X-550-69-477
PREPRINT

NASA TM X- 63770

LONG PERIOD PERTURBATIONS IN THE ELEMENTS OF ARTIFICIAL SATELLITES DUE TO HIGH ORDER ZONAL HARMONICS

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NOVEMBER 1969

GSFC

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ABSTRACT

General formulas for the long period perturbations in the elements of artificial satellites due to high order zonal harmonics are presented. The formulas derived here are based on a more general determining function than Brouwer's determining function in order to permit indirect gravitational effects such as those due to the sun and the moon to be included in the perturbations.

The perturbations are given in the form of Fourier series in the argument of perigee and are associated with each of the zonal harmonics of the earth's gravitational field. The coefficients of the terms of the Fourier series are separable into two functions, one a power series in eccentricity and the other a power series in the inclination as shown by Kaula, Reference 1. The expression for each of these types of series is given.

In order to simplify the calculations, recursion formulas are presented. These formulas show that the evaluation of the terms of the power series can be made in cascade fashion, each successive term being a simple multiple of the preceding one.

Finally the contributions to the Lagrange planetary equations due to the long period even and odd zonal harmonics are given.

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CONTENTS

LONG PERIOD PERTURBATIONS IN THE ELEMENTS OF ARTIFICIAL SATELLITES DUE TO HIGH ORDER ZONAL HARMONICS

INTRODUCTION

In Reference 2, the force function for the zonal harmonics of the earth's gravitational field is given by

$$U = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty} J_n \frac{b^n}{r^n} P_n(\sin\phi)\right] \tag{1}$$

where $\mu = k^2 m_\oplus$, k is the Gaussian constant, $m_\oplus$ the mass of the earth, r the radius from the earth's center to the satellite, J_n the zonal coefficient, ϕ the latitude of the satellite in equatorial geocentric coordinates.

The definition of the Legendre Polynomials is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n} \tag{2}$$

Brouwer, Reference 2, expanded the force function up to order 5. He then deduced the long period perturbations using a determining function by von Zeipel's method. Kaula, Reference 1, and Giacaglia, Reference 3, extended the work of Brouwer to include the effects of zonal harmonics of any arbitrary order.

It is shown in References 1 and 2 that the long period part of the Hamiltonian derived from the force function (1) can be represented in the form

$$\sum_n \sum_q F(i)\, G(e) \cos 2qg\ , \tag{3}$$

for even n, and

$$\sum_n \sum_q F(i)\, G(e) \sin (2q + 1)\, g \tag{4}$$

for n odd. F(i) and G(e) are the inclination and eccentricity functions which are described below.

In References 2 and 3 the corresponding determining functions are given by

$$S = \sum \sum \frac{F(i)\, G(e) \sin 2qg}{2q \left(- \frac{3}{4} \frac{\mu^4 J_2 b^2 \left(1 - 5 \cos^2 i\right)}{L^3 G^4} \right)} \tag{5}$$

for even zonal harmonics, and

$$S = - \sum \sum \frac{F(i)\, G(e) \cos (2q + 1)\, g}{(2q + 1) \left(- \frac{3}{4} \frac{\mu^4 J_2 b^2 \left(1 - 5 \cos^2 i\right)}{L^3 G^4} \right)} \tag{6}$$

for odd zonal harmonics, where L and G are Delaunay variables with $L = \sqrt{\mu a}$ and $G = L \sqrt{1 - e^2}$, a and e are the semimajor axis and eccentricity of the satellite respectively.

In this report the results of References 2 and 3 are extended by introducing more general determining functions than those given by Equations 5 and 6.

Thus we define

$$S = \sum \sum \frac{F(i)\, G(e) \sin 2qg}{2q\dot{\alpha}} \tag{7}$$

for even n and

$$S = \frac{\sum\sum F(i)\, G(e) \cos(2q+1)\, g}{(2q+1)\,\dot{\alpha}} \tag{8}$$

for n odd, where

$$\dot{\alpha} = -\frac{\partial F_s}{\partial G} . \tag{9}$$

In Equation (9) the quantity F_s represents the secular part of the Hamiltonian. If it is desired to include indirect solar and lunar effects F_s must include the secular part of the solar and lunar gravitational potential. The expression for $\dot{\alpha}$ including solar and lunar effects as well as the zonal harmonics J_2, J_2^2 and J_4 is given in Appendix C. The quantity

$$-\frac{3}{4}\frac{\mu^4 J_2 b^2 \left(1 - 5\cos^2 i\right)}{L^3 G^4}$$

appearing in Equations 5 and 6 is a truncation of $\dot{\alpha}$ and is valid for many of the artificial satellites which have been launched. However for satellites more than one or two earth radii from the surface of the earth the indirect solar and lunar effects may be quite significant. In such cases direct solar and lunar effects (Reference 4) should also be included.

In order to shorten calculations, recursion formulas are given so that simple multiplications are involved in generating the required series.

The formulas given in Reference 1, are very compact and include not only the formulations for the zonal harmonics but formulations for tesseral harmonics as well. Consequently, in order to display the characteristics of the various components of the disturbing function, the formulations of Reference 1 are modified and formulas are given describing separately the secular, the long period even, and the long period odd zonal harmonics.

THE HAMILTONIAN

The Hamiltonian, References 2 and 3, for the high order zonals, is in this report divided into secular, long period even, and long period odd portions. The contributions to the mean motion are derived from the secular part of the Hamiltonian while the perturbations in the elements are derived from the long period parts via a determining function.

Secular Part of the Hamiltonian

From the results of References 2 and 3 we find that the secular part of the Hamiltonian is given by

$$\Delta_n F = -\frac{\mu^{n+2} J_n b^n}{2^n L^3 G^{2n-1}} PT \qquad (10)$$

where

$$P = \sum_{j=0}^{\frac{n-2}{2}} K_{0,j} e^{2j} \qquad K_{0,j} = \frac{1}{2^{2j}} \binom{n-1}{2j} \binom{2j}{j}$$

$$T = \sum_{k=0}^{\frac{n}{2}} B_{0,k} \sin^{2k} i \qquad B_{0,k} = \frac{(-1)^{\frac{n}{2}-k}}{2^{2k}} \binom{n}{\frac{n}{2}-k} \binom{n+2k}{2k} \binom{2k}{k} .$$

To illustrate this formula we note for $n = 4$

$$K_{0,0} = 1, \qquad K_{0,1} = \frac{3}{2}$$

so that

$$P = \left(1 + \frac{3}{2} e^2\right)$$

$$B_{0,0} = 6, \qquad B_{0,1} = -30, \qquad B_{0,2} = \frac{105}{4}$$

hence

$$T = 6 - 30\sin^2 i + \frac{105}{4}\sin^4 i,$$

and consequently

$$\Delta_4 F = \frac{-\mu^6 J_4 b^4}{16 L^3 G^7}\left(1 + \frac{3}{2}e^2\right)\left(6 - 30\sin^2 i + \frac{105}{4}\sin^4 i\right)$$

When we write

$$J_4 = -\frac{8}{3}k_4, \qquad e^2 = 1 - \frac{G^2}{L^2}, \qquad \sin^2 i = 1 - \frac{H^2}{G^2}$$

we find that this expression agrees with the result of Reference 2.

Periodic Part of Hamiltonian (n even)

The contribution to the periodic part of the Hamiltonian due to the n^{th} even zonal harmonic J_n, is given by the Fourier series

$$\Delta_n F = \frac{-\mu^{n+2} J_n b^n}{2^n L^3 G^{2n-1}} \sum_{q=1}^{\frac{n-2}{2}} KB\cos 2qg \tag{11}$$

where

$$K = \sum_{j=q}^{\frac{n-2}{2}} K_{q,j}\, e^{2j},$$

$$K_{q,j} = \frac{1}{2^{2j}} \binom{n-1}{2j} \binom{2j}{j-q}$$

$$B = \sum_{k=q}^{n/2} B_{q,k} \sin^{2k} i \,,$$

and

$$B_{q,k} = \frac{(-1)^{\frac{n}{2}+q-k}}{2^{2k-1}} \binom{n}{\frac{n}{2}-k} \binom{n+2k}{2k} \binom{2k}{k-q}$$

for the case j = 4, the Fourier series (11) consists of the single term

$$\Delta_4 F = \frac{-\mu^6 J_4 b^4}{16 L^3 G^7} K B \cos 2g$$

where

$$K = K_{1,1} e^2 = \frac{3}{4} e^2$$

and

$$B = B_{1,1} \sin^2 i + B_{1,2} \sin^4 i$$

$$B = 5 \sin^2 i \left(6 - 7 \sin^2 i\right)$$

making the substitutions for J_4, e^2 and $\sin^2 i$ indicated in the example illustrating the secular part of the Hamiltonian we obtain agreement with Reference 2.

Periodic Part of Hamiltonian (n odd)

If we combine the results of the formulas given above, we find that the odd zonal harmonics define the Fourier series

$$\Delta_n F = \frac{-\mu^{n+2} J_n b^n}{2^n L^3 G^{2n-1}} \sum_{q=0}^{\frac{n-3}{2}} C D \sin\left[(2q+1) g\right] \tag{12}$$

where

$$C = \sum_{j=q}^{\frac{n-3}{2}} C_{q,j} e^{2j+1},$$

$$C_{q,j} = \frac{1}{2^{2j+1}} \binom{n-1}{2j+1} \binom{2j+1}{n-q}$$

$$D = \sum_{k=q}^{\frac{n-1}{2}} D_{q,k} \sin^{2k+1} i,$$

and

$$D_{q,k} = \frac{(-1)^{\frac{n-1}{2}+q-k}}{2^{2k}} \binom{n}{\frac{n-1}{2}-k} \binom{n+1+2k}{1+2k} \binom{1+2k}{k-q}$$

To illustrate Equation (12), we apply it to the case $n = 5$. For the fifth zonal harmonic the Fourier series given by Equation 12 consists of the sum

$$\Delta_5 F = \frac{-\mu^7 J_5 b^5}{2^5 L^3 G^{11}} \left\{ (CD)_{q=0} \sin g + (CD)_{q=1} \sin 3g \right\}.$$

We find that for $q = 0$

$$C = C_{0,0}\, e + C_{0,1}\, e^3 = e\left(2 + \frac{3}{2} e^2\right)$$

$$D = D_{0,0} \sin i + D_{0,1} \sin^3 i + D_{0,2} \sin^5 i$$

$$= \sin i \left(60 - 210 \sin^2 i + \frac{315}{2} \sin^4 i\right)$$

Therefore

$$(CD)_{q=0} = e \sin i \left(2 + \frac{3}{2} e^2\right)\left(60 - 210 \sin^2 i + \frac{315}{2} \sin^4 i\right)$$

For $q = 1$,

$$C = C_{1,1}\, e^3 = \frac{e^3}{2}$$

$$D = D_{1,1} \sin^3 i + D_{1,2} \sin^6 i = \sin^3 i \left(70 - \frac{315}{4} \sin^2 i\right)$$

So that

$$(CD)_{q=1} = \frac{e^3 \sin^3 i}{2} \left(70 - \frac{315}{4} \sin^2 i\right)$$

These results are in agreement with Reference 2.

PERTURBATIONS IN THE ELEMENTS

The method of canonical variables is used in this report. Thus if S is a determining function we find from Reference 2, that

$$G' = G'' + \frac{\partial S}{\partial g'}$$

$$\ell' = \ell'' - \frac{\partial S}{\partial L''}$$

$$g' = g'' - \frac{\partial S}{\partial G''}$$

$$h' = h'' - \frac{\partial S}{\partial H''} \qquad (13)$$

The determining function S is considered to be a function of the new variables L'', G'', H'' and the odd variable g'. The variables ℓ' and h' do not appear in S since we are dealing with long period effects in the zonal harmonics.

We also have from Reference 2 that the mean motions of ℓ, g and h are given by

$$\frac{d\ell''}{dt} = -\frac{\partial F_s}{\partial L''}$$

$$\frac{dg''}{dt} = -\frac{\partial F_s}{\partial G''}$$

$$\frac{dh''}{dt} = -\frac{\partial F_s}{\partial H''} \qquad (14)$$

where F_s is the secular part of the Hamiltonian

Long Period Perturbations - Even Zonal Harmonics

We first find the determining function S by applying the definitions of Equations (3), (4), (7) and (8) to Equations (11). Thus for n even the contribution of the n^{th} harmonic to the determining function is given by

$$\Delta_n S = \frac{-\mu^{n+2} J_n b^n}{2^n L^3 G^{2n-1}} \sum_{q=1}^{\frac{n-2}{2}} \frac{KB \sin 2qg}{2q\dot{\alpha}} \tag{15}$$

The quantity $\dot{\alpha}$ is the mean motion of g. In Appendix C a formula for $\dot{\alpha}$ is given which includes solar and lunar effects. A more general definition of $\dot{\alpha}$ for zonal harmonics is given by the second of Equation 14.

The symbol δ is introduced to denote the difference between an element and its mean value. From Equations (13) and (15) we find

$$\delta_n G = \frac{-\mu^{n+2} J_n b^n}{2^n L^3 G^{2n-1}} \sum_{q=1}^{\frac{n-2}{2}} \frac{KB}{\dot{\alpha}} \cos 2qg \tag{16}$$

Similarly we find

$$\delta_n \ell = \frac{\mu^{n+2} J_n b^n}{2^n L^3 G^{2n-1}} \sum_{q=1}^{\frac{n-2}{2}} \frac{KB}{2q\dot{\alpha}} \left(\nu_e - \frac{1}{\dot{\alpha}} \frac{\partial \dot{\alpha}}{\partial L}\right) \sin 2qg \tag{17}$$

an expression for $\partial\dot{\alpha}/\partial L$ is given in Appendix C. The quantity ν_e is given by

$$\nu_e = \frac{L^3}{K} \frac{\partial}{\partial L} \left(\frac{K}{L^3}\right)$$

$$= \frac{1}{L} \left\{ -3 + \frac{2\left(1 - e^2\right)}{K} \sum_{j=q}^{\frac{n-2}{2}} j K_{q,j} e^{2j-2} \right\} . \tag{18}$$

$$\delta_n g = \frac{\mu^{n+2} J_n b^n}{2^n L^3 G^{2n}} \sum_{q=1}^{\frac{n-2}{2}} \frac{KB}{2q\dot{\alpha}} \left(\lambda_e - \frac{1}{\dot{\alpha}} \dot{\alpha}_G\right) \sin 2qg \tag{19}$$

$$\lambda_e = \frac{G^{2n}}{KB} \frac{\partial}{\partial G} \left(\frac{KB}{G^{2n-1}}\right)$$

$$= 1 - 2n + \frac{2\cos^2 i}{B} \sum_{k=q}^{n/2} kB_{q,k} \sin^{2k-2} i - \frac{2(1-e^2)}{K} \sum_{j=q}^{n-2/2} jK_{q,j}\, e^{2j-2} \tag{20}$$

$$\dot{\alpha}_G = G \frac{\partial \dot{\alpha}}{\partial G}$$

An expression for $\dot{\alpha}_G$ is given in Appendix C.

$$\delta_n h = \frac{\mu^{n+2} J_n b^n}{2^n L^3 G^{2n}} \sum_{q=1}^{\frac{n-2}{2}} \frac{KB}{2q\dot{\alpha}} \left(\mu_e - \frac{1}{\dot{\alpha}} \dot{\alpha}_H\right) \sin 2qg \tag{21}$$

where

$$\mu_e = \frac{G}{B} \frac{\partial B}{\partial H}$$

$$= -\frac{2\cos i}{B} \sum_{k=q}^{n/2} k\, B_{q,k} \sin^{2k-2} i \tag{22}$$

$$\dot{\alpha}_H = G \frac{\partial \dot{\alpha}}{\partial H} ;$$

$\dot{\alpha}_H$ is given in Appendix C.

Long Period Perturbations - Odd Zonal Harmonics

The determining function for the odd zonal harmonics is found by applying the definitions of Equations (3), (4), (7) and (8) to Equation (12). We then obtain, for n odd, the contribution of the n^{th} zonal harmonic

$$\Delta_n S = \frac{\mu^{n+2} J_n b^n}{2^n L^3 G^{2n-1}} \sum_{q=0}^{\frac{n-3}{2}} \frac{CD \cos\left[(2q+1) g\right]}{(2q+1)\dot{\alpha}} \tag{23}$$

Again, by applying the formulas (13) to Equation (23) then results

$$\delta_n G = \frac{-\mu^{n+2} J_n b^n}{2^n L^3 G^{2n-1}} \sum_{q=0}^{\frac{n-3}{2}} \frac{CD \sin\left[(2q+1) g\right]}{\dot{\alpha}} \tag{24}$$

$$\delta_n \ell = \frac{-\mu^{n+2} J_n b^n}{2^n L^3 G^{2n-1}} \sum_{q=0}^{\frac{n-3}{2}} \frac{CD}{(2q+1)\dot{\alpha}} \left(\nu_0 - \frac{1}{\dot{\alpha}} \frac{\partial \dot{\alpha}}{\partial L}\right) \cos\left[(2q+1) g\right] \tag{25}$$

where

$$\nu_0 = \frac{L^3}{C} \frac{\partial}{\partial L} \left(\frac{C}{L^3}\right)$$

$$= \frac{1}{2}\left\{-3 + \frac{2\left(1 - e^2\right)}{C} \sum_{j=q}^{\frac{n-3}{2}} (2j+1)\, C_{q,j}\, e^{2j-2}\right\}$$

$$\delta_n g = \frac{-\mu^{n+2} J_n b^n}{2^n L^3 G^{2n}} \sum_{q=0}^{\frac{n-3}{2}} \frac{CD}{(2q+1)\dot{\alpha}} \left(\lambda_0 - \frac{1}{\dot{\alpha}} \dot{\alpha}_G\right) \cos\left[(2q+1) g\right] \tag{26}$$

$$\lambda_0 = \frac{G^{2n}}{CD}\frac{\partial}{\partial G}\left(\frac{CD}{G^{2n-1}}\right)$$

$$= 1 - 2n + \frac{\cos^2 i}{D}\sum_{k=q}^{\frac{n-1}{2}}(2k+1)\,D_{q,k}\sin^{2k-1} i$$

$$- \frac{(1-e^2)}{C}\sum_{j=q}^{\frac{n-3}{2}}(2j+1)\,C_{q,j}\,e^{2j-1}$$

$$\delta_n h = \frac{-\mu^{n+2}J_n b^n}{2^n L^3 G^{2n}}\sum_{q=0}^{\frac{n-3}{2}}\frac{CD}{(2q+1)\,\dot{\alpha}}\left(\mu_0 - \frac{1}{\dot{\alpha}}\dot{\alpha}_H\right)\cos\left[(2q+1)\,g\right] \qquad (27)$$

$$\mu_0 = \frac{G}{D}\frac{\partial D}{\partial H} = \frac{-\cos i}{D}\sum_{k=q}^{\frac{n-1}{2}}(2k+1)\,D_{q,k}\sin^{2k-1} i$$

THE MEAN MOTIONS

The mean motions of the angular elements are given by Equations (14) where F_s is defined by Equation (10). We then find that

$$\Delta_n\left(\frac{d\ell''}{dt}\right) = \frac{\mu^{n+2}J_n b^n}{2^n L^4 G^{2n-1}}\,T\left\{-3P + 2\left(1-e^2\right)\sum_{j=1}^{\frac{n-2}{2}} jK_{0,j}\,e^{2j-2}\right\} \qquad (28)$$

$$\Delta_n \left(\frac{dg''}{dt}\right) = \frac{\mu^{n+2} J_n b^n}{2^n L^3 G^{2n}} \left\{ (1 - 2n) PT + 2P \cos^2 i \sum_{k=1}^{n/2} k B_{0,k} \sin^{2k-2} i \right.$$

$$\left. - 2T \left(1 - e^2\right) \sum_{j=1}^{\frac{n-2}{2}} j K_{0,j} e^{2j-2} \right\} \quad (29)$$

$$\Delta_n \left(\frac{dh''}{dt}\right) = \frac{-\mu^{n+2} J_n b^n}{2^{n-1} L^3 G^{2n}} (\cos i) P \sum_{k=1}^{n/2} k B_{0,k} \sin^{2k-2} i \quad (30)$$

LONG PERIOD CONTRIBUTIONS TO LAGRANGE'S PLANETARY EQUATIONS

The long period contributions of the zonal harmonics are found by forming the derivatives of Equations 10, 11, and 12. The derivatives of Equation (10) are given by Equations (28) - (30), and should be added to the corresponding derivatives below. It is convenient to use the following formulas for the elements e and i

$$\Delta_n \left(\frac{de}{dt}\right) = -\frac{1}{G} \frac{\left(1 - e^2\right)}{e} \Delta_n \left(\frac{dG}{dt}\right) \quad (31)$$

$$\Delta_n \left(\frac{di}{dt}\right) = \frac{1}{G} \cot i \, \Delta_n \left(\frac{dG}{dt}\right) \quad (32)$$

The equations for the elements G, ℓ, g, and h are given by the theory of canonical variables

$$\frac{dG}{dt} = \frac{\partial F}{\partial g}, \quad \frac{d\ell}{dt} = -\frac{\partial F}{\partial L},$$

$$\frac{dg}{dt} = -\frac{\partial F}{\partial G}, \quad \frac{dh}{dt} = -\frac{\partial F}{\partial H} \quad (33)$$

Long Period Contributions To Lagrange's Planetary Equations - Even Zonal Harmonics

For n even we find

$$\Delta_n \left(\frac{dG}{dt}\right) = \frac{\mu^{n+2} J_n b^n}{2^{n-1} L^3 G^{2n-1}} \sum_{q=1}^{\frac{n-2}{2}} q\, KB \sin 2qg \tag{34}$$

$$\Delta_n \left(\frac{d\ell}{dt}\right) = \frac{\mu^{n+2} J_n b^n}{2^n L^4 G^{2n-1}} \sum_{q=1}^{\frac{n-2}{2}} B \left[-3K + 2\left(1 - e^2\right) \sum_{j=q}^{\frac{n-2}{2}} j\, K_{q,j}\, e^{2j-2} \right] \cos 2qg \tag{35}$$

$$\Delta_n \left(\frac{dg}{dt}\right) = \frac{\mu^{n+2} J_n b^n}{2^n L^3 G^{2n-1}} \sum_{q=1}^{\frac{n-2}{2}} \Bigg[\left(1 - 2n\right) KB + 2K \cos^2 i \sum_{k=q}^{n/2} k\, B_{q,k} \sin^{2k-2} i$$
$$- 2B\left(1 - e^2\right) \sum_{j=q}^{\frac{n-2}{2}} j\, K_{q,j}\, e^{2j-2} \Bigg] \cos 2qg \tag{36}$$

$$\Delta_n \left(\frac{dh}{dt}\right) = \frac{\mu^{n+2} J_n b^n}{2^{n-1} L^3 G^{2n}} K \cos i \sum_{q=1}^{\frac{n-2}{2}} \sum_{k=q}^{n/2} k\, B_{q,k} \sin^{2k-2} i \cos 2qg \tag{37}$$

The functions K, B, and the coefficients $K_{q,j}$ and $B_{q,k}$ are defined by Equation (11).

Long Period Contributions To Lagrange's Planetary Equations - Odd Zonal Harmonics

For n odd we obtain

$$\Delta_n \left(\frac{dG}{dt}\right) = \frac{-\mu^{n+2} J_n b^n}{2^n L^3 G^{2n-1}} \sum_{q=0}^{\frac{n-3}{2}} (2q+1) CD \cos\left[(2q+1) g\right] \tag{38}$$

$$\Delta_n \left(\frac{d\ell}{dt}\right) = \frac{\mu^{n+2} J_n b^n}{2^n L^4 G^{2n-1}} \sum_{q=0}^{\frac{n-3}{2}} D\left[-3C + \left(1-e^2\right) \sum_{j=q}^{\frac{n-3}{2}} (2j+1) C_{q,j} e^{2j-1}\right]\left[\sin(2q+1) g\right] \tag{39}$$

$$\Delta_n \left(\frac{dg}{dt}\right) = \frac{\mu^{n+2} J_n b^n}{2^n L^3 G^{2n}} \sum_{q=0}^{\frac{n-3}{2}} \left\{(1-2n) CD + C \cos^2 i \sum_{k=q}^{\frac{n-1}{2}} (2k+1) D_{q,k} \sin^{2k-1} i\right.$$

$$\left. - D\left(1-e^2\right) \sum_{j=q}^{\frac{n-3}{2}} (2j+1) C_{q,j} e^{2j-1}\right\} \sin\left[(2q+1) g\right] \tag{40}$$

$$\Delta_n \left(\frac{dh}{dt}\right) = \frac{-C \cos i \, \mu^{n+2} J_n b^n}{2^n L^3 G^{2n}} \sum_{q=0}^{\frac{n-3}{2}} \sum_{k=q}^{\frac{n-1}{2}} (2k+1) D_{q,k} \sin^{2k} i \sin\left[(2q+1) g\right] \tag{41}$$

The functions C and D, and the coefficients $C_{q,j}$ and $D_{q,k}$ are defined by Equation (12).

REFERENCES

1. Kaula, William M., Theory of Satellite Geodesy, Blaisdell Publishing Co., 1966.

2. Brouwer, Dirk, Solution of the Problem of Artificial Satellite Theory Without Drag, Astronomical Journal, Nov. 1959, Vol. 64, No. 9, p. 378.

3. Giacaglia, Giorgio, E. O., The Influence of High-Order Zonal Harmonics On The Motion of an Artificial Satellite Without Drag, Astronomical Journal, 69, No. 4, May 1964, p. 303.

4. Fisher, David, Formulas For Long Period Motion of Artificial Satellites, Goddard Space Flight Center Document X-550-69-36, Feb. 1969.

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APPENDIX A

NOMENCLATURE

B	inclination function
C	eccentricity function
D	inclination function
F	Hamiltonian
F_s	Secular part of Hamiltonian
F(i)	inclination function
G	Delaunay variable $G = L\sqrt{1-e^2}$
G(e)	eccentricity function
H	Delaunay variable $H = G \cos i$
J_n	the n^{th} zonal harmonic
K	eccentricity function
L	Delaunay variable $L = \sqrt{\mu a}$
P	eccentricity function
P_n	zonal harmonic of n^{th} degree
S	determining function
T	inclination function
U	force function
a	semimajor axis
b	mean radius of earth
e	eccentricity

g Delaunay variable - argument of perigee

h Delaunay variable - right ascension of the node of the satellite

i inclination of satellite orbit

j summation index

k summation index

n appears as an index of the degree of the zonal harmonic

ℓ the Delaunay mean anomaly

q summation index

r radius of the satellite

$\dot{\alpha}$ mean motion of argument of perigee

$\mu = k^2 m_{\oplus}$

k = Gaussian constant

$m_{\oplus}$ = mass of earth

$e_{☾}$ eccentricity of moon's orbit (0.054900489)

i_{ec} inclination of moon's orbit plane to ecliptic $\sim 5^{\circ}.1453964$

$m_{\odot}'$ ratio of mass of sun to combined mass of sun and earth (0.999997)

$m_{☾}'$ ratio of mass of moon to combined mass of moon and earth (0.012150668)

$n_{\odot}$ mean motion of sun 0.98560027 degrees per day

$n_{☾}$ mean motion of moon 13.064999 degrees per day

ϵ obliquity of the ecliptic

APPENDIX B

RECURSION FORMULAS

When the Hamiltonian and perturbations in the elements for many zonal harmonics are required it is convenient to use recursion formulas to compute sequences of coefficients of the inclination and eccentricity functions.

In deriving the recursion formulas the indices n, q, j and k are varied. It should be noted that although the coefficients in Equations 10, 11, and 12 depend on n this index does not appear as a subscript in order to simplify the notation. However when the index n is varied the additional subscript n is added to the coefficient as in the formulas below.

The recursion formulas given below follow readily from the definitions in Equations 10, 11, and 12.

Recursion Formulas For the Inclination and Eccentricity Functions (n even)

$$K_{q,j,n+2} = \frac{(n+1)\,n}{(n+1-2j)(n-2j)} K_{q,j,n} \tag{B1}$$

$$B_{q,k,n+2} = -\frac{4(n+1+2k)}{n+2-2k} B_{q,k,n} \tag{B2}$$

$$K_{q,j+1} = \frac{1}{4}\frac{(n-2j-1)(n-2j-2)}{(j+1-q)(j+1+q)} K_{q,j} \tag{B3}$$

$$B_{q,k+1} = -\frac{1}{4}\frac{(n-2k)(n+2k+1)}{(k+1-q)(k+1+q)} B_{q,k} \tag{B4}$$

The above four formulas also apply to the secular part of the functions by setting $q = 0$.

$$K_{q+1,j} = \frac{j-q}{j+q+1} K_{q,j} \tag{B5}$$

$$B_{q+1,k} = \frac{q-k}{k+q+1} B_{q,k} \tag{B6}$$

<u>Recursion Formulas For the Inclination and Eccentricity Functions (n odd)</u>

$$C_{q,j,n+2} = \frac{n(n+1)}{(n-2j)(n-2j-1)} C_{q,j,n} \tag{B7}$$

$$D_{q,k,n+2} = \frac{-4(n+2k+2)}{n-2k+1} D_{q,k,n} \tag{B8}$$

$$C_{q,j+1} = \frac{1}{4} \frac{(n-2j-2)(n-2j-3)}{(j+1-q)(j+2+q)} C_{q,j} \tag{B9}$$

$$D_{q,k+1} = -\frac{1}{4} \frac{(n-2k-1)(n+2k+2)}{(k-q+1)(k+q+2)} D_{q,k} \tag{B10}$$

$$C_{q+1,j} = \frac{j-q}{j+q+2} C_{q,j} \tag{B11}$$

$$D_{q+1,k} = \frac{-(k-q)}{k+q+2} D_{q,k} \tag{B12}$$

APPENDIX C

THE MEAN MOTION OF THE ARGUMENT OF PERIGEE AND SOME USEFUL DERIVATIVES

The mean motion $\dot{\alpha}$ of the argument of perigee is given by

$$\dot{\alpha} = n\left\{\frac{3}{4}\frac{\mu^2 J_2 b^2}{G^4}\left(-1+5\frac{H^2}{G^2}\right)+\frac{3}{128}\frac{\mu^4 J_2^2 b^4}{G^8}\left[-35+24\frac{G}{L}+25\frac{G^2}{L^2}\right.\right.$$

$$\left.+\left(90-192\frac{G}{L}-126\frac{G^2}{L^2}\right)\frac{H^2}{G^2}+\left(385+360\frac{G}{L}+45\frac{G^2}{L^2}\right)\frac{H^4}{G^4}\right]$$

$$\left.-\frac{15}{128}\frac{\mu^4 J_4 b^4}{G^8}\left[21-96\frac{G^2}{L^2}+\left(-270+126\frac{G^2}{L^2}\right)\frac{H^2}{G^2}+\left(385-189\frac{G^2}{L^2}\right)\frac{H^4}{G^4}\right]\right\}$$

$$-\frac{3}{8}\frac{a^2}{G}\left(1-\frac{3}{2}\sin^2\epsilon\right)\left(\frac{G^2}{L^2}-5\frac{H^2}{G^2}\right)\left[\frac{n_\odot^2 m_\odot'}{\left(1-e_\odot^2\right)^{3/2}}+\frac{n_{\mathbb{C}}^2 m_{\mathbb{C}}'}{\left(1-e_{\mathbb{C}}^2\right)^{3/2}}\left(1-\frac{3}{2}\sin^2 i_{ec}\right)\right].$$

The quantity G $\partial\dot{\alpha}/\partial G$ and denoted by $\dot{\alpha}_G$ is

$$\dot{\alpha}_G = n\left\{\frac{3}{2}\frac{J_2 b^2}{a^2}\frac{L^4}{G^4}\left(2-15\cos^2 i\right)+\frac{3}{128}\frac{J_2^2 b^4}{a^4}\frac{L^8}{G^8}\left[\left(280-168\frac{G}{L}-150\frac{G^2}{L^2}\right)\right.\right.$$

$$\left.+36\left(-25+48\frac{G}{L}+28\frac{G^2}{L^2}\right)\cos^2 i-30\left(154+132\frac{G}{L}+15\frac{G^2}{L^2}\right)\cos^4 i\right]$$

$$\left.-\frac{15}{128}\frac{J_4 b^4}{a^4}\frac{L^8}{G^8}\left[\left(-168+54\frac{G^2}{L^2}\right)+36\left(75-28\frac{G^2}{L^2}\right)\cos^2 i+30\left(-154+63\frac{G^2}{L^2}\right)\cos^4 i\right]\right\}$$

$$-\frac{3}{8}\frac{a^2}{G}\left(1-\frac{3}{2}\sin^2\epsilon\right)\left[\frac{n_\odot^2 m_\odot'}{\left(1-e_\odot^2\right)^{3/2}}+\left(1-\frac{3}{2}\sin^2 i_{ec}\right)\frac{n_{\mathbb{C}}^2 m_{\mathbb{C}}'}{\left(1-e_{\mathbb{C}}^2\right)^{3/2}}\right]\left(\frac{G^2}{L^2}+15\cos^2 i\right).$$

The formula for G $\partial \dot{\alpha}/\partial H$ and denoted by $\dot{\alpha}_H$ is

$$\dot{\alpha}_H = n \left\{ \frac{15}{2} \frac{J_2 b^2}{a^2} \frac{L^4}{G^4} \cos i + \frac{3}{32} \frac{J_2^2 b^4}{a^4} \frac{L^8}{G^8} \left[\left(45 - 96 \frac{G}{L} - 63 \frac{G^2}{L^2}\right) \cos i \right. \right.$$

$$\left. + \left(385 + 360 \frac{G}{L} + 45 \frac{G^2}{L^2}\right) \cos^3 i \right]$$

$$\left. - \frac{15}{32} \frac{J_4 b^4}{a^4} \frac{L^8}{G^8} \left[9 \left(-15 + 7 \frac{G^2}{L^2}\right) \cos i + 7 \left(55 - 27 \frac{G^2}{L^2}\right) \cos^3 i \right] \right\}$$

$$+ \frac{15}{4} \frac{a^2}{G} \left(1 - \frac{3}{2} \sin^2 \epsilon\right) \left[\frac{n_\odot^2 m_\odot'}{\left(1 - e_\odot^2\right)^{3/2}} + \left(1 - \frac{3}{2} \sin^2 i_{ec}\right) \frac{n_☾^2 m_☾'}{\left(1 - e_☾^2\right)^{3/2}} \right] \cos i .$$