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16. Abstract

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THIRD-ORDER CONTRIBUTIONS TO ELECTRICAL CONDUCTION IN PLASMAS

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SUMMARY

The electrical conductivity is calculated for a highly nonequilibrium plasma corresponding to the application of an electric field varying linearly with time. Two methods are employed: (1) the direct approach consisting of an exact solution to a modification of the third-order kinetic equation of Chapman and Enskog, and (2) Everett's technique for closing out the macroscopic equations of change with the Grad 13-moment velocity distribution function. Comparisons between the two sets of results indicate substantial third-order differences for most interparticle interaction potentials of practical interest; hence, the Grad 13-moment approximation does not appear to describe adequately certain higher order contributions to plasma transport coefficients.

INTRODUCTION

Highly nonequilibrium plasmas corresponding to large electron currents are of considerable interest in such potential applications as high current pinches, arcs, and discharges and in the description of phenomena occurring near electrodes and in low-pressure discharges. Departures in the formulation of problems of this type from those satisfying the ordinary linear flux theory are usually reflected in the dependence of transport coefficients upon the electron diffusion velocity.

Examples of the more rigorous treatments of high current plasmas include the use of Grad's 13-moment velocity distribution functions by Everett (ref. 1) and Yen (ref. 2) in closing out the macroscopic equations of change for the number densities, diffusion velocities, heat fluxes, and pressure tensors. Such procedures yield relations between the 13 moments and the applied force fields and also between the moments themselves, the accuracies of which depend in large measure upon the ability of the Grad approximation to predict the various collisional transfer terms adequately. Previous investigations of this ability have been performed by Meador (ref. 3) for first-order relations between the heat flux, the entropy density, the entropy density production rate, and the electron diffusion velocity, and for second-order contributions to the electron pressure tensor. Similar tests of Everett's and Yen's higher order corrections to the electrical conductivity have not been reported.

The purpose of the present research is to derive the high velocity electrical conductivity of a simple plasma from an exact solution to a modification of the third-order kinetic equation of Chapman and Enskog (ref. 4). Comparisons with similar calculations employing the aforementioned Everett procedure yield the following results: (1) significant differences occur in the contributions to the current density which are proportional to the cube of the electron diffusion velocity, and (2) the reliability of the Grad 13-moment approximation is strongly dependent upon the interparticle interaction potentials in that it is especially poor for very soft and very hard molecules. The second result is understandable in light of the fact that the Grad approximation is an expansion (ref. 5, p. 22) in the eigenfunctions of Boltzmann's binary elastic collision operator for Maxwellian molecules; consequently, a rapid convergence for more general force laws is not guaranteed.

The particular plasmas chosen for this study are those for which the following conditions are applicable: the electron collisions obey the collision model recently developed by Meador (ref. 6), the heavy particles are infinitely massive and at rest relative to the laboratory, the applied and induced magnetic fields are zero (the latter in strict violation of Maxwell's equations, but consistent with a nonrelativistic treatment), all macroscopic quantities are spatially homogeneous, and the applied electric field is proportional to the time.

SYMBOLS

a	electric field parameter defined in equation (3)
A	ratio of collision integrals defined by equation (21)
b	impact parameter
$\vec{c}_e$	electron particle velocity relative to laboratory frame of reference
$\vec{c}_h$	heavy-particle particle velocity
e	magnitude of electron charge
$\vec{E}$	applied electric field
f_e	electron distribution function

$f_e^{(o)}$ Maxwellian contribution to electron distribution function

f_h heavy-particle distribution function

$g_1, g_2, \ldots, g_5$ trial functions appearing in equation (18)

i,j index numbers

$\vec{j}$ electron current density

k Boltzmann's constant

$\hat{k}$ unit vector in z-direction

m_e electron particle mass

n_e electron number density

n_h number density of heavy particles

p_e electron partial pressure

$\overset{o}{\overleftrightarrow{P}}_e$ traceless electron pressure tensor relative to laboratory frame of reference

$\overset{o}{\overleftrightarrow{P}}{}'_e$ traceless electron pressure tensor relative to electron frame of reference

R_{ij} integral defined by equation (10)

s entropy density

$s^{(o)}$ equilibrium entropy density

$\dot{s}_c$ collisional production rate of entropy density

t time

T_e^o electron temperature corresponding to zero electron diffusion velocity

T_e electron temperature relative to laboratory frame of reference

T_e'	electron temperature relative to electron frame of reference
$\vec{u}$	reduced electron particle velocity relative to electron frame of reference
$\vec{v}_e$	electron diffusion velocity
x,y,z	Cartesian coordinates; also used as subscripts to indicate vector and tensor components; in addition, x is used as an integration variable
$\vec{X}_e$	electron body force per unit mass
α	third-order electrical conductivity parameter
$\vec{\beta}$	reduced electron diffusion velocity
$\vec{\beta}_1'$	reduced heat flux relative to electron frame of reference
$\vec{\gamma}$	reduced electron particle velocity relative to laboratory frame of reference
ϵ	azimuthal angle for collisions
η	conductivity parameter defined by equation (55)
ξ	interparticle interaction parameter
σ_o	electrical conductivity for zero electron diffusion velocity
σ	effective electrical conductivity through third order
τ_o	collision time for electron diffusion
τ_s	collision time for entropy production
ϕ_1,ϕ_2,ϕ_3	first-, second-, and third-order electron perturbation functions, respectively
χ	scattering or deflection angle
$\left(\partial f_e/\partial t\right)_c$	collisional time derivative of f_e

Primed quantities in collision integrals refer to conditions after a collision; unprimed quantities, before a collision. When vector symbols appear without an arrow, the magnitude of the vector is denoted. The symbol $\langle\ \rangle$ indicates an average over velocity space.

KINETIC THEORY

The derivations of high-velocity electrical conductivities and other plasma transport coefficients are usually based in some manner upon the electronic Boltzmann equation (ref. 4)

$$\frac{\partial f_e}{\partial t} + \left(\frac{2kT_e^o}{m_e}\right)^{1/2} \vec{\gamma} \cdot \nabla f_e + \left(\frac{m_e}{2kT_e^o}\right)^{1/2} \vec{X}_e \cdot \frac{\partial f_e}{\partial \vec{\gamma}} = \left(\frac{\partial f_e}{\partial t}\right)_c \tag{1}$$

where $\vec{\gamma}$ is the reduced electron particle velocity defined by

$$\vec{\gamma} = \left(\frac{m_e}{2kT_e^o}\right)^{1/2} \vec{c}_e \tag{2}$$

and T_e^o is the electron temperature at zero electron diffusion velocity.

If the time-dependent electric field and electron velocity distribution function are expressed as

$$\vec{E} = \hat{k}at \tag{3}$$

and

$$f_e = f_e^{(o)}\left(1 + \phi_1 + \phi_2 + \phi_3\right) = n_e\left(\frac{m_e}{2\pi kT_e^o}\right)^{3/2} e^{-\gamma^2}\left(1 + \phi_1 + \phi_2 + \phi_3\right) \tag{4}$$

respectively, the following simplification of equation (1) is obtained from the application of the special conditions outlined in the Introduction and the resulting stipulation from the electron equation of continuity that n_e is constant:

$$\frac{\partial(\phi_1 + \phi_2 + \phi_3)}{\partial t} + \frac{eat}{m_e}\left(\frac{m_e}{2kT_e^o}\right)^{1/2}\left[2(1 + \phi_1 + \phi_2 + \phi_3)\gamma_z - \frac{\partial(\phi_1 + \phi_2 + \phi_3)}{\partial\gamma_z}\right]$$

$$= \frac{1}{f_e^{(o)}}\left(\frac{\partial f_e}{\partial t}\right)_c = -\frac{1}{f_e^{(o)}}\int\left(f_e f_h - f_e' f_h'\right)\left|\vec{c}_e - \vec{c}_h\right| b\ db\ d\epsilon\ d\vec{c}_h$$

$$= -\frac{1}{f_e^{(o)}}\int c_e\left(f_e - f_e'\right) f_h b\ db\ d\epsilon\ d\vec{c}_h = -n_h\gamma\left(\frac{2kT_e^o}{m_e}\right)^{1/2}\int\left(\phi_1 + \phi_2 + \phi_3 - \phi_1' - \phi_2' - \phi_3'\right) b\ db\ d\epsilon \tag{5}$$

More specifically, the successive simplifications of the collision integral in equation (5) correspond to the assumptions that the heavy particles are fixed scattering centers for the electrons and that electron-electron collisions can be explicitly neglected at this stage. The latter assumption is part of Meador's collision model (ref. 6), the remainder of which outlines the method whereby the single electron—heavy-particle interaction potential can be generalized semiempirically to include the effects of multiple heavy species as well as electron-electron encounters. These more detailed aspects of the collision model will appear subsequently in the form of an effective interaction parameter ξ, which mathematically (but not physically) assumes the role of the exponent in an inverse-power electron—heavy-particle interaction potential. No restrictions are yet placed upon the characteristics of the unknown functions ϕ_1, ϕ_2, and ϕ_3.

The method adopted in the present research for the direct solution of equation (5) involves the following three statements: (1) the velocity distribution function is analytic in the small electric field parameter a, (2) the unknown function ϕ_i, which is also small and satisfies $\phi_i > \phi_{i+1}$, contains only the ith power of a, and (3) the index number i of ϕ_i designates the order of solution. Hence, the retention of functions through ϕ_3 implies a third-order solution in the sense that contributions to f_e of all a^3 terms are included. Of particular significance in this procedure is the fact that the corresponding first-order form of equation (5), which is written as

$$\frac{\partial\phi_1}{\partial t} + \frac{2eat}{m_e}\left(\frac{m_e}{2kT_e^o}\right)^{1/2}\gamma_z = -n_h\gamma\left(\frac{2kT_e^o}{m_e}\right)^{1/2}\int\left(\phi_1 - \phi_1'\right) b\ db\ d\epsilon \tag{6}$$

differs from that of Chapman and Enskog (ref. 4) in the explicit appearance of ϕ_1 on the left-hand side.

Equation (6) is further simplified by the assumption that ϕ_1 is γ_z multiplied by a function of γ and possibly the time. The following equation results

$$\frac{\partial \phi_1}{\partial t} + \frac{2eat}{m_e}\left(\frac{m_e}{2kT_e^o}\right)^{1/2} \gamma_z = -\frac{R_{13}}{R_{04}\tau_o} \gamma^{1-\frac{4}{\xi}} \phi_1 \tag{7}$$

when the T_e^o-dependent collision time

$$\tau_o = \frac{m_e \sigma_o}{e^2 n_e} \tag{8}$$

and the collision integral (ref. 6)

$$\int \left(\gamma_z - \gamma_z'\right) b \; db \; d\epsilon = \frac{R_{13}}{n_h R_{04} \tau_o}\left(\frac{m_e}{2kT_e^o}\right)^{1/2} \gamma^{-\frac{4}{\xi}} \gamma_z \tag{9}$$

are employed. The R_{ij} integrals are defined by

$$R_{ij} = \int_0^\infty x^{\frac{4i}{\xi}+j} e^{-x^2} dx \tag{10}$$

As mentioned previously and explained in detail in reference 6, the semiempirical interaction parameter ξ in equations (7), (9), and (10) appears formally as the exponent in an inverse-power electron—heavy-particle interaction potential and can be so chosen as to make the present theory reliable for many real plasmas (electron-electron collisions included). Only in the case of Lorentz plasmas (slight ionization or fully ionized gases with large ionic charges), however, can ξ be physically identified with such an exponent, the value of which may range from unity (Coulomb forces) to infinity (rigid spheres). Even if the generalization is not possible, the present analysis would still accomplish its primary purpose of evaluating the capability of Grad 13-moment distribution functions to describe highly nonequilibrium situations.

An exact solution of equation (7) is readily found to be

$$\phi_1 = -\frac{2eaR_{04}\tau_o}{m_e R_{13}}\left(\frac{m_e}{2kT_e^o}\right)^{1/2} \gamma^{\frac{4}{\xi}-1}\left(t - \frac{R_{04}\tau_o}{R_{13}} \gamma^{\frac{4}{\xi}-1}\right)\gamma_z \tag{11}$$

so that the first-order current density becomes

$$\vec{j} = -en_e\left\langle\vec{c}_e\right\rangle = -\frac{en_e}{\pi^{3/2}}\left(\frac{2kT_e^o}{m_e}\right)^{1/2}\int e^{-\gamma^2}\vec{\gamma}\phi_1\, d\vec{\gamma} = \sigma_o\vec{E}\left(1 - \frac{\tau_s}{t}\right) \tag{12}$$

if τ_s is given by

$$\tau_s = \frac{R_{04}R_{22}}{R_{13}^2}\,\tau_o \tag{13}$$

The τ_s of equation (13) is shown in reference 6 to be never less than τ_o nor more than twice τ_o and to satisfy the entropy-production definition

$$\frac{1}{2}\,\tau_s\dot{s}_c = s^{(o)} - s \tag{14}$$

Accordingly, τ_s is called the characteristic time for the collisional production rate of entropy and appears as a convenient parameter in many applications; for example, the kinetic theory expression of the Hall conductivity for small magnetic fields assumes the simple mean-free-path form if τ_s is substituted for τ_o. The present interest lies, of course, in the fact that τ_s acts as a lower limit for t in the expression for the current density, and thereby signifies the size of the time scale inherent in the Boltzmann collision integral.

As a final comment on the first-order solution, it is immediately obvious from equation (11) that an upper limit on the time (and thus the electric field) must exist if ϕ_1 is to be small compared with unity. This requirement, however, does not create any special problems prior to the consideration of ϕ_3, and will be discussed again at that point.

SECOND-ORDER SOLUTION

The substitution of equation (11) and its velocity derivative

$$\frac{\partial\phi_1}{\partial\gamma_z} = -\frac{2eaR_{04}\tau_o}{m_eR_{13}\xi}\left(\frac{m_e}{2kT_e^o}\right)^{1/2}\gamma^{\frac{4}{\xi}-3}\left\{t\left[\xi\gamma^2 - (\xi - 4)\gamma_z^2\right] - \frac{R_{04}\tau_o}{R_{13}}\,\gamma^{\frac{4}{\xi}-1}\left[\xi\gamma^2 - 2(\xi - 4)\gamma_z^2\right]\right\} \tag{15}$$

into equation (5) yields

$$\frac{\partial \phi_2}{\partial t} + \frac{\partial \phi_3}{\partial t} + \frac{eat}{m_e}\left(\frac{m_e}{2kT_e^o}\right)^{1/2}\left(2\phi_2\gamma_z - \frac{\partial \phi_2}{\partial \gamma_z}\right) + \frac{e^2a^2R_{04}\tau_o t}{m_e kT_e^o R_{13}\xi}\gamma^{\frac{4}{\xi}-3}\left\{t\left[\xi\gamma^2 - \left(2\xi\gamma^2 + \xi - 4\right)\gamma_z^2\right]\right.$$

$$\left. - \frac{R_{04}\tau_o}{R_{13}}\gamma^{\frac{4}{\xi}-1}\left[\xi\gamma^2 - 2\left(\xi\gamma^2 + \xi - 4\right)\gamma_z^2\right]\right\} = -n_h\gamma\left(\frac{2kT_e^o}{m_e}\right)^{1/2}\int\left(\phi_2 + \phi_3 - \phi_2' - \phi_3'\right)b\, db\, d\epsilon \quad (16)$$

The second-order form of equation (16) is obtained as follows by deleting all terms corresponding to a^3:

$$\frac{\partial \phi_2}{\partial t} + \frac{e^2a^2R_{04}\tau_o t}{m_e kT_e^o R_{13}\xi}\gamma^{\frac{4}{\xi}-3}\left\{t\left[\xi\gamma^2 - \left(2\xi\gamma^2 + \xi - 4\right)\gamma_z^2\right] - \frac{R_{04}\tau_o}{R_{13}}\gamma^{\frac{4}{\xi}-1}\left[\xi\gamma^2 - 2\left(\xi\gamma^2 + \xi - 4\right)\gamma_z^2\right]\right\}$$

$$= -n_h\gamma\left(\frac{2kT_e^o}{m_e}\right)^{1/2}\int\left(\phi_2 - \phi_2'\right)b\, db\, d\epsilon \quad (17)$$

If a second-order function of the type

$$\phi_2 = \left[g_1(\gamma) + g_2(\gamma)t + g_3(\gamma)t^2\right]\left(3\gamma_z^2 - \gamma^2\right) + g_4(\gamma)t^2 + g_5(\gamma)t^3 \quad (18)$$

is assumed, the following equation results:

$$\phi_2 = \frac{2e^2a^2R_{04}^2\tau_o^2t^2}{27A^2m_e kT_e^o R_{13}^2\xi}\gamma^{\frac{8}{\xi}-4}\left\{3A\left(2\xi\gamma^2 + \xi - 4\right) - \frac{2R_{04}\tau_o}{R_{13}t}\gamma^{\frac{4}{\xi}-1}\left[(3A + 4)\xi\gamma^2\right.\right.$$

$$\left.\left. + (3A + 2)(\xi - 4)\right]\left(1 - \frac{2R_{04}\tau_o}{3AR_{13}t}\gamma^{\frac{4}{\xi}-1}\right)\right\}\left(3\gamma_z^2 - \gamma^2\right) + \frac{e^2a^2R_{04}\tau_o t^3}{18m_e kT_e^o R_{13}\xi}\gamma^{\frac{4}{\xi}-1}\left[4\left(\xi\gamma^2 - \xi - 2\right)\right.$$

$$\left. - \frac{3R_{04}\tau_o}{R_{13}t}\gamma^{\frac{4}{\xi}-1}\left(2\xi\gamma^2 - \xi - 8\right)\right] \quad (19)$$

by using the collision integral

$$\int\left(\gamma_z^2 - \gamma_z'^2\right) b \, db \, d\epsilon = \frac{AR_{13}}{2n_h R_{04}\tau_o}\left(\frac{m_e}{2kT_e^o}\right)^{1/2} \gamma^{-\frac{4}{\xi}}\left(3\gamma_z^2 - \gamma^2\right) \tag{20}$$

derived in reference 6. The quantity A is the ratio of collision integrals

$$A = \frac{\int_0^\infty \left(1 - \cos^2\chi\right) b \, db}{\int_0^\infty (1 - \cos\chi) b \, db} \tag{21}$$

Equation (19) is automatically normalized in the sense that

$$\int f_e^{(o)} \phi_2 \, d\vec{c}_e = 0 \tag{22}$$

In addition, the temperatures T_e and T_e' relative to the laboratory and electron frames of reference, respectively, are given by

$$T_e = \frac{m_e}{3k}\left\langle c_e^2 \right\rangle = T_e^o + \frac{2a^2\sigma_o t^3}{9n_e k}\left(1 - \frac{3\tau_s}{2t}\right) = T_e^o\left[1 + \frac{4\beta^2 t}{9\tau_o}\left(1 - \frac{\tau_s}{t}\right)^{-2}\left(1 - \frac{3\tau_s}{2t}\right)\right] \tag{23}$$

and

$$T_e' = \frac{m_e}{3k}\left\langle\left(\vec{c}_e - \vec{v}_e\right)^2\right\rangle = T_e - \frac{2T_e^o\beta^2}{3} = T_e^o\left\{1 + \frac{4\beta^2 t}{9\tau_o}\left[\left(1 - \frac{\tau_s}{t}\right)^{-2}\left(1 - \frac{3\tau_s}{2t}\right) - \frac{3\tau_o}{2t}\right]\right\} \tag{24}$$

where β is the magnitude of the reduced electron diffusion velocity defined as

$$\vec{\beta} = \left(\frac{m_e}{2kT_e^o}\right)^{1/2} \vec{v}_e \tag{25}$$

It is further noted that the use of equation (12) in the time derivative of equation (23) yields the principle of conservation of energy

$$\frac{3}{2} n_e k \frac{dT_e}{dt} = \vec{E} \cdot \vec{j} \tag{26}$$

for this problem; thus, all the pertinent auxiliary conditions relative to n_e and T_e are satisfied by the ϕ_2 of equation (19).

Because sizeable values of τ_o/t are not especially important, it is convenient now to examine the second-order solution after τ_o/t has become small compared with unity. The neglect of squares and higher powers of this time ratio in equation (19) gives

$$\phi_2 = \frac{2e^2a^2R_{04}\tau_o t^3}{9m_e kT_e^o R_{13}\xi}\gamma^{\frac{4}{\xi}-1}\left\{\xi\gamma^2 - \xi - 2 + \frac{R_{04}\tau_o}{AR_{13}t}\gamma^{\frac{4}{\xi}-3}\left[\left(2\xi\gamma^2 + \xi - 4\right)\left(3\gamma_z^2 - \gamma^2\right) - \frac{3A\gamma^2}{4}\left(2\xi\gamma^2 - \xi - 8\right)\right]\right\} \tag{27}$$

and thus

$$\overset{o}{P}_{exy} = \overset{o}{P}_{exz} = \overset{o}{P}_{eyz} = 0 \tag{28}$$

and

$$\overset{o}{P}_{ezz} = -2\overset{o}{P}_{exx} = -2\overset{o}{P}_{eyy} = n_e m_e\left\langle c_{ez}^2\right\rangle - p_e = \frac{64(\xi + 1)\tau_s p_e\beta^2}{45A\xi\tau_o}\left(1 - \frac{\tau_s}{t}\right)^{-2} \tag{29}$$

as the components of the traceless electron pressure tensor $\overset{o}{\overleftrightarrow{P}}_e$ relative to the laboratory frame of reference.

Since the traceless electron pressure tensor relative to the electron frame of reference has a zz-component given by

$$\overset{o}{P}{}'_{ezz} = n_e m_e\left\langle\left(c_{ez} - v_e\right)^2\right\rangle - n_e kT'_e = \overset{o}{P}_{ezz} - \frac{4p_e\beta^2}{3} \tag{30}$$

the following equation can be written:

$$\overset{o}{P}{}'_{ezz} \approx \frac{4p_e\beta^2}{3}\left[\frac{16(\xi + 1)\tau_s}{15A\xi\tau_o} - 1\right] \tag{31}$$

from equations (29) and (30) if τ_s/t is neglected. In particular,

$$\overset{o}{P}{}'_{ezz}(\xi = 1) = 1.42p_e\beta^2 \tag{32}$$

as compared with the Grad value of $1.17p_e\beta^2$ from reference 3.

THIRD-ORDER SOLUTION

The magnitude of τ_o/t is next assumed to be the same order as the reduced electron diffusion velocity β for the purpose of solving the third-order kinetic equations. Since τ_o times the third term on the left-hand side of equation (16) can be written as

$$-\beta\left(2\phi_2\gamma_z - \frac{\partial\phi_2}{\partial\gamma_z}\right)$$

with the aid of equations (12) and (25), and since the next to last term in the ϕ_2 of equation (19) similarly becomes

$$\frac{4R_{04}\beta^2 t}{9\xi R_{13}\tau_o}\gamma^{\frac{4}{\xi}-1}\left(\xi\gamma^2 - \xi - 2\right)$$

the combination of these two expressions is proportional to $\beta^3 t/\tau_o$ and is therefore of order β^2 in magnitude and third order in the electric field parameter a. The corresponding contributions from the remaining terms of ϕ_2 have orders of magnitude β^3, β^4, and β^5 and are neglected in the present section.

It is further evident from the contribution to ϕ_2 which is retained, and which includes the factor t/τ_o, that some such establishment of the order of magnitude of τ_o/t (and hence the introduction of an upper limit to the elapsed time and the applied electric field) is necessary in order for ϕ_2 to be kept smaller than ϕ_1 and thus for the present expansion to be convergent. This aspect was previously anticipated.

The appropriate reductions in equation (19) and its velocity derivative for use in equation (16) are

$$\phi_2 = \frac{2e^2a^2R_{04}\tau_o t^3}{9m_e kT_e^o R_{13}\xi}\gamma^{\frac{4}{\xi}-1}\left(\xi\gamma^2 - \xi - 2\right) \tag{33}$$

and

$$\frac{\partial\phi_2}{\partial\gamma_z} = \frac{2e^2a^2R_{04}\tau_o t^3}{9m_e kT_e^o R_{13}\xi^2}\gamma^{\frac{4}{\xi}-3}\left[\xi(\xi + 4)\gamma^2 + (\xi + 2)(\xi - 4)\right]\gamma_z \tag{34}$$

Thus, the third-order kinetic equation becomes

$$\frac{\partial \phi_3}{\partial t} + \frac{2e^3 a^3 R_{04} \tau_o t^4}{9m_e^2 kT_e^o R_{13} \xi^2}\left(\frac{m_e}{2kT_e^o}\right)^{1/2} \gamma^{\frac{4}{\xi} - 3}\left[2\xi^2\gamma^4 - \xi(3\xi + 8)\gamma^2 - (\xi + 2)(\xi - 4)\right]\gamma_z$$

$$= -n_h \gamma\left(\frac{2kT_e^o}{m_e}\right)^{1/2} \int \left(\phi_3 - \phi_3'\right) b \, db \, d\epsilon \tag{35}$$

Equation (35) can be solved in a similar manner to equation (6) to yield

$$\phi_3 = -\frac{4R_{04}^2 \sigma_o \beta^2 E t}{9en_e R_{13}^2 \xi^2 \tau_o}\left(\frac{m_e}{2kT_e^o}\right)^{1/2} \gamma^{\frac{8}{\xi} - 4}\left[2\xi^2\gamma^4 - \xi(3\xi + 8)\gamma^2 - (\xi + 2)(\xi - 4)\right]\gamma_z \tag{36}$$

through the leading contribution. Accordingly, if the current density is expressed as

$$\vec{j} = -en_e\left\langle \vec{c}_e \right\rangle = \sigma_o\left[1 + \alpha(\xi)\,\beta^2\right]\vec{E}\left(1 - \frac{\tau_s}{t}\right) \tag{37}$$

the electrical conductivity parameter α is derived from the velocity moment of equation (36) to be

$$\alpha = -\frac{4(\xi + 2)(\xi - 4)\tau_s t}{9\xi(\xi + 8)\tau_o^2} \tag{38}$$

A brief analysis of the essential features of this entire development shows the leading term of the fifth-order contribution to the current density to be proportional to $\beta^5\left(t/\tau_o\right)^2$, which is third order in magnitude if τ_o/t and β are regarded as first order in magnitude. Equations (37) and (38) thus represent the complete description of $\vec{j}$ through terms which are third order in a and second order in magnitude. They also comprise the exact third-order result mentioned in the Introduction as a purpose of the present research.

THE GRAD APPROXIMATION

The final effort of the present research is the determination of α using the Grad 13-moment approximation with Meador's collision model (ref. 6) to close out the macroscopic equations of change. Numerical comparisons with equation (38) should yield valuable insight into the applicability of this technique to highly nonequilibrium plasmas because a completely common framework (that is, basic assumptions and expansions in powers of the electric field parameter a) is provided for both methods. Hence, the preceding exact solutions are unquestionably the correct standards for such comparisons and the only critical question is whether the 13-moment function adequately satisfies the pertinent kinetic equations or their moments.

Although not necessary, a direct confirmation of the statement that the preceding exact solutions are the correct ones is obtained from reference 3. Additional moments caused the Grad approximation to converge rather rapidly toward similarly derived exact solutions in that application.

Since Everett's velocity distribution function (ref. 1) can be written as

$$f_e = n_e\left(\frac{m_e}{2\pi kT'_e}\right)^{3/2} e^{-u^2}\left[1 + \frac{4}{5}\left(u^2 - \frac{5}{2}\right)\vec{\beta}'_1 \cdot \vec{u} + \left(n_e kT'_e\right)^{-1}\overset{o}{\overleftrightarrow{P}}{}'_e : \vec{u}\vec{u}\right] \tag{39}$$

when the reduced heat flux and the reduced electron particle velocity relative to the electron frame of reference are defined by

$$\vec{\beta}'_1 = \left\langle u^2\vec{u}\right\rangle \tag{40}$$

and

$$\vec{u} = \left(\frac{m_e}{2kT'_e}\right)^{1/2}\left(\vec{c}_e - \vec{v}_e\right) \tag{41}$$

respectively, the following third-order closed-out expressions are obtained from the $m_e c_{ez}$ and $m_e c_e^2 c_{ez}$ moments of equation (1) and the use of equations (9) and (39) in evaluating the collision integrals:

$$v_e + \frac{(\xi - 4)\beta'_1}{5\xi}\left(\frac{2kT'_e}{m_e}\right)^{1/2} = 0 \tag{42}$$

and

$$v_e + \frac{(3\xi - 4)\beta'_1}{5\xi}\left(\frac{2kT'_e}{m_e}\right)^{1/2} = -\frac{10R_{04}^2\xi t\sigma_o\beta^2 E}{9en_e R_{13}R_{-1,5}(3\xi - 2)\tau_o} \tag{43}$$

As in the preceding exact development which culminated in equations (36) to (38), all but the leading $\beta^3 t/\tau_o$ terms have been deleted from these expressions.

The simultaneous solution of equations (42) and (43) finally gives

$$v_e(\text{third order}) = \frac{5(\xi - 4)R_{04}^2 t\sigma_o\beta^2 E}{9en_e(3\xi - 2)R_{13}R_{-1,5}\tau_o} \tag{44}$$

so that

$$\vec{j}(\text{third order}) = -en_e\vec{v}_e(\text{third order}) = -\frac{5(\xi - 4)R_{04}^2 t\sigma_o\beta^2}{9(3\xi - 2)R_{13}R_{-1,5}\tau_o}\vec{E} \tag{45}$$

A direct determination of α from equation (45), however, is not possible unless the Grad approximation also yields equation (12) through first order in a. In particular, adjustments may have to be made in order to account for differences between the predictions of σ_o by the exact and Grad methods. This calculation starts with the following first-order closed out expressions analogous to equations (42) and (43):

$$\frac{d\beta}{dt} + \frac{eat}{m_e}\left(\frac{m_e}{2kT_e^o}\right)^{1/2} = -\frac{R_{13}R_{-1,5}}{5R_{04}^2\xi\tau_o}\left[5\xi\beta + (\xi - 4)\beta_1'\right] \tag{46}$$

and

$$\frac{d\beta_1'}{dt} + \frac{5}{2}\frac{d\beta}{dt} + \frac{5eat}{2m_e}\left(\frac{m_e}{2kT_e^o}\right)^{1/2} = -\frac{R_{13}R_{-1,5}(3\xi - 2)}{5R_{04}^2\xi^2\tau_o}\left[5\xi\beta + (3\xi - 4)\beta_1'\right] \tag{47}$$

The simultaneous solution of equations (46) and (47) gives

$$\beta = -\frac{R_{04}^2\left(13\xi^2 - 16\xi + 16\right)\sigma_o E}{4en_eR_{13}R_{-1,5}(3\xi - 2)\xi}\left(\frac{m_e}{2kT_e^o}\right)^{1/2}\left[1 - \frac{R_{04}R_{13}\left(179\xi^4 - 496\xi^3 + 832\xi^2 - 512\xi + 256\right)\tau_s}{4R_{22}R_{-1,5}(3\xi - 2)\left(13\xi^2 - 16\xi + 16\right)\xi t}\right] \tag{48}$$

and

$$\beta_1' = \frac{5R_{04}^2(\xi - 4)\sigma_o E}{4en_eR_{13}R_{-1,5}(3\xi - 2)}\left(\frac{m_e}{2kT_e^o}\right)^{1/2}\left[1 - \frac{R_{04}R_{13}\left(23\xi^2 - 16\xi + 16\right)\tau_s}{4R_{22}R_{-1,5}(3\xi - 2)\xi t}\right] \tag{49}$$

Accordingly, equations (45) and (48) combine to yield

$$\vec{j} = \frac{R_{04}^2\left(13\xi^2 - 16\xi + 16\right)\sigma_o}{4R_{13}R_{-1,5}(3\xi - 2)\xi}\left[1 + \alpha(\xi)\,\beta^2\right]\vec{E}\left[1 - \frac{R_{04}R_{13}\left(179\xi^4 - 496\xi^3 + 832\xi^2 - 512\xi + 256\right)\tau_s}{4R_{22}R_{-1,5}(3\xi - 2)\left(13\xi^2 - 16\xi + 16\right)\xi t}\right] \tag{50}$$

and

$$\alpha = -\frac{20\xi(\xi - 4)t}{9\left(13\xi^2 - 16\xi + 16\right)\tau_o} \tag{51}$$

if proper regard is taken of the order of magnitude consistent with this development.

Percentage errors in the ratio of Grad to exact conductivities are obtained from

$$\frac{\sigma_0(\mathrm{Grad})}{\sigma_0} = \frac{R_{04}^2\left(13\xi^2 - 16\xi + 16\right)}{4R_{13}R_{-1,5}(3\xi - 2)\xi} \tag{52}$$

and range from -4.3 at $\xi = 1$ through 0 at $\xi = 4$ to -4.3 at $\xi = \infty$. As mentioned in reference 3, on the other hand, a somewhat different picture prevails for the heat flux because equation (49) must be compared in that case with

$$\beta_1' = \frac{(\xi - 4)\sigma_0 E}{2en_e\xi}\left(\frac{m_e}{2kT_e^o}\right)^{1/2}\left(1 - \frac{2\tau_s}{t}\right) \tag{53}$$

from the energy moment of equation (11). For example, Grad's result is in error by -26 percent at $\xi = 1$, if τ_s/t is neglected when compared with unity.

COMPARISONS OF RESULTS

A convenient summary of the preceding research on electrical conduction (through second-order terms in magnitude and third order in a) is provided by the expression

$$\vec{j} = \sigma\vec{E} \tag{54}$$

where σ is the effective electrical conductivity defined by

$$\sigma = \sigma_0(\text{exact or Grad})\left(1 - \frac{\eta\tau_s}{t} + \alpha\beta^2\right) \tag{55}$$

The parameter η assumes the values

$$\eta(\text{exact}) = 1 \tag{56}$$

from equation (37) and

$$\eta(\mathrm{Grad}) = \frac{R_{04}R_{13}\left(179\xi^4 - 496\xi^3 + 832\xi^2 - 512\xi + 256\right)}{4R_{22}R_{-1,5}(3\xi - 2)\left(13\xi^2 - 16\xi + 16\right)\xi} \tag{57}$$

from equation (50), whereas the formulas for α are given in equations (38) and (51).

Numerical calculations appropriate to equations (54) and (55) are presented in tables I and II for a variety of effective interparticle interaction potentials ranging from the fully ionized Lorentz plasma ($\xi = 1$) to a gas of rigid spheres ($\xi = \infty$). Only in the neighborhood of Maxwellian molecules ($\xi = 4$) is the Grad 13-moment approximation adequate for the determination of α, the error being very significant for both softer and harder force laws. The differences however are more tolerable in the case of η.

TABLE I.- THE ELECTRICAL CONDUCTIVITY PARAMETER α AS A FUNCTION OF THE EFFECTIVE INTERPARTICLE INTERACTION PARAMETER ξ AND THE METHOD OF SOLUTION

ξ	$\alpha\tau_0/t$ (Grad)	$\alpha\tau_0/t$ (exact)
1	0.513	0.859
2	.247	.196
4	.000	.000
∞	-.171	-.524

TABLE II.- THE ELECTRICAL CONDUCTIVITY PARAMETER η AS A FUNCTION OF THE EFFECTIVE INTERPARTICLE INTERACTION PARAMETER ξ AND THE METHOD OF SOLUTION

ξ	η(Grad)	η(exact)
1	0.759	1.000
2	1.011	1.000
4	1.000	1.000
∞	.861	1.000

CONCLUDING REMARKS

Calculations through third order in the electric field have indicated that large errors can occur when the Grad 13-moment velocity distribution function is used to close out the macroscopic equations of change. Except in the neighborhood of Maxwellian force laws, the only transport coefficient (among those considered) for which the Grad approximation yields accurate results is the first-order electrical conductivity. The difficulties begin with the first-order heat flux relative to the electron frame of reference and the leading contribution to the traceless electron pressure tensor, each of which is in error by 20 to 30 percent for very soft or very hard interaction potentials, and are magnified many times in the case of third-order transport phenomena. Although the plasma chosen for the present work is a hypothetical one, this trend of the Grad 13-moment approximation toward greater (and problem-dependent) discrepancies seems to be well established for highly nonequilibrium systems; consequently, some of the past research on large electron diffusion velocities should perhaps be reevaluated.

Langley Research Center,
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Langley Station, Hampton, Va., August 19, 1969.

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