NASA TECHNICAL NOTE

NASA TN D-5591

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NASA TN D-5591

A VARIATIONAL PRINCIPLE FOR MAGNETOHYDRODYNAMIC CHANNEL FLOW

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • DECEMBER 1969

1. Report No.
NASA TN D-5591

2. Government Accession No.

3. Recipient's Catalog No.

4. Title and Subtitle
A VARIATIONAL PRINCIPLE FOR MAGNETOHYDRODYNAMIC CHANNEL FLOW

5. Report Date
December 1969

6. Performing Organization Code

7. Author(s)
Norman C. Wenger

8. Performing Organization Report No.
E-5235

9. Performing Organization Name and Address
Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio 44135

10. Work Unit No.
120-27

11. Contract or Grant No.

12. Sponsoring Agency Name and Address
National Aeronautics and Space Administration
Washington, D.C. 20546

13. Type of Report and Period Covered
Technical Note

14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract
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17. Key Words (Suggested by Author(s))
Magnetohydrodynamic flow
Flow distribution
Calculus of variations
Channel flow

18. Distribution Statement
Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	27	$3.00

*For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151

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SUMMARY

A variational formulation is presented for a class of magnetohydrodynamic (MHD) channel flow problems. This formulation yields solutions for the fluid velocity and the induced electric potential in a channel with a uniform transverse static magnetic field. The channel cross section is constant but arbitrary, and the channel walls can be either insulators or conductors with finite electrical conductivity. Electric currents are permitted to enter and leave the channel walls so that the solutions are suitable for MHD generator and pump applications. An example of a square channel with conducting walls is solved as an illustration.

INTRODUCTION

The study of magnetohydrodynamic (MHD) channel flow has received considerable attention in the past decade. This interest has been motivated by three principle applications; the MHD generator, the MHD pump, and the electromagnetic flowmeter.

The general model that is normally considered in these studies consists of an infinitely long channel of constant cross section with a uniform static magnetic field applied transverse to the axis of the channel. The walls of the channel are either insulators, conductors, or a combination of insulators and conductors depending on the intended application.

For example, in the MHD generator and pump cases, the channel cross section is normally rectangular with insulated walls perpendicular to the magnetic field and conducting walls parallel to the magnetic field. For the electromagnetic flowmeter case, the channel cross section is normally circular with conducting walls.

In order to carry out an analytical solution for MHD channel flow, it is generally necessary to make simplifying assumptions such as requiring the channel walls to be either perfect conductors or perfect insulators or requiring the channel walls to be very thin. These and other simplifications often greatly limit the usefulness of the results.

This is particularly true for the electromagnetic flowmeter case since the thin wall approximation is often not valid for liquid metal applications and the wall conductivity is neither zero nor infinite. In addition, many analytical solutions give results in the form of infinite series which converge poorly for the large values of the static magnetic field that are encountered in practice.

To alleviate some of these difficulties, Tani (ref. 1) developed a variational formulation for the solution of MHD channel flow problems. His formulation gives solutions for the velocity profile and the induced magnetic field distribution in the channel for an arbitrary channel cross section. It requires, however, that the channel walls be either perfect conductors or insulators and that the admissible functions for the velocity and induced magnetic field satisfy appropriate boundary conditions.

In this report, a variational formulation is presented that gives solutions for the velocity profile and the electric potential distribution in a channel of arbitrary cross section. It also gives solutions for the electric potential distribution in the channel walls. The walls of the channel can be a combination of insulators and conductors but the conductors may have a finite conductivity. In addition, the admissible functions for the velocity and potential need not satisfy any prescribed boundary conditions. Moreover, the formulation is sufficiently general to allow electric currents to enter and leave the channel walls so that the solutions obtained are suitable for the MHD generator and pump applications.

The report concludes with an example that consists of a square channel with conducting walls of finite conductivity.

THE MODEL

A cross section of a generalized channel is shown in figure 1. It consists of the fluid duct S_f bounded by the conducting walls S_c and the insulated walls S_i. The contours C_{fc} and C_{fi} denote the fluid-conducting wall interface and the fluid-insulated wall interface, respectively. The contour C_{co} denotes the outer edge of the conducting wall. The vector $\hat{n}$ is the unit normal to the contours with the positive direction as shown.

The applied static magnetic field B_o is parallel to the x-axis and is uniform with respect to y and z. The applied or generated current density at the outer edge of the conducting wall J_a is considered positive when directed outward. It is assumed that the net current entering the channel cross section due to J_a is zero, so that the two-dimensional features of the model are retained.

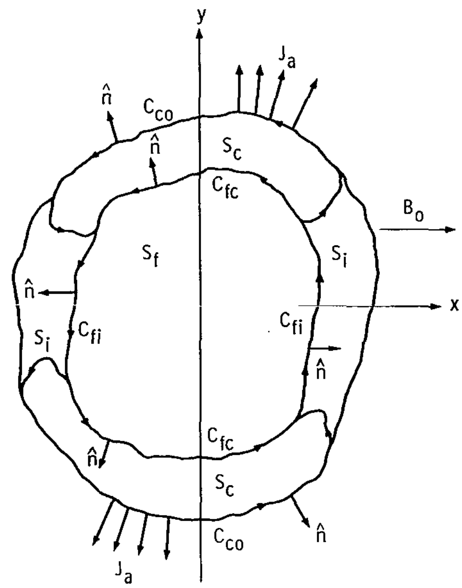

Figure 1. - Cross section of generalized channel.

BASIC EQUATIONS

The basic equations to be used are the standard MHD equations for steady-state conditions which consist of Maxwell's equations, the momentum transport equation, and the generalized Ohm's law. These are

$$\nabla \times \vec{E} = \vec{0} \tag{1a}$$

$$\nabla \times \vec{B} = \mu_o \vec{J} \tag{1b}$$

$$\nabla \cdot \vec{B} = 0 \tag{1c}$$

$$\rho(\vec{V} \cdot \nabla)\vec{V} = -\nabla p + \vec{J} \times \vec{B} + \eta \nabla^2 \vec{V} \tag{1d}$$

$$\vec{J} = \sigma_f(\vec{E} + \vec{V} \times \vec{B}) \tag{1e}$$

where $\vec{E}$, $\vec{B}$, $\vec{J}$, and μ_o are the electric field intensity, magnetic flux density, electric current density, and magnetic permeability of free space, respectively; and $\vec{V}$, ρ, η, σ_f, and p are the fluid velocity, density, viscosity, electrical conductivity, and pressure, respectively.

Equations (1a) to (1e) are based on the assumptions that the fluid is homogeneous and incompressible, the magnetic permeability of the fluid is the same as that of free space, the electric charge density and Hall current are negligible, and the fluid flow is laminar. It may be argued that the last assumption of nonturbulent flow greatly limits the usefulness of the results since many flows in practice are turbulent. However, it has been found experimentally (ref. 2) that the onset of turbulence occurs at a much higher Reynolds number in MHD flow than for ordinary flow due to the suppression of turbulence by the static magnetic field.

The five basic equations (1a) to (1e) can be combined to give two coupled equations of second order. First, an electric potential U which satisfies equation (1a) identically can be defined as

$$\vec{E} = -\nabla U \tag{2a}$$

Next, substituting $\vec{J}$ from equation (1e) into (1b) and then taking the divergence of equation (1b) give

$$\nabla^2 U = \nabla \cdot (\vec{V} \times \vec{B}) \tag{2b}$$

where equation (2a) has been used to eliminate $\vec{E}$. The second equation can be obtained by substituting $\vec{J}$ from equation (1e) into (1d) and then eliminating $\vec{E}$ using equation (2a) giving

$$\rho(\vec{V} \cdot \nabla)\vec{V} = -\nabla p - \sigma_f\left[\nabla U \times \vec{B} - (\vec{V} \times \vec{B}) \times \vec{B}\right] + \eta\nabla^2\vec{V} \tag{2c}$$

Due to the uniformity of the channel cross section and the applied magnetic field with respect to the z-axis, all quantities in the basic equations are independent of z except for the pressure which is linear in z (ref. 3). In addition, it can be shown that the fluid velocity $\vec{V}$ has only a z-component V_z and that $\vec{E}$ and $\vec{J}$ have only x- and y-components (ref. 4). Furthermore, the total magnetic field $\vec{B}$ consists of the applied field B_o in the x-direction and an induced field B_i in the z-direction.

Since the velocity does not vary with z and has only a z-component, the first term in equation (2c) vanishes. Expanding the vector cross products and taking the z-component of equation (2c) along with equation (2b) give the following governing equations:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - B_o \frac{\partial V_z}{\partial y} = 0 \tag{3a}$$

$$\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} - \frac{1}{\eta}\frac{\partial p}{\partial z} + \frac{\sigma_f B_o}{\eta}\frac{\partial U}{\partial y} - \frac{\sigma_f B_o^2}{\eta} V_z = 0 \tag{3b}$$

These equations apply, of course, only in the fluid duct region S_f. In the conducting wall region S_c, equation (3a) applies with $V_z = 0$. In the insulated wall region S_i, equation (3a) also applies with $V_z = 0$ but it need not be solved.

Equations (3a) and (3b) are not unique governing equations in the sense that variables other than the velocity and electric potential can be selected for retention. Tani (ref. 1), for example, eliminated the electric potential but retained the velocity and the induced magnetic field yielding a different but equivalent set of governing equations.

In addition to the basic equations, appropriate boundary conditions must be specified to determine the solution uniquely. These conditions are

$$V_z\Big|_f = 0 \qquad \text{on } C_{fc} \text{ and } C_{fi} \tag{4a}$$

$$U\Big|_f - U\Big|_w = 0 \qquad \text{on } C_{fc} \tag{4b}$$

$$\sigma_f \nabla U \cdot \hat{n}\Big|_f - \sigma_w \nabla U \cdot \hat{n}\Big|_w = 0 \qquad \text{on } C_{fc} \tag{4c}$$

$$\sigma_f \nabla U \cdot \hat{n}\Big|_f = 0 \qquad \text{on } C_{fi} \tag{4d}$$

$$\sigma_w \nabla U \cdot \hat{n}\Big|_w + J_a = 0 \qquad \text{on } C_{co} \tag{4e}$$

where $\Big|_f$ and $\Big|_w$ refer to evaluating the quantity on the fluid or wall side of the contour, respectively.

The boundary condition equations (4a) to (4e) require the following:

(1) The fluid velocity must vanish on the fluid-wall interfaces C_{fc} and C_{fi}.

(2) The electric potential must be continuous across the fluid-conducting wall interface C_{fc}.

(3) The component of the electric current normal to the fluid-conducting wall interface C_{fc} must be continuous.

(4) The component of the electric current normal to the fluid-insulated wall interface C_{fi} must vanish.

(5) The component of the electric current normal to the outer edge of the conducting wall C_{co} must equal the applied or generated current J_a.

In solving the equations, it is convenient to work with dimensionless quantities. This can easily be accomplished by defining L and V_o to be a characteristic length and characteristic velocity of the channel. Let

$$X = \frac{x}{L}, \quad Y = \frac{y}{L}, \quad Z = \frac{z}{L} \qquad \text{dimensionless coordinates} \tag{5a}$$

$$W = \frac{U}{B_o L V_o} \qquad \text{dimensionless potential} \tag{5b}$$

$$V = \frac{V_z}{V_o} \qquad \text{dimensionless velocity} \tag{5c}$$

$$M = B_o L \sqrt{\frac{\sigma_f}{\eta}} \qquad \text{Hartmann number} \tag{5d}$$

$$P_o = \frac{-L^2}{\eta V_o} \frac{\partial p}{\partial z} \qquad \text{dimensionless pressure gradient} \tag{5e}$$

$$J_o = \frac{J_a}{B_o V_o \sigma_w} \qquad \text{dimensionless applied or generated current} \tag{5f}$$

$$\gamma = \frac{\sigma_w}{\sigma_f} \qquad \text{ratio of wall-to-fluid conductivity} \tag{5g}$$

Combining equations (5a) to (5g) with equations (3a) and (3b) yields the following set of equations in dimensionless form:

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} - \frac{\partial V}{\partial Y} = 0 \qquad \text{on } S_f \tag{6a}$$

$$\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + P_o + M^2 \frac{\partial W}{\partial Y} - M^2 V = 0 \qquad \text{on } S_f \tag{6b}$$

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = 0 \qquad \text{on } S_c \tag{6c}$$

Likewise, combining equations (5a) to (5g) with equations (4a) to (4e) gives the following set of dimensionless boundary condition equations:

$$V\Big|_f = 0 \qquad \text{on } C_{fc} \text{ and } C_{fi} \tag{6d}$$

$$W\Big|_f - W\Big|_w = 0 \qquad \text{on } C_{fc} \tag{6e}$$

$$\frac{1}{\sqrt{1+\left(\frac{dY}{dX}\right)^2}}\left(\frac{dY}{dX}\frac{\partial W}{\partial X} - \frac{\partial W}{\partial Y}\right)\Bigg|_f - \frac{\gamma}{\sqrt{1+\left(\frac{dY}{dX}\right)^2}}\left(\frac{dY}{dX}\frac{\partial W}{\partial X} - \frac{\partial W}{\partial Y}\right)\Bigg|_w = 0 \qquad \text{on } C_{fc} \tag{6f}$$

$$\frac{1}{\sqrt{1+\left(\frac{dY}{dX}\right)^2}}\left(\frac{dY}{dX}\frac{\partial W}{\partial X} - \frac{\partial W}{\partial Y}\right)\Bigg|_f = 0 \qquad \text{on } C_{fi} \tag{6g}$$

$$\frac{1}{\sqrt{1+\left(\frac{dY}{dX}\right)^2}}\left(\frac{dY}{dX}\frac{\partial W}{\partial X} - \frac{\partial W}{\partial Y}\right)\Bigg|_w - J_o = 0 \qquad \text{on } C_{co} \tag{6h}$$

The unit normal vector $\hat{n}$ has been replaced by

$$\hat{n} = \frac{-\frac{dY}{dX}\hat{a}_x + \hat{a}_y}{\sqrt{1+\left(\frac{dY}{dX}\right)^2}}$$

where $\hat{a}_x$ and $\hat{a}_y$ are the unit vectors in the X- and Y-directions, respectively. The sign of the square root must be selected so that the positive direction for $\hat{n}$ is as shown in figure 1.

VARIATIONAL EXPRESSION

The goal of this section is to construct a functional of the dependent variables V and W so that the associated Euler-Lagrange equations are the basic governing equations (6a) to (6c) and where the corresponding natural boundary conditions are the prescribed boundary condition equations (6d) to (6h). This construction is performed by summing terms that are obtained by multiplying each governing equation and boundary condition equation by a suitable function and then integrating over the corresponding area or contour where the equation is valid.

Let δV and δW be arbitrary functions of X and Y that are continuous with piecewise continuous first derivatives. The integrals

$$I_1 \equiv 2\int_{S_f}\left[\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + P_o + M^2\frac{\partial W}{\partial Y} - M^2 V\right]\delta V\, dX\, dY \tag{7a}$$

$$I_2 \equiv 2M^2\int_{S_f}\left[\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} - \frac{\partial V}{\partial Y}\right]\delta W\, dX\, dY \tag{7b}$$

$$I_3 \equiv 2\gamma M^2\int_{S_c}\left[\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right]\delta W\, dX\, dY \tag{7c}$$

are identically zero for any δV and δW since the quantities in brackets [ ] are identically zero by virtue of equations (6a) to (6c).

Four additional integral expressions that are identically zero can be obtained from the boundary condition equations (6d) to (6h) by integrating along appropriate contours. Recalling that the differential length along a contour is given by $\sqrt{1 + (dY/dX)^2}\, dX$, these integral expressions can be defined as

$$I_4 \equiv -2M^2\int_{C_{fc}}\left[\left(\frac{\partial W}{\partial X}dY - \frac{\partial W}{\partial Y}dX\right)\Bigg|_f - \gamma\left(\frac{\partial W}{\partial X}dY - \frac{\partial W}{\partial Y}dX\right)\Bigg|_w\right]\delta W \tag{7d}$$

$$I_5 \equiv -2M^2\int_{C_{fi}}\left[\frac{\partial W}{\partial X}dY - \frac{\partial W}{\partial Y}dX\right]\Bigg|_f \delta W \tag{7e}$$

$$I_6 \equiv -2\gamma M^2 \int_{C_{co}} \left[\frac{\partial W}{\partial X} dY - \frac{\partial W}{\partial Y} dX - J_o \sqrt{1 + \left(\frac{dY}{dX}\right)^2} dX \right] \Bigg|_w \delta W \tag{7f}$$

$$I_7 \equiv 2 \int_{C_{fc}+C_{fi}} [V] \left(\frac{\partial \delta V}{\partial X} dY - \frac{\partial \delta V}{\partial Y} dX - M^2 \delta W \, dX \right) \Bigg|_f \tag{7g}$$

These integrals vanish since the quantities in brackets are zero by virtue of the boundary condition equations (6d) to (6h).

The symbol δ can be defined as the variation operator so that the functions δV and δW can be considered as the variation of V and W, respectively. In addition, the δ operator commutes with $\partial/\partial X$ and $\partial/\partial Y$ since X and Y are independent variables. Using these properties along with Green's lemma, the integrals I_1 to I_7 can be integrated by parts and combined to give

$$\sum_{n=1}^{7} I_n = \delta F \tag{8a}$$

where

$$F \equiv \int_{S_f} \left\{ 2P_o V - \left(\frac{\partial V}{\partial X}\right)^2 - \left(\frac{\partial V}{\partial Y}\right)^2 - M^2 \left[V^2 + \left(\frac{\partial W}{\partial X}\right)^2 + \left(\frac{\partial W}{\partial Y}\right)^2 - 2V \frac{\partial W}{\partial Y} \right] \right\} dX \, dY$$

$$- \gamma M^2 \int_{S_c} \left[\left(\frac{\partial W}{\partial X}\right)^2 + \left(\frac{\partial W}{\partial Y}\right)^2 \right] dX \, dY + 2 \int_{C_{fc}+C_{fi}} V \left[\frac{\partial V}{\partial X} dY - \frac{\partial V}{\partial Y} dX \right] \Bigg|_f$$

$$+ 2\gamma M^2 \int_{C_{co}} W J_o \sqrt{1 + \left(\frac{dY}{dX}\right)^2} \Bigg|_w dX \tag{8b}$$

Since each I_n ($n=1, \ldots, 7$) is identically zero, the variation of the functional F is zero. Thus, F is stationary; that is, first order changes in V and W about their true values produce only second order changes in F.

Another way of stating this result is that of all functions that are continuous with piecewise continuous first derivatives, the particular pair of functions for V and W that make F stationary, satisfy both the basic equations (6a) to (6c) and the boundary condition equations (6d) to (6h) and, hence, are the desired solutions.

Even though F was shown to be stationary, it does not necessarily mean that F has a maximum or minimum at the true solution for V and W. For example, as V and W are varied from their true values, F may always increase, always decrease, or either increase or decrease depending on how V and W are varied. To determine which case corresponds to the F under consideration the quantity $F(V + \delta V, W + \delta W) - F(V,W)$ is computed giving

$$\begin{aligned}
F(V+\delta V, W+\delta W) - F(V,W) = {} & 2\int_{S_f}\left\{\left[\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + P_o - M^2 V + M^2\frac{\partial W}{\partial Y}\right]\delta V \right. \\
& \left. + M^2\left[\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} - \frac{\partial V}{\partial Y}\right]\delta W\right\} dX\, dY + 2\gamma M^2\int_{S_c}\left[\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right]\delta W\, dX\, dY \\
& - 2M^2\int_{C_{fc}}\left[\left(\frac{\partial W}{\partial X} dY - \frac{\partial W}{\partial Y} dX\right)\Bigg|_f - \gamma\left(\frac{\partial W}{\partial X} dY - \frac{\partial W}{\partial Y} dX\right)\Bigg|_w\right]\delta W \\
& - 2M^2\int_{C_{fi}}\left[\frac{\partial W}{\partial X} dY - \frac{\partial W}{\partial Y} dX\right]\delta W\Bigg|_f - 2\gamma M^2\int_{C_{co}}\left[\left(\frac{\partial W}{\partial X} dY - \frac{\partial W}{\partial Y} dX\right)\right. \\
& \left. - J_o\sqrt{1 + \left(\frac{dY}{dX}\right)^2} dX\right]\delta W\Bigg|_w + 2\int_{C_{fc}+C_{fi}}[V]\left(\frac{\partial \delta V}{\partial X} dY - \frac{\partial \delta V}{\partial Y} dX - M^2\,\delta W\, dX\right)\Bigg|_f \\
& - \int_{S_f}\left\{\left(\frac{\partial \delta V}{\partial X}\right)^2 + \left(\frac{\partial \delta V}{\partial Y}\right)^2 + M^2\left(\frac{\partial \delta W}{\partial X}\right)^2 + M^2\left(\frac{\partial \delta W}{\partial Y} + \delta V\right)^2\right\} dX\, dY \\
& - \gamma M^2\int_{S_c}\left\{\left(\frac{\partial \delta W}{\partial X}\right)^2 + \left(\frac{\partial \delta W}{\partial Y}\right)^2\right\} dX\, dY + 2\int_{C_{fc}+C_{fi}}\left(\frac{\partial \delta V}{\partial X} dY - \frac{\partial \delta V}{\partial Y} dX\right)\delta V\Bigg|_f \qquad (9)
\end{aligned}$$

A careful examination of equation (9) reveals that the first six integrals vanish because the quantities in brackets [ ] are identically zero. The remaining integrals are of second order in δV and δW. This result is not surprising since F was constructed so that all first order terms in δV and δW vanished. Of the three remaining integrals in equation (9), two are negative definite and the last can be of either sign. Thus, F has neither a minimum nor maximum at the true values for V and W. However, if the class of admissible functions for V and W is restricted so that the last integral must vanish, then F corresponds to a maximum at the true values for V and W since then $F(V + \delta V, W + \delta W) - F(V,W) \leq 0$.

A study of the last integral in equation (9) reveals that the proper restriction to impose is that V must vanish on the contours C_{fc} and C_{fi}. An alternate choice which also makes the last integral vanish is to specify $\partial V/\partial n$ on C_{fc} and C_{fi}. This choice is useless, however, since it would require solving the problem another way first to determine the correct value for $\partial V/\partial n$.

Requiring V to vanish on C_{fc} and C_{fi} may provide a great simplification in many problems in obtaining approximate values for V and W since finding a maximum for F is often much easier than finding a stationary point. Moreover, requiring V to vanish on C_{fc} and C_{fi} completely eliminates one integral in the expression for F given by equation (8b).

Before a solution for V and W can be determined, values for P_o and J_o must be specified. The dimensionless pressure gradient P_o must be a constant as previously noted. The dimensionless current density J_o, however, can be specified as a function of the coordinates along the contour C_{co}. Since the basic equations and boundary condition equations are linear in V, W, P_o, and J_o, solutions for V and W can be obtained by superimposing solutions for $P_o \neq 0$ and $J_o = 0$ with those for $P_o = 0$ and $J_o \neq 0$.

To complete the study of the variational expression, it is desirable to determine its physical significance.

PHYSICAL SIGNIFICANCE OF VARIATIONAL EXPRESSION

Consider the power or energy balance that exists in MHD channel flow. The power per unit length that is supplied to the fluid by the pressure gradient $P_{\Delta p}$ is given by

$$P_{\Delta p} = -\int_{S_f} \frac{\partial p}{\partial z} V_z \, dx \, dy = \eta V_o^2 \int_{S_f} P_o V \, dX \, dY \qquad (10a)$$

If this quantity is negative, it simply means that the channel is acting as a pump.

The dissipative terms consist of the viscous losses in the fluid P_η, the ohmic losses in the fluid P_{σ_f}, and the ohmic losses in the conducting walls P_{σ_w}. Expressing each of these in terms of the power dissipated per unit length along the channel gives

$$P_\eta = -\int_{S_f} \eta \left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} \right) V_z \, dx\, dy = \eta V_o^2 \int_{S_f} \left[\left(\frac{\partial V}{\partial X} \right)^2 + \left(\frac{\partial V}{\partial Y} \right)^2 \right] dX\, dY$$

$$- \eta V_o^2 \int_{C_{fc}+C_{fi}} V \left(\frac{\partial V}{\partial X}\, dY - \frac{\partial V}{\partial Y}\, dX \right) \Bigg|_f \qquad (10b)$$

$$P_{\sigma_f} = \sigma_f \int_{S_f} \left| -\frac{\partial U}{\partial x} \hat{a}_x - \frac{\partial U}{\partial y} \hat{a}_y + V_z B_o \hat{a}_y \right|^2 dx\, dy$$

$$= \eta V_o^2 M^2 \int_{S_f} \left[\left(\frac{\partial W}{\partial X} \right)^2 + \left(\frac{\partial W}{\partial Y} - V \right)^2 \right] dX\, dY \qquad (10c)$$

$$P_{\sigma_w} = \sigma_w \int_{S_c} \left| -\frac{\partial U}{\partial x} \hat{a}_x - \frac{\partial U}{\partial y} \hat{a}_y \right|^2 dx\, dy$$

$$= \eta V_o^2 \gamma M^2 \int_{S_c} \left[\left(\frac{\partial W}{\partial X} \right)^2 + \left(\frac{\partial W}{\partial Y} \right)^2 \right] dX\, dY \qquad (10d)$$

Equation (10b) reveals that the viscous losses in the fluid can be split into two parts; the volume losses P_{η_v} and the surface losses P_{η_s} where

$$P_{\eta_v} = \eta V_o^2 \int_{S_f} \left[\left(\frac{\partial V}{\partial X}\right)^2 + \left(\frac{\partial V}{\partial Y}\right)^2\right] dX\, dY \tag{10e}$$

$$P_{\eta_s} = -\eta V_o^2 \int_{C_{fc}+C_{fi}} V\left(\frac{\partial V}{\partial X}\, dY - \frac{\partial V}{\partial Y}\, dX\right)\Bigg|_f \tag{10f}$$

The surface losses are zero when the boundary condition is imposed that requires V to vanish on C_{fc} and C_{fi}.

The remaining term to be considered is the loss due to the current J_a. Since J_a is positive by definition when it is directed outward, the power that is supplied to an external load per unit length of the channel P_{J_a} is given by

$$P_{J_a} = -\int_{C_{co}} U\, J_a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = -\eta V_o^2 \gamma M^2 \int_{C_{co}} W\, J_o \sqrt{1 + \left(\frac{dY}{dX}\right)^2}\, dX \tag{10g}$$

If P_{J_a} is negative, it simply indicates that power is being supplied by an external source.

Since power is conserved, the power balance for the channel can be expressed as

$$P_{\Delta p} = P_{\eta_v} + P_{\eta_s} + P_{\sigma_f} + P_{\sigma_w} + P_{J_a} \tag{11}$$

Comparing the expression for the functional F given by equation (8b) with the various power dissipation terms given by equations (10a) to (10g) reveals that F can be expressed as

$$\eta V_o^2 F = 2P_{\Delta p} - P_{\eta_v} - P_{\sigma_f} - P_{\sigma_w} - 2P_{\eta_s} - 2P_{J_a} \tag{12}$$

A word of caution is in order at this point. The expression for F given by equation (8b) is defined and valid for an arbitrary choice for V and W. Likewise, the power dissipation terms P_{η_v}, P_{σ_f}, etc., given by equations (10a) to (10g) are valid for arbitrary values for V and W. Thus, equation (12) is valid, in general. However, equation (11) which is the power balance for the channel is only valid for the correct solutions for V and W.

As shown in the previous section, the stationary point for F corresponds to the true solutions for V and W. For these values only, equations (11) and (12) can be combined to give

$$F_{st} = \frac{P_{\Delta p} - P_{J_a}}{\eta V_o^2} \tag{13}$$

where it has been recognized that P_{η_s} vanishes for the true V. Thus, the stationary value of F is proportional to the difference between the power supplied to the fluid by the pressure gradient and the electrical power delivered to an external load.

An important special case occurs for $J_a = 0$ which yields

$$F_{st} = \int_{S_f} P_o V \, dX \, dY \tag{14}$$

where $P_{\Delta p}$ has been replaced using equation (10a). Since the dimensionless pressure gradient P_o is a constant, the stationary value for F is proportional to the average fluid velocity in the channel. This is a very important result since the average velocity, which is often the main quantity of interest, is proportional to a stationary quantity which can be computed to good accuracy.

If the boundary condition $V = 0$ on C_{fc} and C_{fi} is satisfied by all admissible functions, F has a maximum at its stationary point as noted in the previous section. The maximum for F using a subset of the class of admissible functions for V and W will be less than the maximum for F using the entire class of admissible functions. Thus, a lower limit for the average velocity can be easily found by using any admissible function.

EXAMPLE: SQUARE CHANNEL WITH CONDUCTING WALLS

Variational Solution

A square channel is shown in figure 2 using the dimensionless coordinates. The characteristic length for the channel L has been chosen as one-half its height or width so that its inner corners are located at (1, 1), (1, -1), (-1, 1), and (-1, -1). The normalized wall thickness t is the actual wall thickness divided by L.

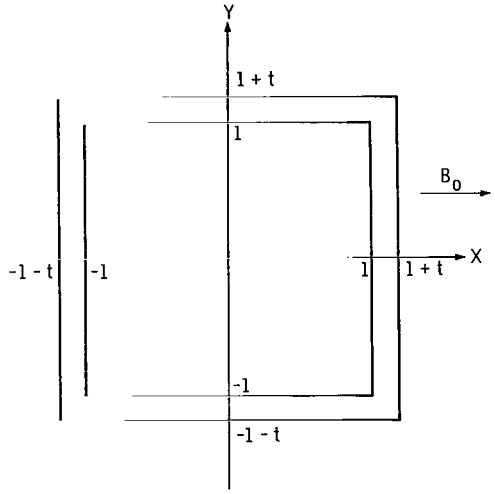

Figure 2. - Cross section of square channel with conducting walls.

Approximate solutions for the velocity and electric potential will be obtained using the Ritz technique. In this technique, the velocity and potential are expressed in terms of known functions of X and Y that approximate the true solution but contain adjustable parameters $\lambda_1, \ldots, \lambda_n$. The approximate solutions for V and W are then substituted into the expression for F given by equation (8b) and the indicated integrations with respect to X and Y are performed. This leaves F as a function of the parameters $\lambda_1, \ldots, \lambda_n$ and the characteristic parameters of the channel P_o, γ, M, and J_o. Assuming that V vanishes at $X = \pm 1$ and $Y = \pm 1$ for all values of $\lambda_1, \ldots, \lambda_n$, the stationary value of F can be found by maximizing F with respect to $\lambda_1, \ldots, \lambda_n$. The corresponding values for $\lambda_1, \ldots, \lambda_n$ when substituted into the approximate functions for V and W will yield the closest approximations to the velocity and potential that are possible for the class of functions used.

In order to determine the accuracy of the solutions obtained using the Ritz technique, a sequence of approximations is normally used. In this method, a suitable, complete, infinite set of functions is selected such as sine or cosine harmonics, Bessel functions, etc., so that some linear combination of them is capable of representing the solution. For the MHD channel flow problem these functions must be capable of representing any continuous function with a piecewise continuous derivative over the channel cross section.

The procedure consists of first obtaining an approximate solution using a linear combination of a finite number of these functions where the adjustable parameters $\lambda_1, \ldots, \lambda_n$ correspond to the coefficients of these functions. The problem is then repeatedly solved, each time increasing the number of functions used. By comparing the results from successive approximations, an estimate of the accuracy and convergence rate can be obtained.

In many problems, however, much is already known about the solution so that the trial functions can be tailored to more accurately approximate the solution thereby reducing the number of adjustable parameters required. This alternate procedure does not allow the error in the approximation to be easily estimated but it is much easier to use computationally since fewer adjustable parameters are involved. This alternate procedure will be followed in this example.

The solution for the square channel will be determined for $P_o = 1$ and $J_o = 0$. Let the trial functions for V and W be given by

$$V(X,Y) = A_1\left(1 - X^{\alpha_1}\right)\left(1 - Y^{\alpha_2}\right) \qquad \begin{matrix} 0 \le X \le 1 \\ 0 \le Y \le 1 \end{matrix} \tag{15a}$$

$$W(X,Y) = \left(C_1Y^{\beta_1} + C_2Y^{\beta_2}\right)\left(1 + C_3X^{\beta_3}\right) \qquad \begin{matrix} 0 \le X \le 1 + t \\ 0 \le Y \le 1 + t \end{matrix} \tag{15b}$$

where A_1, C_1, C_2, C_3, α_1, α_2, β_1, β_2, and β_3 are adjustable parameters. Because of the symmetry of the problem, it is only necessary to specify V and W in the first quadrant. For other quadrants, V and W can be found using the relations $V(X,Y) = V(X,-Y) = V(-X,Y)$ and $W(X,Y) = W(-X,Y) = -W(X,-Y)$. Since the admissible functions for V and W must be continuous with piecewise continuous derivatives, all exponents in equations (15a) and (15b) must be 1 or greater.

Trial functions of the form given by equations (15a) and (15b) have proven to be very useful in solving MHD channel flow problems of this type since the velocity profile, for example, can go from a parabolic ($\alpha_1 = \alpha_2 = 2$) to nearly slug flow (α_1, α_2 large) by merely varying two parameters.

Substituting equations (15a) and (15b) into the expression for F given by equation (8b) yields after performing the integrations

$$F = [\psi]^T[A][\psi] - [\psi]^T[D] \tag{16}$$

where

$$[\psi] = \begin{bmatrix} A_1 \\ C_1 \\ C_2 \end{bmatrix} \qquad [A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \qquad [D] = \begin{bmatrix} d_1 \\ 0 \\ 0 \end{bmatrix}$$

$$a_{11} = -\frac{8\alpha_1^2\alpha_2^2}{(\alpha_2 + 1)(2\alpha_1 - 1)(2\alpha_2 + 1)} - \frac{8\alpha_1^2\alpha_2^2}{(\alpha_1 + 1)(2\alpha_2 - 1)(2\alpha_1 + 1)}$$

$$- \frac{16M^2\alpha_1^2\alpha_2^2}{(\alpha_1 + 1)(2\alpha_1 + 1)(\alpha_2 + 1)(2\alpha_2 + 1)}$$

$$a_{12} = \frac{4M^2\alpha_1\alpha_2}{\beta_1 + \alpha_2}\left[\frac{1}{\alpha_1 + 1} + \frac{C_3}{(\beta_3 + 1)(\beta_3 + \alpha_1 + 1)}\right]$$

$$a_{13} = \frac{4M^2\alpha_1\alpha_2}{\beta_2 + \alpha_2}\left[\frac{1}{\alpha_1 + 1} + \frac{C_3}{(\beta_3 + 1)(\beta_3 + \alpha_1 + 1)}\right]$$

$$a_{22} = -4M^2\left(\frac{\beta_3^2C_3^2}{(2\beta_3 - 1)(2\beta_1 + 1)} + \frac{\beta_1^2}{2\beta_1 - 1}\left\{1 + \frac{2C_3}{\beta_3 + 1} + \frac{C_3^2}{2\beta_3 + 1}\right\}\right.$$

$$\left. + \gamma t\left[\frac{2(\beta_1 + \beta_3)\beta_3^2C_3^2}{(2\beta_3 - 1)(2\beta_1 + 1)} + \frac{2\beta_1^2}{2\beta_1 - 1}\left\{\beta_1 + \frac{(2\beta_1 + \beta_3)C_3}{\beta_3 + 1} + \frac{(\beta_1 + \beta_3)C_3^2}{2\beta_3 + 1}\right\}\right]\right)$$

$$a_{23} = -4M^2\left(\frac{\beta_3^2C_3^2}{(2\beta_3 - 1)(\beta_1 + \beta_2 + 1)} + \frac{\beta_1\beta_2}{\beta_1 + \beta_2 - 1}\left\{1 + \frac{2C_3}{\beta_3 + 1} + \frac{C_3^2}{2\beta_3 + 1}\right\}\right.$$

$$+ \gamma t\left[\frac{(\beta_1 + \beta_2 + 2\beta_3)\beta_3^2C_3^2}{(2\beta_3 - 1)(\beta_1 + \beta_2 + 1)}\right.$$

$$\left.\left. + \frac{\beta_1\beta_2}{\beta_1 + \beta_2 - 1}\left\{(\beta_1 + \beta_2) + \frac{2(\beta_1 + \beta_2 + \beta_3)C_3}{\beta_3 + 1} + \frac{(\beta_1 + \beta_2 + 2\beta_3)C_3^2}{2\beta_3 + 1}\right\}\right]\right)$$

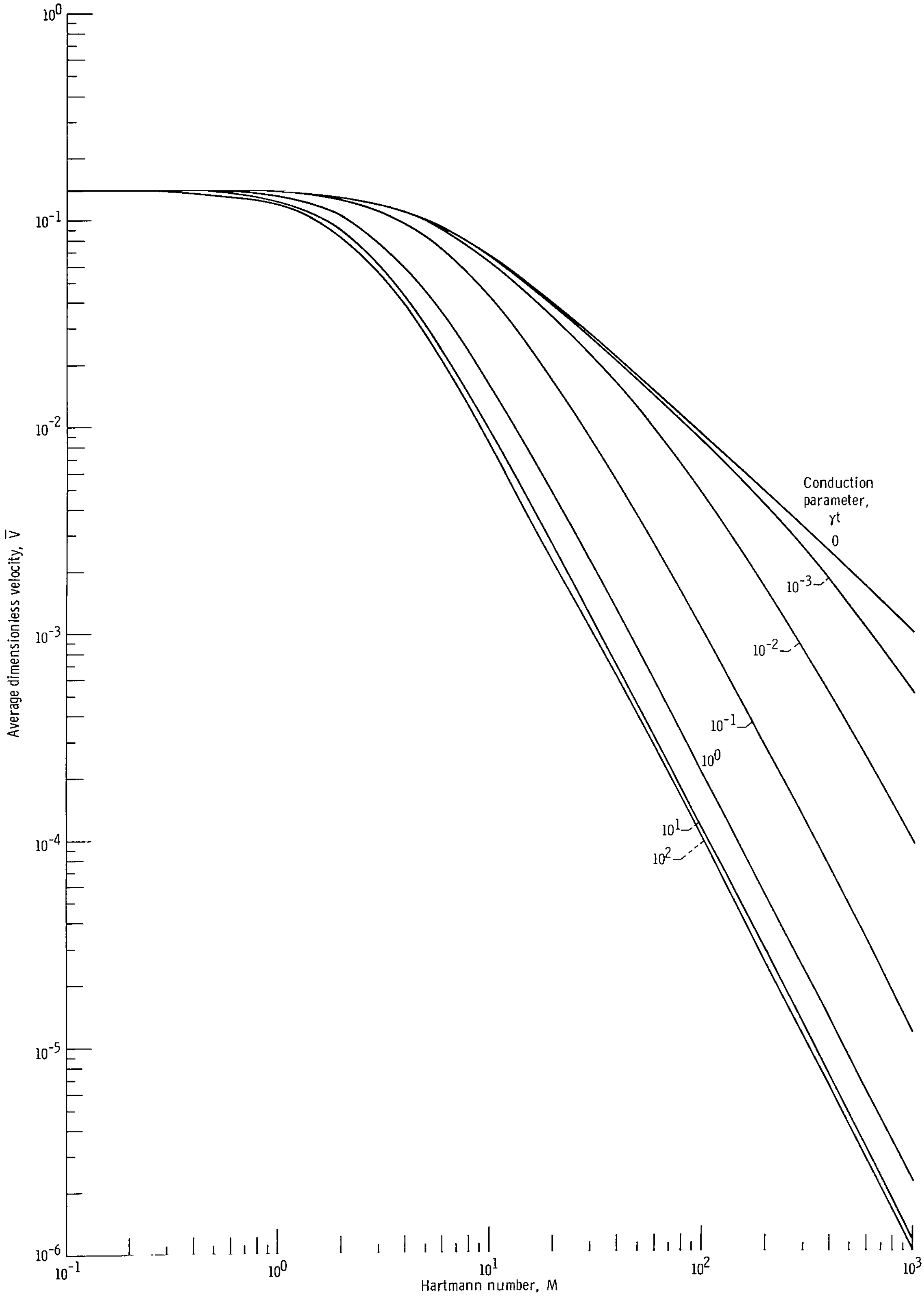

Figure 3. - Variational solution for average dimensionless velocity in square channel.

$$a_{33} = -4M^2\left(\frac{\beta_3^2C_3^2}{(2\beta_3 - 1)(2\beta_2 + 1)} + \frac{\beta_2^2}{2\beta_2 - 1}\left\{1 + \frac{2C_3}{\beta_3 + 1} + \frac{C_3^2}{2\beta_3 + 1}\right\}\right.$$

$$\left. + \gamma t\left[\frac{2(\beta_2 + \beta_3)\beta_3^2C_3^2}{(2\beta_3 - 1)(2\beta_2 + 1)} + \frac{2\beta_2^2}{2\beta_2 - 1}\left\{\beta_2 + \frac{(2\beta_2 + \beta_3)C_3}{\beta_3 + 1} + \frac{(\beta_2 + \beta_3)C_3^2}{2\beta_3 + 1}\right\}\right]\right)$$

$$d_1 = -\frac{8\alpha_1\alpha_2}{(\alpha_1 + 1)(\alpha_2 + 1)}$$

In obtaining equation (16), the "thin wall" approximation $t \ll 1$ was made so that the results are directly comparable with published values. This approximation did not have to be made in order to use this formulation but was done since only thin wall results are available for comparison.

The maximum value for F and the corresponding values for the parameters were found using a computer. Since the normalized cross-sectional area of the channel is 4 and $\overline{P}_o = 1$, the proportionality constant between the average dimensionless fluid velocity $\overline{V}$ and F_{st} is 1/4 (see eq. (14)). Values for $\overline{V}$ as a function of the Hartmann number M are shown in figure 3 for various values of conduction parameter γt.

The results show that the average fluid velocity decreases as M and γt increase. This decrease can be explained as follows: For $M \neq 0$, an electric current is induced in the fluid due to the fluid motion. If $\gamma t = 0$, the wall is an insulator and, hence, any induced current must form a closed path entirely in the fluid. Thus, the total electromagnetic force on the fluid is zero. However, since the electromagnetic force is not identically zero everywhere, the velocity profile of the fluid is distorted which, in turn, increases the viscous force and consequently reduces the average fluid velocity. For $\gamma t \neq 0$, the induced current path is located partially in the wall so that there is a net electromagnetic force in addition to the viscous force acting to reduce the fluid velocity.

Comparison of Results

The variational solution can be compared with other solutions for some limiting cases. For $M = 0$, the average dimensionless velocity from the variational solution is 0.1403, independent of γt, as compared with the exact value of 0.1406, which can be computed using Fourier expansion techniques.

The exact solution for the average flow in a rectangular channel with insulated walls ($\gamma t = 0$) and perfectly conducting walls ($\gamma t = \infty$) has been obtained by Shercliff (ref. 3) and Chang and Lundgren (ref. 5), respectively. Each of these solutions is in the form of a series which converges poorly for large M. Williams (ref. 6), however, transformed these solutions and obtained asymptotic forms for the average velocity for large M. These solutions, simplified for $P_o = 1$ and the square channel, are as follows:

$$\overline{V} = \frac{1}{M}\left[1 - \frac{32}{15\sqrt{2\pi M}} - \frac{1}{M} + \frac{4}{3\sqrt{2\pi M^3}} + \mathcal{O}\left(\frac{1}{M^2}\right)\right] \qquad \gamma t = \infty$$

$$\overline{V} = \frac{1}{M^2}\left[1 - \frac{1}{M} - \frac{1}{M^{3/2}}\left(2.43 - \frac{3.03}{M^{1/2}} + \frac{0.48}{M}\right) + \mathcal{O}\left(\frac{1}{M^3}\right)\right] \qquad \gamma t = \infty$$

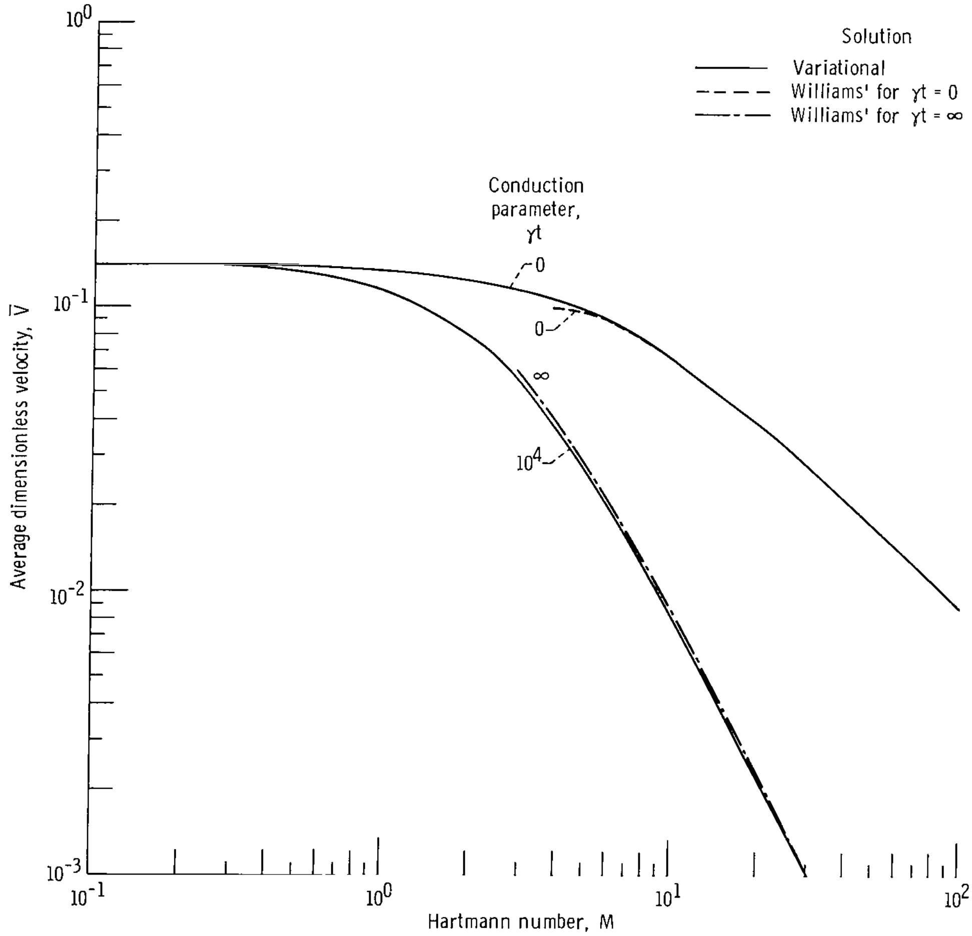

Figure 4. - Comparison of variational solution with Williams (ref. 6) asymptotic solutions for average dimensionless velocity.

A comparison between the variational solutions and these asymptotic forms is shown in figure 4. The agreement between the two solutions for $\gamma t = 0$ is excellent for $M \geq 10$. The variational solution for $\gamma t = 10^4$ is always slightly less than the asymptotic form for $\gamma t = \infty$. The difference, however, decreases to less than 0.1 percent at $M = 1000$.

A Fourier expansion type solution for the rectangular channel with thin walls of finite conductivity has recently been obtained by Chu (ref. 7). A comparison of his solution with the variational solution is shown in figure 5. The agreement between these two solutions is also quite good. As shown, the variational solution for the average velocity is always slightly less than the series solution value. This is due to the fact that the computed maximum for F, and, hence, the average velocity, is always less than or equal to the true maximum for F since the trial functions used are a subset of the entire class of admissible functions.

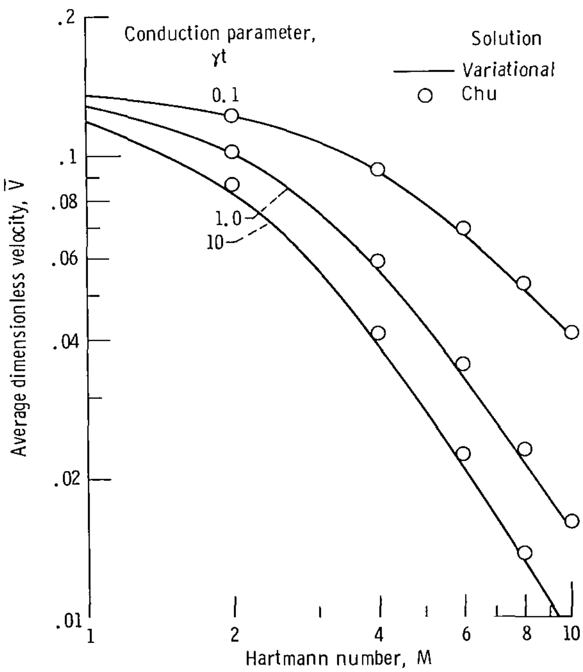

Figure 5. - Comparison of variational solution with Chu's (ref. 7) solution for average dimensionless velocity.

CONCLUDING REMARKS

A variational formulation was presented for a class of MHD channel flow problems. A stationary expression was developed that yielded solutions for the fluid velocity and the induced electric potential in a generalized channel. An example of a square channel with conducting walls was solved as an illustration. Very good agreement was obtained between the variational solution for the average velocity in the square channel and other published values.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, September 24, 1969,
120-27.

APPENDIX - SYMBOLS

$[A]$	matrix (see eq. (16))
A_1	adjustable parameter
a_{ij}	ij^{th} element of matrix $[A]$
$\hat{a}_x, \hat{a}_y$	unit vectors in x- and y-directions
$\vec{B}$	magnetic flux density
B_i	induced magnetic flux density
B_o	uniform applied magnetic flux density
C_{fc}, C_{fi}, C_{co}	contours in generalized channel cross section (see fig. 1)
C_1, C_2, C_3	adjustable parameters
$[D]$	matrix (see eq. (16))
d_1	element of matrix $[D]$
$\vec{E}$	electric field intensity
F	stationary functional (see eq. (8b))
F_{st}	stationary value of F
$I_1, \ldots, I_7$	integrals (see eqs. (7a) to (7g))
$\vec{J}$	electric current density
J_a	applied or generated electric current density
J_o	dimensionless applied or generated electric current density
L	characteristic length of channel
M	Hartmann number
$\hat{n}$	unit normal vector to contours
$\mathcal{O}$	order of
P_{J_a}	power delivered to external load per unit length of channel
P_o	dimensionless pressure gradient
$P_{\Delta p}$	power supplied to fluid by pressure gradient per unit length of channel
P_η	power dissipated by viscous force per unit length of channel
P_{η_s}	power dissipated by surface viscous force per unit length of channel
P_{η_v}	power dissipated by volume viscous force per unit length of channel
P_{σ_f}	power dissipated by ohmic loss in fluid per unit length of channel
P_{σ_w}	power dissipated by ohmic loss in walls per unit length of channel
p	pressure
S_f, S_c, S_i	surfaces in generalized channel cross section (see fig. 1)
t	dimensionless wall thickness of square channel (see fig. 2)

U	electric potential
V	dimensionless fluid velocity
$\overline{V}$	average dimensionless fluid velocity
$\vec{V}$	fluid velocity
V_o	characteristic velocity
V_z	z-component of $\vec{V}$
W	dimensionless potential
X, Y, Z	dimensionless coordinates
x, y, z	rectangular coordinates
α_1, α_2	adjustable parameters
$\beta_1, \beta_2, \beta_3$	adjustable parameters
δ	variational operator
η	fluid viscosity
$\lambda_1, \ldots, \lambda_n$	generalized adjustable parameters
μ_o	magnetic permeability of free space
ρ	fluid density
σ_f, σ_w	fluid and conducting wall electrical conductivities
γ	σ_w/σ_f
$[\psi]$	matrix (see eq. (16))
$\Big\vert_f \; \Big\vert_w$	evaluation of quantity on fluid or wall side of contour

Superscript:

T	transpose of matrix

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