Final Report

RF Project 2829

THE OHIO STATE UNIVERSITY

RESEARCH FOUNDATION

1314 KINNEAR ROAD COLUMBUS, OHIO 43212

METHODS OF SOLUTION OF THE INTEGRAL EQUATION FOR THE FOUCAULT TEST

Dr. B. E. Gatewood

Department of Aeronautical and
Astronautical Engineering

September 1969

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Washington D. C. 20546

Grant No. NGR 36-008-131

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For the period July 1, 1969 - September 30, 1969

Submitted by Dr. B. E. Gatewood

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ABSTRACT

The linear integral equation is solved by collocation, iteration, Fourier series, power series, and an inversion integral for the solid mirror. The collocation and inversion integral solutions appear to be the best and these are used to solve the linear equation of the mirror with a central hole. Numerical integration of the nonlinear equation is made for selected surface deviations to show that results start to change from the linear solution at a surface deviation of 0.025λ (λ is wavelength of light). Several possible procedures for calibrating and scaling the input data for the integral equation are described.

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METHODS OF SOLUTION OF THE INTEGRAL EQUATION FOR THE FOUCAULT TEST

I. INTRODUCTION

The Foucault Test, or knife-edge test, is a widely used method of testing astronomical mirrors and other high-quality optical systems of large aperture during the process of adjustment to correct surface errors in the mirror. Dr. Samuel Katzoff, chief scientist of NASA-Langley, has suggested that it may be possible to use the Foucault Test for adjustment of astronomical mirrors in satellite orbit. These adjustments would correct for small temperature gradients in the mirror.

The purpose of the investigation described in this report is to make use of the work of Linfoot[1] to (a) derive the integral equation for the Foucault Test for the case of a concentric hole in the mirror, (b) investigate methods of solution of the resulting linear integral equation with and without hole, (c) investigate procedures for scaling the input data for the integral equation solution, and (d) prepare a digital computer program for the best method of solution for as highly an automated system as possible.

II. DERIVATION OF INTEGRAL EQUATIONS

Linfoot[1] derives the following basic equations for the light intensity distribution I as seen under the knife-edge test (diameter = 2.00):

$$I = 4\pi^2 \left| D(x) \right|^2 \tag{1}$$

and

$$2\pi D(x) = \pi E(x) + i \int_{-1}^{1} \frac{E(t)dt}{t - x} , \tag{2}$$

where D(x) is the complex displacement on the image surface, E(x) is the wave form function of the mirror, and Eqs. (1) and (2) are for the diameter of the mirror with the knife-edge centrally located. The conditions on Eqs. (1) and (2) are (page 138 of Ref. 1)

(a) the diameter of the mirror subtends only a small angle at the focal point;

(b) the errors of figure of the mirror, though they may amount to many wavelengths, are small compared with the focal distance, s;

(c) the errors of slope on the mirror surface are small; and

(d) the function $E(x)$ is continuous and differentiable, except at the boundary.

Under these conditions, the equations are valid for mirrors of arbitrary edge contour, including central piercings, and of variable reflecting power.

The function $E(x)$ can be taken in the form

$$E(x) = \left|E(x)\right| \exp\left[-\frac{2\pi i}{\lambda}\varphi(x)\right], \tag{3}$$

where $\exp\left[-\frac{2\pi i}{\lambda}\varphi(x)\right]$ is a phase function which describes the errors on the surface of the mirror; λ is the wavelength of light on the mirror. For the perfect mirror, $\varphi(x) = 0$ and $E(x) = 1$, so that

$$2\pi D_0(x) = \pi + i \log\left(\frac{1 - x}{1 + x}\right), \text{ perfect mirror,} \tag{4}$$

and

$$I_0 = 4\pi^2\left|D_0(x)\right|^2 = \pi^2 + \log^2\left(\frac{1 - x}{1 + x}\right), \ -1 < x < 1. \tag{5}$$

For the case of errors, take

$$\left.\begin{aligned} \eta(x) &= -\varphi(x)/\lambda \\ E(x) = \exp[2\pi i\eta(x)] &= \cos 2\pi\eta(x) + i \sin 2\pi\eta(x) \end{aligned}\right\}, \tag{6}$$

where $\eta(x)$ is the surface error in wavelengths on any diameter of the mirror. Equations (1) and (2) become

$$2\pi D(x) = \pi\left[\cos 2\pi\eta(x) + i \sin 2\pi\eta(x)\right]$$

$$+ i \int_{-1}^{1} \frac{\cos 2\pi\eta(t)dt}{t - x} - \int_{-1}^{1} \frac{\sin 2\pi\eta(t)dt}{t - x} \tag{7}$$

and

$$I = \pi^2 + \left[\int_{-1}^{1} \frac{\cos 2\pi\eta(t)dt}{t - x}\right]^2 + \left[\int_{-1}^{1} \frac{\sin 2\pi\eta(t)dt}{t - x}\right]^2$$

$$+ 2\pi \sin 2\pi\eta(x) \int_{-1}^{1} \frac{\cos 2\pi\eta(t)dt}{t - x}$$

$$- 2\pi \cos 2\pi\eta(x) \int_{-1}^{1} \frac{\sin 2\pi\eta(t)dt}{t - x} \tag{8}$$

If $\eta(x)$ is sufficiently small so that

$$\left.\begin{array}{c} \cos 2\pi\eta(x) \sim 1.0, \ \sin 2\pi\eta(x) \sim 2\pi\eta(x) \\ [\eta(x)]^2 \sim 0.0, \end{array}\right\} \tag{9}$$

then Eq. (8) can be linearized approximately to give

$$I - I_0 = 4\pi^2\eta(x) \log\left(\frac{1 - x}{1 + x}\right) - 4\pi^2 \int_{-1}^{1} \frac{\eta(t)dt}{t - x} \tag{10}$$

This is the basic linear integral equation for $\eta(x)$. When I is read on the diameter by the Foucault test, then Eq. (10) can be solved for $\eta(x)$. Take

$$f(x) = 4\pi^2\eta(x), \ F(x) = I - I_0 \tag{11}$$

so that the basic Eq. (10) is

$$F(x) = f(x) \log\left(\frac{1 - x}{1 + x}\right) - \int_{-1}^{1} \frac{f(t)dt}{t - x} \tag{12}$$

When the mirror has a central hole of radius R, Eq. (8) can be linearized to give

$$F(x) = I - I_o = f(x) \log\left|\frac{1 - x}{1 + x} \frac{x + R}{x - R}\right|$$

$$- \int_{-1}^{-R} \frac{f(t)dt}{t - x} - \int_{R}^{1} \frac{f(t)dt}{t - x} \tag{13}$$

and

$$I_o = \pi^2 + \log^2\left|\frac{1 - x}{1 + x} \frac{x + R}{x - R}\right| \tag{14}$$

III. RESTRAINT CONDITIONS

In Eq. (12), if $f(x) = c_2$, then $F(x) = 0$; and if $f(x) = c_1x$, then $F(x) = - 2c_1$. Thus c_2 cannot be determined from $F(x)$ and c_1 may not be determinate from $F(x)$, depending upon the scale for $F(x)$ and the method of solution for $f(x)$ from $F(x)$. To determine c_1 and c_2 in

$$f(x) = y(x) + c_1x + c_2 \tag{15}$$

two boundary conditions at the edge of the mirror (or very near the edge of the mirror) can be specified. These points can be reference points for the errors $f(x)$ on the mirror so that the conditions may be taken as

$$f(1) = 0, \text{ or } f(1 - \Delta x) = 0 \tag{16}$$

and

$$f(-1) = 0, \text{ or } f(-1 + \Delta x) = 0 \tag{17}$$

These conditions can be imposed as restraints in the solution method for $f(x)$ or as boundary conditions for the solution $y(x)$ in Eq. (15). In both cases

$$c_1 = \frac{y(-1 + \Delta x) - y(1 - \Delta x)}{2(1 - \Delta x)} \tag{18}$$

$$c_2 = \frac{y(-1 + \Delta x) + y(1 - \Delta x)}{-2} \tag{19}$$

IV. NUMERICAL INTEGRATION

Since many of the methods of solution of Eqs. (12) and (13) involve numerical integration, a comparison will be made between an exact integration of Eq. (12) for a selected f(x) and several numerical integrations of Eq. (12) for the same f(x). Take

$$\left.\begin{aligned} f(x) &= (1 - z^2)^2 , \quad z^2 \leq 1 \\ &= 0 \qquad\quad\;\; , \quad z^2 > 1 \\ z = \frac{x - a}{b} , &\qquad a \text{ and } b = \text{selected constants} \end{aligned}\right\} \tag{20}$$

Integration of Eq. (12) using the Cauchy principal value gives

$$\left.\begin{aligned} F(x) = I - I_o &= -2z^3 + \frac{10}{3} z - (1 - z^2)^2 \log\left|\frac{1 - z}{1 + z} \, \frac{1 + x}{1 - x}\right| , \quad z^2 < 1 \\ &= \pm \frac{4}{3} , \quad z = \pm 1 , \\ &= \frac{2}{3z} + \frac{2}{3z^3} - (z^2 - 1)^2\left[\frac{2}{z} + \frac{2}{3z^3} + \log\left|\frac{z - 1}{z + 1}\right|\right] , \quad z^2 > 1 . \end{aligned}\right\} \tag{21}$$

Figures 1 and 2 are graphs of Eq. (21) for the exact case, with $a = 0.6$, $b = 0.2$.

For numerical integration, Eq. (12) becomes

$$F(x_i) = f(x_i) \log\left|\frac{1 - x_i}{1 + x_i}\right| - \sum_{j=1}^{N} \frac{f(x_j)(\Delta x)_j w_j}{x_j - x_i} \tag{22}$$

In matrix form, Eq. (22) is

$$[F] = [H][f] , \quad [H] = [D] - [A] , \tag{23}$$

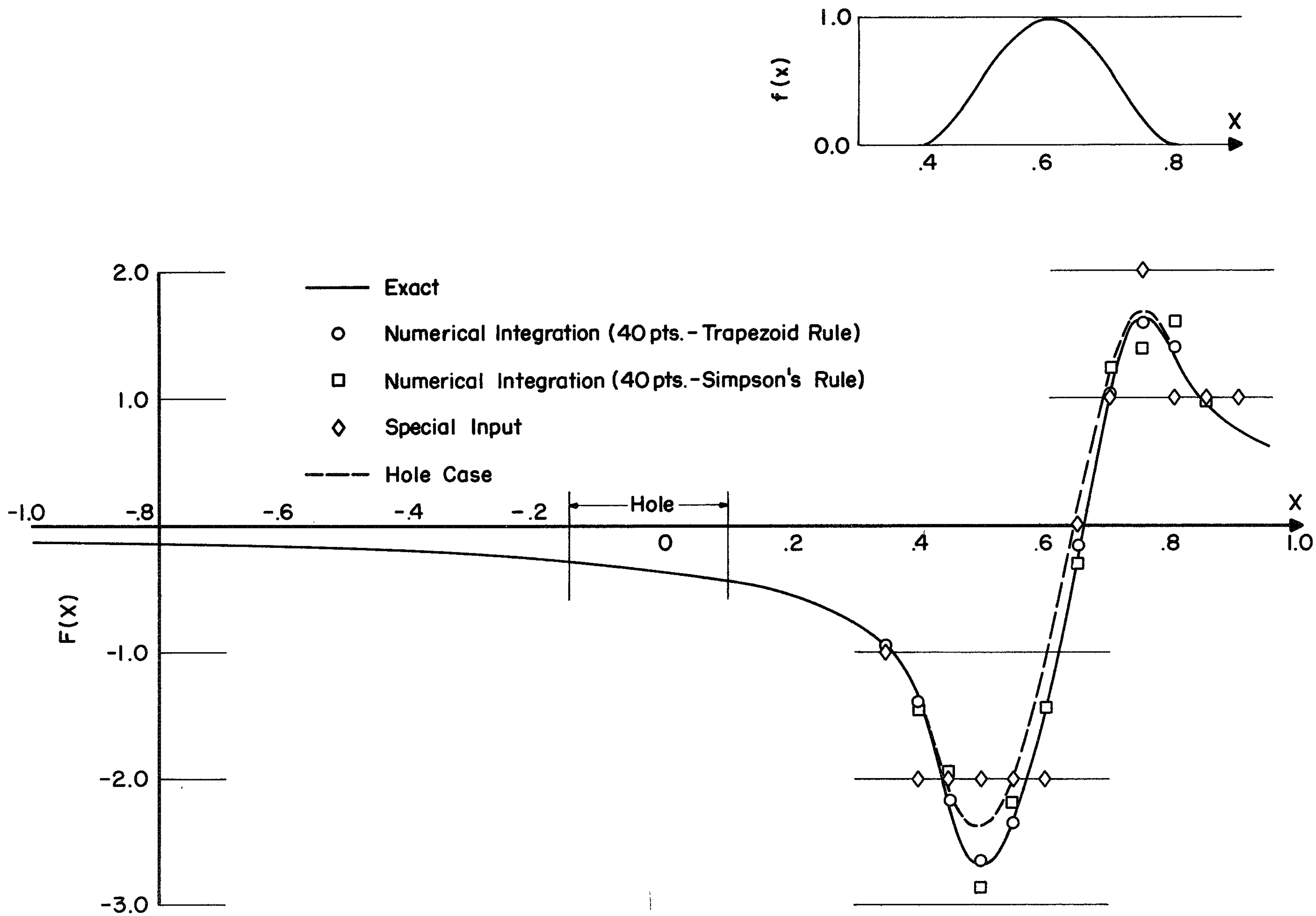

Fig. 1 - Numerical Integration

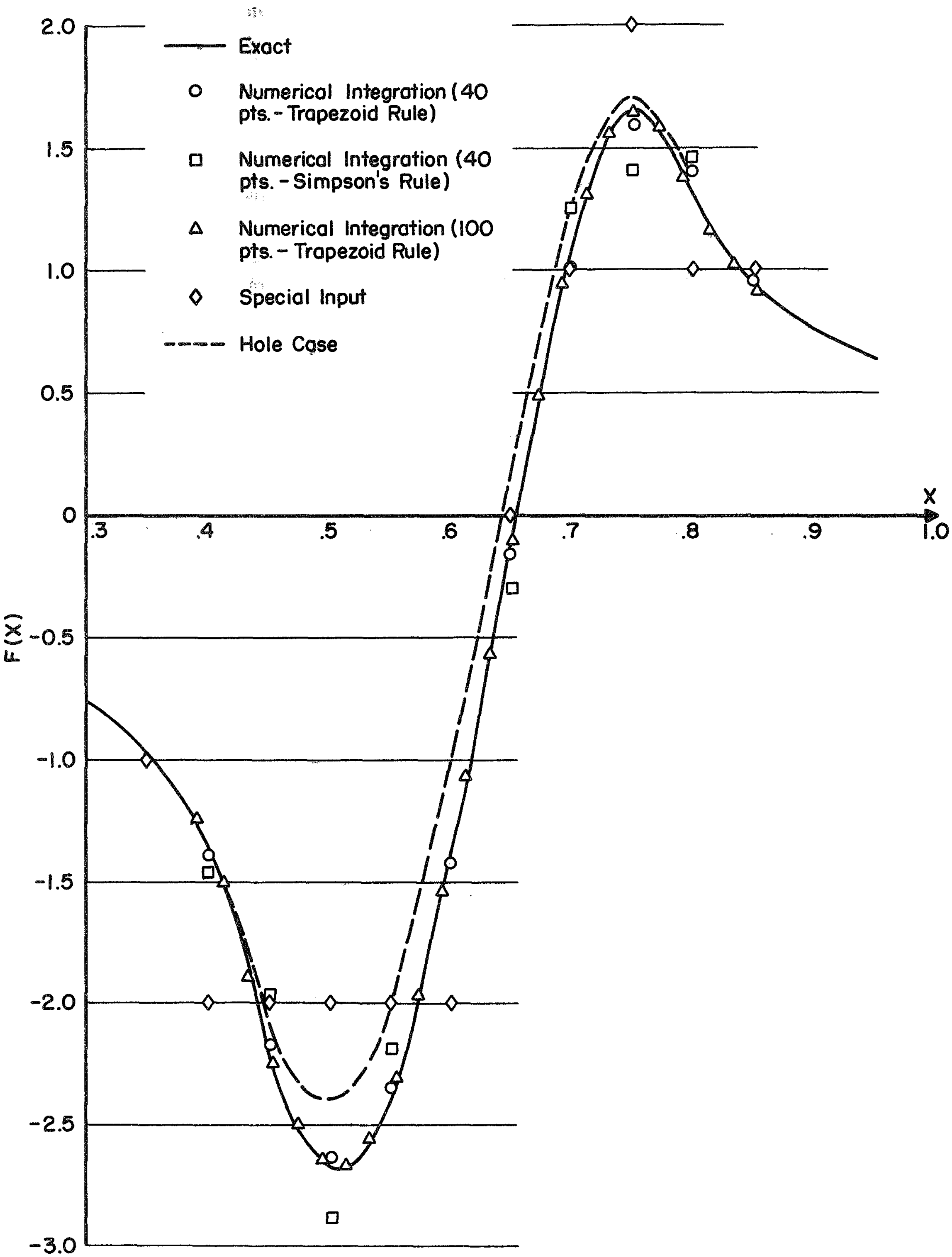

Fig. 2 - Numerical Integration

where [D] is a diagonal matrix with

$$D_{ii} = \log\left|\frac{1 - x_i}{1 + x_i}\right|, \quad D_{ij} = 0 \ , \quad i \neq j \tag{24}$$

and [A] is the numerical integration matrix with elements of the form

$$A_{ij} = \frac{w_j(\Delta x)_j}{x_j - x_i} \ , \tag{25}$$

where w_j is a weighting factor for the particular type of numerical integration.

Since D_{ii} becomes infinite at $x_i = \pm 1$ and A_{ij} becomes infinite at $x_i = x_j$, the end points $x = \pm 1$ must be omitted from Eq. (22) and the diagonal of [A] must be zero; i.e.,

$$A_{ii} = 0. \tag{26}$$

The areas lost in the numerical integration by these restrictions can be approximated as

$$\frac{(\Delta x)_1}{2} \ \frac{f(x_1)}{x_1 - x_i} \ \text{and} \ \frac{(\Delta x)_N}{2} \ \frac{f(x_N)}{x_N - x_i} \tag{27}$$

at the points next to the ends, and

$$\frac{(\Delta x)_i}{2}\left[\frac{f(x_{i+1}) - f(x_{i-1})}{(\Delta x)_i}\right] \tag{28}$$

at the point $x_j = x_i$.

For the <u>trapezoid</u> <u>rule</u> of numerical integration with N + 1 equal spaces over a diameter of 2.0 and with $x = \pm 1$ omitted, the elements of the above matrices become (with some refinements near the ends)

$$
\left.\begin{aligned}
D_{ii} &= \log\left(\frac{N + 1 - i}{i}\right), \quad i = 1,2, \ldots, N \\
D_{ij} &= 0 , \quad i \neq j
\end{aligned}\right\} \quad (29)
$$

and

$$
\left.\begin{aligned}
&A_{ii} = 0 \\
&A_{ij} = \frac{1}{j - i}, \quad \text{except} \\
&A_{21} = -1.76, \quad A_{31} = -0.68, \quad A_{41} = -0.47, \\
&A_{i1} = \frac{1.5}{1 - i}, \quad i = 5,6, \ldots, \quad N \\
&A_{iN} = \frac{1.5}{N - i}, \quad i = 1,2, \ldots, \quad N - 4, \\
&A_{N-1,\,N} = 1.76, \quad A_{N-2,\,N} = 0.68, \quad A_{N-3,\,N} = 0.47 \\
&A_{12} = 0.43, \quad A_{i,\,i+1} = 1.5, \quad i = 2,3, \ldots, \quad N - 2 \\
&A_{N,\,N-1} = -0.43, \quad A_{i,\,i+1} = -1.5, \quad i = 3,4, \ldots, \quad N - 1 \\
&\text{Trapezoid rule}
\end{aligned}\right\} \quad (30)
$$

where the areas from Eqs. (27) and (28) have been included.

For Simpson's rule, the diagonal weighting matrix $[W_s]$ in $[A][W_s]$ is

$$
\text{Diagonal of } [W_s] = \frac{4}{3}, \; \frac{2}{3}, \; \frac{4}{3}, \; \frac{2}{3}, \; \ldots, \; \frac{2}{3}, \; \frac{4}{3}
$$

for end points omitted and odd number of points. To allow for the areas in Eqs. (27) and (28), the [A] matrix must have the form

$$\left.\begin{array}{l} A_{ii} = 0 \\[2ex] A_{ij} = \dfrac{1}{j - } , \quad \text{except} \\[2ex] A_{i1} = \dfrac{1.25}{1 - i} , \quad A_{iN} = \dfrac{1.25}{N - i} , \\[2ex] A_{i,\ i+1} = 2.0, \quad A_{i,\ i-1} = -2.0, \quad i \text{ odd}, \\[2ex] A_{i,\ i+1} = 1.25, \quad A_{i,\ i-1} = -1.25, \quad i \text{ even}, \end{array}\right\} \quad (31)$$

Simpson's rule, N odd

Figures 1 and 2 show the results for numerical integration using $N = 40$ in the trapezoid rule in Eqs. (29) and (30). Simpson's rule with [A] in Eq. (31) gave the same results as the trapezoid rule (within ± 0.01 for some points in Fig. 2). The results in Figs. 1 and 2 (marked Simpson's rule) were obtained by using the [A] in Eq. (30) for $[A][W_s]$. It is evident that the areas in Eqs. (27) and (28) must be allowed for in such a manner as to fit the particular method of numerical integration.

The 40-point numerical integration with the trapezoid rule would appear to be sufficiently accurate for most cases. If $b < 0.2$ in Eq. (20), which is unlikely in actual cases of deviations on the mirror surface caused by temperature, more points may be needed.

V. COLLOCATION SOLUTION

A solution of the linear integral Eq. (12) can be made by collocation by inverting Eq. (22) or Eq. (23). However, from Section III, the H matrix in Eq. (23) is singular. To obtain a nonsingular matrix, the boundary condition [Eq. (17)] can be added to the set of equations in Eq. (23) to get

$$\left.\begin{array}{l} [F_i] = [H][f_i] , \quad i = 1,2, \ldots, \quad N \\[1ex] 0 = f_1 \end{array}\right\} \quad (32)$$

Multiply the N + 1 equations by

$$\begin{bmatrix} 1 & 0 & 0 & \underset{H}{.} & . & . & 0 \end{bmatrix}' = [H_a]' \tag{33}$$

to get

$$[H_a]'[H_a][f] = [H_a]'[F] \tag{34}$$

or

$$[B][f] = [G],\ [B] = [H_a]'[H_a],\ [G] = [H_a]'[F]\ . \tag{35}$$

This operation with the matrix transpose is equivalent to least squaring the rectangular system and gives [B] as a symmetrical matrix with positive diagonal terms.

The solution of Eq. (35) can be made by the Choleski sequencing method for a symmetrical matrix. This method appears to be more accurate than most inversion methods. The matrix B can be split into a diagonal matrix D and a upper triangular matrix S as

$$B = S'DS, \tag{36}$$

where

$$\left.\begin{aligned}
&D_{11} = B_{11};\\
&S_{ii} = 1.0, \quad i = 1,2,\ .\ .\ .\ N;\\
&S_{1j} = B_{1j}/D_{11}, \quad j \geq 2;\\
&D_{ii} = B_{ii} - \sum_{k=1}^{i-1} S_{ki}{}^2 D_{kk}, \quad i \geq 2; \text{ and}\\
&S_{ij} = \frac{1}{D_{ii}}\left[B_{ij} - \sum_{k=1}^{i-1} S_{ki}S_{kj}D_{kk}\right], \quad i \geq 2, \quad j \geq i+1\ .
\end{aligned}\right\} \tag{37}$$

Put Eq. (36) into Eq. (35) to get

$$[S'DS][f] = [G] .$$

Define [DS][f] = [P] so that

$$[S]'[P] = [G]$$

or

$$P_i = G_i - \sum_{k=1}^{i-1} S_{ki}P_k , \quad i = 1,2, \ldots, N \tag{38}$$

and

$$f_i = \frac{P_i}{D_{ii}} - \sum_{k=i+1}^{N} S_{ik}f_k , \quad i = N, N - 1, \ldots, 1 . \tag{39}$$

The D_{ii} terms are positive and $D_{ii} \leq B_{ii}$. Any $D_{ii} = 0$, or small relative to the other terms, indicates a singular system. Note that the inverse of B can be obtained by this procedure by using G = I and $f = B^{-1}$.

Figure 3 shows the results of solving for f_i by the above procedure for the exact [F] of Fig. 1 using 40 points in the trapezoid rule with [H] in Eq. (30). The agreement with the exact curve, Eq. (20), is very close with a maximum error of ± 0.03. Using the 40-point solution for F in Fig. 2, it was found that inversion gave the exact f, thus checking the solution method. Also shown in Fig. 3 is a solution for a selected F shown in Fig. 2. It shows that a rapid change in F(x) in a region indicates a hump or surface deviation for f(x) in the same region. When the area in Eq. (28) was omitted in the H matrix, the maximum error in f(x) was ± 0.13 for exact F.

It should be noted that the results shown in Fig. 3 were calculated using two restraints in Eq. (32), $f_1 = 0$ and $f_N = 0$. When one restraint is used, the numerical integration introduces a small rotation into f(x), which is equivalent to a small translation in F(x)(see Sec. III). If this rotation of slope -0.02 is removed by rotating f(x), the errors are within the ± 0.03 shown in Fig. 3, except at the unrestrained end point. Since a real translation of F(x) gives a real rotation to f(x), two restraints are not permissible and Eq. (32) should be used for the general case when it is not known whether F(x) has a translation term. Any rotation obtained can be removed to give the desired f(x).

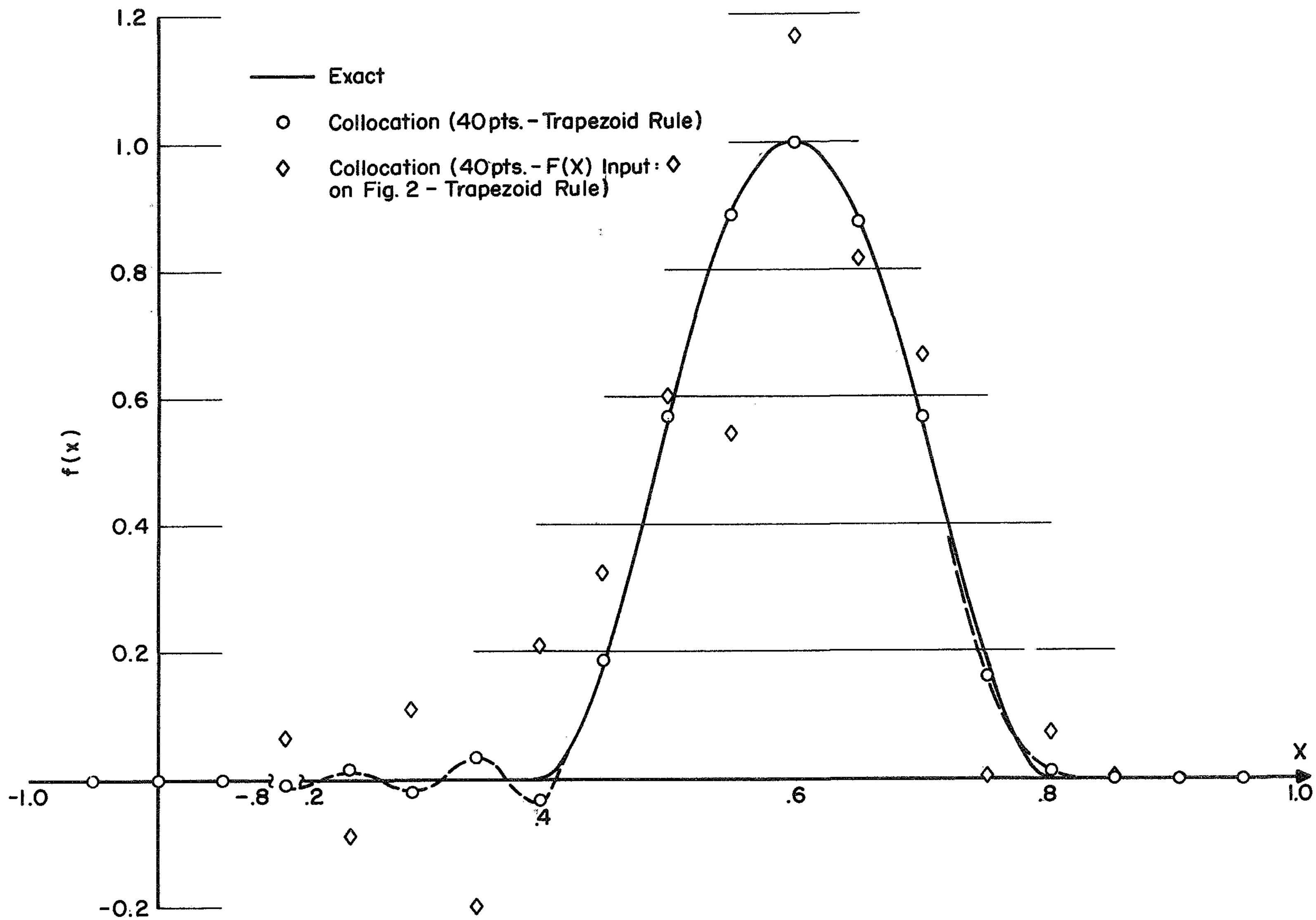

Fig. 3 - Collocation Solutions

VI. ITERATION SOLUTION A

Take Eq. (23) in the form

$$D^{-1}F = D^{-1}[D - A]f = If - D^{-1}Af ,$$

whence

$$f_m = D^{-1}F + D^{-1}Af_{m-1} \tag{40}$$

which can be used as an iteration solution; i.e.,

$$\left.\begin{aligned}
&f_0 = 0 , \\
&f_1 = D^{-1}F , \\
&f_2 = [I + S]f_1 , \quad S = D^{-1}A , \\
&f_3 = [I + S + S^2]f_1 , \quad \text{and} \\
&\cdot \cdot \cdot \cdot \cdot \cdot \\
&f_m = [I + S + . . . + S^{m-1}]D^{-1}F = J_m F .
\end{aligned}\right\} \tag{41}$$

For the trapezoid rule and numerical integration with D in Eq. (29) and A in Eq. (30) and N = 40 points, the J_m matrix in Eq. (41) was found to be divergent. The f_m solution was divergent for the exact F of Fig. 1, although f_2 did show a hump similar to Fig. 3 but not to scale. Because of the form of D, all the f_m have a discontinuity at x = 0.

VII. ITERATION SOLUTION B

Take Eq. (12) in the form

$$\begin{aligned}
\int_{-1}^{1} \frac{f_m(t)dt}{t - x} &= -\left[F(x) - f_{m-1}(x) \log\left|\frac{1 - x}{1 + x}\right|\right] \\
&= -G_{m-1}(x) \quad .
\end{aligned} \tag{42}$$

If the right side of this integral equation is known, then Eq. (42) has an inversion integral solution of the form

$$f_m(x) = \frac{Z(x)}{\pi^2} \left[\int_{-1}^{1} \frac{G_{m-1}(t)dt}{Z(t)(t - x)} + C_o \right] , \tag{43}$$

where, from Muskhelishvili (Chap. 11, Ref. 2), $Z(x)$ depends upon the restrictions on $f_m(x)$ and $G_{m-1}(t)$. The possible forms of $Z(x)$ are

$$Z(x) = \frac{1}{\sqrt{1 - x^2}} , \quad f_m(x) \text{ may be unbounded at } x = \pm 1; \tag{44}$$

$$Z(x) = \sqrt{\frac{1 - x}{1 + x}} , \quad f_m(x) \text{ may be unbounded at } x = -1,\ C_o = 0; \tag{45}$$

$$Z(x) = \sqrt{\frac{1 + x}{1 - x}} , \quad f_m(x) \text{ may be unbounded at } x = 1,\ C_o = 0; \tag{46}$$

and

$$Z(x) = \sqrt{1 - x^2} , \quad f_m(x) \text{ is bounded}, \quad C_o = 0, \quad \int_{-1}^{1} \frac{G_{m-1}(t)dt}{Z(t)} = 0. \tag{47}$$

The form in Eq. (45) is usually used for the pressure distribution on an airfoil with the leading edge at $x = -1$.

Use D and A in Eqs. (29) and (30) for a trapezoid rule numerical integration of Eq. (43), whence

$$\left.\begin{aligned}
&f_o = 0 , \\
&f_1 = \frac{1}{\pi^2} Z_D A Z_D^{-1} F = Z_D R F , \\
&f_2 = Z_D[I + Q]RF , \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
&f_m = Z_D[I + Q + Q^2 + \ldots + Q^{m-1}]RF = L_m F , \\
&Z_D(i,i) = Z(x_i), \quad Z_D(i,j) = 0, \quad i \neq j , \\
&R = \frac{1}{\pi^2} A Z_D^{-1} , \quad Q = \frac{-1}{\pi^2} AD , \quad \text{and} \\
&D \text{ in Eq. (29)}, \quad A \text{ in Eq. (30)}.
\end{aligned}\right\} \tag{48}$$

The Q^{m-1} matrix tends toward zero so that the iteration is convergent.

Calculations were made by the procedure of Eq. (48) for the F in Fig. 1 using 40 points for the numerical integration. Two forms of $Z(x)$ were used, Eqs. (44) and (45). The $Z(x)$ in Eq. (45) makes the integrand in Eq. (43) infinite at $x = 1$ so that the numerical integration must be modified to allow approximately for the area at the end $x = 1$. The $Z(x)$ of Eq. (44) gives a bounded integrand at the end. Table I shows results for both cases. It appears that in spite of correcting the A matrix for the approximate end area due to Eq. (45), the results using Eq. (45) are more erratic than those using Eq. (44). Because of the numerical integration errors, the iteration using $Z(x)$ in Eq. (45) may not converge to the correct result for $f(x)$. However, the iteration using $Z(x)$ in Eq. (44) is practically converged on the fifth iteration. Note that both of these solutions have been translated and rotated by using Eqs. (18) and (19) at $y(-1 + 2\Delta x)$ and $y(1 - 2\Delta x)$.

Table I - f(x) for Iteration Solution B
(see Fig. 3)

Point	f exact	Equation (45)					Equation (44)				
		f_1	f_2	f_3	f_4	f_5	f_1	f_2	f_3	f_4	f_5
26	0	-0.13	-0.01	0.01	0.01	0.00	-0.08	0.00	0.01	0.00	-0.01
27	0	-0.14	0.00	0.02	0.01	0.00	-0.10	0.00	0.01	0.00	-0.01
28	0	-0.16	0.00	0.03	0.02	0.01	-0.13	0.00	0.01	0.00	-0.01
29	0	-0.16	0.05	0.08	0.06	0.05	-0.17	0.02	0.05	0.00	-0.01
30	0.19	-0.07	0.22	0.24	0.21	0.18	-0.08	0.22	0.24	0.18	0.18
31	0.56	0.18	0.53	0.52	0.47	0.46	0.25	0.62	0.60	0.54	0.53
32	0.88	0.50	0.86	0.79	0.73	0.72	0.63	1.01	0.92	0.84	0.85
33	1.00	0.75	1.04	0.91	0.84	0.84	0.90	1.19	1.02	0.95	0.97
34	0.88	0.85	1.00	0.81	0.76	0.78	0.97	1.08	0.87	0.82	0.86
35	0.56	0.76	0.74	0.53	0.51	0.56	0.80	0.72	0.51	0.52	0.57
36	0.19	0.51	0.37	0.19	0.23	0.27	0.48	0.26	0.11	0.18	0.21
37	0	0.26	0.09	-0.02	0.03	0.07	0.19	-0.01	-0.07	0.01	0.03
38	0	0.14	0.04	-0.08	0.00	0.05	0.08	-0.07	-0.06	0.04	0.02
39	0	0.21	-0.01	-0.07	0.08	0.07	0.02	-0.10	-0.03	0.10	0.00
Max. error	0	0.38	0.18	±0.09	0.16	0.16	0.31	0.20	0.08	0.10	±0.03

VIII. FOURIER SERIES SOLUTION

In Eq. (12) take

$$x = \cos\theta \ , \quad f(x) = \sum_{n=1}^{\infty} A_n \sin n\theta \tag{49}$$

so that Eq. (12) becomes

$$\begin{aligned} F(\theta) &= 2 \log\left(\frac{\sin\theta}{1+\cos\theta}\right) \sum_{n=1}^{\infty} A_n \sin n\theta \\ &\quad - \sum_{n=1}^{\infty} A_n \int_0^{\pi} \frac{\sin\theta_t \sin n\theta_t \, d\theta_t}{\cos\theta_t - \cos\theta} \\ &= 2 \log\left(\frac{\sin\theta}{1+\cos\theta}\right) \sum_{n=1}^{\infty} A_n \sin n\theta \\ &\quad + \pi \sum_{n=1}^{\infty} A_n \cos n\theta \end{aligned} \tag{50}$$

Multiply Eq. (50) by cos $m\theta$ and integrate to get

$$\left.\begin{aligned} A_m + \sum_{n=1}^{\infty} K_{mn} A_n &= E_m \\ E_m &= \frac{2}{\pi^2} \int_0^{\pi} F(\theta) \cos m\theta \, d\theta \\ K_{mn} &= \frac{4}{\pi^2} \int_0^{\pi} \log\left(\frac{\sin\theta}{1+\cos\theta}\right) \sin n\theta \cos m\theta \, d\theta \ . \end{aligned}\right\} \tag{51}$$

Integrate* K_{mn} to get

$$\left.\begin{aligned}
K_{mn} &= \frac{2}{\pi^2}\int_0^{\pi} \log\left(\frac{\sin\theta}{1+\cos\theta}\right)\Big[\sin(m+n)\theta - \sin(m-n)\theta\Big]\,d\theta; \\
&= 0 \text{ for } m \text{ odd, } n \text{ even;} \\
&= 0 \text{ for } m \text{ even, } n \text{ odd;} \\
&= -\frac{4}{\pi^2 m}\sum_{k=1}^{m}\frac{1}{2k-1}, \quad n = m; \quad \text{and} \\
&= -\frac{8}{\pi^2(m+n)}\sum_{k=1}^{(m+n)/2}\frac{1}{2k-1} + \frac{8}{\pi^2(m-n)}\sum_{k=1}^{|(m-n)/2|}\frac{1}{2k-1}, \quad \text{for} \\
&\qquad m \text{ odd, } n \text{ odd; } m \text{ even, } n \text{ even, } m \neq n
\end{aligned}\right\} \quad (52)$$

In matrix form, Eq. (51) becomes $[I + K][A] = [E]$, M constants A_m, or

$$[I + K]'[I + K][A] = [I + K]'[E]$$

$$BA = E_1 \ ,$$

whence

$$[A] = [B]^{-1}[E_1] = [B]^{-1}[I + K]'[E]. \qquad (53)$$

From Eq. (49) for N points, $n = 1, 2, \ldots, N$,

*Thanks are due to Dr. Keith Stewartson, Visiting Professor at The Ohio State University, Summer 1969, and Goldsmid Professor of Mathematics and Joint Head of Mathematics Department, University of London, for this integration.

$$f(x_n) = \sum_{m=1}^{M} A_m \sin m\theta_n ,$$

$$[f]^{N\times 1} = [\sin m\theta_n]^{N\times M}[A]^{M\times 1} ,$$

$$\left.\begin{aligned}
&[f] = [G][F] , \\
&[G] = \frac{2}{\pi^2} [\sin m\theta_n][B]^{-1}[I + K]'[W][\cos m\theta_n] , \\
&[W] = \text{diagonal weighting matrix}, \\
&W_{11} = \tfrac{1}{2}(\theta_2 - \theta_1), \quad W_{NN} = \tfrac{1}{2}(\theta_N - \theta_{N-1}), \quad \text{and} \\
&W_{nn} = \tfrac{1}{2}(\theta_{n+1} - \theta_{n-1}), \quad n = 2, 3, \ldots, N - 1 .
\end{aligned}\right\} \tag{54}$$

Equation (54) was solved for the same F as above (Fig. 1) using $M = 20$ constants A_m and $N = 42$ points (end points included). $[B]^{-1}$ was obtained by the procedure of Eqs. (36) - (39). After translating and rotating the results by Eqs. (18) and (19), the solution for f(x) (see Fig. 3 for exact f(x)) had a maximum error of ± 0.04 for 42 equal Δx spaced points and a maximum error of -0.06 for 42 equal $\Delta\theta$ spaced points. Both solutions have a small wave of about ± 0.02 amplitude.

IX. POWER SERIES SOLUTION

Equation (12) can be written in the form

$$F(x) = - \int_{-1}^{1} \frac{f(t) - f(x)}{t - x} dt \tag{55}$$

Take f(x) in the form

$$f(x) = \sum_{n=1}^{\infty} A_n x^n, \quad -1 \le x \le 1, \tag{56}$$

whence

$$F(x) = -\sum_{n=1}^{\infty} A_n \int_{-1}^{1} \frac{t^n - x^n}{t - x}\, dt$$

$$= -\sum_{n=1}^{\infty} A_n \int_{-1}^{1} \left(t^{n-1} + t^{n-2}x + \ldots + tx^{n-2} + x^{n-1}\right) dt$$

$$= -\sum_{n=1}^{\infty} A_n \left[\frac{1 - (-1)^n}{n} + \frac{1 - (-1)^{n-1}}{n - 1}\, x + \ldots + 2x^{n-1}\right]. \tag{57}$$

Multiply by x^m and integrate to get

$$\left.\begin{aligned} &\sum_{n=1}^{\infty} K_{mn} A_n = E_m , \\ &E_m = \int_{-1}^{1} x^m F(x)\, dx , \end{aligned}\right\} \tag{58}$$

$$K_{mn} = -\left[\frac{\{1 - (-1)^n\}\{1 - (-1)^{m+1}\}}{n(m + 1)} + \frac{\{1 - (-1)^{n-1}\}\{1 - (-1)^{m+2}\}}{(n - 1)(m + 2)} + \ldots + \frac{2\{1-(-1)^{m+n}\}}{m + n}\right]$$

or

$$
\left.\begin{aligned}
K_{mn} &= 0, \ m \text{ even and } n \text{ even}, \\
&= 0, \ m \text{ odd and } n \text{ odd}, \\
&= 0, \ m = n, \\
&= -4 \sum_{k=1}^{(n+1)/2} \frac{1}{(n + 2 - 2k)(m - 1 + 2k)}, \quad m \text{ even, } n \text{ odd}, \\
&= -4 \sum_{k=1}^{n/2} \frac{1}{(n + 1 - 2k)(m + 2k)}, \quad m \text{ odd, } n \text{ even} .
\end{aligned}\right\} \tag{59}
$$

In matrix form, as in the previous Fourier series solution,

$$
\left.\begin{aligned}
&[f] = [G][F] , \quad M \text{ constants } A_m , \\
&[G] = \frac{2}{N - 1} [x_n^{\,m}][K'K]^{-1}[K]'[x_n^{\,m}]' , \quad N \text{ points} , \\
&\qquad \text{with end points included.}
\end{aligned}\right\} \tag{60}
$$

Equation (60) was solved for the same F as above using M = 20 constants and N = 42 points (end points included). The system of equations was very ill-conditioned, giving D_{ii} in Eq. (37) as small as 10^{-14} . Also, two of the D_{ii} were negative, indicating only round-off values were left in the calculations, as the D_{ii} must be positive. In spite of this, the final solution for f(x) gave a hump but with maximum errors of ± 0.12. For M = 12 constants, the solution was essentially the same with maximum errors of ± 0.11. The smallest D_{ii} was ± 10^{-12}.

The power series method is considered to be unsatisfactory.

X. INVERSION INTEGRAL SOLUTION

After investigating the above listed solutions, it was found that an inversion integral for Eq. (12) could be obtained from a solution given in Chapter 14 of Ref. 2. The equation

$$F(x) = A(x)f(x) + \frac{B(x)}{\pi i}\int_L \frac{f(t)dt}{t - x}\,, \tag{61}$$

where L is a union of p disconnected smooth arcs, A(x) and B(x) satisfy a Holder condition H on each closed arc, f(x) and F(x) can satisfy H* (tend to infinity at order less than 1), has the solution

$$f(x) = \frac{1}{A^2(x) - B^2(x)}\Big[A(x)F(x) + B(x)Z(x)P_{q-1}(x) - \frac{B(x)Z(x)}{\pi i}\int_L \frac{F(t)dt}{Z(t)(t - x)}\Big]\,; \tag{62}$$

$$\left.\begin{aligned} Z(x) &= \sqrt{A^2(x) - B^2(x)}\prod_{k=1}^{2p}(x - c_k)^{\gamma_k}\exp[\Gamma_o(x)] \\ \Gamma_o(x) &= \sum_{k=1}^{2p}\frac{1}{2\pi i}\int_L \frac{\log G(t) - \log G(c_k)}{t - x}\,dt; \quad \text{and} \end{aligned}\right\} \tag{63}$$

$$\left.\begin{aligned} G(x) &= \frac{A(x) - B(x)}{A(x) + B(x)}\,, \\ \int_L \frac{t^k F(t)dt}{Z(t)} &= 0, \quad q < 0, \quad k = 0,1,\ldots,-q-1\,, \\ c_k &= \text{end points of arcs,} \end{aligned}\right\} \tag{64}$$

$$\left.\begin{aligned}
\gamma_k &= \mp \frac{\log G(c_k)}{2\pi i} + \lambda_k = \alpha_k + i\beta_k + \lambda_k \,, \\
P_{q-1}(x) &= \text{polynomial of order} \leqslant q - 1 \,, \\
q &= -\sum_{k=1}^{2p} \lambda_k = p - r \,, \\
r &= \text{number of ends with bounds on } f(x), \text{ and} \\
\lambda_k &= \text{integer for end conditions.}
\end{aligned}\right\} \qquad \begin{matrix}(64)\\ \text{(cont.)}\end{matrix}$$

For <u>special ends</u>, $f(x)$ is bounded, $G(c_k)$ is positive real, α_k is integer, $\lambda_k = -\alpha_k$, and Real $\gamma_k = 0$. For <u>nonspecial ends</u>, $f(x)$ is bounded if $0 < \alpha_k + \lambda_k < 1$, and $f(x)$ is infinite at an end to order less than 1 if $-1 < \alpha_k + \lambda_k < 0$. q is an index for the classes of solution.

Now, Eq. (61) is the same as Eq. (12) if

$$A(x) = \log \frac{1 - x}{1 + x} \,, \quad B(x) = -\pi i \,. \tag{65}$$

Although $A(x)$ does not satisfy the H condition at the ends, it does satisfy H*, and it appears that if $f(x)$ and $F(x)$ satisfy H and $f = 0$ at the ends, then the solution would exist. Such restrictions on $F(x)$ and $f(x)$ are permissible in the present case. Also, it would appear that if L is from $-1 + \epsilon$ to $1 - \epsilon$, then $A(x)$ is bounded and the conditions would be met.

Use Eq. (65) in Eqs. (63) and (64) to get

$$G(x) = \frac{\log\left(\frac{1-x}{1+x}\right) + \pi i}{\log\left(\frac{1-x}{1+x}\right) - \pi i} = \frac{\log\left|\frac{1-x}{1+x}\right|}{\log\left|\frac{1-x}{1+x}\right|} = 1;$$

$$G(1) = G(-1) = 1 \;, \quad \log G = 0, \quad \Gamma_o(x) = 0;$$

$$\gamma_1 = \gamma_2 = 0 \;, \quad \text{special ends;}$$

$$\lambda_1 = 1 \;, \quad \lambda_2 = 0 \;, \quad q = 1 \;, \quad r = 2;$$

$P_{q-1}(x) = 0$; and

$$Z(x) = \sqrt{\pi^2 + \log^2\left(\frac{1-x}{1+x}\right)} \;. \tag{66}$$

Since $F(x)$ may not meet the integral restriction in Eq. (64) for the case of $q = 1$, it is necessary to use a superposition of solutions to obtain $f(x)$. Take

$$F(x) = F_1(x) + K_1 \;, \tag{67}$$

where $F_1(x)$ meets the condition

$$\int_L \frac{F_1(t)dt}{Z(t)} = 0 \;. \tag{68}$$

This gives

$$K_1 = \frac{\int_L \frac{F(t)dt}{Z(t)}}{\int_L \frac{dt}{Z(t)}} \;. \tag{69}$$

From Sec. III, the K_1 translation of $F(x)$ gives a rotation of $-K_1x/2$ to $f(x)$ so that Eq. (62) becomes

$$f(x) = -\frac{K_1 x}{2} + \frac{1}{Z(x)}\left[\left(\frac{F(x) - K_1}{Z(x)}\right)\log\frac{1 - x}{1 + x} + \int_{-1}^{1}\frac{F(t) - K_1}{Z(t)(t-x)}dt\right]. \quad (70)$$

Here $f(x)$ is bounded and the integrand in the integral is bounded at the ends so that numerical integration should be satisfactory.

A 40-point numerical integration of Eq. (70) using the trapezoid rule and Eq. (30) was made for the same $F(x)$, Fig. 1, as above to get the $f(x)$ in Fig. 3. After using Eqs. (18) and (19), the maximum errors in the solution were + 0.02 at one point and + 0.01 at another point. Errors at all other points were 0.00 to two places.

XI. SOLUTIONS FOR CENTRAL HOLE

All of the solutions described above can be applied to Eq. (13) for the case of the central hole. However, some of the methods are easier to apply than others.

The <u>collocation solution</u> (Sec. V), the <u>iteration solution B</u> (Sec. VII), and the <u>inversion integral solution</u> (Sec. X) can be applied directly by modifying the numerical integration matrix to omit the hole region.

The <u>Fourier Series Solution</u> (Sec. VIII) cannot be applied directly because of the integrations in Eqs. (50) and (52). Consideration was given to applying the Fourier Series on each side of the hole and then iterating between the two sides. However, the integral in Eq. (52) has an additional log term (see Eq. 13) on each side and apparently has no closed form as in Eq. (52). Since the collocation and inversion integral solutions are simpler and more accurate than the Fourier Series, there seems little justification to try to apply the Fourier Series solution to the hole case.

The integrations in the <u>Power Series Solution</u> (Sec. IX) can be made for the hole case but the K matrix in Eq. (59) becomes considerably more complicated. As in Sec. IX, the method is considered unsatisfactory.

For the hole case, the restraint conditions in Sec. III need to be modified by applying Eq. (15) to both sides of the hole. Thus Eqs. (18) and (19) become

$$\left.\begin{aligned}
C_{1L} &= \frac{y(-1 + \Delta x) - y(-R - \Delta x)}{1 - R - 2\Delta x} \,, \\
2C_{2L} &= -y(-1 + \Delta x) - y(-R - \Delta x) \,, \\
C_{1R} &= \frac{y(R + \Delta x) - y(1 - \Delta x)}{1 - R - 2\Delta x} \,, \quad \text{and} \\
2C_{2R} &= -y(R + \Delta x) - y(1 - \Delta x) \,.
\end{aligned}\right\} \qquad (71)$$

To apply the selected $f(x)$ in Eq. (20) to the hole case, it is necessary to modify Eq. (21) to

$$\left.\begin{aligned}
F(x) &= -2z^3 + \frac{10}{3} z - (1 - z^2)^2 \log\left|\frac{1 - z}{1 + z} \; \frac{1 + x}{1 - x} \; \frac{x - R}{x + R}\right| \,, \quad z^2 < 1 \,, \\
&= \pm \frac{4}{3} \,, \quad z = \pm 1 \\
&= \frac{2}{3z} + \frac{2}{3z^3} - (z^2 - 1)^2 \left[\frac{2}{z} + \frac{2}{3z^3} + \log \frac{z - 1}{z + 1}\right] , \quad z^2 > 1 \,.
\end{aligned}\right\} \quad (72)$$

For the numerical integration in Eq. (23), the H = D-A matrices become

$$D_{ii} = \log\left|\frac{1 - x_i}{1 + x_i} \; \frac{x_i + R}{x_i - R}\right| \,, \quad D_{ij} = 0, \quad i \neq j; \qquad (73)$$

$$
\left.
\begin{aligned}
&P = \Delta x = \frac{2(1-R)}{N+2},\ N \text{ points, } N \text{ even; } x = \pm 1,\ x = \pm R \text{ omitted;}\\
&A_{ii} = 0;\\
&A_{ij} = \frac{P}{x_j - x_i}, \quad \text{except :}\\
&A_{21} = -1.76, \quad A_{31} = -0.68, \quad A_{41} = -0.47;\\
&A_{i1} = \frac{1.5P}{x_1 - x_i}, \quad i = 5, 6, \ldots, \quad N;\\
&A\left(\frac{N-2}{2}, \frac{N}{2}\right) = 1.76 = -A\left(\frac{N+4}{2}, \frac{N+2}{2}\right) = A(N-1, N);\\
&A\left(\frac{N-4}{2}, \frac{N}{2}\right) = 0.68 = -A\left(\frac{N+6}{2}, \frac{N+2}{2}\right) = A(N-2, N);\\
&A\left(\frac{N-6}{2}, \frac{N}{2}\right) = 0.47 = -A\left(\frac{N+8}{2}, \frac{N+2}{2}\right) = A(N-3, N);\\
&A_{i,N/2} = \frac{1.5P}{x_{N/2} - x_i}, \quad i = 1, \ldots, \frac{N}{2}, \frac{N}{2} + 5, \ldots, \quad N;\\
&A_{iN} = \frac{1.5P}{x_N - x_i}, \quad i = 1, 2, \ldots, \quad N - 4;\\
&A_{12} = A\left(\frac{N+2}{2}, \frac{N+4}{2}\right) = -A\left(\frac{N}{2}, \frac{N-2}{2}\right) = -A(N, N-1) = 0.43;\\
&A_{i,\,i-1} = 1.5, \quad i = 2, \ldots, \quad \frac{N-2}{2}, \quad \frac{N+4}{2}, \quad \ldots, \quad N-1;\\
&A_{i,\,i+1} = -1.5, \quad i = 2, \ldots, \quad \frac{N-2}{2}, \quad \frac{N+4}{2}, \quad \ldots, \quad N-1;\\
&\text{hole between points } \frac{N}{2} \text{ and } \frac{N+2}{2}, \text{ trapezoid rule.}
\end{aligned}
\right\} \quad (74)
$$

In the following solutions the hole radius is $R = \frac{5}{41}$ and F is given in Eq. (72) and Figs. 1, 2. $\Delta x = \frac{2}{41}$ as above for the 40-point solution. There are 34 points on the mirror diameter.

XII. COLLOCATION SOLUTION FOR CENTRAL HOLE

Equation (32) becomes

$$\left.\begin{aligned} [F_i] &= [H][f_i] \ , \quad i = 1,2, \ldots, \quad N \ , \\ 0 &= f_1 \ , \quad \text{and} \\ 0 &= f_N \ , \end{aligned}\right\} \qquad (75)$$

where H = D - A is given by Eqs. (73) and (74). This system of equations can be solved for f_i by the procedure of Eqs. (33) - (39) in Sec. V.

For the exact [F] of Fig. 1 for hole case with 34 points on the mirror diameter (4 points in the hole and 2 points at edges of hole are omitted from the 40 point solution used above), the collocation solution of Eq. (75) gave f(x) to within ± 0.03 of the exact f(x) in Fig. 3, except at edges of the hole. A rotation of slope (-0.02) was made on f(x). When the edges of the hole were also restrained, the solution was within ± 0.03 everywhere. This is not permissible if F(x) has a real translation.

XIII. INVERSION INTEGRAL SOLUTION FOR CENTRAL HOLE

The inversion integral solution in Eq. (62) - (64) can be applied to Eq. (13) with

$$A(x) = \log\left|\frac{1 - x}{1 + x}\ \frac{x + R}{x - R}\right| , \qquad B(x) = -\pi i \ , \qquad (76)$$

whence Eqs. (66) and (70) become

$$Z(x) = \sqrt{\pi^2 + \log^2\left|\frac{1 - x}{1 + x}\ \frac{x + R}{x - R}\right|} \qquad (77)$$

and

$$f(x) = -\frac{k_1 x}{2(1-R)} - \frac{k_2 x^2}{2(1-R)} + \frac{1}{Z(x)}\left[\left(\frac{F(x)-k_1-k_2 x}{Z(x)}\right)\log\left|\frac{1-x}{1+x}\ \frac{x+R}{x-R}\right|\right.$$

$$\left. + \left(\int_{-1}^{-R} + \int_{R}^{1}\right)\frac{\{F(t) - k_1 - k_2 t\}dt}{Z(t)(t - x)}\right] \qquad (78)$$

where

$$k_1 = \frac{\int_L \frac{F(t)dt}{Z(t)}}{\int_L \frac{dt}{Z(t)}}, \quad k_2 = \frac{\int_L \frac{tF(t)dt}{Z(t)}}{\int_L \frac{t^2dt}{Z(t)}} . \tag{79}$$

Numerical integration of Eq. (78) using Eqs. (73) and (74) gives

$$\left.\begin{aligned} [Zf_o] &= [D + A]\left[\frac{F_o}{Z}\right] = [g] \\ f_i &= \frac{g_i}{Z_i} - \frac{k_1 x_i}{2(1 - R)} - \frac{k_2 x_i^2}{2(1 - R)} . \\ F_{oi} &= F_i - k_1 - k_2 x_i \end{aligned}\right\} \tag{80}$$

Results for the same F(x) as above for hole case with 34 points, after translation and rotation on each side of the hole by Eq. (71), gave a maximum error in f(x) of ± 0.02.

XIV. SUMMARY OF SOLUTIONS FOR LINEAR CASE

Table II shows the summary of maximum errors of the above solutions for the linear case without and with a central hole. The best and also the simplest solutions appear to be the collocation and inversion integral solutions. Even though the inversion integral solution is the most direct, the solution obtained must be translated and rotated to fit the edge conditions. The use of Eq. (71) to calculate these values may not be satisfactory, since these end values are affected the most by the numerical integration with the edges omitted. This difficulty is partially avoided in the collocation solution in Eq. (75) by forcing two edge conditions into the system of equations and spreading the numerical integration errors over the entire diameter. However, the matrix has to be inverted and this introduces some errors.

Table II - Summary of Solutions for Linear Case

Solution Method	Section	Maximum error from f(x) in Fig. 3 without hole	with hole
Collocation	5,12	± 0.03	± 0.03
Iteration A	6	divergent	---
Iteration B	7	± 0.09 (Eq. 45) ± 0.03 (Eq. 44)	---
Fourier Series	8	± 0.04 (Equal Δx); -0.06, +0.01 (Equal Δθ)	---
Power Series	9	± 0.11	---
Inversion Integral	10,13	+ 0.02, -0.00	± 0.02

Both the collocation and the inversion integral methods of solution are recommended. Each could be used to check the other.

XV. NONLINEAR NUMERICAL INTEGRATION

In the nonlinear Eq. (8), take

$$g(x) = 2\pi\eta(x) \ , \qquad \int_L = \int_{-1}^{-R} + \int_R^1$$

and

$$I_o = \pi^2 + \log^2\left|\frac{1 - x}{1 + x} \ \frac{x + R}{x - R}\right| , \qquad \text{hole of radius } R \ , \tag{81}$$

whence

$$F(x) = I - I_o = \left[\int_L \frac{\cos\ g(t)dt}{t - x}\right]^2 - \log^2\left|\frac{1 - x}{1 + x} \ \frac{x + R}{x - R}\right|$$

$$+ 2\pi \sin g(x) \int_L \frac{\cos\ g(t)dt}{t - x} - 2\pi \cos g(x) \int_L \frac{\sin\ g(t)dt}{t - x}$$

$$+ \left[\int_L \frac{\sin\ g(t)dt}{t - x}\right]^2 . \tag{82}$$

For a specified $\eta(x)$ take

$$\left.\begin{aligned} g(x_i) &= 2\pi\eta(x_i) \ , \\ S_i &= \sin g(x_i) \ , \\ C_i &= \cos g(x_i) \ , \\ [T_i] &= [A][S_i] \ , \quad \text{and} \\ [U_i] &= [A][C_i] \ , \end{aligned}\right\} \tag{83}$$

where the matrix A is the numerical integration matrix in Eq. (74). At any point x_i, Eq. (82) gives

$$F(x_i) = U_i^2 - \log^2\left|\frac{1 - x_i}{1 + x_i} \quad \frac{x_i + R}{x_i - R}\right|$$

$$+ 2\pi S_i U_i - 2\pi C_i T_i \ . \tag{84}$$

Assume $g(x)$ to have the form of Eq. (20) as used in the linear case above

$$\left.\begin{aligned} g(x) &= p(1 - z^2)^2 \ , \quad z^2 \leq 1; \\ &= 0 \qquad\qquad\ , \quad z^2 > 1; \quad \text{and} \\ z &= \frac{x - a}{b} \ , \end{aligned}\right\} \tag{85}$$

where p is a factor for the magnitude of the error. Note that

$$p = (2\pi)(\text{max. error in wavelengths}) \tag{86}$$

or

$$\eta_{max} = \text{max. error in wavelengths} = p/2\pi. \tag{87}$$

For the linear case used above $p = 1/2\pi$ and this case was used in Eq. (84) to compare to the linear results. Figure 4 shows the comparison for $F(x)$ using 34-point numerical integration. The linear numerical integration and the nonlinear numerical are quite close.

Figure 5 shows results of Eq. (84) for several values of p; i.e.,

$$\left.\begin{aligned} p &= \frac{1}{2\pi} \ , \quad \eta_{max} = \frac{1}{4\pi^2} \sim 0.025, \quad \text{linear;} \\ p &= \frac{10}{2\pi} \ , \quad \eta_{max} = \frac{10}{4\pi^2} \sim 0.25; \\ p &= 2\pi \ , \quad \eta_{max} = 1; \quad \text{and} \\ p &= 6.25\pi \ , \quad \eta_{max} = 3.08 \ . \end{aligned}\right\} \tag{88}$$

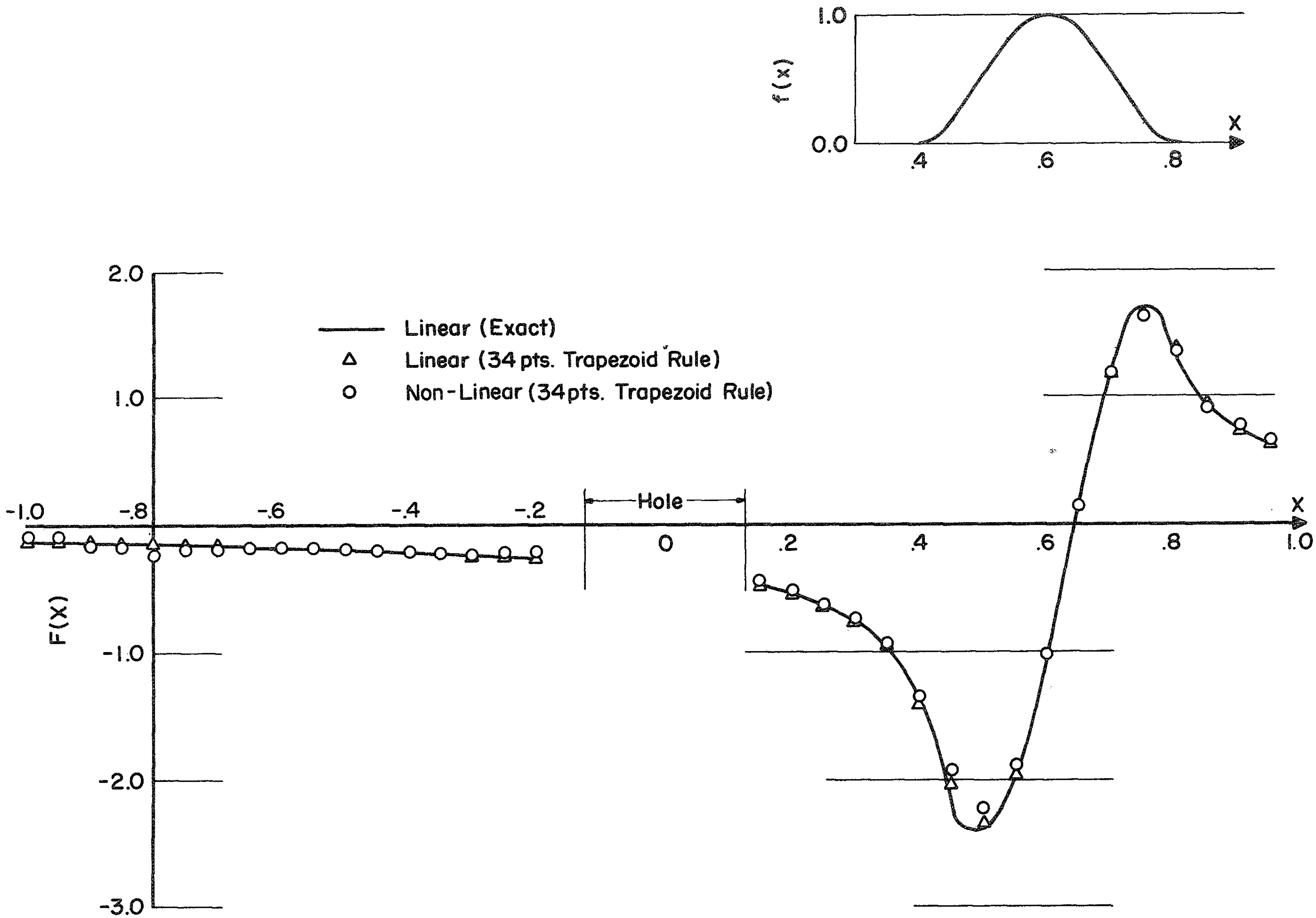

Fig. 4 - Numerical Integration (Nonlinear Case)

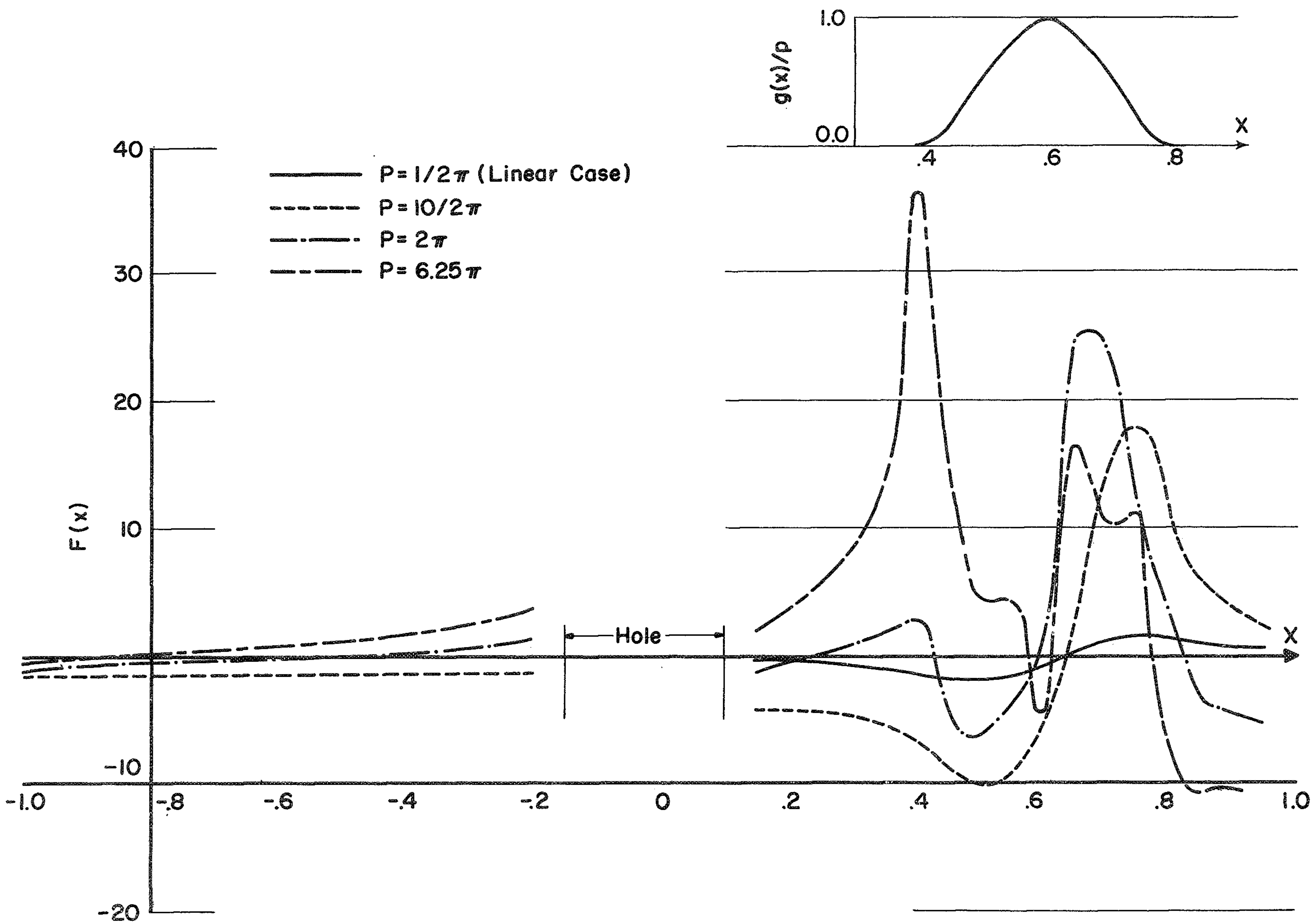

Fig. 5 - Numerical Integration (Nonlinear Case)

It can be concluded that large changes occur in $F(x)$ in the neighborhood of a $f(x)$ hump.

In Sec. III for the linear case it was found a linear $f(x) = C_1x$ produces a constant $F(x) = -2C_1$. Two cases for the nonlinear equation were calculated by Eq. (84) with

$$(a)\ g(x) = \frac{10}{2\pi}\,x \text{ and } (b)\ g(x) = -\frac{10}{2\pi}\,x\ , \tag{89}$$

which is equivalent to $\eta_{max} = \pm\frac{1}{4}$ at the ends. For case (a), $F(x)$ varied from -8.0 at edges of the hole to -14.8 at the ends, being approximately constant over the middle half of the mirror. For case (b), $F(x)$ varied from +19.6 at edges of the hole to +4.3 at the ends, being approximately constant over the middle half of the mirror.

XVI. SCALE FACTOR BETWEEN I AND I_o

Above, $F(x) = I - I_o$ is assumed known. I_o is calculated for the perfect mirror. I is measured on the actual mirror. If K converts I to scale of I_o, then

$$F(x) = \frac{I}{K} - I_o\ . \tag{90}$$

It appears that K depends on the intensity of the light source and on the instrumentation to read I. To calibrate the perfect mirror with a given light of intensity, Q_c,

$$F(x) = 0\ , \quad K_c = \frac{\int_L I_p dt}{\int_L I_o dt} \tag{91}$$

For a different light intensity Q

$$K = \frac{Q}{Q_c}\,K_c\ , \tag{92}$$

Thus, if K_c can be determined for the finished mirror (as perfect as possible) using the specified instrumentation and known light Q_c, then if the light Q can be determined then Eq. (92) would seem to hold, whence

$$F = \frac{Q_c}{Q}\,\frac{I}{K_c} - I_o\ . \tag{93}$$

A procedure to check K may be based on Eq. (90). If

$$F(x) = F_1(x) - C$$

$$\int_L F_1(x)dx = 0 \ , \quad C = \frac{\int_L F(x)dx}{2(1 - R)} \tag{94}$$

then from Sec. III, a rotation of the mirror through a slope change of $-C/2(1 - R)$ will change I and make the new F be F_1. Thus a check K is (I_R read after rotation),

$$K = \frac{\int_L I_R dx}{\int_L I_o dx} \tag{95}$$

It may be possible to determine K directly by giving a small known rotation to the mirror or an equivalent displacement of the knife-edge and measuring the change in I at several points. Since the change in F(x) can be calculated (linear case) as $-2C_1$, C_1 is slope, then

$$K = \frac{(\Delta I)_{meas}}{(\Delta F)_{cal}} \tag{96}$$

at each point. If ΔI is constant, then the K may be satisfactory.

It should be noted that an error in the origin for I shows as a rotation of f(x) and also an error in K shows a rotation in f(x). Thus, if f(x) shows a large rotation, it probably means I and K are not compatible, or the knife-edge is not central.

Finally, in the Eqs. (91) and (95)

$$\left.\begin{aligned} \int_{-1}^{1} I_o dx &= \frac{8}{3}\pi^2 \ , \quad \text{no hole,} \\ &= 2\pi^2(1 - R) + 2\int_R^1 \log^2\left|\frac{1 - x}{1 + x}\ \frac{x + R}{x - R}\right| dx \ , \\ &\quad \text{for hole of radius R.} \end{aligned}\right\} \tag{97}$$

XVII. CONCLUSIONS AND RECOMMENDATIONS

The linear integral equation for the Foucault Test can be solved, with or without a hole, by several different methods. The collocation and inversion integral methods appear to give the best results.

There appears to be several possible procedures for calibrating and scaling the input data for the integral equation solutions. The simplest procedure may be to give the knife-edge a small displacement and measure the change in the light intensity. Since the change can be calculated in the linear case, the scaling can be made by comparing the measured and calculated results.

For various assumed functions for the errors on the mirror, the light distribution on the mirror can be calculated from the nonlinear equation by numerical integration. Large changes occur in the light intensity in the neighborhood of a local surface deviation, whether linear or nonlinear. However, for a large nonlinear deviation, large changes may occur in other regions somewhat removed from the local deviation.

It is recommended that further investigation of the nonlinear equation be conducted, particularly with regard to

(a) input of a nonlinear $F(x)$ solution into the linear equation and comparison of solution to original assumed $f(x)$ in the nonlinear equation;

(b) scaling and calibration in the nonlinear case; and

(c) possible iteration solutions of the nonlinear equation.

XVII. REFERENCES

1. Linfoot, E. H., Recent Advances in Optics, Oxford, Clarendon Press, 1955.

2. Muskhelishvili, N. L., Singular Integral Equations, Trans. by J. R. M. Radok, 1953, P. Noordhoff N. V., Gronigen, Holland.

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C     **A COMPUTER PROGRAM USING THE INVERSION INTEGRAL METHOD OF SOLUTION
C       OF THE INTEGRAL EQUATION FOR THE FOUCAULT TEST**


        IMPLICIT REAL*8 (P,Z,A,C,B)
        DOUBLE PRECISION H(34,34),F(34),OII(34),Y(34),X(34)
      1 FORMAT (//,1X,'F ARRAY')
      2 FORMAT (9F11.6/9F11.6/9F11.6/7F11.6)
      3 FORMAT (1X,'N=',I5,/,1X,'A=',F10.8,/,1X,'B=',F10.8,/,1X,'R=',F10.8
       1)
      4 FORMAT (//,1X,'H MATRIX')
      5 FORMAT (//,1X,'IO ARRAY')
      6 FORMAT (//,1X,'YY ARRAY')
      7 FORMAT (//,1X,'Y ARRAY')
      8 FORMAT (I5,3F20.8)
      9 FORMAT (//,1X,'X ARRAY')
C     **N=THE NUMBER OF DATA POINTS ON THE MIRROR**
C     **A=THE LOCATION OF THE HUMP**
C     **B1=THE WIDTH OF THE HUMP**
C     **C=THE SIZE OF THE HOLE IN THE MIRROR**
        READ(5,8)N,A,B1,C
        WRITE(6,3) N,A,B1,C
        N1=N/2
        N2=N1+1
        N3=N1+2
        N4=N1+3
        N11=N1+4
        N5=N1-1
        N6=N1-2
        N7=N1-3
        N8=N-1
        N9=N-2
        N10=N-3
C     **CALCULATION OF THE LOCATION OF THE EQUALLY SPACED DATA POINTS**
        P=2.0D0*(1.0D0-C)/(N+2.0D0)
        X(1)=-1.0D0
        DO 30 I=1,N1
        I1=N-I+1
        I2=I-1
        IF (I .EQ. 1) I2=1
        X(I)=X(I2)+P
     30 X(I1)=-X(I)
        WRITE(6,9)
        WRITE(6,2) (X(I),I=1,N)
        ADD1=0.0D0
        ADD2=0.0D0
        ADD3=0.0D0
        ADD4=0.0D0
        DO 33 I=1,N
        I1=I+1
        I2=I-1
```

```
C     **CALCULATION OF THE H MATRIX**
      DO 32 J=1,N
      IF (I .EQ. J) GO TO 36
      IF (J .EQ. 1 .OR. J .EQ. N1 .OR. J .EQ. N2 .OR. J .EQ. N) GO TO 37
      IF (J .EQ. I1 .OR. J .EQ. I2) GO TO 38
      H(I,J)=P/(X(J)-X(I))
      GO TO 32
   36 H(I,J)=DLOG(DABS(((1.0D0-X(I))/(1.0D0+X(I)))*((X(I)+C)/(X(I)-C))))
      GO TO 32
   37 H(I,J)=1.5D0*P/(X(J)-X(I))
      GO TO 32
   38 H(I,J)=(J-I)*1.5D0
   32 CONTINUE
C     **CALCULATION OF THE EXACT F**
      Z=(X(I)-A)/B1
      IF (Z**2 .GT. 1.0001) GO TO 34
      F(I)=-2.0D0*Z**3+(10.0D0/3.0D0)*Z-((1.0D0-Z**2)**2)*DLOG(DABS(((1.
     10D0-Z)/(1.0D0+Z))*((1.0D0+X(I))/(1.0D0-X(I)))*((X(I)-C)/(X(I)+C)))
     1)
      GO TO 31
   34 IF (Z**2 .LT. .9998) GO TO 35
      F(I)=(2.0D0/3.0D0)*(1.0D0/Z+1.0D0/(Z**3))-((Z**2-1.0D0)**2)*(2.0D0
     1/Z+2.0D0/(3.0D0*(Z**3))+DLOG((Z-1.0D0)/(Z+1.0D0)))
      GO TO 31
   35 F(I)=DSIGN(4.0D0/3.0D0,Z)
   31 OII(I)=DSQRT(9.869604403667D0+(H(I,I))**2)
      ADD1=ADD1+1.0D0/OII(I)
      ADD2=ADD2+F(I)/OII(I)
      ADD3=ADD3+(X(I)*F(I))/OII(I)
   33 ADD4=ADD4+(X(I)**2)/OII(I)
      H(1,2)= 0.43
      H(2,1)=-1.76
      H(3,1)=-0.68
      H(4,1)=-0.47
      H(N7,N1)= 0.47
      H(N6,N1)= 0.68
      H(N5,N1)= 1.76
      H(N1,N5)=-0.43
      H(N2,N3)= 0.43
      H(N3,N2)=-1.76
      H(N4,N2)=-0.68
      H(N11,N2)=-0.47
      H(N10,N)= 0.47
      H(N9,N)= 0.68
      H(N8,N)= 1.76
      H(N,N8)=-0.43
      WRITE(6,4)
      WRITE(6,2) ((H(I,J),J=1,N),I=1,N)
      WRITE(6,5)
      WRITE(6,2) (OII(I),I=1,N)
      WRITE(6,1)
```

```
      WRITE(6,2) (F(I),I=1,N)
C    **THE INVERSION INTEGRAL METHOD**
      C1=(0.5D0*(F(1)/OII(1)+F(N1)/OII(N1)+F(N2)/OII(N2)+F(N)/OII(N))+AD
     1D2)/(1.0D0/OII(1)+1.0D0/OII(N1)+ADD1)
      C2=(0.5D0*((X(1)*F(1))/OII(1)+(X(N1)*F(N1))/OII(N1)+(X(N2)*F(N2))/
     1OII(N2)+(X(N)*F(N))/OII(N))+ADD3)/((X(1)**2)/OII(1)+(X(N1)**2)/OII
     1(N)+ADD4)
      DO 40 I=1,N
   40 F(I)=(F(I)-C1-C2*X(I))/OII(I)
      CALL DGMPRD (H,F,Y,N,N,1)
      WRITE(6,6)
      WRITE(6,2) (Y(I),I=1,N)
      DO 41 I=1,N
   41 Y(I)=Y(I)/OII(I)
      D5=-Y(1)+(0.5D0*X(1)*(C1+C2*X(1)))/(1.0D0-C)
      DO 42 I=1,N
   42 Y(I)=Y(I)+D5-(0.5D0*X(I)*(C1+C2*X(I)))/(1.0D0-C)
      WRITE(6,7)
      WRITE(6,2) (Y(I),I=1,N)
      STOP
      END
```