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By

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ABSTRACT

Differential dynamic programming is a technique, based on dynamic programming rather than the calculus of variations, for determining the optimal control function of a nonlinear system. Unlike conventional dynamic programming where the optimal cost function is considered globally, differential dynamic programming applies the principle of optimality in the neighborhood of a nominal, possibly nonoptimal, trajectory. This allows the coefficients of a linear or quadratic expansion of the cost function to be computed in reverse time along the trajectory: these coefficients may then be used to yield a new improved trajectory (i.e., the algorithms are of the "successive sweep" type).

A class of nonlinear control problems, linear in the control variables, is studied using differential dynamic programming. It is shown that for the free-end-point problem, the first partial derivatives of the optimal cost function are continuous throughout the state space, and the second partial derivatives experience jumps at switch points of the control function. A control problem that has an analytic solution is used to illustrate these points. The fixed-end-point problem is converted into an equivalent free-end-point problem by adjoining the end-point constraints to the cost functional using Lagrange multipliers: a useful interpretation for Pontryagin's adjoint variables for this type of problem emerges from this treatment.

The above results are used to devise new second- and first-order algorithms for determining the optimal bang-bang control by successively improving a nominal guessed control function. The usefulness of the proposed algorithms is illustrated by the computation of a number of control problem examples.

The correspondence provides by means of an error analysis justification for the neglect of certain terms in the derivation of these algorithms.

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Differential Dynamic Programming Methods for Solving Bang-Bang Control Problems

DAVID H. JACOBSON

Abstract—**Differential dynamic programming is a technique, based on dynamic programming rather than the calculus of variations, for determining the optimal control function of a nonlinear system. Unlike conventional dynamic programming where the optimal cost function is considered globally, differential dynamic programming applies the principle of optimality in the neighborhood of a nominal, possibly nonoptimal, trajectory. This allows the coefficients of a linear or quadratic expansion of the cost function to be computed in reverse time along the trajectory: these coefficients may then be used to yield a new improved trajectory (i.e., the algorithms are of the "successive sweep" type).**

A class of nonlinear control problems, linear in the control variables, is studied using differential dynamic programming. It is shown that for the free-end-point problem, the first partial derivatives of the optimal cost function are continuous throughout the state space, and the second partial derivatives experience jumps at switch points of the control function. A control problem that has an analytic solution is used to illustrate these points. The fixed-end-point problem is converted into an equivalent free-end-point problem by adjoining the end-point constraints to the cost functional using Lagrange multipliers: a useful interpretation for Pontryagin's adjoint variables for this type of problem emerges from this treatment.

The above results are used to devise new second- and first-order algorithms for determining the optimal bang-bang control by successively improving a nominal guessed control function. The usefulness of the proposed algorithms is illustrated by the computation of a number of control problem examples.

NOTATION

The following notation denotes the inner product of two n-dimensional vectors x and y:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i.$$

A power series expansion to second order of a scalar $V(x)$ about $\bar{x}$ (x an n-vector) is represented in the following way:

$$V(\bar{x} + \delta x) = V(\bar{x}) + \langle V_x, \delta x\rangle + \tfrac{1}{2}\langle \delta x, V_{xx}\, \delta x\rangle$$

where $V_x = \partial V/\partial x$ evaluated at $\bar{x}$ and $V_{xx} = \partial^2 V/\partial x^2$ evaluated at $\bar{x}$. V_x is an n-dimensional column vector and V_{xx} is an $n \times n$ symmetric matrix.

Further notational details are described in the text, when required.

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The author is with the Division of Engineering and Applied Physics, Harvard University, Cambridge, Mass.

The following abbreviations are used:

DDP differential dynamic programming
PDE partial differential equation
rhs right-hand side
l.h.s. left-hand side
w.r.t. with respect to.

INTRODUCTION

DIFFERENTIAL dynamic programming is a technique, based on dynamic programming rather than the calculus of variations, for determining optimal control for nonlinear systems [1]–[4]. In each iteration, the system equations (which yield the coefficients of a linear or quadratic expansion of the cost function in the neighborhood of the system trajectory) are integrated in reverse time to yield an improved control function.

In [3] and [4], it was shown that some advantages could be gained over existing methods of solving non-bang-bang control problems [1], [5]–[10] by allowing global variations in control (strong variations in the system trajectory). In particular, control inequality constrained problems of a certain class were solved.

This paper extends the strong variation techniques described in [3] and [4] to bang-bang control problems. In Section I, it is shown that for the free-end-point problem, DDP can be applied piecewise between switch times of the control function. At switch points, a special analysis is necessary to determine whether the partial derivatives of the optimal cost functional w.r.t. the state variables x are continuous. In general, it is found that the first partial derivatives are continuous, but that the second derivatives experience jumps. An example which has an analytic solution is used to illustrate these points.

End-point constraints are introduced into the control problem and again it is shown that dynamic programming can be used, if the end-point constraints are adjoined to the cost functional by Lagrange multipliers. A comparison is made between the DDP results and Pontryagin's Principle, and a new interpretation for Pontryagin's adjoint variables is given for this class of problems. The analysis confirms the results obtained by Berkovitz [11] and Dreyfus [12] and extends these by obtaining new second-order conditions for bang-bang problems.

In Section II, the preceding results are used to devise algorithms for determining the optimal bang-bang control by successively improving a nominal trajectory. New second-order and first-order algorithms for both free- and fixed-end-point problems are obtained; the usefulness of these algorithms is demonstrated by the computation of a number of control problem examples.

An advantage of these successive approximation algorithms over existing methods [13]–[16] is that the integration of the differential equations is done in the stable direction of motion, if the dynamic system is stable. This feature is present in the algorithms [1]–[7] and [10] for non-bang-bang problems (i.e., successive sweep algorithms). Up until the present time, algorithms designed for solving bang-bang control problems have been of the boundary value iteration type, requiring the integration of both system and adjoint equations in the *same* direction in time.

Recently, and independently, Dyer and McReynolds [17], [18] have obtained results similar to those described in this paper. However, the algorithms presented in this paper are simpler to implement in that the nominal assumed control function is not required to contain an a priori specified number of switchings. (In fact, the nominal control need not even be of bang-bang form.)

I. Theoretical Considerations

A. *The Problem*

The system under consideration is assumed to be governed by the following set of ordinary differential equations:[1]

$$\dot{x} = f_1(x; t) + f_2(x; t)u; \quad x(t_0) = x_0 \tag{1}$$

where f_1 is a nonlinear n-dimensional vector function of the state vector x and time t, and f_2 is a nonlinear $n \times m$ matrix function of x and t. u is an m-dimensional control vector.

The controls u_j; $j=1 \cdots m$, are assumed to be constrained in the following way:

$$u_j^b \leq u_j \leq u_j^a; \qquad j = 1 \cdots m \tag{2}$$

where the u_j^b and u_j^a are constants.

The problem is to choose $u(t)$; $t \in [t_0, t_f]$ to satisfy (2) and minimize

$$V(x_0; t_0) = \int_{t_0}^{t_f} L(x; t)dt + F(x(t_f); t_f) \tag{3}$$

where L and F are nonlinear scalar functions of x; t and $x(t_f)$; t_f, respectively. f_1, f_2, L, and F are assumed to be three times continuously differentiable.

It is well known that the optimal control function for the preceding problem is of bang-bang form. Bellman *et al.* [19], Lasalle [20], and Pontryagin *et al.* [21] are some who have studied conditions of optimality for problems linear in control. The problem of time optimal control of linear systems has been well researched, and [13]–[16] are some of the available computational techniques for solving this problem.[2]

Kelley *et al.* [22] have recently considered bang-bang thrust in rocket trajectory problems.

Friedland and Sarachik [23] have attempted to obtain neighboring optimal feedback control laws for the minimum time problem by straightforward differentiation of the adjoint and system equations. Their approach is, therefore, second variational in nature and is, in this sense, similar to the DDP approach used in this paper. However, the question of differentiability of the adjoint variables was not satisfactorily resolved in [23].

Define the Hamiltonian H in the following way:

$$H(x, u, V_x; t) = L(x; t) + \langle V_x, f_1(x; t) + f_2(x; t)u \rangle. \tag{4}$$

A necessary condition of optimality is that H be minimized w.r.t. u; this condition is satisfied if

$$\begin{aligned} u_j &= u_j^a; \qquad (f_2^T V_x)_j < 0; \quad j = 1 \cdots m \\ &= u_j^b; \qquad (f_2^T V_x)_j > 0; \quad j = 1 \cdots m. \end{aligned} \tag{5}$$

The control law (5) is called bang-bang because the u_j switch between their upper and lower bounds depending on the sign of $(f_2^T V_x)_j$. It is assumed that there is a finite number of such switching

$$t_{s_i}; \qquad i = 1 \cdots n_s. \tag{6}$$

Further, it is assumed that the $(f_2^T V_x)_j$; $j=1 \cdots m$ are not zero on a finite time interval, as otherwise the minimizing u_j cannot be determined using (5). Problems in which the $(f_2^T V_x)_j$ are zero on a finite time interval are referred to as "singular" and are not considered in this paper.

The following question arises when attempting to study the preceding problem using dynamic programming:

Do the partial derivatives of the optimal cost $V^o(x; t)$ w.r.t. x; t exist everywhere in the state space, and if not, how does one use the Bellman PDE which is derived, using dynamic programming, on the assumption that the optimal cost $V^o(x; t)$ has continuous partial derivatives up to and including the second?

Consider (3) and let $u_j = u_j^a$; $j=1 \cdots m$, say, for the whole interval $[t_0, t_f]$, i.e., constant controls. The resulting cost $V(x; t)$ has continuous partial derivatives w.r.t. x; t.[3] Now consider the case when the controls u_j are determined by (5); it is not clear that $V(x; t)$ will have continuous partial derivatives at switch points of these controls.

In view of the preceding, one can conclude that if the u_j are constant then Bellman's PDE is valid since $V(x; t)$ has continuous partial derivatives. If there are switch times t_{s_i} present where the u_j change discontinuously, then further investigation is required to discover exactly what happens to $V(x; t)$ and its partial derivatives at these times.

[1] A semicolon is used to separate the time t from other arguments.

[2] Numerous additional references are quoted in [13]–[16].

[3] If the control is constant, it can be shown that the partial derivatives of $V(x; t)$ obey linear differential equations that have continuous bounded solutions.

B. Differential Dynamic Programming

Differential dynamic programming is described fully in [2]–[4].[4] For completeness, a brief resume of these detailed descriptions is given here.

The Bellman PDE for the optimal cost is

$$-\frac{\partial V^o}{\partial t}(x; t) = \min_{\substack{u \\ u\in U}} [L(x; t) + \langle V_x{}^o(x; t), f_1(x; t) + f_2(x; t)u\rangle]. \quad (7)$$

Assume that the optimal control and state trajectory are unknown but that nominal values $\bar{u}(t)$, $\bar{x}(t)$; $t\in[t_0, t_f]$ are available. x and u may be written in terms of these nominal or reference trajectories:

$$\begin{aligned} u &= \bar{u} + \delta u \\ x &= \bar{x} + \delta x \end{aligned} \quad (8)$$

where δx and δu are state and control variables measured w.r.t. $\bar{x}$ and $\bar{u}$. Substituting (8) into (7),

$$\begin{aligned} -\frac{\partial V^o}{\partial t}(\bar{x} + \delta x; t) &= \min_{\substack{\delta u \\ \bar{u}+\delta u\in U}}[L(\bar{x} + \delta x, \bar{u} + \delta u; t) \\ &\quad + \langle V_x{}^o(\bar{x} + \delta x; t), f_1(\bar{x} + \delta x; t) \\ &\quad + f_2(\bar{x} + \delta x; t)(\bar{u} + \delta u)\rangle]. \end{aligned} \quad (9)$$

Assume that δx is, in some way, kept sufficiently small to justify a power series expansion for $V^o(\bar{x}+\delta x; t)$ up to second order in δx:

$$V(\bar{x} + \delta x; t) = V(\bar{x}; t) + \langle V_x, \delta x\rangle + \tfrac{1}{2}\langle \delta x, V_{xx}\delta x\rangle.$$

Writing this equation in terms of the nominal cost $\bar{V}(x; t)$,

$$\begin{aligned} V(\bar{x} + \delta x; t) &= \bar{V}(\bar{x}; t) + a(\bar{x}; t) + \langle V_x, \delta x\rangle \\ &\quad + \tfrac{1}{2}\langle \delta x, V_{xx}\delta x\rangle \end{aligned} \quad (10)$$

where $a(\bar{x}, t)$ is the change in cost from the nominal when starting in state $\bar{x}$; t and using a control function

$$u(\tau) = \bar{u}(\tau) + \delta u(\tau); \qquad \tau \in [t, t_f]. \quad (11)$$

The superscript o has been dropped in (10) because (10) is an expression for the cost *subject to the proviso that δx remains small*.[5] A method for keeping δx small is described in detail in [3] and [4] and briefly in Section II-B of this paper.

Substituting (10) into (9), the following equation is obtained:

$$\begin{aligned} -\frac{\partial \bar{V}}{\partial t} - \frac{\partial a}{\partial t} - \left\langle\frac{\partial V_x}{\partial t}, \delta x\right\rangle - \frac{1}{2}\left\langle \delta x, \frac{\partial V_{xx}}{\partial t}\delta x\right\rangle &= \min_{\substack{\delta u \\ \bar{u}+\delta u\in U}}[L(\bar{x} + \delta x; t) + \langle V_x + V_{xx}\delta x, \\ &\quad \cdot f_1(\bar{x} + \delta x; t) + f_2(\bar{x} + \delta x; t)(\bar{u} + \delta u)\rangle]. \end{aligned} \quad (12)$$

This is the basic PDE satisfied by the locally valid quadratic cost (10). Equation (12) can be used to develop algorithms for finding the optimal trajectory by successively improving a nominal guessed trajectory.[6]

C. Piecewise Differential Dynamic Programming

Consider conditions at $x=\bar{x}$ at time t; i.e., set $\delta x=0$ in (12). The r.h.s. of (12) becomes

$$\min_{\substack{\delta u \\ \bar{u}+\delta u\in U}}[L(\bar{x}; t) + \langle V_x, f_1(\bar{x}; t) + f_2(\bar{x}; t)(\bar{u} + \delta u)\rangle]. \quad (13)$$

Carrying out the minimization in expression (13) using the rules of (5), one obtains the minimizing control

$$u^* = \bar{u} + \delta u^* = u^+, \qquad \text{say} \quad (14)$$

where the components of u^+ are given by (5). Substituting (14) into (13) one obtains

$$L(\bar{x}; t) + \langle V_x, f_1(\bar{x}; t) + f_2(\bar{x}; t)u^+\rangle. \quad (15)$$

Now consider allowing variations δx about $\bar{x}$, i.e., reintroduce δx into (15). Assume that the minimizing u^* remains u^+, except in the neighborhood of switch points of u^+ (these are studied in Section I-D), so expression (15) becomes

$$\begin{aligned} &L(\bar{x} + \delta x; t) \\ &\quad + \langle V_x + V_{xx}\delta x, f_1(\bar{x} + \delta x; t) + f_2(\bar{x} + \delta x; t)u^+\rangle. \end{aligned} \quad (16)$$

Expanding (16) to second order in δx and recalling that this is equal to the l.h.s. of (12), the following equation is obtained:

$$\begin{aligned} -\frac{\partial \bar{V}}{\partial t} - \frac{\partial a}{\partial t} - \left\langle\frac{\partial V_x}{\partial t}, \delta x\right\rangle - \frac{1}{2}\left\langle \delta x, \frac{\partial V_{xx}}{\partial t}\delta x\right\rangle &= H + \langle H_x, \delta x\rangle + \langle V_{xx}f, \delta x\rangle \\ &\quad + \frac{1}{2}\langle \delta x, (H_{xx} + f_x{}^T V_{xx} + V_{xx}f_x)\delta x\rangle. \end{aligned} \quad (17)$$

Equation (17) holds for all δx sufficiently small, so coefficients of like powers of δx may be equated to obtain

$$\begin{aligned} -\frac{\partial \bar{V}}{\partial t} - \frac{\partial a}{\partial t} &= H \\ -\frac{\partial V_x}{\partial t} &= H_x + V_{xx}f \\ -\frac{\partial V_{xx}}{\partial t} &= H_{xx} + f_x{}^T V_{xx} + V_{xx} + V_{xx}f_x \end{aligned} \quad (18)$$

[4] McReynolds [7] has used a similar approach to optimal control problems.

[5] If $\bar{x}(t),\bar{u}(t)$ are sufficiently close to optimal then this proviso is automatically satisfied; (10) then becomes an expression for the optimal cost $V^o(\bar{x}+\delta x; t)$ in the neighborhood of the optimal trajectory.

[6] See [1]–[4].

where all the above quantities are evaluated at $\bar{x}$, u^*, and

$$f = f_1(\bar{x}; t) + f_2(\bar{x}; t)u^*. \tag{19}$$

The quantities $V = \bar{V} + a$, V_x, and V_{xx} are functions of $\bar{x}$ and t so

$$\begin{aligned} \frac{d}{dt}(\bar{V} + a) &= \frac{\partial}{\partial t}(\bar{V} + a) + \langle V_x, f(\bar{x}, \bar{u}; t)\rangle \\ \dot{V}_x &= \frac{\partial V_x}{\partial t} + V_{xx} f(\bar{x}, \bar{u}; t) \\ \dot{V}_{xx} &= \frac{\partial V_{xx}}{\partial t}, \end{aligned} \tag{20}$$

since higher-order terms in the expansion for V have been truncated.

Using (20) in (18) and noting that $-\dot{\bar{V}} = L(\bar{x}; t)$,

$$\begin{aligned} -\dot{a} &= H - H(\bar{x}, \bar{u}, V_x; t) \\ -\dot{V}_x &= H_x + V_{xx}(f - f(\bar{x}, \bar{u}; t)) \\ -\dot{V}_{xx} &= H_{xx} + f_x{}^T V_{xx} + V_{xx} f_x \end{aligned} \tag{21}$$ [7]

where all the quantities are evaluated at $\bar{x}$, u^* unless otherwise stated. At $t = t_f$, from (3) one obtains

$$\begin{aligned} a(t_f) &= 0 \\ V_x(t_f) &= F_x(\bar{x}(t_f) : t_f) \\ V_{xx}(t_f) &= F_{xx}(\bar{x}(t_f); t_f). \end{aligned} \tag{22}$$

Equations (21) can be integrated backward from t_f using boundary conditions (22) and $u^* = u^+$.

There will probably come a time t_s where a component of u^* changes discontinuously (i.e., switches) so that u^* changes from u^+ to u^- say.[8] Conditions at t_s must now be studied to determine what happens to a, V_x, and V_{xx}.

D. Conditions at Switch Points

Using (21) and (22), one is able to calculate $a(t)$, $V_x(t)$, and $V_{xx}(t)$ for $t \in [t_s, t_f]$. Let the cost for $t \geq t_s$ be denoted by

$$V^+(\bar{x} + \delta x; t) = \bar{V}^+ + a^+ + \langle V_x{}^+, \delta x\rangle + \tfrac{1}{2}\langle \delta x, V_{xx}{}^+ \delta x\rangle \tag{23}$$

where superscript + denotes quantities for $t \geq t_s$. For $t \in [t_0, t_s]$, the cost is given by

$$V^-(x; t) = \int_t^{t_s} L(x(\tau); \tau)d\tau + V^+(x(t_s); t_s) \tag{24}$$

where superscript − denotes the cost for $t \leq t_s$ in the region where $u^* = u^-$. One can consider V^- to be function of the switch time t_s, so let us consider t_s to be a parameter of V^- and write it explicitly as such:

$$V^-(x, t_s; t) = \int_t^{t_s} L(x(\tau); \tau)d\tau + V^+(x(t_s); t_s). \tag{25}$$

Now let us allow variations in this switch time of δt_s, and let us write $x(\tau)$ and $x(t_s)$ in terms of the nominal value $\bar{x}(t)$:

$$\begin{aligned} V^-(\bar{x}(t) + \delta x(t), t_s + \delta t_s; t) &= \int_t^{t_s + \delta t_s} L(\bar{x}(t) + \delta x(t) + \Delta x(\tau); \tau)d\tau \\ &\quad + V^+(\bar{x}(t) + \delta x(t) + \Delta x(t_s + \delta t_s); t_s + \delta t_s) \end{aligned} \tag{26}$$

where $\Delta x(\tau)$ is the change in state over the time interval $\tau - t$ starting from $\bar{x}(t) + \delta x(t)$, i.e.,

$$\Delta x(\tau) = \int_t^{\tau} f(\bar{x}(\tau_1) + \delta x(\tau_1), u^-; \tau_1)d\tau_1. \tag{27}$$

Conditions are to be studied at time $t = t_s$, so let us observe (26) at $t = t_s$:[9]

$$\begin{aligned} V^-(\bar{x}(t_s) + \delta x(t_s), t_s + \delta t_s; t_s) &= \int_{t_s}^{t_s + \delta t_s} L(\bar{x}(t_s) + \delta x(t_s) + \Delta x(\tau); \tau)d\tau \\ &\quad + V^+(\bar{x}(t_s) + \delta x(t_s) + \Delta x(t_s + \delta t_s); t_s + \delta t_s) \end{aligned} \tag{28}$$

Expanding the r.h.s. of (28) to second order in δx and δt_s about $\bar{x}$; t_s, the following quantity is obtained:[10]

$$\begin{aligned} &V^+ + \langle V_x{}^+, \delta x\rangle + \tfrac{1}{2}\langle \delta x, V_{xx}{}^+ \delta x\rangle + (H^- - H^+)\delta t_s \\ &+ \langle \delta x, H_x{}^- - H_x{}^+ + V_{xx}{}^+(f^- - f^+)\rangle \delta t_s \\ &+ \tfrac{1}{2}\{H_t{}^- - H_t{}^+ - \langle H_x{}^-, f^- - f^+\rangle \\ &+ \langle f^-, H_x{}^- - H_x{}^+\rangle + \langle f^- - f^+, V_{xx}{}^+(f^- - f^+)\rangle\}\delta t_s{}^2 \end{aligned} \tag{29}$$

(quantities evaluated at $\bar{x}$; t_s) where

$$\left.\begin{aligned} &H^{\mp}(\bar{x}, u^{\mp}, V_x{}^+; t_s) = L(\bar{x}; t_s) + \langle V_x{}^+, f(\bar{x}, u^{\mp}; t_s)\rangle \\ &\text{and} \\ &f^{\mp} = f(\bar{x}, u^{\mp}; t_s) = f_1(\bar{x}; t_s) + f_2(\bar{x}; t_s)u^{\mp}. \end{aligned}\right\} \tag{30}$$

Expanding the l.h.s. of (28) to second-order in δx and δt_s about $\bar{x}$, t_s, one obtains

$$\begin{aligned} &V^-(\bar{x}, t_s; t_s) + \langle V_x{}^-, \delta x\rangle + V_{t_s}{}^- \delta t_s + \langle V_{xt_s}{}^-, \delta x\rangle \delta t_s \\ &\quad + \tfrac{1}{2}\langle \delta x, V_{xx}{}^- \delta x\rangle + \tfrac{1}{2}V_{t_s t_s}{}^- \delta t_s{}^2. \end{aligned} \tag{31}$$

Equating coefficients of like powers of δx and δt_s in ex-

[7] The fact that the $\dot{V}_{xx}$ equation is linear means that there is no conjugate point condition for these problems. A way in which V_{xx} could become unbounded is if strict equality holds in (35).

[8] In general, if more than one component (say r, where $r \leq m$) of u^* changes at the same time t_s, it may be necessary to consider the separate changes in switch times $\delta t_{s_1}, \delta t_{s_2} \cdots \delta t_{s_r}$, for each control in the analysis to follow; this complicates the derivation. This paper treats only the case where one component of the control switches at time t_s.

[9] $V^-(\bar{x}(t_s) + \delta x(t_s), t_s + \delta t_s; t_s)$ is the cost at time $t = t_s$ as a function of the parameters $\bar{x}(t_s) + \delta x(t_s)$ and $t_s + \delta t_s$. The semicolon separates these parameters from the actual time $t = t_s$ at which V^- is observed. Thus, $V_{t_s}{}^-(\bar{x}(t), t_s; t)$ is the partial derivative of V^- w.r.t. the parameter t_s evaluated at $\bar{x}(t)$, t_s; t. So $V_{t_s}{}^-(\bar{x}(t_s), t_s; t_s)$ is this derivative observed at time t_s. $V_t{}^-(\bar{x}(t), t_s; t)$ is the partial derivative of V^- w.r.t. the time t evaluated at $\bar{x}(t)$, t_s; t. So $V_t{}^-(\bar{x}(t_s), t_s; t_s)$ is this derivative observed at time $t = t_s$. For convenience, and where the meaning is clear, $\bar{x}(t)$ will be written as $\bar{x}$.

[10] Details of the algebra of this expansion are to be found in the Appendix.

pressions (30) and (31) yields

$$\begin{aligned} V^-(\bar{x}, t_s; t_s) &= V^+(\bar{x}; t_s) \\ V_x^- &= V_x^+ \\ V_{xx}^- &= V_{xx}^+ \\ V_{t_s}^- &= H^- - H^+ \\ V_{xt_s}^- &= H_x^- - H_x^+ + V_{xx}^+(f^- - f^+) \\ V_{t_st_s}^- &= H_t^- - H_t^+ - \langle H_x^-, f^- - f^+ \rangle \\ &\quad + \langle f^-, H_x^- - H_x^+ \rangle \\ &\quad + \langle f^- - f^+, V_{xx}^+(f^- - f^+)\rangle. \end{aligned} \quad (32)^{11}$$

The quantities in (32) are sensitivities of the cost V^- w.r.t. the parameters x and t_s at time $t=t_s$.

From (32)

$$V_{t_s}^- = H^- - H^+ = \langle V_x^+, f_2[u^- - u^+]\rangle. \quad (33)$$

From (33)

$$V_{t_s}^- = 0 \quad (34)$$

since either $u_j^- = u_j^+$ or $(f_2^T V_x)_j = 0$.

Equation (34) is a necessary condition for V^- to be minimized w.r.t. the switch time t_s. A further necessary condition is

$$V_{t_st_s}^-(\bar{x}, t_s; t_s) \geq 0. \quad (35)$$

From (31)

$$\frac{\partial V^-}{\partial t_s}(\bar{x} + \delta x, t_s + \delta t_s; t_s) = V_{t_s}^- + \langle V_{xt_s}^-, \delta x\rangle + V_{t_st_s}^- \delta t_s. \quad (36)$$

In order to maintain the necessary condition of optimality (34) for the case where variations δx are present, it is required that

$$V_{t_s}^-(\bar{x} + \delta x, t_s + \delta t_s; t_s) = 0. \quad (37)$$

Using (37), (36), and (34), the following relationship is obtained:

$$\delta t_s = -\frac{1}{V_{t_st_s}^-}\langle V_{xt_s}^-, \delta x\rangle. \quad (38)$$

Equation (38) is an optimal local linear feedback controller relating the required change in switch time δt_s to the δx appearing at t_s.

Substituting (38) into (31) in order to eliminate δt_s, one obtains

$$\begin{aligned} V^-\left(\bar{x} + \delta x, t_s - \frac{1}{V_{t_st_s}^-}\langle V_{xt_s}^-, \delta x\rangle; t_s\right) &= V^-(\bar{x}, t_s; t_s) + \langle V_x^-, \delta x\rangle \\ &\quad + \frac{1}{2}\left\langle \delta x, \left(V_{xx}^- - \frac{V_{xt_s}^- \cdot V_{t_sx}^-}{V_{t_st_s}^-}\right)\delta x\right\rangle. \end{aligned} \quad (39)$$

[11] Obtained independently by Dyer and McReynolds [17], [18]. For simplicity of presentation, arguments are omitted. It should be understood that the quantities are evaluated at $\bar{x}(t_s); t_s$.

Recalling (32) and renaming the l.h.s. of (39) as $\hat{V}(\bar{x}+\delta x; t_s)$, the following relations result:

$$\begin{aligned} \hat{V}(\bar{x}; t_s) &= V^+(\bar{x}; t_s) \\ \hat{V}_x &= V_x^+ \\ \hat{V}_{xx} &= V_{xx}^+ - \frac{V_{xt_s}^- \cdot V_{t_sx}^-}{V_{t_st_s}^-}. \end{aligned} \quad (40)$$

With no loss of correctness, the superscripts + and − may now be dropped to yield

$$\begin{aligned} \hat{V}(\bar{x}; t_s) &= V(\bar{x}; t_s) \\ \hat{V}_x &= V_x \\ \hat{V}_{xx} &= V_{xx} - \frac{V_{xt_s} \cdot V_{t_sx}}{V_{t_st_s}}. \end{aligned} \quad (41)$$

The following points are noteworthy:

1) at $t=t_s$, $a(\bar{x}; t)$ is continuous since $\hat{V}=V$ and $\overline{V}(\bar{x}; t)$ is continuous
2) at $t=t_s$, V_x is continuous
3) at $t=t_s$, V_{xx} experiences a jump of magnitude

$$\Delta V_{xx} = -V_{xt_s} \cdot V_{t_sx}/V_{t_st_s}. \quad (42)$$

At a switch point, the jump in V_{xx} is readily computed using (32) and (42); (41) then becomes the new boundary condition for (21) which can continue to be integrated backwards, noting now that $u^* = u^-$.

The preceding analysis is clearly applicable to a control function u^* having any finite number of switch times.

The preceding theory was developed for $\bar{x}$, $\bar{u}$ nonoptimal. On an optimal trajectory (a special case) all the results hold.

E. An Example

Consider the following control problem:

$$\begin{aligned} \dot{x}_1 &= x_2; \qquad x_1(t_0) = x_{10} \\ \dot{x}_2 &= u; \qquad x_2(t_0) = x_{20} \\ |u| &\leq 1. \end{aligned} \quad (43)$$

Minimize

$$V = \int_{t_0}^{\infty} x_1^2 dt. \quad (44)$$

Fuller [24] has analytic expressions for the optimal cost $V^o(x_1, x_2)$. (V^o is independent of t because $t_f = \infty$.)

The question now is whether the predictions that V_x^o is continuous at a switch point and V_{xx}^o experiences a jump hold for this problem.

Differentiation of the analytic expressions for $V^o(x_1, x_2)$ does, indeed, confirm these predictions and, moreover, the jump in V_{xx}^o agrees with that predicted by (42).

For this problem,

$$H(x, u, V_x; t) = x_1^2 + V_{x_1}x_2 + V_{x_2}u \quad (45)$$

where the minimizing control is

$$u^* = -\operatorname{sign} V_{x_2}. \tag{46}$$

(The optimal superscript o is dropped, for convenience.)

Fuller has obtained the following expressions for the optimal cost. $V^N(x_1, x_2)$ (the cost surface when $u=-1$) is given by

$$V^N(x_1, x_2) = x_1^2x_2 - \frac{2}{3}x_1x_2^3 + \frac{2}{15}x_2^5 + 0.764\left(x_1 + \frac{1}{2}x_2^2\right)^{5/2} \tag{47}$$

$V^P(x_1, x_2)$ (the cost surface when $u=+1$) is given by

$$V^P(x_1, x_2) = -x_1^2x_2 + \frac{2}{3}x_1x_2^3 - \frac{2}{15}x_2^5 + 0.764\left(-x_1 + \frac{1}{2}x_2^2\right)^{5/2}. \tag{48}$$

On the switching curve,

$$\begin{aligned} x_1 &= -0.4446x_2^2; \qquad x_2 > 0 \\ &= 0.4446x_2^2; \qquad x_2 < 0. \end{aligned} \tag{49}$$

Consider only the case where $x_2<0$. Substituting (49) into (47) and (48),

$$\begin{aligned} V^N(0.4446x_2^2, x_2) &= V^P(0.4446x_2^2, x_2) \\ &= -0.035x_2^5, \end{aligned} \tag{50}$$

.e., V is continuous at a switch point of the control.

From (47) and (48),

$$V_{x_1}{}^N(x_1, x_2) = 2x_1x_2 + \frac{2}{3}x_2^3 + \frac{5}{2}\cdot 0.764\left(x_1 + \frac{1}{2}x_2^2\right)^{3/2} \tag{51}$$

$$V_{x_1}{}^P(x_1, x_2) = -2x_1x_2 + \frac{2}{3}x_2^5 - \frac{5}{2}\cdot 0.764\left(-x_1 + \frac{1}{2}x_2^2\right)^{3/2}. \tag{52}$$

Using (51), (52), and (49),

$$V_{x_1}{}^N = V_{x_1}{}^P = -0.2x_2^3 \text{ at a switch point,} \tag{53}$$

i.e., V_{x_1} is continuous at a switch point. It can be shown easily that V_{x_2} is also continuous.

$$\begin{aligned} V_{x_1x_1}{}^N(x_1, x_2) &= 2x_2 + \frac{15}{4}\cdot 0.764\left(x_1 + \frac{1}{2}x_2^2\right)^{1/2} \\ &= -0.79x_2 \text{ at a switch point} \end{aligned} \tag{54}$$

$$\begin{aligned} V_{x_1x_1}^P(x_1, x_2) &= -2x_2 + \frac{15}{4}\cdot 0.764\left(-x_1 + \frac{1}{2}x_2^2\right)^{1/2} \\ &= -2.67x_2 \text{ at a switch point.} \end{aligned} \tag{55}$$

So, crossing the switch point along an optimal path, from N to P, $V_{x_1x_1}$ experiences a jump of

$$\Delta V_{x_1x_1} = -0.79x_2 + 2.67x_2 = 1.88x_2. \tag{56}$$

Consider the expression for V_{xt_s} and $V_{t_st_s}$ given by (32):

$$V_{xt_s} = V_{xx}{}^P(f^N - f^P) = -2V_{xx}{}^P\cdot\begin{bmatrix}0\\1\end{bmatrix}. \tag{57}$$

So

$$V_{xt_s}\cdot V_{t_sx} = 4\begin{bmatrix} V_{x_1x_2}^2 & V_{x_1x_2}\cdot V_{x_2x_2} \\ V_{x_2x_2}\cdot V_{x_1x_2} & V_{x_2x_2}^2 \end{bmatrix} \tag{58}$$

$$\begin{aligned} V_{t_st_s} &= -H_x{}^P(f^N - f^P) + \langle f^N - f^P, V_{xx}{}^P(f^N - f^P)\rangle \\ &= 2V_{x_1}{}^P + 4V_{x_2x_2}{}^P. \end{aligned} \tag{59}$$

Now

$$\begin{aligned} &V_{x_1x_2}{}^P \\ &= -2x_1 + 2x_2^2 - \frac{15}{4}\cdot 0.764\left(-x_1 + \frac{1}{2}x_2^2\right)^{1/2} \\ &= 1.785x_2^2 \text{ at a switch point} \end{aligned} \tag{60}$$

and

$$\begin{aligned} &V_{x_2x_2}{}^P \\ &= 4x_1x_2 - \frac{8}{3}x_2^3 + 0.764\left(-x, + \frac{1}{2}x_2^2\right)^{1/2}x_2^2 \\ &\quad + \frac{15}{4}\cdot 0.764\left(-x_1 + \frac{1}{2}x_2^2\right)^{3/2} \\ &= -1.59x_2^3 \text{ at a switch point.} \end{aligned} \tag{61}$$

Substituting (60) and (61) into (58) and (59), and using (42),

$$\Delta V_{x_1x_1} = -\frac{(1.785x_2^2)^2}{-1.59x_2^3 - .1x_2^3} = \frac{(1.785)^2}{1.69}x_2 = 1.88x_2. \tag{62}$$

The predicted jump given by (62) is the same as the actual jump given by (56). Similar agreement can be obtained for the other elements of V_{xx} and for parts of the state space where $x_2>0$.

F. End-Point Equality Constraints

In this section, constraints of the following form are treated:

$$\psi(x(t_f); t_f) = 0 \tag{63}$$

where ψ is an $s\leq n$ vector function. It is assumed that t_f is given explicitly. The case where t_f is given implicitly is discussed later.

Assume, for argument's sake, that $u_j=u_j{}^a$; $j=1 \cdots m$ for the whole time interval, and further that (63) is satisfied using this control. In Section I-A, it was asserted that $V(x; t)$ has continuous partial derivatives for this constant control in the absence of constraints (63). Now, however, these constraints are present and so there may be variations δx at time t, say, which are not allowed because they cause violation of these constraints. This is the same as saying that the partial derivatives of V w.r.t. x may not be defined everywhere in the state space, even for the case where the control is held constant and there are no switchings.

One may, however, convert this constrained problem into a free-end-point problem by adjoining (63) to the cost functional using a vector Lagrange multiplier k of dimension s:[12]

$$V(x, k; t_0) = \int_{t_0}^{t_f} L(x; t)dt + F(x(t_f); t_f) + \langle k, \psi(x(t_f); t_f)\rangle. \quad (64)$$

For a nominal $k=\bar{k}$, this is a free-end-point problem; assume that it has a solution.

G. Piecewise Differential Dynamic Programming for End-Point Constrained Bang-Bang Problems

Consider V to be a function of the multipliers k and assume a second-order expansion for V about the nominal $\bar{x}(t)$ trajectory and the nominal multipliers $\bar{k}$:

$$V(\bar{x} + \delta x, \bar{k} + \delta k; t) = \bar{V} + a + \langle V_x, \delta x\rangle + \langle V_k, \delta k\rangle + \langle \delta x, V_{xk}\delta k\rangle + \tfrac{1}{2}\langle \delta k, V_{kk}\delta k\rangle + \tfrac{1}{2}\langle \delta x, V_{xx}\delta x\rangle. \quad (65)$$

Substituting (65) into the Bellman PDE (7) and carrying out a derivation similar to that of Section I-C, the following equations for the parameters of V are obtained:

$$\left.\begin{aligned} -\dot{a} &= H - H(\bar{x}, \bar{u}, V_x; t)\\ -\dot{V}_x &= H_x + V_{xx}(f - f(\bar{x}, \bar{u}; t))\\ -\dot{V}_{xx} &= H_{xx} + f_x^T V_{xx} + V_{xx} f_x\\ -\dot{V}_k &= V_{xk}^T(f - f(\bar{x}, \bar{u}; t))\\ -\dot{V}_{xk} &= f_x^T V_{xk}\\ -\dot{V}_{kk} &= 0 \end{aligned}\right\} \quad (66)$$

and at $t=t_f$, from (64),

$$\left.\begin{aligned} a(t_f) &= 0\\ V_x(t_f) &= F_x(\bar{x}(t_f); t_f) + \psi_x^T(\bar{x}(t_f); t_f)\bar{k}\\ V_{xx}(t_f) &= F_{xx}(\bar{x}(t_f); t_f) + \bar{k}\psi_{xx}(\bar{x}(t_f); t_f)\\ V_k(t_f) &= \psi(\bar{x}(t_f); t_f)\\ V_{xk}(t_f) &= \psi_x^T(\bar{x}(t_f); t_f)\\ V_{kk}(t_f) &= 0. \end{aligned}\right\} \quad (67)$$

H. Conditions at Switch Points

Following the same approach as that of Section I-D, the following equation in the vicinity of a switch point results:

$$V^-(\bar{x} + \delta x, \bar{k} + \delta k, t_s + \delta t_s; t_s) = \int_{t_s}^{t_s+\delta t_s} L(\bar{x} + \delta x + \Delta x(\tau); \tau)d\tau + V^+(\bar{x} + \delta x + \Delta x(t_s + \delta t_s), \bar{k} + \delta k; t_s + \delta t_s). \quad (68)$$

Expanding both sides of (68) to second order, and equating coefficients of like powers of δx, δk, and δt_s, the following relationships are obtained:[13]

$$\left.\begin{aligned} \hat{V}(\bar{x}, \bar{k}; t_s) &= V(\bar{x}, \bar{k}; t_s)\\ \hat{V}_x &= V_x\\ V_k &= V_k\\ \hat{V}_{xk} &= V_{xk} - V_{xt_s}\cdot V_{t_s k}/V_{t_s t_s}\\ \hat{V}_{xx} &= V_{xx} - V_{xt_s}\cdot V_{t_s x}/V_{t_s t_s}\\ \hat{V}_{kk} &= V_{kk} - V_{kt_s}\cdot V_{t_s k}/V_{t_s t_s} \end{aligned}\right\} \quad (69)$$

where

$$V_{kt_s} = V_{xk}^T(f^- - f^+) \quad (70)$$

and V_{xt_s} and $V_{t_s t_s}$ are given by (32).

The following local linear controller relates δt_s to δk and $\delta x(t_s)$:[14]

$$\delta t_s = - V_{t_s t_s}^{-1}[\langle V_{kt_s}, \delta k\rangle + \langle V_{xt_s}, \delta x\rangle]. \quad (71)$$

I. The Case Where Final Time t_f Is Given Implicitly

Here t_f is treated in the same way as k was in Section I-F, i.e., t_f is imbedded in V:

$$V(x_0, k, t_f; t_0) = \int_{t_0}^{t_s} L(x; t)dt + F(x(t_f); t_f) + \langle k, \psi(x(t_f); t_f)\rangle. \quad (72)$$

In a way similar to that demonstrated in previous sections, differential equations and jump conditions can be obtained for V_{t_f}, V_{xt_f}, V_{kt_f}, and $V_{t_f t_f}$.

A relationship of the form[15]

$$\delta t_s = - V_{t_s t_s}^{-1}[V_{t_f t_s}\delta t_f + \langle V_{kt_s}, \delta k\rangle + \langle V_{xt_s}, \delta x\rangle] \quad (73)$$

relates the changes in switch times to changes in x, k and t_f.

The implicitly given final time problem is mentioned only briefly here, because similar non-bang-bang, problems and their solutions are to be found in [5], [6], and [9].

J. Dynamic Programming and Pontryagin's Minimum Principle

In this section, certain comparisons are drawn between dynamic programming and Pontryagin's principle; in particular, a new interpretation for Pontryagin's adjoint variables λ is given for the class of problems treated in this paper.

[12] Sufficient conditions to guarantee the existence of a set of multipliers k which results in the satisfaction of (63) are given in Section II-I.

[13] Section I-D.

[14] Equation (71) is obtained in a fashion exactly analogous to (38).

[15] Equation (73) is obtained in a fashion exactly analogous to (38).

For the free-end-point problem, it is clear that $\lambda = V_x^o$ because V_x^o has been shown to be continuous and to satisfy the same differential equation as λ. The case of fixed-end-point problems is not as simple. Consider the cost functional[16]

$$V(x_0, k; t_0) = \int_{t_0}^{t_f} L(x; t)dt + F(x(t_f); t_f) + \langle k, \psi(x(t_f); t_f)\rangle. \quad (74)$$

Equation (74) describes the optimal cost for the free-end-point problem obtained by adjoining ψ to (3), i.e., V is minimized and k is chosen such that

$$\psi(x(t_f); t_f) = 0. \quad (75)$$

Consider also the following cost functional:

$$V^o(x_0; t_0) = \int_{t_0}^{t_f} L(x; t)dt + F(x(t_f); t_f) \quad (76)$$

$$\psi(x(t_f); t_f) = 0. \quad (77)$$

Equation (76) describes the optimal cost[17] for the fixed-end-point problem obtained without adjoining (77). Both in (74) and (76), t_f is assumed to be given explicitly.

Dynamic programming in its continuous form requires the optimal cost surfaces $V(x, k; t)$ and $V^o(x; t)$ to have continuous partial derivatives w.r.t. x, k, and t for the derivation of the Bellman PDE to be valid. In this paper, it has been shown that $V(x, k; t)$ has continuous partial derivatives, except at switch points of the control function where special jump conditions have been investigated.

It is well known that in problems with bang-bang control and fixed-end-point, the optimal cost surface V^o is nonsmooth across some switching surfaces in the state space. In fact, V_x^o the first partial derivative of V^o w.r.t. x is not continuous across some switching surfaces.[18] It so happens that the optimal trajectory never crosses these surfaces (where V_x^o is not defined) but travels along them [12].

It is the purpose of this section to obtain an interpretation for Pontryagin's adjoint variable λ for this class of problems where the system trajectory is tangent to the manifold of discontinuous control.

Berkovitz [11] has related Pontryagin's principle to dynamic programming, but does not consider this tangency case.

One is able to convert the free-end-point formulation (74) into the fixed-end-point formulation (76) and (77) locally in the following way.

[16] In this section, optimal control and state values are denoted by u and x with no superscripts or bars.

[17] From (76) onward, superscript o on V denotes optimal cost for the problem with fixed-end-point. No superscript denotes optimal cost for the problem where the end constraints have been adjoined to the cost functional by Lagrange multipliers k.

[18] Some references of direct relevance are [12], [15], [24]–[26].

In the neighborhood of an optimal trajectory, one has to second order

$$V(x + \delta x, k + \delta k; t) = V(x, k; t) + \langle V_x, \delta x\rangle + \langle V_k, \delta k\rangle + \langle \delta x, V_{xk}\delta k\rangle + \tfrac{1}{2}\langle \delta k, V_{kk}\delta k\rangle + \tfrac{1}{2}\langle \delta x, V_{xx}\delta x\rangle. \quad (78)$$

Because the trajectory is optimal,

$$V_k(x, k; t) = \psi(x(t_f); t_f) = 0. \quad (79)$$

Now let us eliminate δk from (78) in such a way that the end-point condition is still satisfied, i.e., determine δk such that

$$\psi(x(t_f) + \delta x(t_f); t_f) = 0. \quad (80)$$

From (78), (79), and (80),

$$V_k(x + \delta x, k + \delta k; t) = V_{xk}^T\delta x + V_{kk}\delta k = 0 \quad (81)$$

whence

$$\delta k = - V_{kk}^{-1}V_{xk}^T\delta x. \quad (82)$$

Substituting (82) into (78),

$$V(x + \delta x, k - V_{kk}^{-1}V_{xk}^T\delta x; t) = V + \langle V_x, \delta x\rangle + \tfrac{1}{2}\langle \delta x, (V_{xx} - V_{xk}V_{kk}^{-1}V_{kx})\delta x\rangle. \quad (83)$$

This is an expression for the optimal cost locally for the fixed-end-point formulation, so in view of (76) and (77) which represent just this problem, one may write (83) as

$$V^o(x + \delta x; t) = V^o + \langle V_x^o, \delta x\rangle + \tfrac{1}{2}\langle \delta x, (V_{xx} - V_{xk}V_{kk}^{-1}V_{kx})\delta x\rangle. \quad (84)$$

Note that V_x^o for this optimal cost function is the same as the V_x for the optimal free-end-point problem, but that

$$V_{xx}^o = V_{xx} - V_{xk}V_{kk}^{-1}V_{kx}. \quad (85)$$

However, (85) holds only if V_{kk} is invertible; so the transformation (82) cannot be done unless V_{kk} is invertible.[19] This means that one cannot convert the free-end-point formulation (74) into the fixed-end-point formulation (76) and (77) in regions of the state space where V_{kk} is not invertible, and so V_x^o cannot be identified with V_x. Since the equations for Pontryagin's λ are exactly the same as those for V_x for the free-end-point problem (and also the same for V_x^o when (82) and (83) are valid), one has the following simple interpretation for λ.

In regions where V_{kk} is invertible, $\lambda = V_x^o = V_x$. In regions where V_{kk} is not invertible, $\lambda = V_x$.

[19] Conditions for this are given in Section II-I. At this point, it is sufficient to note that V_{kk} can be invertible only after at least S switchings of the control function. This is because $\dot{V}_{kk}=0$ and V_{kk} experiences a jump at switch points of the control function.

The preceding section does, it is believed, help to close the conceptual gap between dynamic programming and Pontryagin's principle. The difference in approach between the preceding study and that of others (notably [26]) is that λ is identified with a quantity V_x which is the partial derivative of the optimal cost for the equivalent free-end-point problem. The usual approach is to try and identify λ with a quantity associated with the optimal cost function V^o; this is difficult. Dreyfus [12, p. 204] has this to say when considering the case of trajectories tangent to the switching manifold:

> The classical multipliers cannot be interpreted as partial derivatives of the optimal value function. In fact, the classical multipliers apparently have no physical or geometrical interpretation.

It has been shown in this section that λ does have a simple physical and geometrical meaning.

II. Computational Algorithms

A. *A New Second-Order Algorithm for Free-End-Point Bang-Bang Problems*

The parameters $a(t)$, $V_x(t)$, and $V_{xx}(t)$ of the quadratic expansion (10) for $V(\bar{x}+\delta x; t)$ are easily computed backwards along a nominal trajectory using equations (21), (22), (32), and (41).[20] What is now required is a method of using these data to improve the current nominal trajectory. A procedure which comes immediately to mind is the following.

Apply a new control function computed from the expression

$$u_j(t) = -\operatorname{sign}\,[f_2{}^T(\bar{x}+\delta x; t)(V_x + V_{xx}\delta x)]_j; \quad j = 1 \cdots m. \tag{86}$$

However, this procedure is unsound because $V_{xx}(t)$ experiences a jump at each switch time of $u^*(t)$; thus, $V_x + V_{xx}\delta x$ experiences a jump at switch times of $u^*(t)$ and this may cause $u(t)$ to switch when it should not.[21]

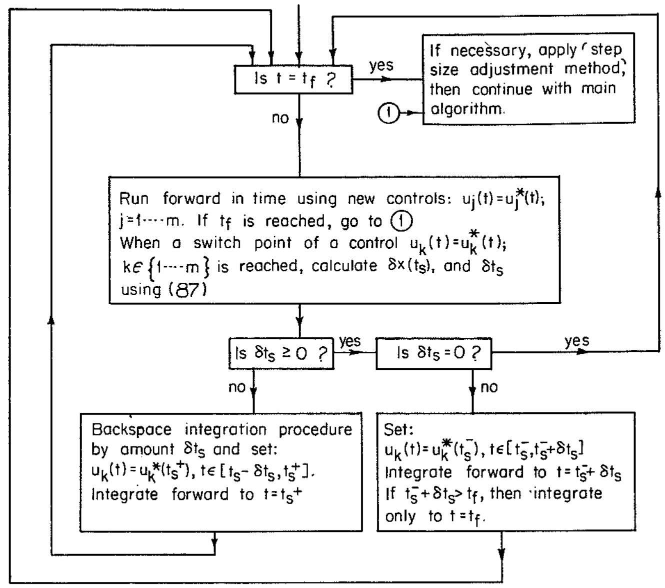
The preceding difficulty is overcome by using the local linear controller (38) to compute changes in the switch times. The controller is given below in general form:

$$\delta t_{s_i} = -V_{t_s t_s}{}^{-1}(\bar{x}, t_{s_i}; t_{s_i})\langle V_{x t_s}(\bar{x}, t_{s_i}; t_{s_i}), \delta x(t_{s_i})\rangle \tag{87}$$

where t_{s_i} is the time of the ith switch of the control $u^*(t)$.

Equation (87) is used in the following way. Run forward in time using the new controls

$$u_j(t) = u_j{}^*(t); \qquad j = 1 \cdots m. \tag{88}$$



Superscripts $^-$ and $^+$ denote times immediately before and after t_s, respectively.

Fig. 1. Flow chart I: obtaining a new control function.

When a switch time t_s, say, of a control $u_k(t) = u_k{}^*(t)$; $k \in \{1 \cdots m\}$ is reached, measure $\delta x(t_s)$ and calculate δt_s using (87). If $\delta t_s > 0$, hold $u_k(t) = u_k{}^*(t_s{}^-)$[22] for the time interval δt_s; after this, once again set $u_k(t) = u_k{}^*(t)$ and continue. If, however, $\delta t_s < 0$, then backspace the integration routine by the amount δt_s, and starting at this time $t_s - \delta t_s$, set $u_k(\tau) = u_k{}^*(t_s{}^+)$ for $t_s - \delta t_s \leq \tau \leq t_s{}^+$ and integrate forward again. After time $t = t_s{}^+$, once again set $u_k(t) = u_k{}^*(t)$ and continue.

The preceding procedure implements the local feedback controller (87) directly; there is thus no chance of discontinuities appearing in $u(t)$ where there should be none, as happens if (86) is used. Fig. 1 summarizes the preceding procedure.

Applying the new control on the whole time interval $[t_0, t_f]$ may produce δx which are too large.[23] The step size adjustment method of the next section must then be used to limit the size of δx to suitable values.

B. *A Step Size Adjustment Method*

The step size adjustment method has been described in detail in [3] and [4]. A brief description and flow chart are given here for completeness.

The improvement in cost on application of a new control function is given by

$$\Delta V = \overline{V}(x_0; t_0) - V(x_0; t_0) \tag{89}$$

where $\overline{V}(x_0; t_0)$ is the current nominal cost and $V(x_0; t_0)$ is the cost produced by the new control function.

[20] It is assumed that t_f is given explicitly.

[21] On the forward run, the position of the switch times of $u(t)$ should not coincide with those of $u^*(t)$ unless $u^*(t)$ happens to be optimal. Friedland and Sarachik [23] used (86) for local feedback control. The fact that it is an unsound scheme is illustrated by their numerical results which show numerous switching in the control functions for examples where there should be only one switch.

[22] In this section, superscripts $-$ and $+$ denote the time instants immediately before and after t_s, respectively.

[23] The theory of Section I is based on the assumption that δx is sufficiently small to justify the use of second-order expansions for V, L, and f w.r.t. x.

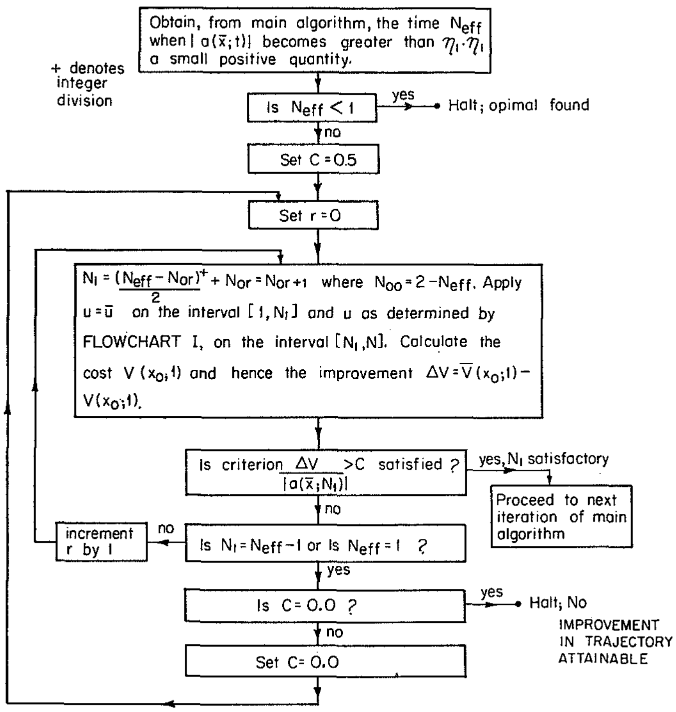


Fig. 2. Flow chart II: step size adjustment method.

The predicted improvement in cost, using a new control on the interval $[t, t_f]$, is given by

$$|a(\bar{x}; t)|. \tag{90}$$

If the old nominal control $\bar{u}(t)$ is used on the interval $[t_0, t_1]$, say, and the new control $u(t)$ is applied on the interval $[t_1, t_f]$, in the way outlined in Section II-A, then the new trajectory is acceptable[24] if the following condition is satisfied:

$$\frac{\Delta V}{|a(\bar{x}; t_1)|} > C; \qquad 0 \le C \le 1. \tag{91}$$

Fig. 2 summarizes the procedure used for determining a suitable time t_1 by subdivision of $[t_0, t_f]$.[25]

C. The Overall Computational Procedure

The procedure is summarized in Fig. 3. The step size adjustment method of Fig. 2 halts the computation if no further reduction in cost can be achieved or if optimality is attained. A necessary condition for the latter is

$$|a(\bar{x}; t_0)| < \eta; \qquad t \in [t_0, t_f] \tag{92}$$

where η is a small positive quantity determined from numerical stability considerations.

[24] That is, the δx produced are small enough.

[25] Because of the necessity of numerical integration, a discretized time scale $[1, N]$ is used. That is, $[t_0, t_f]$ is divided into N-1 steps; a time N_1 analogous to t_1 is then chosen. In practice, it is necessary to ensure that the basic time interval $t_f - t_0/(N-1)$ is small enough to guarantee that a time N_1 can be found such that (91) is satisfied. Normally, $C=0.5$, but if no N_1 can be found using this value, C is set to zero. In practice (see [3], [4]), it is necessary to subdivide the interval $[1, N_{eff}]$ instead of $[1, N]$ where N_{eff} is the time at which $|a(\bar{x}; t)|$ becomes greater than zero.

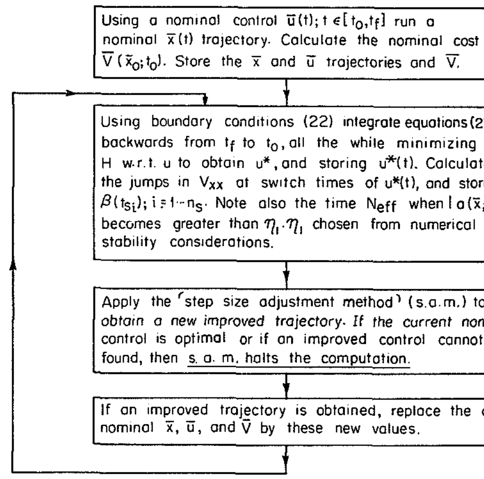


Fig. 3. Flow chart III: the overall computational procedure.

D. Characteristics of the Algorithm

1) It is believed that this is the first time that dynamic programming has been used to derive a second-order algorithm for solving bang-bang control problems.[26] The approach explicitly uses the bang-bang structure to find a local linear feedback controller between switch time changes and small differences in the state from the nominal. This controller is then used to generate a new improved trajectory.

2) If a nominal control is given whose switch times are sufficiently close to the optimal ones, then convergence is quadratic in the sense that (87) is valid.

E. Sufficient Conditions for a Predicted Reduction in Cost at Each Iteration

Assume in this section that $\bar{u}(t); t \in [t_0, t_f]$ satisfies the constraints (2). For $t \in [t_0, t_f]$

1) $[f_2^T(\bar{x}; t) \cdot V_x(x; t)]_j \neq 0$ on a finite time interval; $j = 1 \cdots m$;
2) except at switch points,
$$H(\bar{x}, u^*, V_x; t) < H(\bar{x}, \bar{u}, V_x; t); \qquad u^* \neq \bar{u};$$
3) $V_{t_s t_s}(\bar{x}, t_{s_i}; t_{s_i}) > 0; i = 1 \cdots n_s$;
4) the solutions of the differential equations (21) are bounded.

Proof: In order that u^* be determined by (5), it is necessary that condition 1) be satisfied, since otherwise H is insensitive to u_j on a finite time interval.

For the jumps in V_{xx} at $t_{s_i}; i = 1 \cdots n_s$ to be bounded, it is necessary by inspection that

$$V_{t_s t_s}(\bar{x}, t_{s_i}; t_{s_i}) \neq 0. \tag{93}$$

Further, since V is to be minimized w.r.t. the switch times t_{s_i}, it is necessary that

$$V_{t_s t_s}(\bar{x}, t_{s_i}; t_{s_i}) \geq 0. \tag{94}$$

Equations (93) and (94) yield condition 3).

[26] Dyer and McReynolds [17], [18] have independently obtained similar results.

The predicted change in cost is

$$a(x_0; t_0) = \int_{t_f}^{t_0} [H(\bar{x}, u^*, V_x; t) - H(\bar{x}, \bar{u}, V_x; t)]dt. \quad (95)$$

A condition for the negativity of $a(x_0; t_0)$ is clearly that, except at switch points,

$$H(\bar{x}, u^*, V_x; t) < H(\bar{x}, \bar{u}, V_x; t); \qquad u^* \neq \bar{u}. \quad (96)$$

In order that the quantities above be bounded, it is necessary that the solutions of (21) be bounded.

F. A New First-Order Algorithm

A first-order algorithm is easily derived if expansion (10) is truncated after the first-order terms, i.e.,

$$V(\bar{x} + \delta x; t) = \bar{V} + a + \langle V_x, \delta x \rangle. \quad (97)$$

The following equations result:

$$\begin{aligned} -\dot{a} &= H - H(\bar{x}, \bar{u}, V_x; t); \qquad a(t_f) = 0 \\ -\dot{V}_x &= H_x; \qquad V_x(t_f) = F_x(\bar{x}(t_f); t_f) \end{aligned} \quad (98)$$

where the above quantities are evaluated at $\bar{x}$, u^* unless otherwise stated; u^* is given by (5) as before.

The new control is given by

$$u(t) = u^*(t); \qquad t \in [t_0, t_f]. \quad (99)$$

The step size adjustment method of Section II-B is used to ensure that δx remains small enough.

The algorithm is the same as the first-order method described in [3]; it is, therefore, not discussed further here.

G. A New Second-Order Algorithm for Fixed-End-Point Bang-Bang Problems

First, k in Section I-G, H, and I is set to a nominal value of $\bar{k}$. The $\dot{a}$, $\dot{V}_x$, and $\dot{V}_{xx}$ equations are integrated backward, and jumps in V_{xx} are calculated at switch times. The computational procedure of Section II-C is used to solve this free-end-point problem. Along the optimal trajectory, the following equations hold:

$$\begin{aligned} -\dot{a} &= 0 \\ -\dot{V}_x &= H_x \\ -\dot{V}_k &= 0 \\ -\dot{V}_{xk} &= f_x^T V_{xk} \\ -\dot{V}_{kk} &= 0 \\ -\dot{V}_{xx} &= H_{xx} + f_x^T V_{xx} + V_{xx} f_x. \end{aligned} \quad (100)$$

The differential equation for V_{xk} may now be integrated backward along this trajectory. At switch points, (69) determines the jumps in V_{xk} and V_{kk}. Not that V_{kk} is a piecewise constant function of time.

At $t = t_0$,

$$\begin{aligned} V(x_0, \bar{k} + \delta k; t_0) \\ = V(x_0, \bar{k}; t_0) + \langle V_k, \delta k \rangle + \tfrac{1}{2}\delta k, V_{kk}\langle \delta k \rangle. \quad (101) \end{aligned}$$

From (67) and (100),

$$V_k(x_0, \bar{k}; t_0) = \psi(\bar{x}(t_f); t_f). \quad (102)$$

In order to reduce $V_k(t_0)$ and hence Ψ to zero, differentiate (101) w.r.t. δk and equate to zero:

$$V_{k|t_0} + V_{kk|t_0}\delta k = 0 \quad (103)$$

whence

$$\delta k = -\epsilon V_{kk}^{-1} V_{k|t_0} \quad (104)$$

where ϵ, $0 < \epsilon \leq 1$, is present to ensure that δk is not so large that it invalidates the above expansions. For ϵ sufficiently small, (104) ensures a reduction in V_k.

δk given by (104) is then used in (71) to compute the required changes in switch times which will result in a reduction in the end-point error Ψ.

Equation (71) becomes

$$\begin{aligned} \delta t_{s_i} = -V_{t_s t_s}^{-1}(\bar{x}, \bar{k}, t_{s_i}; t_{s_i})[\langle V_{kt_s}(\bar{x}, \bar{k}, t_{s_i}; t_{s_i}), \\ -\epsilon V_{kk}^{-1} V_{k|t_0}\rangle + \langle V_{xt_s}(\bar{x}, \bar{k}, t_{s_i}; t_{s_i}), \delta x(t_{s_i})\rangle]. \quad (105) \end{aligned}$$

H. The Overall Computational Procedure

1) Set $k = \bar{k}$ and solve the resulting free-end-point problem using the method of Section II-C; go to step 2).

2) Integrate the $\dot{V}_{xk}$ equation backward and compute the jumps in V_{xk} and V_{kk} at switch times of $u^*(t)$. Use (105) to compute the new control function.[27] For ϵ sufficiently small,[28] there will be an improvement in the terminal condition error. Check $|a(x_0; t_0)|$; if this is less than η_1, repeat this step to reduce the terminal error further; if $|a(x_0; t_0)|$ is greater than η_1 repeat step 1).

When $|a(x_0; t_0)| < \eta$, and $|V_k(x_0, k; t_0)| < \eta_2$, stop the computation. η_1 and η_2 are small positive quantities.

I. Sufficient Conditions for a Predicted Improved Trajectory at Each Iteration

Sufficient conditions to guarantee $a(x_0, \bar{k}; t_0) < 0$, and hence a predicted reduction in cost for the free-end-point problem with $k = \bar{k}$, have been given in Section II-E.

A sufficient condition for a predicted reduction in terminal error is, from (104), that

$$V_{kk}(x_0, \bar{k}; t_0) \quad (106)$$

be invertible. Sufficient conditions to ensure (106) are

1) there must be at least s switchings[29] of the control $u^*(t)$ in the interval $[t_0, t_f]$, i.e., $n_s \geq s$;
2) s of the vectors $V_{kt_s}(\bar{x}, \bar{k}, t_{s_j}; t_{s_j})$ must be linearly independent: $j \in \{1 \cdots n_s\}$.

Proof: From (66), (67), and (69),

[27] Using the procedure of Section II-A.
[28] ϵ is chosen experimentally.
[29] k is s-dimensional.

$$V_{kk}(x_0, \bar{k}; t_0) = -\sum_{i=1}^{n_s} \left\{ \frac{V_{kt_s}(\bar{x}, \bar{k}, t_{s_i}; t_{s_i}) \cdot V_{t_sk}(\bar{x}, \bar{k}, t_{s_i}; t_{s_i})}{V_{t_s t_s}(\bar{x}, \bar{k}, t_{s_i}; t_{s_i})} \right\}. \quad (107)$$

Because $V_{t_st_s}$ is assumed positive (Section II-E) and the dyads $V_{kt_s} \cdot V_{t_sk}$ are positive semidefinite, V_{kk} is at worst negative semidefinite. If V_{kk} is negative definite, then it will surely be invertible. Clearly, for V_{kk} to be negative definite, it must have full rank s; thus, in the above summation, n_s must be greater or equal to s, and s of the dyads must be linearly independent.

J. Computed Examples

The computed examples in this section serve to illustrate the usefulness of the second-order algorithms.

Example 1: The example is that of Section I-E. However, the upper limit of integration in (44) is taken as 3 seconds:

$$V = \int_0^3 x_1^2 dt. \quad (108)$$

Initial conditions for (43) are

$$x_1(0) = 1$$
$$x_2(0) = 0. \quad (109)$$

The new second-order algorithm of Section II-C was programmed; a fourth-order Runge–Kutta integration routine was used, and the interval [0, 3] was divided into 500 steps. A nominal control

$$\bar{u}(t) = +1; \qquad t \in [t_0, t_f] \quad (110)$$

was used; this produced a nominal cost of 12.1.

Fig. 4 illustrates the cost as a function of iteration number. The reduction from 12.1 to the optimal value of 0.383 was accomplished in five iterations. Fig. 5 shows phase plane portraits for some iterations, illustrating the movement of switch points from iteration to iteration.

Example 2: This example is one tried by Plant and Athans [16] using a boundary value iteration method. Plant and Athans considered the problem of hitting the unit sphere centered on the origin of the state space of a linear system in minimum time.

Here it is convenient to consider the following problem formulation:

$$\dot{x}_1 = -0.5x_1 + 5x_2; \qquad x_1(0) = 10.$$
$$\dot{x}_2 = -5x_1 - 0.5x_2 + u; \qquad x_2(0) = 10.$$
$$\dot{x}_3 = -0.6x_3 + 10x_4; \qquad x_3(0) = 10. \quad (111)$$
$$\dot{x}_4 = -10x_3 - 0.6x_4 + u; \qquad x_4(0) = 10.$$
$$|u| \leq 1.$$

$$\text{Minimize } V = \langle x(t_f), x(t_f) \rangle; \; t_f = 4.2 \text{ seconds}. \quad (112)$$

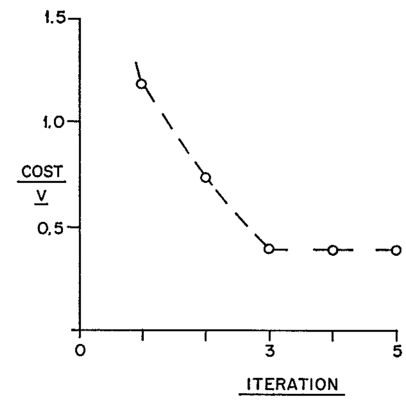


Fig. 4. Example 1: cost versus iteration.

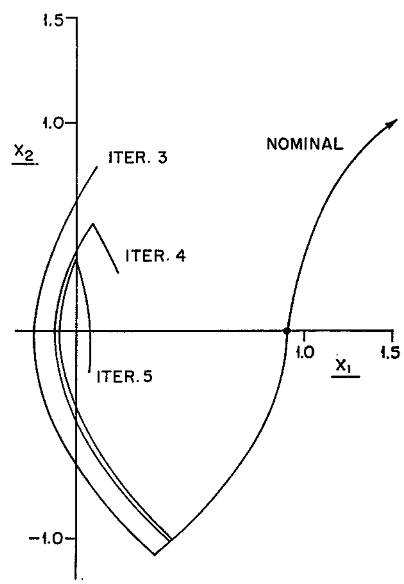


Fig. 5. Example 1: phase plane portraits.

A Runge–Kutta fourth-order integration routine was used with 300 integration steps. A nominal control

$$\bar{u}(t) = +1; \qquad t \in [0, 4.2] \quad (113)$$

was used and this produced a cost of 4.12. The second-order algorithm reduced this cost to the minimum value of 0.996 in two iterations. Fig. 6 shows the cost as a function of iteration number, and Fig. 7 shows the nominal and optimal control functions; note the difference in structure between these two control functions.

Example 3: For a fixed-end-point problem, using the same dynamics as (111) minimize

$$V = x_4(t_f) \quad (114)$$

subject to

$$x_1(t_f) = 2.3$$
$$x_2(t_f) = 2.4 \quad (115)$$
$$x_3(t_f) = 1.5$$

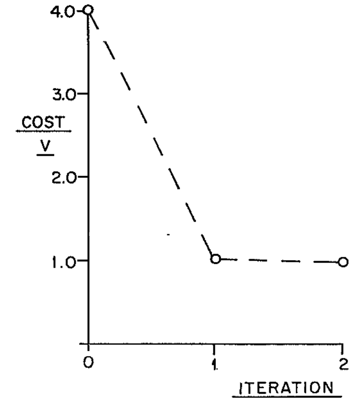


Fig. 6. Example 2: cost versus iteration.

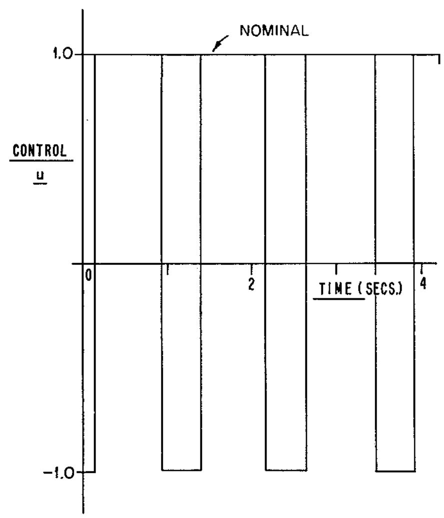


Fig. 7. Example 2: nominal and optimal controls.

where t_f is set at 2.5 seconds.

The algorithm of Section II, G and H was programmed. The following nominal values were used:

$$\bar{u}(t) = +1\,; \qquad t \in [0, 2.5]$$
$$\bar{k}_1 = \bar{k}_2 = \bar{k}_3 = 0.50. \tag{116}$$

Using the preceding nominal control, it was found that

$$x_1(t_f) = 2.81; \qquad x_2(t_f) = 3.06;$$
$$x_3(t_f) = 1.99; \qquad x_4(t_f) = 2.51. \tag{117}$$

After seven iterations of the procedure, the following data were obtained:

$$x_1(t_f) = 2.30; \qquad x_2(t_f) = 2.40;$$
$$x_3(t_f) = 1.50; \qquad x_4(t_f) = 2.31. \tag{118}$$

The optimal values of the Lagrange multipliers are

$$k_1 = 1.07; \qquad k_2 = 0.98; \qquad k_3 = 0.99. \tag{119}$$

The optimal control has six switchings, i.e.,

$$n_s = 6 > s = 3. \tag{120}$$

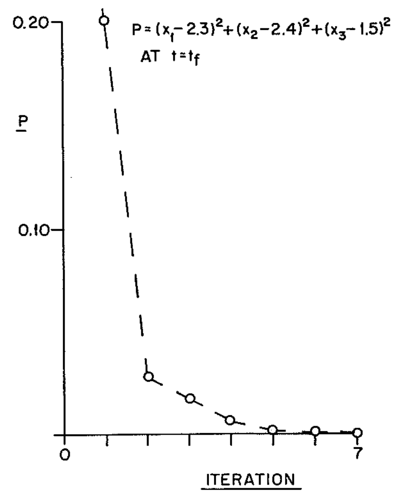


Fig. 8. Example 3: $(x_1-2.3)^2+(x_2-2.4)^2+(x_3-1.5)^2|t_f$ versus iteration.

Fig. 8 shows the quantity

$$(x_1(t_f) - 2.3)^2 + (x_2(t_f) - 2.4)^2 + (x_3(t_f) - 1.5)^2 \tag{121}$$

versus iteration number; this illustrates the reduction in terminal error at each iteration.

Conclusion

In this paper, a class of nonlinear control problems, linear in the control variables, was studied using differential dynamic programming. The approach yields new insight into the behavior of this class of problems. In particular, for the free-end-point problem, it was shown that V_x is continuous throughout the state space and that V_{xx} experiences jumps at switch points of the control function. Equation (35) is a new necessary condition for optimality in bang-bang problems. The fixed-end-point problem was converted into an equivalent free-end-point problem by adjoining the end-point constraints to the cost functional using Lagrange multipliers; a useful interpretation for Pontryagin's adjoint variables for this class or problems emerges from this treatment.

The approach yields differential equations and jump conditions satisfied by the partial derivatives of V along nonoptimal trajectories. In Section II, these results were used to devise algorithms for determining the optimal bang-bang control by successively improving a nominal guessed control function. The algorithms may be thought of as generalizations of successive sweep type methods to control problems with discontinuities. The ability of these algorithms to solve actual numerical examples rapidly was demonstrated by the computation of three control problems.

It is hoped that in a future paper computational methods for solving singular and state inequality constrained problems will be described.

Appendix

A. Derivation of Expression (29)

For convenience, (28) and (27) are repeated here:

$$V^-(\bar{x}(t_s) + \delta x(t_s), t_s + \delta t_s; t_s) = \int_{t_s}^{t_s+\delta t_s} L(\bar{x}(\tau) + \delta x(\tau); \tau) d\tau + V^+(\bar{x}(t_s) + \delta x(t_s) + \Delta x(t_s + \delta t_s); t_s + \delta t_s) \quad (122)$$

where

$$\Delta x(\tau) = \int_{t_s}^{\tau} f(\bar{x}(\tau_1) + \delta x(\tau_1), u^-; \tau_1) d\tau_1. \quad (123)$$

It is required that the r.h.s. of (122) be expanded to second order in δx and δt_s about $\bar{x}(t_s); t_s$.

Consider the first term on the r.h.s. of (122):

$$\int_{t_s}^{t_s+\delta t_s} L(\bar{x}(\tau) + \delta x(\tau); \tau) d\tau = L(\bar{x}(t_s) + \delta x(t_s); t_s)\delta t_s + \frac{1}{2}\frac{dL}{dt}(\bar{x}(t_s) + \delta x(t_s); t_s)\delta t_s^2. \quad (124)$$

The r.h.s. of (124) becomes, to second order in $\delta x(t_s)$ and δt_s,

$$[L + \langle L_x, \delta x\rangle]\delta t_s + \tfrac{1}{2}[L_t + \langle L_x, f^-\rangle]\delta t_s^2. \quad (125)$$

quantities in (125) are evaluated at $\bar{x}(t_s)$; t_s, and $\delta x(t_s)$ is written as δx where

$$f^- = f(\bar{x}(t_s), u^-; t_s) = f_1(\bar{x}(t_s); t_s) + f_2(\bar{x}(t_s); t_s)u^-. \quad (126)$$

Consider the second term on the r.h.s. of (122):

$$\begin{aligned} &V^+(\bar{x}(t_s) + \delta x(t_s) + \Delta x(t_s + \delta t_s); t_s + \delta t_s) \\ &= V^+ + \langle V_x^+, [\delta x + \Delta x(t_s + \delta t_s)]\rangle + V_t^+\delta t_s \\ &\quad + \langle V_{xt}^+, [\delta x + \Delta x(t_s + \delta t_s)]\rangle\delta t_s + \tfrac{1}{2}V_{tt}^+\delta t_s^2 \\ &\quad + \tfrac{1}{2}\langle \delta x + \Delta x(t_s + \delta t_s), V_{xx}^+[\delta x + \Delta x(t_s + \delta t_s)]\rangle \end{aligned} \quad (127)$$

where

$$\Delta x(t_s + \delta t_s) = [f^- + f_x^-\delta x]\delta t_s + \tfrac{1}{2}[f_t^- + f_x^- f^-]\delta t_s^2. \quad (128)$$

[Quantities in (127) and (128) are, unless otherwise specified, evaluated at $\bar{x}(t_s)$; t_s, and the superscript $-$ on f and its derivatives has the same meaning as indicated in (126).]

Using (128) in (127), the r.h.s. of (127) becomes, to second order,

$$\begin{aligned} &V^+ + \langle V_x^+, \delta x + f^-\delta t_s + f_x^-\delta x\delta t_s + \tfrac{1}{2}f_t^-\delta t_s^2 \\ &\quad + \tfrac{1}{2}f_x^- f^-\delta t_s^2\rangle + V_t^+\delta t_s + \langle V_{xt}^+, \delta x + f\delta t_s\rangle\delta t_s \\ &\quad + \tfrac{1}{2}\langle\delta x, V_{xx}^+\delta x\rangle + \langle\delta x, V_{xx}^+ f^-\rangle\delta t_s \\ &\quad + \tfrac{1}{2}\langle f^-, V_{xx}^+ f^-\rangle\delta t_s^2 + \tfrac{1}{2}V_{tt}^+\delta t_s^2. \end{aligned} \quad (129)$$

Define

$$H^{\mp}(\bar{x}(t_s), u^{\mp}, V_x^+(\bar{x}(t_s); t_s); t_s) = L(\bar{x}(t_s); t_s) + \langle V_x^+(\bar{x}(t_s); t_s), f(\bar{x}(t_s), u^{\mp}; t_s)\rangle. \quad (130)$$

Adding (125) and (129) and using (130), the following expression for the r.h.s. of (122) (to second order in $\delta x(t_s)$; δt_s) is obtained:

$$\begin{aligned} &V^+ + \langle V_x^+, \delta x\rangle + \tfrac{1}{2}\langle\delta x, V_{xx}^+\delta x\rangle + (H^- + V_t^+)\delta t_s \\ &\quad + \langle H_x^- + V_{xx}^+ f^- + V_{xt}^+, \delta x\rangle\delta t_s \\ &\quad + \tfrac{1}{2}[H_t^- + V_{tt}^+ + \langle H_x^-, f^-\rangle + 2\langle V_{xt}^+, f^-\rangle \\ &\quad + \langle f^-, V_{xx}^+ f^-\rangle]\delta t_s^2 \end{aligned} \quad (131)$$

[quantities evaluated at $\bar{x}(t_s)$; t_s.] From (18),

$$V_t^+ = \overline{V}_t^+ + a_t^+ = -H^+ \quad (132)$$

$$V_{xt}^+ = -H_x^+ - V_{xx}^+ f^+. \quad (133)$$

Differentiating (131) w.r.t. time,

$$\frac{d}{dt}(V_t^+ + H^+) = \frac{\partial}{\partial t}(V_t^+ + H^+) + \left\langle\frac{\partial}{\partial x}(V_t^+ + H^+), f^+\right\rangle = 0 \quad (134)$$

whence

$$V_{tt}^+ = -H_t^+ - 2\langle V_{xt}^+, f^+\rangle - \langle H_x^+, f^+\rangle - \langle f^+, V_{xx}^+ f^+\rangle. \quad (135)$$

Substituting (133) into (135),

$$\begin{aligned} V_{tt}^+ &= -H_t^+ + 2\langle H_x^+, f^+\rangle + 2\langle f^+, V_{xx}^+ f^+\rangle \\ &\quad - \langle H_x^+, f^+\rangle - \langle f^+, V_{xx}^+ f^+\rangle \\ &= -H_t^+ + \langle H_x^+, f^+\rangle + \langle f^+, V_{xx}^+ f^+\rangle. \end{aligned} \quad (136)$$

Substituting (132), (133), and (136) into (131),

$$\begin{aligned} &V^+ + \langle V_x^+, \delta x\rangle + \tfrac{1}{2}\langle\delta x, V_{xx}^+\delta x\rangle + (H^- - H^+)\delta t_s \\ &\quad + \langle H_x^- - H_x^+ + V_{xx}^+(f^- - f^+), \delta x\rangle\delta t_s \\ &\quad + \tfrac{1}{2}\{H_t^- - H_t^+ - \langle H_x^-, f^- - f^+\rangle \\ &\quad + \langle f^-, H_x^- - H_x^+\rangle \\ &\quad + \langle f^- - f^+, V_{xx}^+(f^- - f^+)\rangle\}\delta t_s^2 \end{aligned} \quad (137)$$

[all quantities evaluated at $\bar{x}(t_s)$; t_s]. Expression (137) is exactly (29).

References

[1] D. Q. Mayne, "A second-order gradient method of optimizing non-linear discrete time systems," *Internat'l J. Control*, vol. 3, no. 1, 1966.

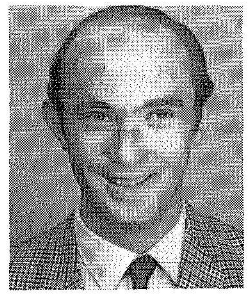
[2] D. H. Jacobson, "Second-order and second-variation methods for determining optimal control: a comparative study using differential dynamic programming," *Internat'l J. Control*, vol. 7, no. 2, pp. 175–196, 1968.

[3] ——, "New second-order and first-order algorithms for determining optimal control: a differential dynamic programming approach," Harvard University, Cambridge, Mass., Tech. Rept. 551, January 1968; also *J. Opt. Theory Appl.*, vol. 2, no. 6, 1968.

[4] ——, "Differential dynamic programming methods for determining optimal control of non-linear systems," Ph.D. dissertation, University of London, London, England, November 1967.

[5] S. K. Mitter, "Successive approximation methods for the solution of optimal control problems," *Automatica*, vol. 3, pp. 135–149, 1966.

[6] S. R. McReynolds and A. E. Bryson, Jr., "A successive sweep method for solving optimal programming problems," *Proc. 1965 Joint Automatic Control Conf.*, no. 6, p. 551.
[7] S. R. McReynolds, "The successive sweep method and dynamic programming," *J. Math. Anal. Appl.*, vol. 19, pp. 565–598, 1967.
[8] H. J. Kelley, R. E. Kopp, and H. Gardner Moyer, "Successive approximation techniques for trajectory optimization," IAS Vehicle Systems Optimization Symp., Garden City, N. Y., November 28–29, 1961.
[9] ——, "A trajectory optimization technique based upon the theory of the second variation," AIAA Astrodynamics Conf., New Haven, Conn., August 19–21, 1963. Also, *Progress in Astronautics and Aeronautics*, vol. 14. New York: Academic Press, 1964, pp. 559–582.
[10] T. E. Bullock and G. F. Franklin, "A second-order feedback method for optimal control computations," *IEEE Trans. Automatic Control*, vol. AC-12, pp. 666–673, December 1967.
[11] L. D. Berkovitz, "Variational methods in problems of control and programming," *J. Math. Anal. Appl.*, vol. 3, pp. 145–169, 1961.
[12] S. E. Dreyfus, *Dynamic Programming and the Calculus of Variations.* New York: Academic Press, 1965.
[13] L. W. Neustadt, "On synthesizing optimal control," *Proc. 2nd Internat'l IFAC Conf.*, Basle, Switzerland, 1963, pp. 283–291.
[14] B. Paiewonsky, "Time optimal control of linear systems with bounded control," *Internat'l Symp. on Non-Linear Differential Equations and Non-Linear Mechanics.* New York: Academic Press, 1963.
[15] H. K. Knudsen, "An iterative procedure for computing time-optimal controls," *IEEE Trans. Automatic Control*, vol. AC-9, pp. 23–30, January 1964.
[16] J. B. Plant and M. Athans, "An iterative technique for the computation of time optimal controls," *Proc. 3rd Internat'l IFAC Conf.*, London, England, June 1966.
[17] P. Dyer and S. R. McReynolds, "On optimal control problems with discontinuities," *J. Math. Anal. Appl.*, 1968.
[18] ——, "On a new necessary condition for optimal control problems with discontinuities," Space Programs Summary 37-48, vol. 3, December 31, 1967.
[19] R. Bellman, I. Glicksberg, and O. Gross, "On the bang-bang control problem," *Quart. J. Appl. Math.*, vol. 14, pp. 11–18, 1956.
[20] J. P. Lasalle, "The time optimal control problem," in *Contributions to the Theory of Non-Linear Oscillations*, vol. 5. Princeton, N. J.: Princeton University Press, 1960, pp. 1–24.
[21] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mischenko, *The Mathematical Theory of Optimal Processes.* New York: Interscience, 1962.
[22] H. J. Kelley, B. R. Uzzell, and S. S. McKay, "Rocket trajectory optimization by a second-order numerical technique," AIAA/NASA-MSC Astrodynamics Conf., Houston, Tex., December 12–14, 1967.
[23] B. Friedland and P. E. Sarachik, "A unified approach to sub-optimal control," *Proc. 3rd Internat'l IFAC Conf.*, London, England, June 1966.
[24] A. T. Fuller, "Study of an optimal non-linear control system," *J. Electronics and Control*, vol. 15, no. 1, 1963.
[25] R. E. Kalman, "The theory of optimal control and the calculus of variations," in *Mathematical Optimization Techniques*, R. Bellman, Ed. Berkeley, Calif.: University of California Press, 1963, ch. 16.
[26] S. Shapiro, "A geometric approach to the theory of optimal control," Ph.D. dissertation, University of London, London, England, 1965.



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A Note on Error Analysis in Differential Dynamic Programming

Abstract—In [1]–[4] new algorithms for determining optimal control of nonlinear systems are described. This correspondence provides by means of an error analysis alternative justification for the neglect of certain terms in the derivation of those algorithms.

ERROR ANALYSIS

We shall consider the class of optimal control problems discussed in [3]. Similar conclusions can be drawn for those described in [1], [2], and [4].

Consider the following control problem formulation

$$\dot{x} = f_1(x,t) + f_u(x,t)u, \quad x(t_0) = x_0 \tag{1}$$

$$V(x_0,t_0) = \int_{t_0}^{t_f} L(x,t)\, dt + F(x(t_f)) \tag{2}$$

$$|u| \le 1. \tag{3}$$

Here, x is an n-dimensional state vector, and u is a scalar control. f_1 and f_u are n-dimensional vector functions of x at time t. The functions L and F are scalar.

The problem is to choose the control function $u(\cdot)$ to satisfy (3) and minimize $V(x_0,t_0)$. Assuming that the preceding optimal control problem is nonsingular, it is clear that the optimal control function is bang-bang.

In [1]–[4] a nominal control function $\bar{u}(\cdot)$ is presumed known; a nominal trajectory $\bar{x}(\cdot)$ results on the application of $\bar{u}(\cdot)$ in (1). The corresponding nominal cost is denoted by $\overline{V}(x_0,t_0)$. The derivation of the algorithms proceeds by writing the Bellman equation[1] in terms of the nominal functions $\bar{x}(\cdot)$ and $\bar{u}(\cdot)$, i.e.,

$$-\frac{\partial V}{\partial t}(\bar{x} + \delta x,t) = \min_{\delta u} [L(\bar{x} + \delta x,t) + \langle V_x(\bar{x} + \delta x,t), f(\bar{x} + \delta x, \bar{u} + \delta u,t)\rangle] \tag{4}$$

where, for convenience,

$$f(x,u,t) \equiv f_1(x,t) + f_u(x,t)u. \tag{5}$$

The cost $V(\bar{x}+\delta x,t)$ is expanded in a Taylor series as follows:

$$V(\bar{x} + \delta x,t) = V(\bar{x},t) + \langle V_x,\delta x\rangle + \frac{1}{2}\langle \delta x, V_{xx}\delta x\rangle + \text{higher order terms} \tag{6}$$

and $V(\bar{x},t)$ is written as

$$V(\bar{x},t) = \overline{V}(\bar{x},t) + a(\bar{x},t) \tag{7}$$

where $\overline{V}(\bar{x},t)$ is the cost incurred when starting the system in state $\bar{x}$ at time t and using the nominal controls $\bar{u}(\tau)$, $\tau \in [t,t_f]$. The quantity $a(\bar{x},t)$ is thus the change in cost produced by the control variation $\delta u(\tau)$, $\tau \in [t,t_f]$.

In [1]–[4] it is assumed that δx is kept sufficiently small to allow one to truncate (6) after quadratic terms, yielding

$$V(\bar{x} + \delta x,t) = \overline{V} + a + \langle V_x,\delta x\rangle + \frac{1}{2}\langle \delta x, V_{xx}\delta x\rangle. \tag{8}$$

Differentiating (8), one obtains

$$V_x(\bar{x} + \delta x,t) = V_x + V_{xx}\delta x. \tag{9}$$

Equations (8) and (9) are substituted into (4), yielding

$$-\frac{\partial \overline{V}}{\partial t} - \frac{\partial a}{\partial t} - \left\langle \frac{\partial V_x}{\partial t}, \delta x\right\rangle - \frac{1}{2}\left\langle \delta x, \frac{\partial V_{xx}}{\partial t}\delta x\right\rangle = \min_{\delta u} [L(\bar{x} + \delta x,t) + \langle V_x + V_{xx}\delta x, f(\bar{x}+\delta x,\bar{u}+\delta u,t)\rangle]. \tag{10}$$

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[1] The Bellman equation is valid in regions where the control is not switching, i.e., between switch points (see [3]).

The algorithms are derived from (10). It should be noted that the term

$$\left\langle \frac{1}{2} V_{xxx}\delta x\delta x, f(\bar{x} + \delta x, \bar{u} + \delta u, t) \right\rangle \quad (11)$$

is omitted from (10) owing to the fact that $V_x(\bar{x}+\delta x,t)$, given by (9), is accurate only to first order in δx. However, the term omitted is second order in δx+higher order terms, and so it may appear as though the omission of this term causes the resulting algorithms, developed in [1]–[4], not to be accurate to second-order terms. This correspondence demonstrates by the use of a straightforward error analysis that the neglect of the term (11) is legitimate by showing that the error introduced into $a(t)$ is third order.

If the term (11) is included in (10), then it is easy to show that the following differential equations hold:

$$\begin{aligned} -\dot{a} &= H(\bar{x},u^*,V_x,t) - H(\bar{x},\bar{u},V_x,t) \\ -\dot{V}_x &= H_x(\bar{x},u^*,V_x,t) + V_{xx}[f(\bar{x},u^*,t) - f(\bar{x},\bar{u},t)] \\ -\dot{V}_{xx} &= H_{xx}(\bar{x},u^*,V_x,t) + f_x^T(\bar{x},u^*,t)V_{xx} + V_{xx}f_x(\bar{x},u^*,t) + \frac{1}{2}V_{xxx}[f(\bar{x},u^*,t) - f(\bar{x},\bar{u},t)] + \frac{1}{2}[f(\bar{x},u^*,t) - f(\bar{x},\bar{u},t)]^T V_{xxx} \end{aligned} \quad (12)$$

where

$$H(x,u,V_x,t) = L(x,t) + \langle V_x, f(x,u,t)\rangle \quad (13)$$

and

$$u^* = \arg\min_u H(\bar{x},u,V_x,t).$$

The preceding equations are identical to those obtained in [3] except for the appearance of V_{xxx} in the $\dot{V}_{xx}$ equation. Note that

$$f(\bar{x},u^*,t) - f(\bar{x},\bar{u},t) = f_u(u^* - \bar{u}). \quad (14)$$

If the V_{xxx} terms are neglected, then $\Delta a(t)$, the error in the predicted change in cost, is clearly of order

$$\int_{t_f}^{t}\int_{t_f}^{t_3}\int_{t_f}^{t_2} |u^*(t_1) - \bar{u}(t_1)|\,dt_1\,|u^*(t_2) - \bar{u}(t_2)|\,dt_2\,|u^*(t_3) - \bar{u}(t_3)|\,dt_3 \quad (15)$$

while $a(t)$ is of order

$$\int_{t_f}^{t} |u^*(t_4) - \bar{u}(t_4)|\,dt_4 \quad (16)$$

because the problem is assumed to be nonsingular. For $|u^*-\bar{u}|$ of order ϵ, $\Delta a(t)$ is of order ϵ^3 and $a(t)$ is of order ϵ. Alternatively, for $(t_f - t)$ of order ϵ, $\Delta a(t)$ is of order ϵ^3 and $a(t)$ is of order ϵ. In either case $a(t)$ is an estimate of the true predicted change in cost, the error in $a(t)$ being third order.

Owing to the neglect of the V_{xxx} terms, an error in the switch times of u^* of magnitude

$$\Delta t_s = -V_{t_s t_s}^{-1} f_u^T \Delta V_x \Big|_t \quad (17)$$

is introduced.[2] However, ΔV_x, the error in V_x, is clearly second order in ϵ. The error introduced into $a(t_s)$ is, to second order in Δt_s, from [3],

$$\Delta t_s V_{t_s} + \frac{1}{2} V_{t_s t_s}\Delta t_s^2. \quad (18)$$

At a switch time, $V_{t_s}=0$ (see [3]) and so the error introduced into $a(t_s)$ by Δt_s is of order ϵ^4. The upshot of the preceding argument is that the neglect of the V_{xxx} terms in the $\dot{V}_{xx}$ equation introduces a third-order error into the estimate of the predicted reduction in cost. The step size adjustment method described in [2], [3], and [5] ensures that this third-order error term is negligible, and hence that a reduction in cost occurs by applying the new control function at each iteration only over a time interval $[t_1,t_f]$, where t_1 is chosen to be sufficiently close to t_f.

Conclusion

It has been demonstrated that the a priori neglect of certain terms in the development of the differential dynamic programming algorithms described in [3] introduces a third-order error into the estimate of the predicted change in cost. The preceding error analysis extends (see [5]) to the algorithms given in [1], [2], and [4].

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References

[1] D. H. Jacobson, "Second-order and second-variation methods for determining optimal control: a comparative study using differential dynamic programming," *Internat'l J. Control*, vol. 7, no. 2, pp. 175–196, 1968.
[2] ——, "New second-order and first-order algorithms for determining optimal control: a differential dynamic programming approach," *J. Optimization Theory and Applications*, vol. 2, no. 6, 1968.
[3] ——, "Differential dynamic programming methods for solving bang-bang control problems," *IEEE Trans. Automatic Control*, vol. AC-13, pp. 661–675, December 1968.
[4] D. H. Jacobson and D. Q. Mayne, "Differential dynamic programming," submitted to the 1969 IFAC Cong., Warsaw, Poland.
[5] ——, *Differential Dynamic Programming*. New York: American Elsevier, 1969.

[2] $V_{t_s t_s}$ is the second partial derivative of V with respect to the switch time (see [3]).

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13. ABSTRACT

Differential dynamic programming is a technique, based on dynamic programming rather than the calculus of variations, for determining the optimal control function of a nonlinear system. Unlike conventional dynamic programming where the optimal cost function is considered globally, differential dynamic programming applies the principle of optimality in the neighborhood of a nominal, possibly nonoptimal, trajectory. This allows the coefficients of a linear or quadratic expansion of the cost function to be computed in reverse time along the trajectory: these coefficients may then be used to yield a new improved trajectory (i.e., the algorithms are of the "successive sweep" type).

A class of nonlinear control problems, linear in the control variables, is studied using differential dynamic programming. It is shown that for the free-end-point problem, the first partial derivatives of the optimal cost function are continuous throughout the state space, and the second partial derivatives experience jumps at switch points of the control function. A control problem that has an analytic solution is used to illustrate these points. The fixed-end-point problem is converted into an equivalent free-end-point problem by adjoining the end-point constraints to the cost functional using Lagrange multipliers: a useful interpretation for Pontryagin's adjoint variables for this type of problem emerges from this treatment.

The above results are used to devise new second- and first-order algorithms for determining the optimal bang-bang control by successively improving a nominal guessed control function. The usefulness of the proposed algorithms is illustrated by the computation of a number of control problem examples.

The correspondence provides by means of an error analysis justification for the neglect of certain terms in the derivation of those algorithms.

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