UNIVERSITY OF MINNESOTA

SPACE SCIENCE CENTER

CONTROL OF DISTRIBUTED-PARAMETER SYSTEMS

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BY
DAVID L. RUSSELL
CONTROL SCIENCE CENTER

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ON THE COVER — The area of the heavens around the Orion Constellation, shown in the cover photograph made through the 120-inch telescope of the Lick observatory, is also the region of observations with an infrared telescope developed by University of Minnesota astro-physicists. The infrared sensory equipment reveals stellar bodies that could not be studied by conventional telescopes, and it is expected to provide data on the birth of stars.

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13. ABSTRACT

This paper deals with the control of distributed-parameter systems of hyperbolic and parabolic type. The concept of approximate controllability is developed and applied to various physical systems including the string and membrane vibrators. The optimal control problem for linear differential equations in Hilbert Space is explored and related to the bang-bang principle.

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TABLE OF CONTENTS

TABLE OF CONTENTS

(continued)

INTRODUCTION

Dr. David Russell of Wisconsin University delivered a series of lectures on the control of distributed parameter systems as part of a seminar course on Topics in Control Theory jointly organized by the Departments of Mathematics and Electrical Engineering, University of Minnesota, during the Winter Quarter of 1969. These notes are based on the above lectures and were prepared under Contract ONR 3776-00 by M. Balachandra.

In the first two chapters partial differential equation models of control systems are discussed. First, it is shown that these exhibit features that do not appear in discretized models, no matter how many degrees of freedom the latter may be assumed to have. The first chapter deals with the vibrating string, for which the method of characteristics may be applied conveniently, making use of geometric arguments. Conditions for complete controllability are derived and a typical optimization problem is formulated. The second chapter concerns the generalization of the above to higher dimensions and it is shown that it is necessary to introduce the concept of approximate controllability. The case of a vibrating circular membrane is discussed and conditions for approximate controllability are obtained.

In Chapter III, a different viewpoint is taken and the problem of the vibrating string is posed as one involving a self-adjoint, unbounded operator in Hilbert space. The control problem then reduces to a trigonometric moment problem and conditions of controllability can be derived in terms of the density and asymptotic gap of the eigenvalues of the operator.

Chapter IV deals with time-optimal control with bounded control variables. It is shown that the bang-bang principle of the finite-dimensional system may be generalized to the infinite-dimensional case with slight modifications. Specific results are obtained for hyperbolic and parabolic problems.

The last two chapters are also concerned with generalizing results of finite-dimensional systems. Chapter V deals with the stabilization of a linear oscillator by means of a control force depending linearly on velocity. It is shown that the known results for the finite-dimensional

system can be generalized to the infinite-dimensional case using the perturbation theory of linear operators. In the last chapter, the optimal control of a linear system with a quadratic performance index is considered. For the finite-dimensional case, the control law is obtained by solving the Kalman-Riccati differential equation and it is shown that the controls so obtained converge in the limit to that for the infinite-dimensional problem, as the number of dimensions is increased without limit.

CHAPTER I

CONTROL OF DISTRIBUTED PARAMETER SYSTEMS WITH ONE VARIABLE

1. Introduction

Most of the plants which the control engineers work with can be represented in a number of ways. If we consider a stretched string of non-uniform density $\rho(x)$, fixed at the left hand end $x = 0$ and free to move vertically at the right hand end $x = 1$, one can represent small motions by solutions of the linear second order partial differential equation

$$\rho(x) \frac{\partial^2 y}{\partial t^2} - \tau \frac{\partial^2 y}{\partial x^2} = 0 \tag{1.1}$$

subject to boundary conditions

$$y(0,t) \equiv 0, \quad \frac{\partial y}{\partial x}(1,t) = \frac{1}{\tau} u(t) \tag{1.2}$$

Here τ is the applied tension and $u(t)$ a controlling force which acts in the vertical direction at the boundary point $x = 1$.

It is not absolutely essential that a partial differential equation model be used, however. One might conceive of the string as composed of n particles located at points $x_k = \frac{k}{n}$, $k = 1, 2, \ldots, n$ having mass $\frac{1}{n}\rho(x_n)$ and connected by massless cords which are, nevertheless, capable of sustaining the tension τ. Letting y_k denote the vertical displacement of the k-th particle we have equations

$$\frac{1}{n}\rho(x_1)\frac{d^2 y_1}{dt^2} = \tau\left(\frac{y_2 - 2y_1 + 0}{\frac{1}{n}}\right)$$

$$\frac{1}{n}\rho(x_k)\frac{d^2 y_k}{dt^2} = \tau\left(\frac{y_{k+1} - 2y_k + y_{k-1}}{\frac{1}{n}}\right), \quad k = 2,\ldots,n-1$$

$$\frac{1}{n}\rho(x_n)\frac{d^2 y_n}{dt^2} = \tau\left(\frac{-y_n + y_{n-1}}{\frac{1}{n}}\right) + u(t) \tag{1.3}$$

or, in matrix notation

$$\frac{d^2y}{dt^2} = n^2\tau\begin{pmatrix} -\frac{2}{\rho(x_1)} & \frac{1}{\rho(x_1)} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\rho(x_2)} & \frac{-2}{\rho(x_2)} & \frac{1}{\rho(x_2)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\rho(x_3)} & \frac{-2}{\rho(x_3)} & \frac{1}{\rho(x_3)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{\rho(x_n)} & \frac{-1}{\rho(x_n)} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ n\,\frac{1}{\rho(x_n)} \end{pmatrix} u(t) \quad \text{or} \quad \frac{d^2y}{dt^2} = Ay + bu(t)$$

A - nxn matrix, b - n vector (1.4)

Other models may also be envisioned, e.g., one composed of n harmonic oscillators corresponding to n normal modes of vibrators of the string. Since ordinary differential equations are so much easier to treat than partial differential equations we might well ask -- why use partial differential equations at all. We shall try to give at least a partial answer to that question here.

2. Discretized Model of a Vibrating String

A first order linear control system

$$\frac{dw}{dt} = Fw + gu \qquad F \text{ mxm matrix } \; g \text{ m-vector} \qquad (2.1)$$

is completely controllable if the vectors g, Fg, F^2g, ... $F^{m-1}g$ form the columns of a non-singular mxm matrix. To study equation (1.4), we let $m = 2n$,

$$F = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix},\; g = \begin{pmatrix} 0 \\ b \end{pmatrix} \text{ and compute } Fg = \begin{pmatrix} b \\ 0 \end{pmatrix},\; F^2g = \begin{pmatrix} 0 \\ Ab \end{pmatrix},\; F^3g = \begin{pmatrix} Ab \\ 0 \end{pmatrix}, \ldots$$

$$F^{2k}g = \begin{pmatrix} 0 \\ A^kb \end{pmatrix},\quad F^{2k+1}g = \begin{pmatrix} A^kb \\ 0 \end{pmatrix}$$

We conclude that $\frac{d^2y}{dt^2} = Ay + bu$ is completely controllable if b, Ab, A^2b, ..., $A^{n-1}b$ form the columns of a non-singular matrix. Now for the matrix A and vector b occurring in our discrete model of the string

$$b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{n}{\rho(x_n)} \end{pmatrix}, \quad Ab = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{n}{\rho(x_n)\rho(x_{n-1})} \\ \frac{-n}{\rho(x_n)^2} \end{pmatrix}; \quad A^2b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{n}{\rho(x_n)\rho(x_{n-1})\rho(x_{n-2})} \\ * \\ * \end{pmatrix}$$

and continuing, we find that $(A^{n-1}b, A^{n-2}b, \ldots, Ab, b)$ is a lower triangular matrix with non-zero entries on the main diagonal and hence non-singular. Thus the discrete model of the string is completely controllable. Among other things, this means:

(i) Given any initial state $y_1(0), \ldots, y_n(0), \frac{dy_1}{dt}(0), \ldots, \frac{dy_n}{dt}(0)$ and any time $T > 0$ there is a control function $u(t)$ which causes the resulting solution of $\frac{d^2y}{dt^2} = Ay + bu$ to satisfy $y_1(T) = \ldots = y_n(T) = \frac{dy_2}{dt}(T) = \ldots = \frac{dy_n}{dt}(T) = 0$ Thus control is possible for arbitrarily small T.

(ii) If we impose an a priori constraint $|u(t)| \leq r$ on the control force and, for some initial condition, pose the problem of bringing the system into the equilibrium configuration in the least possible time T then there is exactly one control force u(t) which solves this minimum time problem and u(t) is a bang-bang control, i.e. u(t) is piecewise continuous on $[0,T]$ and assumes only the values $\pm r$.

Both (i) and (ii) are true no matter how many particles we take the discretized string to be composed of. One of the first notable features of the control theory of partial differential equations is that neither (i) nor (ii) is true for the partial differential equation

$$\rho(x)\frac{\partial^2 y}{\partial t^2} - \tau\frac{\partial^2 y}{\partial x^2} = 0 \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2} - c^2(x)\frac{\partial^2 y}{\partial x^2} = 0$$

$$y(0,t) \equiv 0, \quad \frac{\partial y}{\partial x}(1,t) \equiv \frac{1}{\tau}u(t), \tag{2.2}$$

3. Continuous Model of Vibrating String with Single Boundary Control

To prove this we introduce two families of curves in the (x,t) plane.

$$C_1 = \{(x(t),t) \mid \frac{dx}{dt} = c(x(t))\}, \quad C_2 = \{(x(t),t) \mid \frac{dx}{dt} = -c(x(t))\} \tag{3.1}$$

These are the two families of characteristics of the partial differential equation in question.

Let y(x,t) be a solution of equation (2.2) and put

$$\eta_1(x,t) = \frac{\partial y}{\partial t} - c(x)\frac{\partial y}{\partial x}, \quad \eta_2(x,t) = \frac{\partial y}{\partial t} + c(x)\frac{\partial y}{\partial x} \tag{3.2}$$

Let $x = \xi_1(t)$ describe a curve in C_1 and compute

$$\frac{d}{dt}(\eta_1(\xi_1(t),t)) = \frac{\partial \eta_1}{\partial t} + \frac{\partial \eta_1}{\partial x}\frac{d\xi_1}{dt}$$

$$= \frac{\partial^2 y}{\partial t^2} - c(x)\frac{\partial^2 y}{\partial t \partial x} + c(x)\left(\frac{\partial^2 y}{\partial x \partial t} - c(x)\frac{\partial^2 y}{\partial x^2} - c'(x)\frac{\partial y}{\partial x}\right)$$

$$= \frac{\partial^2 y}{\partial t^2} - [c(x)]^2\frac{\partial^2 y}{\partial x^2} - c'(x)c(x)\frac{\partial y}{\partial x}$$

$$= \frac{c'(\xi_1(t))(\eta_1(\xi_1(t),t) - \eta_2(\xi_1(t),t))}{2}$$

and $$\frac{d}{dt}(\eta_2(\xi_2(t),t)) = \frac{\partial \eta_2}{\partial t} + \frac{\partial \eta_2}{\partial x}\frac{d\xi_2}{dt}$$

$$= \frac{\partial^2 y}{\partial t^2} + c(x)\frac{\partial^2 y}{\partial t \partial x} + \left(\frac{\partial^2 y}{\partial x \partial t} + c(x)\frac{\partial^2 y}{\partial x^2} + c'(x)\frac{\partial y}{\partial x}\right)(-c(x))$$

$$= \frac{c'(\xi_2(t))\ (\eta_1(\xi_2(t),t) - \eta_2(\xi_2(t),t))}{2}$$

Thus η_1 and η_2 satisfy ordinary differential equations along $x = \xi_1(t)$, $x = \xi_2(t)$, respectively but these equations are coupled in a rather unusual way. These equations are very useful. They can be used to prove the existence of solutions of equation (2.2) and, by applying numerical integration techniques to them, they yield a method, called the method of characteristics, for approximating solutions numerically.

We consider (in Figure 1) in the (x,t)-plane the curve $x = \xi_2(t)$

which solves $\frac{d\xi_2}{dt} = -c(\xi_2(t)), \quad \xi_2(0) = 1.$

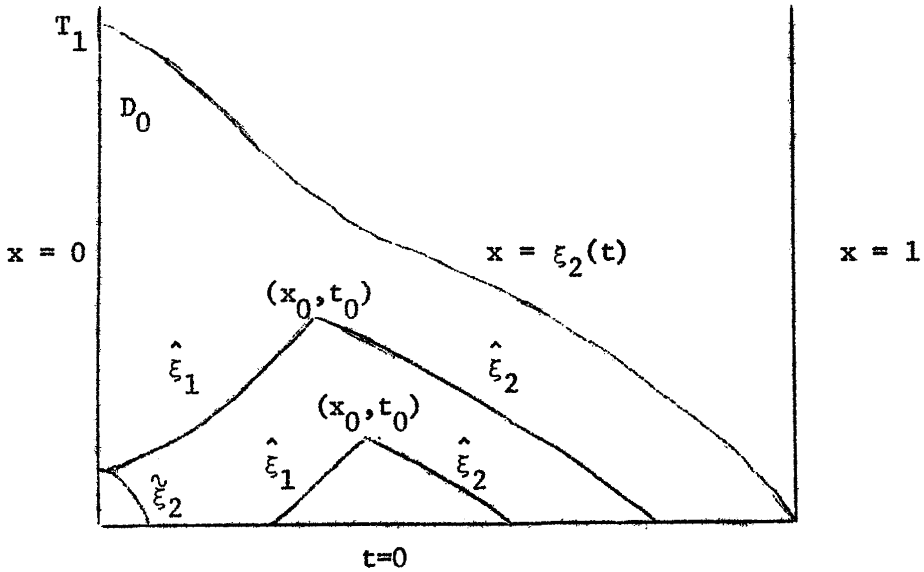


FIGURE 1.

This characteristic from the family C_2 meets the line $x = 0$ at point $(0,T_1)$ and $T_1 > 0$ since $c(x)$ is everywhere positive. Moreover, the curves $x = \xi_2(t)$, $x = 0$ and $t = 0$ bound a triangular domain which we shall call D_0.

Let (x_0,t_0) be an arbitrary point in D_0. Then (x_0,t_0) can be joined to two points on the line segment $t = 0$, $0 \leq x \leq 1$ by two curves, $x = \hat{\xi}_2(t)$, $x = \hat{\xi}_1(t)$ from C_2 and C_1 respectively or else, can be joined to two points on $t = 0$, $0 \leq x \leq 1$ by $x = \hat{\xi}_2(t)$ from C_2 and a composite path formed from a curve $x = \hat{\xi}_1(t)$ of C_1, which connects (x_0,t_0) to the line $x = 0$, and a curve $x = \tilde{\xi}_2(t)$ from C_2 connecting that point on $x = 0$ with a point on $t = 0$. (See Figure 1). Combining this fact with the result that η_1 and η_2 satisfy differential equations

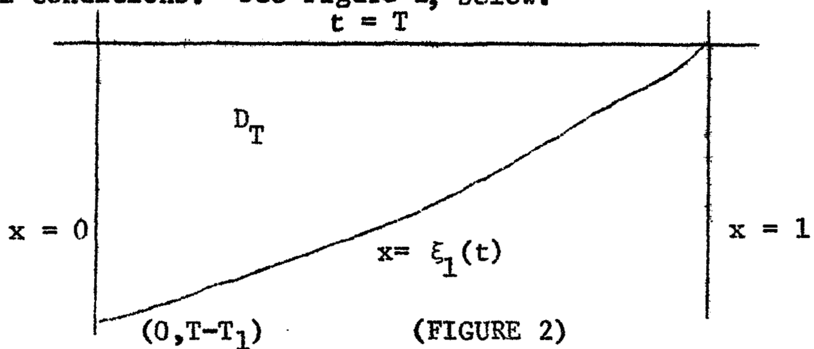
$$\frac{d\eta_1}{dt} = \frac{c'}{2}(\eta_1 - \eta_2), \quad \frac{d\eta_2}{dt} = \frac{c'}{2}(\eta_1 - \eta_2) \tag{3.3}$$

on curves in C_1, C_2, respectively, we conclude that η_1 and η_2 are completely determined in D_0 by the initial data given on $t = 0$, $0 \leq x \leq 1$. For this reason we refer to D_0 as the <u>domain of determinacy of the initial conditions</u>. The control $u(t)$, applied at $x = 1$, has no influence on the solution $y(x,t)$ in D_0.

In the same way a characteristic curve $x = \xi_1(t)$ with

$$\frac{d\xi_1}{dt} = c(\xi_1(t)), \qquad \xi(T) = 1$$

crosses from $(1,T)$ to a point $(0,T-T_1)$ on $x = 0$ and cuts off a triangular domain D_T in which η_1 and η_2, and hence $y(x,t)$, is completely determined by the conditions which we impose at $t = T$. Since we want $y(x,T) = \frac{\partial y}{\partial t}(x,T) = 0$ we must have $y(x,t) = 0$ in D_T. D_T will be called the domain of determinacy of the terminal conditions. See Figure 2, below.



(FIGURE 2)

4. Controllability of the Vibrating String

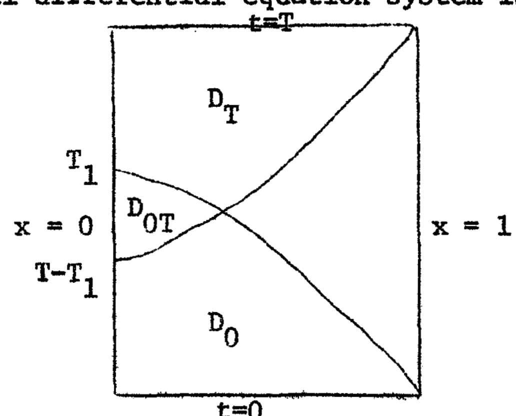
Now let us put all of the above information together and study the situation in the rectangle

$$D = \{(x,t) \mid 0 \le x \le 1, \quad 0 \le t \le T\} \tag{4.1}$$

It will be necessary to consider three cases, depending upon the relationship of T to T_1.

Case (i) $T < 2T_1$. In this case D_0 and D_T intersect in a domain D_{OT} as shown in Figure 3. In D_{OT} $y(x,t)$ is completely determined by the initial data and also completely determined by the terminal data. Since we have taken zero terminal data these determinations are consistent if and only if the initial data are also such that $y(x,t)$ vanishes in D_{OT}. This is not generally the case. Thus in general there is no solution $y(x,t)$ of the partial differential equation satisfying both the initial and the terminal data. We say then that our partial differential equation system is not controllable in time T if $T < 2T_1$.

FIGURE 3
$T < 2T_1$



Case (ii) $T = 2T_1$. In this case D_0 and D_T do not overlap but have exactly one common boundary point $(0,T_1) = (0,T-T_1)$. The problem now is to extend the solution from $D_0 \cup D_T$ into the triangular region $\hat{D} = D - (D_0 \cup D_T)$. This is again done by the method of characteristics. Every point $(\hat{x}_0, \hat{t}_0)$ in $\hat{D}$ can be connected to D_0 by a path $x = \hat{\xi}_1(t)$ and to D_T by a path $x = \hat{\xi}_2(t)$. Using this together with the differential equations satisfied by η_1 and η_2 we are able to form a system of integral equations in $\hat{D}$ whose solution yields a solution of the equation (2.2) and, by showing that the integral equations have a unique solution we obtain a unique extension of $y(x,t)$ from $D_0 \cup D_T$ into $\hat{D}$ and thus obtain a solution of the equation (2.2) in the complete rectangular region D. When this extension is complete we will also have available the function $\frac{\partial y}{\partial x}(1,t)$ of the variable t and this gives us the boundary control function u(t), since

$$u(t) = \tau \frac{\partial y}{\partial x}(1,t).$$

Now $\frac{\partial y}{\partial x}(1,t)$ has the same smoothness properties on the interval $[0,T]$ as $\frac{\partial y}{\partial x}(x,0)$, $\frac{\partial y}{\partial t}(x,0)$ on $0 \leq x \leq 1$. Thus in the case $T = 2T_1$ we have in general a uniquely determined smooth control u(t) on [0,T] bringing the given initial state into equilibrium at time $T = 2T_1$. It should be noted that this time $T = 2T_1$ is the time required for a wave to travel from one end of the string to the other and return.

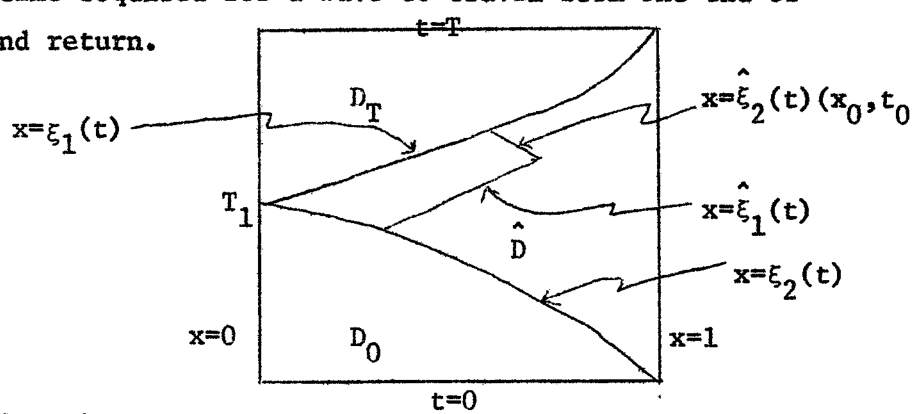


FIGURE 4
$T = 2T_1$

Let $\gamma_0 = \max_{t \in [0,T]} |u(t)|$. If we impose an a priori constraint

$|u(t)| \leq r$,

and if it should happen that $r > r_0$, which is certainly conceivable, we see that u(t) is the unique control bringing the given initial state to rest in least possible time and yet u(t) is not a "bang-bang" control—in fact it nowhere assumes the values $\pm$ r.

Although D_0 and D_T do not intersect in a domain D_{0T} they do meet at the point $(0,T_1) = (0,T-T_1)$. There is no reason to believe that the values of η_1 and η_2, as determined separately in D_0 and D_T, will agree at this point.

As a result we can expect discontinuities in η_1 and η_2 along the boundaries of D_T and D_0, respectively. This means that controls are instantly turned on and instantly shut off.

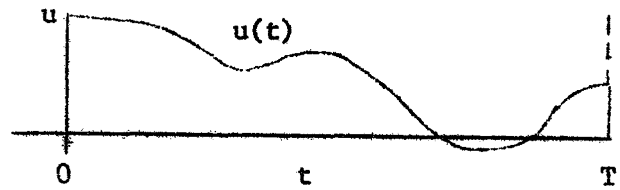


FIGURE 5

Case (iii) $T > 2T_1$

In this case D_0 and D_T are separated in time by an interval $T_1 < t < T-T_1$ of length $T - 2T_1$. For points (0,t) with t in this interval we have thus far assumed only the condition

$$y(0,t) \equiv 0$$

which of course implies $\frac{\partial y}{\partial t}(0,t) \equiv 0$. Since

$$\eta_1 = \frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x}$$

$$\eta_2 = \frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x}$$

this alone is not enough to provide initial values for η_1 and η_2 when integrated along characteristics $x = \hat{\xi}_1(t)$, $x = \hat{\xi}_2(t)$ which initiate on $\{(0,t) \mid T_1 < t < T-T_1\}$. One may specify $\frac{\partial y}{\partial x}$ any way one wishes on this interval and then extend the solution y(x,t) into $\hat{D}$, ultimately obtaining $u(t) = \tau \frac{\partial y}{\partial x}(1,t)$ as a control bringing the initial state into equilibrium at time t = T. There are infinitely many controls now corresponding to the infinitely many possible choices of $\frac{\partial y}{\partial x}(0,t)$ on $T_1 < t < T-T_1$.

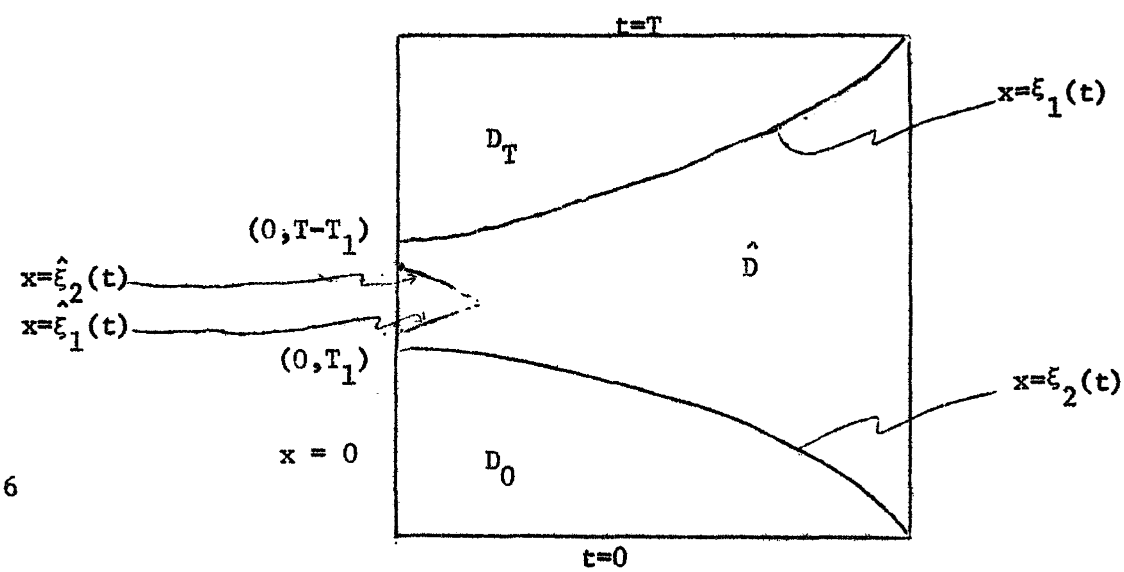


FIGURE 6
$T>2T_1$

There are a number of uses which we can make of the undetermined function $\frac{\partial y}{\partial x}(0,t)$. We note that $\frac{\partial y}{\partial t}(0,t) \equiv 0$ is already continuous for $0 \le t \le T$. If we select $\frac{\partial y}{\partial x}(0,t)$, $T_1 < t < T-T_1$ so as to continuously join $\frac{\partial y}{\partial x}(0,t)$, $0 \le t \le T$ with $\frac{\partial y}{\partial x}(0,t)$, $T-T_1 \le t \le T$ then all data on $0 \le t \le T$, $x = 0$ are continuous and there will no longer be discontinuities on the characteristics $x = \xi_1(t)$, $x = \xi_2(t)$ bounding D_T and D_0, respectively. The control u(t) starts off with u(0+) = 0 and ends with u(T-) = 0. This avoids sharp stresses in the physical medium.

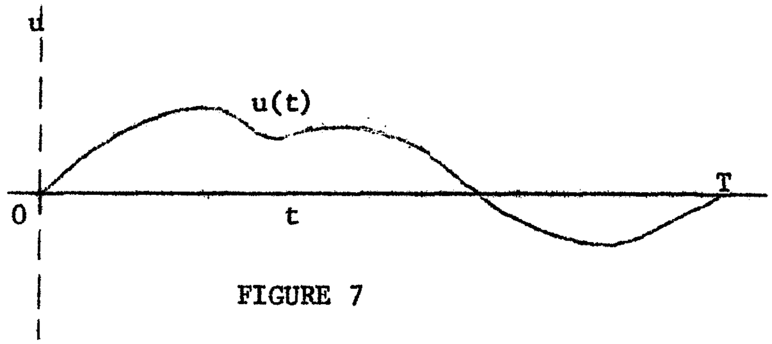


FIGURE 7

5. Optimization Problem for the Vibrating String

Since in Case (iii) the control function u(t) is not unique, it makes sense to pose optimizations problems. The "minimum energy" problem, for instance, requires one to select, among controls u(t) bringing the initial conditions into equilibrium at time T, that control for which

$$\int_0^T u(t)^2 dt$$

is as small as possible. We treat this problem in the following way.

It is quite straightforward to prove that this problem has a solution, so let us assume that $\hat{u}(t)$ is the optimal control and that it results from a choice

$$\frac{\partial y}{\partial x}(0,t) = \hat{w}(t), \qquad T_1 < t < T-T_1 \tag{5.1}$$

Other controls arise from different choices.

$$\frac{\partial y}{\partial x}(0,t) = w(t) = \hat{w}(t) + \epsilon\tilde{w}(t) \tag{5.2}$$

These other choices lead to solutions

$$y(x,t) = \hat{y}(x,t) + \varepsilon\tilde{y}(x,t) \tag{5.3}$$

where $\hat{y}(x,t)$ is the solution resulting from $\frac{\partial y}{\partial x}(0,t) = \hat{w}(t)$ and the given initial and terminal data while $\tilde{y}(x,t)$ is a solution of the equation (2.2) which vanishes identically in D_0 and D_T and satisfies

$$\left.\begin{aligned} \frac{\partial\tilde{y}}{\partial t}(0,t) &\equiv 0 \\ \frac{\partial\tilde{y}}{\partial x}(0,t) &\equiv \tilde{w}(t) \end{aligned}\right\} \qquad T_1 < t < T-T_1 \tag{5.4}$$

From $y(x,t)$ we obtain a control $u(t) = \hat{u}(t) + \varepsilon\tilde{u}(t)$ with

$$\tilde{u}(t) = \tau \frac{\partial\tilde{y}}{\partial x}(1,t) \tag{5.5}$$

We compute

$$\int_0^T u(t)^2 dt = \int_0^T (\hat{u}(t) + \varepsilon\tilde{u}(t))^2 dt$$

$$= \int_0^T \hat{u}(t)^2 dt + 2\varepsilon \int_0^T \hat{u}(t)\tilde{u}(t)dt + \varepsilon^2 \int_0^T \tilde{u}(t)^2 dt$$

and conclude that $\hat{u}(t)$ is optimal if and only if

$$\int_0^T \hat{u}(t)\tilde{u}(t)dt = 0$$

for all $\tilde{u}(t)$ arising in the manner described above, i.e., from choices $\frac{\partial y}{\partial x}(o,t)$ as in equation (5.2).

Let $Z(x,t)$ be the solution of

$$\rho(x)\frac{\partial^2 Z}{\partial t^2} - \tau\frac{\partial^2 Z}{\partial x^2} = 0 \tag{5.6}$$

determined in $\hat{D}$ by data $Z(1,t)$, $\frac{\partial Z}{\partial x}(1,t)$ given on the line $x = 1$, $0 \leq t \leq T$. (Note that $\hat{D}$ is the intersection of the rectangle D: $0 \leq t \leq T$, $0 \leq x \leq 1$, with the domain of determinacy of $x = 1$, $0 \leq t \leq T$.)

Then we compute

$$0 = \int_{\hat{D}} \left\{ \frac{\partial Z}{\partial t} \left(\rho \frac{\partial^2 \tilde{y}}{\partial t^2} - \tau \frac{\partial^2 \tilde{y}}{\partial x^2} \right) + \tau \frac{\partial Z}{\partial x} \left(\frac{\partial^2 \tilde{y}}{\partial t \partial x} - \frac{\partial^2 \tilde{y}}{\partial x \partial t} \right) \right.$$

$$\left. + \left(\rho \frac{\partial^2 Z}{\partial t^2} - \tau \frac{\partial^2 Z}{\partial x^2} \right) \frac{\partial \tilde{y}}{\partial t} + \left(\frac{\partial^2 Z}{\partial t \partial x} - \frac{\partial^2 Z}{\partial x \partial t} \right) \tau \frac{\partial \tilde{y}}{\partial x} \right\} \, dxdt$$

$$= \int_{\hat{D}} \underset{(x,t)}{\text{div}} \begin{pmatrix} -\tau \frac{\partial Z}{\partial t} \frac{\partial \tilde{y}}{\partial x} - \tau \frac{\partial \tilde{y}}{\partial t} \frac{\partial Z}{\partial x} \\ \rho \frac{\partial Z}{\partial t} \frac{\partial \tilde{y}}{\partial t} + \tau \frac{\partial Z}{\partial x} \frac{\partial \tilde{y}}{\partial x} \end{pmatrix} dxdt$$

Now apply the divergence theorem.

$$0 = \int_0^T \left(-\tau \frac{\partial Z}{\partial t}(1,t) \frac{\partial \tilde{y}}{\partial x}(1,t) - \tau \frac{\partial \tilde{y}}{\partial t}(1,t) \frac{\partial Z}{\partial x}(1,t) \right) dt$$

$$- \int_0^T \left(-\tau \frac{\partial Z}{\partial t}(0,t) \frac{\partial \tilde{y}}{\partial x}(0,t) - \tau \frac{\partial \tilde{y}}{\partial t}(0,t) \frac{\partial Z}{\partial x}(0,t) \right) dt$$

$$+ \int_{x=\xi_1(t)} \begin{pmatrix} -\tau \frac{\partial Z}{\partial t} \frac{\partial \tilde{y}}{\partial x} - \tau \frac{\partial \tilde{y}}{\partial t} \frac{\partial Z}{\partial x} \\ \rho \frac{\partial Z}{\partial t} \frac{\partial \tilde{y}}{\partial t} + \tau \frac{\partial Z}{\partial x} \frac{\partial \tilde{y}}{\partial x} \end{pmatrix}^T \begin{pmatrix} -1 \\ \sqrt{\frac{\tau}{\rho}} \end{pmatrix} d\sigma$$

$$+ \int_{x=\xi_2(t)} \begin{pmatrix} -\tau \frac{\partial Z}{\partial t} \frac{\partial \tilde{y}}{\partial x} - \tau \frac{\partial \tilde{y}}{\partial t} \frac{\partial Z}{\partial x} \\ \rho \frac{\partial Z}{\partial t} \frac{\partial \tilde{y}}{\partial t} + \tau \frac{\partial Z}{\partial x} \frac{\partial \tilde{y}}{\partial x} \end{pmatrix}^T \begin{pmatrix} -1 \\ -\sqrt{\frac{\tau}{\rho}} \end{pmatrix} d\sigma$$

We now put

$$\frac{\partial Z}{\partial t}(1,t) = -\hat{u}(t), \quad \frac{\partial Z}{\partial x}(1,t) = 0$$

On $x = \xi_1(t)$ we are looking at the integral of

$$\left(\tau \frac{\partial Z}{\partial t} \frac{\partial \tilde{y}}{\partial x} + \tau \frac{\partial \tilde{y}}{\partial t} \frac{\partial Z}{\partial x}\right) + \frac{\sqrt{\tau}}{\rho}\left(\rho \frac{\partial Z}{\partial t} \frac{\partial \tilde{y}}{\partial t} + \tau \frac{\partial Z}{\partial x} \frac{\partial \tilde{y}}{\partial x}\right)$$

$$= \rho\left[\left(c^2 \frac{\partial Z}{\partial t} \frac{\partial \tilde{y}}{\partial x} + c^2 \frac{\partial \tilde{y}}{\partial t} \frac{\partial Z}{\partial x}\right) + c \frac{\partial Z}{\partial t} \frac{\partial \tilde{y}}{\partial t} + c^3 \frac{\partial Z}{\partial x} \frac{\partial \tilde{y}}{\partial x}\right]$$

$$= \rho c^2 \frac{\partial \tilde{y}}{\partial x}\left[\frac{\partial Z}{\partial t} + c \frac{\partial Z}{\partial x}\right] + \rho c \frac{\partial \tilde{y}}{\partial t}\left[c \frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial t}\right]$$

$$= \rho c\left[c \frac{\partial \tilde{y}}{\partial x} + \frac{\partial \tilde{y}}{\partial t}\right]\left[c \frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial t}\right]$$

which is identically zero becuase $c \frac{\partial \tilde{y}}{\partial x} + \frac{\partial \tilde{y}}{\partial t}$ must be continuous across $x = \xi_1(t)$ and it vanishes in D_T.

Similar considerations apply on $x = \xi_2(t)$ and thus we have

$$0 = \int_0^T \tau \frac{\partial Z}{\partial t}(1,t) \frac{\partial \tilde{y}}{\partial x}(1,t)dt - \int_0^T \tau \frac{\partial Z}{\partial t}(0,t) \frac{\partial \tilde{y}}{\partial x}(0,t)dt$$

whence

$$\int_0^T \hat{u}(t)\tilde{u}(t)dt = \int_0^T \tau \frac{\partial Z}{\partial t}(0,t)\tilde{w}(t)dt$$

For $\hat{u}$ to be optimal the left hand side must be zero for all $\tilde{u}$ hence

$$\int_0^T \frac{\partial Z}{\partial t}(0,t)\, \tilde{w}(t)dt = 0$$

for all $\tilde{w}$. But $\tilde{w}$ is abritrary on $(T_1, T-T_1)$ so we conclude

$$\frac{\partial Z}{\partial t}(0,t) \equiv 0, \qquad t\, \varepsilon (T_1,T-T_1)$$

Thus $\hat{u}$ is optimal if and only if

$$\frac{\partial Z}{\partial t}(1,t) = -\hat{u}(t),\ \frac{\partial Z}{\partial x}(1,t) = 0, \qquad \rho \frac{\partial^2 Z}{\partial t^2} - \tau \frac{\partial^2 Z}{\partial x^2} = 0$$

leads to $\frac{\partial Z}{\partial t}(0,t) \equiv 0, \quad t\ \varepsilon\ (T_1,T-T_1).$

6. Vibrating String with Two-Boundary Control

One may also consider two-boundary control of the vibrating string:

$$\rho(x) \frac{\partial^2 u}{\partial t^2} - \tau \frac{\partial^2 u}{\partial x^2} = 0$$

$$u_0(t) = -\tau \frac{\partial u}{\partial x}(0,t) \qquad u_1(t) = \tau \frac{\partial u}{\partial x}(1,t) \qquad (6.1)$$

Considerations similar to those presented above show that in this case we have controllability if $T \geq T_1$, i.e., the time interval is halved compared to that for single boundary control. The relevant diagrams are shown here. The controls are unique if $T = T_1$ and non-unique if $T > T_1$.

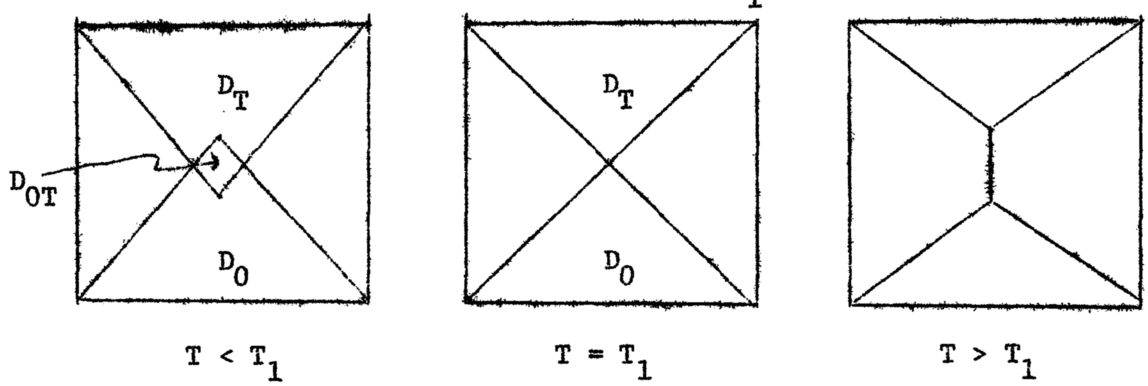


FIGURE 8

CHAPTER II

CONTROL OF DISTRIBUTED PARAMETER SYSTEMS WITH MORE THAN ONE SPACE VARIABLE

7. Approximate Controllability

When we pass to more complicated control problems, in particular higher dimensional ones, we find that the constructive methods of the last chapter fail us completely. In fact, to obtain any results at all we have to content ourselves with what may be called approximate controllability rather than the type of controllability discussed above for the string.

The concept of approximate controllability may be illustrated by the example of the circular membrane. To make matters reasonably simple we shall assume uniform density and elasticity properties. The relevant equations then are

$$\rho \frac{\partial^2 w}{\partial t^2} - \tau \frac{\partial^2 w}{\partial x^2} - \tau \frac{\partial^2 w}{\partial y^2} = 0$$

$$u(\xi,t) = \tau\left(\frac{\partial w}{\partial x}(\xi,t),\ \frac{\partial w}{\partial y}(\xi,t)\right)\begin{pmatrix}\eta_1(\xi,t)\\ \eta_2(\xi,t)\end{pmatrix} \tag{7.1}$$

where $\eta(\xi,t) = \begin{pmatrix}\eta_1(\xi,t)\\ \eta_2(\xi,t)\end{pmatrix}$ is the unit outward normal to the cylinder

$$D\ \begin{cases} ||\xi||^2 = x^2 + y^2 \le 1 \\ 0 \le t \le T \end{cases}$$

at a point (ξ,t) on its boundary.

We will give initial conditions

$$w(x,y,0) \equiv 0, \quad \frac{\partial w}{\partial t}(x,y,0) \equiv 0. \tag{7.2}$$

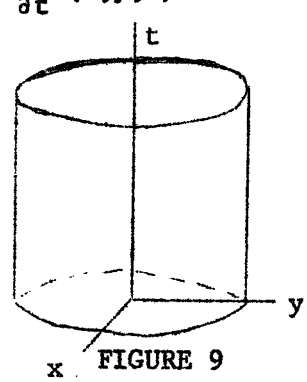


FIGURE 9

We consider the real Hilbert space H_E consisting of states $w(x,y,T)$, $\frac{\partial w}{\partial t}(x,y,T)$ having finite energy

$$E(w) = \frac{1}{2} \int_{x^2+y^2 \leq 1} \left\{ \rho \left(\frac{\partial w}{\partial t}\right)^2 + \tau \left(\frac{\partial w}{\partial x}\right)^2 + \tau \left(\frac{\partial w}{\partial y}\right)^2 \right\} dxdy \tag{7.3}$$

In H_E we use the inner product

$$(v,w)_E = \int_{x^2+y^2 \leq 1} \left\{\rho \frac{\partial v}{\partial t} \frac{\partial w}{\partial t} + \tau \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} + \tau \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} \right\} dxdy \tag{7.4}$$

We let $R(T)$ be the set of all states $w(x,y,T)$, $\frac{\partial w}{\partial t}(x,y,T)$ in H_E which are terminal states for solutions (in the generalized sense) of equation (7.1) with initial conditions (7.2). $R(T)$ is a subspace of H_E called the reachable set for time T.

We will say that our system is <u>approximately controllable</u> in time T if $R(T)$ is dense in H_E relative to the topology induced by the energy norm.

Now $R(T)$ fails to be dense in H_E if and only if there is a state

$$v(x,y,T), \quad \frac{\partial v}{\partial t}(x,y,T) \quad \text{in } H_E$$

orthogonal to all states in $R(T)$, i.e.

$$\int_{x^2+y^2 \leq 1} \left\{ \rho \frac{\partial v}{\partial t} \frac{\partial w}{\partial t} + \tau \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} + \tau \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} \right\} dxdy = 0 \tag{7.5}$$

for all $w, \frac{\partial w}{\partial t} \in R(T)$.

Let $v(x,y,t)$ have terminal values $v(x,y,T)$, $\frac{\partial v}{\partial t}(x,y,T)$ and solve

$$\rho \frac{\partial^2 v}{\partial t^2} - \tau \frac{\partial^2 v}{\partial x^2} - \tau \frac{\partial^2 v}{\partial y^2} = 0$$

$$\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0 \tag{7.6}$$

for $x^2 + y^2 \leq 1$, $t \leq T$. Again we can only expect the equation to be satisfied in a generalized sense.

We then compute

$$0 = \int_D \{ \frac{\partial v}{\partial t} \left(\rho \frac{\partial^2 w}{\partial t^2} - \tau \frac{\partial^2 w}{\partial x^2} - \tau \frac{\partial^2 w}{\partial y^2} \right) + \tau \frac{\partial v}{\partial x} \left(\frac{\partial^2 w}{\partial t \partial x} - \frac{\partial^2 w}{\partial x \partial t} \right)$$

$$+ \tau \frac{\partial w}{\partial x} \left(\frac{\partial^2 v}{\partial t \partial x} - \frac{\partial^2 v}{\partial x \partial t} \right) + \tau \frac{\partial v}{\partial y} \left(\frac{\partial^2 w}{\partial t \partial y} - \frac{\partial^2 w}{\partial y \partial t} \right)$$

$$+ \tau \frac{\partial w}{\partial y} \left(\frac{\partial^2 v}{\partial t \partial y} - \frac{\partial^2 v}{\partial y \partial t} \right) + \frac{\partial w}{\partial t} \left(\rho \frac{\partial^2 v}{\partial t^2} - \tau \frac{\partial^2 v}{\partial x^2} - \tau \frac{\partial^2 v}{\partial y^2} \right) \} \, dxdydt$$

$$= \int_D \operatorname{div}_{x,y,t} \begin{pmatrix} - \tau \frac{\partial v}{\partial t} \frac{\partial w}{\partial x} - \tau \frac{\partial w}{\partial t} \frac{\partial v}{\partial x} \\ - \tau \frac{\partial v}{\partial t} \frac{\partial w}{\partial y} - \tau \frac{\partial w}{\partial t} \frac{\partial v}{\partial y} \\ \rho \frac{\partial v}{\partial t} \frac{\partial w}{\partial t} + \tau \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} + \tau \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} \end{pmatrix} dxdydt$$

Now, using the divergence theorem, we get

$$0 = \int_{x^2+y^2 \le 1} \{ \left(\rho \frac{\partial v}{\partial t} \frac{\partial w}{\partial t} + \tau \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} + \tau \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} \right) \Big|_{t=T}$$

$$- \left(\rho \frac{\partial v}{\partial t} \frac{\partial w}{\partial t} + \tau \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} + \tau \frac{\partial v}{\partial x} \frac{\partial w}{\partial y} \right) \Big|_{t=0} \} \, dxdy$$

$$+ \int_{\partial D} \{ - \tau \frac{\partial v}{\partial t} \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - \tau \frac{\partial w}{\partial t} \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \quad d\sigma$$

The second term in the first integral vanishes because $w = 0$ at $t = 0$ and the second term in the second integral vanishes because of the boundary conditions on v.

Then, since $\tau \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \right) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = u$

$$\int_{\partial D} \left(\frac{\partial v}{\partial t} \, u \right) d\sigma = \int_{x^2+y^2 \le 1} \left(\rho \frac{\partial v}{\partial t} \frac{\partial w}{\partial t} + \tau \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} + \tau \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} \right) \Big|_{t=T} dxdy$$

is true for all admissible u.

Thus, R(T) fails to be dense in H_E if and only if there is a terminal state

$$v(x,y,T), \quad \frac{\partial v}{\partial t}(x,y,T)$$

with H_E-norm different from zero such that $\frac{\partial v}{\partial t} \equiv 0, \quad x^2 + y^2 = 1, \quad 0 \leq t \leq T.$

$$\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \equiv 0, \quad x^2 + y^2 = 1, \; 0 \leq t \leq T.$$

where v(x,y,t) satisfies the equation (7.1) and the prescribed terminal conditions. So the next question is -- is such a solution of equation (7.1) possible?

8. Uniqueness Theorem of Holmgren (as extended by Fritz John).

The tool which we need in order to examine this problem is the uniqueness theorem of Holgren as extended by Fritz John.

A surface $\psi(x,y,t) = c$ is a characteristic surface for the partial differential equation

$$\rho \frac{\partial^2 w}{\partial t^2} - \tau \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0 \tag{8.1}$$

if and only if ψ satisfies

$$\rho(\psi_t)^2 - \tau \left((\psi_x)^2 + (\psi_y)^2 \right) = 0 \tag{8.2}$$

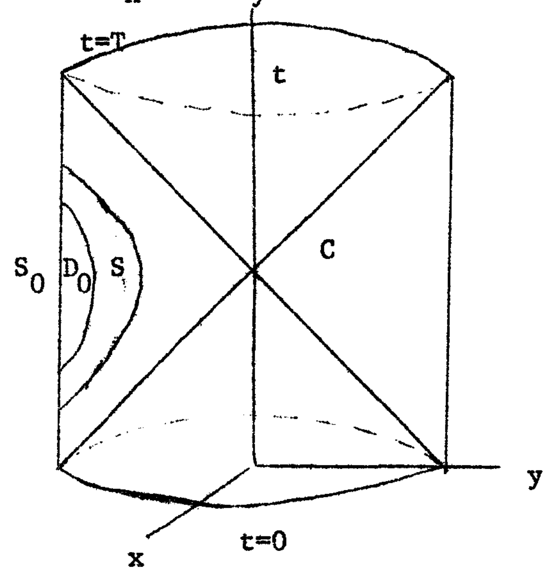


FIGURE 10

The conic surface C in the figure can be described in this manner. The figure illustrates the case $T = 2\left(\frac{\rho}{\tau}\right)$, the time required for a wave to cross the membrane.

Consider now a domain D_0 bounded by two surfaces $S_0 \subseteq \{(x,y,t) \mid x^2+y^2 = 1\}$ and $S = \{(x,y,t) \mid \phi(x,y,t) = c\}$ where ϕ is smooth.

We will suppose that S is uniformly non-characteristic: there is some $\varepsilon > 0$ such that

$$\rho(\psi_t)^2 - \tau((\psi_x)^2 + (\psi_y)^2) < -\varepsilon \qquad (8.3)$$

for all (x,y,t) on S. Holmgren's uniqueness theorem then states:

Let v(x,y,t) and $\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ be given for $(x,y,t) \varepsilon S_0$. Then there is at most one solution (there may be none) v(x,y,t) of (8.1) in D_0 assuming these given values on S_0.

9. Approximate Controllability of the Vibrating Membrane

From this result we conclude that v(x,y,t) and $\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ in a set $\{(x,y,t) \mid x^2 + y^2 = 1,\ t_0 \leq t \leq t_1\}$ determine any solution v(x,y,t) of equation (8.1) in the region which is the solid of revolution formed by rotating the two triagles shown in Figure 11.

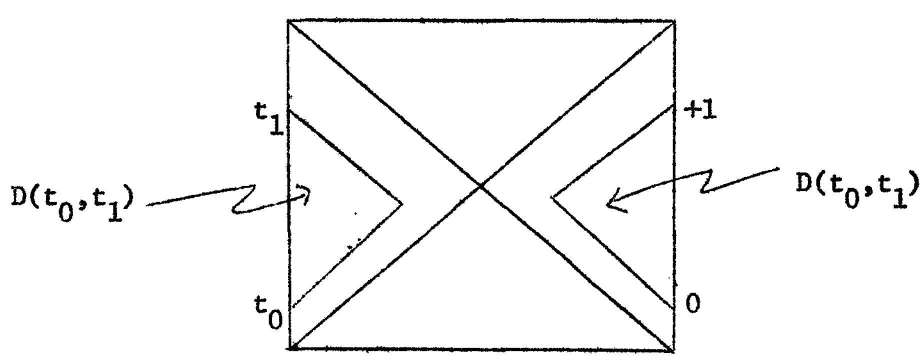


FIGURE 11

The fact that for the v(x,y,t) of interest to us we have $\frac{\partial v}{\partial t} \equiv 0$ on $x^2 + y^2 = 1$ shows that for $x^2 + y^2 = 1$, δ real

$$v(x,y,t) = v(x,y,t+\delta).$$

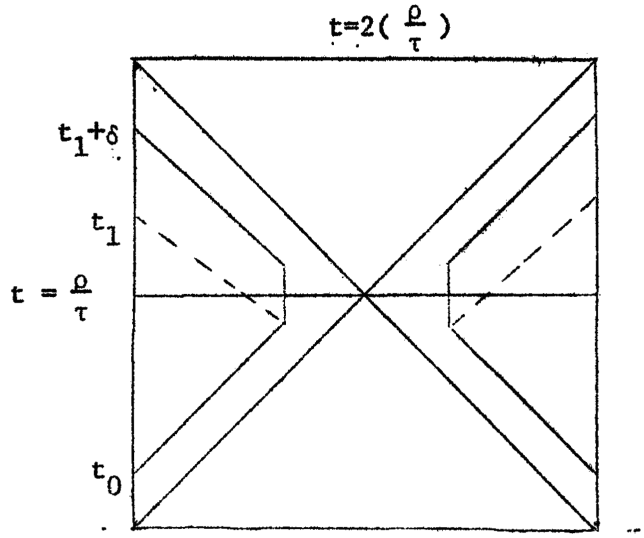


FIGURE 12

Then in regions D_{t_0,t_1} , $D_{t_0+\delta,\ t_1+\delta}$, our uniqueness theorem shows that

$$v(x,y,t+\delta) \equiv v(x,y,t).$$

Letting δ vary we see that $\frac{\partial v}{\partial t} \equiv 0$ in a region $D_{t_0,t_1+\delta}$ whose cross section consists of two trapezoids as shown in Figure 12.

Letting $t_1 \to 2\ (\frac{\rho}{\tau})$, $t_0 \to 0$, $\delta \to 0$ we can show that $\frac{\partial v}{\partial t} \equiv 0$ in some neighborhood of every point $(x,y,\frac{\rho}{\tau})$, $0 < x^2 + y^2 \leq 1$.

We conclude that $\frac{\partial^2 y}{\partial t^2} = 0$ a.e. for $t = \frac{\rho}{\tau}$. But then $v(x,y,\frac{\rho}{\tau}) = \tilde{v}(x,y)$.

$$\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2} = 0 \qquad \left(\frac{\partial \tilde{v}}{\partial x}, \frac{\partial \tilde{v}}{\partial y} \right) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = 0$$

$\Rightarrow \tilde{v}$ is a constant, i.e., $v(x,y,\frac{\rho}{\tau})$ is a constant. But such a state has zero energy. Since energy is conserved for $v(x,y,t)$ we conclude that the energy at time T is also zero. Thus $v(x,y,T)$ is actually not in the Hilbert space H_E but is a multiple of the vector which we excluded from $L^2\ (\{\{(x,y)(x^2 + y^2 \leq 1\})$ to form H_E.

Thus no non-zero vector in H_E can be orthogonal to all $w \in R(T) \Rightarrow R(T)$ is dense in H_E for $T = 2\ (\frac{\rho}{\tau})$. The same is true for $T > 2\ (\frac{\rho}{\tau})$, of course.

Now consider $T < 2\ (\frac{\rho}{\tau})$.

In this case the appropriate diagram is the one shown below . Note the configuration at $t = \frac{T}{2}$. The characteristic surfaces emanating from the boundaries of the sets $x^2 + y^2 \leq 1$, $t = 0$ and $x^2 + y^2 = 1$, $t = T$, cut a smaller disk, which we shall call K, out of the disc $x^2 + y^2 \leq 1$, $t = \frac{T}{2}$.

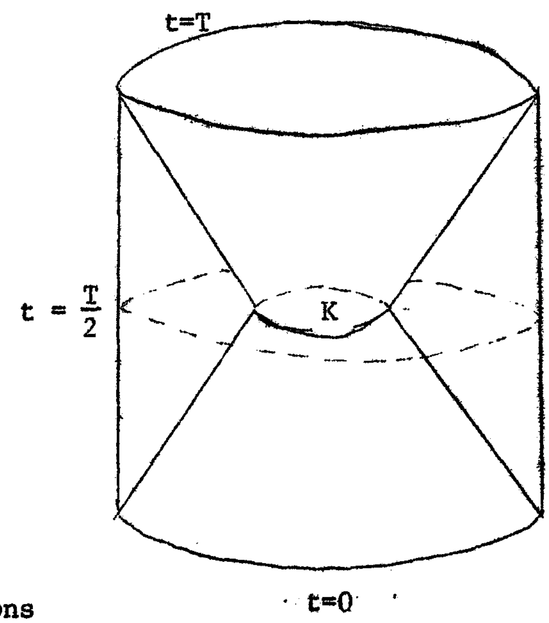


FIGURE 13

Let us pose conditions

$$v(x,y,\tfrac{T}{2}) \equiv 0 \qquad \text{for all } x,\ y,\ x^2 + y^2 \leq 1$$

$$\frac{v}{t}(x,y,\tfrac{T}{2}) \equiv 0 \qquad \text{outside K}$$

but $\frac{\partial v}{\partial t}(x,y,\frac{T}{2})$ is not the zero function inside K. Then the energy $E(v,\frac{T}{2})$ is clearly non-zero. We let this "initial" state evolve forward and backward in time using equation (8.1) (which is reversible) and obtain a solution $v(x,y,t)$ of equation (8.1) which satisfies

$$v(x,y,t) \equiv \frac{\partial v}{\partial t}(x,y,t) \equiv 0, \qquad x^2 + y^2 = 1$$

On the other hand energy is conserved so the H_E-norm of $v(x,y,t)$, $\frac{\partial v}{\partial t}(x,y,T)$ is not zero. Thus, according to our earlier results, we see that R(T) cannot be dense in H_E for $T < 2(\frac{\rho}{\tau})$.

A number of interesting questions present themselves in connection with this problem of the vibrating membrane. For example:

(i) If we allow the controls to be any functions in L^2 ($\{x^2 + y^2 = 1$, $0 \leq t \leq T\}$) then does R(T) actually coincide with $L^2[E,T]$, i.e., is R(T) closed, when $T \geq 2 (\frac{\rho}{\tau})$. For the string the answer to the analogous question is "yes."

(ii) If controls are exercised only on some arc of the circle $x^2 + y^2 = 1$ for all t, what then is the control time which replaces $2 (\frac{\rho}{\tau})$. Is it still finite? These are questions still to be answered.

CHAPTER III.

HILBERT SPACE FORMULATION OF THE DISTRIBUTED-PARAMETER CONTROL PROBLEM

10. Problems With Control Function Occurring in the Differential Equation.

We turn now from this "geometric" point of view to one based primarily on Hilbert space theory. We shall also pass, at least temporarily, from boundary value control problems to problems where the control forces have the form

$$u(x,t) = g(x)f(t). \tag{10.1}$$

The function $g(x)$, which we call the force distribution function, determines the manner in which the control force is applied to the physical medium. The function $f(t)$ is real valued and can be prescribed at will by the operator of the plant.

Consider, for example, a stretched string attached to a rigid rod, as shown in the diagram. The rod is pivoted at the end to some support and we are free to move the other end.

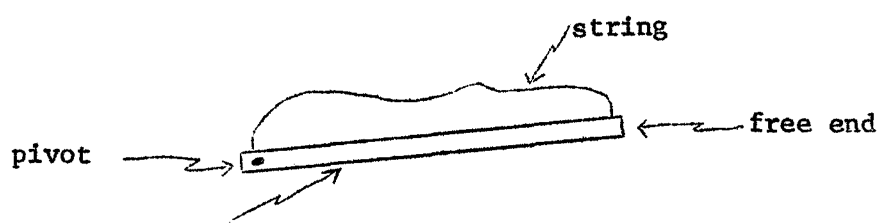


Figure 14

If we assume that the rod is moved only through small angles, then its configuration can be described by

$$xh(t), \quad 0 \leq x \leq 1, \quad t \geq 0.$$

Let $w(x,t)$ denote the string displacement relative to an inertial frame of reference and let $y(x,t)$ denote its displacement relative to the rod. Then, w satisfies

$$\rho \frac{\partial^2 w}{\partial t^2} - \tau \frac{\partial^2 w}{\partial x^2} = 0, \quad \begin{array}{l} w(0,t) \equiv 0 \\ w(1,t) \equiv h(t) \end{array} \tag{10.2}$$

Now

$$w(x,t) = y(x,t) + xh(t) \tag{10.3}$$

and so, by substituting in (10.2),

$$\rho\left(\frac{\partial^2 y}{\partial t^2} + xh''(t)\right) - \tau\left(\frac{\partial^2 y}{\partial x^2}\right) = 0$$

yielding

$$\rho \frac{\partial^2 y}{\partial t^2} - \tau \frac{\partial^2 y}{\partial x^2} = xf(t) \tag{10.4}$$

where

$$f(t) = -\rho h''(t). \tag{10.5}$$

The new boundary conditions are

$$y(o,t) \equiv 0, \; y(1,t) \equiv 0. \tag{10.6}$$

Let us consider the partial differential equation

$$\rho(x) \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}(p(x) \frac{\partial u}{\partial x}) = g(x)f(t) \tag{10.7}$$

with boundary conditions

$$\begin{aligned} A_0 u(0,t) + B_0 \frac{\partial u}{\partial x}(0,t) &\equiv 0, \qquad A_0^2 + B_0^2 \neq 0 \\ A_1 u(1,t) + B_1 \frac{\partial u}{\partial x}(1,t) &\equiv 0 \qquad A_1^2 + B_1^2 \neq 0 \end{aligned} \tag{10.8}$$

We give initial conditions

$$u(x,0) \equiv u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) \equiv v_0(x) \tag{10.9}$$

and stipulate that both $\frac{d^2u_0}{dx^2}$ and $\frac{dv_0}{dx}$ lie in $L^2[0,1]$. It should be noted that this is a requirement somewhat stronger than finite energy. For finite energy all we need is that $\frac{du_0}{dx}$ and v_0 be in $L^2[0,1]$.

Again we wish to find $f(t) \in L^2[0,T]$ such that the resulting solution of the partial differential equation will satisfy

$$u(x,T) \equiv 0, \qquad \frac{\partial u}{\partial t}(x,T) \equiv 0. \tag{10.10}$$

11. Formulation of Problem in Hilbert Space.

One considers first the linear operator

$$L(u) = -\frac{1}{\rho(x)} \frac{\partial}{\partial x} (p(x) \frac{\partial u}{\partial x}) \tag{11.1}$$

defined on the domain in $L^2[0,1]$ consisting of functions $u(x)$ whose second derivatives lie in $L^2[0,1]$ and satisfy the boundary conditions (10.10).

If in $L^2[0,1]$ we employ the inner product

$$(u,v) = \int_0^1 u(x)v(x)\rho(x)dx \tag{11.2}$$

we make $L^2[0,1]$ into a Hilbert space H and, with respect to this inner product the operator L can be shown to be an unbounded self-adjoint operator which is positive definite or positive semi-definite, depending upon the boundary conditions. In order to make our presentation simpler we will assume that we are dealing with a case where the operator L is positive definite. In this case L possesses a sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1} < \dots \tag{11.3}$$

and corresponding eigenfunctions $\phi_1(x)$, $\phi_2(x)$, ... which form an orthonormal basis for H relative to the inner product described above, i.e.,

$$\int_0^1 \phi_k(x)\phi_\ell(x)\rho(x)\,dx = \delta_{k1} \tag{11.4}$$

Given any $\psi(x) \in L^2[0,1]$ we have the unique representation

$$\psi(x) = \sum_{k=1}^{\infty} \psi_k \phi_k(x) \tag{11.5}$$

where

$$\psi_k = \int_0^1 \psi(x)\phi_k(x)\rho(x)\,dx. \tag{11.6}$$

We let u(x,t) be the solution of the equation (10.7) with the given initial and boundary conditions. We put

$$u(x,t) = \sum_{k=1}^{\infty} \beta_k(t)\phi_k(x),$$

$$u_0(x) = \sum_{k=1}^{\infty} \mu_k\phi_k(x),$$

$$v_0(x) = \sum_{k=1}^{\infty} \nu_k\phi_k(x),$$

$$g(x) = \sum_{k=1}^{\infty} \gamma_k\phi_k(x). \tag{11.7}$$

Proceeding formally we find that

$$\frac{d^2\beta_k}{dt^2} + \lambda_k\beta_k = \gamma_k f(t), \qquad k = 1,2,\dots \tag{11.8}$$

with initial conditions $\beta_k(0) = \mu_k$, $\frac{d\beta_k}{dt}(0) = \nu_k$. We use the variation of parameters formula to integrate these equations and we find that, with

$\omega_k = \sqrt{\lambda_k}$,

$$\beta_k(T) = \mu_k \cos(\omega_k T) + \frac{\nu_k}{\omega_k} \sin(\omega_k T)$$

$$+ \int_0^T \frac{\gamma_k}{\omega_k} \sin(\omega_k (T-s)) f(s) ds$$

$$\frac{d\beta_k}{dt}(T) = - \mu_k \omega_k \sin(\omega_k T) + \nu_k \cos(\omega_k T)$$

$$+ \int_0^T \gamma_k \cos(\omega_k (T-s)) f(s) ds. \tag{11.9}$$

Consequently, if we wish to have

$$\beta_k(T) = \frac{d\beta_k}{dt}(T) = 0, \quad k = 1,2,3,\ldots$$

we must have

$$\int_0^T \sin (\omega_k (T-s)) f(s) ds = -\frac{\mu_k \omega_k}{\gamma_k} \cos(\omega_k T) - \frac{\nu_k}{\gamma_k} \sin(\omega_k T)$$

$$\int_0^T \cos(\omega_k (T-s)) f(s) ds = \frac{\mu_k \omega_k}{k} \sin(\omega_k T) - \frac{\nu_k}{\gamma_k} \cos(\omega_k T). \tag{11.10}$$

12. A Trigonometric Moment Problem.

We see then that control can be effected if we can solve a certain trigonometric moment problem, namely the one given above, for a function $f(t)$ in $L^2[0,T]$. It is easily seen that the above problem is equivalent to

$$\int_0^T \sin(\omega_k s) f(s) ds = \frac{\mu_k \omega_k}{\gamma_k}$$

$$\int_0^T \cos(\omega_k s) f(s) ds = - \frac{\nu_k}{\gamma_k}$$

or

$$\int_0^T e^{i\omega_k s} f(s) ds = \frac{\mu_k \omega_k}{\gamma_k} - i \frac{\nu_k}{\gamma_k} = c_k$$

$$\int_0^T e^{-i\omega_k s} f(s) ds = \frac{\mu_k \omega_k}{\gamma_k} + i \frac{\nu_k}{\gamma_k} = d_k. \tag{12.1}$$

With the assumption which we have made on the initial conditions we readily see that $\Sigma\ |c_k|^2 + |d_k|^2 < \infty$ provided

$$\lim_{k\to\infty} \inf \quad k|\gamma_k| > 0 \tag{12.2}$$

which assumption we now make.*

In the abstract a moment problem has the form

$$(p_k, f) = c_k, \qquad k = 1,2,3,\ldots \tag{12.3}$$

where the p_k are certain elements of a separable Hilbert space H and f is a fixed element of that space. The c_k are square-summable.

Now if the p_k form a complete orthonormal set we can solve this problem quite readily by setting

$$f = \sum_{k=1}^{\infty} c_k p_k,$$

for then

$$(p_k, f) = (p_k, \sum_{\ell=1}^{\infty} c_\ell p_\ell) = \sum_{\ell=1}^{\infty} c\ (p_k, p_\ell) = c_k.$$

In the moment problem at hand, this is the situation we would face if we had $\omega_k = k$ and we took $T = 2\pi$. This, however, is true only when we deal with a uniform string.

The elements $p_k\ \varepsilon H$ form a <u>Riesz</u> <u>basis</u> in H if there is a complete orthonormal set $\{\tilde{p}_k\} \subseteq H$ and a linear transformation $T:H \to H$ with both T and T^{-1} bounded such that

$$T\tilde{p}_k = p_k, \qquad k = 1,2,\ldots \tag{12.4}$$

Suppose now that the p_k form a Riesz basis and we have a moment problem

$$(p_k, f) = c_k, \quad k = 1.2,\ldots$$

Then

$$(T\tilde{p}_k, f) = c_k =$$
$$(\tilde{p}_k, T^*f) = c_k$$

This last problem has the solution

$$T^*f = \sum_{k=1}^{\infty} c_k \tilde{p}_k \tag{12.5}$$

and thus we obtain for f

*See D. L. Russell: "Nonharmonic Fourier Series in the Control Theory of Distributed Parameter Systems". J. Math. Anal. and Appl., Vol. 18, No. 3, 1967, pp. 542-60.

$$f = (T^*)^{-1} \sum_{k=1}^{\infty} c_k \tilde{p}_k = \sum_{k=1}^{\infty} c_k (T^*)^{-1} \tilde{p}_k$$

We put $(T^*)^{-1}\tilde{p}_k = q_k$. Then

$$(p_k, q_\ell) = (T\tilde{p}_k, (T^*)^{-1} p_\ell) = (\tilde{p}_k, T^*(T^*)^{-1} p_\ell)$$

$$= (p_k, p_\ell) = \delta_{k\ell}.$$

The $\{q_k\}$ are a biorthogonal set for $\{p_k\}$. Our control problem now reduces to the question of whether or not the functions $\sin(\omega_k t)$, $\cos(\omega_k t)$, $k = 1,2,3,\ldots$, or $e^{\pm i\omega_k t}$, $k = 1,2,3,\ldots$ form a Riesz basis in $L^2[0,T]$.

13. Density, Asymptotic Gap and the Moment Problem.

This problem has been studied in great detail by a number of prominent mathematicians, among them Paley and Wiener, Laurent, Schwartz, Levinson, to name only a few. We shall summarize their results without giving proofs.

Let $\{\Theta_k\}$, k assuming integer values between $-\infty$ and $+\infty$, be a double ended sequence of real numbers. If

$$\lim_{|k|\to\infty} \frac{k}{\Theta_k} \text{ exists and } D = \lim_{|k|\to\infty} \frac{k}{\Theta_k} > 0 \tag{13.1}$$

the sequence $\{\Theta_k\}$ is said to possess a density D. If

$$\lim_{|k|\to\infty} \inf (\Theta_{k+1} - \Theta_k) = \Gamma > 0 \tag{13.2}$$

the sequence is said to have an asymptotic gap Γ. The properties of the set of functions $\{e^{i\Theta_k t} \mid -\infty < k < \infty\}$ in $L^2[0,T]$ depend decisively upon the relationship which T bears to the density D and the gap Γ. We shall assume for the moment that such a gap and density exist.

If $T < 2\pi D$ it is known that the set $e^{i\theta_k t}$ is excessive in $L^2[0,T]$. It is not in general possible to solve the moment problem

$$\int_0^T e^{i\Theta_k t} f(t)dt = c_k, \quad \sum_{k=-\infty} |c_k|^2 < \infty \tag{13.3}$$

by any choice of $f(t) \varepsilon L^2[0,T]$.

If $T > 2\pi D$ the set $\{e^{i\Theta_k t}\}$ is deficient in $L^2[0,T]$, the functions $e^{i\Theta_k t}$ span a proper subspace H of $L^2[0,T]$. Whether or not they form a Riesz basis for this subspace H depends upon the gap Γ. If $\Gamma > \frac{2\pi}{T}$, which is certainly true if $\Gamma \geq \frac{1}{D}$, this is true. In this case the moment problem (13.3) can be solved by a function $f(t) \in H$ and there is exactly one such solution in H. However, $H^{\perp}$ is non-empty in this case (in fact, it is infinite-dimensional) and any function $f(t) + \hat{f}(t)$, $\hat{f}(t) \in H^{\perp}$, also solves the moment problem.

The case $T = 2\pi D$ is by far the most interesting. It has been shown that the set $\{e^{i\Theta_k t}\}$ forms a Riesz basis for $L^2[0,2\pi D]$ provided

$$\limsup_{|k|\to\infty} \left|\Theta_k - \frac{k}{D}\right| < \frac{1}{4D} . \tag{13.4}$$

The constant on the right cannot be replaced by any larger number. If this holds then the moment problem (13.3) has exactly one solution in $L^2[0,2\pi D]$.

14. Reformulation of the Control Problem as an Eigenvalue Problem.

Now, we must relate all of this to the moment problem

$$\begin{aligned} \int_0^T e^{i\omega_k t} f(t)\,dt &= \frac{\mu_k \omega_k}{\gamma_k} - i\,\frac{\nu_k}{\gamma_k} = c_k \\ \int_0^T e^{-i\omega_k t} f(t)\,dt &= \frac{\mu_k \omega_k}{\gamma_k} + i\,\frac{\nu_k}{\gamma_k} = d_k \end{aligned} \tag{14.1}$$

which, as we have seen, is equivalent to the control problem originally posed. Clearly what we need is more information about the ω_k, i.e., the frequencies associated with the normal modes of vibration of solutions of the partial differential equation

$$\rho(x)\,\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\,\frac{\partial u}{\partial x}\right) = 0 \tag{14.2}$$

with boundary conditions

$$\begin{aligned} A_0 u(0,t) + B_0\,\frac{\partial u}{\partial x}(0,t) &\equiv 0, \qquad A_0^2 + B_0^2 \neq 0, \\ A_1 u(1,t) + B_1\,\frac{\partial u}{\partial x}(1,t) &\equiv 0, \qquad A_1^2 + B_1^2 \neq 0. \end{aligned} \tag{14.3}$$

If we put

$$u^* = \sqrt[4]{p(x)\rho(x)}\;u, \qquad x^* = \int_0^x \sqrt{\frac{\rho(\xi)}{p(\xi)}}\,d\xi \tag{14.4}$$

one can see, with a little calculation, that a new partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - r(x)u = \gamma(x)f(t) \tag{14.5}$$

is obtained on an interval $0 \leq x \leq \ell$, where

$$\ell = \int_0^1 \sqrt{\frac{\rho(x)}{p(x)}} \, dx. \tag{14.6}$$

The new boundary conditions are

$$\begin{aligned} a_0 u(0,t) + b_0 \frac{\partial u}{\partial x}(0,t) &\equiv 0 \qquad a_0^2 + b_0^2 \neq 0 \quad b_i = 0 \\ a_1 u(1,t) + b_1 \frac{\partial u}{\partial x}(1,t) &\equiv 0 \qquad a_1^2 + b_1^2 \neq 0 \quad \text{if } B_i = 0. \end{aligned} \tag{14.7}$$

This transformation has the effect of "straightening out" the characteristics of the partial differential equation.

The eigenvalues of the operator

$$L(u) = -\frac{\partial^2 u}{\partial x^2} - r(x)u, \qquad \begin{aligned} a_0 u(0) + b_0 u'(0) &= 0 \\ a_1 u(1) + b_1 u'(1) &= 0 \end{aligned} \tag{14.8}$$

on $L^2[0,\ell]$ have been studied in great detail. See, for instance, the books by Birkhoff, Rota and Tricomi. Three types of problems should be distinguished:

(i) $b_0 = b_1 = 0$ (equivalent to $B_0 = B_1 = 0$)

The "prototype" operator is $-\frac{\partial^2 u}{\partial x^2}$, $u(0) = u(\ell) = 0$, with eigenvalues

$$\tilde{\lambda}_k = \frac{k^2\pi^2}{\ell^2}, \text{ frequency } \tilde{\omega}_k = \frac{k\pi}{\ell} \tag{14.9}$$

$k = 1,2,\ldots$

(ii) $b_0 = 0$, $b_1 \neq 0$ (equivalent to $B_0 = 0$, $B_1 \neq 0$)

Prototype operator:

$$-\frac{\partial^2 u}{\partial x^2}, \qquad \begin{aligned} u(0) &= 0 \\ u'(\ell) &= 0 \end{aligned}$$

Eigenvalues:

$$\tilde{\lambda}_k = \frac{(k + \frac{1}{2})^2 \pi^2}{\ell^2}, \; k = 0,1,2,\ldots \tag{14.10}$$

(iii) $b_0 \neq 0$, $b_1 \neq 0$ (equivalent to $B_0 \neq 0$, $B_1 \neq 0$)

Prototype operator: $-\frac{\partial^2 u}{\partial x^2}$, $u'(0) = u'(\ell) = 0$

Eigenvalues: $$\tilde{\lambda}_k = \frac{k^2\pi^2}{\ell^2}, \; k = 0,1,2,\ldots \tag{14.11}$$

In all of these cases one can prove that the eigenvalues λ_k of the original operator $L[u]$ are related to those of the prototype operator by asymptotic relations of the form

$$\lambda_k = \tilde{\lambda}_k + o(1)$$

whence

$$\omega_k = \sqrt{\tilde{\lambda}_k + o(1)} = \tilde{\omega}_k \sqrt{1 + O(\frac{1}{\tilde{\lambda}_k})} = \tilde{\omega}_k(1 + O(\frac{1}{\tilde{\lambda}_k}))$$

$$= \tilde{\omega}_k + O(\frac{1}{\tilde{\omega}_k}) = \tilde{\omega}_k + O(\frac{1}{k}).$$

This shows that the set $\{\pm\omega_k\}$ has the same asymptotic gap and density as the set $\{\pm i\tilde{\omega}_k\}$, i.e., $D = \frac{\ell}{\pi}$, $\Gamma = \frac{\pi}{\ell}$.

The fact that the remainder term is $O(\frac{1}{k})$ enables one to prove that $\{e^{\pm i\omega_k t}\}$ is a Riesz basis for $L^2[0,T]$ if and only if $\{e^{\pm i\tilde{\omega}_k t}\}$ is a Riesz basis for that space and if there is any excess or deficiency it must be the same in both cases.

15. Discussion of the Three Cases of the Eigenvalue Problem.

Now let us look at the three cases individually:

(i) $\tilde{\omega}_k = \frac{k\pi}{\ell}$

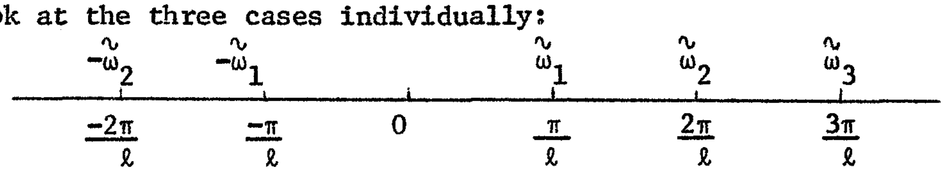


Figure 15

Gap $= \frac{\pi}{\ell}$, density $= \frac{\ell}{\pi}$.

Critical interval length $= 2\pi D = 2\pi \frac{\ell}{\pi} = 2\ell$. Now it is well known that $\{1, e^{\pm i\tilde{\omega}_k t}\}$ is an orthonormal basis for $L^2[0,2\ell]$. We are missing one element, namely 1. So, in this case $\{e^{\pm i\omega_k t}\}$ spans a subspace H of $L^2[0,2\ell]$ whose orthogonal complement has dimension 1.

Let $\{q_k,\ k = \pm 1, \pm 2, \ldots\}$ be biorthogonal to $\{e^{\pm i\omega_k t}\}$ in H. Then the moment problem

$$(e^{i\omega_k t}, f(t)) = c_k$$

$$(e^{-i\omega_k t}, f(t)) = d_k \tag{15.1}$$

is solved by $f(t) = \sum_{k=1}^{\infty} c_k q_k + d_k q_{-k}$ and we obtain thereby the control of least

$L^2[0,2\ell]$ norm bringing the given initial conditions to zero at time $T = 2\ell$. However, if H is spanned by $q(t)$, then

$$f(t) = \sum_{k=1}^{\infty} c_k q_k + d_k q_{-k} + \alpha q(t) \tag{15.2}$$

α real

is likewise a solution. This corresponds to the fact that a constant force acting on the uniform string fixed at both ends over a time interval 2ℓ accomplishes exactly nothing. For the non-uniform string $q(t)$ need not be a constant, however.

(ii) $\omega_k = \dfrac{(k + \frac{1}{2})\pi}{\ell}$

$k = 0,1,2,\ldots$

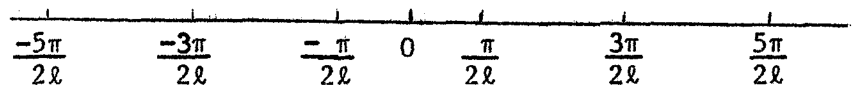


Figure 16

The unitary operator $f(t) \to f(t)e^{i\frac{\pi}{2\ell}t}$ carries $\{e^{\pm i\tilde{\omega}_k t}\}$ into $\{1, e^{\pm i\frac{k\pi}{\ell}t}\}$ which is an orthonormal basis for $L^2[0,2\ell]$. It follows that $\{e^{\pm i\omega_k t}\}$ is an orthonormal basis for $L^2[0,2\pi\ell]$ and that $\{e^{\pm i\omega_k t}\}$ is a Riesz basis for $L^2[0,2\ell]$.

In this case the desired controls exist and are unique.

(iii) $\tilde{\omega}_k = \dfrac{k\pi}{\ell}$, $k = 0,1,2,\ldots$

In this case the $\pm\omega_k$ look like

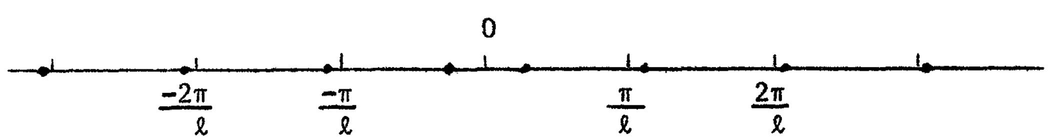


Figure 17

provided none of the λ_k are equal to zero. In this case we have an excess of one.

It turns out that we can solve all but one of the moment equations. What this means in practice for the prototype problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = g(x)f(t)$$

$$\frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial x}(\ell) = 0 \qquad (15.3)$$

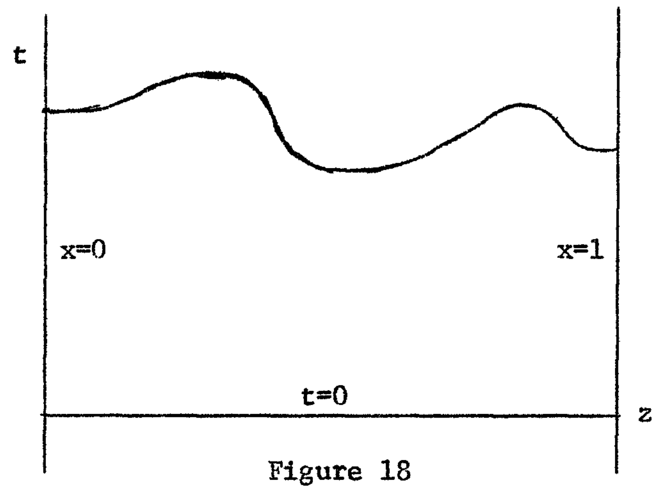


Figure 18

is that all vibration can be stopped in time $T = 2\ell$ but the string, as a body, may have moved to another location, over which we have no control. It is not hard to show that if we take $T > 2\ell$ we can determine the final position as we wish.

What is the meaning of the critical interval length $T = 2\ell$? We remarked earlier that the change of variables taking us from equation (14.2) to equation (14.5) strightens out the characteristics. What is more, the slope of the characteristics is made equal to 1.

In all of this the time variable is left unchanged. So the picture is something like this:

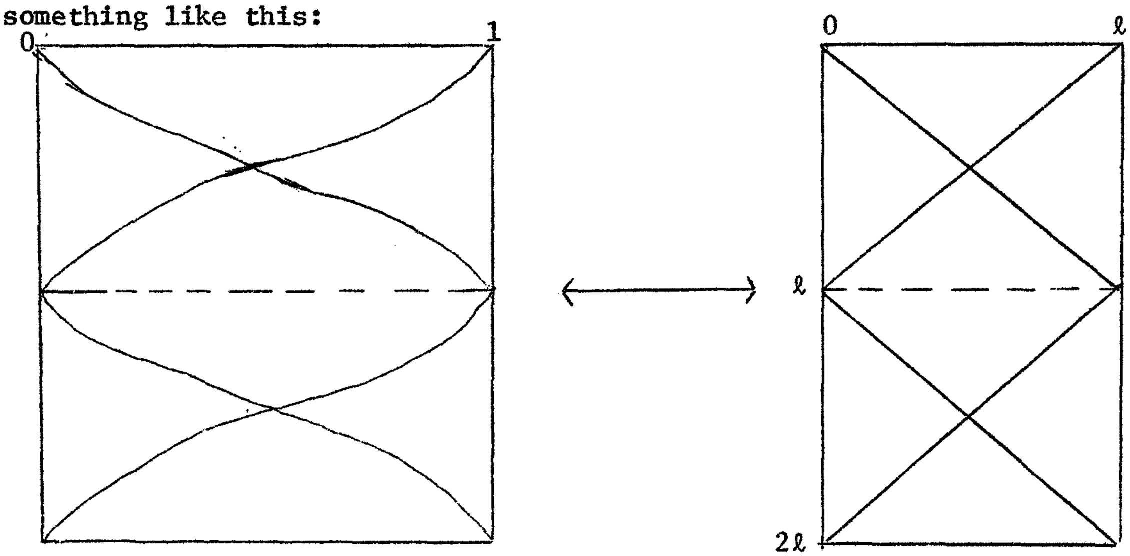


Figure 19

$$\rho(x) \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}(p(x) \frac{\partial u}{\partial x}) = g(x)f(t) \qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + r(x)u = \tilde{g}(x)f(t)$$

Thus the control time, $T = 2\ell$ is just $T = 2T_1$, the same time as was required for control in our earlier theory.

16. Correlation of the Present Results with Those of Chapter I.

It seems appropriate therefore to ask: what is the relationship between these two theories--the one being geometric, the other algebraic?

We can readily answer this question for the partial differential equation

$$\rho(x) \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}(p(x) \frac{\partial u}{\partial x}) = g(x)f(t) \tag{16.1}$$

with boundary conditions

$$a_0 u(0,t) + b_0 \frac{\partial u}{\partial x}(0,t) \equiv 0, \quad a_1 u(1,t) + b_1 \frac{\partial u}{\partial x}(1,t) \equiv 0 \tag{16.2}$$

with the assumption that $b_1 \neq 0$. We will relate this to

$$\rho(x) \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}(p(x) \frac{\partial u}{\partial x}) = 0 \tag{16.3}$$

with boundary conditions

$$a_0 u(o,t) + b_0 \frac{\partial u}{\partial x}(0,t) \equiv 0, \quad a_1 u(1,t) + b_1 \frac{\partial u}{\partial x}(1,t) = f(t) \tag{16.4}$$

From our study up to this point we know that (16.1,2) is controllable in time T if $T \geq 2\pi D = 2\ell$ and that we can find a control $f(t)$ reducing initial conditions

$$u_0(x) = \sum_{k=1}^{\infty} \mu_k \phi_k(x),$$

$$\frac{\partial u}{\partial t}(x,0) = \nu_0(x) = \sum_{k=1}^{\infty} v_k \phi_k(x) \tag{16.5}$$

to the zero state at time T if and only if we can solve the moment problem

$$\int_0^T \sin\,(\omega_k(T - s))f(s)ds = \frac{-\mu_k\omega_k}{\gamma_k}\cos(\omega_k T) - \frac{\nu_k}{\gamma_k}\sin(\omega_k T) = \hat{c}_k$$

$$\int_0^T \cos(\omega_k(T - s))f(s)ds = \frac{\mu_k\omega_k}{\gamma_k}\sin(\omega_k T) - \frac{\nu_k}{\gamma_k}\cos(\omega_k T) = \hat{d}_k \qquad (16.6)$$

Moreover, we have seen that if $\hat{p}_k(t)$, $\hat{q}_k(t)$ are functions forming a biorthogonal set relative to $\sin(\omega_k(T - s))$, $\cos\,(\omega_k(T - s))$, i.e., if

$$\int_0^T \sin(\omega_k(T - s))\hat{p}_\ell(s)ds = \delta_{k\ell}$$

$$\int_0^T \sin(\omega_k(T - s))\hat{q}_\ell(s)ds = 0$$

$$\int_0^T \cos(\omega_k(T - s))\hat{p}_\ell(s)ds = 0$$

$$\int_0^T \cos(\omega_k(T - s))\hat{q}_\ell(s)ds = \delta_{k\ell} \qquad (16.7)$$

then the series

$$f(t) = \sum_{k=1}^{\infty} (\hat{c}_k\hat{p}_k(t) + \hat{d}_k\hat{q}_k(t)) \qquad (16.8)$$

converges in the $L^2[0,T]$ norm and yields the desired solution of the moment problem.

Thus far we have given no constructive means whereby the biorthogonal functions $\hat{p}_k(t)$, $\hat{q}_k(t)$ can be constructed. We will do this now, showing that they arise out of solution of the boundary value control problem (16.3,4) for particular initial conditions.

Let $u(x,t)$ be a solution of the boundary value control problem (16.3,4)

$$u(x,0) = u_0(x) = \sum_{k=1}^{\infty} \mu_k\phi_k(x),\quad \frac{\partial u}{\partial t}(x,0) = v_0(x) = \sum_{k=1}^{\infty}\nu_k\phi_k(x)$$

$$u(x,T) = \frac{\partial u}{\partial t}(x,T) = 0, \qquad (16.9)$$

and let $w(x,t)$ be a solution of

$$\rho(x)\ \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial w}{\partial x}\right) = 0$$

$$a_0 w(0,t) + b_0\ \frac{\partial w}{\partial x}(0,t) \equiv 0, \quad a_1 w(1,t) + b_1\ \frac{\partial w}{\partial x}(1,t) = 0 \tag{16.10}$$

$$w(x,0) = \sum_{k=1}^{\infty} \zeta_k \phi_k(x), \quad \frac{\partial w}{\partial t}(x,0) = \sum_{k=1}^{\infty} \eta_k \phi_k(x). \tag{16.11}$$

Now we compute

$$0 = \int\int_D w(x,t)\left[\rho(x)\ \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right)\right] dxdt$$

$$= \int_0^1 \rho(x) \int_0^T w(x,t)\ \frac{\partial^2 u}{\partial t^2}\ dtdx - \int_0^T\int_0^1 w(x,t)\ \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) dxdt$$

$$= \int_0^1 \rho(x) \int_0^T \frac{\partial^2 w}{\partial t^2}\ u(x,t)dtdx + \int_0^1 \rho(x)\left[\frac{\partial u}{\partial t} w - u\frac{\partial w}{\partial t}\right]_{t=0}^{t=T} dx$$

$$- \int_0^T \int_0^1 \frac{\partial}{\partial x}\left(p(x)\ \frac{\partial w}{\partial x}\right)(u(x,t))\ dxdt$$

$$- \int_0^T \left[p(x)\ w\ \frac{\partial u}{\partial x} - p(x)\ \frac{\partial w}{\partial x}\ u\right]_{x=0}^{x=1} dt$$

on integrating twice by parts, the integral with respect to t in the first term and that with respect to x in the second term. Now using the conditions $u = \frac{\partial u}{\partial t} = 0$ at $t = T$, and the boundary conditions (16.4, 16.10) on u and w, we get

$$0 = \int\int_D \left[\rho(x)\ \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x)\ \frac{\partial w}{\partial x}\right)\right] u(x,t)\ dxdt$$

$$- \int_0^1 \rho(x)\left[\frac{\partial u}{\partial t}(x,0)w(x,0) - u(x,0)\ \frac{\partial w}{\partial t}(x,0)\right] dx$$

$$- \int_0^T p(1)\left[\frac{1}{b_1}\left(a_1 u(1,t) + b_1\ \frac{\partial u}{\partial x}(1,t)\right)\right] w(1,t)dt$$

From this we have

$$\int_0^T p(1)f(t)w(1,t)dt$$

$$= \int_0^1 \rho(x) \left[u(x,0) \frac{\partial w}{\partial t}(x,0) - \frac{\partial u}{\partial t}(x,0)w(x,0) \right] dx$$

whence, using the expansions noted earlier,

$$\int_0^T p(1)f(t)w(1,t)dt$$

$$= \int_0^1 \rho(x) \left[\left(\sum_{k=1}^{\infty} \mu_k\phi_k(x) \right) \left(\sum_{j=1}^{\infty} \eta_j\phi_j(x) \right) \right.$$

$$\left. - \left(\sum_{k=1}^{\infty} \nu_k\phi_k(x) \right) \left(\sum_{j=1}^{\infty} \zeta_j\phi_j(x) \right) \right] dx = \sum_{k=1}^{\infty} (\mu_k\eta_k + \nu_k\zeta_k) \tag{16.12}$$

Now let us put $\mu_{k_0} = 1$, all other μ_k and all $\nu_k = 0$. Then

$$\int_0^T p(1)f(t)w(1,t)dt = \eta_{k_0}$$

We put $\eta_j = 1$, all other $\eta_k = 0$ and all $\zeta_k = 0$. Then

$$w(x,t) = \cos(\omega_j t)\phi_j(x)$$

and thus

$$\int_0^T p(1)\ \phi_j(1)f(t)\cos(\omega_j t)dt$$

$$= \begin{cases} 1 \text{ if } j = k_0 \\ 0 \text{ if } j \neq k_0 \end{cases} \tag{16.13}$$

If we put all $\eta_k = 0$, $\zeta_j = 1$ and all other $\zeta_k = 0$ we have

$$\omega(x,t) = \frac{1}{\omega_j} \sin(\omega_j t)\phi_j(x)$$

and we see that

$$\int_0^T \frac{p(1)\phi_j(1)}{\omega_j} f(t) \sin(\omega_j t)dt = 0 \text{ for all } j.$$

We conclude that if f(t) is a boundary value control which brings the initial conditions

$$u(x,0) = \phi_{k_0}(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0 \tag{16.14}$$

to rest at time t = T, then the function $h_{k_0}(t) = p(1)\phi_{k_0}(1)f(t)$ has the property

$$\left.\begin{aligned} \int_0^T h_{k_0}(t) \cos(\omega_j t)dt &= \delta_{k_0 j} \\ \int_0^T h_{k_0}(t) \sin(\omega_j t)dt &= 0 \end{aligned}\right\} \text{ for all } j \tag{16.15}$$

We can obtain a function $\hat{h}_{k_0}(t)$ satisfying

$$\left.\begin{aligned} \int_0^T \hat{h}_{k_0}(t)\cos(\omega_j t)dt &= 0 \\ \int_0^T \hat{h}_{k_0}(t)\sin(\omega_j t)dt &= \delta_{k_0 j} \end{aligned}\right\} \text{ for all } j \tag{16.16}$$

by finding the boundary value control f(t) which reduces initial conditions

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = \phi_{k_0}(x) \tag{16.17}$$

to zero at time t = T. In this case we have

$$\hat{h}_{k_0}(t) = \frac{p(1)\phi_{k_0}(1)}{\omega_{k_0}} f(t). \tag{16.18}$$

Now the boundary value controls can be computed by applying numerical integration as indicated in an earlier section.

CHAPTER IV.

THE BANG-BANG PRINCIPLE

17. Linear Differential Equations in Hilbert Space

Let H be a Hilbert space and let y, b be vectors in H, u scalar. We consider a process described by a differential equation

$$\frac{dy}{dt} = Ay + bu \tag{17.1}$$

where A is an operator, in general unbounded, defined on a domain Δ which is dense in H.

We are going to assume that A is a normal operator, which means that $AA^* = A^*A$. In addition we will assume that all of the eigenvalues λ of A are of single multiplicity, i.e. they correspond to exactly one eigenvector $\phi \varepsilon H$, and we will assume that all of these eigenvalues lie in some left half-plane of the complex plane. The normality of A ensures that the eigenvectors form an orthonormal basis for H. Thus each $y \varepsilon H$ has a unique representation

$$y = \sum_{k=1}^{\infty} \eta_k \phi_k \tag{17.2}$$

$\{\eta_k\}$ square summable and conversely each such series represents an element of H. Δ is the set of all y defined by (17.2) such that $Ay \varepsilon H$, i.e. such that the sequence $\{\lambda_k \eta_k\}$ is square summable.

Let us consider some examples. We take H to be $L^2[0,1]$; i.e. all functions $y(x)$, $0 \leq x \leq 1$, such that $\int_0^1 |\, y(x) \,|^2 dx < \infty$. If we take $A = \frac{\partial}{\partial x} (p(x) \frac{\partial y}{\partial x})$ on a domain Δ consisting of functions y(x) having L^2 second derivatives ($y'(x)$ is absolutely continuous and $y''(x)$, defined almost everywhere, is square integrable) and satisfying appropriate boundary conditions (e.g. $y'(0) = y'(1) = 0$) we obtain a self-adjoint, and hence normal, operator defined on L^2, all of whose eigenvalues lie in the non-positive real axis.

If we wish to consider something like a string

$$\frac{\partial^2 w}{\partial t^2} - \frac{1}{\rho(x)} \frac{\partial}{\partial x} \left(p(x) \frac{\partial w}{\partial x} \right) = g(x)u(t) \tag{17.3}$$

we let

$$Tw = - \frac{1}{\rho(x)} \frac{\partial}{\partial x} \left(p(x) \frac{\partial w}{\partial x} \right) \tag{17.4}$$

Then, with appropriate boundary conditions T becomes a positive self-adjoint operator and has a positive self-adjoint square root $T^{1/2}$. We put $w_1 = w$, $w_2 = \frac{dw}{dt}$ and obtain

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -T & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix} u(t) \tag{17.5}$$

The letting

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} I & I \\ iT^{1/2} & -iT^{1/2} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{17.6}$$

we have

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} iT^{1/2} & 0 \\ 0 & -iT^{1/2} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \frac{-iT^{-1/2}g}{2} \\ \frac{iT^{-1/2}g}{2} \end{pmatrix} u \tag{17.7}$$

which is of the form (17.1) with

$$A = \begin{pmatrix} iT^{1/2} & 0 \\ 0 & -iT^{1/2} \end{pmatrix} \tag{17.8}$$

being anti-hermitian, and hence normal. It should be emphasized that solutions of this first order equation may only represent generalized solutions of the original partial differential equation, (17.3).

Consider now the solutions of (17.1). If A is normal and all of its eigenvalues lie in some left half plane the operator equation

$$\frac{dY}{dt} = AY, \qquad Y(0) = I \tag{17.9}$$

has a unique solution Y(t) which we denote by e^{At}, defined for all $t \geq 0$ and strongly continuous, i.e. $e^{At}y_0$ is a continuous vector valued function for each $y_0 \varepsilon H$. The precise sense in which $Y(t) = e^{At}$ satisfies (17.9) need not concern us here. However, if $y_0 \varepsilon \Delta = \text{dom } A$, $e^{At}y_0$ does provide us with a bona fide solution of

$$\frac{dy}{dt} = Ay, \qquad y(0) = y_0 \tag{17.10}$$

For the first order equation derived from the string partial differential equation we obtain a solution of the homogeneous equation if the initial state y_0 has finite energy.

For the inhomogeneous equation, if u(t) is integrable we can form an integral in H:

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}bu(s)\,ds \tag{17.11}$$

and call it a solution of (17.1). If $b\varepsilon \mathrm{dom}A$ (true, for instance in the case of the string equation) then the right hand side of the above equation can be differentiated and eq. (17.1) is actually satisfied.

18. Optimal Control Problem for Differential Equations in Hilbert Space

So much, then, for a little background on linear differential equations in Hilbert space. We will now pose an optimal control problem, in fact a time optimal control problem, for such systems and see if we can establish a "bang-bang" principle.

Let a point $y_0 \varepsilon H$ be given and let $N(\varepsilon,y_0)$ denote the set

$$N(\varepsilon,y_0) = \{y\varepsilon H \mid \|y-y_0\| \leq \varepsilon\}. \tag{18.1}$$

We will assume $0 \notin N(\varepsilon,y_0)$. We place constraints on the scalar control function $u(t)$

$$-1 \leq u(t) \leq 1 \tag{18.2}$$

and we pose the problem: Find $\hat{u}(t)$ defined and measurable on an interval $[0,T]$ so that the solution $\hat{y}(t)$ of

$$\frac{d\hat{y}}{dt} = A\hat{y} + b\hat{u}, \qquad \hat{y}(0) = 0 \tag{18.3}$$

satisfies $\hat{y}(T)\varepsilon N(\varepsilon,y_0)$, $\hat{u}$ obeys the above constraints, and no control u obeying these constraints brings $y(T)$ to $N(\varepsilon,y_0)$ in a time shorter than T.

We remark that the time optimal control problem can be posed in many other ways. We could ask that $\hat{y}(T) = 0$, for instance. This problem is somewhat more difficult as we will indicate later.

Our development now follows a familiar track. We let $R(t) \subseteq H$ be the set of attainability from 0 at time t: i.e.

$$R(t) = \{y\varepsilon H \mid y = \int_0^t e^{A(t-s)}bu(s)\,ds, \quad -1 \leq u(s) \leq 1\} \tag{18.4}$$

It is easy to verify that $R(t)$ is closed and convex for each t.

Our main interest lies in characterizing the optimal control $\hat{u}(t)$ and so let us assume that our time optimal control problem has a solution. If we know there is some control $u(t)$ bringing 0 into $N(\varepsilon,y_0)$ in time T_1 (the problem of controllability, discussed earlier) then it is not too hard to show that there is a time optimal control in our sense. But this would take us somewhat away from what we really want to do.

So we assume $\hat{y}(T)\epsilon N(\varepsilon,y_0)$, so that $R(T)\cap N(\varepsilon,y_0)$ is not empty and we assume $R(t)\cap N(\varepsilon,y_0)$ is empty if $t < T$, $\hat{y}(T)\epsilon\partial N(\varepsilon,y_0)$, otherwise we could reach $N(\varepsilon,y_0)$ earlier.

<u>Proposition</u>. Let $y\epsilon R(T)$. Then

$$\mathrm{Re}(\hat{y}(T) - y_0,\ y-y_0) \geq ||\hat{y}(T) - y_0||^2 = \varepsilon^2. \tag{18.5}$$

<u>Proof</u>. Suppose y were such that $\mathrm{Re}(\hat{y}(T) - y_0,\ y-y_0) = ||\hat{y}(T) - y_0||^2 - \delta, \delta > 0$. Let $\eta = \lambda y + (1-\lambda)\hat{y}(T)$, $0 \leq \lambda \leq 1$. Because $R(t)$ is convex $\eta\epsilon R(t)$.
We compute

$$||\eta - y_0||^2 = (\lambda(y-y_0) + (1-\lambda)(\hat{y}(T) - y_0),$$

$$\lambda(y - y_0) + (1-\lambda)(\hat{y}(T) - y_0)) = \lambda^2||y-y_0||^2 + 2\lambda(1-\lambda)\mathrm{Re}(\hat{y}(T) - y_0,\ y-y_0)$$

$$+ (1-\lambda)^2||\hat{y}(T) - y_0||^2.$$

$$= \lambda^2(||y-y_0||^2 - ||\hat{y}(T) - y_0||^2) + \lambda^2||\hat{y}(T) - y_0||^2 + 2\lambda(1-\lambda)\ [||\hat{y}(T)-y_0||^2 - \delta$$

$$+ (1-\lambda)^2||\hat{y}(T) - y_0||^2 = -2\lambda\delta + \lambda^2(2\delta + ||\hat{y}(T) - y_0||^2) + ||\hat{y}(T) - y_0||^2$$

$$= -2\lambda\delta + \lambda^2(2\delta + ||\hat{y}(T) - y_0||^2) + \varepsilon^2.$$

For λ near 0 and positive we have

$$||\eta - y_0||^2 < \varepsilon^2$$

and thus η lies in the interior of $N(\varepsilon,y_0)$. But then we can show very readily that $R(t)\cap N(\varepsilon,y_0) \neq \emptyset$ for some $t < T$, a contradiction. Therefore,

$$\mathrm{Re}(\hat{y}(T) - y_0,\ y - y_0) \geq ||\hat{y}(T) - y_0||^2$$

as claimed, whenever $y\epsilon R(T)$. Thus

$$\mathrm{Re}(\hat{y}(T) - y_0,\ (y-y_0) - (\hat{y}(T) - y_0)) = \mathrm{Re}(\hat{y}(T) - y_0,\ y-\hat{y}(T)) \geq 0,\ y\epsilon R(t) \quad (18.$$

Now let $u(t)$ be an arbitrary admissible control and $y = y(T)$ the point in $R(T)$ corresponding to this control, i.e.,

$$y = \int_0^T e^{A(T-s)}bu(s)ds \tag{18.7}$$

Then

$$\mathrm{Re}(\hat{y}(T) - y_0,\ y-\hat{y}(T)) = \mathrm{Re}(\hat{y}(T) - y_0,\ \int_0^T e^{A(T-s)}b(u(s)-\hat{u}(s))ds)$$

$$= \mathrm{Re} \int_0^T (\hat{y}(T) - y_0,\ e^{A(T-s)}b)(u(s) - \hat{u}(s))ds \geq 0$$

If this is to be true for all admissible u(s), we must have

$$u(s) = -\mathrm{sgnRe}(\hat{y}(T) - y_0, e^{A(T-s)}b) \tag{18.8}$$

for almost all s in [0,T]. Thus, loosely speaking, $\hat{u}(s)$ assumes extreme values whenever

$$\mathrm{Re}(\hat{y}(T) - y_0, e^{A(T-s)}b) \neq 0.$$

Now let A have eigenvalues

$$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k, \ldots \quad \lambda_k = \mu_k + i\nu_k, \tag{18.9}$$

and a complete orthonormal basis of eigenvectors $\phi_1, \phi_2, \ldots, \phi_k, \ldots$ in H.

We let

$$\hat{y}(T) - y_0 = \sum_{k=1}^{\infty} \zeta_k \phi_k$$

$$b = \sum_{k=1}^{\infty} \beta_k \phi_k$$

$$T - s = \sigma \tag{18.10}$$

Then

$$(\hat{y}(T) - y_0, e^{A(T-s)}b) = \sum_{k=1}^{\infty} \zeta_k \beta_k e^{(\mu_k + i\nu_k)\sigma}$$

and we are talking therefore, of the set of points σ where

$$p(\sigma) = \sum_{k=1}^{\infty} \zeta_k \beta_k e^{\mu_k \sigma} (\cos \nu_k \sigma + i \sin \nu_k \sigma)$$

has a non-vanishing real part.

Now let

$$\zeta_k = \zeta_k^r + i\zeta_k^i, \ \beta_k = \beta_k^r + i\beta_k^i, \ \mathrm{Re}(p(\sigma)) = (\zeta_k^r \beta_k^r - \zeta_k^i \beta_k^r) e^{\mu_k \sigma} \cos \nu_k \sigma$$

$$- (\zeta_k^r \beta_k^i + \zeta_k^i \beta_k^r) e^{\mu_k \sigma} \sin \nu_k \sigma. \tag{18.11}$$

To get more explicit results now we have to begin to treat particular cases.

19. Hyperbolic Problems

The typical hyperbolic problem comes from a second order equation

$$\frac{d^2 w}{dt^2} + Tw = gu \tag{19.1}$$

where T is positive and self-adjoint. We will suppose that T has eigenvalues

$$0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots \tag{19.2}$$

all of single multiplicity with corresponding eigenvectors ϕ_k.

As we have seen, we can go to a first order equation

$$\frac{dy}{dt} = \begin{pmatrix} iT^{1/2} & 0 \\ 0 & -iT^{1/2} \end{pmatrix} y + \begin{pmatrix} \frac{-iT^{-1/2}g}{2} \\ \frac{iT^{-1/2}g}{2} \end{pmatrix} u \tag{19.3}$$

The eigenvalues now are $\pm\, i(\lambda_k{}^{1/2}) = \pm\, i\omega_k$ and the eigenvectors are

$\begin{pmatrix} \phi_k \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_k \end{pmatrix}, \ldots \quad k = 1,2,\ldots.$ Thus $\mu_k = 0,\ \nu_k = \omega_k$. If we let

$g = \sum_{k=1}^{\infty} \gamma_k \phi_k$ then

$$b = \begin{pmatrix} \frac{-iT^{-1/2}g}{2} \\ \frac{iT^{-1/2}g}{2} \end{pmatrix} = \sum_{k=1}^{\infty} \left\{ \frac{-i\gamma_k}{2\omega_k} \begin{pmatrix} \phi_k \\ 0 \end{pmatrix} + \frac{i\gamma_k}{2\omega_k} \begin{pmatrix} 0 \\ \phi_k \end{pmatrix} \right\} \tag{19.4}$$

If we let

$$\hat{y}(T) - y_0 = \sum_{k=1}^{\infty} \zeta_k \begin{pmatrix} \phi_k \\ 0 \end{pmatrix} + \bar{\zeta}_k \begin{pmatrix} 0 \\ \phi_k \end{pmatrix} \tag{19.5}$$

(which it must be if it is to correspond to a <u>real</u> state for the original partial differential equation) then

$$(\hat{y}(T) - y_0,\ e^{A(T-s)}b) = \sum_{k=1}^{\infty} \left\{ \frac{-i\gamma_k}{2\omega_k} \zeta_k e^{i\omega_k\sigma} + \frac{i\gamma_k}{2\omega_k} \bar{\zeta}_k e^{-i\omega_k\sigma} \right\} \tag{19.6}$$

and the real part is given by

$$\mathrm{Re}(p(\sigma)) = \frac{-\gamma_k \zeta_k^i}{\omega_k} \cos \omega_k\sigma - \frac{\gamma_k \zeta_k^r}{\omega_k} \sin \omega_k\sigma \tag{19.7}$$

Thus, whenever the above expression does not vanish

$$\hat{u}(s) = \hat{u}(T-\sigma) = \mathrm{sgn}\left(\frac{\gamma_k \zeta_k^2}{\omega_k} \cos \omega_k\sigma + \frac{\gamma_k \zeta_k^r}{\omega_k} \sin\omega_k\sigma \right) \tag{19.8}$$

Now the theory of non-harmonic Fourier series, used previously, enters the picture again.

Let us assume that no $\gamma_k = 0$. This is the condition for approximate controllability. Then, since $\hat{Y}(T) - y_0 \neq 0$, not all ζ_k are zero and thus not all of the coefficients $\frac{\gamma_k \zeta_k^i}{\omega_k}$, $\frac{\gamma_k \zeta_k^r}{\omega_k}$ are zero.

Let us suppose that the frequencies ω_k have a finite non-zero density D (e.g. in the case of the string we have already seen that $D = \frac{\ell}{\pi}$). (This type of density is typical for hyperbolic equations with one space dimension.) It is then known (see Paley and Wiener, Levinson, etc.) that on any interval $[0,T]$ such that $T > 2\pi D$ it is not possible that

$$\frac{\gamma_k \zeta_k^i}{\omega_k} \cos \omega_k \sigma + \frac{\gamma_k \zeta_k^r}{\omega_k} \sin \omega_k \sigma$$

vanishes identically. Thus, if it is not possible to reach $N(\varepsilon, y_0)$ in time $T \leq 2\pi D$, but this is possible in some time $T > 2\pi D$, then the time optimal control $\hat{u}(t)$ must assume extreme values in some set of positive measure contained in $[0,T]$. More than this we cannot say. In some cases this result can be slightly strengthened to $T \geq 2\pi D$ but if $T < 2\pi D$ we cannot be sure that a time optimal control ever assumes extreme values.

Hyperbolic problems in two or more space variables frequently have the property that D is infinite. So in these cases we do not obtain anything that we could really call a bang-bang principle since we cannot be sure that $\operatorname{Re} p(\sigma)$ does not vanish on an interval, no matter how long that interval is.

There are oscillatory systems that are not hyperbolic in the sense used in the theory of partial differential equations, (i.e. the equations do not have distinct real characteristics, etc.). For example, consider the simple beam equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{1}{\rho(x)} \frac{\partial^2}{\partial x^2} \left(p(x) \frac{\partial^2 w}{\partial x^2}\right) = g(x)u(t) \qquad (19.9)$$

$$Tw = \frac{1}{\rho(x)} \frac{\partial^2}{\partial x^2} \left(p(x) \frac{\partial^2 w}{\partial x^2}\right) \qquad (19.10)$$

is positive self-adjoint if appropriate boundary conditions are given (e.g., $w(0) = w'(0) = 0$, $w(1) = w'(1) = 0$). So the analysis proceeds as before and we look at

$$\operatorname{Re}(p(\sigma)) = \left(\frac{\gamma_k \zeta_k^i}{\omega_k} \cos \omega_k \sigma + \frac{\gamma_k \zeta_k^r}{\omega_k} \sin \omega_k \sigma\right) \qquad (19.11)$$

In this particular case, however, we have $\omega_k = O(k^2)$, $k \to \infty$ which gives a density $D = 0$. This implies that $\mathrm{Re}(p(\sigma))$ cannot vanish identically on any interval. So in this case we see that a time optimal control $\hat{u}(t)$ must have extreme values on a dense set in $[0,T]$. However, despite strenuous efforts it has not been possible to show that the set of extreme values has full measure. (It clearly has positive measure because $\mathrm{Re}(p(\sigma))$ is a continuous function.)

20. Parabolic Problems

The usual example cited for a parabolic equation is

$$\frac{dy}{dt} = Ay + bu \tag{20.1}$$

with A negative self-adjoint, as in the heat equation

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial t^2} = g(x)u(t) \tag{20.2}$$

If we look at the eigenvalues of $A = -\frac{\partial^2 y}{\partial t^2}$ in the case of the heat equation, or any negative self-adjoint A, they lie wholly on the negative real axis.

This is an extreme case of a more general situation which we also call parabolic. Let us suppose that A is normal and has all of its eigenvalues in a sector:

$$\lambda \text{ an eigenvalue of } A \Rightarrow \lambda \in \{\mu \in C \mid |\arg(\mu-\mu_0) - \pi| \leq \frac{\pi}{2} - \delta\} \tag{20.3}$$

where $\delta > 0$. Graphically things look like this:

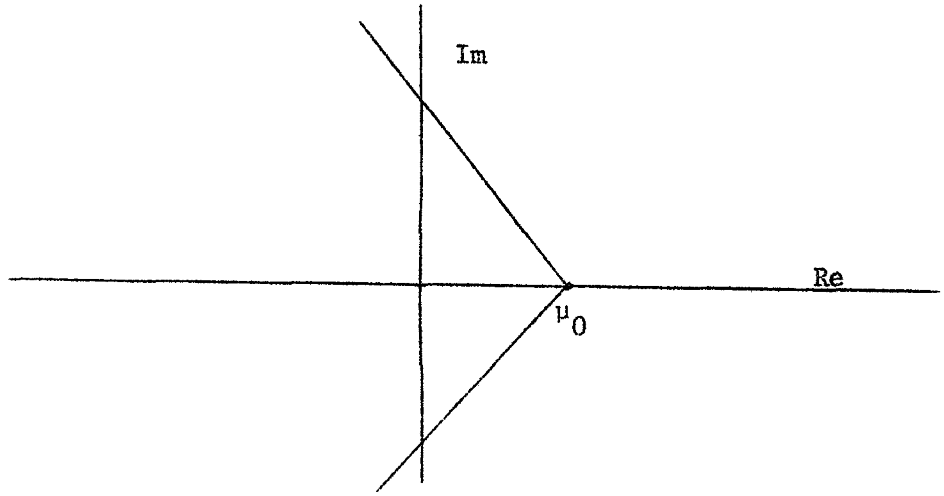


Figure 20

If we assume a particular mode of energy dissipation for certain oscillatory systems we obtain equations parabolic in this sense. Suppose T is self-adjoint and positive. Consider a second order equation

$$\frac{d^2w}{dt^2} + 2aT^{1/2}\frac{dw}{dt} + T = gu \tag{20.4}$$

(a 0 but small)

which with $w = w_1$, $\frac{dw}{dt} = w_2$ becomes

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -T & -aT^{1/2} \end{pmatrix}\begin{pmatrix} w_1 \\ \dot{w}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix} u \tag{20.5}$$

Now put

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} I & I \\ -aT^{1/2}+i\sqrt{1-a^2}\;T^{1/2} & -aT^{1/2}-i\sqrt{1-a^2}T^{1/2} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{20.6}$$

and we obtain

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -aT^{1/2}+i\sqrt{1-a^2}\;T^{1/2} & 0 \\ 0 & -aT^{1/2}-i\sqrt{1-a^2}\;T^{1/2} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \frac{-iT^{-1/2}}{2\sqrt{1-a^2}}\,g \\ \frac{iT^{-1/2}}{2\sqrt{1-a^2}}\,g \end{pmatrix} u(t) = Ay + bu. \tag{20.7}$$

Now if T has eigenvalues $\lambda_k > 0$, $T^{1/2}$ has eigenvalues $\omega_k = \sqrt{\lambda_k}$ and A, as just indicated, has eigenvalues

$$-a\omega_k \pm i\sqrt{1-a^2}\;\omega_k$$

which lie in a sector as described earlier.

It is interesting that this type of damping is well known to engineers (though they do not use the equation we have just developed) and is known as structural damping.

So let us now consider a system

$$\frac{dy}{dt} = Ay + bu \tag{20.8}$$

where A is a normal (unbounded) operator whose spectrum lies in a sector

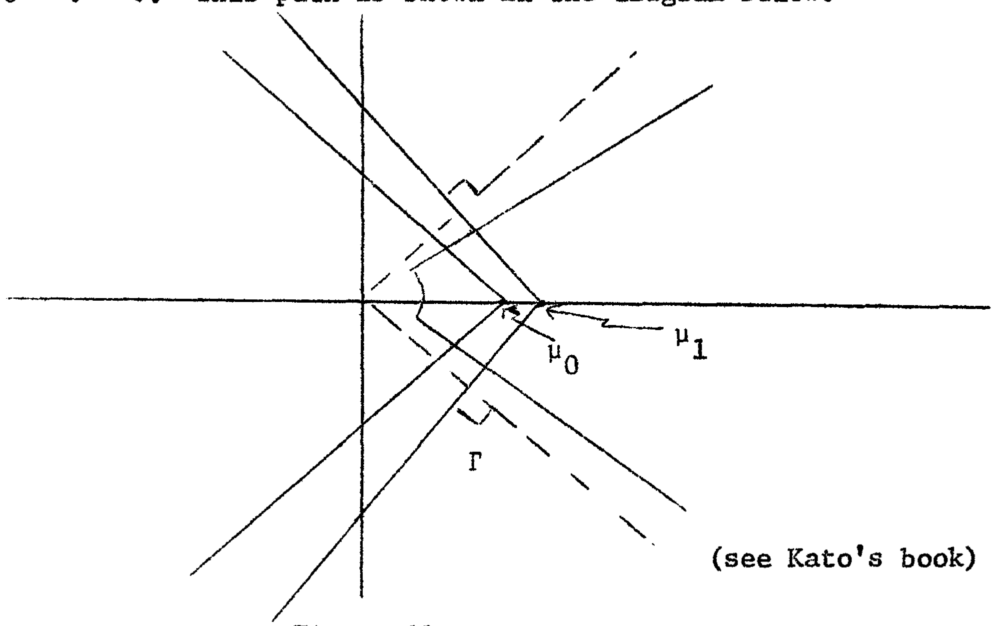
$$\{\mu \,|\, |\arg(\mu-\mu_0) - \pi| \leq \frac{\pi}{2} - \delta,\ \mu_0 \text{ real},\ \delta > 0\} \tag{20.9}$$

This general equation will include both the heat equation and the structurally damped oscillator which we have described above.

We now take a path

$$\Gamma = \Gamma(\theta,\mu_1) = \{\mu \,|\, \arg(\mu-\mu_1) = \frac{\pi}{2} + \theta \text{ or } \arg(\mu-\mu_1) = \frac{3\pi}{2} - \theta\} \tag{20.10}$$

where $\mu_1 > \mu_0$ and $0 < \theta < \delta$. This path is shown in the diagram below.



(see Kato's book)

Figure 21

The normality of A can be used to show that if μ is a point on the path Γ:

$$||(\mu I-A)^{-1}|| \leq K(1+|\mu|) \tag{20.11}$$

where K is some positive constant. Using this one can see that the operator-valued integral

$$\frac{1}{2\pi i} \int_{\Gamma} e^{\mu\sigma} (\mu I-A)^{-1} d\mu$$

is uniformly convergent to $e^{A\sigma}$ if σ lies in a set

$$\{\sigma \,|\, |\sigma| \geq \sigma_0,\ |\arg \sigma| \leq \theta - \psi\}$$

where $\sigma_0 > 0$ and $\psi > 0$.

The function

$$p(\sigma) = (\vartheta(T) - y_0,\ e^{A\sigma}b) = \frac{1}{2\pi i}\int_\Gamma e^{\mu\sigma}(\vartheta(T) - y_0,\ (\mu I-A)^{-1}b)\,d\sigma \qquad (20.12)$$

is uniformly convergent in the same subset of the complex plane and therefore represents a holomorphic function there. Now we let σ_0, ψ tend to zero, let θ tend to δ and we have $p(\sigma)$ holomorphic in the interior of

$$\{\sigma \mid |\arg \sigma| < \delta,\ |\sigma| > 0\}$$

which includes all of the positive real axis.

Then $\mathrm{Re}(p(\sigma))$ is real analytic for real σ and cannot vanish on a set of positive measure unless it vanishes on the whole positive real axis.

But if $(\vartheta(T) - y_0,\ e^{A\sigma}b) = (\vartheta(T) - y_0,\ e^{A\sigma}b)$ has real part identically zero for $\sigma > 0$ then the attainable points

$$y(t) = \int_0^t e^{A(t-s)}bu(s)\,ds$$

all have the property that $(\vartheta(T) - y_0,\ y(t))$ is purely imaginary, and thus denies that the system is even approximately controllable (i.e., that one can achieve a dense set of points in H).

So if we have approximate controllability (and this question has been studied at length by Fattorini and others) then a time-optimal control, (in our sense) is always a bang-bang control for a parabolic (in our sense) system.

I will indicate only very briefly what happens if we replace our target set, a neighborhood of y_0, by the point y_0 itself. To get results of the type we have been studying one then must establish the existence of a vector $\eta \varepsilon H$ such that for all $y \varepsilon R(T)$

$$(\eta, \vartheta(T) - y) \geq 0$$

Then $(\eta, \vartheta(T) - z) = 0$ defines a supporting hyperplane.

In finite dimensional spaces the convexity of R(T) implies the existence of η right away. But if H is an infinite dimensional space as it is for distributed systems, this is no longer true in general and it is a rather difficult problem to determine just when it is true. In our problem we do not need a supporting hyperplane to R(T) because we have one for the target set $N(\varepsilon, y_0)$.

CHAPTER V

LINEAR STABILIZATION OF THE LINEAR OSCILLATOR

21. The n-dimensional linear oscillator

The control techniques which we have discussed so far have one serious disadvantage. The logical and mathematical steps from the measurement of the state of the system to the actual implementation of the control force are quite complicated. In this chapter we will present a stabilization technique whose implementation is very simple.

Consider a one dimensional oscillator

$$\ddot{x} + \alpha x = u, \qquad \alpha > 0 \tag{21.1}$$

When $u = 0$ it is well known that the solutions oscillate indefinitely maintaining constant energy. To damp out these oscillations it is often convenient to put

$$u = - \gamma\dot{x}, \tag{21.2}$$

thus providing a control force proportional to velocity but oppositely directed. The resulting closed loop system is

$$\ddot{x} + \gamma\dot{x} + \alpha x = 0 \tag{21.3}$$

for which $x = \dot{x} = 0$ is an asymptotically stable critical point.

Let us generalize somewhat on this theme. An n-dimensional linear oscillator is represented by a second order system

$$\ddot{x} + Ax = 0 \qquad x \varepsilon R^n \qquad A, \text{ an } n\text{x}n \text{ matrix} \tag{21.4}$$

where A is symmetric and positive definite. It is easy to verify for such a system that the energy

$$E(x,\dot{x}) = \frac{1}{2}(\dot{x},\dot{x}) + \frac{1}{2}(x,Ax) \tag{21.5}$$

is conserved in the motion. Suppose now that $b_1, b_2,\ldots,b_m$, $m \leq n$, are unit vectors in R^n and $u^1, u^2,\ldots,u^m$ are scalars representing how much control force we exert in the directions $b_1,b_2,\ldots,b_m$, respectively. The controlled system is then

$$x + Ax = Bu \qquad B = (b_1,b_2,\ldots,b_m),\ u = \begin{pmatrix} u^1 \\ u^2 \\ \vdots \\ u^m \end{pmatrix} \tag{21.6}$$

Now let us measure the component of the oscillator's velocity in the direction b_i, $i = 1,2,\ldots,m$. Calling this quantity v^i we have

$$v^i = (b_i,\dot{x})$$

and, putting $v = \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{pmatrix}$ we have $v = B^T\dot{x}$.

Suppose we decide now to employ the control policy

$$u = -\gamma v = -\gamma B^T\dot{x} \tag{21.7}$$

The resulting closed loop system is

$$\ddot{x} + \gamma BB^T\dot{x} + Ax = 0 \tag{21.8}$$

If the equation (21.4) is a finite dimensional approximation to a distributed oscillator we may consider the possiblity of measuring the velocity at various stations on this distributed object. Each such measurement yeilds the quantity $(b, \dot{x})$ for some vector b which depends upon the station in question. On the other hand b u represents a force of magnitude u applied to the distributed oscillator at that same point. Thus one interpretation of the control policy we have described is that we measure velocity at m different stations and apply forces at those same stations which are megatively proportional to the measured velocities. Engineers call this an ILAF (Identical Location of Accelercmeter and Forces) control system.

Theorem If rank $[B, AB, \ldots, A^rB] = n$ for some positive integer r, then $x = x = 0$ is an asymptotically stable critical point for equation (21.8)

Proof. Put $E(x,\dot{x}) = \frac{1}{2}(\dot{x},\dot{x}) + \frac{1}{2}(x,Ax)$. We compute the time derivative of this quantity along solutions x(t) of our differential equation:

$$\frac{d}{dt}E(x,x) = \frac{1}{2}(\ddot{x},\dot{x}) + \frac{1}{2}(\dot{x},\ddot{x}) + \frac{1}{2}(\dot{x},Ax) + \frac{1}{2}(x,A\dot{x}) = (\dot{x},\ddot{x}) + (\dot{x},Ax)$$

(since A is symmetric)

$$= (\dot{x}, -\gamma BB^T\dot{x} - Ax) + (\dot{x},Ax) = -\gamma(\dot{x},BB^T\dot{x}) = -\gamma||B^T\dot{x}||^2 \leq 0$$

and equality holds only on that linear subspace of the 2n-dimensional state space R^{2n} where $B^T\dot{x} = 0$.

There is a theorem due to LaSalle which states that, under the above conditions, we have

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \dot{x}(t) = 0$$

if no non-trivial solution x(t) of equation (21.8) can satisfy $B^T\dot{x}(t) \equiv 0$, i.e. if no nontrivial solution of the differential equation can remain in ꞰB.

Suppose it were true that $B^T\dot{x}(t) \equiv 0$. Then $BB^T\dot{x}(t) \equiv 0$ also and such a solution must satisfy

$$\ddot{x} + Ax = 0$$

also, so that $\ddot{x}(t) \equiv -Ax(t)$.

Now if $B^T\dot{x}(t) \equiv 0$ then $\frac{d^j}{dt^j} B^T\dot{x}(t) \equiv 0$, $j = 1,2,\ldots,2v$. Now

$$\frac{d}{dt} B^T\dot{x}(t) = B^T\ddot{x}(t) = -B^TAx(t) \equiv 0$$

$$\frac{d^2}{dt^2} B^T\dot{x}(t) = -B^TA\dot{x}(t) \equiv 0.$$

Continuing in this way we see that

$$\frac{d^{2j}}{dt^{2j}} B^T\dot{x}(t) \equiv (-1)^j B^TA^j\dot{x}(t) \equiv 0, \; j = 1,\ldots,r.$$

Thus

$$\begin{bmatrix} B^T \\ -B^TA \\ \vdots \\ (-1)^r B^TA^r \end{bmatrix} \dot{x}(t) \equiv 0$$

which implies $x(t) \equiv 0$ since rank $\begin{bmatrix} B^T \\ \vdots \\ (-1)^r B^TA^r \end{bmatrix}$ = rank $[B, AB, \ldots, A^rB] = n$.

But if $\dot{x}(t) \equiv 0$ also $\ddot{x}(t) \equiv 0$ and, since $\ddot{x}(t) + Ax(t) \equiv 0$ and A is positive definite we have $x(t) \equiv 0$. Thus $x(t) \equiv \dot{x}(t) \equiv 0$ and we have the trivial solution. It follows that $x = \dot{x} = 0$ must be asymptotically stable.

22. Generalization to the infinite-dimensional oscillator.

Our main purpose in this section will be to obtain an infinite dimensional analog of the result which we have just proved for an n-dimensional oscillator. This will not be particularly easy for LaSalle's theorem does not carry over into the infinite dimensional case. Our proof will be based on perturbation theory of linear operators and we will have to make a number of assumptions having no counterpart in the above finite dimensional theory.

We consider

$$\ddot{x} + Ax = gu \tag{22.1}$$

for x, g lying in a Hilbert space H, $||g|| = 1$, and A self-adjoint and positive, i.e.

$$(x,Ax) \geq \alpha||x||^2 \text{ for some } \alpha > 0. \tag{22.2}$$

We assume A has eigenvalues $0 < \lambda_1 < \lambda_2$ ... and corresponding eigenvectors $\phi_1, \phi_2, \ldots$ forming a complete orthonormal set in H. Expanding g in terms of these eigenvectors we have

$$g = \sum_{k=1}^{\infty} \gamma_k \phi_k \tag{22.3}$$

and we assume no $\gamma_k = 0$ so that g is not orthogonal to any eigenvector of A.

The control policy which we will employ is

$$u(t) = -\varepsilon(\dot{x}, g), \qquad \varepsilon > 0, \tag{22.4}$$

yielding the closed loop system

$$\ddot{x} + \varepsilon G\dot{x} + Ax = 0 \tag{22.5}$$

where G is a linear operator from H into itself given by

$$G(x) = (x,g)g. \tag{22.6}$$

It should be noted that G is a projection, i.e.,

$$G^2 = G, \text{ for } G(G(x)) = ((x,g)g,g)g = (x,g)(g,g)g = G(x).$$

Theorem. If equation (22.5) has any solution of the form

$$x(t) = e^{\nu t}\phi, \qquad \phi \in H, \quad ||\phi|| = 1, \quad [(\text{w.l.o.g.})]$$

then $\mathrm{Re}(\nu) < 0$.

Proof. If there is such a solution then

$$e^{\nu t}(\nu^2 I + \varepsilon\nu G + A)\phi = 0$$

so that ν, ϕ provide a solution of the quadratic eigenvalue problem

$$(\nu^2 I + \varepsilon\nu G + A)\phi = 0. \tag{22.7}$$

Then $((\nu^2 I + \varepsilon\nu G + A)\phi, \phi) = 0$ so that

$$\nu^2 + \varepsilon\nu(G\phi,\phi) + (A\phi,\phi) = 0, \tag{22.8}$$

a quadratic equation in ν. Using the quadratic formula we have

$$\nu = \frac{-\varepsilon(G\phi,\phi) \pm \sqrt{\varepsilon^2(G\phi,\phi)^2 - 4(A\phi,\phi)}}{2} \tag{22.9}$$

From the positivity of A it is clear that $\mathrm{Re}(\nu) < 0$ if $(G\phi,\phi) > 0$. Now since G is a projection we clearly have $(G\phi,\phi) \geq 0$ so all we need to do is to show that $(G\phi,\phi) \neq 0$. Now

$$(G\phi,\phi) = ((\phi,g)g,\phi) = (\phi,g)(g,\phi) = |(\phi,g)|^2$$

so $(G\phi,\phi) = 0 \Rightarrow (\phi,g) = 0$. But if $(\phi,g) = 0$ then $G\phi = (\phi,g)g = 0$ and

$$(\nu^2 I + \varepsilon\nu G + A)\phi = 0 \Longleftrightarrow (\nu^2 I + A)\phi = 0$$

which implies that ϕ is an eigenvector of A corresponding to the eigenvalue $-\nu^2$. But then $\phi = \phi_k$ for some k and we have

$$0 = (\phi,g) = (\phi_k,g) = \overline{\gamma_k} \Rightarrow \gamma_k = 0$$

which contradicts our supposition that no $\gamma_k = 0$. Thus $(G\phi,\phi) > 0$ and we must have

$$\mathrm{Re}(\nu) < 0$$

as claimed.

(Note that we made no assumption on the size of ν)

Thus, if we could assert that every solution of our closed loop system, e.g. (22.5), is a linear combination of solutions of the above form it would be clear that every such solution x(t) satisfies

$$\lim_{t\to\infty} \| x(t) \| = 0.$$

It all sounds very easy, but in order to follow this route we will have to introduce quite a bit more in the way of mathematical machinery.

23. A Perturbation Result in Hilbert Space.

We consider the differential equation in H:

$$\ddot{x} + \varepsilon G\dot{x} + Ax = 0, \quad Gx = (x,g)g, \quad g = \sum_{k=1}^{\infty} \gamma_k\phi_k, \quad \gamma_k \neq 0, \quad k = 1,2,\ldots \tag{23.1}$$

For convenience now we put

$$w_k = \begin{cases} \lambda_k^{1/2} & k = 1,2,\ldots \\ -\lambda_{-k}^{1/2} & k = -1,-2,\ldots \end{cases} \qquad \begin{aligned} \phi_k &= \phi_{-k}, \; k = 1,2,\ldots \\ \lambda_k &= \lambda_{-k}, \; k = 1,2,\ldots \\ \gamma_k &= \gamma_{-k}, \; k = 1,2,\ldots \end{aligned} \tag{23.2}$$

Now put $x = A^{-1/2}x^1, \; \dot{x} = x^2$ (23.3)

where $A^{-1/2}$ is the inverse of the unique positive square root of A, and we obtain

$$\frac{d}{dt}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -\varepsilon G \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \widehat{A}(\varepsilon) \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \tag{23.4}$$

which is a first order equation in $H \oplus H$. For a given initial vector $\begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$ the solution of the above first order equation is

$$\exp\left[\begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -\varepsilon G \end{pmatrix} t\right] \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$$

It is clear that the properties of such solutions will depend upon the nature of the spectrum of $\tilde{A}(\varepsilon)$ and the associated eigenvectors, if there are any.

Suppose now we could prove that $\tilde{A}(\varepsilon)$ has eigenvalues $\nu_k(\varepsilon)$, $k = \pm 1, \pm 2, \ldots$ and associated eigenvectors $\psi_k(\varepsilon)$, $k = \pm 1, \pm 2, \ldots$ with the property that the $\psi_k(\varepsilon)$ form a Riesz basis in $H + H$, i.e. each vector $Z \varepsilon H \oplus H$ has an expansion

$$Z = \sum_{k=1}^{\infty} [\zeta_k(\varepsilon)\psi_k(\varepsilon) + \zeta_{-k}(\varepsilon)\psi_{-k}(\varepsilon)] \tag{23.5}$$

and there are positive numbers M and m, independent of Z, such that

$$m \sum_{k=1}^{\infty} [\,|\zeta_k|^2 + |\zeta_{-k}|^2] \leq \|Z\|^2 \leq M \sum_{k=1}^{\infty} [\,|\zeta_k|^2 + |\zeta_{-k}|^2] \tag{23.6}$$

If this is true, expand $\begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$ in terms of the ψ_k

$$\begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} = \sum_{k=1}^{\infty} \xi_k\psi_k + \xi_{-k}\psi_{-k} \tag{23.7}$$

and we have $\exp(\tilde{A}(\varepsilon)t)\begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} =$

$$\sum_{k=1}^{\infty} \xi_k \exp(\nu_k t)\psi_k + \xi_{-k} \exp(\nu_{-k}t)\psi_{-k} \tag{23.8}$$

and we claim that this vector-valued function approaches zero as $t\to\infty$. For, let $\rho > 0$ be given. Since

$$\sum_{k=1}^{\infty} [\,|\xi_k|^2 + |\xi_{-k}|^2] \leq \frac{1}{m} \left\|\begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}\right\|^2$$

the sum on the left converges and we can find k_ρ such that

$$\sum_{k=k_\rho}^{\infty} [\,|\xi_k|^2 + |\xi_{-k}|^2] < \frac{\rho^2}{2M}$$

Then, since all ν_k have negative real parts, $|\exp(\nu_k t)| < 1$ for all k and we have

$$||\exp(\tilde{A}(\varepsilon))\begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}||^2 \le M \sum_{k=1}^{\infty} \left(|\xi_k \exp(\nu_k t)|^2 + |\xi_{-k} \exp(-\nu_k t)|^2 \right)$$

$$\le M \sum_{k=1}^{k_\rho - 1} (|\xi_k \exp(\nu_k t)|^2 + |\xi_{-k} \exp(\nu_{-k} t)|^2) + \frac{\rho^2}{2}$$

Now let $\nu_\rho = \max_{k=\pm 1,\ldots,\pm k_\rho - 1} \{\mathrm{Re}(\nu_k)\} < 0$

and we have

$$M \sum_{k=1}^{k_\rho - 1} (|\xi_k \exp(\nu_k t)|^2 + |\xi_{-k} \exp(\nu_{-k} t)|^2)$$

$$\le M \exp(2\nu_\rho t) \sum_{k=1}^{k_\rho - 1} (|\xi_k|^2 + |\xi_{-k}|^2)$$

Taking t_ρ so large that $\exp(2\nu_\rho t_\rho) \le \dfrac{\rho^2}{2 \sum_{k=1}^{k_\rho - 1} (|\xi_k|^2 + |\xi_{-k}|^2)}$

we see that for all $t \ge t_\rho$

$||\exp(\tilde{A}(\varepsilon)t)|| \le \rho$. Hence, $\lim_{t\to\infty} ||\exp(\tilde{A}(\varepsilon)t)|| = 0$.

Thus it remains only to show that $\tilde{A}(\varepsilon)$ possesses a Riesz basis of eigenvectors $\nu_k(\varepsilon)$. This is not particularly easy but we will be able to indicate the main ideas of the proof.

<u>Theorem</u>. If

(i) $\exists \hat{M} > 0$ such that $0 < |\gamma_k| \le \hat{M} \dfrac{1}{w_k}$, $k = 1,2,\ldots$

(ii) $\exists \tilde{M} > 0$ such that $\dfrac{k}{\lambda_k - \lambda_{k-1}} \le \tilde{M}$, $k = 2,3,\ldots$

(iii) $\varepsilon > 0$ is sufficiently small

then $\tilde{A}(\varepsilon)$ has eigenvalues

$$\nu_k(\varepsilon) = i\omega_k - \frac{\varepsilon}{2} |\gamma_k|^2 + O(\varepsilon^2 \frac{1}{|\omega_k|^2}), \quad k = \pm 1, \pm 2, \ldots$$

and corresponding eigenvectors $\psi_k(\varepsilon)$ forming a Riesz basis in $H \oplus H$.

Remark. Condition (ii) above implies that $\omega_k \geq M_0 k$ for some $M_0 > 0$. It also implies a separation between ω_{k-1} and ω_k which remains bounded away from zero for all k = 2, 3, ... It would seem that the theorem should be true under more general conditions but as yet we have no proof of this.

Let ν be a complex number and consider the identity

$$\begin{pmatrix} -\nu I & A^{1/2} \\ -A^{1/2} & -\nu I-\varepsilon G \end{pmatrix} \begin{pmatrix} A^{1/2}\phi \\ \nu\phi \end{pmatrix} = \begin{pmatrix} 0 \\ -(A + \varepsilon\nu G + \nu^2 I)\phi \end{pmatrix}$$

valid for $\phi \varepsilon \mathrm{dom}\, A$. If there exists a non-zero $\phi\varepsilon\Delta$ such that

$$(A + \varepsilon\nu G + \nu^2 I)\phi = 0$$

then ν is an eigenvalue of $\tilde{A}(\varepsilon)$ and

$$\psi = c \begin{pmatrix} A^{1/2}\phi \\ \nu\phi \end{pmatrix}, \text{ c arbitrary scalar,}$$

is an associated eigenvector. If we take

$$\nu = iw_k, \quad c = \frac{1}{\sqrt{2}\,|w_k|}, \quad \phi = \phi_k \qquad k = \pm 1, \pm 2, \ldots$$

we obtain the orthonormal eigenvectors of the "unperturbed" operator $\tilde{A}(0)$, namely

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \phi_k \\ i\phi_k \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_k \\ -i\phi_k \end{pmatrix} \quad k = 1,2,\ldots$$

The idea is to show that if $\varepsilon > 0$ is sufficiently small the eigenvectors of $\tilde{A}(\varepsilon)$ remain close enough to those of $\tilde{A}(0)$ so that there will exist a bounded and invertible linear transformation taking the eigenvectors of $\tilde{A}(0)$ into those of $\tilde{A}(\varepsilon)$. This is accomplished in the following way. For k = ± 1, ± 2, we put

$$\nu_k(\varepsilon) = i\omega_k - \frac{\varepsilon}{2}|\gamma_k|^2 + \varepsilon\mu_k(\varepsilon)$$

$$\phi_k(\varepsilon) = \phi_k - \varepsilon i\omega_k(A_k - \lambda E_k)^{-1}E_kG\phi_k + \varepsilon(A_k - \lambda_k E_k)^{-1}\phi_k(\varepsilon) \qquad (23.9)$$

w.l.o.g

$$(\theta_k(\varepsilon), \phi_k) = 0$$

(Note: $A_k = AE_k = E_kA$)

where E_k is the orthogonal projection from H onto the subspace of H.

spanned by $\phi_1, \phi_2, \ldots, \phi_{k-1}, \phi_{k+1}, \phi_{k+2}, \ldots$.

Now consider the equation satisfied by $\nu_k(\varepsilon)$ and $\phi_k(\varepsilon)$

$$(A + \varepsilon\nu_k(\varepsilon)G + \nu_k(\varepsilon)^2 I)\phi_k(\varepsilon) = 0 \tag{23.10}$$

Abbreviate this by writing $x_k(\varepsilon) = 0$. Such an equation is true if and only if

$$E_k x_k(\varepsilon) = 0$$

$$(x_k(\varepsilon), \phi_k) = 0.$$

Thus we must have

$$E_k(A + \varepsilon\nu_k(\varepsilon)G + \nu_k(\varepsilon)^2 I)\phi_k(\varepsilon) = 0$$

$$((A + \varepsilon\nu_k(\varepsilon)G + \nu_k(\varepsilon)^2 I)\phi_k(\varepsilon), \phi_k) = 0 \tag{23.11}$$

Now substitute the expressions (23.9) in place of $\nu_k(\varepsilon)$ and $\phi_k(\varepsilon)$ in (23.11). So doing we obtain a pair of very complicated equations which we will not reproduce here. Suffice it to say that they have the form

$$\mu_k(\varepsilon)[1 + \varepsilon F_k(\varepsilon, \mu_k(\varepsilon), \theta_k(\varepsilon))] = \varepsilon G_k(\varepsilon, \theta_k(\varepsilon))$$

$$\theta_k(\varepsilon) + \varepsilon H_k(\varepsilon, \mu_k(\varepsilon), \theta_k(\varepsilon)) = \varepsilon J_k(\varepsilon, \mu_k(\varepsilon)) \tag{23.12}$$

The first equation is a scalar equation. The second is a vector equation in the space $E_k H$. Together they may be considered as a vector equation in H itself. To this equation we apply the implicit function theorem as it is stated for equations in Hilbert spaces (see for instance, the book by Dieudonné). Using the assumptions of the theorem and this implicit function theorem we can show that

$$|\mu_k(\varepsilon)| \le K_1 |\varepsilon| \frac{1}{|\omega_k|^2}$$

$$||\theta_k(\varepsilon)|| \le K_2 (\varepsilon) \tag{23.13}$$

uniformly for all $k = \pm 1, \pm 2, \pm 3, \ldots$ and $|\varepsilon| \le \varepsilon_0$ for some $\varepsilon_0 > 0$. Then, going back to (23.9) we have

$$\nu_k(\varepsilon) = i\omega_k - \frac{\varepsilon}{2} |\gamma_k|^2 + 0\left(\varepsilon \frac{1}{|\omega_k|^2}\right)$$

$$\phi_k(\varepsilon) = \phi_k + 0\left(\varepsilon \frac{1}{|\omega_k|}\right) \tag{23.14}$$

Because $\sum_{k=1}^{\infty} \frac{1}{|\omega_k|^2} < \infty$, a theorem of Paley and Wiener applies to show that the $\theta_k(\varepsilon)$ form a Riesz basis in H. Then

$$\frac{1}{\sqrt{2}\ |\omega_k|} \begin{pmatrix} A^{1/2}\ \phi_k(\varepsilon) \\ \nu_k(\varepsilon)\phi_k(\varepsilon) \end{pmatrix} \qquad k = \pm 1, \pm 2, \ldots$$

can easily be shown to form a Riesz basis of eigenvectors of $\tilde{A}(\varepsilon)$ in $H \oplus H$ and we have the desired result.

CHAPTER VI.

OPTIMAL CONTROL OF A DISTRIBUTED OSCILLATING SYSTEM WITH RESPECT TO A QUADRATIC COST CRITERION

24. Formulation of the Optimization Problem.

Our basic control system is described by a second order ordinary differential equation in a Hilbert space H:

$$\frac{d^2y}{dt^2} + Ly = \hat{B}u(t) \tag{24.1}$$

L is a self-adjoint positive operator (in general unbounded) defined on a dense domain $\Delta \subseteq H$ with eigenvalues $\lambda_i \geq \lambda_0 > 0$, $i = 1,2,\ldots$ and associated eigenvectors $\phi_1, \phi_2, \ldots$. Whether or not there are multiple eigenvalues is unimportant as long as that multiplicity remains finite. The control u(t) will be taken as an element of m-dimensional Euclidian space.

A system state consists of a pair $w = \begin{pmatrix} y \\ \frac{dy}{dt} \end{pmatrix}$ in the basic state space $H \oplus H$. We will only be concerned with states possessing finite energy

$$E(w) = \frac{1}{2}\left(||L^{1/2}y||^2 + ||\frac{dy}{dt}||^2 \right) \tag{24.2}$$

If

$$y = \sum_{k=1}^{\infty} \eta_k \phi_k, \quad \frac{dy}{dt} = \sum_{k=1}^{\infty} \zeta_k \phi_k \tag{24.3}$$

then

$$E(w) = \frac{1}{2}\left[\sum_{k=1}^{\infty} \lambda_k (\eta_k)^2 + (\zeta_k)^2 \right] \tag{24.4}$$

We will let $W_E \subseteq H \oplus H$ denote all finite energy states. With the inner product

$$(w,\hat{w})_E + \left(\begin{pmatrix} y \\ \frac{dy}{dt} \end{pmatrix}, \begin{pmatrix} \hat{y} \\ \frac{d\hat{y}}{dt} \end{pmatrix} \right)_E = (L^{1/2}y, L^{1/2}\hat{y}) + \left(\frac{dy}{dt}, \frac{d\hat{y}}{dt} \right) \tag{24.5}$$

(the usual inner product in H)

W_E becomes a Hilbert space with norm

$$||w||_E = \sqrt{(w,w)} = \sqrt{2E(w)} \tag{24.6}$$

It can be shown quite easily that if the initial state w(0) lies in W_E and if we select a control u(t) such that

$$\int_0^T ||u(t)||^2 dt < \infty$$

for all T > 0 then the response w(t) to u(t) via the differential equation $\frac{dy}{dt} + Ly = \hat{B}u(t)$ (w(t) may be a generalized solution of this equation) will lie in W_E for all $t \geq 0$ and will have uniformly bounded energy on compact intervals.

It therefore makes sense to assign to each initial state w(0) and each control u(t) defined on $0 \leq t \leq T$, a quadratic cost

$$C(w(0),u) = \int_0^T [||w(s)||_E^2 + (u(s),\mathcal{U}u(s)]ds + ||w(T)||_E^2 \qquad (24.7)$$

where $\mathcal{U}$ is a symmetric positive definite mxm matrix. More generally, for $0 \leq t \leq T$ we define

$$C(w(t),u) = \int_t^T [||w(s)||_E^2 + (u(s),\mathcal{U}u(s))]ds + ||w(T)||_E^2 \qquad (24.8)$$

Our basic control objective will be to choose a square integrable control u*(t) so that

$$C(w(0),u^*) \leq C(w(0),u) \qquad (24.9)$$

for all other square integrable controls u.

We will approach this infinite dimensional optimization problem through a series of finite dimensional problems. We will let W_r denote the 2r-dimensional subspace of $H \oplus H$ spanned by

$$\begin{pmatrix} \phi_1 \\ 0 \end{pmatrix} \begin{pmatrix} \phi_2 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} \phi_r \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \phi_r \end{pmatrix}$$

and we will let E_r denote the orthogonal projection from $H \oplus H$ onto W_r. It should be noted that

$$W_r \subset W_E \subset H \qquad \text{for all positive integers r.}$$

Our differential equation (24.1) can be written in first order form:

$$\frac{dw}{dt} = Aw + Bu(t), \quad A = \begin{pmatrix} 0 & I \\ -L & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ \hat{B} \end{pmatrix} \qquad (24.10)$$

We project this entire system onto W_r

$$E_r \frac{dw}{dt} = E_r Aw + E_r Bu(t)$$

We let $w_r = E_r w$, $B_r = E_r B$. Because W_r is spanned by

$$\begin{pmatrix} \phi_1 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} \phi_r \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \phi_r \end{pmatrix}$$

we readily see that $E_r Aw = E_r^2 Aw = E_r AE_r w = A_r w_r$. Thus the projected system is

$$\frac{dw_r}{dt} = A_r w_r + B_r u(t) \tag{24.11}$$

We define a cost for this finite dimensional system by

$$C_r(w_r(t),u) = \int_t^T [||w_r(s)||_E^2 + (u(s), Uu(s))]ds + ||w_r(T)||_E^2$$

$$= \int_t^T [(w_r(s), V_r w_r(s)) + (u(s),Uu(s))]ds + (w_r(T),V_r w_r(T)),$$

$$V_r = \begin{pmatrix} L_r & 0 \\ 0 & E_r \end{pmatrix} \tag{24.12}$$

commensurate with the cost $C(w(t),u)$ for the original infinite dimensional system, and consider the problem of minimizing $C_r(w_r(0),u)$, i.e. of finding a square integrable $u_r^*(t)$ such that

$$C_r(w_r(0),u_r^*) \leq C_r(w_r(0),u) \tag{24.13}$$

for all square integrable u.

25. Review of the Optimization Problem in the Finite-Dimensional Case.

Now let us refresh our memory concerning the solution of this finite dimensional optimization problem. This problem was first treated by Kalman and has been worked over several times since, notably by W. M. Wonham and D. L. Lukes.

We consider a matrix differential equation

$$-\frac{dQ_r}{dt} = A_r^T Q_r + Q_r A + V_r - Q_r^T B_r U^{-1} B_r^* Q_r \tag{25.1}$$

with terminal condition $Q_r(T) = V_r$. We let u(t) be any square integrable control on [0,T] and we compute

$$C_r(w_r(0),u) - (w_r(0),Q_r(0)w_r(0)) = \int_0^T [(w_r(s),V_r w_r(s))$$

$$+ (u)s),Uu(s))]ds + (w_r(T),V_r w_r(T)) - (w_r(0),Q_r(0)w_r(0))$$

$$= \int_0^T [(w_r(s),V_r w_r(s)) + (u(s),Uu(s))]ds$$

$$+ \int_0^T \frac{d}{ds}(w_r(s), Q_r(s)w_r(s))ds = \int_0^T [(w_r(s), V_r w_r(s)) + (u(s), Uu(s))$$

$$+ (A_r w_r(s) + B_r u(s), Q_r(s)w_r(s)) + (w_r(s), Q_r(s)(A_r w_r(s) + B_r u(s)))$$

$$+ (w_r(s), [-A_r^T Q_r(s) - Q_r(s)A_r - V_r + Q_r(s)B_r U^{-1} B_r^* Q_r(s)]w_r(s))]ds$$

$$= \int_0^T [(u(s), Uu(s)) + (B_r u(s), Q_r(s)w_r(s)) + (w_r(s), Q_r(s)B_r u(s))$$

$$+ (w_r(s), [Q_r(s)B_r U^{-1} B_r^* Q_r(s)]w_r(s))]ds$$

$$= \int_0^T [(Uu(s) + B_r^* Q_r(s)), U^{-1}(Uu(s) + B_r^* Q_r(s))]ds$$

Thus

$$C_r(w_r(0), u) - (w_r(0), Q_r(0)w_r(0))$$

$$= \int_0^T [(Uu(s) + B_r^* Q_r(s)), U^{-1}(Uu(s) + B_r^* Q_r(s))]ds \geq 0$$

for any choice of u. Moreover, for the choice

$$u_r^*(t) = -U^{-1} B_r^* Q_r(s) \tag{25.2}$$

we have

$$C_r(w_r(0), u) - (w_r(0), Q_r(0)w_r(0)) = 0 \tag{25.3}$$

Thus we conclude (i) the optimal control for the finite dimensional problem is generated by the linear feedback law (equation (25.2)) and this is the unique solution of that problem; (ii) The optimal cost for the finite dimensional problem is $(w_r(0), Q_r(0)w_r(0))$ where $Q_r(t)$ satisfies the Kalman-Riccati differential equation (25.1); (iii) At any intermediate time $0 < t < T$, u_r^* is the unique control minimizing $C_r(w_r(t), u)$ and the optimal cost is $(w_r(t), Q_r(t)w_r(t))$.

26. Generalization to the Infinite-Dimensional Problem.

Now the idea is to increase the dimension, letting r tend to infinity, and show that the controls u_r^* converge to the optimal control u^* for the original infinite dimensional problem.

Our first step will be to establish a certain monotonicity. We claim that the following relationship holds:

Whenever $w_r(t) = E_r w_{r+1}(t)$, we have

$$C_r(w_r(t),u_r^*) \leq C_{r+1}(w_{r+1}(t),u_{r+1}^*), \quad 0 \leq t \leq T \tag{26.1}$$

The proof is quite easy. By optimality of u_r^* we have

$$C_r(w_r(t),u_r^*) \leq C_r(w_r(t),u_{r+1}^*)$$

But $C_r(w_r(t),u_{r+1}^*) = \int_t^T [\tilde{w}_r(s),V_r\tilde{w}_r(s)) + (u_{r+1}^*(s),Uu_{r+1}^*(s))]ds$ $+ (\tilde{w}_r(T),V_r\tilde{w}_r(T))$ where $\tilde{w}_r(s)$ solves $\frac{d\tilde{w}_r}{ds} = A_r\tilde{w}_r + B_r u_{r+1}^*(s)$ with $\tilde{w}_r(t) = w_r(t)$. Thus $\tilde{w}_r(s) = E_r w_{r+1}(s)$ where $w_{r+1}(s)$ satisfies $\frac{dw_{r+1}}{ds} = A_{r+1}w_{r+1} + B_{r+1}u_{r+1}^*(s)$. Then

$$C_{r+1}(w_{r+1}(t),u_{r+1}^*) - C_r(w_r(t),u_{r+1}^*)$$

$$= \int_0^T [(w_{r+1}(s),V_{r+1}w_{r+1}(s)) - (\tilde{w}_r(s),V_r\tilde{w}_r(s))ds$$

$$+ (w_{r+1}(T),V_{r+1}w_{r+1}(T)) - (\tilde{w}_r(T),V_r\tilde{w}_r(T))$$

$$= \int_0^T [(w_{r+1}(s),V_{r+1}w_{r+1}(s)) - (w_{r+1}(s),E_rV_{r+1}E_rw_{r+1}(s))]ds$$

$$+ (w_{r+1}(T),V_{r+1}w_{r+1}(T)) - (w_{r+1}(T),E_rV_{r+1}E_rw_{r+1}(T)) \geq 0$$

since E_r commutes with V_{r+1}.

Thus, what we have actually shown, since w(t) could be any element in W_E, is that with $w \epsilon W_E$, $w_r = E_r w$, $w_{r+1} = E_{r+1}w$,

$$(w_r,Q_r(t)w_r) \leq (w_{r+1},Q_{r+1}(t)w_{r+1}) \tag{26.2}$$

for each positive integer r and each t, $0 \leq t \leq T$.

Now let $w \epsilon W$ and let $\tilde{w}(s)$ be the solution of $\frac{d\tilde{w}}{ds} = A\tilde{w} + Bu$ for $u(s) \equiv 0$, $t \leq s \leq T$,

i.e. $$\frac{d\tilde{w}}{ds} = A\tilde{w}, \quad \tilde{w}(t) = w. \tag{26.3}$$

In this case energy is conserved and we have

$$||\tilde{w}(s)||_E = ||\tilde{w}(t)||_E = ||w||_E, \quad t \leq s \leq T.$$

Consequently,

$$C(\tilde{w}(t),0) = \int_t^T ||\tilde{w}(s)||_E^2 ds + ||\tilde{w}(T)||_E^2$$

$$= ||\tilde{w}(t)||_E^2 \left(\int_t^T 1\, ds + 1\right) = (1 + (T-t))||w||_E^2$$

$$\leq (1 + T)||w||_E^2 \quad \text{for } 0 \leq t \leq T.$$

By more or less the same arguments given previously we see that for the r-dimensional system

$$C_r(w_r,0) \leq C(w,0) \tag{26.4}$$

so that $(w_r, Q_r(t)w_r) \leq (1 + T)||w||_E^2, \quad w_r \doteq E_r w.$

Thus, for any $t\varepsilon[0,T]$ and for any $w\varepsilon W_E$ we have

$$(w_r, Q_r(t)w_r) \leq (w_{r+1}, Q_{r+1}w_{r+1}) \leq (1 + T)||w||_E^2 \tag{26.5}$$

whence, with V_r as defined previously and

$$V = \begin{pmatrix} L & 0 \\ 0 & I \end{pmatrix} \tag{26.6}$$

we have

$$(V_r^{1/2}w_r,\ V_r^{-1/2}Q_r(t)V_r^{-1/2}\ V_r^{1/2}\ w_r)$$

$$\leq (V_{r+1}^{1/2}\ w_{r+1},\ V_{r+1}^{-1/2}\ Q_{r+1}(t)V_{r+1}^{-1/2}\ V_{r+1}^{1/2}\ w_{r+1})$$

$$\leq (1 + T)||V^{1/2}\ w||^2.$$

We extend $V_r^{-1/2}Q_r(t)V_r^{-1/2}$ to an operator $\hat{Q}_r(t)$ on $H \oplus H$ by setting

$$\hat{Q}_r(t)Z \doteq V_r^{-1/2}\ Q_r(t)V_r^{-1/2}\ Z_r \tag{26.7}$$

where $Z = Z_r + \hat{Z}_r$, $Z_r \varepsilon W_r$, $\hat{Z}_r \varepsilon W_r^{\perp}$.

Further, we note that $V^{1/2}$ and $V_r^{1/2}$ agree on W_r and that $V^{1/2}$ maps W_E onto $H + H$ with

$$||w||_E^2 \doteq ||V^{1/2}w||^2 = ||Z||^2, \text{ where } Z = V^{1/2}w.$$

From this we conclude that for any $Z\varepsilon H + H$ we have

$$(Z,\hat{Q}_r(t)Z) \leq (Z,\hat{Q}_{r+1}(t)Z) \leq (1 + T)||Z||^2 \tag{26.8}$$

so that the $\hat{Q}_r(t)$ form a sequence of self-adjoint positive semi-definite operators which are monotone increasing and bounded above. We then cite a familiar theorem of functional analysis to the effect that there is a self-adjoint positive semi-definite operator $\hat{Q}(t)$ defined on $H \oplus H$ with the property that for each $Z \; H \oplus H$

$$\lim_{r\to\infty} \hat{Q}_r(t)Z = \hat{Q}(t)Z \tag{26.9}$$

Moreover $(Z,\hat{Q}(t)Z) \le (1 + T)||Z||^2$, $Z\varepsilon H \oplus H$, $0 \le t \le T$. Put

$$Q(t) = V^{1/2}\hat{Q}(t)V^{1/2} \tag{26.10}$$

and we have $(w,Q(t)w) \le (1 + T)||w||_E^2$ for all $w\varepsilon W_E$. (26.11)

We now claim: the control $u^*(t)$ which minimizes the cost $C(w(0),u)$ is uniquely determined by the feedback law

$$u^*(t) = -U^{-1}B^*Q(t)w(t) \tag{26.12}$$

and the optimal cost is $C(w(0),u^*) = (w(0),Q(t)w(0))$ for each initial state $w(0)$ in W_E.

To prove this we first note that

$$B^*V^{-1/2} = (0,\hat{B}^*)\begin{pmatrix} L^{1/2} & 0 \\ 0 & I \end{pmatrix} = (0,\hat{B}^*) = B^*$$

and, in the same way, $B_r^*V^{-1/2} = B_r^*V_r^{-1/2} = B_r^*$. Consequently

$$U^{-1}B_r^*Q_r(t)V_r^{-1/2} = U^{-1}B_r^*V_r^{-1/2}Q_r(t)V_r^{-1/2} = U^{-1}B_r^*\hat{Q}_r(t).$$

Let $w(0)$ be chosen as a finite energy state and let $w_r(0) = E_r w(0)$. We note from earlier work that for any square integrable u

$$C_r(w_r(0),u) = \int_0^T [(w_r(t),V_r w_r(t)) + (u(t),Uu(t))]dt$$

$$= (w_r(0),Q_r(0)w_r(0))$$

$$+ \int_0^T ((Uu(t) + B_r^*Q_r(t)w_r(t)),\ U^{-1}(Uu(t) + B_r^*Q_r(t)w_r(t)))dt$$

We let $r \to \infty$. The term $(w_r(t),V_r w_r(t))$ tends to $||w(t)||_E^2$ for each t. The term $(w_r(0),Q_r(0)w_r(0))$ can be rewritten as

$$(V_r^{1/2} w_r(0),\ V_r^{-1/2} Q_r(0) V_r^{-1/2} V_r^{1/2} w_r(0))$$

$$= (Z_r(0),\ V_r^{-1/2} Q_r(0) V_r^{-1/2} Z_r(0)) \text{ with } Z_r(0) = V_r^{1/2} w_r(0).$$

As r tends to ∞ this quantity approaches $(Z(0), \hat{Q}(0)Z(0)) = (w(0), Q(0)w(0))$.

Consider $B_r^* Q_r(t) w_r(t) = B_r^* V_r^{-1/2} Q_r(t) V_r^{-1/2} Z_r(t)$ which converges for each fixed t to $B^*\hat{Q}(t)Z(t) = B^*Q(t)w(t)$.

Using the above-noted pointwise convergence results together with the Lebesque dominated convergence theorem we have

$$C(w(0),u) = (w(0),Q(0)w(0))$$

$$+ \int_0^T ((Uu(t) + B^*Q(t)w(t)),\ U^{-1}(Uu(t) + B^*Q(t)w(t)))dt$$

from which it is clear that the optimal control is

$$\begin{aligned} u^*(t) &= -U^{-1}B^*Q(t)w(t) \\ &= \lim_{r\to\infty} -U^{-1}B_r^*Q_r(t)w_r(t) = \lim_{r\to\infty} u_r^*(t) \end{aligned}$$

and our proof is complete.

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