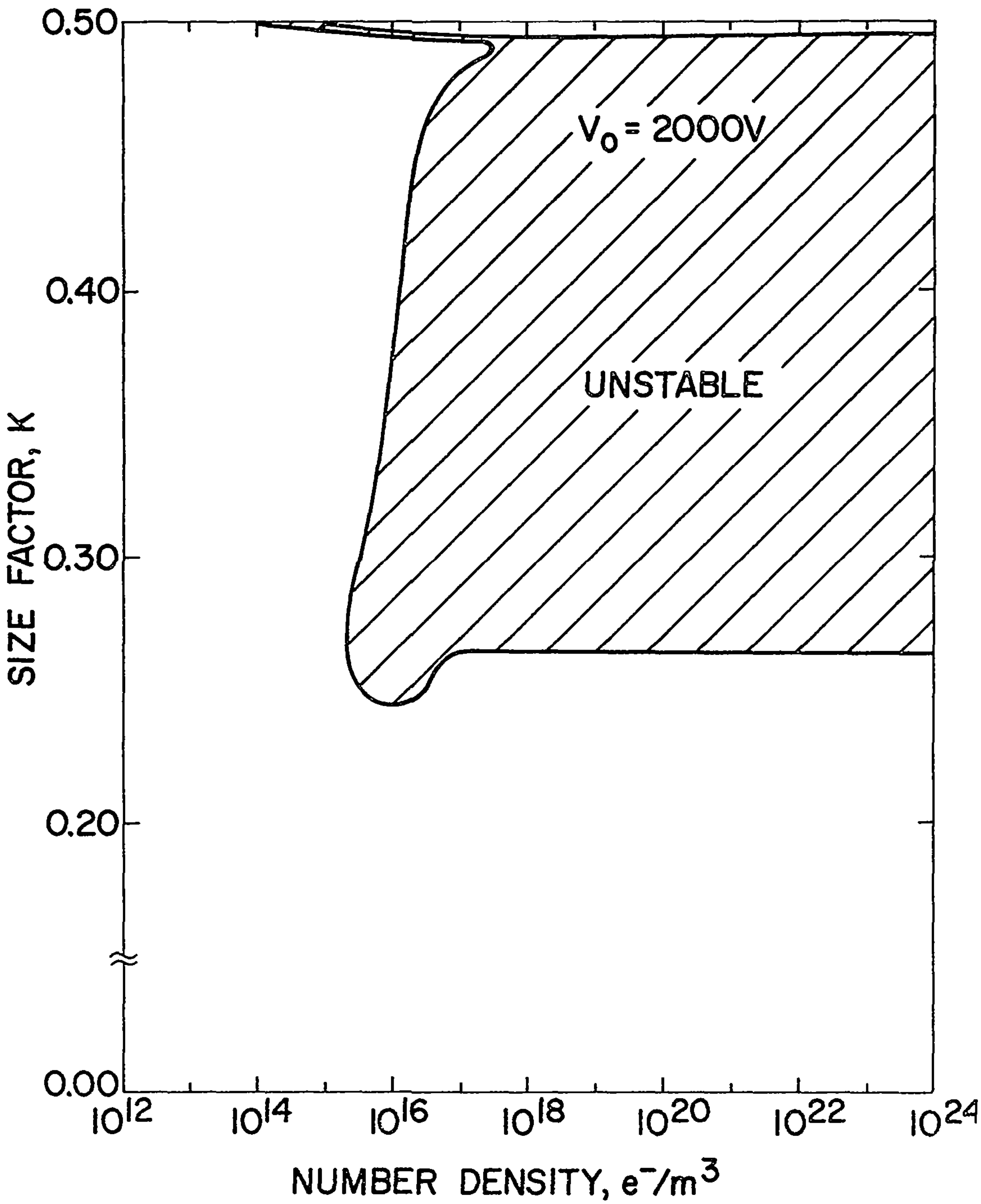

Fig. 5
Stability diagram for the magnetron V_o = 2000V.

TABLE IV

STABILITY CRITERION FOR THE MAGNETRON

DISPERSION EQUN --P*X↑2+Q*X+R=0--
A-IN CYL   B-IN CLOUD   C-OUT CLOUD   D-OUT CYL
K-F(B)-SIZE FACTOR FOR CLOUD-K=0,CLOUD FILLS CHAMBER
NUMBER DENSITY OF ELECTRONS (M↑3) - N
MAGNETIC FIELD (GAUSS)   BO= 1000
APPLIED VOLTAGE (VOLT)   VO= 1*10↑4
ELECTRON CHARGE (COULOMB)   EL= 1.6*10↑ -19
PERMITIVITY (FARAD/M)   EO= 8.8*10↑ -12

A= 1.4605*10↑ -3
D= .014986

K= .500	COMPLEX	N=	1*10↑12
K= .490	COMPLEX	N=	1*10↑19
K= .480	COMPLEX	N=	1*10↑18
K= .470	COMPLEX	N=	1*10↑18
K= .460	COMPLEX	N=	1*10↑18
K= .450	COMPLEX	N=	1*10↑18
K= .440	COMPLEX	N=	1*10↑18
K= .430	COMPLEX	N=	1*10↑18
K= .420	COMPLEX	N=	1*10↑17
K= .410	COMPLEX	N=	1*10↑17
K= .400	COMPLEX	N=	1*10↑17
K= .390	COMPLEX	N=	1*10↑17
K= .380	COMPLEX	N=	1*10↑17
K= .370	COMPLEX	N=	1*10↑17

TABLE IV (CONT.)

K= .360	COMPLEX	N=	1*10↑17
K= .350	COMPLEX	N=	1*10↑17
K= .340	COMPLEX	N=	1*10↑17
K= .330	COMPLEX	N=	1*10↑17
K= .320	COMPLEX	N=	1*10↑17
K= .310	COMPLEX	N=	1*10↑17
K= .300	COMPLEX	N=	1*10↑17
K= .290	COMPLEX	N=	1*10↑17
K= .280	COMPLEX	N=	1*10↑17
K= .270	COMPLEX	N=	1*10↑17
K= .260	COMPLEX	N=	1*10↑17
K= .250			
K= .240			
K= .230			
K= .220			
K= .210			
K= .200			
K= .190			
K= .180			
K= .170			
K= .160			
K= .150			
K= .140			
K= .130			
K= .120			
K= .110			
K= .100			
K= .090			
K= .080			
K= .070			
K= .060			
K= .050			

TABLE IV (CONT.)

K= .Ø4Ø
K= .Ø3Ø
K= .Ø2Ø
K= .Ø1Ø
K= .ØØØ
Ø6ØT Ø6/27/68 11:Ø1:19 ØØ.71 HRS

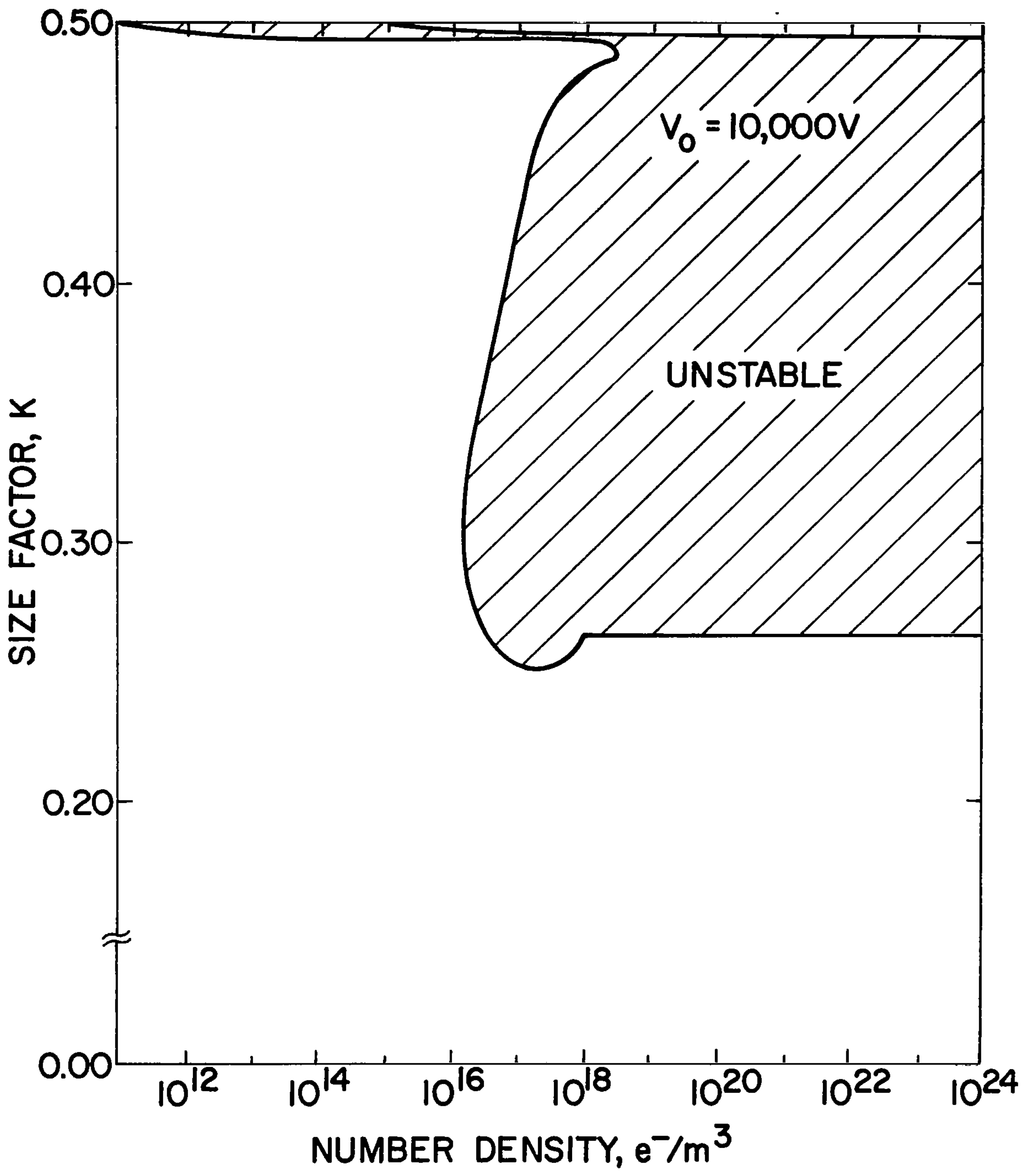

Fig. 6

Stability diagram for the magnetron V_o = 10,000V.

ROUGH FREQUENCY CALCULATIONS IN THE MAGNETRON WITH THE FUNCTIONAL RELATIONSHIP OF $\zeta = f\,(N, K, V_0)$

It is possible, taking the operating voltage for the standard Redhead gage $V_0 = 6000$ volts, to construct a table of the actual oscillation frequencies $\omega = eN/\varepsilon_0 B\,\zeta$ for values of N and K over the allowed range in K and a realistic range in N to determine if these conditions are at least roughly consistent with experiment. Such a calculation was performed and the computer program is shown in Table V and the results in Table VI. From the relation between frequency, anode voltage, and magnetic field of f = 11.2 V/B (MHZ) taken from the paper by Redhead (ref. 7) we, for this operating point, should obtain a frequency of 27.2 MHZ. In comparison with the table, however, this figure should be multiplied by 2π since the calculations are in angular frequency, so we should want a angular frequency of 4.22×10^8 HZ to agree with Redhead. Looking at the table we see that this frequency is within the stable region in K and a reasonable value for N . In fact this rough calculation gives a very small value for K indicating that the charge cloud nearly fills the chamber. A relation between N and K for this observed frequency could be worked out, but this exercise will be left until better limitations can be put on N .

TABLE V

```
1.01; ROUGH FREQUENCY CALCULATIONS FOR THE MAGNETRON
1.02 PAGE
1.10 PRINT"        ","FREQUENCY CALCULATIONS FOR THE MAGNETRON"
1.11 TYPE #,#
1.20 TYPE "DISPERSION EQUN --P*X↑2+Q*X+R=0--"
1.30 PRINT"     ","A-IN CYL","          ","B-IN CLOUD","
1.31 PRINT"C-OUT CLOUD","        ","D-OUT CYL"
1.32 TYPE #
1.41 TYPE "K-F(B)-SIZE FACTOR FOR CLOUD-K=0,CLOUD FILLS CHAMBER"
1.42 TYPE "0<=K=>0.267, STABILITY AND CLOUD SIZE"
1.43 TYPE #
1.431 PRINT"PHYSICAL DIMENSIONS (M)  A-B-C-D"
1.432 TYPE #
1.44 PRINT"NUMBER DENSITY OF ELECTRONS (M↑3) - N"
1.45 TYPE #
1.50 PRINT"MAGNETIC FIELD (GAUSS)", BO FOR BO=1000
1.51 TYPE #
1.52 PRINT"APPLIED VOLTAGE (VOLT)",VO FOR VO=6000
1.53 TYPE #
1.54 PRINT"ELECTRON CHARGE (COULOMB)",EL FOR EL=1.6*10↑-19
1.55 TYPE #
1.56 PRINT"PERMITTIVITY (FARAD/M)", EO FOR EO=8.8*10↑-12
1.57 TYPE #,#
1.60 PRINT A FOR A=((0.115/2)*2.54)*(1/100)
1.61 TYPE #
1.62 PRINT D FOR D=((1.180/2)*2.54)*(1/100)
1.63 TYPE #
1.71 L=D-A
1.72 PAGE
1.81 PRINT"            ","TABLE OF NUMBER DENSITY AND FREQUENCIES"
1.82 TYPE #,#
1.83 PRINT"        ","N","         ","W[1]","        ","W[2]"
1.84 TYPE #,#
1.99 DO PART 2

2.10 K=0.450
2.11 B=A+K*L
2.12 C=D-K*L
2.14 PRINT K
2.15 TYPE #
2.21 N=10↑12
2.22 TO STEP 3.20 IF N>10↑24
2.31 Q5=($PI*N*EL)/(2*LN(D/A))
```

```
2.32 Q6=((C↑2-B↑2)+2*(C↑2)*LN(D/C)-2*(B↑2)*LN(D/B))
2.33 Q7=(2*$PI*EO*VO)/(LN(D/A))
2.34 Q4=Q5*Q6-Q7
2.35 E=(Q4)/($PI*N*EL)
2.41 P=-4*(D↑2-A↑2)
2.42 Q1=P/(-4)
2.43 R1=(E/B↑2)*(1-(B↑2/C↑2)-(E/C↑2))*(D↑2-A↑2)
2.44 Q2=(1-(B↑2/C↑2))-(E/B↑2)*(1+(B↑2/C↑2))
2.45 R2=(-E/B↑2)*(C↑2-A↑2)*(D↑2-C↑2)*C↑-2
2.46 Q3=(B↑2*C↑2-A↑2*D↑2)*(C↑2-B↑2)*(B↑-2)*(C↑-2)
2.47 R3=-(1-(B↑2/C↑2)-(E/C↑2))*(D↑2-B↑2)*(B↑2-A↑2)*B↑-2
2.48 R4=(C↑2-B↑2)*(D↑2-C↑2)*(B↑2-A↑2)*(B↑-2)*(C↑-2)
2.51 Q=2*(Q1*Q2+Q3)
2.52 R=R1+R2+R3+R4
2.53 DIS=Q↑2-4*P*R
2.60 TO STEP 3.10 IF DIS<0
2.61 TO STEP 2.70 IF DIS>0
2.70 X[I]=(-Q+((-1)↑I)*SQRT(DIS))/(2*P) FOR I=1,2
2.71 W[1]=((EL*N)/(EO*BO))*X[1]
2.72 W[2]=((EL*N)/(EO*BO))*X[2]
2.73 PRINT N IN FORM 1
2.74 PRINT W[1] IN FORM 1
2.75 PRINT W[2] IN FORM 1
2.76 TYPE #
2.99 DO PART 3

3.10 SET N=100*N
3.11 TO STEP 2.22
3.20 SET K=K-0.050
3.21 TYPE #
3.22 LOGOUT IF K<0
3.23 TO STEP 2.11


FORM        1
     #.#↑↑↑

←
```

TABLE VI

FREQUENCY CALCULATIONS FOR THE MAGNETRON

DISPERSION EQUN --P*X↑2+Q*X+R=0--
A-IN CYL B-IN CLOUD C-OUT CLOUD D-OUT CYL
K-F(B)-SIZE FACTOR FOR CLOUD-K=0,CLOUD FILLS CHAMBER
0<=K=>0.267, STABILITY AND CLOUD SIZE

PHYSICAL DIMENSIONS (M) A-B-C-D
NUMBER DENSITY OF ELECTRONS (M↑3) - N
MAGNETIC FIELD (GAUSS) BO= 1000
APPLIED VOLTAGE (VOLT) VO= 6000
ELECTRON CHARGE (COULOMB) EL= 1.6*10↑ -19
PERMITTIVITY (FARAD/M) EO= 8.8*10↑ -12

A= 1.4605*10↑ -3
D= .014986

TABLE OF NUMBER DENSITY AND FREQUENCIES

	N	W[1]	W[2]
K= .450			
	1.0+12	4.5+04	3.3+04
	1.0+14	4.6+04	3.2+04
	1.0+16	7.2+04	2.4+04
K= .400			
	1.0+12	5.5+04	2.8+04
	1.0+14	5.5+04	2.8+04
	1.0+16	8.4+04	2.6+04
K= .350			
	1.0+12	6.7+04	2.5+04
	1.0+14	6.7+04	2.4+04
	1.0+16	8.8+04	3.1+04
K= .300			
	1.0+12	8.5+04	2.2+04
	1.0+14	8.5+04	2.2+04
	1.0+16	8.5+04	3.8+04

TABLE VI (CONT.)

K= .250			
	1.0+12	1.1+05	1.9+04
	1.0+14	1.1+05	1.9+04
	1.0+16	7.3+04	4.5+04
	1.0+18	1.7+06	-.3+07
	1.0+20	1.6+08	-.3+09
	1.0+22	1.6+10	-.3+11
	1.0+24	1.6+12	-.3+13
K= .200			
	1.0+12	1.5+05	1.7+04
	1.0+14	1.5+05	1.7+04
	1.0+18	3.1+06	-.9+07
	1.0+20	3.1+08	-.9+09
	1.0+22	3.1+10	-.9+11
	1.0+24	3.1+12	-.9+13
K= .150			
	1.0+12	2.1+05	1.5+04
	1.0+14	2.1+05	1.6+04
	1.0+16	5.7+04	1.3+04
	1.0+18	4.2+06	-.2+08
	1.0+20	4.2+08	-.2+10
	1.0+22	4.2+10	-.2+12
	1.0+24	4.2+12	-.2+14
K= .100			
	1.0+12	3.3+05	1.4+04
	1.0+14	3.2+05	1.4+04
	1.0+16	6.6+04	-.7+05
	1.0+18	5.2+06	-.4+08
	1.0+20	5.2+08	-.4+10
	1.0+22	5.2+10	-.4+12
	1.0+24	5.2+12	-.4+14
K= .050			
	1.0+12	5.6+05	1.3+04
	1.0+14	5.6+05	1.3+04
	1.0+16	7.4+04	-.2+06
	1.0+18	6.2+06	-.8+08
	1.0+20	6.2+08	-.8+10
	1.0+22	6.2+10	-.8+12
	1.0+24	6.2+12	-.8+14

TABLE VI (CONT.)

K= .ØØØ

1.Ø+12	1.2+Ø6	1.1+Ø4
1.Ø+14	1.2+Ø6	1.2+Ø4
1.Ø+16	8.3+Ø4	-.7+Ø6
1.Ø+18	7.2+Ø6	-.2+Ø9
1.Ø+20	7.2+Ø8	-.2+11
1.Ø+22	7.2+1Ø	-.2+13
1.Ø+24	7.2+12	-.2+15

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LIMITATIONS ON THE VALUE OF N

Looking at Eq. (28) we see that the applied potential is a measure of the charge residing on the inner electrode and the charge contained between the cylinders.

$$\phi_o(d) = -\frac{Q}{2\pi\varepsilon_o} \ln \frac{d}{a} + \frac{Ne}{4\varepsilon_o}\left[(c^2-b^2) + 2c^2 \ln \frac{d}{c} - 2b^2 \ln \frac{d}{b}\right] \quad (28)$$

The first term on the right hand side is positive since Q is a negative quantity. The second term is also positive since e is the specific charge and the expression in brackets is always positive. Thus as N increases $|Q|$ decreases. However Q cannot go positive otherwise the field at the cathode would change sign and repel the incoming ions effectively turning off the primary supply of electrons for the discharge. Thus we see that N must be equal to or less than that number that would make the charge on the inner electrode go to zero. Substituting V_o for $\phi_o(d)$ we can write an equation for the maximum N.

$$N_{max} \leq \frac{4\varepsilon_o V_o}{e} \frac{1}{\left[(c^2-b^2) + 2c^2\ln\frac{d}{c} - 2b^2\ln\frac{d}{b}\right]} \quad (40)$$

For this geometry the expression in brackets varies from 2.7×10^{-5} for $K = 0.45$ to 2.2×10^{-4} for $K = 0$. The maximum value of N can be calculated from Eq. (40) for various values of

V_o and K . Calculations of N_{max} for various values of K are tabulated in Table Vii and graphed for several voltages in Fig. 7.

Physically this limitation on N corresponds to putting just enough electrons into the discharge to reduce the field at the cathode to zero. Further addition of electrons would correspond to driving the field (at the cathode) positive which would draw electrons from the discharge. Thus this boundary is a self limiting one.

TABLE VII

CALCULATIONS OF N_{max} AND THE EXPRESSION IN THE DENOMINATOR OF EQ. (40)

$$D = \left[(c^2-b^2) + 2c^2 \ln\frac{d}{c} - 2b^2 \ln\frac{d}{b}\right]$$

FOR VARIOUS K VALUES AND V_o = 6000 volt. N_{max} VALUES FOR OTHER APPLIED VOLTAGES ARE A SIMPLE MULTIPLE OF THIS VALUE

K	D	N_{max}
0.45	2.7×10^{-5}	4.9×10^{16} $1/m^3$
0.40	5.3	2.5
0.35	7.9	1.7
0.30	1.0×10^{-4}	1.3
0.25	1.3	1.0
0.20	1.5	8.8×10^{15}
0.15	1.7	7.8
0.10	1.9	7.0
0.05	2.0	6.6
0	2.2	6.0

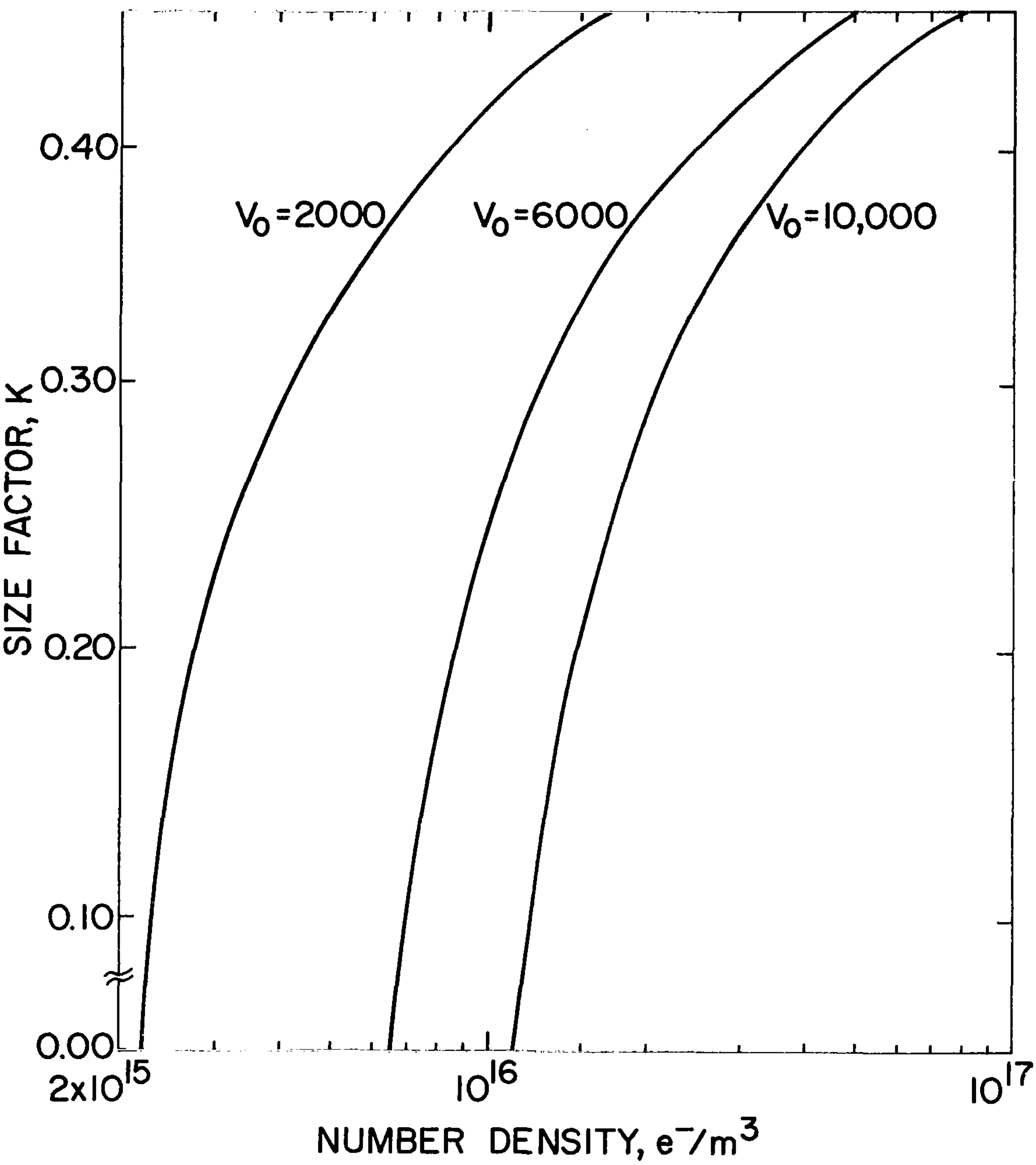

Fig. 7 Graph of Maximum Number Density, N_{max} for Applied Voltages of 2000, 6000, 10000V.

POTENTIAL AND ELECTRIC FIELD PROFILES IN THE GAGE FOR THE EXTREME CASES: CHARGE FREE AND MAXIMUM CHARGE

Charge Free Case

The charge free field and potential are determined from Eqs. (25).

$$E_o = \frac{Q}{2\pi\varepsilon_o r}$$

$$\phi_o = \frac{Q}{2\pi\varepsilon_o}\ln\frac{r}{a} \qquad (25)$$

The condition that $\phi_o = 0$ at $r = a$ and $\phi_o = V_o$ at $r = d$ requires that $-Q/2\pi\varepsilon_o = V_o/\ln\frac{d}{a}$ so these expressions are now written as

$$E_{cf} = -\frac{V_o}{r\ln\frac{d}{a}}$$

$$\phi_{cf} = \frac{V_o}{\ln\frac{d}{a}}\ln\frac{r}{a}$$

where the subscripts cf indicate charge free. Putting in the values of $V_o = 6000$ volt and d and a for the gage we can compute the charge free potential and field in the gauge. This calculation is shown Table VIII.

Maximum Charge Case

The condition for maximum charge was derived from Eq. (28) relating the applied potential to the charge on the cathode and the number of electrons in the cloud. The curves of Fig. 7 relate the number density of electrons to K value to reduce the field at the cathode to zero. Since the maximum charge must be for the cloud filling the entire chamber we take K = 0.001 and a value of $N = 6.0 \times 10^{15}$. The calculations of the potential and the field for this situation are obtained from Eqs. (26).

$$E_o = \frac{Q}{2\pi\varepsilon_o r} - \frac{Neb}{\varepsilon_o}\left(\frac{r}{b} - \frac{b}{r}\right)$$

$$\phi_o = \frac{Q}{2\pi\varepsilon_o}\ln\frac{r}{a} + \frac{Neb^2}{4\varepsilon_o}\left[\frac{r^2}{b^2} - 1 - 2\ln\frac{r}{b}\right] \qquad (26)$$

where for these values of K and N, $\dot{Q} = 0$. This calculation is shown in Table IX.

TABLE VIII

CALCULATION OF POTENTIAL AND FIELD FOR THE CHARGE FREE CASE

A= 1.4605*10↑ -3
D= .014986
VO= 6000

RADIUS	POTENTIAL	FIELD
2.81-03	1.69+03	-9.16+05
4.17-03	2.70+03	-6.19+05
5.52-03	3.43+03	-4.67+05
6.87-03	3.99+03	-3.75+05
8.22-03	4.45+03	-3.13+05
9.58-03	4.85+03	-2.69+05
1.09-02	5.19+03	-2.36+05
1.23-02	5.49+03	-2.10+05
1.36-02	5.76+03	-1.89+05
1.50-02	6.00+03	-1.72+05

TABLE IX

CALCULATION OF POTENTIAL AND FIELD FOR THE MAXIMUM CHARGE CASE

A= 1.4605*10↑ -3
D= .014986
VO= 6000

RADIUS	POTENTIAL	FIELD
2.81-03	8.26+01	-2.30+05
4.17-03	3.01+02	-4.11+05
5.52-03	6.35+02	-5.78+05
6.87-03	1.08+03	-7.39+05
8.22-03	1.63+03	-8.97+05
9.58-03	2.29+03	-1.05+06
1.09-02	3.06+03	-1.21+06
1.23-02	3.93+03	-1.36+06
1.36-02	4.90+03	-1.52+06
1.50-02	5.98+03	-1.67+06

COMPARISON OF THE CHARGE FREE AND MAXIMUM CHARGE CONDITIONS

In the following two figures we show the maximum charge and charge free profiles for the field (Fig. 8) and the potential (Fig. 9).

BUILDUP AND MAINTAINANCE OF THE DISCHARGE

In one of the previous sections we put an upper limit on the total number of electrons that could be maintained in the discharge. In this section we consider how the electron cloud builds up and then sustains itself.

When the discharge starts to build up the ions that strike the cathode have a relatively high energy (several KeV as seen in Fig. 9) and at this energy the yield (number of electrons liberated per ion striking the cathode) is approximately 3 based on the measurements of A^+ ions on Nichrome V (ref. 8). Argon ions are typical and the commerical gages as well as the original ones by Redhead are constructed of Nichrome V. However as the discharge builds up the electron cloud everywhere lowers the potential so that the average ion energy and likewise the yield decreases.

The number of electrons pumped into the cloud depends on the number of ions striking the cathode, the energy of these ions, the yield of the cathode material and the relative number of electrons liberated from the cathode that enter the cloud. We can look at the effect of each of these parameters in turn on the buildup and maintance of the discharge..

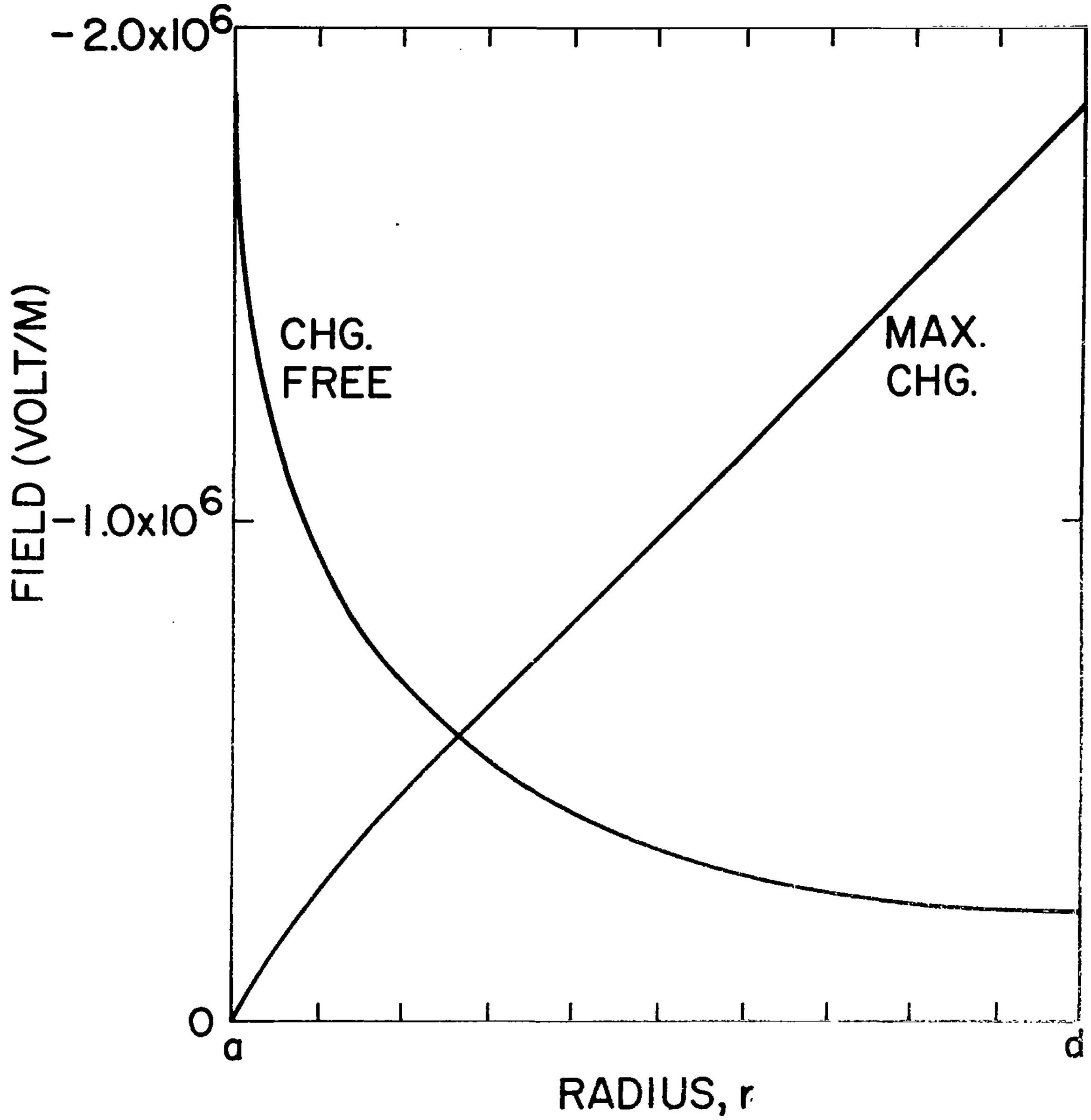

Fig. 8

Graphs of electric field as a function of radius for the charge free and maximum charge conditions in the magnetron.

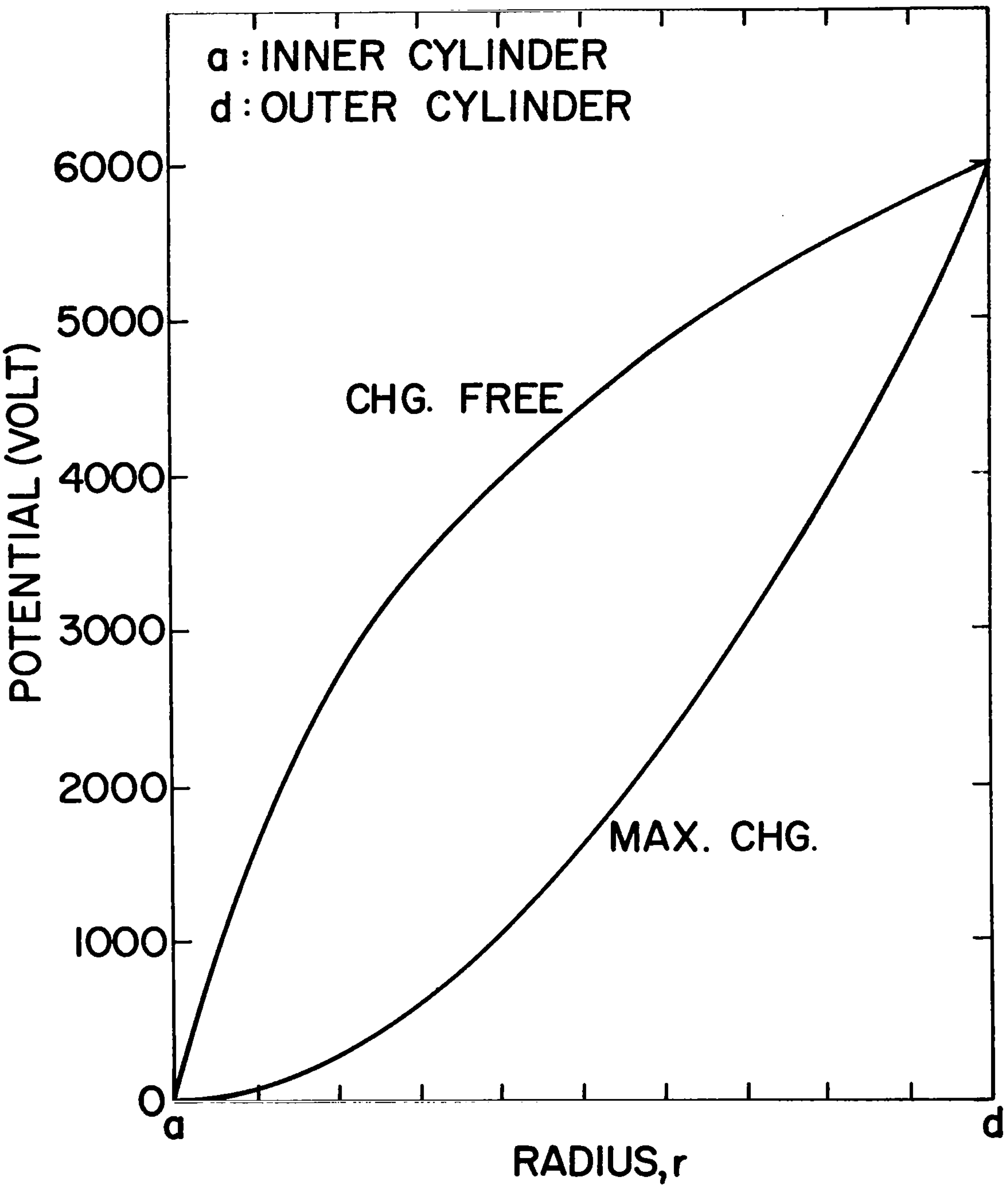

Fig. 9

Graphs of potential as a function of radius for the charge free and maximum charge conditions in the magnetron.

The number of ions striking the cathode will decrease with pressure and experimentally we know this is a linear decrease. As the energy of the ions decreases the yield decreases though looking at the curve of Fig. 9 and the yield curves (ref. 8) the number of electrons liberated per ion cannot decrease much less than unity. Assuming no effects due to gas adsorption the yield must reamin constant. The relative number of electrons that escape the cathode depends also on the pressure.

At this point without making any calculations we can make a very educated guess as to the mechanism that makes the gage linear over a wide range. If we assume that the charge cloud is feed by the mechanism described then the limitation must be via the cloud itself and in particular the limitation on N described in a previous section. This predicts a constant number of electrons in the discharge and a constant potential and field profile which implies a constant sensitivity of the gage. This implies that in the linear range the supply of electrons back into the discharge is greater than the loss rate due to collisions.

At the point where the gage becomes non-linear the rate at which electrons are feed into the discharge must just balance some loss mechanism. Since the non-linearity begins at a constant current and not constant pressure it is safe to assume that the loss mechanism is independent of pressure.

Physically the situation is the following. The charge cloud in the gage is constant in numbers of electrons and as the pressure decreases the number of electrons necessary to maintain this discharge decreases. Though this primary mechanism for supplying electrons to the discharge is more than sufficient to keep the number of electrons in the cloud at its maximum over a wide pressure range at sufficiently low pressures the ion current which controls the number of electrons going into the discharge goes so low that the small "unknown for the present" loss rate begins to sap electrons from the discharge.

CALCULATION OF THE SENSITIVITY OF THE GAGE

First Rough Estimate Of The Sensitivity

Using some of the results of the previous sections it is possible to make an estimate of the sensitivity of the gauge when it is operating in the linear region; corresponding to the maximum stored charge. For this first estimate we take an average electron energy and calculation the ion current as a function of pressure for a beam of electrons of this energy and density, as taken from Fig. 7, passing through a gas where we know the ionization cross section and the pressure density relation.

Average Energy of the Electrons.- We make a very rough estimate of the average electron energy by looking at the electron ballistics. Electrons liberated from the central electrode start out with approximately zero energy and execute cycloidal motion according to the following parametric equations which are written in Cartesian coordinates (ref. 2, 3)

$$x = \frac{E}{\omega_c B}(1 - \cos\omega_c t)$$

$$y = -\frac{E}{\omega_c B}\sin\omega_c t + \frac{E}{B}t \qquad (41)$$

where $\omega_c = |e|B/m$ is the cyclotron frequency. The dimensions involved are sufficiently small to justify a switch to the more convenient Cartesian coordiantes. The x-direction corresponds to the direction of the electric field and the y-direction, the azimuthal direction. The energy of the electron in its cycloidal path is determined by the radial potential profile and the radial position of the electron. Now this initial electron begets others by ionizing collisions and so on across the discharge with the subsequent electrons also executing this cycloidal motion. The cycloid height or maximum excursion in the direction of the field is

$$D = \frac{2E}{\omega_c B} = \frac{2mE}{eB^2} \qquad (42)$$

Now if we look at the radial field profile, Fig. 8 and Table IX, we see that halfway across the discharge the field is 9.0×10^5 volt/m. In the preceding calculations the end caps have not been taken into account. The field must go to zero (or nearly so) at the end caps as well as the cathode so as a first approximation we assume that the field goes linearly, in the axial direction, from a value of zero at the end caps to the value given in Fig. 8 and Table XI for any particular radius.

Halfway across the discharge (in the radial direction) then the field goes from zero at one end cap to 9.0×10^5 volt/m midway between the end caps and back to zero at the other end cap. Thus the average field that an electron will encounter is 4.5×10^5 volt/m.

We have not taken into account the changes in N due to the influence of the end caps however the electric field is more important since it enters the calculation of the sensitivity as the square while N enters linearly. For a field of 4.5×10^5 volt/m which is what the electrons encounter on average the excursion is 5.0×10^{-4} m. This value is sufficiently small compared with the dimensions of the device to justify use of Eqs. (41) at least for this gross approximation.

With the assumption of a uniform density of electrons executing these cycloidal motions the average electron energy must be the average energy of an electron executing this motion midway across the discharge. Note that the field increases roughly linearly across the discharge both in the radial (Fig. 8) and axial directions.

With a cycloidal height of 5.0×10^{-4} m and a field of 4.5×10^5 volt/m the electron in its cycloid goes from zero to 225 eV in energy. As a first estimate then we simply take one half this value or 112 eV as the average electron energy.

Model for the Calculation of Sensitivity.- Now that we have a first estimate of the electron energy we can set up a simple model to calculate the ionization rate in the gauge. Consider a unit volume of a gas whose number density is simply related to the pressure by

$$\frac{N}{V} = 3.2 \times 10^{16}\ P \tag{43}$$

where P is measured in Torr and N/V has the units of molecules/cm^3. This relation is valid at room temperature. Note that this calculation is carried out in cgs units because interaction cross sections important to the calculation are given in cm^2. We also can ascribe certain properties to the electron gas namely a density of 6.0×10^9 $1/cm^3$ and velocity of 6.3×10^8 cm/sec. (corresponding to an energy of 112 eV). The number of electrons passing through a unit volume per second is just the density in the stream times the velocity which turns out to be 3.8×10^{18} $1/cm^2$-sec. This is the number that pass through unit area in 1 sec. We take this number as the number of electrons that can cause an ionizing collision.

Calculation of the Sensitivity.- The number of ions or particle current produced per unit volume is

$$N_i = \frac{N}{V} \times A \times N_e \tag{44}$$

where A is the interaction cross section and N_e is the number of electrons that can cause an ionization. The ionization cross section for electrons of 112 eV on nitrogen is 2.5×10^{-16} cm^2 (ref. 9) so Eq. (44) gives

$$N_i = 2.5 \times 10^{-16} cm^2 \times 3.8 \times 10^{18} \frac{1}{cm^2\text{-}sec} \times 3.2 \times 10^{16}\, P \frac{1}{cm^3}$$

$$N_i = 3.0 \times 10^{19}\, P\ 1/cm^3\text{-}sec.$$

The sensitivity of the gage is just this expression taking into account the volume of the discharge (V_g) and the charge per particle or

$$\frac{I_+}{P} = N_i \times |e| \times V_g \tag{45}$$

where we take for the volume of the discharge the volume enclosed by the anode ring, $V_g = 10\ cm^3$. Carrying out the calculations we have for the sensitivity.

$$\frac{I_+}{P} = 3.0 \times 10^{19} \frac{1}{cm^3\text{-}sec}\ 1.6 \times 10^{-19}\ C \times 10\ cm^3$$

$$\frac{I_+}{P} = 48 \frac{Amp}{Torr}$$

This calculated value is a factor of 5 times the value of 9 $\frac{amp}{Torr}$ measured by Redhead, (ref. 10). In the next section we will explore some more suitable estimates for electron energy that will give a better estimate of the sensitivity.

More Refined Calculations Of The Sensitivity

The calculation of the sensitivity of the gage is inaccurate primarily because the average electron energy coincides with the peak of ionization. We can make a better estimate by paying closer attention to the electron ballistics particularly how much time the electron spends in a particular energy range and its cross section over that range. The first of Eqs. (41) describes the path of the electron in the direction of the field as a function of time. Multiplying this expression by the field gives an expression for the electron energy as a function of time. Thus we can write

$$\mathcal{E} = Ex = \frac{E^2}{\omega_c B}(1 - \cos\omega_c t) \tag{46}$$

The time for the electron to reach its maximum excursion is $t_o = \pi/\omega_c = 1.8 \times 10^{-10}$ sec. Using this relation or the simpler one

$$\mathcal{E} = 112\ (1 - \cos\omega_c t)\ \text{eV}$$

we have an expression for the electron energy as a function of time. This curve is sufficiently close to linear to justify our taking a time factor for the electron in a particular energy range equal to the fraction of the energy range. Thus we say the electron spends 0.10 of its time between 0 and 22.5 eV.

Now if we compare this curve to the ionization cross section curve we see that some matching of cross sections, average energy, and time spent in that energy range is most appropriate. We can approximate by breaking up the ionization curve into two sections. Since the cross section falls off rapidly for low energies this region is neglected. The total particle current is the number density relation x electron flux x time factor x average corss section.

In the range from 20 to 45 eV:

average energy = 32 eV

average velocity = 3.3×10^{8} cm/sec.

electron flux = 2.0×10^{18} $1/cm^2$-sec.

time factor = 0.11

average cross section = 1.1×10^{-16} cm^2

particle current = 3.2×10^{16} P $1/cm^3 \times 2.0 \times 10^{18} \frac{1}{cm^2\text{-sec.}}$

$0.11 \times 1.1 \times 10^{-16}$ cm^2

particle current = $7.8 \times 10^{17} \frac{1}{cm^3\text{-sec.}}$

In the range from 45 to 225 eV:

average energy = 135 eV

average velocity = 6.8×10^{8} cm/sec.

electron flux = 4.1×10^{18} $1/cm^2$-sec.

time factor = 0.80

average cross section = 2.5×10^{-16} cm^2

$$\text{particle current} = 3.2 \times 10^{16}\ P\ 1/\text{cm}^3 \times 4.1 \times 10^{18}\ \frac{1}{\text{cm}^2\text{-sec.}} \times 0.80 \times 2.5 \times 10^{-16}\ \text{cm}^2$$

particle current = 2.5×10^{19} 1/cm^3-sec.

Now looking at these particle currents we see that the particle current is due primarily to the second region. The total ion current from the discharge is this particle current x charge x volume of the discharge.

$$I_+ = 2.6 \times 10^{19}\ \frac{1}{\text{cm}^3\text{-sec.}} \times 1.6 \times 10^{-19}\ \text{C} \times 10\ \text{cm}^3\ P$$

The sensitivity of the gage is then $I_+/P = 42$ amp/Torr which compares a little more favorably with the value of 9 amp/torr measured by Redhead (ref. 10).

CONCLUSIONS

The principal effort of this program was the development of a theoretical model for the magnetron gage that would provide for the observed oscillations as well as the linear relationship between pressure and ion current. The physical mechanism of slipping stream instability was shown to be an approximate model of the oscillations, wherein the oscillation frequency and variation of frequency with 1/B are consistent with experimental evidence. These calculations, in turn, yielded expressions for the electric field, potential and dependance of the electron cloud density on the applied voltage. The linear pressure-current relation which holds experimentally over a wide pressure range, follows from the model when a simple limiting mechanism is employed to hold the field at the cathode at zero. Calculations of the sensitivity of the gage based on the assumption of maximum charge stored in the gage are sufficiently close to experimental values, considering the approximations involved, to indicate that the theory is basically correct. It now appears evident that the mechanism leading to the non-linearity or loss in sensitivity at low pressures is an electron loss mechanism and that this mechanism is closely connected with the character of the discharge.

APPENDIX A

In Region 1

$$\phi_o = A \ln r + B$$

$$E_o = \frac{d\phi_o}{dr} = \frac{A}{r}$$

at r=a $E_o = \frac{Q}{2\pi a\varepsilon_o} = \frac{A}{a} \therefore A = \frac{Q}{2\pi\varepsilon_o}$

at r=a $\phi_o = 0 \quad A \ln a = -B \therefore B = -A \ln a$

$$\phi_o = A(\ln r - \ln a)$$

$$\phi_o = \frac{Q}{2\pi\varepsilon_o} \ln \frac{r}{a}$$

$$E_o = \frac{Q}{2\pi\varepsilon_o r}$$

In Region 2

$$E_o = \frac{Q}{2\pi\varepsilon_o r} - \frac{Ne\pi}{2\pi\varepsilon_o}\left(\frac{r^2-b^2}{r}\right)$$

$$\frac{r^2-b^2}{r} = \frac{r^2b-b^3}{rb} = \frac{br}{b} - \frac{b^2}{r} = b\left(\frac{r}{b} - \frac{b}{r}\right)$$

$$E_o = \frac{Q}{2\pi\varepsilon_o r} - \frac{Ne}{2\pi\varepsilon_o}\pi b\left(\frac{r}{b} - \frac{b}{r}\right)$$

$$\phi_o = -\int\left[\frac{Q}{2\pi\varepsilon_o r} - \frac{Ne\pi b}{2\pi\varepsilon_o}\left(\frac{r}{b} - \frac{b}{r}\right)\right] dr$$

$$\phi_o = -\frac{Q}{2\pi\varepsilon_o}\ln r + \frac{Ne\pi b}{2\pi\varepsilon_o}\frac{r^2}{2b} - \frac{Ne\pi b^2}{2\pi\varepsilon_o}\ln r$$

$$\phi_o = -\left[\frac{Q}{2\pi\varepsilon_o} + \frac{Ne\pi b^2}{2\pi\varepsilon_o}\right]\ln r + \frac{Ne\pi b}{4\pi\varepsilon_o} r^2 + C_1$$

at $r=b$ $\quad \phi_o = -\frac{Q}{2\pi\varepsilon_o}\frac{b}{a}$

$$-\frac{Q}{2\pi\varepsilon_o}\ln\frac{b}{a} = -\left[\frac{Q}{2\pi\varepsilon_o} + \frac{Ne\pi b^2}{2\pi\varepsilon_o}\right]\ln b + \frac{Ne\pi b^2}{4\pi\varepsilon_o} + C_1$$

$$C_1 = -\frac{Q}{2\pi\varepsilon_o}\ln\frac{b}{a} + \frac{Q}{2\pi\varepsilon_o}\ln b + \frac{Ne\pi b}{2\pi\varepsilon_o}\ln b - \frac{Ne\pi b^2}{4\pi\varepsilon_o}$$

$$C_1 = -\frac{Q}{2\pi\varepsilon_o}\ln a + \frac{Ne\pi b}{2\pi\varepsilon_o}\ln b - \frac{Ne\pi b^2}{2\pi\varepsilon_o}$$

$$\phi_o = -\left[\frac{Q}{2\pi\varepsilon_o} + \frac{Ne\pi b^2}{2\pi\varepsilon_o}\right]\ln r + \frac{Ne\pi}{4\pi\varepsilon_o} + \frac{Q}{2\pi\varepsilon_o}\ln a + \frac{Ne\pi b^2}{2\pi\varepsilon_o}\ln b - \frac{Ne\pi b^2}{4\pi\varepsilon_o}$$

$$\phi_o = -\frac{Q}{2\pi\varepsilon_o}\ln\frac{r}{a} + \frac{Ne\pi b^2}{4\pi\varepsilon_o}\left[\frac{r^2}{b^2} - 1 - 2\ln\frac{r}{b}\right]$$

In Region 3

$$E_o = \frac{Q}{2\pi\varepsilon_o r} - \frac{Ne\pi(c^2-b^2)}{2\pi\varepsilon_o r}$$

$$E_o = \frac{Q-Ne\pi(c^2-b^2)}{2\pi\varepsilon_o r}$$

$$\phi_o = -\int \frac{\left[Q-Ne\pi\cdot(c^2-b^2)\right]}{2\pi\varepsilon_o} \frac{dr}{r} = - \frac{Q-Ne\pi(c^2-b^2)}{2\pi\varepsilon_o} \int \frac{dr}{r}$$

$$\phi_o = \frac{Ne\pi(c^2-b^2)-Q}{2\pi\varepsilon_o} \ln r + B$$

at r=c $\phi_o = - \frac{Q}{2\pi\varepsilon_o} \ln \frac{c}{a} + \frac{Neb^2}{4\varepsilon_o} \left[\frac{c^2}{b^2} - 1 - 2 \ln \frac{c}{b}\right]$

$$- \frac{Q}{2\pi\varepsilon_o} \ln \frac{c}{a} + \frac{Neb^2}{4\varepsilon_o} \left|\frac{c^2}{b^2} - 1 - 2 \ln \frac{c}{b}\right| = \frac{Ne\pi(c^2-b^2)-Q}{2\pi\varepsilon_o} \ln c + B$$

$$B = \frac{Neb^2}{4\varepsilon_o} \left[\frac{c^2}{b^2} - 1 - 2 \ln \frac{c}{d}\right] - \frac{Q}{2\pi\varepsilon_o} \ln \frac{c}{a} - \frac{Ne\pi(c^2-b^2)-Q}{2\pi\varepsilon_o} \ln C$$

$$\phi_o = \frac{Ne(c^2-b^2)}{2\varepsilon_o} \ln r - \frac{Q}{2\pi\varepsilon_o} \ln r + \frac{Neb^2}{4\varepsilon_o} \left(\frac{c^2}{b^2} - 1 - 2 \ln \frac{c}{d}\right)$$

$$- \frac{Q}{2\pi\varepsilon_o} \ln \frac{c}{a} - \frac{Ne\pi(c^2-b^2)}{2\pi\varepsilon_o} \ln c$$

$$+ \frac{Q}{2\pi\varepsilon_o} \ln c$$

$$\phi_o = -\frac{Q}{2\pi\varepsilon_o}\ln\frac{r}{a} + \frac{Ne}{4\varepsilon_o}\left[c^2-b^2-2b^2\ln\frac{c}{b} + 2(c^2-b^2)\ln r - 2(c^2-b^2)\ln c\right]$$

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aside

$$-2b^2\ln c+2b^2\ln b+2c^2\ln r-2b^2\ln r-2c^2\ln c+2b^2\ln c = -2b^2\ln\frac{r}{b} + 2c^2\ln\frac{r}{c}.$$

$$\phi_o = \frac{Q}{2\pi\varepsilon_o}\ln\frac{r}{a} + \frac{Ne}{4\varepsilon_o}\left[(c^2-b^2) + 2c^2\ln\frac{r}{c} - 2b^2\ln\frac{r}{b}\right]$$

APPENDIX B

$$\phi_2 = \beta r^{\ell} + \gamma r^{-\ell}$$

$$\phi_2(b) = \beta b^{\ell} + \gamma b^{-\ell}$$

$$\frac{d\phi_2}{dr} = \beta\ell r^{\ell-1} - \gamma\ell r^{-\ell-1}$$

$$\left.\frac{d\phi_2}{dr}\right|_b = \beta\ell b^{\ell-1} - \gamma\ell b^{-\ell-1} = \ell(\beta b^{\ell-1} - \gamma b^{-\ell-1})$$

$$\phi_1 = \frac{(\beta b^{2\ell} + \gamma)}{(b^{2\ell} - a^{2\ell})}(r^{2\ell} - a^{2\ell})r^{-\ell}$$

$$\frac{d\phi_1}{dr} = \frac{(\beta b^{2\ell} + \gamma)}{(b^{2\ell} - a^{2\ell})}\left[-\ell(r^{2\ell} - a^{2\ell})r^{-\ell-1} + r^{-\ell}(2\ell)r^{2\ell-1}\right]$$

$$\frac{d\phi_1}{dr} = \frac{(\beta b^{2\ell} + \gamma)}{(b^{2\ell} - a^{2\ell})}\ell r^{-1}\left[2r^{\ell} - r^{-\ell}(r^{2\ell} - a^{2\ell})\right]$$

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aside

$$\left[2r^{\ell} - r^{\ell} + r^{-\ell}a^{2\ell}\right]$$

$$\shortparallel$$

$$\left[r^{\ell} + r^{-\ell}a^{2\ell}\right]$$

$$\shortparallel$$

$$r^{-\ell}\left[r^{2\ell} + a^{2\ell}\right]$$

$$\frac{d\phi_1}{dr} = \frac{(\beta b^{2\ell} + \gamma)}{(b^{2\ell} - a^{2\ell})} \ell r^{-\ell-1}(r^{2\ell} + a^{2\ell})$$

$$\left.\frac{d\phi_1}{dr}\right|_b = \frac{(\beta b^{2\ell} + \gamma)}{(b^{2\ell} - a^{2\ell})} \ell b^{-\ell-1}(b^{2\ell} + a^{2\ell})$$

APPENDIX C

The potentials of interest are

$$\phi_2 = \beta r^\ell + \gamma r^{-\ell}$$

and

$$\phi_3 = \frac{(\beta c^{2\ell}+\gamma)(d^{2\ell}-r^{2\ell})}{(d^{2\ell}-c^{2\ell})} r^{-\ell} + \frac{\phi_o(d) d^\ell (r^{2\ell}-c^{2\ell})}{d^{2\ell}-c^{2\ell}} r^{-\ell}$$

First $\phi(c)$ which can be evaluated from either expression

$$\phi(c) = (\beta c^\ell + \gamma c^{-\ell})$$

The derivative of ϕ_3 is

$$\frac{d\phi_3}{dr} = \frac{(\beta c^{2\ell}+\gamma)}{(d^{2\ell}-c^{2\ell})}(-\ell d^{2\ell} r^{-\ell-1} - \ell r^{\ell-1}) + \frac{d^\ell \phi_o(d)}{(d^{2\ell}-c^{2\ell})}(\ell r^{\ell-1} + \ell c^{2\ell} r^{-\ell-1})$$

$$\frac{d\phi_3}{dr} = \ell r^{-\ell-1}\left[\frac{d^\ell \phi_o(d)(r^{2\ell}+c^{2\ell})}{(d^{2\ell}-c^{2\ell})} - \frac{(\beta c^{2\ell}+\gamma)}{(d^{2\ell}-c^{2\ell})}(r^{2\ell}+d^{2\ell})\right]$$

Evaluating this derivative at r=c we have

$$\left.\frac{d\phi_3}{dr}\right|_c = \ell c^{-\ell-1}\left[\frac{2d^\ell c^{2\ell}\phi_o(d)}{(d^{2\ell}-c^{2\ell})} - \frac{(\beta c^{2\ell}+\gamma)}{(d^{2\ell}-c^{2\ell})}(d^{2\ell}+c^{2\ell})\right]$$

The derivative of ϕ_2 is

$$\frac{d\phi_2}{dr} = \beta\ell^{\ell-1} - \gamma\ell r^{-\ell-1} = \ell r^{\ell-1}(\beta-\gamma r^{-2\ell})$$

and at r=c

$$\frac{d\phi_2}{dr}\Big| = \ell c^{\ell-1}(\beta-\gamma c^{-2\ell})$$

Now we can also evaluate $v_o(c)$

$$v_o(c) = -\frac{Q-Ne\pi(c^2-b^2)}{2\pi\varepsilon_o cB}$$

APPENDIX D

We start with Eqs. (37) and (38).

$$(37) \qquad 2\left(\zeta + \frac{Q\ell}{2\pi Neb^2}\right)(\beta a^{2\ell}+\gamma) = (\beta+\gamma b^{-2\ell})(b^{2\ell}-a^{2\ell})$$

$$(38) \qquad \left[2\left(\zeta + \frac{Q\ell}{2\pi Nec^2}\right) - \ell\left(1 - \frac{b^2}{c^2}\right)\right](\beta d^{2\ell}+\gamma-d^{\ell}\phi_o(d)) = -(\beta+\gamma c^{-2\ell})(d^{2\ell}-c^{2\ell})$$

Working first with Eq. (37) this can be written

$$\left[2\left(\zeta + \frac{Q\ell}{2\pi Neb^2}\right)a^{2\ell}-b^{2\ell}+a^{2\ell}\right]\beta+\left[2\left(\zeta + \frac{Q\ell}{2\pi Neb^2}\right)-1+a^{2\ell}b^{-2\ell}\right]\gamma = 0$$

or

$$\left[2\left(\zeta + \frac{Q\ell}{2\pi Neb^2} + \frac{1}{2}\right)a^{2\ell}-b^{2\ell}\right]\beta+\left[2\left(\zeta + \frac{Q\ell}{2\pi Neb^2}\right)-1+a^{2\ell}b^{-2\ell}\right]\gamma = 0$$

Equation (38) becomes

$$\left\{\left[2\left(\zeta + \frac{Q\ell}{2\pi Nec^2}\right) - \ell\left(1 - \frac{b^2}{c^2}\right)\right]d^{2\ell} + d^{2\ell} - c^{2\ell}\right\}\beta + \left\{2\left(\zeta + \frac{Q\ell}{2\pi Nec^2}\right) - \ell\left(1 - \frac{b^2}{c^2}\right)\right.$$

$$\left. + c^{-2\ell}d^{2\ell} - 1\right\}\gamma = \frac{d^{\ell}\phi_o(d)}{2\left(\zeta + \frac{Q\ell}{2\pi Nec^2}\right) - \ell\left(1 - \frac{b^2}{c^2}\right)}$$

These two equations written in the form of the characteristic determinant if $\phi_o(d)$ is taken as zero are,

$$\begin{vmatrix} (\quad)\beta + (\quad)\gamma \\ (\quad)\beta + (\quad)\gamma \end{vmatrix} = 0$$

Writing this all out we have

$$\left[2\left(\zeta + \frac{Q\ell}{2\pi Neb^2}\right)a^{2\ell} - b^{2\ell} + a^{2\ell}\right]\left[2\left(\zeta + \frac{Q\ell}{2\pi Nec^2}\right) - \ell\left(1 - \frac{b^2}{c^2}\right) + c^{-2\ell}d^{2\ell} - 1\right] -$$

$$\left[2\left(\zeta + \frac{Q\ell}{2\pi Neb^2}\right) - 1 + a^{2\ell}b^{-2\ell}\right]\left[\left[2\left(\zeta + \frac{Q\ell}{2\pi Nec^2}\right) - \ell\right]1 - \frac{b^2}{c^2}\right]$$

or

$$\left[2\zeta a^{2\ell} + \frac{Q\ell a^{2\ell}}{\pi Neb^2} - b^{2\ell}+a^{2\ell}\right]\left[2\zeta + \frac{Q\ell}{\pi Nec^2} - \frac{\ell(c^2-b^2)}{c^2} + c^{-2\ell}d^{2\ell}-1\right] -$$

$$\left[2\zeta + \frac{Q\ell}{\pi Neb^2} - 1+a^{2\ell}b^{-2\ell}\right]\left[2\zeta d^{2\ell} + \frac{Q\ell d^{2\ell}}{\pi Nec^2} - \ell d^{2\ell}\frac{(c^2-b^2)}{c^2} + d^{2\ell}-c^{2\ell}\right]=0$$

Now we can multiply this out.

$$4\zeta^2 a^{2\ell} + \frac{2\zeta a^{2\ell}Q\ell}{\pi Nec^2} - \frac{2\zeta a^{2\ell}\ell(c^2-b^2)}{c^2} + 2\zeta a^{2\ell}c^{-2\ell}d^{2\ell} - 2\zeta a^{2\ell}$$

$$+ \frac{2\zeta Q\ell a^{2\ell}}{\pi Neb^2} + \left(\frac{Q\ell}{\pi Ne}\right)^2 \frac{a^{2\ell}}{b^2c^2} - \frac{Q\ell^2 a^{2\ell}(c^2-b^2)}{\pi Neb^2c^2} + \frac{Q\ell a^{2\ell}c^{-2\ell}d^{2\ell}}{\pi Neb^2} - \frac{Q\ell a^{2\ell}}{\pi Neb^2}$$

$$-2b^{2\ell}\zeta - \frac{Q\ell b^{2\ell}}{\pi Nec^2} + \frac{b^{2\ell}\ell(c^2-b^2)}{c^2} - b^{2\ell}c^{-2\ell}d^{2\ell} + b^{2\ell}$$

$$+ 2a^{2\ell}\zeta + \frac{Q\ell a^{2\ell}}{\pi Nec^2} - \frac{a^{2\ell}\ell(c^2-b^2)}{c^2} + a^{2\ell}c^{-2\ell}d^{2\ell} - a^{2\ell}$$

$$- 4\zeta^2 d^{2\ell} - \frac{2\zeta Q\ell d^{2\ell}}{\pi Nec^2} + \frac{2\zeta\ell d^{2\ell}(c^2-b^2)}{c^2} - 2\zeta d^{2\ell} + 2\zeta c^{2\ell}$$

$$- \frac{2\zeta d^{2\ell}Q\ell}{\pi Neb^2} - \left(\frac{Q\ell}{\pi Ne}\right)^2 \frac{d^{2\ell}}{b^2c^2} + \frac{Q\ell^2 d^{2\ell}(c^2-b^2)}{\pi Neb^2c^2} - \frac{Q\ell d^{2\ell}}{\pi Neb^2} + \frac{Q\ell c^2}{\pi Neb^2}$$

$$2\zeta d^{2\ell} + \frac{Q\ell d^{2\ell}}{\pi Nec^2} - \frac{\ell d^{2\ell}(c^2-b^2)}{c^2} + d^{2\ell} - c^{2\ell}$$

$$- 2a^{2\ell}b^{-2\ell}\, d^{2\ell} - \frac{Q\ell d^{2\ell}a^{2\ell}b^{-2\ell}}{\pi Nec^2} + \frac{\ell d^{2\ell}a^{2\ell}b^{-2\ell}(c^2-b^2)}{c^2} - a^{2\ell}b^{-2\ell}d^{2\ell}$$

$$+ a^{2\ell}b^{-2\ell}c^{2\ell} = 0$$

Combining terms we have

$$- 4\zeta^2(d^{2\ell}-a^{2\ell})$$

$$+ 2\zeta\Bigg[\frac{a^{2\ell}Q\ell}{\pi Nec^2} - \frac{a^{2\ell}\ell(c^2-b^2)}{c^2} + a^{2\ell}c^{-2\ell}d^{2\ell} - a^{2\ell} + \frac{Q\ell a^{2\ell}}{\pi Neb^2}$$

$$- b^{2\ell}+a^{2\ell} - \frac{Q\ell d^{2\ell}}{\pi Nec^2} + \frac{\ell d^{2\ell}(c^2-b^2)}{c^2} - d^{2\ell} + c^{2\ell}$$

$$- \frac{d^{2\ell}Q\ell}{\pi Neb^2} + d^{2\ell} - a^{2\ell}b^{-2\ell}d^{2\ell}\Bigg]$$

$$+ \left[\left(\frac{Q\ell}{\pi Ne}\right)^2 \left(\frac{a^{2\ell}}{b^2c^2} - \frac{d^{2\ell}}{b^2c^2}\right) + \left(\frac{Q\ell^2}{\pi Ne}\right) \left(\frac{d^{2\ell}(c^2-b^2)}{b^2c^2} - \frac{a^{2\ell}(c^2-b^2)}{b^2c^2}\right) \right.$$

$$+ \left(\frac{Q\ell}{\pi Ne}\right) \left(\frac{a^{2\ell}c^{-2\ell}d^{2\ell}}{b^2} - \frac{a^{2\ell}}{b^2} - \frac{b^{2\ell}}{c^2} + \frac{a^{2\ell}}{c^2} - \frac{d^{2\ell}}{b^2} + \frac{c^{2\ell}}{b^2} + \frac{d^{2\ell}}{c^2} - \frac{a^{2\ell}b^{-2\ell}d^{2\ell}}{c^2}\right)$$

$$+ \ell \left(\frac{b^{2\ell}(c^2-b^2)}{c^2} - \frac{a^{2\ell}(c^2-b^2)}{c^2} - \frac{d^{2\ell}(c^2-b^2)}{c^2} + \frac{d^{2\ell}a^{2\ell}b^{-2\ell}(c^2-b^2)}{c^2}\right)$$

$$- b^{2\ell}c^{-2\ell}d^{2\ell} + b^{2\ell} + a^{2\ell}c^{-2\ell}d^{2\ell} - a^{2\ell} + d^{2\ell} - c^{2\ell}$$

$$\left. - a^{2\ell}b^{-2\ell}d^{2\ell} + a^{2\ell}b^{-2\ell}c^{2\ell} \right] = 0$$

$$- 4\zeta^2(d^{2\ell}-a^{2\ell})$$

$$+ 2\zeta \left[\ell(d^{2\ell}-a^{2\ell}) \left\{ \left(1 - \frac{b^2}{c^2}\right) - \frac{Q}{\pi Neb^2} \left(1 + \frac{b^2}{c^2}\right) \right\} \right.$$

$$\left. + b^{-2\ell}c^{-2\ell} \left\{ (b^{2\ell}c^{2\ell} - a^{2\ell}d^{2\ell})(c^{2\ell}-b^{2\ell}) \right\} \right]$$

$$+ \left[\frac{\ell^2 Q}{\pi N e b^2} \left(1 - \frac{b^2}{c^2} - \frac{Q}{\pi N e c^2} \right) (d^{2\ell} - a^{2\ell}) \right.$$

$$- \left(\frac{\ell Q}{\pi N e b^2} \right) (-a^{2\ell} c^{-2\ell} d^{2\ell} + a^{2\ell} + d^{2\ell} - c^{2\ell})$$

$$- \ell \left\{ \frac{Q}{\pi N e} \left(\frac{b^{2\ell}}{c^{2\ell}} - \frac{a^{2\ell}}{c^2} - \frac{d^{2\ell}}{c^2} + \frac{a^{2\ell} b^{-2\ell} d^{2\ell}}{c^2} \right) + \frac{(c^2 - b^2)}{c^2} \left(\frac{-b^{2\ell}}{c^2} + \frac{a^{2\ell}}{c^2} \right. \right.$$

$$\left. \left. \frac{d^{2\ell}}{c^2} + \frac{d^{2\ell} a^{2\ell} b^{-2\ell}}{c^2} \right) \right\}$$

$$\left. + (c^{2\ell} - b^{2\ell})(d^{2\ell} - c^{2\ell})(b^{2\ell} - a^{2\ell}) b^{-2\ell} c^{-2\ell} \right] = 0$$

Continuing the combination we have

$$- 4\zeta^2 (d^{2\ell} - a^{2\ell})$$

$$+ 2\zeta \left[\ell (d^{2\ell} - a^{2\ell}) \left\{ \left(1 - \frac{b^2}{c^2} \right) - \frac{Q}{\pi N e b^2} \left(1 + \frac{b^2}{c^2} \right) \right\} + (b^{2\ell} c^{2\ell} - a^{2\ell} d^{2\ell}) \right.$$

$$\left. (c^{2\ell} - b^{2\ell}) b^{-2\ell} c^{-2\ell} \right]$$

$$+ \left[\frac{\ell^2 Q}{\pi N e b^2} \left(1 - \frac{b^2}{c^2} - \frac{Q}{\pi N e c^2} \right) (d^{2\ell} - a^{2\ell}) \right.$$

$$- \frac{\ell Q}{\pi N e b^2} (c^{2\ell} - a^{2\ell})(d^{2\ell} - c^{2\ell}) c^{-2\ell}$$

$$- \ell \left(1 - \frac{b^2}{c^2} - \frac{Q}{\pi N e c^2} \right) (d^{2\ell} - b^{2\ell})(b^{2\ell} - a^{2\ell}) b^{-2\ell}$$

$$\left. + (c^{2\ell} - b^{2\ell})(d^{2\ell} - c^{2\ell})(b^{2\ell} - a^{2\ell}) b^{-2\ell} c^{-2\ell} \right] = 0$$

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